January 2014

The Effectiveness Of Manipulatives In A High School Algebra II Class

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THE EFFECTIVENESS OF

MANIPULATIVES IN A HIGH SCHOOL

ALGEBRA II CLASS

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Submitted to the Faculty of the Graduate School of
Eastern Kentucky University
in partial fulfillment of the requirements
for the degree of
MASTER OF ARTS
May, 2014

ACKNOWLEDGEMENTS

I would like to thank each member of the committee, Dr. Daniel Mundfrom, Dr. Robert Thomas, Dr. Lisa Kay, Dr. Rachel Bishop-Ross, and Dr. Ken Dutch for taking the time to review my research and offer constructive criticism. Special thanks go to Dr. Mundfrom for having the patience to walk me through the statistical analysis and answer my many questions and to Dr. Kay for offering guidance and help in coordinating many different aspects of this process since last fall. I would also like to thank Dr. Cheryll Crowe for assisting me as I began this research. Additional thanks go to Ashley Isaacs for partnering with me in this research. She generously agreed to complete the necessary training, teach her classes with the two instructional methods, share assessment data, and offer professional insights on the teaching and learning process.

Abstract

This study explores the use of manipulatives in high school Algebra II. The effectiveness of the Concrete-Representational-Abstract (CRA) Model is compared to explicit instruction. The participants in this study are students from six high school Algebra II classes –two honors classes, and four standard classes. One honors class and two standard classes were randomly selected as the treatment groups receiving CRA instruction. The other three classes learned through abstract explicit instruction. Each class learned two new mathematical concepts, domain and range of quadratic functions and transformations of quadratic functions, through the selected method of instruction. At the end of instruction, student comprehension, accuracy, and retention of the mathematical content were analyzed through the use of pre-, post-, and follow-up tests. The results of the treatment and non-treatment groups will be used to determine if the use of manipulatives is beneficial for higher level high school algebra classes.

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CHAPTER I

INTRODUCTION

Definition of Manipulatives

Manipulatives are tools that make learning new mathematical skills a hands-on process. Swan and Marshall (2010) have developed the following definition of *manipulative*: "A mathematics manipulative material is an object that can be handled by an individual in a sensory manner during which conscious and unconscious mathematical thinking will be fostered" (p 14). This definition includes materials designed specifically for use in the math classroom such as Base Ten Blocks and Algebra Tiles, but it is also flexible enough to incorporate creative mathematical uses of common objects such as popsicle sticks, beads, and dice. The most critical component to consider when selecting manipulatives is the ease with which students will be able to associate the tools with mathematical concepts and transfer their understanding of the manipulatives to abstract thought (Ojose, 2008).

Teaching Philosophy behind Manipulatives

The concept of manipulatives dates back to Piaget's theory on the stages of cognitive development (1977). Piaget believed that students progress through four stages of development beginning with the sensorimotor stage in infancy, the preoperational stage in early childhood, the concrete operations phase, and finally the formal operations stage. Ojose (2008) summarizes Piaget's theory as it relates to mathematics instruction; he describes how students need concrete experiences to "lay the foundation for more advanced mathematical thinking" as they move from the concrete operational stage to the formal operations stage (p. 28). In addition to helping students advance to higher levels of cognitive development, manipulatives can help students who already possess the ability to think abstractly. Modeling a mathematical concept with manipulatives leads students to think about the mathematics in a different way and attain a higher level of understanding (Cooper, 2012). Teaching with manipulatives incorporates a multi-representational approach to mathematics which meets the needs of students with a variety of learning styles (McNeil & Jarvin, 2007).

CRA Method

While there is evidence that suggests using manipulatives is an effective strategy across grade levels and developmental levels, simply giving students manipulatives to work with does not guarantee a concept will be understood (Maccini & Gagnon, 2000). The use of concrete learning tools must be combined with carefully planned instruction and well-executed transitions. The recommended model for incorporating manipulatives into mathematics instruction is called the concrete-representational-abstract (CRA) approach (Sousa, 2008).

The CRA method begins by introducing students to a new topic using hands-on materials or manipulatives. The concrete objects engage kinesthetic learners and lead students to develop a conceptual understanding of how the different components of an algebraic expression or equation can be combined. Manipulatives give meaning to numbers and symbols. Since manipulatives will not always be available to students in problem-solving situations, students must learn to progress beyond the concrete stage.

The next phase in the CRA model is the representational phase. The representational phase simply means that students can draw pictures of the manipulatives to represent the same algebraic situations when the manipulatives are not available. Finally, students reach the abstract level. In this phase, students learn how symbols can be substituted for the manipulatives to more efficiently solve the numerical problem. In traditional instruction, the abstract level is where most algebra teachers begin (Witzel, 2005). For the CRA method to be effective, students must clearly understand the connection between the real objects and the symbolic manipulation of numbers (Sousa, 2008, p187).

Are Manipulatives Beneficial for Secondary Students?

The majority of the research related to manipulatives focuses on elementary classrooms; however, studies have emerged that explore the use of manipulatives in middle grades (Witzel & Allsopp, 2007). A multi-representational approach can help students make the pivotal transition from arithmetic to abstract thought that occurs in Pre-Algebra and Algebra I courses (Witzel, 2005). Middle school students are only beginning to develop the cognitive ability to engage in abstract reasoning (Sousa, 2008). Therefore, when abstract algebra skills are introduced, middle school students need a link between the tangible and the abstract. Manipulatives provide that connection. Manipulatives can also serve as a motivational tool to engage students in learning. In addition, there are readily available materials and lesson plans that incorporate manipulatives into Pre-Algebra and Algebra I content. The question remains as to whether the same benefits apply to high school mathematics such as Algebra II.

In the experience of the author, manipulative materials and lesson plans are less commonly available in higher level mathematics courses than in the lower grades. Maccini and Hughes (2000) state, “Little is known about strategy instruction and the use of manipulatives on the performance of students with [learning disabilities] at the secondary level with more complex mathematics tasks” (p. 11). By the time students reach Algebra II, they have experience with abstract algebraic symbols and variables from Algebra I. High school Algebra II students also have a more developed frontal lobe of the brain, which makes them better able to engage in abstract reasoning than students who are in the middle grades (Sousa, 2008). Furthermore, it is more difficult to connect the increasingly complex concepts learned in Algebra II to the same tactile models that work well with basic linear algebra skills. Despite these arguments against the use of concrete models in high school, it may still be beneficial to allow older students the opportunity to work through a multi-representational model when introduced to new skills.

One justification as to why manipulatives may be appropriate in Algebra II is that more low-achieving students are being required to take this course. Beginning in the 2011-2012 school year, all high school graduates in Kentucky are required to pass an Algebra II mathematics course (Kentucky Legislature, 2012). At the same time that the course requirement has increased, the Algebra II curriculum has become more rigorous as demanded by the Common Core standards (Common Core, 2012). Furthermore, all students are state mandated to take an end of course assessment over the Algebra II curriculum. The increase in mathematical requirements for all students is intended to prepare high school graduates for a future in higher education and the workplace, but raising the rigor and number of required courses poses a challenge for low-achieving students who have struggled with mathematics since the early grades. Manipulatives may be tools that make the rigorous and abstract content of the Algebra II curriculum accessible to low-achievers.

Guiding Questions

This research seeks to uncover the most appropriate uses of manipulatives and to apply those tools to higher level algebra topics. The central question this research seeks to answer is, "Can manipulatives improve the mathematical understanding of students studying the Algebra II curriculum?"

It is likely that the use of manipulatives will look different in the high school classroom than in the lower grades since the high school curriculum involves more complex procedures. However, it is important to remain focused on the goal of manipulatives: to build abstract understanding of mathematical concepts by first exploring relationships with physical objects. In this study, lessons were developed that emphasize the connection between underlying mathematical concepts and the manipulatives, then the effectiveness of the CRA model was compared to traditional explicit instruction. Student comprehension, accuracy, and memory were analyzed after students learned from each of the two methods of instruction.

CHAPTER II

REVIEW OF THE LITERATURE

Advantages and Disadvantages of Manipulatives

Cooper (2012) summarizes a literature review written by Suydam and Higgins in 1977 that reported the results of twenty-three studies comparing achievement of students who learned using concrete materials to students who learned without them. The primary grades study yielded mixed results: Eleven studies reported that manipulatives improved performance, tw studies reported decreased performance, and 10 studies indicated there was no significant difference in performance. Conflicting research on the effectiveness of manipulatives indicates that concrete learning is not the answer for every student in all situations. The challenge for the mathematics teacher is to evaluate the skills and learning styles of the class and determine if manipulatives can engage students in the curriculum in a way that deepens their understanding.

An abundance of evidence can be found as to the benefits of teaching through a multi-representational approach. The National Council of Teachers of Mathematics Principles and Standards for Mathematics encourage the use of manipulatives in the mathematics classroom, and the Common Core Standards describe concrete objects as appropriate tools for assisting in problem solving (NCTM, 2000 Common Core, 2012). Manipulatives introduce variety to class activities and capture the interest of students which can increase student motivation (Cooper, 2012). Multi-representational teaching builds on students' innate understanding of physical objects, which can lead to a better foundation for abstract representations of algebraic expressions and equations (McNeil & Jarvin, 2007). In addition, it has been demonstrated that when students are physically active throughout learning, memory and understanding are improved (McNeil & Jarvin, 2007). It is a widely accepted belief in education that when multiple learning styles are used to teach the same concept, a larger audience will be reached and students will acquire greater depth of knowledge by thinking about a problem in different ways.

McNeil and Jarvin (2007) point out that even if a research study yields positive results, those results may not be able to be replicated in other classroom environments. Often, teachers or students view manipulatives as toys and fail to make a significant mathematical connection to the activity (Cooper, 2012). Teachers may be to blame for misunderstanding the purpose of manipulatives and failing to help students make meaningful connections between the objects and mathematics. In an Australian study, it was discovered that while classroom teachers believed manipulatives are useful, the same teachers could not identify what made the manipulatives helpful in understanding mathematics (Swan & Marshall, 2010). If teachers do not understand the philosophy behind manipulatives, it is unlikely they will communicate the meaning effectively to their students.

Failure of concrete instruction occurs when students cannot transfer the meaning of the hands-on activity to the abstract level (Cooper, 2012). It is easy for the students to miss the intended purpose of the lesson without explicit instruction or a carefully developed sequence of discovery steps. Students are more likely to misunderstand the mathematical connection to the manipulatives if the objects are too complicated or if the students associate the objects with other meanings outside of school (McNeil & Jarvin, 2007). McNeil and Jarvin describe a class activity in which toy cars were used as manipulatives. While the toys captured the attention of the students, the children had trouble moving past their previous experiences with the objects as toys and were not able to associate the toys with numerical quantities. The process of effectively making the connection between hands-on activities and abstract algebra concepts takes skillful planning on the part of the teacher and a larger investment of instructional time than traditional instructional methods.

Another barrier for high school teachers is finding time for multiple representations of a skill when there is already limited time to teach the required standards (Witzel, Smith, & Brownell, 2001). Finally, it is possible that when students are required to think about the procedure for working with the manipulatives, the procedure for working with abstract symbols, and the connection between the two mediums, they may not have the mental capacity to process all of the information (McNeil & Jarvin, 2007). Such a mental overload may prevent students from grasping the intended purpose of the activity.

The implications of the conflicting research can be confusing to a classroom teacher who is considering whether manipulatives can improve student understanding. McNeil and Jarvin (2007) recommend that teachers ask this question before using a concrete activity in class: “Does it effectively build students’ conceptual understanding of mathematical equivalence and help students prepare for writing and solving equations, or does it divert students’ attention away from the symbolic notation of mathematics to something else?" (p. 310). If the teacher feels that manipulatives clearly establish a foundation for mathematical learning, the use of the CRA method can enhance student interest and understanding. However, if the manipulatives are solely for entertainment, other methods of instruction would better serve the students.

Are Manipulatives Only for Students with Learning Disabilities?

Past research by Witzel and Allsopp (2007) suggests "the use of manipulatives is especially effective for students with high-incidence disabilities, such as learning disabilities (LD), attention deficit hyperactivity disorder (ADHD), and mild to moderate mental disabilities (MD)" (p. 244). A great deal of the research continues to focus on students with disabilities. Maccini and Hughes (2000) conducted a study of problem-solving strategies through the use of manipulatives with six LD students in various high school Algebra I courses. Witzel and Allsopp focused one study on a class of 23 low achieving 6^{th} grade students, some of whom were diagnosed with LD or ADHD.

Teacher testimony supports the use of manipulatives with special education students. Special education and general education math teachers were surveyed about teaching strategies they find to be beneficial for implementing the NCTM standards with LD and ED students. The top response from general education teachers was the use of manipulatives (Maccini & Gagnon, 2000). In inclusive classrooms, manipulatives are a strategy that put gifted students and low achievers at an equal starting point when being introduced to new concepts. Weak math students are not immediately overwhelmed because a new topic is introduced with confusing symbols that they have failed to master in the past. Instead, those students can understand how algebra tiles or number chips can be arranged to represent a given situation and develop enough confidence to “buy in” to the mathematics.

Research in inclusive classrooms suggests that manipulatives may also be beneficial for average and high-achieving students. Thomas Cooper (2012) states, “Even for students capable of using symbolic procedures, concrete models can increase their conceptual understanding by requiring them to look at mathematics in a different way” (p. 106). Another study demonstrated that at every ability level, middle school students in a Pre-Algebra class who learned with manipulatives outperformed students who learned through explicit instruction (Witzel, 2005).

A Middle Grades Success Story

A more realistic classroom environment was the target of a study on manipulatives conducted by Witzel (2005). He investigated a full-sized inclusive Pre-Algebra class taught by the regular classroom teacher. Twelve general education math teachers participated in the study. Each teacher taught two classes as part of the study; one class was taught with the CRA method, and the other was taught with abstract explicit instruction. For each teacher, one of the two classes was randomly assigned to be the CRA class. Every class contained students with and without learning disabilities.

Each pair of classes studied the same five topics ranging from simplifying expressions to solving equations with variables on each side. All classes took exactly 19 50-minute class periods to learn the material. The CRA group proceeded through one day of concrete instruction, one day of pictorial instruction, and two days of abstract instruction for each of the five topics. The non-treatment group was taught with researched-based strategies through explicit instruction for each day of the unit. The students in the non-treatment group still received high quality instruction that was probably similar to a typical math class taught in middle and high schools. Witzel (2005) references an article titled “Using Explicit and Strategic Instruction to Teach Division Skills to Students with Learning Disabilities” by Bryant, Hartman, and Kim when he states, “Explicit instruction has long been the accepted means to math instruction for students with disabilities” (p. 53). Each teacher was observed throughout the process to ensure they correctly followed the teaching model for both the CRA and explicit instruction lessons.

Each student in the study was assessed with the same pre-, post-, and follow-up test three weeks after the unit. The explicit instruction group outperformed the CRA group on the pre-test, yet on both the post-test and the follow-up, the CRA group surpassed the explicit instruction group (Witzel, 2005). Thus, the multi-representational CRA model appeared to have strong benefits on initial learning and retention of abstract algebra topics. Of equal importance was the result that students in every ability group made greater improvements when taught with CRA rather than with explicit instruction (Witzel, 2005). This study indicates that manipulatives can be a powerful tool in the middle school Pre-Algebra curriculum; the question remains as to whether similar results can be achieved with high school students in higher level Algebra courses.

Virtual Manipulatives

Virtual manipulatives are an alternative to physical manipulatives. One source of virtual manipulatives is the online National Library of Virtual Manipulatives (2010). The site organizes resources by grade level and by topic. Some of the tools available for high school are equation scales, algebra tiles, and visual problem solving activities. SMART Exchange (2012) is another valuable online resource where lessons involving virtual manipulatives created by other teachers are shared.

Swan and Marshall (2010) suggest that students should have experience with physical manipulatives before moving on to virtual manipulatives. However, at the high school level, virtual manipulatives could be beneficial if adequate technology resources are available. Classrooms that have access to tablets or iPods can easily take advantage of the online resources because all students would have the ability to interact with the manipulatives. Interactive white boards can be helpful for demonstration but they limit the ability of individuals in the class to explore on their own. A benefit of virtual manipulatives is that some sources such as Java applets allow students to save their work so that it can be assessed by the teacher (Cooper, 2012). With physical manipulatives, the only way to assess student understanding is by observing each student.

Factors to Consider When Using Manipulatives

Manipulatives have potential to deliver excitement and a higher level of conceptual knowledge to a math class at any level if the tools are part of carefully sequenced instruction that makes the mathematical meaning of the objects understandable to students. Before introducing a lesson with manipulatives, there are several factors to consider. First, manipulatives on their own do not impart mathematical knowledge. Swan and Marshall (2010) contend, “Without the appropriate discussion and teaching to make the links to the mathematics explicit, the very opposite may be true: children may end up with mathematical misconceptions" (p. 19). The CRA method can assist in making the transition from concrete to abstract. Secondly, manipulatives are not just toys to make math fun; if they do not assist in learning mathematics, then the activity is not worthwhile. Finally, when deciding on which manipulatives to use, the teacher should ensure that the tools do not require a complex set of rules to follow and the objects are not familiar to the students in other non-school settings (McNeil & Jarvin, 2007). Teachers should always keep in mind the purpose of manipulatives is to help students understand the underlying concepts of abstract mathematics. The end goal should be for students to be proficient in the abstract calculation apart from the manipulatives.

CHAPTER III

METHODS

Purpose

There is compelling evidence in support of using manipulatives to teach mathematics. However, to be effective, concrete objects must be applied with intentional focus on mathematical content. Most of the research on multi-representational instruction focuses on the elementary level, which may leave doubt in the minds of high school teachers about the value of using manipulatives. The purpose of this action research is to discover if the use of manipulatives can improve learning and retention of the Algebra II curriculum. The guiding question that motivates this research is: "Can manipulatives improve mathematical understanding of students studying the Algebra II curriculum?"

Participants

The design of this research is modeled after the study conducted by Bradley Witzel (2005), which is described in "Using CRA to Teach Algebra to Students with Math Difficulties in Inclusive Settings." This study was selected as a guide because it describes whole-class instruction with students of varying abilities. While Witzel examined twenty-four classrooms taught by twelve middle school math teachers, this study investigated six Algebra II courses taught by two different teachers at Scott County High School in Georgetown, Kentucky.

The effectiveness of two teaching models was compared: CRA (concrete-representational-abstract) and abstract explicit instruction. Four Standard Algebra II classes and two Honors Algebra II classes were the target of the investigation. One teacher instructed two Standard Algebra II classes and two Honors Algebra II classes, and the second teacher instructed two Standard Algebra II classes. One Standard Algebra II class from each of the two teachers and one Honors Algebra II class were selected randomly to be the treatment group–the class that receives CRA instruction. The other three classes were taught using explicit instruction as illustrated in Figure 1: Assigning Treatment and Non-Treatment Groups, Appendix A[1].

Classroom Instruction

Lessons on two different topics were taught to each pair of classes. The topics were domain and range of quadratic functions and transformations of quadratic functions. These two topics were selected because each skill can be illustrated using concrete objects, this is the students' first exposure to the skills, and the topics are taught near the beginning of the academic year as part of the same unit. Not all topics in Algebra II are well suited for learning through physical manipulation of objects. Manipulatives should not be considered if students have past experience with the abstract level of a skill. For example, systems of linear equations is a topic that is taught in Algebra I but reviewed and further developed in Algebra II. It should not be necessary to begin at the concrete level when students already know how to use the abstract methods of elimination and substitution. Other abstract Algebra II standards may be difficult to clearly illustrate with manipulatives in a way that deepens students' understanding.

[1] All figures and tables can be found in Appendix A

In the treatment classes, the students worked through the CRA model. Students began to explore domain and range of quadratic functions concretely by placing craft beads along points on the graph of a parabola. Each pair of students proceeded to slide the beads vertically to the x-axis to identify the x-values of the domain; they slid the beads horizontally to the y-axis to help visualize y-values in the range. The next phase of the lesson still involved thinking about the manipulatives, but rather than handling the objects, students only used the pictorial representation of the quadratic graph. Finally, students reflected on the results they obtained from the concrete and pictorial examples and tried to devise a strategy for finding the domain and range of a quadratic function without looking at the graph. This portion of the lesson required class discussion and guidance by the teacher to lead students to understand how the y-coordinate of the vertex can be used to abstractly determine the range of a quadratic function. A complete lesson plan that further describes the three-phase process can be found in Appendix B.

Class activities for the CRA lesson on transformations of quadratic functions followed the same three-phase format. For this lesson, students used wax sticks as the hands-on tool for exploring graphs of parabolas. After the wax sticks were shaped to form the graphs of two different quadratic functions, the students were able to physically move the first parabola to transform it into the second graph. In the representational phase, students used a graphing calculator to view the graphs of two different parabolas and describe the transformation from one graph to the next. The abstract phase involved recognizing patterns that enable students to predict the transformations that occur in the graph using only the vertex form of the equation. A complete description of the lesson and lesson materials is available in Appendix B.

The non-treatment classes learned through abstract explicit instruction. For both topics, the teacher modeled the thought process and algebraic skills through whole-class instruction. Students in these classes worked through the same examples as the treatment classes, but rather than working with physical objects, they were asked to answer questions about domain and range and the transformations of quadratic functions by observing algebraic equations and their corresponding graphs. The teacher used scaffolding throughout the examples until the students could solve similar problems independently. Corrective and positive feedback was provided to the students throughout the process. A more in-depth lesson plan for each skill is provided in Appendix B.

The treatment and non-treatment classes that were paired together spent the same amount of class time on each topic even though they learned in different ways. The classes that were not paired together spent slightly different amounts of class time developing the targeted skills. For example, the Honors Algebra II classes did not require as much time as the standard Algebra II classes to master the skills at the abstract level. In Witzel's (2005) study, each topic was developed over four class periods. In the CRA class, the first day was spent on concrete instruction, the second on representational, and the last two on abstract. The explicit instruction class also spent four days on each topic. Due to the fast paced nature of the Algebra II curriculum, four days could not be allotted for the mastery of one skill. The instruction sequence was completed in two class periods. Additional time was allotted to review all skills in the unit before the assessment.

All classes followed the same procedure and sequence of examples that is outlined in the lesson plan for initial instruction. Teachers were allowed flexibility to remediate the abstract skills as necessary based on formative assessment, provided that the same class activities and same amount of class time were used in the paired treatment and non-treatment classes. This flexibility allowed the teacher to best meet the needs of the students while also maintaining consistency. It is important that any differences in test scores are a reflection of the two instructional methods that are the focus of this study, not a result of different remediation activities.

Assessment

Student learning was measured by pre-, post-, and follow-up assessments. Both topics described previously are a part of a unit on graphs of parabolas. Before instruction began, all six classes took a pre-test to determine prior knowledge. Following the unit, students completed a post-test to measure learning that occurred as a direct result of the recent instruction. The pre-test and post-test both contained the same number and style of questions on each topic. Students answered three short-answer questions on domain and range. In the first question, students identified the domain and range from the graph of a parabola; in the next two questions the domain and range were identified from the equation of a parabola. The transformations portion of the assessment contained one question in which students identified the vertex, determined the direction of opening, and concluded whether the graph had vertical stretch or compression from the vertex form of the equation of a parabola. The next two questions required students to describe and graph the transformations from the parent function to the graph of a second parabola in vertex form. Post-test results were compared to determine whether the treatment and non-treatment groups had different levels of success. Several weeks after the unit had been completed and assessed, an abbreviated follow-up assessment was administered to four of the classes to determine retention of the skills.

Scoring

Each question on the pre-test and post-test had a maximum score of five points. Since the assessment contained three questions on each skill, (three questions on domain and range of functions and three questions on transformations of functions), the maximum score for each standard is fifteen points.

For each domain and range question, two points were awarded for correctly identifying the domain as all real numbers, and three points were awarded for correctly stating the range. As part of the three-point score for the range, students earned one point if the *y*-variable and the correct inequality symbol were used and two additional points if the correct *y*-coordinate of the vertex was stated in the range. Students who demonstrated the correct process for calculating the *y*-coordinate of the vertex with a calculation error received one of the two points for the calculation.

The transformations portion of the assessment contained two different types of questions. For the first question, students earned one point for correctly identifying each coordinate of the vertex from vertex form, one point for correctly identifying the direction of opening, and two points for correctly identifying the stretch or compression of the parabola. The other two questions on this section required students to list the transformation and graph the function. One point was awarded for correctly identifying each of the four types of transformations that could occur (vertical translation, horizontal translation, reflection across the *x*-axis, and vertical stretch or compression). The final point was awarded for correctly graphing the transformed parabola.

The follow-up test was an abbreviated version of the pre- and post-test. This assessment contained two short-answer questions on domain and range and one question in which students described the transformations that occurred to a quadratic function. Each question was scored according to the same guidelines as the pre-and post-test.

Possible Implications

The results of this small-scale study can be used to help teachers determine whether manipulatives are a tool that is useful to incorporate into Algebra II instructional plans. Data that support the CRA method could be used as justification for investing in more concrete materials and teacher training. If the results do not favor CRA instruction, teachers can focus professional development time on high quality explicit instruction or different student-centered approaches. The results of this research could also give insight into which level of mathematics courses should be taught using manipulatives. It is possible that manipulatives are best suited for inclusive classes in which special education students and low achievers need to establish a foundation for new concepts before working at the abstract level. Another possibility is that manipulatives can provide an opportunity for learners of all abilities to develop more depth in mathematical understanding. Whether the results favor CRA or explicit instruction, the data can be instrumental in future instructional planning.

CHAPTER IV

RESULTS

Student Participants

A total of 143 students were enrolled in the classes included in this study. Some student scores were eliminated from the reported data due to absence from class. Assessment scores were not included if a student missed the primary day of CRA or explicit instruction, or if the student missed enough days of class that they were delayed in taking the post-test by a week or more. The numbers of reported scores are recorded in Table 1: Number of Reported Scores by Class and Standard. If a student's post-test score was eliminated from the data set, the pre-test and follow-up scores for that student were also eliminated from the data.

Absence of Follow-up Scores

The intended methodology was to report pre-, post-, and follow-up test scores for each student. It was not possible to obtain follow-up scores for the two classes taught by teacher #2. Teacher #2 spent more time on initial instruction and remediation. There was no time remaining at the end of the term for a follow-up test. Scheduled vacation time and weather-related school cancelations prevented follow-up scores from being collected in a timely manner after the term ended. Follow-up scores are reported for the four classes taught by teacher #1.

Mean Scores and Growth by Treatment

Mean post-test scores were calculated for the three treatment and the three non-treatment classes. The results are recorded in Table 2: Treatment vs. Non-treatment Post-Test Scores for all 6 Classes. Mean values indicate that the treatment group scored 0.12 points lower on the domain and range skill, and 0.13 points higher on the transformations skill on a 15-point assessment. The differences between the means of the classes taught by the two different instructional methods were small and did not consistently favor one method over the other.

Both the treatment and non-treatment groups demonstrated significant growth from the pre-test to the post-test. The treatment classes had growth scores of 12.39 and 11.38 from the pre-test to the post-test on each of the two skills, while the non-treatment classes had growth scores of 12.51 and 11.09. Growth from the pre-test to the post-test is also reported in Table 2. Growth was calculated by subtracting the mean pre-test score from the mean post-test score. The pre-test scores of 0 indicate that students had no prior knowledge on the domain and range skill. Pre-test scores on the transformation skill averaged 0.567 and 0.730 out of 15 possible points for the treatment and non-treatment groups respectively. Some students earned a small number of points on the transformations pre-test. Those points likely came from multiple-choice questions in which students may have guessed correctly. Most students had little knowledge of the content before classroom instruction began, and they demonstrated considerable growth as the result of instruction.

Mean follow-up scores from the four tested classes are reported in Table 3: Treatment vs. Non-treatment Follow-up Scores for 4 classes. Follow-up scores were originally calculated out of 10 possible points for the domain and range skill and out of 5 possible points for the transformations skills. Scores for both skills were scaled to a maximum score of 15 points to make comparison of follow-up and pre-test scores consistent. Follow-up scores followed the same pattern as the post-test scores. The mean follow-up score was 0.49 points lower for the treatment group on the domain and range skill and 0.44 points higher for the treatment group on the transformations skill.

Analysis of Variance

ANOVA was used to analyze the data from this study. A total of seven effects were tested as part of the ANOVA. The primary effects include the treatment (CRA or explicit instruction), the skill (domain and range or transformations), teacher (teacher #1 or teacher #2), and the time of the assessment (pre-, post-, or follow-up). The interaction effects are treatment-by-time, skill-by-time, and teacher-by-time. The ANOVA results are reported in Table 4: Analysis of Variance.

To determine if an effect was significant, α was calculated by taking 0.05 divided by seven tests which results in $\alpha = 0.007$. Dividing the standard α-value by the number of tests in the experiment helps to avoid inflating the amount of Type 1 error in the combined results of all seven tests. With a threshold of $\alpha = 0.007$, it is clear that timing of the assessment ($p < 0.001$) impacts the mean test performance. The importance of the timing of the assessment should not come as a surprise considering the growth that occurred from pre-test to post-test as reported in Table 2. The contrast value of pre-test vs. post-test (p <0.001) further supports the claim that after instruction, there is a significant gain in test scores across both skills, both teachers, and both treatments.

An unintended discovery that resulted from this experiment is that there is a difference in students' ability to learn the two skills of domain and range of quadratic functions and identifying transformations of quadratic functions. The ANOVA p-value of 0.0014 for skill meets the significance level which indicates that it is unlikely that the difference in means is unrelated to the skills that were taught. In addition, the mean test scores on the post-test for domain and range were 12.39 for the treatment group and 12.51 for the non-treatment group, while the mean post-test scores on the transformations skill were 11.95 and 11.82. Higher means occurred in both the treatment and non-treatment groups on the domain and range skill compared to the transformation skill, which suggests that students found the domain and range skill easier to learn than the transformations skill.

The time-by-skill interaction also met the level of significance ($p < 0.001$). The significance level of the time-by-skill interaction is consistent with the differences in growth scores by skill that are reported in Table 2. Pre- to post-test growth for the treatment group was 12.39 points on the domain and range skill and 11.38 points on the transformations skill, while the non-treatment group had pre- to post-test growth of 12.51 points and 11.09 points on the two different skills. Gains in performance were not the same across each skill over time.

While time and skill impact student performance, the teacher ($p = 0.3666$) and the main focus of this study—the treatment ($p = 0.7455$) was not significant at the 0.007

level. There is not enough evidence to reject the null hypothesis that the instructional method has no impact on mean scores. Some caution should be used when using ANOVA alone to determine significance since the small sample size limits the results.

Effect Size

To put the differences in means between the treatment and non-treatment groups into perspective, Cohen’s d-statistic was used to measure effect size. The results are reported in Table 5: Estimate of the Effect Size. The effect size is the best measure of variation between the two groups for this study because effect size is an accurate reflection of differences in means even when data are collected from a small sample—in this case only 6 classes.

Negative d-values were calculated for the skill of domain and range on both the post-test (d = -0.115) and the follow-up test (d = -0.696). D-values that are negative indicate that the non-treatment group outperformed the treatment group on that skill. However, on the post-test, the difference in means between the two groups is small enough that it could be attributed to random sampling error. The d-values were positive for the post- test (0.376) and follow-up test (0.247) for the skill of transformations.

The effect size does not indicate that either of the two methods of instruction led to dramatically higher test performance. However, there are two d-values that warrant some consideration. On the follow-up test for domain and range, the difference in means is more than 2/3 of a standard deviation in favor of the non-treatment group. This may provide an indication that students retained the ability to perform abstract problems about domain and range of quadratic functions better when they learned through explicit instruction. A smaller d-value that may still highlight some importance is the statistic that reflects a difference of more than 1/3 of a standard deviation in favor of the treatment group on the post-test for the transformations skill. While it is possible that a difference in means of this size could be attributed to random error, it is also possible that there was a small benefit in using the CRA method for this skill.

Summary

The statistics reported do not strongly favor either the CRA method of instruction or explicit instruction. While the ANOVA results do not suggest that the treatment had a significant effect on the test scores, the low level of significance could be attributed to the small sample size. The differences in means and the effect size indicate that there may be small benefits to teaching the domain and range skill with explicit instruction and the transformations skill with the CRA method.

CHAPTER V

DISCUSSION

Answer to Research Question

This quasi-experimental research sought to discover whether manipulatives can help high school students learn abstract mathematical skills that are part of the Algebra II curriculum. The data indicate that student learning occurred for both skills through CRA instruction and explicit instruction. Large increases in mean test scores were reported for the treatment and non-treatment groups from the pre-test to the post-test. The analysis of variance indicates that the most significant changes in test scores occurred as a result of the time that was spent on classroom instruction for each skill. This growth was consistent across both skills, both teachers, and both treatments.

While learning took place in the treatment classes, the analysis of variance indicates that there is no evidence that supports that teaching was more effective when manipulatives were used in place of explicit instruction. The estimate of effect size suggests that students may have retained the ability to solve abstract problems about domain and range of quadratic functions better when they learned through explicit instruction and it is possible that the CRA method was slightly more effective on the transformations skill. These inconsistent results may lead to more confusion in the mind of the teacher who is considering using manipulatives in an Algebra II class. The following sections seek to offer the reader more insight into the classroom environment created when the CRA method of instruction is used in comparison to explicit instruction.

Unmeasured Value of Manipulatives

The data in this study suggest that there is no significant improvement in student performance on two different abstract skills when CRA method of instruction is used in place of traditional explicit instruction in several Algebra II classes. What this research may have failed to measure is the development of students' problem-solving abilities. In addition to mathematical content, the Common Core Standards (2012) outline eight standards for mathematical practice that describe the thought processes students should be engaging in as part of a meaningful mathematical curriculum. Two of these skills include persevering while solving meaningful and challenging problems (MP1) and using inductive reasoning to make and defend mathematical conjectures (MP3).

The CRA method of instruction engages students in problem-solving and building mathematical conjectures on a deeper level than explicit instruction. As part of the CRA lessons, each pair of students was involved in recognizing patterns and trying to develop generalizations. Students shared and defended their observations with their classmates. When explicit instruction is used, the intention is to guide students through similar thought processes, but when the steps occur more quickly and as a whole class, not all students make the same connections. While it is important to engage students in higher-order thinking and mathematical communication, these skills cannot always be measured by traditional assessment methods. The pre-, post-, and follow-up assessments used in this study do not measure growth in problem-solving strategies.

Teacher Observations of CRA and Explicit Instruction

Two teachers participated in this research process. Their observations may give insight to other teachers who are considering how the classroom structure differs when CRA instruction is used in comparison to explicit instruction.

For both skills and all three classes in which manipulatives were used, students were actively involved in the learning process. Nearly every student sought to follow the directions, worked with the manipulatives, and recorded their observations. The activity sparked meaningful mathematical discussion between classmates. Throughout the process, there was some struggle to make sense of the activity and generalize the results of the series of examples, yet most students persisted. At the end of the concrete and pictorial phases of the lesson, there was still some confusion and misconceptions that needed to be corrected. However, by this point in the lesson, students were more invested in discovering the solutions to the questions in the activity and questions of their own. A whole-class discussion and additional examples eliminated most of the confusion.

For the classes in which explicit instruction was used, most students were able to comprehend the underlying concept and process for identifying domain and range and transformations of quadratic functions. There was less interaction between students and more students were prone to lose focus during whole-class instruction. A pictorial representation of the functions seemed to be sufficient for students to understand the meaning of the new algebraic concepts. During explicit instruction, students were still guided to use prior knowledge to generalize abstract strategy. For example, in the domain and range lesson, students were able to recognize that calculating the vertex would allow them to determine the range. However, fewer students were actively making connections.

A major difference between the two styles of instruction is the length of class time required. More time is needed to guide students through all phases of the CRA process than when students learn through explicit instruction that begins with the pictorial or abstract phase. For the purpose of this study, both the treatment and non-treatment classes spent the same amount of class time on instruction. Students were slightly rushed through the CRA process, and some groups did not have time to finish all of the questions on the handout before beginning the class summary discussion. Students in the explicit instruction classes had time to begin the homework assignment in class, while students in the CRA instruction classes had to complete the majority of the assignment at home. With students who do not have the discipline to complete assignments at home, losing in-class work time to refine strategies independently can be detrimental to students' skill comprehension.

The CRA method serves as a means of differentiating instruction for a wide variety of learners more easily than explicit instruction. When the CRA method is used, students can work through the activity at a pace that allows them to individually make connections between the different phases of the lesson. Advanced students can work ahead and share generalizations with other members of the class. During the CRA lesson, some students abandoned the manipulatives early because they were able to quickly make the connection between the concrete and pictorial representations, while other students felt more comfortable using the manipulatives for all of the examples in the first phase of the activity.

After completing the instructional unit, both teachers agreed that the use of manipulatives was more appropriate for the lesson on transformations of functions than for the lesson on domain and range. The beads seemed to complicate the process of identifying domain and range. Students using manipulatives had more trouble recognizing that the domain and range extend beyond the boundaries of the graph paper when they were asked to list the coordinates of the beads, than the students who only used the pictorial representation of domain and range as part of the explicit lesson. This lesson seemed to be a situation in which students had the necessary skills to move to the abstract level quickly, and the manipulatives only complicated the learning process and demanded more class time. The wax sticks served the purpose of illustrating transformations of functions more clearly than a pictorial representation alone. Especially when trying to understand the concept of vertical stretch and compression, having the ability to pick up and lay the wax parabolas on top of each other was instrumental.

Conclusions and Future Research

The data collected as part of this research do not conclusively support that either the CRA method or explicit method of instruction is more effective in teaching Algebra II students abstract skills. Learning occurred with both methods of instruction. If manipulatives engage students, spark meaningful discussion, and allow for differentiation without inhibiting learning, teachers may feel there is value in using the CRA method in Algebra II classes. Other teachers may seek other student-centered class activities in an effort to promote student engagement and higher-order thinking while simultaneously improving comprehension of abstract skills. The teacher must make the decision, keeping in mind personal teaching style, the needs and skills of the students, and the algebraic skills the lesson communicates.

While the results of this small, quasi-experimental research study do not favor the CRA method, it certainly should not be used as a reason to dismiss the possibility that manipulatives could be beneficial in an Algebra II classroom. Future studies with a larger sample may reveal more evidence against the null hypothesis. There may also be other topics in the Algebra II curriculum that are better suited to learning with the CRA method of instruction or other manipulatives that more clearly illustrate abstract Algebra II concepts. Results may also differ with other groups of students who possess less ability to reason abstractly. Finally, an assessment that measures problem-solving ability may illustrate greater benefits from using manipulatives.

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Appendix A

Figure and Tables

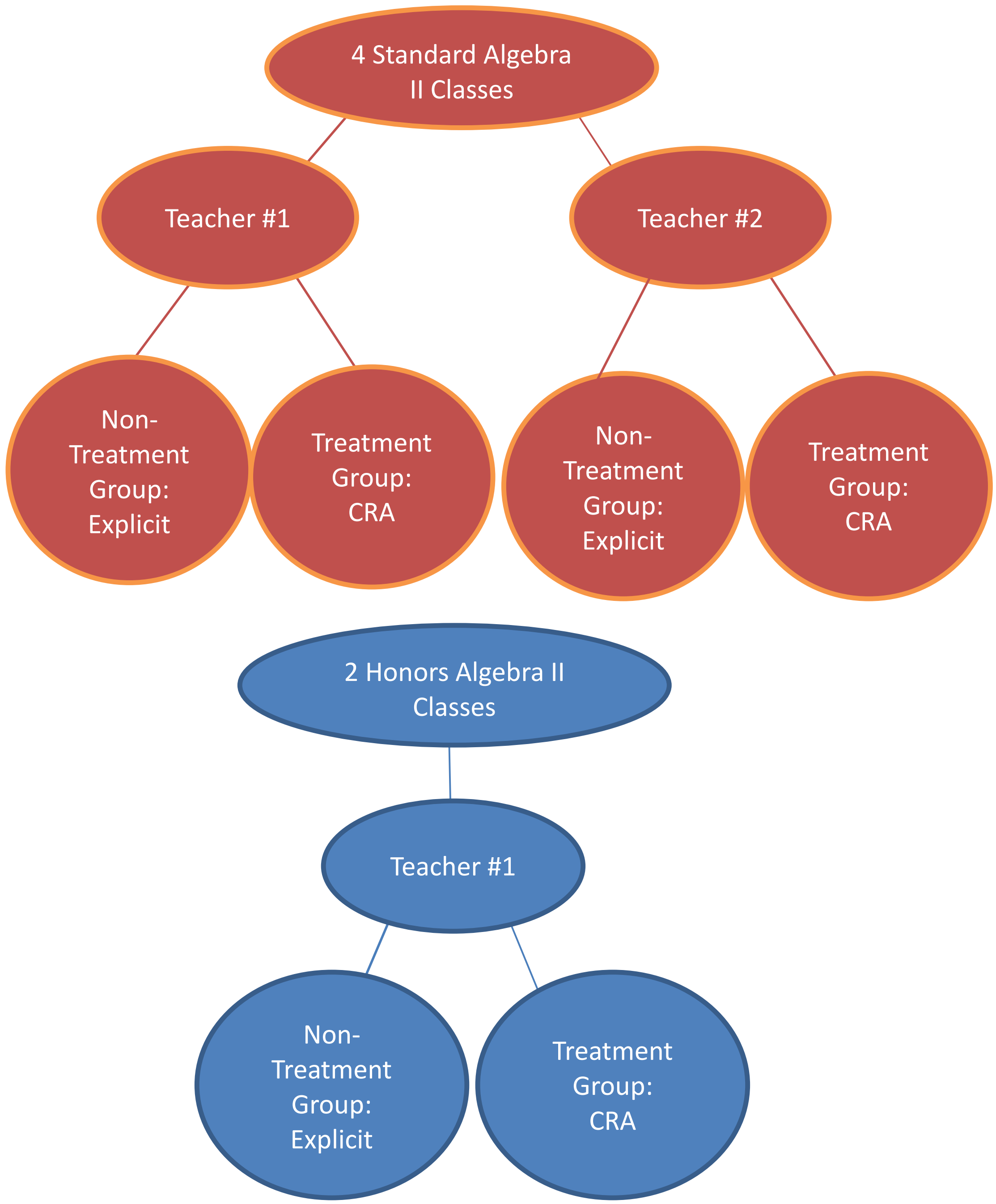


Figure 1: Assigning Treatment and Non-Treatment Groups

Table 1: Number of Reported Scores by Class and Standard

Class Description	Total Number of Students	Number of Reported Scores for Domain and Range	Number of Reported Scores for Transformations
Honors Treatment	28	22	25
Honors Non-treatment	26	21	25
Teacher #1 Treatment	26	21	24
Teacher #1 Non-treatment	26	19	18
Teacher #2 Treatment	17	16	16
Teacher #2 Non-treatment	20	19	19

Table 2: Treatment vs. Non-treatment Post-Test Scores for all 6 Classes

Skill: Domain and Range

	Mean	Standard Deviation	Growth: Pre to Post
Treatment	12.390	0.624	12.390
Non-treatment	12.510	1.042	12.510

Skill: Transformations

	Mean	Standard Deviation	Growth: Pre to Post
Treatment	11.947	0.537	11.380
Non-treatment	11.816	0.346	11.086

Max Score = 15

Table 3: Treatment vs. Non-treatment Follow-up Scores for 4 classes

Skill: Domain and Range

	Mean	Standard Deviation	Growth: Pre to Follow-up
Treatment	12.953	1.347	12.953
Non-treatment	13.440	0.700	13.440

Skill: Transformations

	Mean	Standard Deviation	Growth: Pre to Follow-up
Treatment	11.070	1.655	10.504
Non-treatment	10.635	1.761	9.905

Max Score = 15

Table 4: Analysis of Variance

Primary Effects	F	p
Time	512.05	<0.0001
Skill	14.19	0.0014
Teacher	0.86	0.3666
Treatment	0.11	0.7455
Interaction Effects		
Treatment- by-Time	0.04	0.965
Skill-by-Time	17.36	<0.0001
Teacher-by-Time	0.22	0.6443
Contrast Post vs. Pre	2431.63	<0.0001

Time = When that test was administered (pre-instruction, post-instruction, or follow-up)
Skill = Standard that was assessed (domain and range or transformations)
Teacher = Who delivered instruction (teacher #1 or #2)
Treatment = Method of instruction (CRA or direct instruction)

Table 5: Estimate of the Effect Size

Skill	Time	Cohen's d
Domain and Range	Post-test	-0.117
	Follow-up	-0.696
Transformations	Post-test	0.379
	Follow-up	0.247

Appendix B

Lesson Plans and Instructional Materials

Concrete-Representational-Abstract Lesson Plan for Domain and Range of Quadratic Functions

Title: Domain and Range of Quadratic Functions

Standards
Quality Core: E.2.a. Determine the domain and range of a quadratic function; graph the function with and without technology

Lesson Objective
I can determine the domain and range of a quadratic function from the graph or from the equation

Prerequisite Skills
Students need to be able to graph linear and quadratic functions by completing a table and plotting points. A graphing calculator can be used to fill in a table of values and to view the graph of parabolas.

Materials
Each pair of students will need 11 craft beads, a plastic page protector with a copy of the large coordinate plane provided, a dry erase marker with an eraser, the Domain and Range of Functions Handout provided, and a graphing calculator.
The teacher will need board space and writing utensils to display student answers.

Preparation
Assign students to pairs before they arrive; group students with similar abilities together.
Direct students to sit with their partner as they arrive to class.
Prepare bags of 11 or more beads for each group
Copy the coordinate plane on 8x11 paper or cardstock and insert the graphs into page protectors.
Copy the handout for each group (or each student).

Lesson Outline

I. Warm-up – Match linear and quadratic equations to their graphs. Pass out materials to students while they complete the warm-up independently. Check answers as a group.

II. Domain and Range of Functions Activity

a. The teacher will model the process of using beads to help identify the domain and range of a function. The entire class will work through example 1 together on the handout. A graphing calculator can be used to complete the table. Place beads on the graph page at each of the ordered pairs from the table. Use the dry erase marker to draw a curve that connects the dots (the first example is a line). Slide the beads vertically to the x-axis and answer the questions on the handout. Return the beads to the original position, and then repeat the process on the y-axis. Be sure students understand that even though there are not beads at every integer y-coordinate, there is still a point on the line at each y-coordinate. Clear the beads and erase the line from the coordinate plane.
b. Students will work in pairs to complete questions 2 – 8. Advanced students may continue to 9 and 10 if after the teacher checks their answers to question 8. Questions 2-5 lead students to use concrete objects (craft beads) to display the graph of a function and explore the relationship between the graph and the domain and range of the function. Questions 6 and 7 use a graphing calculator to display and draw the graph of quadratic functions and then students will identify the domain and range from the pictorial representation of each function. Question 8 leads students to make generalizations between the direction of opening of the parabola, the vertex, and the steps to abstractly determine the domain and range of the function.
c. While students work, the teacher will circulate the room, assist students, and engage students in conversation that encourages students to explain the reasoning behind their answers.

III. Class Discussion

a. Students will write the domain and range for questions 2-7 on the board. The class will discuss the accuracy of the answers and strategies for determining the domain and range.
b. The class will discuss question 8 in detail. Students will share their observations and generalizations. The teacher will highlight useful observations and correct any misconceptions. The following concepts need to be emphasized as part of the discussion. The domain of any quadratic function is all real numbers because the graph extends toward both negative and positive infinity on the x-axis. The vertex is the most critical point when determining the range of a quadratic function. If the *a* value of the equation is positive, then the vertex is a minimum, and the range is all y-values greater than or equal to the y-coordinate of the vertex. If the *a* value of the equation is negative, then the vertex is a maximum, and the range is all y-values less than or equal to the y-coordinate of the vertex.

c. Complete questions 9 and 10 out loud as a class. In these examples, students will determine the domain and range of a quadratic function abstractly without graphing. Formatively assess students' understanding through questioning as the class works through the examples together. Clarify student misconceptions before students begin the assignment independently.

IV. Independent reinforcement and assessment. Assign additional practice problems that involve calculating the domain and range of quadratic functions abstractly.

Names:____

Domain and Range of Functions

Part 1: Lines

1. $f(x) = -2x + 4$

a. Complete the table of values for the function. On your large coordinate plane, place a bead at each of the points in the table. Connect the points with a marker.

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
f(x)											

b. Slide all of the beads vertically until they are all on the x-axis.

c. The **domain of a function is all of the possible x-values of the function**. Write the domain of this function by describing the x-values with beads.

d. Return the beads to each of the points in the table. Then slide the beads horizontally until they are all on the y-axis. If a y-value does not have a bead, does that mean that there is not a point on the graph for that y-value? Explain.

e. The **range of a function is all of the possible y-values of the function**. Write the range of this function by describing y-values with beads.

2 $f(x) = x - 2$

a. Complete the table of values for the function $f(x) = x - 2$. On your coordinate plane, place a bead at each of the points in the table. Connect the points with a marker.

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
f(x)											

b. Slide all of the beads vertically until they are all on the x-axis.

c. Write the domain of this function by describing the x-values with beads.

d. Return the beads to each of the points in the table. Then slide the beads horizontally until they are all on the y-axis.

e. Write the range of this function by describing y-values with beads.

f. Can you make any generalizations about the domain and range of diagonal lines?

Part 2: Parabolas

3. $f(x) = x^2$

a. Complete the table of values for the function $f(x) = x^2$. Place a bead at each of the points in the table. Connect the points.

x	-3	-2	-1	0	1	2	3
f(x)							

b. Slide all of the beads vertically until they are all on the x-axis.

c. Write the domain of this function by describing the x-values with beads.

d. Return the beads to each of the points in the table. Then slide the beads horizontally until they are all on the y-axis.

e. Write the range of this function by describing y-values with beads.

4. $f(x) = x^2 - 2x + 4$

a. Complete the table of values for the function $f(x) = x^2 - 2x + 4$. Place a bead at each of the points in the table. Connect the points.

x	-3	-2	-1	0	1	2	3
f(x)							

b. Slide all of the beads vertically until they are all on the x-axis.

c. Write the domain of this function by describing the x-values with beads.

d. Return the beads to each of the points in the table. Then slide the beads horizontally until they are all on the y-axis.

e. Write the range of this function by describing y-values with beads.

5. $f(x) =-2x^2 + 1$

a. Complete the table of values for the function $f(x) = -2x^2 + 1$. Place a bead at each of the points in the table. Connect the points.

x	-3	-2	-1	0	1	2	3
f(x)							

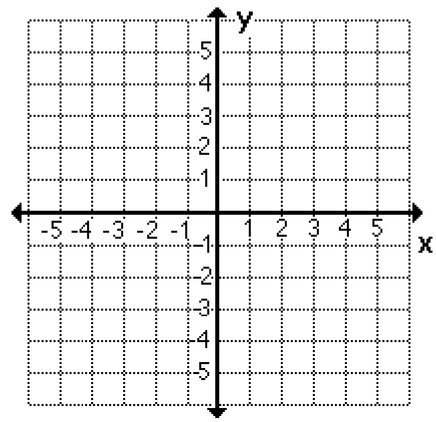
b. Slide all of the beads vertically until they are all on the x-axis.

c. Write the domain of this function by describing the x-values with beads.

d. Return the beads to each of the points in the table. Then slide the beads horizontally until they are all on the y-axis.

e. Write the range of this function by describing y-values with beads.

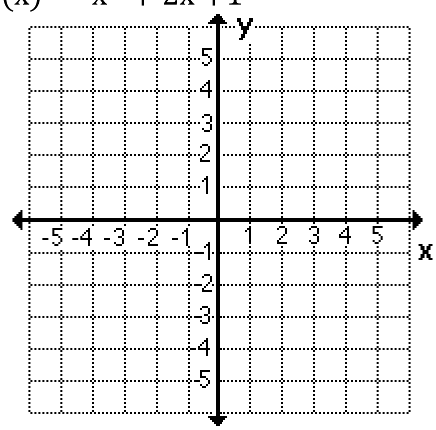
6a. Use a graphing calculator to sketch a graph of $f(x) = 2x^2 + 4x$



b. Describe the domain of f(x)

c. Describe the range of f(x)

7a. Use a graphing calculator to sketch a graph of $f(x) = -x^2 + 2x + 1$



b. Describe the domain of f(x)

c. Describe the range of f(x)

8a. What generalizations can you make about the domain of a quadratic function?

b. For quadratic functions, will the domain be the same as the range?

c. What is the most important point on the graph when determining the range?

d. How can you tell if the y-values of a quadratic function will be below the vertex by looking at the equation? (hint: look at #5 and #7)

e. How can you tell if the y-values of a quadratic function will be above the vertex by looking at the equation?

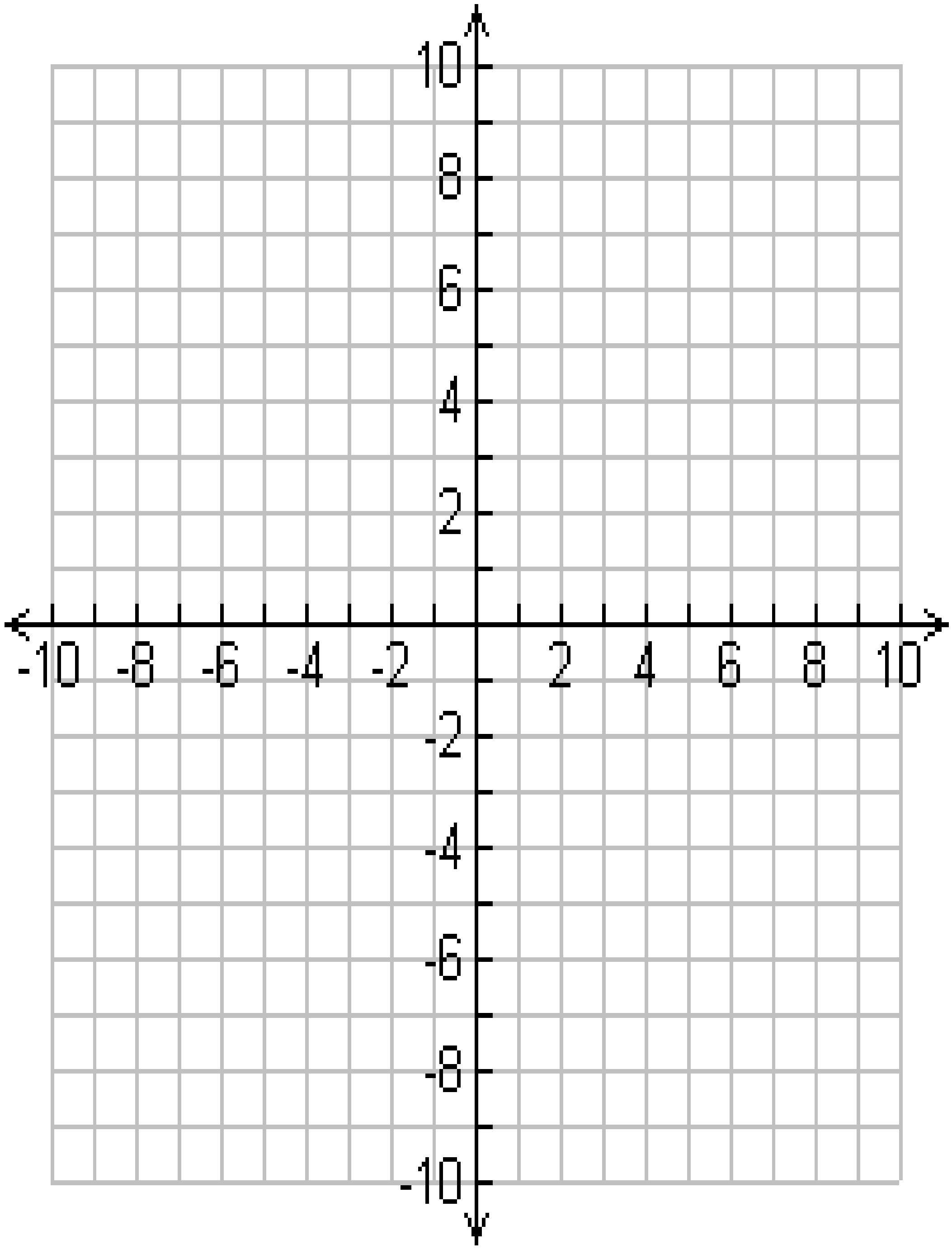
9. Find the domain and range of each function without graphing. Show calculation

a. $y = 2x^2 + 4x - 8$

b. $y = -x^2 - 6x$

10. Find the maximum or minimum value of the function. Then state the domain and range of the function. $f(x) = 3x^2 + 2x$

Large Coordinate Plane



Explicit Lesson Plan for Domain and Range

Title: Domain and Range of Quadratic Functions

Standards

Quality Core: E.2.a. Determine the domain and range of a quadratic function; graph the function with and without technology

Lesson Objective

I can determine the domain and range of a quadratic function from the graph or from the equation

Prerequisite Skills

Students need to be able to graph linear and quadratic functions by completing a table and plotting points. A graphing calculator can be used to fill in a table of values and to view the graph of parabolas.

Materials

Students will need paper and pencil

Teacher will need SMARTboard and projector or other device for presenting the lesson

Preparation

Create slides of lesson definitions and examples in advance

Lesson Outline

I. Warm-up – Match linear and quadratic equations to their graphs. Check answers as a group.

II. Domain and Range of Functions Notes
   a. Define domain and range
   b. Domain and range from graphs: Display the equations and graphs of each of the following functions: $f(x) = -2x + 4$, $f(x) = 1/3x – 2$ $f(x) = x^3$, $f(x) = x^2$, $f(x) = x^2 - 2x + 4$, $f(x) = -2x^2 + 1$. Use think aloud and student questioning strategies as the class identifies the domain and range of each function from the graph.
   c. Generalizations: Begin by asking students, "What is true about the domain of the quadratic functions we have seen?" Be sure that all students understand that the domain of any quadratic function is all real numbers because the graph extends toward both negative and positive infinity on the x-axis. Continue by asking, "What is the most critical point of the graph when you find the range of a quadratic function?" The

following concepts need to be emphasized as part of the discussion: The vertex is the most critical point when determining the range of a quadratic function. If the *a* value of the equation is positive, then the vertex is a minimum, and the range is all y-values greater than or equal to the y-coordinate of the vertex. If the *a* value of the equation is negative, then the vertex is a maximum, and the range is all y-values less than or equal to the y-coordinate of the vertex.

d. Domain and range from equations: Use the generalization from part c to determine the domain and range of the following functions without graphing: $f(x) = 2x^2 + 4x - 8$, $y = -x^2 - 6x$, $f(x) = 3x^2 + 2x$. Formatively assess students' understanding through questioning as the class works through the examples together. Clarify student misconceptions before students begin the assignment independently.

III. Independent reinforcement and assessment. Assign additional practice problems that involve calculating the domain and range of quadratic functions abstractly.

Concrete-Representational-Abstract Lesson Plan for Transformations of Quadratic Functions

Title: Transformations of Quadratic Functions

Standards

Quality Core: E.2.b Use transformations to draw the graph of a relation and to determine the relation that fits a graph

Lesson Objective

I can use the vertex form of a quadratic function to identify transformations of the graph and to draw the graph of the parabola

Prerequisite Skills

Students need to be able to graph parabolas by making a table of values. A graphing calculator can be used to obtain the table. Students need to understand the meaning of the following vocabulary: transformation, translation, reflection, compression, and stretch.

Materials

Each pair of students will need 2 wax sticks (available at craft stores), a dry erase marker, a plastic page protector with a copy of the large coordinate plane, the Transformations of Functions Handout provided in the lesson materials, and a graphing calculator.

Preparation

Assign students to pairs before they arrive; group students with similar abilities together. Direct students to sit with their partner as they arrive to class. Copy the coordinate plane on 8x11 paper or cardstock and insert the graphs into page protectors (use the same materials as the domain and range lesson). Insert 2 wax sticks and a dry erase marker with eraser into each page protector along with the coordinate plane. Copy the handout for each group.

Lesson Outline

I. Warm-up – Review vocabulary about transformations (reflection, translation, compression, stretch)

II. Check and collect homework from the previous day. Pass out materials while students check their answers.

III. Transformations of Quadratic Functions Activity: Students will work in pairs through the Transformations of Quadratic Functions Handout.

a. For the concrete phase of the lesson, students will use a graphing calculator to make a table of values for two quadratic functions, use a marker to plot the points on the coordinate plane, and then bend the wax sticks to fit the shape of the parabola. Students will be asked to describe the transformations that would change the first parabola into the second parabola. As they try to identify the transformations, they will be able to lift the first wax parabola off of the page and move it around to match the second parabola while the wax maintains the original parabolic shape. Physically moving the parabola is intended to clarify the meaning of translating, reflecting, and stretching a parabola. Since the wax sticks are the same length, students are more easily able to compare the widths of the two parabolas. As students identify the transformations, the teacher will check for accuracy and engage the students in discussion to clarify their answers or correct misunderstandings.

b. For the representational phase of the activity, students will attempt to predict the transformations that will occur between two parabolas, and then they will check and modify their work using the graphical display on a graphing calculator. The teacher will check work and transition students to the abstract phase by asking how the equation could be used to recognize the transformations that occur.

c. To begin the abstract phase of the lesson, students will use their observations from questions 1-3 on the handout to generalize how the *a*, *h*, and *k* values of the vertex form $f(x) = a(x - h)^2 + k$ affect transformations of a parabola from the parent function. It is likely that not all students will be able to accurately describe how each value transforms the graph. Clarification will occur during the class discussion.

IV. Class Discussion

a. Students will share their discoveries and generalizations from question 4 on the handout.

b. The teacher will encourage students to discuss, defend their ideas, and clarify misunderstandings. The following topics need to be emphasized as part of the discussion. The graph $f(x) = x^2$ is the parent function for parabolas. The vertex of the parent function is at the origin and the graph opens up. When the *a* value is negative, the parabola is reflected across the x-axis. When $|a| > 1$ the parabola is vertically stretched; when $|a| < 1$ the parabola is vertically compressed. (The stretch and compression are the most difficult for students to understand. The concept can be illustrated by sticking the wax sticks up on the board or under a document camera. The teacher or a student can illustrate a vertical stretch by holding the vertex in place and pulling on the ends of the parabola until it is narrower. Similarly, a vertical compression can be illustrated by pushing the ends of the parabola toward the vertex.) The *k* value determines the vertical translation of the parabola. When *k* is positive the graph shifts up *k* units; when *k* is negative the graph shifts down. The *h* value determines horizontal translation. When the sign in parenthesis is negative the graph shifts *h* units to the right; when the sign in parenthesis in positive, the graph shifts *h* units to the left.

V. Independent reinforcement and assessment. Assign additional practice problems that involve graphing quadratics in vertex form and describing the transformations from the parent function.

Names:____

Transformations of Quadratic Functions

1a. Make a table of values that can be used to graph each function

$f(x) = x^2$ $\qquad$ $g(x) = (x - 3)^2 + 4$

b. Bend 2 different colored wax stix to fit the shape of the graph on your large coordinate plane. Record the color that you used for each function on the right.

c. Describe **two** transformations of f(x) that would be completed to obtain g(x) (Be specific and use correct mathematical Vocabulary such as: translation, reflection, compression, and stretch.)

Wait for your answers to be checked before removing the wax sticks from your coordinate plane. You can begin completing the tables for question 2 while you wait.

2a. Make a table of values that can be used to graph each function

$f(x) = x^2$ $\qquad$ $g(x) = -2(x + 1)^2 - 3$

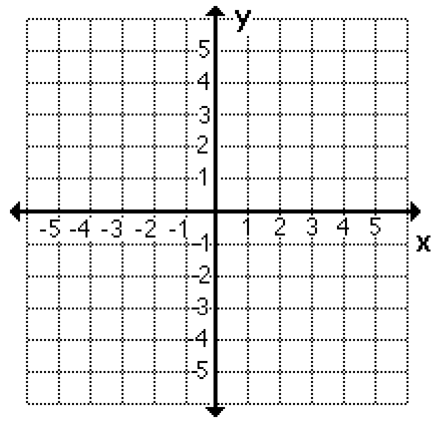
b. Bend 2 different colored wax stix to fit the shape of the graph on your coordinate plane. Record the color that you used for each function on the right.

c. Describe **four** transformations of f(x) that would be completed to obtain g(x) (Be specific and use correct mathematical Vocabulary such as: translation, reflection, compression, and stretch.)

Wait for your answers to be checked before removing the wax sticks from your large coordinate plane. You can begin making predictions for question 3 while you wait.

3a. Using the graph of $f(x) = x^2$ as a guide, predict the transformations that will occur from f(x) to $h(x) = \frac{1}{4}(x + 2)^2 + 3$. There are a total of 3 transformations.

b. Use a graphing calculator to generate a graph of each function. Sketch the graph to the right.



c. Check your predictions from part a. Record additional transformations or changes here.

4. Each of the quadratic functions on this page is written in vertex form. Vertex form looks like: $f(x) = a(x - h)^2 + k$

a. How can the *a* value be used to predict the transformations of the graph?

b. How can the *k* value be used to predict the transformations of the graph?

c. How can the *h* value be used to predict the transformations of the graph?

d. What are the *a*, *h*, and *k* values of the parent function $f(x) = x^2$

Explicit Lesson Plan for Transformations of Quadratic Functions

Title: Transformations of Quadratic Functions

Standards
Quality Core: E.2.b Use transformations to draw the graph of a relation and to determine the relation that fits a graph

Lesson Objective
I can use the vertex form of a quadratic function to identify transformations of the graph and to draw the graph of the parabola

Prerequisite Skills
Students need to be able to graph parabolas by making a table of values. A graphing calculator can be used to obtain the table. Students need to understand the meaning of the following vocabulary: transformation, translation, reflection, compression, and stretch.

Materials
Students will need paper, pencil, and a graphing calculator
Teacher will need SMARTboard and projector or other device for presenting the lesson

Preparation
Create slides of lesson definitions and examples in advance

Lesson Outline

I. Warm-up – Review vocabulary about transformations (reflection, translation, compression, stretch)

II. Check and collect homework from the previous day

III. Transformation of Functions Notes
   a. Compare the benefits of quadratic functions written in standard form to quadratic functions written in vertex form. Include the following:
      Useful properties of standard form: $f(x) = ax^2 + bx + c$
         a tells the direction of opening and the width
         c is the y-intercept
         $-b/2a$ can be used to find the vertex
      Another useful form is called vertex form
      Useful properties of vertex form: $f(x) = a(x - h)^2 + k$
         a tells the direction of opening and the width

(h, k) is the vertex
This form makes it easier to identify transformations of the parent function $f(x) = x^2$

b. Describing transformation of quadratic functions examples. For each pair of functions, use a table to graph each function and describe the transformations that occur from f(x) to g(x)
    i. $f(x) = x^2$ $g(x) = (x - 3)^2 + 4$
    ii. $f(x) = x^2$ $g(x) = -2(x + 1)^2 - 3$
    iii. $f(x) = x^2$ $g(x) = 1/4(x + 2)^2 + 3$

   Explain that $f(x) = x^2$ is the parent function for quadratics. That is the reason the function $f(x) = x^2$ is used repeatedly for comparison.

c. Generating functions to match the description of a transformation
    i. example 1: The graph of $f(x) = x^2$ is translated 5 units to the right and down 2 units. Then it is reflected across the x-axis. Write the equation of the transformation. Call it g(x)
    ii. example 2: The graph of $f(x) = (x - 3)^2$ is reflected across the y-axis and then translated 1 unit up. Write the equation of the transformation. Call it h(x)

IV. Independent reinforcement and assessment

a. Exit slip: Describe as many transformations as you can from the graph of $f(x) = x^2$ to the graph of $h(x) = -1/2\ (x - 5)^2 - 4$
b. Assign additional practice problems that involve graphing quadratics in vertex form and describing the transformations from the parent function.