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Applications Of Leggett Williams Type Fixed Point Theorems To A Second Order Difference Equation

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APPLICATIONS OF LEGGETT WILLIAMS TYPE FIXED POINT THEOREMS TO A SECOND ORDER DIFFERENCE EQUATION

By

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DEDICATION

This thesis is dedicated to my parents for all their support.

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ABSTRACT

Using extensions of the LeggettWilliams fixed-point theorem, we prove the existence of solutions for a class of second-order difference equations with Dirichlet boundary conditions. We present these fixed point theorems and then show what conditions have to be met in order to satisfy the theorem. Finally, we provide specific examples to show the hypotheses of the theorems do not contradict one another.

Contents

Chapter 1

Introduction

For years, fixed point theory has found itself at the center of study of boundary value problems. Many fixed point theorems have provided criteria for the existence of positive solutions or multiple positive solutions of boundary value problems. Some of these results can be seen in the works of Guo [12], Krosnosel'skii [14], Leggett and Williams [15], and Avery et al. [4, 8].

Applications of the aforementioned fixed point theorems have been seen in works dealing with ordinary differential equations [3, 7, 11] and dynamic equations on time scales [10, 16, 19]. Most relevant to this research, these theorems have been utilized for results that involve finite difference equations [6, 9, 13, 18].

In this paper, we give an application of two recent Avery *et al.* fixed point theorems to obtain at least one positive solution of the difference equation

$$\Delta^2 u(k) + f(u(k)), \quad k \in \{0, 1, ..., N\}, \tag{1.1}$$

with boundary conditions

$$u(0) = u(N+2) = 0, \tag{1.2}$$

Here $f : \mathbb{R} \to [0, \infty)$ is any continuous function and Δ^2 is the second-difference operator defined by $(\Delta^2 u)(k) = u(k+2) - 2u(k+1) + u(k)$.

The original Leggett-Williams theorem states the following.

Theorem 1.1 (Leggett-Williams [15]). *Let K be a cone in a Banach space E and define the sets*

$$K_{\epsilon_1} := \{x \in K \ : \ ||x|| \leq \epsilon_1\}$$

and

$$S(\beta, \epsilon_2, \epsilon_3) := \{x \in K \ : \ \epsilon_2 \leq \beta(x) \text{ and } ||x|| \leq \epsilon_3\}$$

for $\epsilon_1 > 0$ and $\epsilon_3 > \epsilon_2 > 0$ and any concave positive functional β defined on the cone K, with $\beta(x) \leq ||x||$.

Suppose that $c \geq b > a > 0$, α is a concave positive functional with $\alpha(x) \leq ||x||$ and $A : K_c \to K$ is a completely continuous operator such that

(i) $\{x \in S(\alpha, a, b) \ : \ \alpha(x) > a\} \neq \emptyset$, *and* $\alpha(Ax) > a$ *if* $x \in S(\alpha, a, b)$,

(ii) $Ax \in K_c$ *if* $x \in S(\alpha, a, c)$, *and*

(iii) $\alpha(Ax) > a$ *for all* $x \in S(\alpha, a, c)$ *with* $||Ax|| > b$.

Then A has a fixed point in $S(\alpha, a, c)$.

The Leggett-Williams fixed point theorem has been modified by many [1, 2, 4, 5, 17] in order to generalize the class of boundary value problem the fixed point theorem can be applied to. In the two fixed point theorems used here, the conditions involving the norm in the Leggett-Williams fixed point theorem are replaced by a more general convex positive functional. Also, only subsets of the cone are required to map inward and outward.

In Chapter 2, we provide some background definitions. In Chapter 3, we present the first fixed point theorem and show how this fixed point theorem can be applied to show the existence of a positive symmetric solution of (1.1),(1.2). A nontrivial example is then provided. In Chapter 4, similar results are obtained while using a second fixed point theorem.

Chapter 2

Preliminaries

2.1 Definitions

Definition 2.1. Let E be a real Banach space. A nonempty closed convex set $\mathcal{P} \subset E$ is called a cone provided:

(i) $u \in \mathcal{P}$, $\lambda \geq 0$ implies $\lambda u \in \mathcal{P}$;

(ii) $u \in \mathcal{P}$, $-u \in \mathcal{P}$ implies $u = 0$.

Definition 2.2. A map α is said to be a nonnegative continuous concave functional on a cone $\mathcal{P}$ of a real Banach space E if

$$\alpha : \mathcal{P} \to [0, \infty)$$

is continuous and

$$\alpha(tu + (1-t)v) \geq t\alpha(u) + (1-t)\alpha(v),$$

for all $u, v \in \mathcal{P}$ and $t \in [0, 1]$.

Definition 2.3. Similarly, the map β is a nonnegative continuous convex functional on a cone $\mathcal{P}$ of a real Banach space E if

$$\beta : \mathcal{P} \to [0, \infty)$$

is continuous and

$$\beta(tu + (1-t)v) \leq t\beta(u) + (1-t)\beta(v),$$

for all $u, v \in \mathcal{P}$ and $t \in [0, 1]$.

2.2 The Green's Function

It is well known that the Green's function for $-\Delta^2 u = 0$ satisfying the boundary conditions (1.2) is given by

$$H(k,l) = \frac{1}{N+2} \begin{cases} k(N+2-l), & k \in \{0, \ldots, l\}, \\ \\ l(N+2-k), & k \in \{l+1, \ldots, N+2\}, \end{cases}$$

So u solves (1.1),(1.2) if and only if $u(k) = \sum_{l=1}^{N+1} H(k,l) f(u(l))$. Notice that $H(k,l) \geq 0$.

Lemma 2.1. *For* $(k,l) \in \{0, \ldots, N+2\} \times \{0, \ldots, N+2\}$, $H(N+2-k, N+2-l) = H(k,l)$.

Proof. For $(k,l) \in \{0,\dots,N+2\} \times \{0,\dots,N+2\}$,

$$
\begin{aligned}
&H(N+2-k, N+2-l)\\
&= \frac{1}{N+2}\begin{cases} (N+2-k)(N+2-(N+2-l)), & 0 \le N+2-k \le N+2-l \le N+1,\\ (N+2-l)(N+2-(N+2-k)), & 1 \le N+2-l \le N+2-k \le N+2, \end{cases}\\
&= \frac{1}{N+2}\begin{cases} l(N+2-k), & 1 \le l \le k \le N+2,\\ k(N+2-l), & 0 \le k \le l \le N+1, \end{cases}\\
&= H(k,l).
\end{aligned}
$$

$\square$

Lemma 2.2. *$H(k,l)$ has the property that $\frac{H(y,l)}{H(w,l)} \ge \frac{y}{w}$ for all $l, w, y \in \{0,\dots,N+2\}$ with $w \ge y$.*

Proof. When $y \le w \le l$,

$$\frac{H(y,l)}{H(w,l)} = \frac{\frac{1}{N+2}(y(N+2-l))}{\frac{1}{N+2}(w(N+2-l))} = \frac{y}{w}.$$

When $y \le l \le w$,

$$\frac{H(y,l)}{H(w,l)} = \frac{\frac{1}{N+2}(y(N+2-l))}{\frac{1}{N+2}(l(N+2-w))} \ge \frac{y(N+2-w)}{w(N+2-w)} = \frac{y}{w}.$$

When $l \le y \le w$,

$$\frac{H(y,l)}{H(w,l)} = \frac{\frac{1}{N+2}(l(N+2-y))}{\frac{1}{N+2}(l(N+2-w))} \ge 1 \ge \frac{y}{w}.$$

$\square$

Chapter 3

Application of the First Fixed Point Theorem

3.1 The Fixed Point Theorem

Definition 3.1. Let α and ψ be nonnegative continuous functionals on a cone $\mathcal{P}$ and δ and β be nonnegative continuous convex functionals on $\mathcal{P}$; then, for nonnegative real numbers a, b, c, and d we define the sets

$$A := A(\alpha, \beta, a, d) = \{x \in P : a \leq \alpha(x) \text{ and } \beta(x) \leq d\},$$

$$B := B(\alpha, \delta, \beta, a, b, d) = \{x \in A : \delta(x) \leq b\},$$

and

$$C := C(\alpha, \psi, \beta, a, c, d) = \{x \in A : c \leq \psi(x)\}.$$

Theorem 3.1 (Anderson, Avery, Henderson [2]). *Suppose $\mathcal{P}$ is a cone in a real Banach space E, α and ψ are nonnegative continuous concave functionals on $\mathcal{P}$, β and δ are nonnegative continuous convex functionals on $\mathcal{P}$, and for nonnegative real numbers a, b, c, and d, the sets A, B, and C are defined as above. Furthermore, suppose that A is a bounded subset of $\mathcal{P}$, that $T : A \to \mathcal{P}$ is a completely continuous operator, and that the following conditions hold:*

(A1) $\{x \in A : c < \psi(x) \text{ and } \delta(x) < b\} \neq \emptyset$ *and* $\{x \in P : \alpha(x) < a \text{ and } d < \beta(x)\} = \emptyset$;

(A2) $\alpha(Tx) \geq a$ *for all* $x \in B$;

(A3) $\alpha(Tx) \geq a$ *for all* $x \in A$ *with* $\delta(Tx) > b$;

(A4) $\beta(Tx) \leq d$ *for all* $x \in C$ *and,*

(A5) $\beta(Tx) \leq d$ *for all* $x \in A$ *with* $\psi(Tx) < c$.

Then T *has a fixed point* $x^* \in A$.

3.2 Preliminaries

Define the Banach space E to be

$$E = \{u : \{0, \dots, N+2\} \to \mathbb{R}\}$$

with the norm

$$||u|| = \max_{k \in \{0,1,\dots,N+2\}} |u(k)|.$$

Define the cone $\mathcal{P} \subset E$ by

$$\mathcal{P} := \{u \in E : u(N+2-k) = u(k), u \text{ is nonnegative and nondecreasing on } \\ \{0,1,\dots,\lfloor\tfrac{N+2}{2}\rfloor\}, \text{ and } wu(y) \geq yu(w) \text{ for } w \geq y \text{ with } y,w \in \{0,1,\dots,\lfloor\tfrac{N+2}{2}\rfloor\}\big\}.$$

Define the operator $T : E \to E$ by

$$Tu(k) := \sum_{l=1}^{N+1} H(k,l) f(u(l)),$$

where $H(k,l)$ is the Green's function for $-\Delta^2 u(k) = 0$ satisfying the boundary conditions (1.2). So if u is a fixed point of T, u solves (1.1),(1.2).

Lemma 3.1. *The operator* $T : E \to E$ *is completely continuous.*

Proof. We use the Arzelá Ascoli theorem to show that T is a compact operator, and thus completely continuous. So we must show T is continuous, uniformly bounded, and equicontinuous. First, note that $H(k,l)$ is bounded, so there exists a $K > 0$ such that $|H(k,l)| \leq K$ for all $k, l \in \{0, \ldots, N+2\} \times \{0, \ldots, N+2\}$.

Let $\epsilon > 0$. Let $u \in E$. Since f is continuous, f is uniformly continuous on $[-||u|| - 1, ||u|| + 1]$. So there exists a $\delta > 0$ with $\delta < 1$ such that for all $x, y \in [-||u|| - 1, ||u|| + 1]$, $|f(x) - f(y)| < \epsilon/(N+1)K$. So for all $v \in E$ with $||u - v|| < \delta$, $u(k), v(k) \in [-||u|| - 1, ||u|| + 1]$ and $|u(k) - v(k)| < \delta$ for all $k \in \{0, \ldots, N+2\}$. Thus for all $k \in \{0, \ldots, N+2\}$, $|f(u(k)) - f(v(k))| < \epsilon/(N+1)K$. Thus for $k \in \{0, \ldots, N+2\}$

$$
\begin{aligned}
|Tu(k) - Tv(k)| &\leq \sum_{l=1}^{N+1} |H(k,l)||f(u(l)) - f(v(l))| \\
&< \sum_{l=1}^{N+1} K \cdot \frac{\epsilon}{(N+1)K} = \epsilon.
\end{aligned}
$$

So $||Tu - Tv|| < \epsilon$ and therefore T is continuous.

Now let $\{u_n\}$ be a bounded sequence in E with $||u_n|| \leq K_0$ for all n. Since f is continuous, there exists a $K_1 > 0$ such that $|f(u_n(k))| \leq K_1$ for all $k \in \{0, \ldots, N+2\}$ and for all n. So for $k \in \{0, \ldots, N+2\}$

$$
\begin{aligned}
|Tu_n(k)| &\leq \sum_{l=1}^{N+1} |H(k,l)||f(u_n(l))| \\
&\leq \sum_{l=1}^{N+1} KK_1 = (N+1)KK_1
\end{aligned}
$$

for all n. So $\{Tu_n\}$ is uniformly bounded.

Lastly, choose $\delta < 1$. So if $k_1, k_2 \in \{0, \ldots, N+2\}$ with $|k_1 - k_2| < \delta$, $k_1 = k_2$. Thus for all n, $|Tu_n(k_1) - Tu_n(k_2)| = 0 < \epsilon$. So if $|k_1 - k_2| < \delta$, $|Tu_n(k_1) - Tu_n(k_2)| < \epsilon$. So T is equicontinuous. Hence by the Arzelá Ascoli theorem, T is compact, and thus uniformly continuous. □

Lemma 3.2. *The operator T acting on the set A maps A to P. That is $T : A \to$*

$\mathcal{P}$.

Proof. Let $u \in A$. We first need to show $Tu(N+2-k) = Tu(k)$. By Lemma 2.1 $H(N+2-k, N+2-l) = H(k,l)$. Now

$$Tu(N+2-k) = \sum_{l=1}^{N+1} H(N+2-k, l) f(u(l)).$$

Substitute $r = N+2-l$. So

$$\begin{aligned} Tu(N+2-k) &= \sum_{r=1}^{N+1} H(N+2-k, N+2-r) f(u(N+2-r)) \\ &= \sum_{r=1}^{N+1} H(k,r) f(u(r)) = Tu(k). \end{aligned}$$

So $Tu(N+2-k) = Tu(k)$.

Next we need to show $Tu(k)$ is nonnegative and nondecreasing on $\{0, 1, \dots, \lfloor \frac{N+2}{2} \rfloor\}$. Since $H(k,l) \geq 0$ for $k, l \in \{0, \dots, N+2\}$ and $f : [0, \infty) \to [0, \infty)$, $Tu(k)$ is nonnegative for all $k \in \{0, \dots, N+2\}$.

To show that $Tu(k)$ is nondecreasing on $\{0, 1, \dots, \lfloor \frac{N+2}{2} \rfloor\}$, we show $\Delta Tu(k)$ is nonnegative on $\{0, 1, \dots, \lfloor \frac{N+2}{2} \rfloor\}$. Now

$$\Delta_k H(k,l) = H(k+1,l) - H(k,l) = \frac{1}{N+2} \begin{cases} N+2-l, & k \in \{0, \dots, l\}, \\ \\ -l, & k \in \{l, \dots, N+1\}. \end{cases}$$

So

$$\begin{aligned}
\Delta Tu(k) &= \sum_{l=1}^{N+1} \Delta_k H(k,l) f(u(l)) \\
&= \sum_{l=1}^{k-1} \frac{-l}{N+2} f(u(l)) + \sum_{l=k}^{N+1} \frac{N+2-l}{N+2} f(u(l)) \\
&= \sum_{l=1}^{k-1} \frac{-l}{N+2} f(u(l)) + \sum_{l=k}^{N+1} \frac{N+2-l}{N+2} f(u(N+2-l)) \\
&= \sum_{l=1}^{k-1} \frac{-l}{N+2} f(u(l)) + \sum_{r=1}^{N+2-k} \frac{r}{N+2} f(u(r)) \\
&= \sum_{l=1}^{k-1} \frac{-l}{N+2} f(u(l)) + \sum_{l=1}^{N+2-k} \frac{l}{N+2} f(u(l)).
\end{aligned}$$

Since $k \in \{0, 1, \ldots, \lfloor \frac{N+2}{2} \rfloor\}$,

$$\begin{aligned}
\Delta Tu(k) &= \sum_{l=1}^{k-1} \frac{-l}{N+2} f(u(l)) + \sum_{l=1}^{N+2-k} \frac{l}{N+2} f(u(l)) \\
&= \sum_{l=k}^{N+2-k} \frac{l}{N+2} f(u(l)) \geq 0.
\end{aligned}$$

So $Tu(k)$ is nondecreasing.

Lastly, since by Lemma 2.2, $H(k,l)$ has the property that $\frac{H(y,l)}{H(w,l)} \geq \frac{y}{w}$ for all l and for $w \geq y$, $wTu(y) \geq yTu(w)$. Thus $T : A \to \mathcal{P}$. $\square$

For $u \in \mathcal{P}$, define the concave functionals α and ψ on $\mathcal{P}$ by

$$\alpha(u) := \min_{k \in \{\tau, \ldots, \lfloor \frac{N+2}{2} \rfloor\}} u(k) = u(\tau),$$

$$\psi(u) := \min_{k \in \{\mu, \ldots, \lfloor \frac{N+2}{2} \rfloor\}} u(k) = u(\mu),$$

and the convex functionals δ and β on P by

$$\delta(u) := \max_{k \in \{0, \ldots, \nu\}} u(k) = u(\nu),$$

$$\beta(u) := \max_{k \in \{0, \ldots, \lfloor \frac{N+2}{2} \rfloor\}} u(k) = u(\lfloor \tfrac{N+2}{2} \rfloor).$$

3.3 Positive Symmetric Solutions to (1.1),(1.2)

Theorem 3.2. *Assume $\tau, \mu, \nu \in \{1, \dots, \lfloor \frac{N+2}{2} \rfloor\}$ are fixed with $\tau \le \mu < \nu$, that d and m are positive real numbers with $0 < m < \frac{d\mu}{\lfloor \frac{N+2}{2} \rfloor}$ and $f : [0, \infty) \to [0, \infty)$ is a continuous function such that*

(i) $f(w) \ge \dfrac{2(N+2)d}{(\nu - \tau)(3 + 2N - \tau - \nu)\lfloor \frac{N+2}{2} \rfloor}$ *for* $w \in \left[\dfrac{\tau d}{\lfloor \frac{N+2}{2} \rfloor}, \dfrac{\nu d}{\lfloor \frac{N+2}{2} \rfloor}\right]$,

(ii) $f(w)$ *is decreasing for* $w \in [0, m]$ *and* $f(m) \ge f(w)$ *for* $w \in [m, d]$, *and*

(iii) $2\sum_{l=1}^{\mu} \dfrac{l\lceil \frac{N+2}{2} \rceil}{N+2} f\left(\dfrac{ml}{\mu}\right) \le d$
$-f(m)\frac{1}{N+2}\left(\lceil \frac{N+2}{2} \rceil\right)\left(\lfloor \frac{N+2}{2} \rfloor - \mu\right)\left(\mu + 1 + \lfloor \frac{N+2}{2} \rfloor\right)$.

Then T has a fixed point $x^* \in A$. Thus (1.1), (1.2) *has at least one positive symmetric solution $u^* \in A(\alpha, \beta, \frac{\tau d}{\lfloor \frac{N+2}{2} \rfloor}, d)$.*

Proof. Let $a = \frac{\tau d}{\lfloor \frac{N+2}{2} \rfloor}$, $b = \frac{\nu d}{\lfloor \frac{N+2}{2} \rfloor}$, $c = \frac{\mu d}{\lfloor \frac{N+2}{2} \rfloor}$. By Lemma 3.1, T is completely continuous. By Lemma 3.2, $T : A \to \mathcal{P}$. Let $u \in A$. Then $\beta(u) = u\left(\lfloor \frac{N+2}{2} \rfloor\right) \le d$. But u achieves its maximum at $\lfloor \frac{N+2}{2} \rfloor$, so A is bounded.

First, we show (A1) holds. Let $u \in P$ and let $\beta(u) > d$. Then

$$\begin{aligned} \alpha(u) = u(\tau) &\ge \frac{\tau}{\lfloor \frac{N+2}{2} \rfloor} u(\lfloor \tfrac{N+2}{2} \rfloor) \\ &= \frac{\tau}{\lfloor \frac{N+2}{2} \rfloor} \beta(u) \\ &> \frac{\tau d}{\lfloor \frac{N+2}{2} \rfloor} = a. \end{aligned}$$

So $\{u \in P \ : \ \alpha(u) < a \text{ and } d < \beta(u)\} = \emptyset$.

Now let $K \in \left(\dfrac{2d(N+2)}{\lfloor \frac{N+2}{2} \rfloor(3N+2-\mu)}, \dfrac{2d(N+2)}{\lfloor \frac{N+2}{2} \rfloor(3N+2-\nu)}\right)$. Define

$u_K(k) = K \sum_{l=1}^{N+1} H(k,l) = \dfrac{Kk}{2(N+2)}(3N+2-k)$. Now

$$\begin{aligned}
\alpha(u_k) = u_k(\tau) &= \frac{K\tau}{2(N+2)}(3N+2-\tau) \\
&> \frac{2d\tau(3N+2-\tau)}{2\lfloor \frac{N+2}{2} \rfloor (3N+2-\mu)} \\
&\geq \frac{\tau d}{\lfloor \frac{N+2}{2} \rfloor} = a.
\end{aligned}$$

Also,

$$\begin{aligned}
\beta(u_k) = u_k(\lfloor \tfrac{N+2}{2} \rfloor) &= \frac{K\lfloor \frac{N+2}{2} \rfloor}{2(N+2)}(3N+2-\lfloor \tfrac{N+2}{2} \rfloor) \\
&< \frac{2\lfloor \frac{N+2}{2} \rfloor d(3N+2-\lfloor \frac{N+2}{2} \rfloor)}{2\lfloor \frac{N+2}{2} \rfloor (3N+2-\nu)} \\
&\leq \frac{\lfloor \frac{N+2}{2} \rfloor d}{\lfloor \frac{N+2}{2} \rfloor} = d.
\end{aligned}$$

So $u_k \in A$.

Since

$$\begin{aligned}
\psi(u_k) = u_k(\mu) &= \frac{K\mu}{2(N+2)}(3N+2-\mu) \\
&> \frac{2d\mu(3N+2-\mu)}{2\lfloor \frac{N+2}{2} \rfloor (3N+2-\mu)} \\
&= \frac{\mu d}{\lfloor \frac{N+2}{2} \rfloor} = c,
\end{aligned}$$

and

$$\begin{aligned}
\delta(u_k) = u_k(\nu) &= \frac{K\nu}{2(N+2)}(3N+2-\nu) \\
&< \frac{2d\nu(3N+2-\nu)}{2\lfloor \frac{N+2}{2} \rfloor (3N+2-\nu)} \\
&= \frac{\nu d}{\lfloor \frac{N+2}{2} \rfloor} = b,
\end{aligned}$$

$\{u \in A : c < \psi(u) \text{ and } \delta(u) < b\} \neq \emptyset$. Therefore (A1) holds.

Next, we show (A2) holds. Let $u \in B$ with $\delta(u) < b$. By (i),

$$
\begin{aligned}
\alpha(Tu) &= \sum_{l=1}^{N+1} H(\tau, l) f(u(l)) \\
&\geq \sum_{l=\tau+1}^{\nu} H(\tau, l) f(u(l)) \\
&\geq \frac{2(N+2)d}{(\nu-\tau)(3+2N-\tau-\nu)\lfloor \frac{N+2}{2} \rfloor} \cdot \frac{\tau(\nu-\tau)(3+2N-\tau-\nu)}{2(N+2)} \\
&\geq \frac{\tau d}{\lfloor \frac{N+2}{2} \rfloor} = a.
\end{aligned}
$$

So (A2) holds.

We will now show (A3) holds. Let $u \in A$ with $\delta(Tu) > b$. Then

$$
\begin{aligned}
\alpha(Tu) =& Tu(\tau) \\
=& \sum_{l=1}^{N+1} H(\tau, l) f(u(l)) \\
\geq& \frac{\tau}{\nu} \sum_{l=1}^{N+1} H(\nu, l) f(u(l)) \\
=& \frac{\tau}{\nu} \delta(Tu) \\
>& \frac{\tau}{\nu} b \\
=& \frac{d\tau}{\lfloor \frac{N+2}{2} \rfloor} = a.
\end{aligned}
$$

So (A3) holds.

Now we show (A4) holds. Let $u \in C$. By the concavity of u and since $c = \dfrac{\mu d}{\lfloor \frac{N+2}{2} \rfloor}$, for all $k \in \{0, 1, \ldots, \mu\}$,

$$
u(k) \geq \frac{ck}{\mu} \geq \frac{mk}{\mu}.
$$

So, by (ii) and (iii), we have

$$
\begin{aligned}
\beta(Tu) =& \sum_{l=1}^{N+1} H\left(\lfloor \tfrac{N+2}{2} \rfloor, l\right) f(u(l)) \\
\le& 2 \sum_{l=1}^{\lfloor \frac{N+2}{2} \rfloor} \frac{l\left(N+2-\lfloor \frac{N+2}{2} \rfloor\right)}{N+2} f(u(l)) \\
=& 2 \sum_{l=1}^{\mu} \frac{l\left(\lceil \frac{N+2}{2} \rceil\right)}{N+2} f(u(l)) + 2 \sum_{l=\mu+1}^{\lfloor \frac{N+2}{2} \rfloor} \frac{l\left(\lceil \frac{N+2}{2} \rceil\right)}{N+2} f(u(l)) \\
\le& 2 \sum_{l=1}^{\mu} \frac{l\left(\lceil \frac{N+2}{2} \rceil\right)}{N+2} f\left(u\left(\frac{ml}{\mu}\right)\right) + 2 \sum_{l=\mu+1}^{\lfloor \frac{N+2}{2} \rfloor} \frac{l\left(\lceil \frac{N+2}{2} \rceil\right)}{N+2} f(m) \\
\le& d - f(m) \frac{1}{N+2} \left(\left\lceil \frac{N+2}{2} \right\rceil\right) \left(\left\lfloor \frac{N+2}{2} \right\rfloor - \mu\right) \left(\mu + 1 + \left\lfloor \frac{N+2}{2} \right\rfloor\right) \\
&+ f(m) \frac{1}{N+2} \left(\left\lceil \frac{N+2}{2} \right\rceil\right) \left(\left\lfloor \frac{N+2}{2} \right\rfloor - \mu\right) \left(\mu + 1 + \left\lfloor \frac{N+2}{2} \right\rfloor\right) \\
=& d.
\end{aligned}
$$

So (A4) is satisfied.

Last, we show (A5) is satisfied. Let $u \in A$ with $\psi(Tu) < c$. So

$$
\begin{aligned}
\beta(Tu) =& \sum_{l=1}^{N+1} H(\lfloor \tfrac{N+2}{2} \rfloor, l) f(u(l)) \\
\le& \frac{\lfloor \frac{N+2}{2} \rfloor}{\mu} \sum_{l=1}^{N+1} H(\mu, l) f(u(l)) \\
\le& \frac{\lfloor \frac{N+2}{2} \rfloor}{\mu} \psi(Tu) \\
<& \frac{c \lfloor \frac{N+2}{2} \rfloor}{\mu} = d.
\end{aligned}
$$

Therefore T has a fixed point and (1.1), (1.2) has at least one positive symmetric solution $u^* \in A$.

$\square$

3.4 Example

Example 3.1. Let $N = 18$, $\tau = 1$, $\mu = 9$, $\nu = 10$, $d = 5$, and $m = 4.4$. Notice that $0 < \tau \leq \mu < \nu \leq 10 = \lfloor \frac{N+2}{2} \rfloor$, and $0 < m = 4.4 \leq 4.5 = \frac{d\mu}{\lfloor \frac{N+2}{2} \rfloor}$. Define a continuous function $f : [0, \infty) \to [0, \infty)$ by

$$f(w) = \begin{cases} \frac{45-w}{500}, & 0 \leq w \leq 40 \\ \\ \frac{1}{100}, & \qquad w \geq 40. \end{cases}$$

Then,

(i) for $w \in [\frac{1}{2}, 5]$, $f(w) \geq f(5) = \dfrac{2}{25} > \dfrac{5}{63} = \dfrac{2 \cdot 20 \cdot 5}{(10-1) \cdot (3 + 2 \cdot 18 - 1 - 10)(10)}$,

(ii) $f(w)$ is decreasing for $w \in [0, 4.4]$ and $f(m) \geq f(w)$ for $w \in [4.4, 5]$, and

(iii) $2 \displaystyle\sum_{l=1}^{9} \frac{10l}{20} f\left(\frac{4.4l}{9}\right) = \frac{5657}{1500} < \frac{1047}{250} = 5 - f(4.4)(\tfrac{1}{20})(10)(10-9)(9+1+10)$.

So the hypotheses of Theorem 3.2 are satisfied. Therefore, the difference equatione

$$\Delta^2 u(k) + f(u(k)), \quad k \in \{0, 1, ..., 18\},$$

with boundary conditions

$$u(0) = u(20) = 0,$$

has a positive symmetric solution u^* with $u(1) \geq \dfrac{1}{2}$ and $u(10) \leq 5$.

Chapter 4

Application of the Second Fixed Point Theorem

4.1 The Fixed Point Theorem

Definition 4.1. Let ψ and δ be nonnegative continuous functionals on a cone $\mathcal{P}$; then, for positive real numbers a, and b we define the sets

$$P(\psi, b) := \{x \in P : \psi(x) \leq b\},$$

and

$$P(\psi, \delta, a, b) := \{x \in P : a \leq \psi(x) \text{ and } \delta(x) \leq b\}.$$

Theorem 4.1 (Anderson, Avery, Henderson [4])**.** *Suppose $\mathcal{P}$ is a cone in a real Banach space E, α is a nonnegative continuous concave functional on $\mathcal{P}$, β is a nonnegative continuous convex functional on $\mathcal{P}$, and $T : \mathcal{P} \to \mathcal{P}$ is a completely continuous operator. Assume there exist nonnegative numbers a, b, c, and d such that:*

(A1) $\{x \in \mathcal{P} : a < \alpha(x)$ *and* $\beta(x) < b\} \neq \emptyset$*;*

(A2) if $x \in \mathcal{P}$ *with* $\beta(x) = b$ *and* $\alpha(x) \geq a$*, then* $\beta(Tx) < b$*;*

(A3) if $x \in \mathcal{P}$ with $\beta(x) = b$ and $\alpha(Tx) < a$, then $\beta(Tx) < b$;

(A4) $\{x \in \mathcal{P} : c < \alpha(x)$ and $\beta(x) < d\} \neq \emptyset$;

(A5) if $x \in \mathcal{P}$ with $\alpha(x) = c$ and $\beta(x) \leq d$, then $\alpha(Tx) > c$;

(A6) if $x \in \mathcal{P}$ with $\alpha(x) = c$ and $\beta(Tx) > d$, then $\alpha(Tx) > c$.

If

(H1) $a < c$, $b < d$, $\{x \in \mathcal{P} : b < \beta(x)$ and $\alpha(x) < c\} \neq \emptyset$, $\mathcal{P}(\beta, b) \subset \mathcal{P}(\alpha, c)$, and $\mathcal{P}(\alpha, c)$ is bounded,

then T has a fixed point x^* in $\mathcal{P}(\beta, \alpha, b, c)$.

If

(H2) $c < a$, $d < b$, $\{x \in \mathcal{P} : a < \alpha(x)$ and $\beta(x) < d\} \neq \emptyset$, $\mathcal{P}(\alpha, a) \subset \mathcal{P}(\beta, d)$, and $\mathcal{P}(\beta, d)$ is bounded,

then T has a fixed point x^* in $\mathcal{P}(\alpha, \beta, a, d)$.

4.2 Preliminaries

Define the Banach space E to be

$$E = \{u : \{0, \ldots, N+2\} \to \mathbb{R}\}$$

with the norm

$$||u|| = \max_{k \in \{0,1,\ldots,N+2\}} |u(k)|.$$

Define the cone $\mathcal{P} \subset E$ by

$$\mathcal{P} := \{u \in E : u(N+2-k) = u(k), u \text{ is nonnegative and nondecreasing on } \{0, 1, \ldots, \lfloor \tfrac{N+2}{2} \rfloor\}, \text{ and } wu(y) \geq yu(w) \text{ for } w \geq y \text{ with } y, w \in \{0, 1, \ldots, \lfloor \tfrac{N+2}{2} \rfloor\}\}.$$

Define the operator $T : E \to E$ by

$$Tu(k) := \sum_{l=1}^{N+1} H(k,l) f(u(l)),$$

where $H(k,l)$ is the Green's function for $-\Delta^2 u(k) = 0$ satisfying the boundary conditions (1.2). So if u is a fixed point of T, u solves (1.1),(1.2). By Lemma 3.1, T is completely continuous

Lemma 4.1. *The operator $T : \mathcal{P} \to \mathcal{P}$.*

Proof. The proof of this lemma is very similar to the proof of Lemma 3.2, so it is omitted. $\square$

For $u \in \mathcal{P}$, define the concave functional α on $\mathcal{P}$ by

$$\alpha(u) := \min_{k \in \{\tau, \dots, \lfloor \frac{N+2}{2} \rfloor\}} u(k) = u(\tau),$$

and the convex functional β on P by

$$\beta(u) := \max_{k \in \{0, \dots, \lfloor \frac{N+2}{2} \rfloor\}} u(k) = u(\lfloor \tfrac{N+2}{2} \rfloor).$$

4.3 Positive Symmetric Solutions to (1.1),(1.2)

Theorem 4.2. *If $\tau \in \{1, ..., \lfloor \frac{N+2}{2} \rfloor\}$ is fixed, b and c are positive real numbers with $3b < c$, and $f : [0, \infty) \to [0, \infty)$ is a continuous function such that:*

(i) $f(w) > \dfrac{c(N+2)}{\tau(N+1-\tau)(\lfloor \frac{N+2}{2} \rfloor - \tau)}$ *for* $w \in \left[c, \dfrac{c\lfloor \frac{N+2}{2} \rfloor}{\tau}\right]$,

(ii) $f(w)$ *is decreasing for* $w \in \left[\dfrac{b}{\lfloor \frac{N+2}{2} \rfloor}, \dfrac{b\tau}{\lfloor \frac{N+2}{2} \rfloor}\right]$ *with* $f\left(\dfrac{b\tau}{\lfloor \frac{N+2}{2} \rfloor}\right) \geq f(w)$ *for* $w \in \left[\dfrac{b\tau}{\lfloor \frac{N+2}{2} \rfloor}, b\right]$,

(iii) and $2\sum_{l=1}^{\tau}\frac{l(\lceil\frac{N+2}{2}\rceil)}{N+2}f\left(\frac{bl}{2}\right)$
$\leq b - f\left(\frac{b\tau}{\lfloor\frac{N+2}{2}\rfloor}\right)\frac{1}{N+2}(\lceil\frac{N+2}{2}\rceil)(\lfloor\frac{N+2}{2}\rfloor-\tau)(\tau+1+\lfloor\frac{N+2}{2}\rfloor)$,

then T has a fixed point x^* in $\mathcal{P}(\beta,\alpha,b,c)$. Thus the discrete right-focal problem (1.1), (1.2) *has at least one positive symmetric solution $u^{**}\in\mathcal{P}(\beta,\alpha,b,c)$.*

Proof. Note that by Lemma 4.1, $T:\mathcal{P}\to\mathcal{P}$. By Lemma 3.1, T is completely continuous. First, we let $a=\frac{b\tau}{\lfloor\frac{N+2}{2}\rfloor}$ and $d=\frac{c\lfloor\frac{N+2}{2}\rfloor}{\tau}$. Then, we have $a=\frac{b\tau}{\lfloor\frac{N+2}{2}\rfloor}<\frac{c\tau}{3\lfloor\frac{N+2}{2}\rfloor}<c$ and $b<\frac{c}{3}=\frac{d\tau}{3\lfloor\frac{N+2}{2}\rfloor}<d$.

We proceed to verify properties (A1) and (A4). First, for $K\in\left(\frac{2b(N+2)}{(3N+2-\tau)\lfloor\frac{N+2}{2}\rfloor},\frac{2b(N+2)}{(3N+2-\lfloor\frac{N+2}{2}\rfloor)\lfloor\frac{N+2}{2}\rfloor}\right)$, define the function u_L by

$$u_L(k):=\sum_{l=1}^{N+1}LH(k,l)=\frac{Lk}{2(N+2)}(3N+2-k).$$

Since

$$\alpha(u_L)=u_L(\tau)=\frac{L\tau}{2(N+2)}(3N+2-\tau)>a,$$

and

$$\beta(u_L)=u_L(\lfloor\tfrac{N+2}{2}\rfloor)=\frac{L\lfloor\frac{N+2}{2}\rfloor}{2(N+2)}(3N+2-\lfloor\tfrac{N+2}{2}\rfloor)<b,$$

$u_L\in\{u\in P: a<\alpha(u) \text{ and } \beta(u)<b\}$.

Similarly, for $J\in\left(\frac{2c(N+2)}{\tau(3N+2-\tau)},\frac{2c(N+2)}{\tau(3N+2-\lfloor\frac{N+2}{2}\rfloor)}\right)$, define the function u_J by

$$u_J(k):=\sum_{l=1}^{N+1}JH(k,l)=\frac{Jk}{2(N+2)}(3N+2-k).$$

Since

$$\alpha(u_J)=u_J(\tau)=\frac{J\tau}{2(N+2)}(3N+2-\tau)>c,$$

and

$$\beta(u_J)=u_J(\lfloor\tfrac{N+2}{2}\rfloor)=\frac{J\lfloor\frac{N+2}{2}\rfloor}{2(N+2)}(3N+2-\lfloor\tfrac{N+2}{2}\rfloor)<\frac{c\lfloor\frac{N+2}{2}\rfloor}{\tau}=d,$$

$u_J \in \{u \in P : c < \alpha(u) \text{ and } \beta(u) < d\}$. Hence we have $\{u \in P : a < \alpha(u) \text{ and } \beta(u) < b\} \neq \emptyset$ and $\{u \in P : c < \alpha(u) \text{ and } \beta(u) < d\} \neq \emptyset$. Therefore conditions (A1) and (A4) hold.

Turning to (A2), let $u \in P$ with $\beta(u) = b$ and $\alpha(u) \geq a$. By the concavity of u, for $l \in \{0, ..., \tau\}$, we have

$$u(l) \geq \left(\frac{u(\tau)}{\tau}\right) l \geq \frac{bl}{\lfloor \frac{N+2}{2} \rfloor}.$$

and for all $l \in \{\tau, ..., \lfloor \frac{N+2}{2} \rfloor\}$, we have

$$\frac{b\tau}{\lfloor \frac{N+2}{2} \rfloor} \leq u(l) \leq b.$$

Hence by (ii) and (iii), it follows that

$$\begin{aligned}
\beta(Tv) =& \sum_{l=1}^{N+1} H\left(\lfloor \tfrac{N+2}{2} \rfloor, l\right) f(u(l)) \\
\leq& 2 \sum_{l=1}^{\lfloor \frac{N+2}{2} \rfloor} \frac{l(\lceil \frac{N+2}{2} \rceil)}{N+2} f(u(l)) \\
=& 2 \sum_{l=1}^{\tau} \frac{l(\lceil \frac{N+2}{2} \rceil)}{N+2} f(u(l)) + 2 \sum_{l=\tau+1}^{\lfloor \frac{N+2}{2} \rfloor} \frac{l(\lceil \frac{N+2}{2} \rceil)}{N+2} f(u(l)) \\
\leq& 2 \sum_{l=1}^{\tau} \frac{l(\lceil \frac{N+2}{2} \rceil)}{N+2} f\left(\frac{bl}{2}\right) + 2 \sum_{l=\tau+1}^{\lfloor \frac{N+2}{2} \rfloor} \frac{l(\lceil \frac{N+2}{2} \rceil)}{N+2} f\left(\frac{b\tau}{\lfloor \frac{N+2}{2} \rfloor}\right) \\
\leq& b - f\left(\frac{b\tau}{\lfloor \frac{N+2}{2} \rfloor}\right) \frac{1}{N+2} (\lceil \tfrac{N+2}{2} \rceil)(\lfloor \tfrac{N+2}{2} \rfloor - \tau)(\tau + 1 + \lfloor \tfrac{N+2}{2} \rfloor) \\
&+ f\left(\frac{b\tau}{\lfloor \frac{N+2}{2} \rfloor}\right) \frac{1}{N+2} (\lceil \tfrac{N+2}{2} \rceil)(\lfloor \tfrac{N+2}{2} \rfloor - \tau)(\tau + 1 + \lfloor \tfrac{N+2}{2} \rfloor) \\
=& b.
\end{aligned}$$

So (A2) is satisfied.

Next, we establish (A3) of theorem 3.1, and so we let $u \in \mathcal{P}$ with $\beta(u) = b$

and $\alpha(Tu) < a$. By the properties of $H(k,l)$,

$$\begin{aligned}
\beta(Tu) &= \sum_{l=1}^{N+1} H(\lfloor \tfrac{N+2}{2} \rfloor, l) f(u(l)) \\
&\leq \frac{\lfloor \frac{N+2}{2} \rfloor}{\tau} \sum_{l=1}^{N+1} H(\tau, l) f(u(l)) \\
&= \frac{\lfloor \frac{N+2}{2} \rfloor}{\tau} \alpha(Tu) \\
&< \frac{a \lfloor \frac{N+2}{2} \rfloor}{\tau} = b,
\end{aligned}$$

so (A3) holds.

In dealing with (A5), let $u \in P$ with $\alpha(u) = c$ and $\beta(u) \leq d$. Then for $l \in \{\tau, ..., N+2\}$, we have

$$c \leq u(l) \leq d = \frac{c \lfloor \frac{N+2}{2} \rfloor}{\tau}.$$

Hence by Property (i),

$$\begin{aligned}
\alpha(Tu) = \sum_{l=1}^{N+1} H(\tau, l) f(u(l)) \geq \sum_{l=\tau+1}^{N+1} H(\tau, l) f(u(l)) = \\
= \sum_{l=\tau+1}^{N+1} \frac{\tau(\lfloor \frac{N+2}{2} \rfloor - \tau)}{N+2} f(u(l)) > \sum_{l=\tau+1}^{N+1} \frac{c}{N+1-\tau} = c,
\end{aligned}$$

and so (A5) is valid.

Now we address (A6). So, let $u \in P$ with $\alpha(u) = c$ and $\beta(Tu) > d$. Again, by the properties of H,

$$\begin{aligned}
\alpha(Tu) &= \sum_{l=1}^{N+1} H(\tau, l) f(u(l)) \geq \\
&\geq \frac{\tau}{\lfloor \frac{N+2}{2} \rfloor} \sum_{l=1}^{N+1} H(\lfloor \tfrac{N+2}{2} \rfloor, l) f(u(l)) = \\
&= \frac{\tau}{\lfloor \frac{N+2}{2} \rfloor} \beta(Tu) > \frac{\tau d}{\lfloor \frac{N+2}{2} \rfloor} = c
\end{aligned}$$

and so (A6) of Theorem 2.3 also holds.

Last, we show (H1) holds. Let $K \in \left(\frac{2b}{\lfloor \frac{N+2}{2} \rfloor}, \frac{2c}{3\lfloor \frac{N+2}{2} \rfloor} \right)$. Then define

$$u_K(k) = K \sum_{l=1}^{N+1} H(k,l) = \frac{Kk}{2(N+2)}(3N+2-k).$$

So

$$\begin{aligned} \beta(u_K) &= \frac{K\lfloor \frac{N+2}{2} \rfloor}{2(N+2)}(3N+2-\lfloor \tfrac{N+2}{2} \rfloor) \\ &> \frac{b}{(N+2)}(3N+2-\lfloor \tfrac{N+2}{2} \rfloor) \geq b, \end{aligned}$$

and

$$\begin{aligned} \alpha(u_K) &= \frac{K\tau}{2}(3N+2-\tau) \\ &< \frac{c\tau}{3(N+2)\lfloor \frac{N+2}{2} \rfloor}(3N+2-\tau) \leq c. \end{aligned}$$

Thus $\{u \in \mathcal{P} \ : \ b < \beta(u) \text{ and } \alpha(u) < c\} \neq \emptyset$.

If $u \in \mathcal{P}(\beta, b)$, then

$$\alpha(u) \leq \beta(u) \leq b < c,$$

and hence $\mathcal{P}(\beta, b) \subset \mathcal{P}(\alpha, c)$.

Lastly, if $u \in \mathcal{P}(\alpha, c)$, then

$$\frac{\tau}{\lfloor \frac{N+2}{2} \rfloor}\beta(u) \leq \alpha(u) \leq c,$$

and so

$$||u|| = \beta(u) \leq \frac{c\lfloor \frac{N+2}{2} \rfloor}{\tau}.$$

Therefore $\mathcal{P}(\alpha, c)$ is bounded. So (H1) holds. Thus T has a fixed point $u^{**} \in \mathcal{P}(\beta, \alpha, b, c)$. $\square$

4.4 Example

Notice the previous example fails for these new conditions. For $N = 18$ and $\tau = 1$, by (i),

$$f(w) > \frac{10c}{81} \text{ for } w \in [c, 9c].$$

If $9c \leq 40$, $f(9c) > \frac{10c}{81}$, implying $c < 0.6364$. Thus, since $3b < c$, $b < 0.214$. Then (iii) fails, since $b - f\left(\frac{b\tau}{\lfloor \frac{N+2}{2} \rfloor}\right) \frac{1}{N+2}(\lceil \frac{N+2}{2} \rceil)(\lfloor \frac{N+2}{2} \rfloor - \tau)(\tau + 1 + \lfloor \frac{N+2}{2} \rfloor) < 0$.

If $9c > 40$, then for (i) to hold, $\frac{1}{100} > \frac{10c}{81} > \frac{400}{81}$. Therefore (i) does not hold. Thus a new example is needed.

Example 4.1. Example: Let $N = 10$, $\tau = 2$, $b = 2$, and $c = 7$. Notice that $3b < c$. Define a continuous $f : [0, \infty) \to [0, \infty)$ by

$$f(w) = \begin{cases} \frac{1-w}{6} & 0 \leq w < 1 \\ \\ 0 & 1 \leq w < 2 \\ \\ w - 2 & 2 \leq w. \end{cases}$$

Then,

(i) $f(w) > \frac{7 \cdot 12}{2 \cdot 9 \cdot 4} = 7/6$ for $w \in [7, 21]$,

(ii) $f(w)$ is decreasing on $[\frac{1}{3}, \frac{2}{3}]$, and $f(\frac{2}{3}) \geq f(w)$ for $w \in [\frac{2}{3}, 1]$, and

(iii) $\sum_{l=1}^{2} l f(l) = 0 \leq 1 = 2 - f(\frac{2}{3}) \cdot \frac{1}{2} \cdot 4 \cdot 9$

Therefore by Theorem 3.1, the right focal boundary value problem,

$$\Delta^2 u(k) + f(u(k)), \quad k \in \{0, 1, ..., 10\},$$

with boundary conditions

$$u(0) = u(12) = 0,$$

has a positive symmetric solution u^{**} with $u^{**}(6) \geq 2$ and $u^{**}(2) \leq 7$.

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