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Smallest Eigenvalues For A Fractional Boundary Value Problem With A Fractional Boundary Condition

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DEDICATION

I dedicate this thesis to my husband, Klay, who has shown endless love, support, and encouragement throughout graduate school and life. I would also like to dedicate this work to my parents, Robert and Karen. I am grateful for their continuous guidance, love, and support. Without these three, none of this would have been possible.

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Abstract

We establish the existence of and then compare smallest eigenvalues for the fractional boundary value problems $D_{0+}^{\alpha}u + \lambda_1 p(t)u = 0$ and $D_{0+}^{\alpha}u + \lambda_2 q(t)u = 0$, $0 < t < 1$, satisfying boundary conditions when $n-1 < \alpha \leq n$. First, we consider the case when $0 < \beta < n-1$, satisfying $u^{(i)}(0) = 0$, $i = 0, 1, \ldots, n-2$, $D_{0+}^{\beta}u(1) = 0$. Then, the case when $\beta = 0$ is considered, satisfying the conditions $u^{(i)}(0) = 0$, $i = 0, 1, \ldots, n-2$, $u(1) = 0$.

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Chapter 1

Introduction

In the next two sections, we will follow authors Miller and Ross in their book, "An Introduction to the Fractional Calculus and Fractional Differential Equations" [35].

1.1 History of Fractional Calculus

Taking an nth order derivative when n is a positive integer can be easily understood and visualized with many types of functions. However, the question that is a bit more troubling is what if "n be 1/2?" or any other fraction for that matter. This exact question was asked by L'Hôpital in 1695. At that time, Leibniz considered fractional calculus to be a "paradox from which, one day, useful consequences will be drawn."[35]

Throughout the years, fractional calculus has intrigued many mathematicians. In 1730, Euler commented that finding fractional derivatives of the form $\frac{d^n p}{dt^n}$, where p is a function of t and n is a fraction, can be made through interpolation instead of continued differentiation such as the case when n is a positive integer, see [16]. The next year, he was able to extend the relation

$$\frac{d^n z^p}{dz^n} = \frac{p!}{(p-n)!} z^{p-n}$$

to when n is an arbitrary α

$$\frac{d^\alpha z^p}{dz^\alpha} = \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} z^{p-\alpha}.$$

This led Euler to extend the Gamma function for fractional factorial values.

In 1812, Laplace [32] was able to define a fractional derivative by the use of an integral. Then, in 1819, Lacroix [31] published the first mention of a derivative of arbitrary order. In his paper, he was able to generalize the case of integer order. Starting with $y = t^m$, where m is a positive integer, Lacroix developed the nth derivative

$$\frac{d^n y}{dt^n} = \frac{m!}{(m-n)!} t^{m-n}$$

under the condition that $m \geq n$. Using Euler's Gamma function, then

$$\frac{d^n y}{dt^n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} t^{m-n}.$$

He was then able to answer L'Hôpital's question from over a century before to show what happens when $n = 1/2$ if $y = t$. His conclusion was that

$$\frac{d^{1/2} y}{dt^{1/2}} = \frac{2\sqrt{t}}{\sqrt{\pi}}.$$

Joseph Fourier [17] also made mention of derivatives of arbitrary order by use of his integral representation of $f(t)$ in 1822. In 1823, Abel [1] was the first mathematician to apply the fractional derivative to the solutions of integral equations. He was able to solve the integral equation

$$k = \int_0^t (t-s)^{-1/2} f(s) ds$$

by operating on both sides of the equation with $\frac{d^{1/2}}{dt^{1/2}}$ to obtain

$$\frac{d^{1/2}}{dt^{1/2}} k = \sqrt{\pi} f(t). \tag{1.1}$$

It was almost a decade later when Joseph Liouville started contributing to fractional calculus. His first formula for a fractional derivative,

$$D^{\alpha}\sum_{n=0}^{\infty} c_n e^{a_n t} = \sum_{n=0}^{\infty} c_n a_n^{\alpha} e^{a_n t}, \quad \alpha, a_n > 0,$$

was able to generalize the derivative of arbitrary rational order α. However, it restricted the functions under consideration to those of the form $\sum_{n=0}^{\infty} c_n e^{a_n t}$. Liouville's second formula for a fractional derivative,

$$D^{\alpha} t^{-a} = \frac{(-1)^{\alpha}\Gamma(a+\alpha)}{\Gamma(a)} t^{-a-\alpha}, \quad a > 0,$$

restricted the functions under consideration to those of the form t^{-a}. He was also the first to attempt to solve differential equations with fractional operators.

Because of discrepancies between the formulas of Lacroix and Liouville for fractional derivatives, William Center [3], in 1848, used the function t^0 in order to show that the two were not equal. Thus, he found

$$\frac{d^{1/2}}{dt^{1/2}} t^0 = \frac{\Gamma(1)}{\Gamma(1/2)} t^{-1/2} = \frac{1}{\sqrt{\pi t}}.$$

Hence, the question of what the generalized form for a fractional derivative remained. In 1840, De Morgan [6], presented the idea that even though neither Lacroix nor Liouville defined a generalized form for $D^n t^m$, they both may have defined a formula for a more specific fractional derivative. This was proven to be true.

As a student, G.F. Bernhard Riemann [38] was able to contribute to fractional integration. However, his work did not get published until after his death in 1892. His goal was to find a generalized form of a Taylor series, and he was able to derive

$$I^{v} f(t) = \frac{1}{\Gamma(v)} \int_{c}^{t} (t-s)^{v-1} f(s) ds + \Psi(t),$$

where $\Psi(t)$ is a complementary function that Riemann added because of the ambiguity in the lower limit of integration, c. He was trying to provide a measure of deviation from the law of exponents

$$I_{c+}^{u} I_{c+}^{v} f(t) = I_{c+}^{u+v} f(t)$$

for the case $I_{c+}^{u} I_{d+}^{v} f(t)$ when $c \neq d$.

Many mathematicians commented on the existence of the complementary function, including Cayley, Peacock, and Liouville. However, errors among the mathematicians created confusion and distrust for fractional operators.

1.2 Riemann-Liouville Fractional Calculus

In 1869, N. Ya. Sonin [39] published a paper, "On differentiation with arbitrary index," that first led to what we now call the Riemann-Liouville definition. He was able to use Cauchy's integral formula for the nth derivative,

$$D^n f(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta,$$

to guide him in formulating a generalization for other values of n. When n is an integer, it is easy to generalize using the Gamma Function. However, when n is not an integer, the integrand is no longer a pole, but it contains a branch point. Thus, a branch cut is needed to contour, and this was not included in the work of Sonin and Letnikov.

Laurent [33] published a paper in 1884 where he was able to use Cauchy's integral formula as well, but his contour was an open circuit on a Riemann surface, instead of the closed circuit of Sonin and Letnikov. This produced the definition that

$$I_{c+}^{v} f(t) = \frac{1}{\Gamma(v)} \int_c^t (t-s)^{v-1} f(s) ds, \quad v > 0, \tag{1.2}$$

which used for the integration to an arbitrary order. Notice that when, $t > c$, we have Riemann's formula but without the complementary function. However, when $c = 0$, the most common version occurs which is what we refer to as the Riemann-Liouville fractional integral,

$$I_{0+}^{v} f(t) = \frac{1}{\Gamma(v)} \int_0^t (t-s)^{v-1} f(s) ds, \quad v > 0. \tag{1.3}$$

A sufficient condition that (1.3) converges is when

$$f\left(\frac{1}{t}\right) = O(t^{1-\epsilon}), \quad \epsilon > 0.$$

Integrable functions with this property are commonly known as functions of Riemann class.

When $c = -\infty$, (1.2) becomes

$$I_{-\infty}^{v} f(t) = \frac{1}{\Gamma(v)} \int_{-\infty}^t (t-s)^{v-1} f(s) ds, \quad v > 0. \tag{1.4}$$

A sufficient condition that this converge is that

$$f(-t) = O(t^{v-\epsilon}), \quad \epsilon > 0, \quad t \to \infty. \tag{1.5}$$

Integrable functions with this property are referred to as functions of Liouville class. Notice that both formulas found originally by Lacroix and Liouville that started such debate hold true for (1.2) and (1.4).

If the upper limit of integration is infinity, the Weyl fractional integral

$$W_{\infty}^{v} f(t) = \frac{1}{\Gamma(v)} \int_t^{\infty} (s-t)^{v-1} f(s) ds, \quad v > 0, \tag{1.6}$$

is used.

Notice we have yet to define the standard definition for Riemann-Liouville fractional derivatives, $D_{c^+}^{\alpha}$, when $\alpha > 0$. Let n be the smallest integer greater than

α. Then, $0 < n - \alpha \leq 1$. Thus, the fractional derivative of $f(t)$ can be defined as

$$D_{c^+}^{\alpha} f(t) = D^n \left[I_{c^+}^{n-\alpha} f(t) \right] \tag{1.7}$$

with arbitrary order of α where $t > 0$. If $c = 0$, then $D_{0+}^{\alpha} f(t)$ defines the standard Riemann-Liouville fractional derivative.

1.3 u_0-Positive Operators and Smallest Eigenvalues

In this paper, we will consider three boundary value problems consisting of fractional differential equations

$$D_{0+}^{\alpha} u + \lambda_1 p(t) u = 0, \quad 0 < t < 1,$$

$$D_{0+}^{\alpha} u + \lambda_2 q(t) u = 0, \quad 0 < t < 1;$$

the first, for $1 < \alpha \leq 2$ with boundary conditions $u(0) = D_{0+}^{\beta} u(1) = 0$; the second, for $n-1 < \alpha \leq n$ where n is a natural number, with boundary conditions $u^{(i)}(0) = 0$, $i = 0, 1, \ldots, n-2$, $D_{0+}^{\beta} u(1) = 0$; and the last, also for $n - 1 < \alpha \leq n$, satisfying the boundary conditions $u^{(i)}(0) = 0,\ i = 0, \ldots, n-2,\ u(1) = 0$. Boundary value problems are unique in that they may have no solutions, infinitely many solutions, or a unique solution. The real numbers λ_1 and λ_2 such that these boundary value problems yield a nontrivial solution are called eigenvalues.

The purpose of this paper is to show the existence of smallest eigenvalues by using the theory of u_0-positive operators with respect to a cone in a Banach space. Then, a comparison of those eigenvalues can be made. The technique for showing the existence and then comparing these smallest eigenvalues involves the application of sign properties of the Green's function for the specified boundary value problem, followed by the application of u_0-positive operators with respect to a cone in a Banach space. These applications are presented in books by Krasnosel'skii [29] and by Krein and Rutman [30].

These cone-theoretic techniques have been used by many authors to study the existence of smallest eigenvalues of ordinary boundary value problems. See [2, 4, 5, 11, 12, 13, 14, 18, 19, 20, 21, 23, 24, 25, 26, 34, 36, 37, 40, 41]. Recently, Eloe and Neugebauer [15] developed a method for showing the existence of and comparing smallest eigenvalues for fractional boundary value problems. This method has been used in a few papers [7, 8, 9, 10, 22, 28, 42]. Here, we look to extend the results to a fractional boundary value problem with fractional boundary conditions.

Chapter 2

Smallest Eigenvalues of a Fractional Boundary Value Problem

2.1 Introduction to Problem

Let α and β be real numbers with $1 < \alpha \leq 2$ and $0 < \beta < 1$. We will consider the eigenvalue problems

$$D_{0+}^{\alpha}u + \lambda_1 p(t)u = 0, \quad 0 < t < 1, \tag{2.1}$$

$$D_{0+}^{\alpha}u + \lambda_2 q(t)u = 0, \quad 0 < t < 1, \tag{2.2}$$

satisfying the boundary conditions

$$u(0) = D_{0+}^{\beta}u(1) = 0, \tag{2.3}$$

where D_{0+}^{α} and D_{0+}^{β} are the standard Riemann-Liouville fractional derivatives, and $p(t)$ and $q(t)$ are continuous nonnegative functions on $[0,1]$, where neither $p(t)$ nor $q(t)$ vanishes identically on any nondegenerate compact subinterval of $[0,1]$. In this paper, we will show the existence of smallest eigenvalues (2.1),(2.3) and (2.2),(2.3). Assuming $p(t) \leq q(t)$, we will then compare these smallest eigenvalues.

2.2 Preliminary Definitions and Theorems

Definition 2.1. For $0 < t < \infty$, the Gamma Function is defined as

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt.$$

Notice that Γ satisfies the following properties:

(i) $\Gamma(x+1) = x\Gamma(x)$,

(ii) $\Gamma(n+1) = n!$.

Definition 2.2. The α-th Riemann-Liouville fractional derivative of the function $u : [0,1] \to \mathbb{R}$, denoted $D^\alpha_{0+}u$, is defined as

$$D^\alpha_{0+}u(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} u(s) ds,$$

provided the right-hand side exists.

For $1 < \alpha \leq 2$, the Riemann-Liouville fractional derivative we consider

$$\begin{aligned} D^\alpha_{0+}u(t) &= \frac{1}{\Gamma(2-\alpha)} \frac{d^2}{dt^2} \int_0^t (t-s)^{2-\alpha-1} u(s) ds \\ &= \frac{1}{\Gamma(2-\alpha)} \frac{d^2}{dt^2} \int_0^t (t-s)^{1-\alpha} u(s) ds. \end{aligned}$$

Definition 2.3. Let $\mathcal{B}$ be a Banach space over $\mathbb{R}$. A closed nonempty subset $\mathcal{P}$ of $\mathcal{B}$ is said to be a cone provided the following:

(i) $\alpha u + \beta v \in \mathcal{P}$, for all $u, v \in \mathcal{P}$ and all $\alpha, \beta \geq 0$, and

(ii) $u \in \mathcal{P}$ and $-u \in \mathcal{P}$ implies $u = 0$.

Definition 2.4. A cone $\mathcal{P}$ is solid if the interior, $\mathcal{P}^\circ$ of $\mathcal{P}$ is nonempty. A cone $\mathcal{P}$ is reproducing if $\mathcal{B} = \mathcal{P} - \mathcal{P}$; i.e., given $w \in \mathcal{B}$, there exist $u, v \in \mathcal{P}$ such that $w = u - v$.

Remark 2.1. Krasnosel'skii [29] proved that every solid cone is reproducing.

Cones generate a natural partial ordering on a Banach space.

Definition 2.5. Let $\mathcal{P}$ be a cone in a real Banach space $\mathcal{B}$. If $u, v \in \mathcal{B}$, say that $u \leq v$ with respect to $\mathcal{P}$ if $v - u \in \mathcal{P}$. If both $M, N : \mathcal{B} \to \mathcal{B}$ are bounded linear operators, say $M \leq N$ with respect to $\mathcal{P}$ if $Mu \leq Nu$ for all $u \in \mathcal{P}$.

Definition 2.6. A bounded linear operator $M : \mathcal{B} \to \mathcal{B}$ is u_0-positive with respect to $\mathcal{P}$ if there exists $u_0 \in \mathcal{P}\backslash\{0\}$ such that for each $u \in \mathcal{P}\backslash\{0\}$, there exist $k_1(u) > 0$ and $k_2(u) > 0$ such that $k_1 u_0 \leq Mu \leq k_2 u_0$ with respect to $\mathcal{P}$.

The following two results are fundamental to our existence and comparison results and are attributed to Krasnosel'skii [29]. The proof of Theorem 2.1 can be found in [29], and the proof of Theorem 2.2 is provided by Keener and Travis [27] as an extension of Krasonel'skii's results.

Theorem 2.1. *Let $\mathcal{B}$ be a real Banach space, and let $\mathcal{P} \subset \mathcal{B}$ be a reproducing cone. Let $L : \mathcal{B} \to \mathcal{B}$ be a compact, u_0-positive, linear operator. Then L has an essentially unique eigenvector in $\mathcal{P}$, and the corresponding eigenvalue is simple, positive, and larger than the absolute value of any other eigenvalue.*

Theorem 2.2. *Let $\mathcal{B}$ be a real Banach space, and $\mathcal{P} \subset \mathcal{B}$ be a cone. Let both $M, N : \mathcal{B} \to \mathcal{B}$ be bounded linear operators, and assume that at least one of the operators is u_0-positive. If $M \leq N$, then*

(1) $Mu_1 \geq \lambda_1 u_1$ for some $u_1 \in \mathcal{P}$ and some $\lambda_1 > 0$;

(2) $Nu_2 \leq \lambda_2 u_2$ for some $u_2 \in \mathcal{P}$ and some $\lambda_2 > 0$, thus $\lambda_1 \leq \lambda_2$; and

(3) if $\lambda_1 = \lambda_2$, then u_1 is a scalar multiple of u_2.

2.3 Comparison of Smallest Eigenvalues

The Green's function for $-D^{\alpha}_{0+}u = 0$, (2.3) is given by

$$G(\beta; t, s) = \begin{cases} \frac{t^{\alpha-1}(1-s)^{\alpha-1-\beta} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, \\ \\ \frac{t^{\alpha-1}(1-s)^{\alpha-1-\beta}}{\Gamma(\alpha)}, & 0 \leq t \leq s < 1. \end{cases} \tag{2.4}$$

Therefore, $u(t) = \lambda_1 \int_0^1 G(\beta; t, s)p(s)u(s)ds$ if and only if $u(t)$ solves (2.1),(2.3). Similarly, $u(t) = \lambda_2 \int_0^1 G(\beta; t, s)q(s)u(s)ds$ if and only if $u(t)$ solves (2.2),(2.3). Notice that $G(\beta; t, s) \geq 0$ on $[0, 1] \times [0, 1)$ and $G(\beta; t, s) > 0$ on $(0, 1] \times (0, 1)$.

Define the Banach Space

$$\mathcal{B} = \{u : u = t^{\alpha-1}v, v \in C[0, 1]\},$$

with the norm

$$\|u\| = |v|_0,$$

where $|v|_0 = \sup\limits_{t\in[0,1]} |v(t)|$ denotes the usual supremum norm. Notice that for $u \in \mathcal{B}$,

$$|u|_0 = |t^{\alpha-1}v|_0 \leq t^{\alpha-1}\|u\|,$$

implying

$$|u|_0 \leq \|u\|.$$

Define the linear operators

$$Mu(t) = \int_0^1 G(\beta; t, s)p(s)u(s)ds \tag{2.5}$$

and

$$Nu(t) = \int_0^1 G(\beta; t, s)q(s)u(s)ds. \tag{2.6}$$

Now,

$$\begin{aligned} Mu(t) &= \int_0^1 \frac{t^{\alpha-1}(1-s)^{\alpha-1-\beta}}{\Gamma(\alpha)}p(s)u(s)ds - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)u(s)ds \\ &= t^{\alpha-1}\left(\int_0^1 \frac{(1-s)^{\alpha-1-\beta}}{\Gamma(\alpha)}p(s)u(s)ds \right.\\ &\quad \left. -t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)u(s)ds\right). \end{aligned}$$

Notice that, since $\alpha > 1$ and $\beta < 1$,

$$\begin{aligned}
\left| \int_0^1 \frac{(1-s)^{\alpha-1-\beta}}{\Gamma(\alpha)} p(s)u(s)ds \right| &\leq \frac{|p|_0|v|_0}{\Gamma(\alpha)} \left| \int_0^1 s^{\alpha-1}(1-s)^{\alpha-1-\beta} ds \right| \\
&= \frac{|p|_0|v|_0}{\Gamma(\alpha)} |B(\alpha, \alpha-\beta)| \\
&= \frac{|p|_0|v|_0}{\Gamma(\alpha)} \left| \frac{\Gamma(\alpha)\Gamma(\alpha-\beta)}{\Gamma(\alpha+\beta)} \right| \\
&= \frac{|p|_0|v|_0\Gamma(\alpha-\beta)}{\Gamma(\alpha+\beta)} \\
&< \infty,
\end{aligned}$$

where $B(a, b)$ is the Beta Function, defined by

$$B(a, b) = \int_0^1 u^{a-1}(1-u)^{b-1} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

Therefore, the first term inside the parentheses is well-defined.

Set

$$g(t) = \begin{cases} 0, & t = 0, \\ t^{1-\alpha} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p(s)u(s)ds, & 0 < t \leq 1. \end{cases} \tag{2.7}$$

Then, for $|p|_0 = P$ and $\|u\| = L$ we have,

$$\begin{aligned}
|g(t)| &= \left| t^{1-\alpha} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p(s)u(s)ds \right| \\
&= \left| t^{1-\alpha} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p(s)s^{\alpha-1}v(s)ds \right| \\
&\leq PLt^{1-\alpha} \int_0^t (t-s)^{\alpha-1}s^{\alpha-1}ds \\
&\leq PLt^{1-\alpha}t^{\alpha-1} \int_0^t (t-s)^{\alpha-1}ds \\
&= \frac{PLt^\alpha}{\alpha},
\end{aligned}$$

where $\dfrac{PL}{\alpha} \geq 0$. So, $\lim_{t \to 0^+} g(t) = g(0) = 0$. Thus, $g \in C[0, 1]$.

Therefore, $M : \mathcal{B} \to \mathcal{B}$. An argument similar to the one made by Eloe and Neugebauer in [15] shows that M is compact, and the same can be said for N. Thus, we have the following result.

Theorem 2.3. *The operators $M, N : \mathcal{B} \to \mathcal{B}$ are compact.*

Next, we define the cone

$$\mathcal{P} = \{u \in \mathcal{B} \mid u(t) \geq 0 \text{ for } t \in [0,1]\}.$$

Lemma 2.1. *The cone $\mathcal{P}$ is solid in $\mathcal{B}$ and hence reproducing.*

Proof. Define

$$\Omega := \{u = t^{\alpha-1}v \in \mathcal{B} : u(t) > 0 \text{ for } t \in (0,1], v(0) > 0\}. \tag{2.8}$$

We will show $\Omega \subset \mathcal{P}^{\circ}$. Let $u \in \beta$ such that $u = t^{\alpha-1}v$. Since $v(0) > 0$, there exists an $\epsilon_1 > 0$ such that $v(0) - \epsilon_1 > 0$. Since $v \in C[0,1]$, there exists an $a \in (0,1)$ such that $v(t) > \epsilon_1$ for all $t \in (0,a)$. So $u(t) = t^{\alpha-1}v(t) > \epsilon_1 t^{\alpha-1}$ for all $t \in (0,a)$. Also, since $u(t) > 0$ on $[a,1]$, there exists an $\epsilon_2 > 0$ with $u(t) - \epsilon_2 > 0$ for all $t \in [a,1]$.

Let $\epsilon = \min\{\frac{\epsilon_1}{2}, \frac{\epsilon_2}{2}\}$. Define $B_\epsilon(u) = \{\hat{u} \in \mathcal{B} : \|u - \hat{u}\| < \epsilon\}$. Let $\hat{u} \in B_\epsilon(u)$. Then, $\hat{u} = t^{\alpha-1}\hat{v}$, where $\hat{v} \in C[0,1]$. Now, $|\hat{u}(t) - u(t)| \leq t^{\alpha-1}\|\hat{u} - u\| < \epsilon t^{\alpha-1}$. So for $t \in (0,a)$, $\hat{u}(t) > u(t) - t^{\alpha-1}\epsilon > t^{\alpha-1}\epsilon_1 - t^{\alpha-1}\epsilon_1/2 = t^{\alpha-1}\epsilon_1/2$. Thus, $\hat{u}(t) > 0$ for $t \in (0,a)$. Also, $|\hat{u}(t) - u(t)| \leq \|\hat{u} - u\| < \epsilon$. So for $t \in [a,1]$, $\hat{u}(t) > u(t) - \epsilon > \epsilon_2 - \epsilon_2/2 > 0$. So $\hat{u}(t) > 0$ for all $t \in [a,1]$. Hence, $\hat{u} \in \mathcal{P}$ implying $B_\epsilon(u) \subset \mathcal{P}$. Therefore, $\Omega \subset \mathcal{P}^{\circ}$. □

Lemma 2.2. *The bounded linear operators M and N are u_0-positive with respect to $\mathcal{P}$.*

Proof. First, we show $M : \mathcal{P}\backslash\{0\} \to \Omega \subset \mathcal{P}^{\circ}$. Let $u \in \mathcal{P}$. So $u(t) \geq 0$. Then, since $G(\beta;t,s) \geq 0$ on $[0,1] \times [0,1)$ and $p(t) \geq 0$ on $[0,1]$,

$$Mu(t) = \int_0^1 G(\beta;t,s)p(s)u(s)ds \geq 0,$$

for $0 \leq t \leq 1$. So $M : \mathcal{P} \to \mathcal{P}$.

Now, let $u \in \mathcal{P}\backslash\{0\}$. There exists a compact interval $[a,b] \subset [0,1]$ such that $u(t) > 0$ and $p(t) > 0$ for all $t \in [a,b]$. Then, since $G(\beta;t,s) > 0$ on $(0,1] \times (0,1)$,

$$\begin{aligned} Mu(t) &= \int_0^1 G(\beta;t,s)p(s)u(s)ds \\ &\geq \int_a^b G(\beta;t,s)p(s)u(s)ds \\ &> 0 \end{aligned}$$

for $0 < t \leq 1$.

Now,

$$\begin{aligned} &Mu(t) \\ &\quad = t^{\alpha-1}\left(\int_0^1 \frac{(1-s)^{\alpha-1-\beta}}{\Gamma(\alpha)}p(s)u(s)ds - t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)u(s)ds\right). \end{aligned}$$

Let

$$v(t) = \int_0^1 \frac{(1-s)^{\alpha-1-\beta}}{\Gamma(\alpha)}p(s)u(s)ds - t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)u(s)ds.$$

Thus, $v(0) = \int_0^1 \frac{(1-s)^{\alpha-1-\beta}}{\Gamma(\alpha)}p(s)u(s)ds - g(0) > 0$, where $g(t)$ was defined as an equation previously in (2.7). So $M : \mathcal{P}\backslash\{0\} \to \Omega \subset \mathcal{P}^\circ$.

Now, choose $u_0 \in \mathcal{P}\backslash\{0\}$, and let $u \in \mathcal{P}\backslash\{0\}$. So $Mu \in \Omega \subset \mathcal{P}^\circ$. Choose $k_1 > 0$ sufficiently small and k_2 sufficiently large so that $Mu - k_1u_0 \in \mathcal{P}^\circ$ and $u_0 - \frac{1}{k_2}Mu \in \mathcal{P}^\circ$. So $k_1u_0 \leq Mu$ with respect to $\mathcal{P}$, and $Mu \leq k_2u_0$ with respect to $\mathcal{P}$. Thus $k_1u_0 \leq Mu \leq k_2u_0$ with respect to $\mathcal{P}$ and so M is u_0-positive with respect to $\mathcal{P}$. A similar argument shows N is u_0-positive. $\square$

Theorem 2.4. *Let $\mathcal{B}$, $\mathcal{P}$, M, and N be defined as earlier. Then M (respectively, N) has an eigenvalue that is simple, positive, and larger than the absolute value of any other eigenvalue, with an essentially unique eigenvector that can be chosen to be in $\mathcal{P}^\circ$.*

Proof. Since M is a compact linear operator that is u_0-positive with respect to $\mathcal{P}$, by Theorem 2.1, M has an essentially unique eigenvector, say $u \in \mathcal{P}$, and eigenvalue Λ with the above properties. Since $u \neq 0$, we have that $Mu \in \Omega \subset \mathcal{P}^\circ$ and $u = M\left(\frac{1}{\Lambda}u\right) \in \mathcal{P}^\circ$. □

Theorem 2.5. *Let $\mathcal{B}$, $\mathcal{P}$, M, and N be defined as earlier. Let $p(t) \le q(t)$ on $[0,1]$. Let Λ_1 and Λ_2 be the eigenvalues defined in Theorem 2.4 associated with M and N, respectively, with the essentially unique eigenvectors u_1 and $u_2 \in \mathcal{P}^\circ$. Then $\Lambda_1 \le \Lambda_2$, and $\Lambda_1 = \Lambda_2$ if and only if $p(t) = q(t)$ on $[0,1]$.*

Proof. Let $p(t) \le q(t)$ on $[0,1]$. So, for any $u \in \mathcal{P}$ and $t \in [0,1]$,

$$
\begin{aligned}
(Nu - Mu)(t) &= \int_0^1 G(t,s)(q(s)-p(s))u(s)ds \\
&= t^{\alpha-1}\left(\int_0^1 \frac{(1-s)^{\alpha-1-\beta}}{\Gamma(\alpha)}(q(s)-p(s))u(s)ds \right.\\
&\quad \left. - t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(q(s)-p(s))u(s)ds\right) \\
&\ge 0.
\end{aligned}
$$

So $Nu - Mu \in \mathcal{P}$ for all $u \in \mathcal{P}$, or $M \le N$ with respect to $\mathcal{P}$. Then, by Theorem 2.2, $\Lambda_1 \le \Lambda_2$.

If $p(t) = q(t)$, then $\Lambda_1 = \Lambda_2$. Now suppose $p(t) \neq q(t)$. So $p(t) < q(t)$ on some subinterval $[a,b] \subset [0,1]$.

Let

$$
v(t) = \int_0^1 \frac{(1-s)^{\alpha-1-\beta}}{\Gamma(\alpha)}(q(s)-p(s))u(s)ds - t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(q(s)-p(s))u(s)ds.
$$

Since $p(t) < q(t)$, then $v(0) > 0$. So, $(N-M)u_1 \in \Omega \subseteq \mathcal{P}^\circ$. So there exists $\epsilon > 0$

such that $(N - M)u_1 - \epsilon u_1 \in \mathcal{P}$. So $\Lambda_1 u_1 + \epsilon u_1 = Mu_1 + \epsilon u_1 \leq Nu_1$, implying $Nu_1 \geq (\Lambda_1 + \epsilon)u_1$. Since $N \leq N$ and $Nu_2 = \Lambda_2 u_2$, by Theorem 2.2, $\Lambda_1 + \epsilon \leq \Lambda_2$, or $\Lambda_1 < \Lambda_2$. $\square$

Lemma 2.3. *The eigenvalues of* (2.1),(2.3) *are reciprocals of eigenvalues of* M, *and conversely. Similarly, eigenvalues of* (2.2),(2.3) *are reciprocals of eigenvalues of* N, *and conversely.*

Proof. Let Λ be an eigenvalue of M with corresponding eigenvector $u(t)$. Notice that

$$\Lambda u(t) = Mu(t) = \int_0^1 G(\beta; t, s)p(s)u(s)ds,$$

if and only if

$$u(t) = \frac{1}{\Lambda}\int_0^1 G(\beta; t, s)p(s)u(s)ds,$$

if and only if

$$D_{0+}^{\alpha}u(t) + \frac{1}{\Lambda}p(t)u(t) = 0, \quad 0 < t < 1,$$

with

$$u(0) = D_{0+}^{\beta}u(1) = 0.$$

So, $\dfrac{1}{\Lambda}$ is an eigenvalue of (2.1),(2.3), if and only if Λ is an eigenvalue of M. A similar argument can be made that the reciprocals of eigenvalues of N are eigenvalues of (2.2),(2.3) and vice versa. $\square$

Since the eigenvalues of (2.1),(2.3) are reciprocals of eigenvalues of M and conversely, and the eigenvalues of (2.2),(2.3) are reciprocals of eigenvalues of N and conversely, the following theorem is an immediate consequence of Theorems 2.4 and 2.5.

Theorem 2.6. *Assume the hypotheses of Theorem 2.5. Then there exists smallest positive eigenvalues* λ_1 *and* λ_2 *of* (2.1),(2.3) *and* (2.2),(2.3), *respectively, each of which is simple, positive, and less than the absolute value of any other eigenvalue of the corresponding problems. Also, eigenfunctions corresponding to* λ_1 *and* λ_2

may be chosen to belong to $\mathcal{P}^\circ$. *Finally,* $\lambda_1 \geq \lambda_2$, *and* $\lambda_1 = \lambda_2$ *if and only if* $p(t) = q(t)$ *for all* $t \in [0, 1]$.

Chapter 3

An Extension to a Higher Order Problem

3.1 The Extension

Now, we will consider the arbitrary case for all fractional derivatives and show the existence and comparison of these smallest eigenvalues. Since we are showing the arbitrary case of the case we presented in Chapter 2, we will again use the techniques Eloe and Neugebauer developed to show the existence and comparison of smallest eigenvalues for fractional boundary value problems. Thus, the proofs will be similar to those in Chapter 2.

Let $n \in \mathbb{N}$, $n \geq 2$. Let α, β be real numbers such that $n-1 < \alpha \leq n$ and $0 < \beta < n-1$. Consider the eigenvalue problems

$$D_{0+}^{\alpha} u + \lambda_1 p(t) u = 0, \quad 0 < t < 1, \tag{3.1}$$

$$D_{0+}^{\alpha} u + \lambda_2 q(t) u = 0, \quad 0 < t < 1, \tag{3.2}$$

which satisfy the boundary conditions

$$u^{(i)}(0) = 0, \ i = 0, 1, \ldots, n-2, \quad D_{0+}^{\beta} u(1) = 0, \tag{3.3}$$

where D_{0+}^{α} and D_{0+}^{β} are the standard Riemann-Liouville derivatives, and $p(t)$ and $q(t)$ are continuous nonnegative functions on $[0, 1]$, where neither $p(t)$ nor $q(t)$

vanishes identically on any nondegenerate compact subinterval of $[0,1]$. Using the preliminary definitions and Theorem 2.1 and Theorem 2.2, we will show the existence of smallest eigenvalues (3.1),(3.3) and (3.2),(3.3). We will then compare these smallest eigenvalues under the assumption that $p(t) \le q(t)$.

Now, for $n-1 < \alpha \le n$, the α-th Riemann-Liouville fractional derivative of the function $u : [0,1] \to \mathbb{R}$ will be defined as

$$D_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} u(s) ds, \tag{3.4}$$

provided the right-hand side exists.

3.2 Comparison of Smallest Eigenvalues

The Green's function for $-D_{0+}^{\alpha} u = 0$, (3.3) is given by

$$G(\beta; t, s) = \begin{cases} \frac{t^{\alpha-1}(1-s)^{\alpha-1-\beta} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s \le t \le 1, \\ \\ \frac{t^{\alpha-1}(1-s)^{\alpha-1-\beta}}{\Gamma(\alpha)}, & 0 \le t \le s < 1. \end{cases} \tag{3.5}$$

Therefore, $u(t) = \lambda_1 \int_{(0)}^1 G(\beta; t, s) p(s) u(s) ds$ if and only if $u(t)$ solves (3.1),(3.3). Similarly, $u(t) = \lambda_2 \int_0^1 G(\beta; t, s) q(s) u(s) ds$ if $u(t)$ solves (3.2),(3.3). Notice that $G(\beta; t, s) \ge 0$ on $[0,1] \times [0,1)$ and $G(\beta; t, s) > 0$ on $(0,1] \times (0,1)$.

Now, define the Banach Space

$$\mathcal{B} = \{u : u = t^{\alpha-1} v, v \in C[0,1]\},$$

with the norm

$$\|u\| = |v|_0,$$

where $|v|_0 = \sup_{t \in [0,1]} |v(t)|$ denotes the usual supremum norm. Notice that for $u \in \mathcal{B}$,

$$|u|_0 = |t^{\alpha-1} v|_0 \le t^{\alpha-1} \|u\|,$$

implying

$$|u|_0 \leq \|u\|.$$

Define the linear operators

$$Mu(t) = \int_0^1 G(\beta; t, s)p(s)u(s)ds \tag{3.6}$$

and

$$Nu(t) = \int_0^1 G(\beta; t, s)q(s)u(s)ds. \tag{3.7}$$

Now,

$$\begin{aligned} Mu(t) &= \int_0^1 \frac{t^{\alpha-1}(1-s)^{\alpha-1-\beta}}{\Gamma(\alpha)} p(s)u(s)ds - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p(s)u(s)ds \\ &= t^{\alpha-1}\left(\int_0^1 \frac{(1-s)^{\alpha-1-\beta}}{\Gamma(\alpha)} p(s)u(s)ds \right. \\ &\quad \left. -t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p(s)u(s)ds\right). \end{aligned}$$

Notice that, since $n-1 < \alpha \leq n$ and $0 < \beta < n-1$,

$$\begin{aligned} \left|\int_0^1 \frac{(1-s)^{\alpha-1-\beta}}{\Gamma(\alpha)} p(s)u(s)ds\right| &\leq \frac{|p|_0|v|_0}{\Gamma(\alpha)}\left|\int_0^1 s^{\alpha-1}(1-s)^{\alpha-1-\beta}ds\right| \\ &= \frac{|p|_0|v|_0}{\Gamma(\alpha)}\left|B(\alpha,\alpha-\beta)\right| \\ &= \frac{|p|_0|v|_0}{\Gamma(\alpha)}\left|\frac{\Gamma(\alpha)\Gamma(\alpha-\beta)}{\Gamma(\alpha+\beta)}\right| \\ &= \frac{|p|_0|v|_0\Gamma(\alpha-\beta)}{\Gamma(\alpha+\beta)} \\ &< \infty \end{aligned}$$

since $\Gamma(\alpha-\beta) \leq (n-1)!$ and $\Gamma(\alpha+\beta) > (n-1)!$.

Therefore, the first term inside the parentheses is well-defined.

Set

$$g(t) = \begin{cases} 0, & t = 0, \\ \\ t^{1-\alpha} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p(s)u(s)ds, & 0 < t \leq 1. \end{cases}$$

Then, for $|p_0| = P$, $\|u\| = L$,

$$\begin{aligned} |g(t)| &= \left| t^{1-\alpha} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p(s)u(s)ds \right| \\ &= \left| t^{1-\alpha} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p(s)s^{\alpha-1}v(s)ds \right| \\ &\leq PLt^{1-\alpha} \int_0^t (t-s)^{\alpha-1}s^{\alpha-1}ds \\ &\leq PLt^{1-\alpha}t^{\alpha-1} \int_0^t (t-s)^{\alpha-1}ds \\ &= \frac{PLt^\alpha}{\alpha}, \end{aligned}$$

where $\dfrac{PL}{\alpha} \geq 0$. So, $\lim\limits_{t\to 0^+} g(t) = g(0) = 0$. Thus, $g \in C[0,1]$. Therefore, $M : \mathcal{B} \to \mathcal{B}$.

A similar argument to [15] shows that M is compact. This can also be applied to N, to show that N is compact.

Theorem 3.1. *The operators $M, N : \mathcal{B} \to \mathcal{B}$ are compact.*

Next, we define the cone

$$\mathcal{P} = \{u \in \mathcal{B} \mid u(t) \geq 0 \text{ for } t \in [0,1]\}.$$

Lemma 3.1. *The cone $\mathcal{P}$ is solid in $\mathcal{B}$ and hence reproducing.*

Proof. Define

$$\Omega := \{u = t^{\alpha-1}v \in \mathcal{B} : u(t) > 0 \text{ for } t \in (0,1], v(0) > 0\} \tag{3.8}$$

We will show that $\Omega \subset \mathcal{P}^\circ$. Since $v(0) > 0$, there exists an $\epsilon_1 > 0$ such that $v(0) - \epsilon_1 > 0$. Since $v \in C[0,1]$, there exists an $a \in (0,1)$ such that $v(t) > \epsilon_1$ for all $t \in (0, a)$. So $u(t) = t^{\alpha-1}v(t) > \epsilon_1 t^{\alpha-1}$ for all $t \in (0, a)$. Now, on the interval $[a, 1]$, $u(t) > 0$. Thus there exists an $\epsilon_2 > 0$ with $u(t) - \epsilon_2 > 0$ for all $t \in [a, 1]$.

Let $\epsilon = \min\{\frac{\epsilon_1}{2}, \frac{\epsilon_2}{2}\}$. Define $B_\epsilon(u) = \{\hat{u} \in \mathcal{B} : \|u - \hat{u}\| < \epsilon\}$. Let $\hat{u} \in B_\epsilon(u)$. Then, $\hat{u} = t^{\alpha-1}\hat{v}$, where $\hat{v} \in C[0, 1]$. Now, $|\hat{u}(t) - u(t)| \leq t^{\alpha-1}\|\hat{u} - u\| < \epsilon t^{\alpha-1}$. So for $t \in (0, a)$, $\hat{u}(t) > u(t) - t^{\alpha-1}\epsilon > t^{\alpha-1}\epsilon_1 - t^{\alpha-1}\epsilon_1/2 = t^{\alpha-1}\epsilon_1/2$. Thus, $\hat{u}(t) > 0$ for $t \in (0, a)$. Also, $|\hat{u}(t) - u(t)| \leq \|\hat{u} - u\| < \epsilon$. So for $t \in [a, 1]$, $\hat{u}(t) > u(t) - \epsilon > \epsilon_2 - \epsilon_2/2 > 0$. Thus, $\hat{u}(t) > 0$ for all $t \in [a, 1]$. Hence, $\hat{u} \in \mathcal{P}$ and thus $B_\epsilon(u) \subset \mathcal{P}$. Therefore, $\Omega \subset \mathcal{P}^\circ$. □

Lemma 3.2. *The bounded linear operators M and N are u_0-positive with respect to $\mathcal{P}$.*

Proof. First, we show $M : \mathcal{P}\backslash\{0\} \to \Omega \subset \mathcal{P}^\circ$. Let $u \in \mathcal{P}$. So $u(t) \geq 0$. Then since $G(\beta; t, s) \geq 0$ on $[0, 1] \times [0, 1)$ and $p(t) \geq 0$ and from the definition $u(t) \geq 0$ on $[0, 1]$,

$$Mu(t) = \int_0^1 G(\beta; t, s)p(s)u(s)ds \geq 0,$$

for $0 \leq t \leq 1$. So $M : \mathcal{P} \to \mathcal{P}$.

Now, let $u \in \mathcal{P}\backslash\{0\}$. So there exists a compact interval $[a, b] \subset [0, 1]$ such that $u(t) > 0$ and $p(t) > 0$ for all $t \in [a, b]$. Then, since $G(\beta; t, s) > 0$ on $(0, 1] \times (0, 1)$,

$$\begin{aligned} Mu(t) &= \int_0^1 G(\beta; t, s)p(s)u(s)ds \\ &\geq \int_a^b G(\beta; t, s)p(s)u(s)ds \\ &> 0, \end{aligned}$$

for $0 < t \leq 1$.

Now,

$$\begin{aligned} &Mu(t) \\ &\quad = t^{\alpha-1}\left(\int_0^1 \frac{(1-s)^{\alpha-1-\beta}}{\Gamma(\alpha)}p(s)u(s)ds - t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)u(s)ds\right). \end{aligned}$$

Let

$$v(t) = \int_0^1 \frac{(1-s)^{\alpha-1-\beta}}{\Gamma(\alpha)} p(s)u(s)ds - t^{1-\alpha} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p(s)u(s)ds.$$

Thus, $v(0) = \int_0^1 \frac{(1-s)^{\alpha-1-\beta}}{\Gamma(\alpha)} p(s)u(s)ds - g(0) > 0$ where $g(t)$ was defined previously. So $M : \mathcal{P}\backslash\{0\} \to \Omega \subset \mathcal{P}^\circ$.

Now, choose $u_0 \in \mathcal{P}\backslash\{0\}$, and let $u \in \mathcal{P}\backslash\{0\}$. So $Mu \in \Omega \subset \mathcal{P}^\circ$. Choose $k_1 > 0$ sufficiently small and k_2 sufficiently large so that $Mu - k_1u_0 \in \mathcal{P}^\circ$ and $u_0 - \frac{1}{k_2}Mu \in \mathcal{P}^\circ$. So $k_1u_0 \leq Mu$ with respect to $\mathcal{P}$ and $Mu \leq k_2u_0$ with respect to $\mathcal{P}$. Thus $k_1u_0 \leq Mu \leq k_2u_0$ with respect to $\mathcal{P}$ and so M is u_0-positive with respect to P. A similar argument shows N is u_0-positive. □

Theorem 3.2. *Let $\mathcal{B}$, $\mathcal{P}$, M, and N be defined as earlier. Then M (and N) has an eigenvalue that is simple, positive, and larger than the absolute value of any other eigenvalue, with an essentially unique eigenvector that can be chosen to be in $\mathcal{P}^\circ$.*

Proof. Since M is a compact linear operator that is u_0-positive with respect to $\mathcal{P}$, by Theorem 2.1, M has an essentially unique eigenvector, say $u \in \mathcal{P}$, and eigenvalue Λ with the above properties. Since $u \neq 0$, $Mu \in \Omega \subset \mathcal{P}^\circ$ and $u = M\left(\frac{1}{\Lambda}u\right) \in \mathcal{P}^\circ$. □

Theorem 3.3. *Let $\mathcal{B}$, $\mathcal{P}$, M, and N be defined as earlier. Let $p(t) \leq q(t)$ on $[0,1]$. Let Λ_1 and Λ_2 be the eigenvalues defined in Theorem 3.2 associated with M and N, respectively, with the essentially unique eigenvectors u_1 and $u_2 \in \mathcal{P}^\circ$. Then $\Lambda_1 \leq \Lambda_2$, and $\Lambda_1 = \Lambda_2$ if and only if $p(t) = q(t)$ on $[0,1]$.*

Proof. Let $p(t) \leq q(t)$ on $[0,1]$. So for any $u \in \mathcal{P}$ and $t \in [0,1]$,

$$\begin{aligned}(Nu - Mu)(t) &= \int_0^1 G(\beta; t, s)(q(s) - p(s))u(s)ds \\ &= t^{\alpha-1}\left(\int_0^1 \frac{(1-s)^{\alpha-1-\beta}}{\Gamma(\alpha)}(q(s)-p(s))u(s)ds \right.\\ &\quad \left. -t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(q(s)-p(s))u(s)ds\right)\end{aligned}$$

$$\geq 0$$

So $Nu - Mu \in \mathcal{P}$ for all $u \in \mathcal{P}$, or $M \leq N$ with respect to $\mathcal{P}$. Then, by Theorem 2.2, $\Lambda_1 \leq \Lambda_2$.

If $p(t) = q(t)$, then $\Lambda_1 = \Lambda_2$. Now suppose $p(t) \neq q(t)$. So $p(t) < q(t)$ on some subinterval $[a, b] \subset [0, 1]$. Let

$$v(t) = \int_0^1 \frac{(1-s)^{\alpha-1-\beta}}{\Gamma(\alpha)}(q(s)-p(s))u(s)ds - t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(q(s)-p(s))u(s)ds.$$

Since $p(t) < q(t)$, then $v(0) > 0$. So, $(N - M)u_1 \in \Omega \subseteq \mathcal{P}^{\circ}$. So there exists $\epsilon > 0$ such that $(N - M)u_1 - \epsilon u_1 \in \mathcal{P}$. So $\Lambda_1 u_1 + \epsilon u_1 = Mu_1 + \epsilon u_1 \leq Nu_1$, implying $Nu_1 \geq (\Lambda_1 + \epsilon)u_1$. Since $N \leq N$ and $Nu_2 = \Lambda_2 u_2$, by Theorem 2.2, $\Lambda_1 + \epsilon \leq \Lambda_2$, or $\Lambda_1 < \Lambda_2$. □

Lemma 3.3. *The eigenvalues of* (3.1),(3.3) *are reciprocals of eigenvalues of M, and conversely. Similarly, eigenvalues of* (3.2),(3.3) *are reciprocals of eigenvalues of N, and conversely.*

Proof. Let Λ be an eigenvalue of M with corresponding eigenvector $u(t)$. Notice that

$$\Lambda u(t) = Mu(t) = \int_0^1 G(\beta; t, s)p(s)u(s)ds,$$

if and only if

$$u(t) = \frac{1}{\Lambda}\int_0^1 G(\beta; t, s)p(s)u(s)ds,$$

if and only if

$$D_{0+}^{\alpha}u(t) + \frac{1}{\Lambda}p(t)u(t) = 0, \quad 0 < t < 1,$$

with

$$u(0) = D_{0+}^{\beta}u(1) = 0.$$

So $\dfrac{1}{\Lambda}$ is an eigenvalue of (3.1), (3.3), if and only if Λ is an eigenvalue of M. A similar argument can be made that the reciprocals of eigenvalues of N are eigenvalues of (3.2), (3.3) and vice versa. $\square$

Since the eigenvalues of (3.1), (3.3) are reciprocals of eigenvalues of M and conversely, and the eigenvalues of (3.2), (3.3) are reciprocals of eigenvalues of N and conversely, the following theorem is an immediate consequence of Theorems 3.2 and 3.3.

Theorem 3.4. *Assume the hypotheses of Theorem 3.3. Then there exist smallest positive eigenvalues λ_1 and λ_2 of* (3.1), (3.3) *and* (3.2),(3.3), *respectively, each of which is simple, positive, and less than the absolute value of any other eigenvalue of the corresponding problems. Also, eigenfunctions corresponding to λ_1 and λ_2 may be chosen to belong to $\mathcal{P}^\circ$. Finally, $\lambda_1 \geq \lambda_2$, and $\lambda_1 = \lambda_2$ if and only if $p(t) = q(t)$ for all $t \in [0, 1]$.*

Chapter 4

The Case when $\beta = 0$

4.1 A Conjugate Problem

Now, let $n-1 < \alpha \leq n$ denote a real number. We will consider the eigenvalue problems

$$D_{0+}^{\alpha} u + \lambda_1 p(t) u = 0, \quad 0 < t < 1, \tag{4.1}$$

$$D_{0+}^{\alpha} u + \lambda_2 q(t) u = 0, \quad 0 < t < 1, \tag{4.2}$$

satisfying the boundary conditions

$$u^{(i)}(0) = 0, \; i = 0, \ldots, n-2, \quad u(1) = 0, \tag{4.3}$$

where D_{0+}^{α} and D_{0+}^{β} are the standard Riemann-Liouville fractional derivatives, and $p(t)$ and $q(t)$ are continuous nonnegative functions on $[0, 1]$ where neither $p(t)$ nor $q(t)$ vanishes identically on any nondegenerate compact subinterval of $[0, 1]$. We will show the existence of smallest eigenvalues (4.1), (4.3) and (4.2), (4.3). Assuming $p(t) \leq q(t)$, we will then compare these smallest eigenvalues.

Notice the boundary conditions we consider are different than the previous two chapters. Here, we examine when $\beta = 0$. Thus, the techniques used, although similar, will differ when looking at the boundary condition when $t = 1$ since the Green's function equals zero at that point.

4.2 Comparison of Smallest Eigenvalues

The Green's function for the eigenvalue problem (4.1), (4.3) and (4.2), (4.3) is given by

$$G(t,s) = \begin{cases} \frac{t^{\alpha-1}(1-s)^{\alpha-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s \le t \le 1, \\ \\ \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le t \le s < 1. \end{cases} \tag{4.4}$$

Therefore, $u(t) = \lambda_1 \int_0^1 G(t,s)p(s)u(s)ds$ if and only if $u(t)$ solves (4.1),(4.3). Similarly, $u(t) = \lambda_2 \int_0^1 G(t,s)q(s)u(s)ds$ if $u(t)$ solves (4.2),(4.3). Notice that $G(t,s) \ge 0$ on $[0,1] \times [0,1)$ and $G(t,s) > 0$ on $(0,1) \times (0,1)$.

Define the Banach Space

$$\mathcal{B} = \{u : u = t^{\alpha-1}v, v \in C^{(1)}[0,1], v(1) = 0\},$$

with the norm

$$\|u\| = |v'|_0.$$

Notice that for $v \in C^{(1)}[0,1]$, $v(1) = 0$, $0 \le t \le 1$, we have that

$$|v(t)| = |v(t) - v(1)| = \left| \int_1^t v'(s)ds \right| \le (1-t)|v'|_0 \le \|u\|.$$

Therefore, $|v|_0 \le \|u\| = |v'|_0$, and

$$|u|_0 = |t^{\alpha-1}v|_0 \le t^{\alpha-1}\|u\|,$$

implying

$$|u|_0 \le \|u\|.$$

Define the linear operators

$$Mu(t) = \int_0^1 G(t,s)p(s)u(s)ds, \tag{4.5}$$

and

$$Nu(t) = \int_0^1 G(t,s)q(s)u(s)ds. \tag{4.6}$$

Theorem 4.1. *The operators $M, N : \mathcal{B} \to \mathcal{B}$ are compact.*

Proof. First, we show $M : \mathcal{B} \to \mathcal{B}$. Let $u \in \mathcal{B}$. So there is a $v \in C^{(1)}[0,1]$ such that $u = t^{\alpha-1}v$. Since $v \in C^{(1)}[0,1]$ and $p \in C[0,1]$, let $L = |v|_0$ and $P = |p|_0$. Now,

$$\begin{aligned} Mu(t) &= \int_0^1 \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)u(s)ds - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)u(s)ds \\ &= t^{\alpha-1}\left(\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)u(s)ds - t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(alpha)}p(s)u(s)ds\right). \end{aligned}$$

Define

$$g(t) = \begin{cases} 0, & t = 0, \\ t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)u(s)ds, & 0 < t \le 1. \end{cases}$$

Notice $g \in C^{(1)}(0,1]$. Now,

$$\begin{aligned} |g(t)| &= \left|t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)u(s)ds\right| \\ &= \left|t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)s^{\alpha-1}v(s)ds\right| \\ &\le PLt^{1-\alpha}\int_0^t (t-s)^{\alpha-1}s^{\alpha-1}ds \\ &\le PLt^{1-\alpha}t^{\alpha-1}\int_0^t (t-s)^{\alpha-1}ds \\ &= \frac{PLt^\alpha}{\alpha}, \end{aligned}$$

where $\dfrac{PL}{\alpha} \geq 0$. Thus, $\lim_{t\to 0^+} g(t) = g(0) = 0$ and $g \in C[0,1]$. Also, for $t > 0$,

$$\begin{aligned}
|g'(t)| &= \left|(1-\alpha)t^{-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)u(s)ds + (\alpha-1)t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha)}p(s)u(s)ds\right| \\
&\leq \left|(1-\alpha)t^{-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)s^{\alpha-1}v(s)ds\right| \\
&\quad + \left|(\alpha-1)t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha)}p(s)s^{\alpha-1}v(s)(s)ds\right| \\
&\leq (\alpha-1)PLt^{-\alpha}t^{\alpha-1}\int_0^t (t-s)^{\alpha-1}ds + (\alpha-1)PLt^{1-\alpha}t^{\alpha-1}\int_0^t (t-s)^{\alpha-2}ds \\
&= (\alpha-1)PL\left(t^{-1}\int_0^t (t-s)^{\alpha-1}ds + \int_0^t (t-s)^{\alpha-2}ds\right) \\
&= (\alpha-1)PL\left(\frac{t^{\alpha-1}}{\alpha} + \frac{t^{\alpha-1}}{\alpha-1}\right) \\
&= \left(\frac{\alpha-1}{\alpha}+1\right)PLt^{\alpha-1}.
\end{aligned}$$

So, $\lim_{t\to 0^+} g'(t) = 0$. Moreover, using the definition of derivative and L'Hôpital's rule,

$$g'(0) = \lim_{t\to 0^+}\frac{g(t)-g(0)}{t} = \lim_{t\to 0^+}\frac{g(t)}{t} = \lim_{t\to 0^+} g'(t) = 0.$$

So $g' \in C[0,1]$.

Now let

$$\hat{v}(t) = \int_0^t \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)u(s)ds - t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)u(s)ds.$$

Then, $\hat{v}(1) = 0$. Thus, $Mu \in \mathcal{B}$. So, $M : \mathcal{B} \to \mathcal{B}$. A similar argument can be made that $N : \mathcal{B} \to \mathcal{B}$. □

Next, we define the cone

$$\mathcal{P} = \{u \in \mathcal{B} \mid u(t) \geq 0 \text{ for } t \in [0,1]\}.$$

Lemma 4.1. *The cone $\mathcal{P}$ is solid in $\mathcal{B}$ and hence reproducing.*

Proof. Define

$$\Omega := \{u = t^{\alpha-1}v \in \mathcal{B}|\ u(t) > 0,\ \text{for } t \in (0,1),\ v'(1) < 0\} \tag{4.7}$$

We will show $\Omega \subset \mathcal{P}^\circ$. Let $u \in \Omega$. Since $v(0) > 0$, there exists an $\epsilon_1 > 0$ such that $v(0) - \epsilon_1 > 0$. Since $v \in C[0,1]$, there exists an $a \in (0,1)$ such that $v(t) > \epsilon_1$ for all $t \in (0,a)$. Thus, $u(t) = t^{\alpha-1}v(t) > \epsilon_1 t^{\alpha-1}$ for all $t \in (0,a)$. Now, since $v'(1) < 0$, there exists an $\epsilon_2 > 0$ such that $v'(1)+\epsilon_2 < 0$, implying that $-v'(1) > \epsilon_2$. Then, by the definition of derivative, $\lim\limits_{t\to 1^-} \dfrac{-v(t)+v(1)}{t-1} > \epsilon_2$. Since $v(1) = 0$, then $\lim\limits_{t\to 1^-} \dfrac{v(t)}{1-t} > \epsilon_2$. Thus, there exists a $b \in (a,1)$ such that for $t \in (b,1)$, $\dfrac{v(t)}{1-t} > \epsilon_2$. Implying, $v(t) > (1-t)\epsilon_2$. Therefore, $u(t) > b^{\alpha-1}(1-t)\epsilon_2$ for all $t \in (b,1)$. Also, since $u(t) > 0$ on $[a,b]$, there exists an $\epsilon_3 > 0$ such that $u(t) - \epsilon_3 > 0$ for all $t \in [a,b]$.

Let $\epsilon = \min\left\{\dfrac{\epsilon_1}{2}, \dfrac{b^{\alpha-1}\epsilon_2}{2}, \dfrac{\epsilon_3}{2}\right\}$. Define $B_\epsilon(u) = \{\hat{u} \in \mathcal{B} : \|u - \hat{u}\| < \epsilon\}$. Let $\hat{u} \in B_\epsilon(u)$. Thus, $\hat{u} = t^{\alpha-1}\hat{v}$, where $\hat{v} \in C^{(1)}[0,1]$ with $\hat{v}(1) = 0$. Now

$$|\hat{u}(t) - u(t)| \le t^{\alpha-1}\|\hat{u} - u\| < \epsilon t^{\alpha-1}.$$

So, for $t \in (0,a)$, $\hat{u}(t) > u(t) - t^{\alpha-1}\epsilon > t^{\alpha-1}\epsilon_1 - t^{\alpha-1}\epsilon_1/2 = t^{\alpha-1}\epsilon_1/2$. So, $\hat{u}(t) > 0$ for $t \in (0,a)$. By the Mean Value Theorem, there exists $c \in (t,1)$ such that

$$\frac{\hat{v}(1) - v(1) - \hat{v}(t) + v(t)}{1-t} = \hat{v}'(c) - v'(c).$$

Since $\hat{v}(1) = 0$ and $v(1) = 0$, then

$$\left|\frac{v(t) - \hat{v}(t)}{1-t}\right| = |\hat{v}'(c) - v'(c)| \le |\hat{v}' - v'|_0\,.$$

However,

$$\left|\frac{u(t) - \hat{u}(t)}{1-t}\right| \le \left|\frac{v(t) - \hat{v}(t)}{1-t}\right|.$$

So, $|u(t) - \hat{u}(t)| \leq (1-t)\|\hat{u} - u\| < (1-t)\epsilon$, for $t \in (b, 1)$. Thus,

$$\hat{u}(t) > u(t) - (1-t)\epsilon > b^{\alpha-1}(1-t)\epsilon_2 - (1-t)b^{\alpha-1}\epsilon - 2/2 = (1-t)b^{\alpha-1} > 0.$$

Therefore, for $t \in (b, 1)$, $\hat{u}(t) > 0$. Also, $|\hat{u}(t) - u(t)| \leq ||\hat{u} - u|| < \epsilon$. So for $t \in [a, b]$, $\hat{u}(t) > u(t) - \epsilon > \epsilon_3 - \epsilon_3/2 > 0$. So, $\hat{u}(t) > 0$ for all $t \in [a, b]$. So, $\hat{u} \in \mathcal{P}$, and therefore $B_\epsilon(u) \subset \mathcal{P}$. Thus, $\Omega \subset \mathcal{P}^\circ$.

□

Lemma 4.2. *The bounded linear operators M and N are u_0-positive with respect to $\mathcal{P}$.*

Proof. First, we show $M : \mathcal{P}\backslash\{0\} \to \Omega \subset \mathcal{P}^\circ$. Let $u \in \mathcal{P}$. So $u(t) \geq 0$. Then since $G(t, s) \geq 0$ on $[0, 1] \times [0, 1)$ and $p(t) \geq 0$ on $[0, 1]$,

$$Mu(t) = \int_0^1 G(t, s)p(s)u(s)ds \geq 0,$$

for $0 \leq t \leq 1$. So $M : \mathcal{P} \to \mathcal{P}$.

Now, let $u \in \mathcal{P}\backslash\{0\}$. So there exists a compact interval $[a, b] \subset [0, 1]$ such that $u(t) > 0$ and $p(t) > 0$ for all $t \in [a, b]$. Then, since $G(t, s) > 0$ on $(0, 1] \times (0, 1)$,

$$\begin{aligned} Mu(t) &= \int_0^1 G(t, s)p(s)u(s)ds \\ &\geq \int_a^b G(t, s)p(s)u(s)ds \\ &> 0, \end{aligned}$$

for $0 < t \leq 1$. Now,

$$\begin{aligned} &Mu(t) \\ &\quad = t^{\alpha-1}\left(\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)u(s)ds - t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)u(s)ds\right). \end{aligned}$$

Let

$$v(t)=\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)u(s)ds - t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)u(s)ds.$$

Thus, $v(0)=\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)u(s)ds - g(0)>0$ where $g(t)$ was defined previously and

$$\begin{aligned}
v'(1) &= (1-\alpha)\left(\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha)}p(s)u(s)ds - \int_0^t \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)u(s)ds\right)\\
&= (1-\alpha)\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha)}p(s)u(s)\left(1-(1-s)\right)ds\\
&\leq 0.
\end{aligned}$$

So $M:\mathcal{P}\backslash\{0\}\to\Omega\subset\mathcal{P}^\circ$.

Now, choose $u_0\in\mathcal{P}\backslash\{0\}$, and let $u\in\mathcal{P}\backslash\{0\}$. So $Mu\in\Omega\subset\mathcal{P}^\circ$. Choose $k_1>0$ sufficiently small and k_2 sufficiently large so that $Mu-k_1u_0\in\mathcal{P}^\circ$ and $u_0-\frac{1}{k_2}Mu\in\mathcal{P}^\circ$. So $k_1u_0\leq Mu$ with respect to $\mathcal{P}$ and $Mu\leq k_2u_0$ with respect to $\mathcal{P}$. Thus $k_1u_0\leq Mu\leq k_2u_0$ with respect to $\mathcal{P}$ and so M is u_0-positive with respect to P. A similar argument shows N is u_0-positive. □

Lemma 4.3. *The eigenvalues of* (4.1),(4.3) *are reciprocals of eigenvalues of M, and conversely. Similarly, eigenvalues of* (4.2),(4.3) *are reciprocals of eigenvalues of N, and conversely.*

Proof. Let Λ be an eigenvalue of M with corresponding eigenvector $u(t)$. Notice that

$$\Lambda u(t)=Mu(t)=\int_0^1 G(t,s)p(s)u(s)ds,$$

if and only if

$$u(t)=\frac{1}{\Lambda}\int_0^1 G(t,s)p(s)u(s)ds,$$

if and only if

$$D_{0+}^\alpha u(t)+\frac{1}{\Lambda}p(t)u(t)=0,\ \ 0<t<1,$$

with

$$u^{(i)}(0)=u(1)=0\ i=0,1,\ldots,n-2.$$

So, $\dfrac{1}{\Lambda}$ is an eigenvalue of (4.1),(4.3), if and only if Λ is an eigenvalue. A similar argument can be made that the reciprocals of eigenvalues of N are eigenvalues of (4.2),(4.3) and vice versa. $\square$

Theorem 4.2. *Let $\mathcal{B}$, $\mathcal{P}$, M, and N be defined as earlier. Then M (and N) has an eigenvalue that is simple, positive, and larger than the absolute value of any other eigenvalue, with an essentially unique eigenvector that can be chosen to be in $\mathcal{P}^\circ$.*

Proof. Since M is a compact linear operator that is u_0-positive with respect to $\mathcal{P}$, by Theorem 2.1, M has an essentially unique eigenvector, say $u \in \mathcal{P}$, and eigenvalue Λ with the above properties. Since $u \neq 0$, $Mu \in \Omega \subset \mathcal{P}^\circ$ and $u = M\left(\frac{1}{\Lambda}u\right) \in \mathcal{P}^\circ$. $\square$

Theorem 4.3. *Let $\mathcal{B}$, $\mathcal{P}$, M, and N be defined as earlier. Let $p(t) \leq q(t)$ on $[0,1]$. Let Λ_1 and Λ_2 be the eigenvalues defined in Theorem 4.2 associated with M and N, respectively, with the essentially unique eigenvectors u_1 and $u_2 \in \mathcal{P}^\circ$. Then $\Lambda_1 \leq \Lambda_2$, and $\Lambda_1 = \Lambda_2$ if and only if $p(t) = q(t)$ on $[0,1]$.*

Proof. Let $p(t) \leq q(t)$ on $[0,1]$. So for any $u \in \mathcal{P}$ and $t \in [0,1]$,

$$\begin{aligned}
(Nu - Mu)(t) &= \int_0^1 G(t,s)(q(s)-p(s))u(s)ds \\
&= t^{\alpha-1}\left(\int_0^1 \frac{(1-s)^{\alpha-1-\beta}}{\Gamma(\alpha)}(q(s)-p(s))u(s)ds\right. \\
&\quad \left. -t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(q(s)-p(s))u(s)ds\right) \\
&\geq 0.
\end{aligned}$$

So $Nu - Mu \in \mathcal{P}$ for all $u \in \mathcal{P}$, or $M \leq N$ with respect to $\mathcal{P}$. Then, by Theorem 2.2, $\Lambda_1 \leq \Lambda_2$.

If $p(t) = q(t)$, then $\Lambda_1 = \Lambda_2$. Now suppose $p(t) \neq q(t)$. So $p(t) < q(t)$ on some subinterval $[a,b] \subset [0,1]$. Let

$$v(t) = \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}(q(s)-p(s))u(s)ds - t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(q(s)-p(s))u(s)ds.$$

Then,

$$v'(1) = (1-\alpha)\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha)}\left(q(s)-p(s)\right)u(s)ds - \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}.$$

Since $p(t) < q(t)$, then $v(0) > 0$ and $v'(1) < 0$. So, $(N-M)u_1 \in \Omega \subseteq \mathcal{P}^\circ$. So there exists $\epsilon > 0$ such that $(N-M)u_1 - \epsilon u_1 \in \mathcal{P}$. So $\Lambda_1 u_1 + \epsilon u_1 = Mu_1 + \epsilon u_1 \leq Nu_1$, implying $Nu_1 \geq (\Lambda_1 + \epsilon)u_1$. Since $N \leq N$ and $Nu_2 = \Lambda_2 u_2$, by Theorem 2.2, $\Lambda_1 + \epsilon \leq \Lambda_2$, or $\Lambda_1 < \Lambda_2$. □

Since the eigenvalues of (4.1), (4.3) are reciprocals of eigenvalues of M and conversely, and the eigenvalues of (4.2), (4.3) are reciprocals of eigenvalues of N and conversely, the following theorem is an immediate consequence of Theorems 4.2 and 4.3.

Theorem 4.4. *Assume the hypotheses of Theorem 4.3. Then there exist smallest positive eigenvalues λ_1 and λ_2 of* (4.1), (4.3) *and* (4.2), (4.3), *respectively, each of which is simple, positive, and less than the absolute value of any other eigenvalue of the corresponding problems. Also, eigenfunctions corresponding to λ_1 and λ_2 may be chosen to belong to $\mathcal{P}^\circ$. Finally, $\lambda_1 \geq \lambda_2$, and $\lambda_1 = \lambda_2$ if and only if $p(t) = q(t)$ for all $t \in [0,1]$.*

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