January 2013

Positive Solutions, Existence Of Smallest Eigenvalues, And Comparison Of Smallest Eigenvalues Of A Fourth Order Three Point Boundary Value Problem

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POSITIVE SOLUTIONS, EXISTENCE OF SMALLEST EIGENVALUES, AND COMPARISON OF SMALLEST EIGENVALUES OF A FOURTH ORDER THREE POINT BOUNDARY VALUE PROBLEM

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Submitted to the Faculty of the Graduate School of
Eastern Kentucky University
in partial fulfillment of the requirements
for the degree of
MASTER OF SCIENCE
August, 2013

DEDICATION

This thesis is dedicated to my family for all their support.

ACKNOWLEDGEMENTS

I would like to thank my thesis chair, Dr. Jeffrey Neugebauer, for his guidance in this research. I would also like to thank the other committee members, Dr. Steve Szabo and Dr. Jason Gibson, for their time. I would like to express thanks to my parents for always encouraging me and providing me with financial support. I also thank Dr. Talitha Washington for her support of my endeavors while at the University of Evansville.

ABSTRACT

The existence of smallest positive eigenvalues is established for the linear differential equations $u^{(4)} + \lambda_1 q(t)u = 0$ and $u^{(4)} + \lambda_2 r(t)u = 0$, $0 \leq t \leq 1$, with each satisfying the boundary conditions $u(0) = u'(p) = u''(1) = u'''(1) = 0$ where $1 - \frac{\sqrt{3}}{3} \leq p < 1$. A comparison theorem for smallest positive eigenvalues is then obtained. Using the same theorems, we will extend the problem to the fifth order via the Green's Function and again via Substitution. Applying the comparison theorems and the properties of u_0-positive operators to determine the existence of smallest eigenvalues. The existence of these smallest eigenvalues is then applied to characterize extremal points of the differential equation $u^{(4)} + q(t)u = 0$ satisfying boundary conditions $u(0) = u'(p) = u''(b) = u'''(b) = 0$ where $1 - \frac{\sqrt{3}}{3} \leq p \leq b \leq 1$. These results are applied to show the existence of a positive solution to a nonlinear boundary value problem.

Contents

Chapter 1

Introduction

We begin this thesis by considering the eigenvalue problems

$$u^{(4)} + \lambda_1 q(t)u = 0, \tag{1.1}$$

$$u^{(4)} + \lambda_2 r(t)u = 0, \tag{1.2}$$

satisfying the boundary conditions

$$u(0) = u'(p) = u''(1) = u'''(1) = 0, \tag{1.3}$$

where $1 - \frac{\sqrt{3}}{3} \leq p < 1$, and $q(t)$ and $r(t)$ are continuous nonnegative functions on $[0, 1]$, with neither $q(t)$ nor $r(t)$ vanishing identically on any compact subinterval of $[0, 1]$.

The second chapter of this thesis focuses on comparing the smallest eigenvalues for these eigenvalue problems. First, using the theory of u_0-positive operators with respect to a cone in a Banach space, we establish the existence of smallest eigenvalues for (1.1),(1.3), and (1.2),(1.3), and then compare these smallest eigenvalues after assuming a relationship between $q(t)$ and $r(t)$. We will then look at extensions of these theorems, first by exploring the fifth order problem extension using the Green's Function, and then through a substitution and relating the problem closely to the fourth order problem above.

In the third chapter, we will consider extremal points of the equation

$$u^{(4)} + q(t)u = 0 \tag{1.4}$$

for $0 \le t \le 1$ satisfying the boundary conditions

$$u(0) = u'(p) = u''(b) = u'''(b) = 0, \tag{1.5_b}$$

where p is fixed with $1 - \frac{\sqrt{3}}{3} \le p \le b \le 1$, and $q(t)$ is a continuous nonnegative function on $[0, 1]$ that does not vanish identically on any compact subinterval of $[0, 1]$.

We establish the existence of a largest interval, $[0, b)$, such that on any subinterval $[0, c]$ of $[0, b)$, there exists only the trivial solution of (1.4),(1.5_c). We accomplish this by characterizing the first extremal point through the existence of a nontrivial solution that lies in a cone by establishing the spectral radius of a compact operator. We then apply these results to show the existence of a positive solution of a fourth order nonlinear boundary value problem.

The technique for the comparison of these eigenvalues involve the application of sign properties of the Green's function, followed by the application of u_0-positive operators with respect to a cone in a Banach space. These applications are presented in books by Krasnosel'skii [26] and by Krein and Rutman [25].

Several authors applied these techniques in comparing eigenvalues for boundary problems different from those seen here. Previous work has been devoted to boundary value problems for ordinary differential equations involving conjugate, Lidstone, and right focal conditions. For example, Eloe and Henderson have studied smallest eigenvalue comparisons for a class of two point boundary value problems [8] and for a class of multipoint boundary value problems [9]. Most relevant to this work, Neugebauer has also studied smallest eigenvalue comparisons for three-point boundary value problems [29, 30]. In addition, comparison results have been obtained for difference equations [15] and for boundary value problems on time scales [2, 4, 18, 19, 27]. For additional work on this field, see [3, 10, 11, 13, 16, 21, 22, 23, 32, 33].

When characterizing extremal points, we will be defining a family of Banach spaces, cones, and operators. Using the theory of Krein and Rutman [25], we show the existence of a first extremal point is equivalent to properties of the spectral radius of the operators and the existence of solutions of the boundary value problem existing in a cone.

There has been some work done on extremal points. Eloe, Hankerson, and Henderson characterized extremal points for a class of multipoint boundary value problems [6] and for a class of two point boundary value problems [7]. Eloe, Henderson, and Thompson characterized extremal points for impulsive Lidstone boundary value problems [12]. Karna characterized extremal points for a fourth order two point boundary value problem [20]. Neugebauer considered a three point boundary value problem in his work [29].

For the theory used in this thesis, we refer the reader to Amann [1], Deimling [5], Krasnosel'skii [26], Krein and Rutman [25], Schmidt and Smith [31], and Zeidler [34].

Chapter 2

Comparison of Smallest Eigenvalues

2.1 Preliminary Definitions and Theorems

We start with some preliminary definitions and theorems that are crucial to our results.

Definition 2.1. Let $\mathcal{B}$ be a Banach space over $\mathbb{R}$. A closed nonempty subset $\mathcal{P}$ of $\mathcal{B}$ is said to be a cone provided

(i) $\alpha u + \beta v \in \mathcal{P}$, for all $u, v \in \mathcal{P}$ and all $\alpha, \beta \geq 0$, and

(ii) $u \in \mathcal{P}$ and $-u \in \mathcal{P}$ implies $u = 0$.

Definition 2.2. A cone $\mathcal{P}$ is solid if the interior, $\mathcal{P}^{\circ}$, of $\mathcal{P}$, is nonempty. A cone $\mathcal{P}$ is reproducing if $\mathcal{B} = \mathcal{P} - \mathcal{P}$; i.e., given $w \in \mathcal{B}$, there exist $u, v \in \mathcal{P}$ such that $w = u - v$.

Remark 2.1. Krasnosel'skii [26] showed that every solid cone is reproducing.

Definition 2.3. Let $\mathcal{P}$ be a cone in a real Banach space $\mathcal{B}$. If $u, v \in \mathcal{B}$, then $u \leq v$ with respect to $\mathcal{P}$ if $v - u \in \mathcal{P}$. If both $M, N : \mathcal{B} \to \mathcal{B}$ are bounded linear operators, we say $M \leq N$ with respect to $\mathcal{P}$ if $Mu \leq Nu$ for all $u \in \mathcal{P}$.

Definition 2.4. A bounded linear operator $M : \mathcal{B} \to \mathcal{B}$ is u_0-positive with respect to $\mathcal{P}$ if there exists $u_0 \in \mathcal{P}\backslash\{0\}$ such that for each $u \in \mathcal{P}\backslash\{0\}$, there exist $k_1(u) > 0$ and $k_2(u) > 0$ such that $k_1 u_0 \leq Mu \leq k_2 u_0$ with respect to $\mathcal{P}$.

The following three results are fundamental to our comparison results and are attributed to Krasnosel'skii [26]. The proof of Lemma 2.1 is provided, the proof of Theorem 2.1 can be found in Krasnosel'skii's book [26], and the proof of Theorem 2.2 is provided by Keener and Travis [24] as an extension of Krasonel'skii's results.

Lemma 2.1. *Let $\mathcal{B}$ be a Banach space over the reals, and let $\mathcal{P} \subset \mathcal{B}$ be a solid cone. If $M : \mathcal{B} \to \mathcal{B}$ is a linear operator such that $M : \mathcal{P}\backslash\{0\} \to \mathcal{P}^\circ$, then M is u_0-positive with respect to $\mathcal{P}$.*

Proof. Let $u_0 \in \mathcal{P}^\circ$ and let $u \in \mathcal{P}\backslash\{0\}$. It follows $Mu \in \mathcal{P}^\circ$. Choose $k_1 > 0$ sufficiently small so that $Mu - k_1 u_0 \in \mathcal{P}^\circ$. Choose $k_2 > 0$ sufficiently large so that $u_0 - \frac{1}{k_2} Mu \in \mathcal{P}^\circ$. This choice of k_1, k_2 insures that $k_1 u_0 \leq Mu$ and $\frac{1}{k_2} Mu \leq u_0$ with respect to $\mathcal{P}$. Thus $k_1 u_0 \leq Mu \leq k_2 u_0$, establishing the lemma. $\square$

Theorem 2.1. *Let $\mathcal{B}$ be a real Banach space, and let $\mathcal{P} \subset \mathcal{B}$ be a reproducing cone. Let $L : \mathcal{B} \to \mathcal{B}$ be a compact, u_0-positive, linear operator. Then L has an essentially unique eigenvector in $\mathcal{P}$, and the corresponding eigenvalue is simple, positive, and larger than the absolute value of any other eigenvalue.*

Theorem 2.2. *Let $\mathcal{B}$ be a real Banach space, and let $\mathcal{P} \subset \mathcal{B}$ be a cone. Let both $M, N : \mathcal{B} \to \mathcal{B}$ be bounded, linear operators and assume that at least one of the operators is u_0-positive. If $M \leq N$, $Mu_1 \geq \lambda_1 u_1$ for some $u_1 \in \mathcal{P}$ and some $\lambda_1 > 0$, and $Nu_2 \leq \lambda_2 u_2$ for some $u_2 \in \mathcal{P}$ and some $\lambda_2 > 0$, then $\lambda_1 \leq \lambda_2$. Futhermore, $\lambda_1 = \lambda_2$ implies u_1 is a scalar multiple of u_2.*

2.2 The Fourth Order Problem

In this section, we consider the fourth order eigenvalue problems (1.1),(1.3) and (1.2),(1.3). We derive comparison results for these fourth order eigenvalue problems by applying the theorems previously mentioned. To do this, we will define integral operators whose kernel is the Green's function for $-u^{(4)} = 0$ satisfying (1.3) and show these operators are u_0-positive.

This Green's function is given by

$$G(t,s) = \begin{cases} -t[\frac{p^2}{2} - ps] - \frac{t^2 s}{2} + \frac{t^3}{6}, & 0 \le t, p \le s \le 1, \\ \\ -t[\frac{p^2}{2} - ps - \frac{(p-s)^2}{2}] - \frac{t^2 s}{2} + \frac{t^3}{6}, & 0 \le t \le s \le p \le 1, \\ \\ -t[\frac{p^2}{2} - ps] - \frac{t^2 s}{2} + \frac{t^3}{6} - \frac{(t-s)^3}{6}, & 0 \le p \le s \le t \le 1, \\ \\ -t[\frac{p^2}{2} - ps - \frac{(p-s)}{2}] - \frac{t^2 s}{2} + \frac{t^3}{6} - \frac{(t-s)^3}{6}, & 0 \le s \le t, p \le 1. \end{cases}$$

Now, $u(t)$ solves (1.1),(1.3) if and only if $u(t) = \lambda_1 \int_0^1 G(t,s)q(s)u(s)ds$, and $u(t)$ solves (1.2),(1.3) if and only if $u(t) = \lambda_2 \int_0^1 G(t,s)r(s)u(s)ds$.

It was shown in [14] that $G(t,s) \ge 0$ on $[0,1] \times [0,1]$ and $G(t,s) > 0$ on $(0,1] \times (0,1)$.

Lemma 2.2. *For $0 < s < 1$, we have $\frac{\partial}{\partial t} G(t,s)|_{t=0} > 0$.*

Proof. When $t = 0$, we have $t \le s$. Thus, we start by considering $\frac{\partial}{\partial t} G(t,s)|_{t=0}$ for $t \le s$ and $p \le s$. Then

$$\begin{aligned} \left.\frac{\partial}{\partial t} G(t,s)\right|_{t=0} &= \left.\frac{\partial}{\partial t}\left[-t\left[\frac{p^2}{2} - ps\right] - \frac{t^2 s}{2} + \frac{t^3}{6}\right]\right|_{t=0} \\ &= \left.\left[-\left(\frac{p^2}{2} - ps\right) - ts + \frac{t^2}{2}\right]\right|_{t=0} \\ &= -\frac{p^2}{2} + ps \\ &\ge \frac{p^2}{2} - p^2 \\ &= \frac{p^2}{2} \\ &> 0 \end{aligned}$$

for $0 < s < 1$.

Next, we consider $\frac{\partial}{\partial t}G(t,s)|_{t=0} > 0$ for $t \leq s$ and $p \geq s$. Then

$$\begin{aligned}
\frac{\partial}{\partial t}G(t,s)\Big|_{t=0} &= \frac{\partial}{\partial t}\left[-t\left[\frac{p^2}{2} - ps - \frac{(p-s)^2}{2}\right] - \frac{t^2 s}{2} + \frac{t^3}{6}\right]\Big|_{t=0} \\
&= \left[-\left(\frac{p^2}{2} - ps - \frac{(p-s)^2}{2}\right) - ts + \frac{t^2}{2}\right]\Big|_{t=0} \\
&= -\frac{p^2}{2} + ps + \frac{(p-s)^2}{2} \\
&\geq -\frac{s^2}{2} + s^2 \\
&= \frac{s^2}{2} \\
&> 0.
\end{aligned}$$

for $0 < s < 1$. Thus, for $t = 0$ and $0 < s < 1$, we have $\frac{\partial}{\partial t}G(t,s) > 0$. □

To apply Theorems 2.1 and 2.2, we need to define a Banach space $\mathcal{B}$ and a cone $\mathcal{P} \subset \mathcal{B}$. Define the Banach space $\mathcal{B}$ by

$$\mathcal{B} = \{u \in C^1[0,1] \mid u(0) = 0\},$$

with the norm

$$\|u\| = \sup_{0 \leq t \leq 1} |u'(t)|.$$

Define the cone $\mathcal{P}$ to be

$$\mathcal{P} = \{u \in \mathcal{B} \mid u(t) \geq 0 \text{ on } [0,1]\}.$$

Notice that for $u \in \mathcal{B}$, $0 \leq t \leq 1$,

$$\begin{aligned}
|u(t)| = |u(t) - u(0)| &= \left|\int_0^t u'(s)ds\right| \\
&\leq \|u\|t \\
&\leq \|u\|,
\end{aligned}$$

and so $\sup_{0 \le t \le 1} |u(t)| \le \|u\|$.

Lemma 2.3. *The cone $\mathcal{P}$ is solid in $\mathcal{B}$ and hence reproducing.*

Proof. Define

$$\Omega = \{u \in \mathcal{B} \mid u(t) > 0 \text{ on } (0,1] \text{ and } u'(0) > 0\}.$$

Note $\Omega \subseteq \mathcal{P}$. We will show $\Omega \subseteq \mathcal{P}^\circ$. Since $u'(0) > 0$, there exists $\epsilon_1 > 0$ such that $u'(0) - \epsilon_1 > 0$, and so $u'(0) > \epsilon_1$. By the definition of the derivative, $u'(0) = \lim_{t \to 0^+} \dfrac{u(t) - u(0)}{t - 0} > \epsilon_1$, and so there exists an $a \in (0,1)$ such that for all $t \in (0,a)$, $\frac{u(t)-u(0)}{t-0} > \epsilon_1$. It follows that for all $t \in (0,a)$, $u(t) > t\epsilon_1$. Also, since $u(t) > 0$ on $[a,1]$, there exists $\epsilon_2 > 0$ such that $u(t) - \epsilon_2 > 0$ for all $t \in [a,1]$.

Let $\epsilon = \min\{\frac{\epsilon_1}{2}, \frac{\epsilon_2}{2}\}$. Let $B_\epsilon(u) = \{v \in \mathcal{B} \mid ||u - v|| < \epsilon\}$. Let $v \in \mathcal{B}_\epsilon(u)$, and so $|u'(0) - v'(0)| \le ||u - v|| < \epsilon$. Consequently, $v'(0) > 0$. Next, by the Mean Value Theorem, for $t \in (0,a)$, $|u(t) - v(t)| \le t||u - v||$. Thus $|u(t) - v(t)| < t\epsilon$ for $t \in (0,a)$. Thus $v(t) > u(t) - t\epsilon > t\epsilon_1 - t\frac{\epsilon_1}{2} = t\frac{\epsilon_1}{2} > 0$ for all $t \in (0,a)$. Lastly, for all $t \in [a,1]$, $|u(t) - v(t)| \le ||u - v|| < \epsilon$. We obtain that $v(t) > u(t) - \epsilon > \epsilon_2 - \frac{\epsilon_2}{2} > \frac{\epsilon_2}{2}$, and so $v(t) > 0$ on $(0,1]$. Thus, $v \in \Omega$, and so $\mathcal{B}_\epsilon(u) \subseteq \Omega \subseteq \mathcal{P}^\circ$, concluding the proof of the lemma. □

Next, we define our linear operators $M, N : \mathcal{B} \to \mathcal{B}$ by

$$Mu(t) = \int_0^1 G(t,s)q(s)u(s)ds, \ 0 \le t \le 1,$$

and

$$Nu(t) = \int_0^1 G(t,s)r(s)u(s)ds, \ 0 \le t \le 1.$$

Lemma 2.4. *The bounded linear operators M and N are compact.*

Proof. We will prove the statement for M only (the proof for N is similar). We apply the Arzelà-Ascoli Theorem to show that M is a compact operator by showing that M is continuous, and for any bounded sequence $\{u_n\} \in \mathcal{B}$, the sequence $\{Mu_n\}$ is uniformly bounded and equicontinuous.

Let $u, v \in \mathcal{B}$. Since $q(t)$ is a nonnegative continuous function on $[0,1]$, $q(t)$ has a maximum value on $[0,1]$. Let $L = \sup_{0 \le t \le 1} \{q(t)\}$ be this maximum value. Since $\frac{\partial}{\partial t} G(t,s)$ is bounded, let $K = \sup_{(t,s) \in [0,1] \times [0,1]} \left\{ \frac{\partial}{\partial t} G(t,s) \right\}$. Then, for $\epsilon > 0$, choose $\delta = \frac{\epsilon}{LK} > 0$ such that if $||u - v|| < \delta$, for any $t \in [0,1]$,

$$\begin{aligned}
|Mu'(t) - Mv'(t)| &= \left| \int_0^1 \frac{\partial}{\partial t} G(t,s) q(s)(u(s) - v(s)) ds \right| \\
&\le \int_0^1 \frac{\partial}{\partial t} G(t,s) q(s) |u(s) - v(s)| ds \\
&< LK\delta = \epsilon.
\end{aligned}$$

If $\|u - v\| < \delta$, then $\sup_{0 \le t \le 1} |Mu'(t) - Mv'(t)| < \epsilon$. Thus, for $\|u - v\| < \delta$, $\|Mu - Mv\| < \epsilon$, and hence M is continuous.

Let $\{u_n\}$ be a bounded sequence in $\mathcal{B}$. By boundedness there exists a $K_0 > 0$ such that $\|u_n\| \le K_0$ for all n. Since $Mu_n(t) = \int_0^1 G(t,s) q(s) u_n(s) ds$, we have

$$\begin{aligned}
|Mu_n'(t)| &= | \int_0^1 \frac{\partial}{\partial t} G(t,s) q(s) u_n(s) ds | \\
&\le KK_0 L,
\end{aligned}$$

for all n, which shows that $\{Mu_n\}$ is uniformly bounded.

Lastly, since $\frac{\partial}{\partial t} G(t,s)$ is continuous for any fixed s, for $\epsilon > 0$, there exists a $\delta > 0$ such that if $|t_1 - t_2| < \delta$, $|\frac{\partial}{\partial t} G(t_1, s) - \frac{\partial}{\partial t} G(t_2, s)| < \frac{\epsilon}{LK_0}$. Then for any n,

$$\begin{aligned}
|Mu_n'(t_1) - Mu_n'(t_2)| &\le \int_0^1 \left| \frac{\partial}{\partial t} G(t_1, s) - \frac{\partial}{\partial t} G(t_2, s) \right| q(s) u_n(s) ds \\
&< \frac{\epsilon}{LK_0} LK_0 = \epsilon.
\end{aligned}$$

Thus $|Mu_n'(t_1) - Mu_n'(t_2)| < \epsilon$ for any t_1, t_2 such that $|t_1 - t_2| < \delta$ establishing the equicontinuity of $\{Mu_n\}$. Therefore, M is compact by the Arzelá-Ascoli Theorem.

□

Lemma 2.5. *The bounded linear operators M and N are u_0-positive with respect* *to $\mathcal{P}$.*

Proof. We will show that $M : \mathcal{P}\backslash\{0\} \to \Omega \subseteq \mathcal{P}^{\circ}$. First, let $u \in \mathcal{P}$ so that $u(t) \geq 0$ on $[0,1]$, $G(t,s) \geq 0$ on $[0,1] \times [0,1]$, and $q(t) \geq 0$ on $[0,1]$. Thus

$$\begin{aligned} Mu(t) &= \int_0^1 G(t,s)q(s)u(s)ds \\ &\geq 0, \end{aligned}$$

for $0 \leq t \leq 1$. Thus $Mu \in P$ and $M : \mathcal{P} \to \mathcal{P}$.

Next, let $u \in \mathcal{P}\backslash\{0\}$. Consequently, there exists a compact subinterval $[a,b] \subset (0,1)$ such that $u(t) > 0$ and $q(t) > 0$ on $[a,b]$. Since $G(t,s) > 0$ on $(0,1] \times (0,1)$,

$$\begin{aligned} Mu(t) &= \int_0^1 G(t,s)q(s)u(s)ds \\ &\geq \int_a^b G(t,s)g(s)u(s)ds \\ &> 0, \end{aligned}$$

for $0 < t \leq 1$. Also, $\frac{\partial}{\partial t}G(t,s)|_{t=0} > 0$ for $0 < s < 1$, so

$$\begin{aligned} (Mu)'(0) &= \int_0^1 \frac{\partial}{\partial t}G(0,s)q(s)u(s)ds \\ &\geq \int_a^b \frac{\partial}{\partial t}G(0,s)q(s)u(s)ds \\ &> 0. \end{aligned}$$

Hence, $Mu \in \Omega \subseteq \mathcal{P}^{\circ}$. Thus by Lemma 2.1, M is u_0-positive. A similar argument can be made to show N is u_0-positive. $\square$

Remark 2.2. Notice that

$$\Lambda u = Mu = \int_0^1 G(t,s)q(s)u(s)ds,$$

if and only if

$$u(t) = \frac{1}{\Lambda} \int_0^1 G(t,s)q(s)u(s)ds,$$

if and only if

$$-u^{(4)}(t) = \frac{1}{\Lambda} q(t)u(t), \; 0 \leq t \leq 1,$$

with

$$u(0) = u'(p) = u''(1) = u'''(1) = 0.$$

This shows that the eigenvalues of (1.1),(1.3) are reciprocals of eigenvalues of M, and conversely. Similarly, eigenvalues of (1.2),(1.3) are reciprocals of eigenvalues N, and conversely.

Theorem 2.3. *Let $\mathcal{B}$, $\mathcal{P}$, M, and N be defined as earlier. Then M (and, by similar reasoning, N) has an eigenvalue that is simple, positive, and larger than the absolute value of any other eigenvalue, with an essentially unique eigenvector that can be chosen to be in $\mathcal{P}^\circ$.*

Proof. Since M is u_0-positive it has, from Theorem 2.1, an essentially unique eigenvector, namely $u \in \mathcal{P}$, and eigenvalue Λ with the properties stated above. Since $u \neq 0$, we have $Mu \in \Omega \subseteq \mathcal{P}^\circ$ and $\Lambda u = Mu$. Therefore, $u = \frac{1}{\Lambda} Mu = M(\frac{1}{\Lambda}u)$. Notice that $\frac{1}{\Lambda}u \neq 0$ and so $M(\frac{1}{\Lambda}u) \in \mathcal{P}^\circ$. It follows that $u \in \mathcal{P}^\circ$, completing the proof. □

Theorem 2.4. *Let $\mathcal{B}$, $\mathcal{P}$, M, and N be defined as earlier. Let $q(t) \leq r(t)$ on $[0,1]$. Let Λ_1 and Λ_2 be the eigenvalues, defined in Theorem 2.3, associated with M and N, respectively, with the essentially unique eigenvectors u_1 and $u_2 \in \mathcal{P}^\circ$. Then $\Lambda_1 \leq \Lambda_2$, and $\Lambda_1 = \Lambda_2$ if and only if $q(t) = r(t)$ on $[0,1]$.*

Proof. Let $q(t) \leq r(t)$ on $[0,1]$. Thus, for any $u \in \mathcal{P}$ and $t \in [0,1]$,

$$(Nu - Mu)(t) = \int_0^1 G(t,s)(r(s) - q(s))u(s)ds \geq 0,$$

and so $(Nu - Mu) \in \mathcal{P}$. Thus $Mu \le Nu$ for all $u \in \mathcal{P}$, implying that $M \le N$ with respect to $\mathcal{P}$. So, by Theorem 2.2, $\Lambda_1 \le \Lambda_2$. If $q(t) = r(t)$, then $\Lambda_1 = \Lambda_2$.

Suppose now that $q(t) \neq r(t)$. Then there exists some subinterval $[a,b] \subseteq [0,1]$ such that $q(t) < r(t)$ for all $t \in [a,b]$. Through reasoning similar to the proof of Lemma 2.5, we have $N - M \ : \ \mathcal{P}\backslash\{0\} \to \Omega$. Therefore, $(N-M)u_1 \in \Omega \subseteq \mathcal{P}^\circ$ and so there exists some $\epsilon > 0$ such that $(N-M)u_1 - \epsilon u_1 \in \mathcal{P}$. Then $\epsilon u_1 \le (N - M)u_1 = Nu_1 - Mu_1$ with respect to $\mathcal{P}$. We have $\Lambda_1 u_1 + \epsilon u_1 = Mu_1 + \epsilon u_1 \le Nu_1$. This implies that $Nu \ge (\Lambda_1 + \epsilon)u_1$. Since $N \le N$ and $Nu_2 = \Lambda_2 u_2$, $\Lambda_1 + \epsilon \le \Lambda_2$, thus $\Lambda_1 < \Lambda_2$. □

By Remark 2.2, the following theorem is an immediate consequence of Theorems 2.3 and 2.4.

Theorem 2.5. *Assume the hypotheses of Theorem 2.4. Then there exist smallest positive eigenvalues λ_1 and λ_2 of* (1.1),(1.3) *and* (1.2),(1.3)*, respectively, each of which is simple, positive, and less than the absolute value of any other eigenvalue of the corresponding problems. Also, eigenfunctions corresponding to λ_1 and λ_2 may be chosen to belong to $\mathcal{P}^\circ$. Finally, $\lambda_1 \ge \lambda_2$, and $\lambda_1 = \lambda_2$ if and only if $q(t) = r(t)$ for $0 \le t \le 1$.*

2.3 The Fifth Order Extension Using the Green's Function

We now consider the eigenvalue problems

$$u^{(5)} + \lambda_1 q(t) u = 0, \tag{2.1}$$

$$u^{(5)} + \lambda_2 r(t) u = 0, \tag{2.2}$$

satisfying the boundary value conditions

$$u(0) = u'(0) = u''(p) = u'''(1) = u^{(4)}(1), \tag{2.3}$$

where p is fixed with $1-\frac{\sqrt{3}}{3} \leq p \leq 1$, and $q(t)$ and $r(t)$ are continuous nonnegative functions on $[0,1]$ that do not vanish identically on any compact subinterval of $[0,1]$. We will derive comparison theorems for these fifth order eigenvalue problems using a similar technique to that used for the fourth order problem by using the Green's function, $G_5(t,s)$, for $-u^{(5)}=0$ satisfying (2.3). The Green's function is continuous and differentiable, so we know $\frac{\partial}{\partial t}G_5(t,s)=G(t,s)$, where $G(t,s)$ is as defined in previous sections. Therefore,

$$G_5(t,s)=\begin{cases} -t^2[\frac{p^2}{4}-\frac{ps}{2}]-\frac{t^3s}{6}+\frac{t^4}{24}, & 0\leq t,p\leq s\leq 1,\\ \\ -t^2[\frac{p^2}{4}-\frac{ps}{2}-\frac{(p-s)^2}{4}]-\frac{t^3s}{6}+\frac{t^4}{24}, & 0\leq t\leq s\leq p\leq 1,\\ \\ -t^2[\frac{p^2}{4}-\frac{ps}{2}]-\frac{t^3s}{6}+\frac{t^4}{24}-\frac{(t-s)^4}{24}, & 0\leq p\leq s\leq t\leq 1,\\ \\ -t^2[\frac{p^2}{4}-\frac{ps}{2}-\frac{(p-s)}{4}]-\frac{t^3s}{6}+\frac{t^4}{24}-\frac{(t-s)^4}{24}, & 0\leq s\leq t,p\leq 1. \end{cases}$$

Now $u(t)$ solves (2.1), (2.3) if and only if $u(t)=\lambda_1\int_0^1 G_5(t,s)q(s)u(s)ds$, and $u(t)$ solves (2.2), (2.3) if and only if $u(t)=\lambda_2\int_0^1 G_5(t,s)r(s)u(s)ds$. Since $\frac{\partial}{\partial t}G_5(t,s)=G(t,s)$, we have $\frac{\partial}{\partial t}G_5(t,s)\geq 0$ on $[0,1]\times[0,1]$ and $\frac{\partial}{\partial t}G_5(t,s)>0$ on $(0,1)\times(0,1)$. Also, $\frac{\partial^2}{\partial^2 t}G_5(t,s)=\frac{\partial}{\partial t}G(t,s)$. Thus $\frac{\partial^2}{\partial^2 t}G_5(t,s)|_{t=0}>0$ for $0<s<1$.

In order to apply Theorem 2.1 and Theorem 2.2, we define a Banach space $\mathcal{B}$ and a cone $\mathcal{P}\subset\mathcal{B}$. Define the Banach space $\mathcal{B}$ by

$$\mathcal{B}=\{u\in C^2[0,1]\mid u(0)=u'(0)=0\},$$

with the norm

$$\|u\|=\sup_{0\leq t\leq 1}|u''(t)|.$$

Define the cone

$$\mathcal{P}=\{u\in\mathcal{B}\mid u'(t)\geq 0 \text{ on } [0,1]\}.$$

We see that for $u \in \mathcal{B}$, $0 \leq t \leq 1$ we have,

$$\begin{aligned}|u'(t)| = |u'(t) - u'(0)| =& |\int_0^t u''(s)ds| \\ \leq& \|u\| t \\ \leq& \|u\|,\end{aligned}$$

and so $\sup_{0 \leq t \leq 1} |u'(t)| \leq \|u\|$.

Lemma 2.6. *The cone $\mathcal{P}$ is solid in $\mathcal{B}$ and hence reproducing.*

Proof. Define

$$\Omega = \{u \in \mathcal{B} \mid u'(t) > 0 \text{ on } (0,1] \text{ and } u''(0) > 0\}.$$

Note $\Omega \subseteq \mathcal{P}$. We will show $\Omega \subseteq \mathcal{P}^\circ$. Since $u''(0) > 0$, there exists $\epsilon_1 > 0$ such that $u''(0) - \epsilon_1 > 0$ and so $u"(0) > \epsilon_1$. By the definition of the derivative, $u''(0) = \lim_{t \to 0^+} \dfrac{u'(t) - u'(0)}{t - 0} > \epsilon_1$. So there exists an $a \in (0,1)$ such that for all $t \in (0,a)$, $\frac{u'(t)-u'(0)}{t-0} > \epsilon_1$. So for all $t \in (0,a)$, $u'(t) > t\epsilon_1$. Also, since $u'(t) > 0$ on $[a,1]$, there exists $\epsilon_2 > 0$ such that $u'(t) - \epsilon_2 > 0$ for all $t \in [a,1]$.

Let $\epsilon = \min\{\frac{\epsilon_1}{2}, \frac{\epsilon_2}{2}\}$. Let $B_\epsilon(u) = \{v \in \mathcal{B} \mid ||u - v|| < \epsilon\}$. Let $v \in \mathcal{B}_\epsilon(u)$. So $|u''(0) - v''(0)| \leq ||u - v|| < \epsilon$. Thus $v''(0) > 0$. Next, by the Mean Value Theorem, for $t \in (0,a)$, $|u'(t) - v'(t)| \leq t\|u - v\|$. Thus $|u'(t) - v'(t)| < t\epsilon$ for $t \in (0,a)$, yielding $v'(t) > u'(t) - t\epsilon > t\epsilon_1 - t\frac{\epsilon_1}{2} = t\frac{\epsilon_1}{2} > 0$ for all $t \in (0,a)$. Lastly, for all $t \in [a,1]$, we have $|u'(t) - v'(t)| \leq ||u - v|| < \epsilon$. This gives $v'(t) > u'(t) - \epsilon > \epsilon_2 - \frac{\epsilon_2}{2} > \frac{\epsilon_2}{2}$ and so $v'(t) > 0$ on $(0,1]$. We obtain $v \in \Omega$, and hence $\mathcal{B}_\epsilon(u) \subseteq \Omega \subseteq \mathcal{P}^\circ$. □

Now, we define our linear operators M and N by

$$Mu(t) = \int_0^1 G_5(t,s)q(s)u(s)ds, \ 0 \leq t \leq 1,$$

and

$$Nu(t) = \int_0^1 G_5(t,s)r(s)u(s)ds,\ 0 \le t \le 1,$$

Since $G_5(0,s) = \frac{\partial}{\partial t}G_5(t,s)|_{t=0} = 0$, we have $M, N : \mathcal{B} \to \mathcal{B}$. A standard application of the Arzelá-Ascoli Theorem, as in the previous section, shows that M and N are compact.

Lemma 2.7. *The bounded linear operators M and N are u_0-positive with respect to $\mathcal{P}$.*

Proof. We will show that $M : \mathcal{P}\backslash\{0\} \to \Omega \subseteq \mathcal{P}^\circ$. First, let $u \in \mathcal{P}$. Then $u(t) \ge 0$ on $[0,1]$. Also, $\frac{\partial}{\partial t}G_5(t,s) = G(t,s) \ge 0$ on $[0,1] \times [0,1]$, and $q(t) \ge 0$ on $[0,1]$. It follows that

$$\begin{aligned} Mu'(t) &= \int_0^1 \frac{\partial}{\partial t}G_5(t,s)q(s)u(s)ds \\ &\ge 0, \end{aligned}$$

for $0 \le t \le 1$. Thus $Mu \in P$ and $M : \mathcal{P} \to \mathcal{P}$.

Next, let $u \in \mathcal{P}\backslash\{0\}$. There exists a compact subinterval $[a,b] \subset (0,1)$ such that $u(t) > 0$ and $q(t) > 0$ on $[a,b]$. Since $\frac{\partial}{\partial t}G_5(t,s) > 0$ on $(0,1] \times (0,1)$,

$$\begin{aligned} Mu'(t) &= \int_0^1 \frac{\partial^2}{\partial^2 t}G_5(t,s)q(s)u(s)ds \\ &\ge \int_a^b \frac{\partial}{\partial t}G_5(t,s)g(s)u(s)ds \\ &> 0, \end{aligned}$$

for $0 < t \le 1$. Also, $\frac{\partial^2}{\partial^2 t}G_5(t,s)|_{t=0} > 0$ for $0 < s < 1$, so

$$\begin{aligned} (Mu)'''(0) &= \int_0^1 \frac{\partial^2}{\partial^2 t}G_5(0,s)q(s)u(s)ds \\ &\ge \int_a^b \frac{\partial^2}{\partial^2 t}G_5(0,s)q(s)u(s)ds \\ &> 0. \end{aligned}$$

This shows $Mu \in \Omega \subset \mathcal{P}^\circ$. Therefore, by Lemma 2.1, M is a u_0-positive operator with respect to $\mathcal{P}$. A similar argument for N completes the proof. $\square$

Remark 2.3. Note that

$$\Lambda u = Mu = \int_0^1 G_5(t,s)q(s)u(s)ds,$$

if and only if

$$u(t) = \frac{1}{\Lambda}\int_0^1 G_5(t,s)q(s)u(s)ds,$$

if and only if

$$-u^{(5)}(t) = \frac{1}{\Lambda}q(t)u(t),\ 0 \le t \le 1,$$

with

$$u(0) = u'(0) = u''(p) = u'''(1) = u^{(4)}(1) = 0,$$

where p is fixed with $1-\frac{\sqrt{3}}{3} \le p \le 1$. The eigenvalues of (2.1), (2.3) are reciprocals of eigenvalues of M, and conversely. Also, the eigenvalues of (2.2), (2.3) are reciprocals of the eigenvalues of N, and conversely.

Theorem 2.6. *Let $\mathcal{B}, \mathcal{P}, M$, and N be defined as before. Then M (and N) has an eigenvalue that is simple, positive, and larger in absolute value than any other eigenvalue with an essentially unique eigenvector that can be chosen to be in $\mathcal{P}^\circ$.*

Proof. Since M is a compact, linear, u_0-positive operator with respect to $\mathcal{P}$, by Theorem 2.1, M has essentially unique eigenvector, say $u \in \mathcal{P}$, and eigenvalue Λ with the above properties. Since $u \neq 0$, we have $Mu \in \Omega \subset \mathcal{P}^\circ$ and $u = M(\frac{1}{\Lambda}u) \in \mathcal{P}^\circ$. $\square$

Theorem 2.7. *Let $\mathcal{B}, \mathcal{P}, M$, and N be defined as before. Let $q(t) \le r(t)$ on $[0,1]$. Let Λ_1 and Λ_2 be the eigenvalues, defined in Theorem 2.6, associated with M and N, respectively, with the essentially unique eigenvectors u_1 and $u_2 \in \mathcal{P}^\circ$. Then $\Lambda_1 \le \Lambda_2$, and $\Lambda_1 = \Lambda_2$ if and only if $q(t) = r(t)$ on $[0,1]$.*

Proof. Let $q(t) \leq r(t)$ on $[0, 1]$. Then for any $u \in \mathcal{P}$, and $t \in [0, 1]$, we have

$$(Nu - Mu)'(t) = \int_0^1 \frac{\partial}{\partial t} G_5(t, s)(r(s) - q(s))u(s)ds \geq 0.$$

This gives $Nu - Mu \in \mathcal{P}$ for all $u \in \mathcal{P}$, or $M \leq N$ with respect to $\mathcal{P}$. Then, by Theorem 2.2, $\Lambda_1 < \Lambda_2$. Now either $q(t) = r(t)$, or $q(t) \neq r(t)$. If $q(t) = r(t)$, then $\Lambda_1 = \Lambda_2$. Now suppose $q(t) \neq r(t)$, then $q(t) < r(t)$ on some subinterval $[\alpha, \beta] \subset [0, 1]$. Then $(N - M)u_1 \in \Omega \subset \mathcal{P}^\circ$, and so there exists $\epsilon > 0$ such that $(N - M)u_1 - \epsilon u_1 \in \mathcal{P}$. Then $\Lambda_1 u_1 + \epsilon u_1 = Mu_1 + \epsilon u_1 \leq Nu_1$, implying $Nu_1 \geq (\Lambda + \epsilon)u_1$. Since $N \leq N$ and $Nu - 2 = \Lambda_2 u_2$, by Theorem 2.2, $\Lambda_1 + \epsilon \leq \Lambda_2$, or $\Lambda_1 < \Lambda + 2$. □

By Remark 2.3, the following theorem is an immediate consequence of theorems 2.6 and 2.7 .

Theorem 2.8. *Assume the hypotheses of Theorem 2.7. Then there exist smallest positive eigenvalues λ_1 and λ_2 of* (2.1), (2.3) *and* (2.2), (2.3), *respectively, each of which is simple, positive, and smaller in absolute value than any other eigenvalue of the corresponding equations. Also, eigenfunctions corresponding to λ_1 and λ_2 may be chosen to belong to $\mathcal{P}^\circ$. Finally, $\lambda_1 \geq \lambda_2$, and $\lambda_1 = \lambda_2$ if and only if $q(t) = r(t)$ for $0 \leq t \leq 1$.*

2.4 The Fifth Order Extension via Substitution

We now consider the problems (2.1) and (2.2) satisfying (2.3) and the eigenvalue problems

$$v^{(4)} + \lambda_1 q(t) \int_0^t v(s)ds = 0, \tag{2.4}$$

$$v^{(4)} + \lambda_2 r(t) \int_0^t v(s)ds = 0, \tag{2.5}$$

satisfying the boundary conditions

$$v(0) = v'(p) = v''(1) = v'''(1) = 0, \tag{2.6}$$

where p is fixed and $1 - \frac{\sqrt{3}}{3} \leq p < 1$, and $q(t)$ and $r(t)$ are continuous nonnegative functions on $[0, 1]$, where neither $q(t)$ nor $r(t)$ vanishes identically on any compact subinterval of $[0, 1]$.

First, we note that if $u(t)$ is a solution to (2.1), (2.3), then $u'(t)$ solves (2.4), (2.6). Also, if $v(t)$ is a solution to (2.4), (2.6), then $\int_0^t v(s)ds$ is a solution to (2.1), (2.3). Similarly, if $u(t)$ is a solution to (2.2), (2.3), then $u'(t)$ solves (2.5), (2.6), and if $v(t)$ is a solution to (2.5), (2.6), then $\int_0^t v(s)ds$ is a solution to (2.2), (2.3).

Now, let λ be an eigenvalue of (2.1), (2.3) with the corresponding eigenvector $u(t)$. Then $u'(t)$ is a solution to (2.4), (2.6) with the same eigenvalue λ. Also, if λ is an eigenvalue of (2.4), (2.6) with the corresponding eigenvector $v(t)$, then $\int_0^t v(s)ds$ is a solution to (2.1), (2.3) with the corresponding eigenvalue λ. So eigenvalues of (2.1), (2.3) are eigenvalues of (2.4), (2.6), and conversely. Similarly, eigenvalues of (2.2), (2.3) are eigenvalues of (2.5), (2.6), and conversely. Thus any comparison theorems for (2.4), (2.6) will apply to (2.1), (2.3), and (2.5), (2.6) will apply to (2.2), (2.3).

For these reasons, we will derive comparison theorems for the eigenvalue problems (2.4), (2.6), and (2.5), (2.6), and then use these as they apply to (2.1), (2.3), and (2.2), (2.3).

The function $G(t, s)$, as defined earlier, is the Green's Function for $-v^{(4)} = 0$ satisfying (2.6). So $v(t)$ solves solves (2.4), (2.6) if and only if

$$v(t) = \lambda_1 \int_0^1 G(t, s)q(s) \left(\int_0^s v(t)dt \right) ds,$$

and $v(t)$ solves (2.5), (2.6) if and only if

$$v(t) = \lambda_2 \int_0^1 G(t, s)r(s) \left(\int_0^s v(t)dt \right) ds.$$

Note, as before, $G(t, s) \geq 0$ on $[0, 1] \times [0, 1]$, $G(t, s) > 0$ on $(0, 1] \times (0, 1]$, and $\frac{\partial}{\partial t} G(t, s) \mid_{t=0} > 0$ for $0 < s < 1$.

In order to apply Theorems 2.1 and 2.2, we need to define a Banach Space

$\mathcal{B}$ and a cone $\mathcal{P} \subset \mathcal{B}$. First, define $\mathcal{B}$ by

$$\mathcal{B} = \{v \in C^1[0,1] \mid v(0) = 0\}$$

with norm

$$\|v\| = \sup_{0 \le t \le 1} |v'(t)|.$$

Define the cone $\mathcal{P}$ to be

$$\mathcal{P} = \{v \in \mathcal{B} \mid v(t) \ge 0 \text{ on } [0,1]\}.$$

Notice that for $v \in \mathcal{B}$, $0 \le t \le 1$, we have

$$\begin{aligned} |v(t)| = |v(t) - v(0)| &= \left| \int_0^t v'(s) ds \right| \\ &\le \|v\| t \\ &\le \|v\|, \end{aligned}$$

and so $\sup\limits_{0 \le t \le 1} |v'(t)| \le \|v\|$.

Lemma 2.8. *The cone $\mathcal{P}$ is solid in $\mathcal{B}$ and hence reproducing.*

Proof. Define

$$\Omega = \{v \in \mathcal{B} \mid v(t) > 0 \text{ on } (0,1] \text{ and } v'(0) > 0\}.$$

It was shown in Section 2 of this chapter that $\Omega \subset \mathcal{P}^\circ$. Therefore $\mathcal{P}$ is solid in $\mathcal{B}$. □

Next, we define our linear operators $M, N : \mathcal{B} \to \mathcal{B}$ by

$$Mv(t) = \int_0^1 G(t,s) q(s) \left(\int_0^s v(t) dt \right) ds, \ 0 \le t \le 1,$$

and

$$Nv(t) = \int_0^1 G(t,s)r(s)\left(\int_0^s v(t)dt\right)ds,\ 0 \le t \le 1,$$

A standard application of the Arzelá-Ascoli Theorem shows that M and N are compact.

Lemma 2.9. *The bounded linear operators M and N are u_0-positive with respect to $\mathcal{P}$.*

Proof. We will show that $M : \mathcal{P}\backslash\{0\} \to \Omega \subseteq \mathcal{P}^\circ$. First, let $v \in \mathcal{P}$. Then $v(t) \ge 0$ on $[0,1]$, $G(t,s) \ge 0$ on $[0,1]\times[0,1]$, and $q(t) \ge 0$ on $[0,1]$. Thus

$$\begin{aligned} Mv(t) &= \int_0^1 G(t,s)q(s)\left(\int_0^s v(s)dt\right)ds \\ &\ge 0 \end{aligned}$$

for $0 \le t \le 1$. Thus $Mv \in P$ and $M : \mathcal{P} \to \mathcal{P}$.

Next, let $v \in \mathcal{P}\backslash\{0\}$. There exists a compact subinterval $[a,b] \subset (0,1)$ such that $v(t) > 0$ and $q(t) > 0$ on $[a,b]$. Since $G(t,s) > 0$ on $(0,1]\times(0,1)$,

$$\begin{aligned} Mv(t) &= \int_0^1 G(t,s)q(s)\left(\int_0^s v(t)dt\right)ds \\ &\ge \int_a^b G(t,s)g(s)\left(\int_0^s v(t)dt\right)ds \\ &> 0 \end{aligned}$$

for $0 < t \le 1$. Also, $\frac{\partial}{\partial t}G(t,s)|_{t=0} > 0$ for $0 < s < 1$, and so

$$\begin{aligned} (Mv)'(0) &= \int_0^1 \frac{\partial}{\partial t}G(0,s)q(s)\left(\int_0^s v(t)dt\right)ds \\ &\ge \int_a^b \frac{\partial}{\partial t}G(0,s)q(s)\left(\int_0^s v(t)dt\right)ds \\ &> 0. \end{aligned}$$

Then $Mv \in \Omega \subseteq \mathcal{P}^\circ$. Thus, by Lemma 2.1, M is u_0-positive. A similar argument shows that N is u_0-positive as well. $\square$

Remark 2.4. Notice that

$$\Lambda v = Mv = \int_0^1 G(t,s)q(s)\left(\int_0^s v(t)dt\right)ds,$$

if and only if

$$v(t) = \frac{1}{\Lambda}\int_0^1 G(t,s)q(s)\left(\int_0^s v(t)dt\right)ds,$$

if and only if

$$-v^{(4)}(t) = \frac{1}{\Lambda}q(t)\int_0^t v(s)ds,\ 0 \le t \le 1,$$

with

$$v(0) = v'(p) = v''(1) = v'''(1) = 0.$$

This shows that the eigenvalues of (2.4),(2.6) are reciprocals of eigenvalues of M, and conversely. Similarly, eigenvalues of (2.5),(2.6) are reciprocals of eigenvalues N, and conversely.

Theorem 2.9. *Let $\mathcal{B}$, $\mathcal{P}$, M, and N be defined as above. Then M (and N) has an eigenvalue that is simple, positive, and larger than the absolute value of any other eigenvalue, with an essentially unique eigenvector that can be chosen to be in $\mathcal{P}^\circ$.*

Proof. Since M is u_0-positive, from Theorem 2.1, it has an essentially unique eigenvector, namely $u \in \mathcal{P}$, and eigenvalue Λ with the properties stated above. Since $v \neq 0$ we have $Mu \in \Omega \subseteq \mathcal{P}^\circ$ and $\Lambda v = Mv$. Therefore, $v = \frac{1}{\Lambda}Mv = M(\frac{1}{\Lambda}v)$. Notice that $\frac{1}{\Lambda}v \neq 0$, which gives $M(\frac{1}{\Lambda}v) \in \mathcal{P}^\circ$, and so $v \in \mathcal{P}^\circ$. $\square$

Theorem 2.10. *Let $\mathcal{B}$, $\mathcal{P}$, M, and N be defined as before. Let $q(t) \le r(t)$ on $[0,1]$. Let Λ_1 and Λ_2 be the eigenvalues, defined in Theorem 2.9, associated with M and N, respectively, with the essentially unique eigenvectors v_1 and $v_2 \in \mathcal{P}^\circ$. Then $\Lambda_1 \le \Lambda_2$, and $\Lambda_1 = \Lambda_2$ if and only if $q(t) = r(t)$ on $[0,1]$.*

Proof. Let $q(t) \leq r(t)$ on $[0,1]$. Thus, for any $v \in \mathcal{P}$ and $t \in [0,1]$,

$$(Nv - Mv)(t) = \int_0^1 G(t,s)(r(s) - q(s) \left(\int_0^s v(t)dt \right) ds \geq 0.$$

So $(Nv - Mv) \in \mathcal{P}$. We have that $Mv \leq Nv$ for all $v \in \mathcal{P}$, implying that $M \leq N$ with respect to $\mathcal{P}$. So, by Theorem 2.2, $\Lambda_1 \leq \Lambda_2$. So either $q(t) = r(t)$ or $q(t) \neq r(t)$. If $q(t) = r(t)$, then $\Lambda_1 = \Lambda_2$. Suppose that $q(t) \neq r(t)$. Then there exist some subinterval $[a,b] \subseteq [0,1]$, such that $q(t) < r(t)$. Through reasoning similar to the proof of Lemma 2.9, $N - M \ : \ \mathcal{P} \backslash \{0\} \rightarrow \Omega$. Therefore $(N - M)v_1 \in \Omega \subseteq \mathcal{P}^{\circ}$. So there exists some $\epsilon > 0$ such that $(N - M)v_1 - \epsilon v_1 \in \mathcal{P}$. So $\epsilon v_1 \leq (N - M)v_1 = Nv_1 - Mv_1$ with respect to $\mathcal{P}$. Thus $\Lambda_1 v_1 + \epsilon v_1 = Mu_1 + \epsilon v_1 \leq Nv_1$. This implies that $Nv \geq (\Lambda_1 + \epsilon)v_1$. Since $N \leq N$ and $Nv_2 = \Lambda_2 v_2$, $\Lambda_1 + \epsilon \leq \Lambda_2$, thus $\Lambda_1 < \Lambda_2$. □

By Remark 2.4, the following theorem is an immediate consequence of Theorems 2.9 and 2.10.

Theorem 2.11. *Assume the hypotheses of Theorem 2.10. Then there exists smallest positive eigenvalues λ_1 and λ_2 of* (2.4),(2.6) *and* (2.4),(2.6)*(consequently for* (2.1), (2.3), *and* (2.2) (2.3)*), respectively, each of which is simple, positive, and less than the absolute value of any other eigenvalue of the corresponding problems. Also, eigenfunctions corresponding to λ_1 and λ_2 may be chosen to belong to $\mathcal{P}^{\circ}$. Finally, $\lambda_1 \geq \lambda_2$, and $\lambda_1 = \lambda_2$ if and only if $q(t) = r(t)$ for $0 \leq t \leq 1$.*

Chapter 3

Extremal Points

3.1 Preliminary Definitions and Theorems

We will begin this section with a few key definitions and theorems for classifying extremal points of a boundary value problem.

Definition 3.1. We say that b_0 is the first extremal point of the boundary value problem (1.4), (1.5_b) if $b_0 = \inf\{b > p \mid$ (1.4), (1.5_b) has a nontrivial solution $\}$.

Definition 3.2. A bounded linear operator $N : \mathcal{B} \to \mathcal{B}$ is said to be positive with respect to the cone $\mathcal{P}$ if $N : \mathcal{P} \to \mathcal{P}$.

Definition 3.3. The set of eigenvalues of a bounded linear operator N is known as the spectrum. The supremum of the absolute values of the this set is known as the spectral radius and is denoted $r(N)$.

The following four theorems are crucial to our results. The first result can be found in [28] and the other three can be found in [1] or [26]. Assume in each of the following that $\mathcal{P}$ is a reproducing cone and $N, N_1, N_2 : \mathcal{B} \to \mathcal{B}$ are compact, linear, and positive with respect to $\mathcal{P}$.

Theorem 3.1. *Let N_b, $0 \leq b \leq 1$, be a family of compact, linear operators on a Banach space such that the mapping $b \to N_b$ is continuous in the uniform topology. Then the mapping $b \to r(N_b)$ is also continuous.*

Theorem 3.2. *Assume $r(N) > 0$. Then $r(N)$ is an eigenvalue of N, and there is a corresponding eigenvector in P.*

Theorem 3.3. *If $N_1 \leq N_2$ with respect to $\mathcal{P}$, then $r(N_1) \leq r(N_2)$.*

Theorem 3.4. *Suppose there exist $\gamma > 0$, $u \in \mathcal{B}$, $-u \notin \mathcal{P}$ such that $\gamma u \leq Nu$ with respect to $\mathcal{P}$. Then N has an eigenvector in $\mathcal{P}$ which corresponds to an eigenvalue λ with $\lambda \geq \gamma$.*

3.2 Characterization of Extremal Points

Now, we will characterize extremal points of the boundary value problem (1.4), (1.5_b). We will assume throughout that the boundary value problem

$$u^{(4)} + \lambda q(t) = 0 \tag{3.1}$$

with boundary value conditions

$$u(0) = u'(p) = u''(p) = u'''(p) = 0 \tag{3.2}$$

has only the trivial solution for $\lambda \leq 1$.

We define a Banach space $\mathcal{B}$ and cone $\mathcal{P} \subset \mathcal{B}$ in order to apply the above theorems. First, define the Banach space $\mathcal{B}$

$$\mathcal{B} = \{u \in C^1[0,1] \mid u(0) = 0\},$$

with norm

$$\|u\| = \sup_{0 \leq t \leq 1} |u'(t)|.$$

Define the cone $\mathcal{P} \subset \mathcal{B}$ as

$$\mathcal{P} = \{u \in \mathcal{B} \mid u(t) \geq 0 \text{ on } [0,1]\}.$$

Furthermore, for each $b \in [p,1]$, we define the family of Banach spaces $\mathcal{B}_b \subset \mathcal{B}$ and cones $\mathcal{P}_b \subset \mathcal{B}_b$. Define the Banach space $\mathcal{B}_b$ by

$$\mathcal{B}_b = \{u \in C^1[0,b] \mid u(0) = 0\},$$

with norm

$$\|u\| = \sup_{0 \leq t \leq b} |u'(t)|.$$

Define the cone $\mathcal{P}_b \subset \mathcal{B}_b$ as

$$\mathcal{P}_b = \{u \in \mathcal{B}_b \mid u(t) \geq 0 \text{ on } [0,b]\}.$$

For $u \in \mathcal{B}_b$, $0 \leq t \leq b \leq 1$, we have

$$\begin{aligned} |u(t)| = |u(t) - u(0)| &= \left| \int_0^t u'(s)ds \right| \\ &\leq \|u\| t \\ &\leq \|u\|, \end{aligned}$$

and so $\sup_{0 \leq t \leq b} |u(t)| \leq \|u\|$.

Note that for all $b \in [p,1]$, $\mathcal{P}_b^\circ \neq \{\emptyset\}$. In fact,

$$\Omega_b := \{u \in \mathcal{B}_b \mid u(t) > 0 \text{ on } (0,b] \text{ and } u'(t) > 0\} \subset \mathcal{P}_b^\circ.$$

For each $b \in [p, 1]$, the Green's function for $-u^{(4)} = 0$, (1.5_b) is

$$G(b; t, s) = \begin{cases} -t[\frac{p^2}{2} - ps] - \frac{t^2 s}{2} + \frac{t^3}{6}, & 0 \le t, p \le s \le b, \\ \\ -t[\frac{p^2}{2} - ps - \frac{(p-s)^2}{2}] - \frac{t^2 s}{2} + \frac{t^3}{6}, & 0 \le t \le s \le p \le b, \\ \\ -t[\frac{p^2}{2} - ps] - \frac{t^2 s}{2} + \frac{t^3}{6} - \frac{(t-s)^3}{6}, & 0 \le p \le s \le t \le b, \\ \\ -t[\frac{p^2}{2} - ps - \frac{(p-s)}{2}] - \frac{t^2 s}{2} + \frac{t^3}{6} - \frac{(t-s)^3}{6}, & 0 \le s \le t, p \le b. \end{cases}$$

Define the linear operator

$$N_b u(t) = \begin{cases} \int_0^b G(b; t, s) q(s) u(s) ds, & 0 \le t \le b, \\ \int_0^b G(b; b, s) q(s) u(s) ds + (t - b) \int_0^b \frac{\partial}{\partial t} G(b; b, s) q(s) u(s), & b \le t \le 1. \end{cases}$$

From how we defined N_b, we have $N_b u \in C^1[0, 1]$ for $u \in C^1[0, 1]$, and $N_b u(0) = 0$. This yields $N_b : \mathcal{B} \to \mathcal{B}$. Also, when N_b is restricted to $\mathcal{B}_b$, $N_b : \mathcal{B}_b \to \mathcal{B}_b$ by

$$N_b u(t) = \int_0^b G(b; t, s) q(s) u(s) ds,$$

and so $u(t)$ is a solution to (1.4), (1.5_b) if and only if

$$u(t) = N_b u(t) = \int_0^b G(b; t, s) q(s) u(s) ds$$

for $t \in [0, b]$.

Lemma 3.1. *For all $b \in [p, 1]$, the linear operator N_b is positive with respect to $\mathcal{P}$ and $\mathcal{P}_b$. Also, $N_b : \mathcal{P}_b \backslash \{0\} \to \mathcal{P}_b^\circ$.*

Proof. Let $b \in [p, 1]$. For $u \in \mathcal{P}$, we have $G(b; t, s) \ge 0$, $\frac{\partial}{\partial t} G(b; b, s) \ge 0$, and $q(s) u(s) \ge 0$, $N_b u(t) \ge 0$ when $0 \le t \le 1$. We can conclude $N_b : \mathcal{P} \to \mathcal{P}$ and similarly, $N_b : \mathcal{P}_b \to \mathcal{P}_b$.

Now, let $u \in \mathcal{P}_b \backslash \{0\}$. There exists a compact interval $[\alpha, \beta] \subset [0, b]$ such that $q(s)u(s) > 0$ for all $s \in [\alpha, \beta]$. Since $G(b; t, s) > 0$ for $0 < t \leq b$,

$$\begin{aligned} N_b u(t) &= \int_0^b G(b; t, s) q(s) u(s) ds \\ &\geq \int_\alpha^\beta G(b; t, s) q(s) u(s) ds \\ &> 0. \end{aligned}$$

Since $\frac{\partial}{\partial t} G(b; t, s)|_{t=0} > 0$, we can see

$$\begin{aligned} N_b u'(t) &= \int_0^b \frac{\partial}{\partial t} G(b; 0, s) q(s) u(s) ds \\ &\geq \int_\alpha^\beta \frac{\partial}{\partial t} G(b; 0, s) q(s) u(s) ds \\ &> 0, \end{aligned}$$

and so $N_b u \in \Omega_b$. we obtain $N_b : \mathcal{P} \backslash \{0\} \to \Omega_b \subset \mathcal{P}_b^\circ$.

$\square$

Lemma 3.2. *The map $b \to N_b$ is continuous in the uniform topology.*

Proof. First, note from earlier that $\sup_{0 \leq t \leq 1} |u(t)| \leq \|u\|$. Consider the function $f : [p, 1] \to \{N_b\}$, $b \in [p, 1]$, defined by $f(b) = N_b$. Let $p \leq b_1 < b_2 \leq 1$. Let $\epsilon > 0$. Now

$$\begin{aligned} \|f(b_2) - f(b_1)\| &= \|N_{b_2} - N_{b_1}\| \\ &= \sup_{\|u\|=1} \|N_{b_2} u - N_{b_1} u\| \\ &= \sup_{\|u\|=1} \{ \sup_{t \in [0,1]} |(N_{b_2} u)'(t) - (N_{b_1} u)'(t)| \}. \end{aligned}$$

Since $\frac{\partial}{\partial t} G(b; t, s)$ and $q(t)$ are continuous functions in t for $0 \leq t \leq b$, they are bounded above for $0 \leq t \leq b$. Choose K and Q such that $|\frac{\partial}{\partial t} G(b; t, s)| \leq K$ for all $b \in [p, 1]$ and $|q(t)| \leq Q$ for $0 \leq t \leq 1$. Since $G(b; t, s) \in C^1[0, b]$ in t, there exists a $\delta > 0$ with $\delta < \frac{\epsilon}{2KQ}$ such that for $|t_2 - t_1| < \delta$, $|\frac{\partial}{\partial t}G(b; t_2, s) - \frac{\partial}{\partial t}G(b; t_1, s)| < \frac{\epsilon}{2KQ}$. □

Theorem 3.5. *For $p \leq b \leq 1$, $r(N_b)$ is strictly increasing as a function of b.*

Proof. It was previously shown in Theorem 2.3 that if $b = 1$, there is a $\lambda > 0$ and $u \in \mathcal{P}_b \backslash \{0\}$ such that $N_b u(t) = \lambda u(t)$ for $t \in [0, b]$. Similarly, one can show that for $b \in [p, 1)$, there is a $\lambda > 0$ and $u \in \mathcal{P}_b \backslash \{0\}$ such that $N_b u(t) = \lambda u(t)$ for $t \in [0, b]$. Extend u to $[b, 1]$ by $\lambda u(t) = \int_0^b G(b; b, s)q(s)u(s)ds + (x - b)\int_0^b \frac{\partial}{\partial t}G(b; b, s)q(s)u(s)ds$. Then for $t \in [0, 1]$, $N_b u(t) = \lambda u(t)$. Thus for $p \leq b \leq 1$, $r(N_b) \geq \lambda > 0$.

Now let $p \leq b_1 < b_2 \leq 1$. Since $r(N_{b_1}) > 0$, then by Theorem 3.2, there exists a $u_0 \in \mathcal{P}_{b_1} \backslash \{0\}$ such that $N_{b_1} u_0 = r(N_{b_1})u_0$. Let $u_1 = N_{b_1} u_0$ and $u_2 = N_{b_2} u_0$. Then for $t \in (0, b_1]$,

$$(u_2 - u_1)(t) = \int_{b_1}^{b_2} G(b_2; t, s)q(s)u(s)ds > 0.$$

Also,

$$(u_2 - u_1)'(0) = \int_{b_1}^{b_2} \frac{\partial}{\partial t}G(b_2; 0, s)q(s)u(s)ds > 0.$$

Thus the restriction of $u_2 - u_1$ to $[0, b_1]$ belongs to Ω_{b_1}, so there exists $\delta > 0$ such that $u_2 - u_1 \geq \delta u_0$ with respect to $\mathcal{P}_{b_1}$. Since $u_2 \in \mathcal{P}$, it follows that $u_2 - u_1 \geq \delta u_0$ with respect to $\mathcal{P}$. Thus

$$\begin{aligned} u_2 &\geq u_1 + \delta u_0 \\ &= r(N_{b_1})u_0 + \delta u_0 \\ &= (r(N_{b_1}) + \delta)u_0 \end{aligned}$$

with respect to $\mathcal{P}$. We get $N_{b_2} u_0 \geq (r(N_{b_1}) + \delta)u_0$ with respect to $\mathcal{P}$, and so by Theorem 3.4, $r(N_{b_2}) \geq r(N_{b_1}) + \delta$. Then $r(N_{b_2}) > r(N_{b_1})$ and $r(N_b)$ is strictly increasing. □

Theorem 3.6. *The following are equivalent:*

(i) b_0 *is the first extremal point of the boundary value problem corresponding to* (1.4),(1.5$_b$)*;*

(ii) there exists a nontrivial solution u *of the boundary value problem* (1.4),(1.5$_{b_0}$) *such that* $u \in \mathcal{P}_{b_0}$*;*

(iii) $r(N_{b_0}) = 1$.

Proof. First, we show (iii→ii). Assume $r(N_{b_0}) = 1$. By Theorem 3.2, 1 is an eigenvalue of N_{b_0}, so there exists a $u \in \mathcal{P}_{b_0}$ such that $N_{b_0}u = 1u$. So u solves (1.4), (1.5$_{b_0}$).

Next, we show (ii→i). Let u be a nontrivial solution to (1.4), (1.5$_{b_0}$) with $u \in \mathcal{P}_{b_0}$. Extend u to $[b_0, 1]$ by $u(t) = \int_0^{b_0} G(b_0; b_0, s)q(s)u(s)ds + (t-b_0)\int_0^{b_0} \frac{\partial}{\partial t}G(b_0; b_0, s)q(s)u(s)ds$, and hence $N_{b_0}u = u$ and hence 1 is an eigenvalue of N_{b_0}. Thus $r(N_{b_0}) \geq 1$. If $r(N_{b_0}) = 1$, then for all $b \in [p, b_0]$, $r(N_{b_0}) < 1$. Hence b_0 is the first extremal point of (1.4),(1.5$_b$).

Now assume that $r(N_{b_0}) > 1$. By Theorem 3.2, there exists a $w \in \mathcal{P}_{b_0}\backslash 0$ such that $N_{b_0}w = r(N_{b_0})w$. By Lemma 3.1, $N_{b_0}w \in \Omega_{b_0}$. Therefore $r(N_{b_0})w \in \Omega_{b_0}$, and so $u - \delta w \in \mathcal{P}_{b_0}$ for some $\delta > 0$.

For $[t \in b_0, 1]$, extend $w(t)$ by letting $w(t) = \int_0^{b_0} G(b_0; b_0, s)q(s)w(s)ds + (t-b_0)\int_0^{b_0} \frac{\partial}{\partial t}G(b_0; b_0, s)q(s)w(s)ds$. Then $u - \delta w \in \mathcal{P}$, and so $u \geq \delta w$ with respect to $\mathcal{P}$. Assume δ is maximal such that this inequality holds. Then $u = N_{b_0}u \geq N_{b_0}(\delta w) = \delta N_{b_0}w = \delta r(N_{b_0})w$. Since $r(N_{b_0}) > 1$, $\delta r(N_{b_0}) > \delta$. However, $u \geq \delta r(N_{b_0})w$, which contradicts the maximality of δ. Thus $r(N_{b_0}) = 1$.

Lastly, we show (i→iii). Assuming (i), there exists a $u \in \mathcal{P}_b\backslash\{0\}$ such that $u = N_{b_0}u$. This shows that $r(N_{b_0}) \geq 1$. We claim $r(N_{b_0}) < 1$. By way of contradiction, assume $r(N_{b_0}) > 1$. Following in the way of Remark 2.2, we can show that if Λ is an eigenvalue of N_p, then $\frac{1}{\Lambda}$ is an eigenvalue of (3.1),(3.2). By our assumption, (3.1),(3.2) has only the trivial solution for $\lambda \leq 1$. Thus if (3.1),(3.2) has a nontrivial solution, $\Lambda < 1$. So $r(N_p) < 1$. Since $r(N_b)$ is continuous with respect to b, by the Intermediate Value Theorem, there exists an $\alpha \in (p, b_0)$ such that $r(N_\alpha) = 1$. So there exists a nontrivial solution u to (1.4),(1.5$_\alpha$) with $u \in \mathcal{P}_\alpha \backslash \{0\}$, which is a contradiction since b_0 is the first extremal point of (1.4),(1.5$_\alpha$). Therefore $r(N_{b_0}) = 1$. $\square$

3.3 The Nonlinear Problem

In this section, we consider the nonlinear boundary value problem

$$u^{(4)} + f(t, u) = 0 \tag{3.3}$$

for $0 \leq t \leq 1$, satisfying boundary conditions (1.5$_b$), where $f(t, u) : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is continuous and $f(t, 0) \equiv 0$.

Assume $q(t) \equiv \dfrac{\partial f}{\partial u}(t, u)\Big|_{u=0}$ exists, is a nonnegative continuous function on $[0, 1]$, and does not vanish identically on any nondegenerate compact subinterval of $[0, 1]$. Then the variational equation along the zero solution of (3.3) is

$$u^{(4)} + q(t)u = 0. \tag{3.4}$$

For the existence of nontrivial solutions of the boundary value problem (3.3),(1.5$_b$), we apply the following fixed point theorem for nonlinear operator equations; see Deimling [5] or Schmitt and Smith [31].

Lemma 3.3. *Let $\mathcal{B}$ be a Banach space and $\mathcal{P} \subset \mathcal{B}$ a reproducing cone. Let $M : \mathcal{B} \to \mathcal{B}$ be a completely continuous, nonlinear operator such that $M : \mathcal{P} \to \mathcal{P}$ and $M(0) = 0$. Let M be Fréchet differentiable at $u = 0$ whose Fréchet derivative $N = M'(0)$ has the property:*

(A) There exist $w \in \mathcal{P}$ and $\mu > 1$ such that $Nw = \mu w$, and $Nu = u$ implies that $u \notin \mathcal{P}$. Further, there exists $\rho > 0$ such that, if $u = (1/\lambda)Mu$, $u \in \mathcal{P}$ and $\|u\| = \rho$, then $\lambda \leq 1$.

Then, the equation $u = Mu$ has a solution $u \in \mathcal{P} \backslash \{0\}$.

Theorem 3.7. *Assume b_0 is the first extremal point of* (3.4), (1.5_b). *Assume also the following condition holds:*

(A') There exists a $\rho(b) > 0$ such that, if $v(t)$ is a nontrivial solution of $u^{(4)} + (1/\lambda)f(t,u) = 0$ satisfying (1.5_b), and if $u \in \mathcal{P}$, with $\|u\| = \rho(b)$, then $\lambda \leq 1$.

Then, for all b satisfying $b_0 < b \leq 1$, the boundary value problem (3.3),(1.5_b) *has a solution $u \in \mathcal{P}\backslash\{0\}$.*

Proof. For each b satisfying $b_0 < b \leq 1$, let N_b be defined as in the previous section with respect to $q(t) \equiv \left.\dfrac{\partial f}{\partial u}(t,u)\right|_{u=0}$. Define the linear operator M_b by

$$M_b v(x) = \begin{cases} \int_0^b G(t,s)q(s)u(s)ds, & 0 \leq t \leq b, \\ \int_0^b G(b,s)p(s)q(s)ds & \\ \qquad +(t-b)\int_0^b \frac{\partial}{\partial t}G(b,s)q(s)u(s)ds, & b \leq t \leq 1. \end{cases}$$

Then M_b is Fréchet differentiable at $v = 0$ and $M_b'(0) = N_b$.

From Theorem 3.5 and Theorem 3.6, it follows that $r(N_{b_0}) = 1$ and $r(N_b) > 1$ for $b > b_0$. Moreover, since b_0 is the first extremal point of (3.4) corresponding to (1.5_b), it follows from Theorem 3.6 that, for $b > b_0$, if $N_b u = u$ and u is nontrivial, then $u \notin \mathcal{P}$. Thus, *(A')* and Lemma 3.3 imply there exists a $u \in \mathcal{P}\backslash\{0\}$ such that $M_b u = u$. So u is a nontrivial solution of (3.3), (1.5_b), with $u \in \mathcal{P}\backslash\{0\}$. $\square$

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