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On the uniqueness of flow in a recent tsunami model

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We give an elementary proof of uniqueness for the integral curve starting from the vertical axis in the phase-plane analysis of the recent model [A. Constantin and R.S. Johnson, *Propagation of very long water waves, with vorticity, over variable depth, with applications to tsunamis*, Fluid Dyn. Res. 40 (2008), pp. 175–211]. Our technique can be easily applied in circumstances where the reparametrization device from Constantin [A. Constantin, *A dynamical systems approach towards isolated vorticity regions for tsunami background states*, Arch. Ration. Mech. Anal. doi: 10.1007/s00205-010-0347-1] might lead to some serious difficulties.

Keywords: ordinary differential equation; uniqueness of solution; non-Lipschitzian functional term

AMS Subject Classifications: 34A12; 34A34; 37E35

1. Introduction

A recent model [1,2] for studying the currents underlying the water surface before the arrival of large, destructive waves (tsunamis) to the shoreline has been investigated in [3] via a technical phase-plane analysis relying on several sharp estimates and topological techniques.

A key feature in the long series of arguments is regarding the *uniqueness* of solutions to the following initial value problem:

$$\begin{cases} \psi'' + \dfrac{\psi'}{r} + f(\psi) = 0, & r \geq r_0 \geq 1, \\ \psi'(r_0) = \psi_1, \\ \psi(r_0) = 0, \end{cases} \tag{1}$$

where $\psi_1 \neq 0$ and the (vorticity) function $f : \mathbb{R} \to \mathbb{R}$ reads as

$$f(\psi) = \begin{cases} \psi - \dfrac{\psi}{\sqrt{|\psi|}}, & \text{for } \psi \neq 0, \\ 0, & \text{otherwise,} \end{cases} \tag{2}$$

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see [3, Eq. (2.4)]. The author has established the continuity of the solution with respect to the initial data, via a reparametrization of the integral curves of the differential equation from (1) recast in polar coordinates. We recall that the uniqueness of solution to an initial value problem is essential for establishing the continuity with respect to the initial data when the functional terms of an ordinary differential equation are non-Lipschitzian, cf [4, p. 24, Theorem 3.4]. Constantin's technique, however, might lead to some serious difficulties when one is employing a more complicated (though academic) vorticity function, e.g.

$$f(\psi) = \begin{cases} \psi - \dfrac{\psi}{\sqrt{|\psi|}} \left[1 + c_1 - \sin\left(\dfrac{c_2 \psi^2}{\psi^2 + 1} \right) \right], & \psi \neq 0, \\ 0, & \psi = 0, \end{cases} \tag{3}$$

where $0 < c_1 = \sin \frac{c_2}{2} < c_2 < \frac{3 - 2\sqrt{2}}{4 + 3\sqrt{2}} < 0.02$, see [5].

Notice now that both functions from (2), (3) satisfy the restrictions

$$\begin{cases} f(0) = 0, \quad \psi \cdot f(\psi) < 0, \\ |f(\psi_1) - f(\psi_2)| \leq \dfrac{C}{\sqrt{\min\{|\psi_1|, |\psi_2|\}}} \cdot |\psi_1 - \psi_2|, \end{cases} \tag{4}$$

for any numbers ψ, ψ_1, ψ_2 lying in $[-\delta, 0) \cup (0, \delta]$, with $\psi_1 \cdot \psi_2 > 0$, and some $C, \delta > 0$.

In this note, we apply a technique inspired by the approaches from [6,7] for dealing with similar issues to prove that the problem (1) has a unique solution to the right of r_0. Given the preceding formulae of the vorticity functions f, it is enough to establish the uniqueness only for the case where $\psi_1 > 0$ (otherwise, we replace ψ with $-\psi$ and remain in the class of nonlinearities f described by (4)).

2. Uniqueness of flow

THEOREM 2.1 *Assume that the continuous function $f : \mathbb{R} \to \mathbb{R}$ satisfies the hypotheses (4). Then, given $\psi_1 > 0$, the initial value problem (1) has a unique solution to the right of r_0.*

Proof Assume that ψ, Ψ are two solutions of (1) defined in an interval $[r_0, r_1]$ small enough to ensure that $\psi(r)$, $\Psi(r) \in (0, \delta]$ for every $r \in [r_0, r_1]$.

Introduce the function $x \in C^1([r_0, r_1], \mathbb{R})$ with the formula $x(r) = \Psi(r) - \psi(r)$. Notice as well that $x(r_0) = x'(r_0) = 0$ and we have the essential estimate of the behaviour of x close to r_0:

$$\lim_{r \searrow r_0} \frac{x(r)}{\ln(r/r_0)} = \lim_{r \searrow r_0} [r x'(r)] = r_0 \cdot x'(r_0) = 0. \tag{5}$$

The function $y : [r_0, r_1] \to [0, +\infty)$ with the formula

$$y(r) = \begin{cases} \dfrac{|x(r)|}{\ln(r/r_0)}, & \text{for } r \in (r_0, r_1], \\ 0, & \text{otherwise} \end{cases}$$

is continuous by means of (5), so, there exists $r^\star = r^\star(r_1) \in (r_0, r_1]$ such that $y(r^\star) = \sup_{s \in (r_0, r_1]} y(s) > 0$.

Equation from (1) can be recast as

$$[r\psi'(r)]' = -rf(\psi),$$

leading to

$$\psi(r) = r_0\psi_1 \ln\frac{r}{r_0} - \int_{r_0}^r \tau\ln\frac{r}{\tau}\cdot f(\psi(\tau))\mathrm{d}\tau, \quad r\in[r_0, r_1]. \tag{6}$$

The latter integral equation yields

$$|x(r)| \le \int_{r_0}^r \tau\ln\frac{r}{\tau}\cdot\frac{C|x(\tau)|}{\sqrt{\min\{\psi(\tau), \Psi(\tau)\}}}\,\mathrm{d}\tau, \quad r\in[r_0, r_1].$$

Given the sign of f in $(0, \delta]$, we get from (6) that $\min\{\psi(r), \Psi(r)\} \ge r_0\psi_1\ln\frac{r}{r_0}$. As a consequence, we obtain that

$$|x(r)| \le \frac{C}{\sqrt{r_0\psi_1}}\cdot\int_{r_0}^r \tau\ln\frac{r}{\tau}\frac{|x(\tau)|}{\sqrt{\ln(\tau/r_0)}}\,\mathrm{d}\tau$$

$$\le \frac{C}{\sqrt{r_0\psi_1}}\cdot\int_{r_0}^r \tau\frac{|x(\tau)|}{\ln(\tau/r_0)}\,\mathrm{d}\tau\cdot\ln\frac{r}{r_0}, \quad r\in[r_0, r_2], \tag{7}$$

where $r_2\in(r_0, r_1)$ is close enough to r_0 to have $\ln(r/r_0)\in[0, 1)$ and, respectively, $\frac{C}{\sqrt{r_0\psi_1}}\cdot\frac{r^2-r_0^2}{2} \le \frac{1}{2}$ for all $r\in[r_0, r_2]$.

Further, the inequality (7) reads as

$$y(r) \le \frac{C}{\sqrt{r_0\psi_1}}\int_{r_0}^r \tau y(\tau)\mathrm{d}\tau < \frac{C}{\sqrt{r_0\psi_1}}\int_{r_0}^r \tau\,\mathrm{d}\tau\cdot y(r^\star), \quad r^\star = r^\star(r_2),$$

for every $r\in(r_0, r_2)$. Finally, we get that

$$y(r) \le \frac{1}{2}\cdot y(r^\star), \quad r\in(r_0, r_2],$$

which leads to an obvious contradiction by taking $r = r^\star$.

The proof is complete. ∎

3. Comments

The reduction of the comparison of two (hypothetical) solutions to an initial value problem to an analysis of the behaviour of the solutions to some integral inequality has been employed in various uniqueness results that have stemmed from the classical first order, and, respectively, n-th order Nagumo criteria, see [8,9]. Among the far-reaching generalizations of such results we would like to cite the theorems of Athanassov [10] devoted to first-order equations, and of Constantin [6] and Wintner [11] for the n-th order differential equations. A recent reevaluation of the technique has been presented in [12].

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