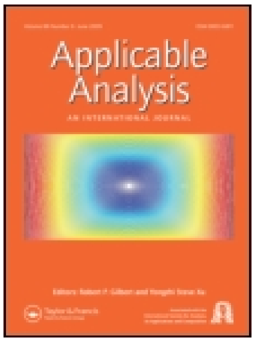

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Exact geophysical waves in stratified fluids

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When studying water waves travelling over an inviscid fluid at the Earth's surface there are additional Coriolis and centrifugal forces which influence the motion of the fluid particles. In particular, for waves propagating near the Equator the geophysical wave problem can be modelled by the so-called *f*-plane approximation. In this paper, we provide an explicit exact solution to the edge wave problem for stratified geophysical flows in the equatorial *f*-plane approximation.

Keywords: edge waves; stratification; coriolis effects; Lagrangian coordinates

AMS Subject Classifications: Primary 76B15; Secondary 74G05; 76B70; 37N10

1. Introduction

Water stratification occurs when water masses with different properties (e.g. salinity, oxygenation, density and temperature) form layers that act as barriers to water mixing. These layers are arranged according to density, with the least dense water masses sitting above the more dense layers. Because of the many effects that may occur even when the density variation is moderate, the stratified flows are of great interest in the field of geophysical fluid dynamics. A rigorous study of stratified water waves has been initiated in [1], where existence of gravity stratified water waves with density increasing with depth is established, and continued in [2,3] for stratified flows driven by surface tension. Exact solutions of the water wave problem, with the stagnation points, have been constructed in [4] for flows which are linearly stratified.

In water of constant density, there exists an explicit solution for gravity waves in deep water which was found first by Gerstner [5] and later on by Rankine [6]. Its features have been analyzed in [7,8]. Gerstner's solution describes the evolution of each individual fluid particle in the flow: particles move on a circles, the radius of the circles decreasing with depth. Moreover, the corresponding flow is rotational and the vorticity decays very fast with depth.

Constantin [9,10], Mollo-Christensen [11] and Yih [12] showed that Gerstner's solution can be modified to describe exact solutions of the edge wave problem, that is

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three dimensional waves which travel along the beach shore and vanish in the direction perpendicular to the shoreline. The edge-waves play an important role in sediment transport in the nearshore, and, while moving, they trace nice sinusoidal run-up patterns in the longshore. The first reference to edge waves dates back to the nineteen century, when Stokes [13], on the basis of linear water theory, gave a simple solution of a system describing a wave which was bounded in amplitude at the shoreline, and which decayed away from the shore. Greenspan [14] found an edge wave solution in the case of stratified fluid with an exponentially varying density over a sloping beach. Other methods for the generation of edge wave are given by Evans [15], who constructed edge waves over a sloping beach on which a mixed boundary condition is satisfied. A one-dimensional model describing edge waves in the presence of strong longshore currents has been analysed in [16], where the authors find criteria for the existence of edge waves over variable seabed profiles. By means of asymptotic analysis, Johnson [17] derived a two-dimensional model equation for the edge wave problem. This mixed type elliptic-hyperbolic equation has been studied in [18], in the context of periodic edge waves, and later on in [19] for solitary waves. An overview of the methods and the results that apply to the edge wave problem can be found in [20].

Beyond the context of constant density, it is shown in [12,21] that Gerstner solution and the related edge wave propagating along a sloping beach, can be adapted to provide explicit free surface flows in incompressible fluids with arbitrary density stratification.

When studying water waves travelling over an inviscid fluid at the Earth's surface, due to the rotation of the Earth around its axis, there are additional Coriolis and centrifugal forces which appear and influence the motion of the fluid particles, cf [22,23]. Particularly, for waves propagating near the Equator, the geophysical wave problem can be modelled by the so-called f-plane approximation, the Coriolis parameter being constant and centrifugal forces neglected. The physical relevance of the f-plane approximation for geophysical edge waves near the Equator has been recently discussed in [24]. We mention that the f-plane approximation for the deep-water wave problem possesses an explicit solution: Gerstner's solution to the deep water wave problem [7,8] can be generalized, cf [25], to describe deep water waves in the geophysical context. We emphasize that any solution of the f-plane approximation which has the property that the pressure is constant along the streamlines has to be a vertical translation of the solution described in [25], cf [26]. Moreover, the solution found in [25] may be adapted to describe geophysical waves travelling over uniform currents, cf [27], where the motion of the fluid particles in dependence of the current's strength and the direction of propagation of the wave is also analysed. It was recently shown in [28] that there exists also an explicit solution for the geophysical edge wave problem in the f-plane approximation if the shoreline is parallel to the Equator.

Concerning geophysical stratified flows, Constantin [29] found an exact solution for geophysical equatorial water waves in the β-plane approximation. This solution describes equatorial trapped waves propagating eastward in a stratified inviscid fluid. The aim of this paper is to show that the solutions found in [25] and [28] for the geophysical deep water and edge wave problem, respectively, may be used to describe also waves propagating over a stratified fluid. In the context of deep water waves, cf Section 3, and in Section 4, for the edge wave problem, we prove that the solution found for homogeneous flows describes waves propagating over a stratified fluid if and only if the density is constant on the isobaric surfaces, that is on the surfaces of constant pressure.

2. The governing equations for geophysical water waves

When considering a rotating frame with the origin at a point O on Earth's surface, the governing equation in the f-plane approximation for a fluid layer localized near the Equator is Euler's equation

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} + 2(\boldsymbol{\Omega} \times \mathbf{u}) = -\frac{\nabla P}{\rho} + \mathbf{g}. \tag{2.1}$$

Here, t represents time, $\mathbf{u} = (u, v, w)$ is the fluid's velocity and $\boldsymbol{\Omega}$ the rotation vector of Earth round the polar axis towards east.[1] We denoted with ρ the density of the water, $\mathbf{g}$ the gravity vector, P the pressure and $\mathrm{d}/\mathrm{d}t$ is the material time derivative

$$\frac{\mathrm{d}h}{\mathrm{d}t} = h_t + h_x u + h_y v + h_z w,$$

which express the rate of change of the quantity h associated with the same fluid particle as it moves about.

Here, we analyse the stratified fluids, therewith we have additionally an equation expressing the fact that the density ρ may vary:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \nabla \cdot (\rho\mathbf{u}) = 0.$$

Assuming that the flow is volume preserving

$$u_x + v_y + w_z = 0, \tag{2.2}$$

and ρ is always non-vanishing and positive, we get that

$$\rho_t + u\rho_x + v\rho_y + w\rho_z = 0. \tag{2.3}$$

Equations (2.1)–(2.3) are the equation of motion within the fluid layer. They are supplemented by suitable boundary conditions at the wave surface, cf [30]:

(i) at the free surface, which decouples the motion of the water from that of the air, we set $P = P_0$, where P_0 is the (constant) atmospheric pressure;
(ii) the kinematic boundary condition expresses the fact that the free surface consists at all times of the same fluid particles.

3. Geophysical two-dimensional deep water waves in a stratified fluid

In this section, we choose the rotating reference frame, with origin in a point O close to the Equator, to have the x-axis chosen horizontally due east, the y-axis horizontally due north and the z-axis upward. Herein, we consider only two-dimensional flows, independent upon the y-coordinate and with the velocity component $v \equiv 0$ throughout the flow. The rotation and the gravity vector have within this frame the following representation

$$\boldsymbol{\Omega} = (2\omega, 0, -2\omega) \quad \text{and} \quad \mathbf{g} = (0, 0, -g),$$

with $\omega = 73 \cdot 10^{-6}\,\mathrm{rad\,s^{-1}}$ denoting the rotational speed of the Earth and $g = 9,8\,\mathrm{m\,s^{-2}}$ the gravitational constant. Letting $z = \eta(t, x)$ be the surface of the ocean, the kinematic boundary condition in this case is given by

$$w = \eta_t + u\eta_x \quad \text{on } z = \eta(t, x). \tag{3.1}$$

Equations (2.1)–(2.3) and the boundary conditions (i) and (ii) are supplemented by the far field boundary condition

$$(u, w) \to (0, 0) \text{ as } z \to -\infty \text{ uniformly for } x \in \mathbb{R},\ t \geq 0, \tag{3.2}$$

expressing the fact that at great depths there is practically no motion. Summarizing, the governing equations for two-dimensional geophysical stratified deep-water waves in the *f*-plane approximation are encompassed by the nonlinear evolution problem

$$\begin{cases} u_t + uu_x + wu_z + 2\omega w = -P_x/\rho & \text{for } z < \eta(t,x), \\ w_t + uw_x + ww_z - 2\omega u = -P_z/\rho - g & \text{for } z < \eta(t,x), \\ u_x + w_z = 0 & \text{for } z < \eta(t,x), \\ P = P_0 & \text{on } z = \eta(t,x), \\ w = \eta_t + u\eta_x & \text{on } z = \eta(t,x), \\ \rho_t + u\rho_x + w\rho_z = 0 & \text{on } z = \eta(t,x), \\ (u, w) \to (0, 0) & \text{as } z \to -\infty. \end{cases} \tag{3.3}$$

In the case when the density is constant, it is proved in [25] that the problem (3.3) has an explicit Gerstner-like solution. Letting $\Sigma_0 := \mathbb{R} \times (-\infty, b_0)$ for some $b_0 \leq 0$, the mapping

$$\begin{cases} x(t,a,b) = a - \dfrac{e^{kb}}{k}\sin(ka - kct), \\ z(t,a,b) = b + \dfrac{e^{kb}}{k}\cos(ka - kct), \end{cases} \quad (a,b) \in \Sigma_0, \tag{3.4}$$

where $k > 0$ is fixed and $c \neq 0$ defines for each $t \geq 0$ a diffeomorphism from Σ_0 into an infinite strip $\Omega(t)$, which is bounded from above by a periodic graph and unbounded from below. Each point $(a, b) \in \Sigma_0$ identifies a fluid particle in the fluid layer $\Omega(t)$ and (3.4) are the equations describing the path of this particle. The wave surface is the curve obtained by setting $b = b_0$ in (3.4). It is shown in [25] that the Equations (3.4) define a solution of the deep water wave problem, if and only if the wave speed c takes one of the values

$$c := \frac{-\omega \pm \sqrt{\omega^2 + kg}}{k}. \tag{3.5}$$

When c is positive, then the wave moves eastwards, and when c is negative then it moves from east to west with velocity $|c|$. The pressure P^{ho}, in the case when the fluid is homogeneous with density $\rho = \rho^{ho} \in \mathbb{R}$, is given by the following expression:

$$P^{ho} = P_0 + \frac{g\rho^{ho}}{2k}\left(e^{2kb} - e^{2kb_0}\right) - g\rho^{ho}(b - b_0). \tag{3.6}$$

When considering that the fluid is heterogeneous, that is $\rho = \rho(t, a, b)$, the system (3.4) defines an exact solution of (3.3) if and only if there exists a function P^{he}, the pressure in the heterogeneous layer, solving the system of equations

$$\begin{cases} P^{he}_a = 0, \\ P^{he}_b = \rho(t, a, b) P^{ho}_b / \rho^{ho}, \end{cases} \tag{3.7}$$

with ρ satisfying (2.3). Requiring that $\partial_a P^{he}_b = \partial_b P^{he}_a$ and since $P^{ho}_b \neq 0$, we conclude that ρ depends only on b, $\rho = \overline{\rho}(b)$, with $\overline{\rho} \in C^1((-\infty, b_0], (0, \infty))$. Therewith, the pressure is given by

$$P^{he} = P_0 + g \int_{b_0}^{b} \overline{\rho}(s)(e^{2ks} - 1)\mathrm{d}s.$$

4. Geophysical edge waves in a stratified fluid

In this section, we restrict our consideration to edge waves travelling over a sloping beach, which forms an angle α with the still fluid surface. The axes of the reference frame are chosen such that the xy-plane coincides with the sloping bed and the z-axis is normal to it, pointing towards the fluid surface. Moreover, the x-axis is parallel to the shoreline and it is tangent to the Equator, pointing in the east direction, while the y-axis and the rotation vector $\mathbf{\Omega}$ form an angle equal to α. In the coordinate system $Oxyz$, the rotation and the gravity vector have the following representation:

$$\mathbf{\Omega} = (0, \omega\cos(\alpha), -\omega\sin(\alpha)) \quad \text{and} \quad \mathbf{g} = (0, -g\sin(\alpha), -g\cos(\alpha)).$$

Therewith, the equation of motion within the fluid layer may be recast in the $Oxyz$ coordinates as the following system

$$\begin{cases} u_t + uu_x + vu_y + wu_z + 2\omega(w\cos(\alpha) + v\sin(\alpha)) = -P_x/\rho, \\ v_t + uv_x + vv_y + wv_z - 2\omega u\sin(\alpha) = -P_y/\rho - g\sin(\alpha), \\ w_t + uw_x + vw_y + ww_z - 2\omega u\cos(\alpha) = -P_z/\rho - g\cos(\alpha), \\ u_x + v_y + w_z = 0, \\ \rho_t + u\rho_x + v\rho_y + w\rho_z = 0. \end{cases} \tag{4.1}$$

The edge wave problem is complete if we impose, additionally to (4.1) and (i)–(ii) one more boundary condition:

(iii) the fluid bed is impermeable and the normal component of the velocity vector is zero at the sloping bed.

Setting $\Sigma := \{(a, b, c) : a \in \mathbb{R},\ b \leq b_0,\ 0 \leq c \leq (b_0 - b)\tan(\alpha)\}$, whereby $b_0 \leq 0$, it is shown in [28] that the map

$$\begin{cases} x(t, a, b, c) := a - \dfrac{1}{k} e^{k(b-c)} \sin(k(a + st)), \\ y(t, a, b, c) := b - c + \dfrac{1}{k} e^{k(b-c)} \cos(k(a + st)) + \dfrac{2\omega s\cot(\alpha)}{g} z(c), \\ z(c) := \dfrac{g(1 + \tan(\alpha))}{g + 2\omega s} c - \dfrac{g}{g + 2\omega s} \dfrac{\tan(\alpha)}{2k} e^{2kb_0}\left(1 - e^{-2kc(1+\cot(\alpha))}\right), \end{cases} \tag{4.2}$$

defines a diffeomorphism on Σ for every $t \geq 0$, the fluid layer $t \geq 0$ being the diffeomorphic image of Σ. The three-dimensional wave surface is the image of the boundary $c = (b_0 - b)$ of Σ. Again, each point (a, b, c) defines a unique particle in the fluid layer and (4.2) define the path of every fixed particle. Moreover, s is the speed at which the wave travels parallel to the shoreline and takes one of the values

$$s_{1/2} := \frac{\omega \sin(\alpha) \pm \sqrt{\omega^2 \sin^2(\alpha) + gk \sin(\alpha)}}{k}. \tag{4.3}$$

If $s = s_1$, then the wave travels eastwards, and if $s = s_2$, it travels from east to west with wave speed $|s_2|$. Moreover, it was shown that the pressure P^{ho} within a homogeneous fluid with density ρ^{ho} is given by

$$\begin{aligned} P^{ho} = P_0 &+ \frac{g\rho^{ho} \sin(\alpha)}{2k} e^{2k(b-c)} - g\rho^{ho}(c \cos(\alpha) + (b - b_0) \sin(\alpha)) \\ &- \frac{g\rho^{ho} \sin(\alpha)}{2k} e^{2k(b_0 - c(1+\cot(\alpha)))}. \end{aligned} \tag{4.4}$$

The system (4.2) describes an edge wave travelling over stratified water with density $\rho = \rho(t, a, b, c)$ if and only if there exists a function P^{he}, the pressure in the heterogeneous fluid, solving the equations

$$\begin{cases} P^{he}_a = 0, \\ P^{he}_b = \rho(t, a, b, c) P^{ho}_b / \rho^{ho}, \\ P^{he}_c = \rho(t, a, b, c) P^{ho}_c / \rho^{ho}, \end{cases} \tag{4.5}$$

with ρ satisfying (2.3). Since, (4.5) is integrable exactly when $\operatorname{curl}(P^{he}_a, P^{he}_b, P^{he}_c) = 0$ and since $P^{ho}_b \neq 0$, we conclude that the density $\rho = \overline{\rho}(P^{ho}(b, c))$, with $\overline{\rho} \in C^1([P_0, \infty), (0, \infty))$. Moreover, for this solution the surfaces of constant pressure (the isobaric surfaces) coincide with those of constant density (the isopycnic surfaces), as the pressure is given by the following relation:

$$P^{he} = P_0 + \frac{1}{\rho^{ho}} \int_{P_0}^{P^{ho}(b,c)} \overline{\rho}(s) \mathrm{d}s.$$

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Note

1. Taken to be a perfect sphere of radius 6371 km.

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