Control of open quantum systems

by

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"Out of Sight", Illustration: Elisabeth Warscheid

I can't control the wind, but I can adjust the sail

Ricky Skaggs

Declaration

This work has not previously been accepted in substance for any degree and is not being concurrently submitted in candidature for any other degree. The thesis is the result of my own investigations, except where otherwise stated. Where correction service has been used, the extent and nature of the corrections is clearly marked in footnotes. Other sources are acknowledged by footnotes giving explicit references. A bibliography is appended.
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Abstract

Known as decoherence, the unavoidable interaction of a quantum system with its surrounding environment is usually considered to be detrimental for quantum information processing. In this thesis the coherent, open loop control of such open systems is studied. Concepts from quantum control theory and the theory of open quantum system are adopted in order to fight decoherence and implement quantum gates in a noiseless manner. In particular, Lie algebraic methods and numerical optimization tools are used to investigate the control properties of a single spin interacting with a spin environment. We show that, independent of the size of the environment, every unitary transformation can be implemented on the system spin through a single control field. We proceed by investigating dynamical decoupling, a method to suppress the interactions with the environment, for finite- and for infinite dimensional systems. We prove that every finite dimensional system can be protected from decoherence, even if the environment is infinite dimensional, whereas for noise described by a Lindblad master equation dynamical decoupling will never succeed. This will lead to a new method to distinguish decoherence from intrinsic noise terms. We further prove that not every infinite dimensional system can be protected from decoherence through dynamical decoupling. Afterwards we investigate dynamical decoupling of systems that are described by quadratic Hamiltonians, showing that such interactions can always be suppressed with two simple operations. In the last part we investigate the coherent control of a Lindblad master equation. We show that a strong noise process exhibiting a decoherence free subspace can substantially increase the number of unitary operations that can be implemented, allowing us to fully control parts of the system. Afterwards we develop a scheme to make Hamiltonians and Lindbladians commutative by adding an auxiliary system. The old, possibly non-commutative dynamics, is recovered through a non-selective measurement.

Some of the results were obtained in collaboration with my supervisor Dr. Daniel Burgarth and other researchers. The following parts of the thesis were published in scientific journals, are still in the peer-review process or are in preparation for submission:

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Contents

1 Introduction

The last decades have witnessed spectacular technological progress, to the extent that now the implementation of high-fidelity quantum technologies can be thought of as a goal belonging to the not-so-distant future. However, the loss of quantum coherence due to the unavoidable interaction of a quantum system with its surrounding environment [1], i.e. decoherence, represents one of the major obstacles. Quantum features one wants to use for quantum information tasks are quickly washed out by the unavoidable coupling with the environment. Analogous to the wind driving a sailing boat in an undesirable direction, the process of decoherence drives a quantum system towards a classical ensemble of states. On the one hand the quest for a fundamental understanding of the sources and mechanisms of decoherence attracts substantial research effort, while on the other the development of strategies to minimize its detrimental effect, in view of practical applications, is also a major research focus. Although distinct, these two research lines are deeply intertwined, since the deeper the understanding of such open systems, the more effective the strategies to fight decoherence can be.

In this respect quantum control theory offers a valuable potential. The general idea behind quantum control is to use the interaction of a quantum system with properly tailored classical control fields to steer its dynamics towards the desired outcome. It has been successfully used for various purposes, for instance in order to drive chemical reactions, atomic and molecular transitions, to control spins for nuclear magnetic resonance and to implement unitary gates for quantum information tasks [2]. In the presence of an environment, control pulses have been calculated

to decrease the induced noise and to study fidelity limits for the implementation of unitary gates that are subject to decoherence [2]. A natural first step is to characterize the operations that can be implemented in the presence of the environment with the help of control pulses. Although substantial progress has recently been made [2], the characterization of the operations that can be implemented in an open system setting remains a challenging task. Additionally not much is known about the relevant timescales needed to achieve a target operation. Therefore the following questions deserve further investigations regarding open systems:

I Which operations can be achieved by applying control fields to the system?

II What are the relevant time scales, and in particular how long does it take to reach a desired target?

Sometimes one is mainly interested in protecting the system from decoherence without additionally implementing a specific operation. Moreover, we might not be able to identify all relevant interactions leading to decoherence. We are thus aiming here for a strategy to protect the system from decoherence, regardless of how the interactions with the environment appear. Such a strategy is dynamical decoupling [3]. The application of frequent, instantaneous control pulses makes it possible to average the system-environment interactions to zero. In this respect it is a specific instance of quantum control. Its particular strength is that it is applicable even if the details of the system-environment coupling are unknown. However, it is not clear how efficient dynamical decoupling can be in the presence of other noise sources. Furthermore, for continuous variable systems, such as quantum harmonic oscillators, no general framework is known to protect an infinite dimensional quantum system from decoherence. We then may ask:

III Can we always suppress the interactions with the environment through dynamical decoupling in order to protect a quantum system from decoherence, even if the system is described by an infinite dimensional system?

IV Can we learn something about the environment, and in particular other noise sources, by observing how the system reacts if we apply decoupling?

Having posed these questions, one may ask as well whether it is it always necessary to suppress the detrimental effect of the environment? One of the milestones in the last decades, regarding the control of decoherence, is the observation that sometimes noise can be beneficial. Rather than fighting against the environment, dissipative state preparation and dissipative quantum computing turned out to be valuable alternatives to unitary gate designs [4]. Inspired by these ideas, can we gain something from a noisy dynamics that is accompanied by some controls? Changing the paradigm:

V Instead of fighting against decoherence can quantum control make use of the environment as a resource?

In order to address these question we first need to think about how we formulate the control of an open quantum system. In this thesis we focus on open loop control schemes, that is we compute the control fields only based on the target we want to achieve and the underlying model describing the system dynamics. Regarding the control of an open quantum system there are two approaches one can follow. In the first approach one begins with an effective description of the reduced dynamics in terms of a Lindblad-type master equation, usually relying on approximations based on time scale arguments. Then the controls are added coherently, such that afterwards one can investigate the control properties of the Lindblad master equation. In the second approach one starts with a full system-environment description through a Hamiltonian and studies the control properties of the total system using well known concepts from closed quantum systems. Following from this, one can conclude the control properties of the reduced dynamics. Unfortunately, when it comes to infinite dimensional environments, the adoption of methods from closed systems is difficult since not much is known about the control of an infinite dimensional system. On the other hand, regarding the first approach, the characterization of the control properties of the Lindblad master equation is equally challenging. Moreover, as we will

explain later, these two approaches lead to different predictions for the controlled system dynamics.

The thesis can be read as going from one approach for describing the control of an open quantum system to another, whereas the middle Chap. 4, about dynamical decoupling, shows in a dramatic way the differences between the two approaches. In the preliminary chapter we start by introducing concepts from open quantum systems and quantum control theory that are used throughout the thesis and are relevant for understanding the presented work. Afterwards we briefly characterize the difficulties in combining the two fields by reviewing the two afore mentioned approaches for formulating the control of an open quantum system. In this section we give a summary of the state of the field. The following chapters focus on the questions I-V. At the beginning of each chapter we give a detailed literature overview of the aspects that are addressed.

Because it is not clear whether the approximations yielding a Lindblad-type master are valid in the presence of control fields, in order to address questions I and II, we begin in Chap. 3 with the examination of the control properties of a specific system-environment Hamiltonian. To avoid the difficulties arising for infinite dimensional environments, we investigate a model that is finite but scalable, that is we characterize the control properties of a single spin interacting with a spin environment. Using Lie algebraic and group theoretical methods, we determine the unitary operations that can be implemented on the system spin. Additionally we numerically study the minimum time needed in order to implement these operations with high precision dependent on the size of the environment. Instead of implementing specific gates through the controls, in Chap. 4 we proceed by investigating the suppression of decoherence through dynamical decoupling. In order to address questions III and IV we begin by studying dynamical decoupling for finite dimensional quantum systems that are subject to noise, either described by a system-environment Hamiltonian or by a Lindblad-type master equation for the reduced dynamics. These investigations will lead to a method which allows us to distinguish between decoherence and intrinsic non-unitary dynamics. Afterwards we establish a framework for dynamical

decoupling of infinite dimensional quantum systems, and we analyze its performance for a quantum harmonic oscillator interacting with a bosonic environment. It will turn out that we cannot suppress all types of noise sources using dynamical decoupling. In particular, not every infinite dimensional system can be decoupled from the environment, moreover, dynamical decoupling will never succeed for non-unitary dynamics described by a Lindblad-type master equation. Since it is hopeless to fight against such noise sources with unitary operations we try to use them, tackling question V. Switching the approach, in Chap. 5 we proceed by studying the coherent control of the Lindblad master equation. In the presence of a strong noise process we analyze the implementation of unitary gates on a subset of states that are free from decoherence. We will see that such a noise process can turn two commuting Hamiltonians into non-commuting ones, allowing us to implement universal unitary operations. We will build upon this idea in Chap. 6 by presenting a framework that allows us to make Lindblad operators and Hamiltonians commutative, before discussing several applications of this scheme.

2 Preliminaries

We begin by introducing the theory of open quantum systems, methods and concepts from quantum control theory, a characterization, and an overview of the difficulties arising in combining the two fields. These introductory sections have the aim of providing an overview to the non-expert reader of the concepts and the terminology used throughout the thesis. They focus on specific aspects relevant for the understanding of the rest of the manuscript. Clearly they are far from complete and there are many good textbooks that cover the different subjects in detail. Regarding the theory of open quantum system the Sec. 2.1.1, 2.1.4, 2.1.4 are along the lines of [1, 5, 6], whereas Sec. 2.1.2 is orientated on [7, 8]. The last Sec. 2.1.5 about collapse models is a relatively new field and we point to review articles within the section. We proceed with an introduction into quantum control theory, where the main concepts can be found in [9] and for a deeper mathematical understanding of control theory in general we refer to [10, 11]. At the end of this chapter we then finally come to the subject of this thesis. We introduce two approaches for controlling an open quantum system and give an overview of the state of the art of the field. Throughout the preliminary chapter we highlight key results by referring to original research papers within each section.

2.1 Open quantum systems

Any realistic quantum system interacts with its surrounding environment, which usually consists of many degrees of freedom. This has the effect that quantum

features one wants to use, for example for quantum information tasks, are quickly washed out so that the implementation of quantum gates becomes noisy. The understanding and the control of this process is therefore of great importance for the development of new quantum technologies, as well as for a fundamental understanding of the quantum-to-classical transition. Unfortunately, in most of the cases, a complete microscopic description of system and environment is not feasible because of the many degrees of freedom involved in the dynamics. However, we are in fact interested in the dynamics of the quantum system itself and thus in a description of the system dynamics that includes the effect of the environment. Such a description can be given in terms of quantum dynamical maps and under certain approximations it is possible to describe the system dynamics with a differential equation that includes the effect of the environment. In the following three sections we review how such dynamical maps arise, which properties they have, and how a description in terms of a differential equation for the system state can be obtained. By introducing the concept of decoherence afterwards, we quantify a bit the lack of "quantumness" due to the interaction with the environment. This brings us to the question whether the quantum-to-classical transition can be explained with this concept. We briefly discuss the problems of this approach and how the introduction of collapse models try to resolve them.

2.1.1 From closed to open quantum systems

According to quantum mechanics the evolution of a state vector $|\psi(t)\rangle \in \mathcal{H}$, with $\mathcal{H}$ being the Hilbert space of the quantum system, is governed by the Schrödinger equation

$$\frac{d}{dt}|\psi(t)\rangle = -iH(t)|\psi(t)\rangle, \tag{2.1.1}$$

where $H(t)$ is the (possibly time dependent) Hamiltonian of the system. We denote by $S(\mathcal{H})$ the state space of $\mathcal{H}$ and by $B(\mathcal{H})$ the space of all bounded operators acting

on $\mathcal{H}$. Here and throughout the thesis we set $\hbar = 1$. Since the Schrödinger equation is a linear first order differential equation its solution is given by

$$|\psi(t)\rangle = U(t, t_0)|\psi(t_0)\rangle, \tag{2.1.2}$$

where the time evolution operator

$$U(t, t_0) = \hat{T} \exp\left(-i \int_{t_0}^{t} dt'\, H(t')\right), \tag{2.1.3}$$

can be obtained by integrating (2.1.1). The Hamiltonian does not necessarily commute with itself at different times, which is the reason why the time ordering operator $\hat{T}$, that orders products of time dependent operators from right to left, was introduced. Because the Hamiltonian is a self-adjoint operator we have that $U(t, t_0)U^\dagger(t, t_0) = U(t, t_0)^\dagger U(t, t_0) = \mathbb{1}$, i.e U is a unitary operator. In fact it can be shown that the only transformations mapping state vectors into state vectors are unitary transformations. We call systems which are described by a unitary time evolution *closed quantum systems*. They are characterized by the fact that their dynamics can always be reversed by applying a unitary transformation U^{-1}. Substituting the solution (2.1.2) into the Schrödinger equation (2.1.1) we obtain a differential equation for the time evolution operator

$$\dot{U}(t, t_0) = -iH(t)U(t, t_0), \quad U(t_0, t_0) = \mathbb{1}, \tag{2.1.4}$$

to which we refer as the Schrödinger equation of the time evolution operator. Here it can already be mentioned that one aim of quantum control theory is to characterize the possible solutions to this equation. If the system is described by a density operator $\rho(t) \in S(\mathcal{H})$, rather than a state vector, the dynamics is described by the Liouville-von Neumann equation

$$\dot{\rho}(t) = -i[H(t), \rho(t)]. \tag{2.1.5}$$

It can easily be checked that its solution reads

$$\rho(t) = U(t, t_0)\rho(t_0)U^\dagger(t, t_0) = \mathcal{U}_{t,t_0}(\rho(t_0)), \tag{2.1.6}$$

where we call the map

$$\mathcal{U}_{t,t_0}(\cdot) = \hat{T} \exp\left(\int_{t_0}^{t} dt' \mathcal{K}_{t'}(\cdot)\right), \tag{2.1.7}$$

which is generated by $\mathcal{K}_t(\cdot) = -i[H(t), \cdot]$, a unitary map.
In order to introduce the notion of an open quantum system we decompose the total system into the system of interest (S) and the environment (B), sometimes called a bath. The total Hilbert space is then decomposed as

$$\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_B, \tag{2.1.8}$$

with d_S and d_B being the dimension of $\mathcal{H}_S$ and $\mathcal{H}_B$ respectively, where we consider finite dimensional quantum systems unless otherwise stated. In general the environment consists of many degrees of freedom so that the solution of the Schrödinger- or the Liouville-von Neumann equation often becomes unfeasible. On the other hand, we are mainly interested in the evolution of observables of the quantum system S. Consider for example an observable $\tilde{M} = M \otimes \mathbb{1}_B$ that acts only non-trivially on the system. Its mean value as a function of time is given by $\langle \tilde{M} \rangle(t) = \mathrm{tr}\{(M \otimes \mathbb{1}_B)\rho(t)\}$ with $\rho(t)$ being the state at time t of the composite system. Fixing a basis, we obtain $\langle \tilde{M} \rangle(t) = \sum_{i=1}^{d_S} \langle \psi_i^S | M \left(\sum_{j=1}^{d_B} \langle \psi_j^B | \rho(t) | \psi_j^B \rangle \right) | \psi_i^S \rangle$. If we introduce the partial trace $\mathrm{tr}_B\{\cdot\} = \sum_{j=1}^{d_B} \langle \psi_j^B | \cdot | \psi_j^B \rangle$ over B, and assume that the state of the total system is initially uncorrelated $\rho(t_0) = \rho_S(t_0) \otimes \rho_B(t_0)$ with $\rho_S \in S(\mathcal{H}_S)$ and $\rho_B(t_0) \in S(\mathcal{H}_B)$, we find $\langle M \rangle(t) = \mathrm{tr}\{M \rho_S(t)\}$. Thus the dynamical properties of the mean value are determined by a dynamical map

$$\mathcal{E}_{t,t_0} : \rho_S(t_0) \to \rho_S(t), \tag{2.1.9}$$

which is of the form

$$\mathcal{E}_{t,t_0}(\cdot) = \mathrm{tr}_B\{U(t,t_0)(\,\cdot \otimes \rho_B(t_0))U^\dagger(t,t_0)\}. \tag{2.1.10}$$

Note that in general the map (2.1.10) is not a unitary map because $U(t,t_0)$ correlates the system with the environment, unless $U(t,t_0) = U_S(t,t_0) \otimes U_B(t,t_0)$. System and environment exchange information, henceforth we call a system whose evolution is

not unitary an open quantum system. One might wonder if the dynamics of an open quantum system can be reversed by applying the inverse of the map $\mathcal{E}$. It can be shown that the inverse of a map $\mathcal{E}$, mapping quantum states into quantum states, exists if and only if the map is a unitary map. In this sense an open quantum system is irreversible. The crucial point is here that the inverted map has to preserve the properties of a quantum state. In the following we review some important properties of these maps, which are used throughout this thesis.

2.1.2 Completely positive trace preserving maps

It was already mentioned that the only transformations which map state vectors into state vectors are unitary transformations. The question arises what characterizes a map $\mathcal{E}$ that maps density operators into density operators? To begin let us recall that the density operator ρ is hermitian, positive, and has trace one. Additionally we have as an inherent quantum mechanical requirement that the map has to preserve convex combinations of density operators. Note that any convexity and positivity preserving map on the set of density operators must necessarily be linear when extended to the full linear space of matrices. Interestingly, if we relax this condition, it can be shown that the non-signaling condition can be violated [7]. Clearly $\mathcal{E}$ has to preserve the properties of the density operator. Therefore we need $\mathrm{tr}\{\mathcal{E}(\rho)\} = 1$ and $\mathcal{E}(\rho) \geq 0$ for all quantum states, whereas positivity implies that hermiticity is preserved[1]. However, positivity alone is not sufficient. If we introduce an auxiliary system with finite dimensional Hilbert space $\mathcal{H}_B$, every quantum state $\rho \in S(\mathcal{H} \otimes \mathcal{H}_B)$ should be mapped into another quantum state, even if a map $\tilde{\mathcal{E}} = \mathcal{E} \otimes \mathrm{id}$ acts only non-trivially on parts of the system. Thus positivity on the extended Hilbert space is required. Maps that satisfy these requirements are called completely positive and trace preserving (CPTP).

[1] Note that here and throughout the thesis we work in the Schrödinger picture. Including the Heisenberg picture, i.e maps acting on effects rather than states, we generally need that positive operator are mapped into positive operators. Furthermore for maps that do not provide a complete description of the process we have the weaker condition $0 \leq \mathrm{tr}\{\mathcal{E}(\rho)\} \leq 1$. These maps are called CP maps.

To summarize, a CPTP map

- preserves the trace:

$$\mathrm{tr}\{\mathcal{E}(\rho)\} = 1, \quad \forall \rho \in S(\mathcal{H}). \tag{2.1.11}$$

- preserves convex combinations (linearity):

$$\mathcal{E}(\sum_i p_i \rho_i) = \sum_i p_i \mathcal{E}(\rho_i) \;\text{ with }\; 0 \leq p_i \leq 1, \; \sum_i p_i = 1. \tag{2.1.12}$$

- is completely positive:

$$(\mathcal{E} \otimes \mathrm{id})(\rho) \geq 0, \quad \forall \rho \in S(\mathcal{H} \otimes \mathcal{H}_B), \tag{2.1.13}$$

  for every finite-dimensional extension of the Hilbert space.

We conclude that maps that map quantum states into quantum states are those that are completely positive and trace preserving, and in the language of quantum information theory they are called *quantum channels*.
We already saw in the last section that a CPTP map arises from a unitary evolution acting on the Hilbert space of system and environment, followed by the partial trace over the degrees of freedom of the environment. There are several ways to represent a CPTP map, which we are going to discuss briefly in the following.
Let $\mathcal{E} : S(\mathcal{H}) \to S(\mathcal{H})$ be a CPTP map with d_S being the dimension of the Hilbert space, then:

Stinespring dilation: there exists an auxiliary Hilbert space $\mathcal{H}_B$, a pure state $|\phi\rangle\langle\phi| \in S(\mathcal{H}_E)$ and a unitary operator U acting on $\mathcal{H} \otimes \mathcal{H}_E$ such that

$$\mathcal{E}(\rho) = \mathrm{tr}_B\{U(\rho \otimes |\phi\rangle\langle\phi|)U^\dagger\}, \quad \forall \rho \in S(\mathcal{H}), \tag{2.1.14}$$

where the dimension of the auxiliary Hilbert space does not need to exceed d_S^2. In words, every CPTP map can be represented as a unitary evolution on a larger Hilbert space followed by a reduction through the partial trace. The Stinespring dilation is unique up to unitary rotations of the ancilla system.

Kraus form: every $\mathcal{E}$ can be written in the form

$$\mathcal{E}(\rho) = \sum_{j=1}^{r} M_j \rho M_j^\dagger, \tag{2.1.15}$$

where the so called Kraus operators $M_j = \langle \psi_j^B | U | \phi \rangle$ can be optained from the Stinespring dilation by taking $\{|\psi_j^B\rangle\}_{j=1}^{d_B}$ as basis for $\mathcal{H}_B$ with d_B being the dimension of the auxiliary Hilbert space and $r \leq d_S^2$ is the minimal number of Kraus operators. The trace preserving condition (2.1.11) yields $\sum_{j=1}^{r} M_j^\dagger M_j = \mathbb{1}$.

Choi-Jamiolkowski Isomorphism for CPTP maps: there is a one-to-one correspondence between $\mathcal{E}$ and a subset of the state space $S(\mathcal{H} \otimes \mathcal{H})$ defined through the mapping

$$J : \mathcal{E} \mapsto J(\mathcal{E}) = (\mathcal{E} \otimes \mathrm{id})(|\Omega\rangle\langle\Omega|), \tag{2.1.16}$$

with $|\Omega\rangle = \frac{1}{\sqrt{d_S}} \sum_{i=1}^{d_S} |i\rangle \otimes |i\rangle$ being the maximally entangled state. One can easily check that J is a density operator, which we call the Choi state, and in which all properties of the map $\mathcal{E}$ are encoded. Its reduced state is the fully mixed state, and for unitary maps $\mathcal{E}(\cdot) = U(\cdot)U^\dagger$ the Choi state is a pure state.

The Choi-Jamiolkowski isomorphism relates CPTP maps to quantum states, making it possible to represent $\mathcal{E}$ as a matrix. There are also several other ways of obtaining a matrix representation for a quantum channel by fixing an operator basis. This procedure will be used in the next section to obtain the *vector of coherence* representation of the Lindblad master equation. Another straightforward way to represent quantum channels as matrices is to vectorize the density operator according to the rule (row vectorization)

$$vec(\rho) := (\rho_{1,1}, \rho_{1,2}, \ldots, \rho_{1,d_s}, \ldots, \rho_{d_s,1}, \ldots, \rho_{d_s,d_s})^T \equiv |\rho\rangle, \tag{2.1.17}$$

with $\rho_{n,m},\ n,m = 1, \ldots, d_S$ being the matrix elements of ρ in a certain basis. Using the identity $vec(ABC) = A \otimes C^T |B\rangle$, we obtain for the Kraus representation (2.1.15)

of a quantum channel

$$\hat{\mathcal{E}} \equiv vec(\mathcal{E}(\rho)) = \sum_{j=1}^{r} M_j \otimes \bar{M}_j |\rho\rangle, \tag{2.1.18}$$

where $\bar{M}_j$ is the complex conjugate of M_j. The CPTP map is now represented by a $d_S^2 \times d_S^2$ matrix acting on a vector of length d_S^2. Note that this representation is basis dependent, and in contrast to the Choi state, it does not have the properties of a quantum state. The matrix representation $\hat{\mathcal{E}}$ of a quantum channel is related to the Choi state $J(\mathcal{E})$ through $\hat{\mathcal{E}} = d_S J^{\Gamma}(\mathcal{E})$, where J^{Γ} is obtained from reshuffling the matrix elements of the Choi state in a certain basis according to the rule $\langle m, n|J^{\Gamma}(\mathcal{E})|k, l\rangle = \langle m, k|J(\mathcal{E})|n, l\rangle$ [7]. Later on the Choi state, as well as the matrix representation $\hat{\mathcal{E}}$, will be used to numerically calculate control fields in an open quantum system setting. Throughout the thesis we will use both representations, whereas from now on we explicitly say when the representation $\hat{\mathcal{E}}$ is used, without indicating it with a hat.

Now we come back to the dynamical properties of the CPTP map $\mathcal{E}_{t,t_0}$ given by (2.1.10). One of the most important problems arising in open quantum systems is the relation between the continuity of time and the CPTP map $\mathcal{E}_{t,t_0}$. Since time is continuous, one would expect that an evolution between t_0 and t can be partitioned in an evolution between $[t_1, t_0]$ and $[t, t_1]$, such that the total CPTP map can be written as a concatenation $\mathcal{E}_{t,t_0} = \mathcal{E}_{t,t_1}\mathcal{E}_{t_1,t_0}$. However, this in general is not true! Indeed the problem arises with $\mathcal{E}_{t,t_1}(\rho_S(t_1)) = \mathrm{tr}_B\{U(t, t_1)\rho(t_1)U^{\dagger}(t, t_1)\}$ where $\rho(t_1)$ is the state of the total system at time t_1. Generally this state cannot be written as a tensor product of the state of the system S and the environment B, because the evolution $U(t, t_0)$ correlates both systems and it depends on what initial states $\rho_S(t_0),\ \rho_B(t_0)$ we are taking. Hence $\mathcal{E}_{t,t_1}$ is in general not a CPTP map.[2] As a consequence the dynamics of an open quantum system can in general not be described by a differential equation. For further discussion of this problem we refer to [5].

Nevertheless, under certain approximations, the correlations that are built can be

[2]Even the definition of $\mathcal{E}_{t,t_1}$ as a map does not really work since it cannot be applied to arbitrary density operators. The state $\rho_S(t_1)$ must come from the specific total system evolution U.

neglected on an appropriate time scale and we will see in the next section that then a description of the dynamics in terms of a differential equation becomes possible.

2.1.3 Lindblad form

Let us suppose that we can decompose the CPTP map $\mathcal{E}_{t_2,t_0}$ into two CPTP maps, so that their concatenation

$$\mathcal{E}_{t_2,t_0} = \mathcal{E}_{t_2,t_1}\mathcal{E}_{t_1,t_0}, \tag{2.1.19}$$

maps a quantum state at time t_0 into a quantum state at time t_2. This is called the *divisibility criterion*. Since the evolution does not depend on previous time steps, divisibility is often taken to be the definition of a *Markovian* evolution [12]. We concentrate on an important special case, assuming that the CPTP maps depend only on time differences, $t = t_2 - t_1$, $s = t_1 - t_0$, which would be the case for time independent Hamiltonians in (2.1.7). Then $\mathcal{E}_t \equiv \mathcal{E}_{t_2,t_1}$ forms a one parameter semigroup $\{\mathcal{E}_t : t \geq 0\}$, i.e,

$$\mathcal{E}_{t+s} = \mathcal{E}_t\mathcal{E}_s, \quad \mathcal{E}_0 = \mathrm{id}, \quad \forall t, s \in \mathbb{R}_+, \tag{2.1.20}$$

and we assume that $\mathcal{E}_t$ depends continuously on time [3] implying differentiability in a finite dimensional setting. Now we consider

$$\begin{aligned} \rho(t + \Delta t) - \rho(t) &= (\mathcal{E}_{t+\Delta t} - \mathcal{E}_t)\rho(0), \\ &= (\mathcal{E}_{\Delta t} - \mathrm{id})\rho(t), \end{aligned} \tag{2.1.21}$$

and we note that the divisibility criterion was used. Dividing by Δt and taking the limit $\Delta t \to 0$ we obtain the differential equation

$$\dot{\rho}(t) = \mathcal{L}\rho(t), \tag{2.1.22}$$

called the *master equation*, where it can be shown that the generator

$$\mathcal{L} = \lim_{\Delta t \to 0} \frac{\mathcal{E}_{\Delta t} - \mathrm{id}}{\Delta t}, \tag{2.1.23}$$

[3] in a sense that when $t \to t_0$, $\|\mathcal{E}_t(\rho) - \mathcal{E}_{t_0}(\rho)\| \to 0$, $\forall \rho \in S(\mathcal{H})$

always exists [13]. The solution of (2.1.22) is given by a family of CPTP maps

$$\mathcal{E}_t = e^{\mathcal{L}t}, \tag{2.1.24}$$

which we denote from now on by Λ_t. Clearly, for a unitary evolution, Λ_t is just given by the solution of the Liouville-von Neumann equation (2.1.5) with a time independent Hamiltonian, but in general the generator $\mathcal{L}$ does not have this commutator form. It can be proven that a map $\mathcal{L}$ is the generator of the solution (2.1.24), i.e. of continuous dynamical semigroups of CPTP maps, if and only if it can be written in the form

$$\mathcal{L}(\rho) = -i[H, \rho] + \sum_j \gamma_j (2L_j \rho L_j^\dagger - (L_j^\dagger L_j \rho + \rho L_j^\dagger L_j)), \tag{2.1.25}$$

with $\gamma_j \geq 0$ and $H = H^\dagger$. This form is called the *Lindblad form*, where $\mathcal{L}$ is called the *Lindbladian* and we refer to

$$\mathcal{D}(\rho) = \sum_j \gamma_j (2L_j \rho L_j^\dagger - (L_j^\dagger L_j \rho + \rho L_j^\dagger L_j)), \tag{2.1.26}$$

as the *dissipative part* of the dynamics. The result (2.1.25) was proven by Gorini, Kossakowski and Sudarshan for finite dimensional systems [14], by Lindblad for infinite dimensional system with bounded operators [15], and independently Christensen and Evans found a more compact form [16], which will be used and discussed later. Note that for the coherently controlled master equation, which will be introduced later, the Hamiltonian and thus the Lindbladian becomes time dependent $\mathcal{L}(t)(\cdot) \equiv \mathcal{L}_t(\cdot)$, so that the solution to the master equation reads $\Lambda_t = \hat{T} \exp(\int_0^t dt' \, \mathcal{L}_{t'})$.

Before we introduce two matrix representations of the Lindbladian in the next section, we first want to introduce some terminology. We call a CPTP map *unital* if it preservers the identity, i.e. $\mathcal{E}(\mathbb{1}) = \mathbb{1}$. In other words, the totally mixed state is unaffected by the open system dynamics. Note that if $\mathcal{L}(\mathbb{1}) = 0$ the solution of the Lindblad master equation (2.1.24) is unital.

Matrix represenations of the Lindblad form

The vectorization procedure of the density operator (2.1.18) can be used to represent the Lindbladian $\mathcal{L}$ as a $d^2 \times d^2$ matrix with d being the dimension of the quantum system. One finds

$$\mathcal{L} = -i[H \otimes \mathbb{1} - \mathbb{1} \otimes \bar{H}] + \sum_j \gamma_j (L_j \otimes \bar{L}_j - (L_j^\dagger L_j \otimes \mathbb{1} + \mathbb{1} \otimes L_j^T \bar{L}_j)), \quad (2.1.27)$$

such that the time evolution of the density operator ρ, obtained from the solution of the Lindblad master equation (2.1.24), is described by the evolution of a vector $|\rho\rangle \in \mathbb{C}^{d^2}$ in a complex vector space of dimension d^2. Since ρ is hermitian and has trace one, it is possible to describe the time evolution of ρ in terms of the time evolution of a vector $\boldsymbol{v} \in \mathbb{R}^{d^2-1}$ in a real, $d^2 - 1$ dimensional vector space to which we refer as the *vector of coherence* representation [13]. More explicitly, if we choose an orthonormal operator basis $\{B_j\}_{j=1}^{d^2-1}$, with

$$\mathrm{tr}\{B_i\} = 0, \quad B_i = B_i^\dagger, \quad \mathrm{tr}\{B_i B_j\} = \delta_{i,j}, \quad (2.1.28)$$

of traceless and hermitian $d \times d$ matrices, the density operator takes the form

$$\rho = \frac{1}{d}\mathbb{1} + \sum_{j=1}^{d^2-1} v_j B_j, \quad (2.1.29)$$

where $v_j = \mathrm{tr}\{B_j \rho\}$. If we collect the coefficients v_i in a vector $\boldsymbol{v} = (v_1, \ldots, v_{d^2-1})^T$, the master equation (2.1.22) with the Lindbladian (2.1.25) can be written as

$$\dot{\boldsymbol{v}} = A\boldsymbol{v} + \boldsymbol{q}_0, \quad (2.1.30)$$

where we refer to [13] for an explicit representation of the $(d^2-1) \times (d^2-1)$ matrix A and the vector $\boldsymbol{q}_0 \in \mathbb{R}^{d^2-1}$. Instead of describing the time evolution of the density operator in terms of a linear transformation given by the CPTP map $\Lambda_t = e^{\mathcal{L}t}$, the vector of coherence representation yields an affine transformation, obtained from solving (2.1.30). The solution $\boldsymbol{v}(t)$ undergoes a rotation and a translation in $\mathbb{R}^{d^2-1}$, whereas shrinking or dilating of the length $\|\boldsymbol{v}(t)\|$ of the vector of coherence can be traced back to the dissipative parts of the Lindblad master equation. In fact, for

purely unitary dynamics $\frac{d}{dt}\|\boldsymbol{v}(t)\| = 0$, the length does not change. The purity of a quantum state $\mathrm{tr}\{\rho^2\} = \frac{1}{d} + \|\boldsymbol{v}\|^2$ is bounded by $\frac{1}{d} \leq \mathrm{tr}\{\rho^2\} \leq 1$ with $\mathrm{tr}\{\rho^2\} = 1$ for a pure state and $\mathrm{tr}\{\rho^2\} = \frac{1}{d}$ for a totally mixed state. Hence the trajectory described by $\boldsymbol{v}(t)$ is confined to a d dimensional sphere with radius $r = \sqrt{1 - 1/d}$, centered at the origin. Pure states are located at the surface and the totally mixed state lies at the origin. Clearly for a single qubit ($d = 2$) this sphere is just given by the Bloch ball and the Bloch vector $\boldsymbol{v} = (v_x, v_y, v_z)^T$ is obtained from choosing the normalized Pauli spin operators $\frac{1}{\sqrt{2}}\sigma_x$, $\frac{1}{\sqrt{2}}\sigma_y$ and $\frac{1}{\sqrt{2}}\sigma_z$ as an operator basis. Note that for $d = 2$ the set of states is the entire Bloch ball, but for $d > 2$ this is no longer true.

Long time behavior

After having introduced two matrix representations of the Lindbladian, we briefly want to characterize in the following the long time behavior of $\Lambda_t = e^{\mathcal{L}t}$. Additionally we give two examples of a Lindbladian that are frequently used in the thesis.
In a finite dimensional space an evolution given by (2.1.24) has always at least one *fixed point* ρ_{ss}, such that

$$e^{\mathcal{L}t}(\rho_{ss}) = \rho_{ss}. \tag{2.1.31}$$

Thus ρ_{ss} is an eigenoperator of $\mathcal{L}$ with eigenvalue 0, i.e

$$\mathcal{L}(\rho_{ss}) = 0. \tag{2.1.32}$$

We call a semigroup $\mathcal{E}_t$ *relaxing* if there exists one unique fixed point [17, 18], that is

$$\lim_{t\to\infty} e^{\mathcal{L}t}(\rho) = \rho_{ss}, \qquad \forall \rho \in S(\mathcal{H}). \tag{2.1.33}$$

Clearly this is a special case, and in general there are other fixed points of the dynamics. The set of fixed points satisfying (2.1.32) is called the steady state manifold, and we call the steady state manifold *attractive* if all initial states are mapped into this set. More precisely, the steady state manifold is attractive if all non zero

eigenvalues of $\mathcal{L}$ have negative real parts [19, 20], so that we have in the long time limit

$$\lim_{t\to\infty} e^{\mathcal{L}t} = \mathcal{P}, \tag{2.1.34}$$

with $\mathcal{P} = \mathcal{P}^2$ being a not necessarily self-adjoint (super) projection that maps all states into the steady state manifold.

So far we have introduced the Lindblad form, its properties, and how it arises. Now we want to present two common examples of a Lindbladian of single qubit that are, for example, obtained from an interaction of a qubit with the quantized free electromagnetic field within the Born-Markov approximation. The first one is the *pure dephasing channel* generated by

$$\mathcal{L}(\cdot) = \frac{\gamma}{4}[\sigma_z, [\sigma_z, \cdot]], \tag{2.1.35}$$

and describing the exponential decay with a rate γ of the off diagonal elements of the density operator in the σ_z basis $\{|0\rangle, |1\rangle\}$. Note that $\mathcal{L}(\mathbb{1}) = 0$, showing that the pure dephasing channel is unital. The fixed point set is given by the set of convex combinations of $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$. In contrast, the *amplitude damping channel*, generated by

$$\mathcal{L}(\cdot) = \gamma(2\sigma_-(\cdot)\sigma_+ - (\sigma_+\sigma_-(\cdot) - (\cdot)\sigma_+\sigma_-)), \tag{2.1.36}$$

with $\sigma_\pm = \frac{1}{2}(\sigma_x \pm i\sigma_y)$ being the Pauli lowering and raising operators, describes the decay of a qubit to its ground state $|0\rangle\langle 0|$, which is the unique fixed point of the dynamics. Both channels describe decoherence, which we will discuss in the next section.

2.1.4 Decoherence

The preceding topics were very standard and a straightforward introduction of necessary mathematical concepts. Decoherence involves a lot of interpretational and conceptual aspects and in the following we focus on those most relevant for the rest

of the thesis.

The superposition principle lies at the heart of quantum mechanics, and it follows directly from the linear character of the Schrödinger equation. Together with entanglement and other non-classical phenomena, it gives rise to new technologies such as the quantum computer. Clearly, however, it does not operate on a macroscopic scale, although there is nothing present in the formulation of quantum mechanics that would prevent macroscopic quantum superpositions to exist. So why do we not observe quantum mechanical effects in our daily life? Besides the Copenhagen interpretation, there are at least two modern approaches that try to explain the emergence of classicality on a macroscopic level. The first one we are going to discuss is decoherence, which attempts to give an explanation within the standard formulation of quantum theory [21, 22].

Decoherence arises from the interaction of a quantum system with its environment. The interaction between the system and the environment creates correlations and we saw in the last section that the resulting system dynamics is in general not unitary any more. As a consequence, the entropy of the system is not conserved, since a leakage of information into the environment takes place. Coherent superpositions of system states are transformed into statistical mixtures, mathematically described by a CPTP map. This process is called decoherence - the decay of the off diagonal elements of the density operator in a preferred basis.

We start with a Hamiltonian description of system and environment. The total Hamiltonian reads

$$H = H_S \otimes \mathbb{1} + \mathbb{1} \otimes H_B + H_{S,B}, \tag{2.1.37}$$

where the interaction of system and environment can always be written in the form

$$H_{S,B} = \sum_{\alpha} S_\alpha \otimes B_\alpha, \tag{2.1.38}$$

with S_α and B_α being hermitian operators.

To get an insight into the effect of decoherence we consider as a simple example a qubit that interacts with the environment through $H_{S,B} = \sigma_z \otimes B$ and we assume

that the energy of system and bath is conserved $[H_S, H] = [H_B, H] = 0$. We take $\rho_B(0) = |\phi\rangle\langle\phi|$ as the initial state of the bath. Using (2.1.10), the qubit state in the interaction picture at time t in the eigenbasis of σ_z is given by

$$\begin{aligned}\rho_S(t) = \mathcal{E}_t(\rho_S(0)) = \langle 0|\rho_S(0)|0\rangle|0\rangle\langle 0| + \langle 1|\rho_S(0)|1\rangle|1\rangle\langle 1| \\ + f(t)\langle 0|\rho_S(0)|1\rangle|0\rangle\langle 1| + \bar{f}(t)\langle 1|\rho_S(0)|0\rangle|1\rangle\langle 0|,\end{aligned} \tag{2.1.39}$$

with $f(t) = \langle\phi|\exp(-i2Bt)|\phi\rangle$ being the overlap of the evolved bath states with the initial one. The first observation is that the diagonal elements of $\rho_S(t)$ stay unaffected by the interaction with the environment, which can be traced back to the assumption that the energy of the system is conserved. If the bath states become orthogonal, $f(t_{\mathrm{DC}}) = 0$, for some time t_{DC}, the system density operator becomes diagonal in the computational basis, described by the CPTP map

$$\mathcal{E}_{t_{\mathrm{DC}}}(\cdot) = M_0(\cdot)M_0 + M_1(\cdot)M_1, \tag{2.1.40}$$

with Kraus operators $M_n = |n\rangle\langle n|,\ n = 0, 1$. The state has fully decohered into a statistical mixture, sometimes called pure dephasing. The time t_{DC} is given by the first time the initial state $|\phi\rangle$ of the environment has evolved under $U(2t) = \exp(-i2tB)$ into an orthogonal state. It is known that this time can be lower bounded [23], yielding for the simple qubit example $t_{\mathrm{DC}} \geq \pi/(2\Delta B)$, with ΔB being the energy dispersion with respect to the initial bath state. One may also ask if coherence is recovered as a function of time. Assuming B has a discrete energy spectrum, due to the properties of almost periodic functions [24], it is always possible to find a time $T_{\mathrm{rec}} > 0$ such that for each ϵ we have

$$\|\mathcal{E}_{T_{\mathrm{rec}}} - \mathrm{id}\|^2_{\mathrm{HS}} = 2|f(0) - f(T_{\mathrm{rec}})|^2 < \epsilon, \tag{2.1.41}$$

where $\mathcal{E}$ was treated as a matrix obtained from (2.1.18). This is just another version of the quantum recurrence theorem [25] in an open system setting. There exists a time T_{rec} for which coherence can be recovered arbitrarily well. See Fig. 2.1.4 for an illustration of this effect. The crucial assumption here was that the energy spectrum of B is discrete. As a consequence, the off diagonal elements cannot decay

exponentially, as it would for semigroup dynamics described by the pure dephasing channel (2.1.35). In fact, more generally, it can be shown [6, 26] that:

> if a function $P(t) = |\langle\phi| \exp(-iHt)|\phi\rangle|^2$ decays exponentially in time, the Hamiltonian H has a continuous spectrum over the whole real line.

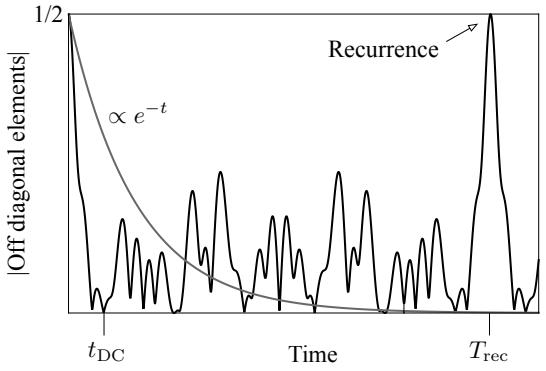


Figure 2.1: Decoherence of a qubit described by an exponential decay (gray); resulting from a qubit-environment interaction (black) with an environment consisting of 4 qubits.

In particular this mathematical subtlety will become important when we study in Sec. 4.1 the performance of dynamical decoupling, which is a method to suppress the interactions with the environment in order to fight decoherence. Within this context we provide two examples, one with a time dependent two-qubit Hamiltonian, and another based on an unbounded interaction that will lead (without approximations) to amplitude damping (2.1.36) and pure dephasing (2.1.35).

To some extent decoherence can explain the emergence of classicality in the macroscopic world [27]. Due to the interaction of a macroscopic object, which is characterized by many degrees of freedom, with its environment, quantum coherence becomes rapidly spread over many more degrees than an observer can have access to. This process becomes faster and faster if more degrees of freedom couple to the environment. Hence the density operator of the macroscopic object rapidly becomes diagonal in a preferred basis and therefore the object is described by a classical ensemble of states. However, decoherence is based on the framework of unitary quantum mechanics on a larger system. For an observer outside system and environment the dynamics remains unitary and decoherence does not appear. Over the years it has been pointed out in the literature that the emergence of classicality is re-

lated to the collapse of the wave function rather than the transition from a coherent superposition into a statistical mixture. For A. Bassi et al. the reason for this lies in the fact that *the statistical operator describing a statistical mixture, describes at the same time infinitely many inequivalent statistical mixtures* [28]. A transformation of the form (2.1.40) transforms the density operator into a statistical mixture of states, which could still consist of coherent superpositions of macro states, instead of being a mixture of states with definite macroscopic properties. Bassi et al. further point out that the loss of coherence can be understood as a consequence of the interaction with the environment, but it does not explain the measurement process causing the collapse of the wave function to one distinct macroscopic state [29]. Moreover the Schrödinger equation is deterministic in the sense that its solution describes the propagation of the initial state to some final state in a deterministic way. Only together with the Born rule, that assigns probabilities to the outcome of an experiment, does quantum mechanics become probabilistic. The role of decoherence in the emergence of classicality is still under debate and an ongoing area of research. For a detailed overview for what decoherence can explain and what not we refer to [21].

2.1.5 Collapse models

To overcome the issues mentioned in the previous sections collapse models have been devised to incorporate the wave function collapse into a single equation, such that it affects directly the wave function and not only the density operator. Most fundamental differential equations which describe physical phenomena are non-linear, with linearity being a convenient approximation in some appropriate limiting cases. Since the superposition principle was demonstrated up to a mesoscopic scale in the laboratory [30, 31], any non linear extension that causes the wave function collapse must be negligible on this scale, but at the same time must be amplified when moving from the micro- into the macroscopic domain. Additionally the basis in which the collapse takes place must be chosen in such a way that macroscopic objects have

a definite position in space. Based on these assumptions a whole zoo of non-linear extensions of the Schrödinger equation have appeared, refer to [29, 32, 33, 34] for an overview. Gisin [35] and Polcinski [36] proved that any non-linear deterministic extension would allow superluminal signaling, so in fact the only collapse models to survive are those based on stochastic differential equations. The derivation and investigation of these equations require a bit of elaboration of stochastic calculus, which is beyond the scope of this thesis. Along the lines of [37] we therefore only sketch here the steps of how a non-linear stochastic modification of the Schrödinger equation incorporates the collapse of the wave function.

In the following we assume some basic knowledge about stochastic processes. We consider a Markov process of the state $|\psi(t)\rangle$ in the Hilbert space described by the Ito differential equation

$$d|\psi\rangle = (Cdt + \boldsymbol{A} \cdot d\boldsymbol{B})|\psi\rangle, \tag{2.1.42}$$

with C some operator, $\boldsymbol{A} = \{A_i\}$ as set of operators and $\boldsymbol{B} = \{B_i\}$ a real Wiener process, where the increments obey

$$\mathbb{E}[dB_i] = 0, \qquad \mathbb{E}[dB_i dB_j] = \delta_{ij}\gamma dt, \tag{2.1.43}$$

with γ being a real constant and the dot product in (2.1.42) has the meaning $\boldsymbol{A} \cdot d\boldsymbol{B} = \sum_k A_i dB_i$. If this term would not appear and if we identify $C = -iH$ as the Hamiltonian, the usual Schrödinger equation is recovered. For an initial state $|\psi(0)\rangle$ the stochastic differential equation (2.1.42) generates at time t an ensemble of state vectors $|\psi(t)\rangle$ depending on the particular realization of the Wiener process. Note that the (2.1.42) is still a linear stochastic differential equation and it is easy to show, using the Ito rules, that the norm of the state vector is not conserved. If we incorperate norm-conserving into (2.1.42), we obtain a non-linear stochastic differential equation

$$d|\psi\rangle = \left([-iH - \frac{1}{2}\gamma(\boldsymbol{A}^\dagger - \boldsymbol{R}) \cdot \boldsymbol{A} + \frac{1}{2}\gamma(\boldsymbol{A} - \boldsymbol{R}) \cdot \boldsymbol{R}]dt + (\boldsymbol{A} - \boldsymbol{R}) \cdot d\boldsymbol{B}\right)|\psi\rangle, \tag{2.1.44}$$

$$\boldsymbol{R} = \frac{1}{2}\langle\phi|(\boldsymbol{A}^\dagger + \boldsymbol{A})|\phi\rangle, \tag{2.1.45}$$

where the non-linearity enters due to the dependency of $\boldsymbol{R}$ on $|\phi\rangle$. If we write $\boldsymbol{A} = \sum_j \boldsymbol{a}_j P_j$, with P_j being orthogonal projections that sum up to the identity, one can show that $|\psi(t)\rangle$ reduces asymptotically to one of the states $P_j|\psi(0)\rangle$ times a normalization factor. In other words, for $t \to \infty$ the initial state has collapsed into a state vector that lies in the common eigenspaces of $\boldsymbol{A}$. Another way of looking at it is to consider the density operator $\rho = \mathbb{E}[|\psi\rangle\langle\psi|]$ that is given by the ensemble average taken of over all realizations of $|\psi\rangle$. Using once more Ito calculus, we obtain a master equation in Lindblad form

$$\dot{\rho}(t) = -i[H, \rho(t)] + \gamma \left(\boldsymbol{A}\rho \cdot \boldsymbol{A}^\dagger - \frac{1}{2}(\boldsymbol{A}^\dagger \cdot \boldsymbol{A}\rho + \rho \boldsymbol{A}^\dagger \cdot \boldsymbol{A}) \right), \tag{2.1.46}$$

where the decay rate γ determines how fast the collapse takes place. Applying this formalism to a system of identical particles with center off mass coordinate Q, one arrives at the CSL model [37] that predicts for the off diagonal elements of the density operator a decay according to $\langle Q''|\rho(t)|Q'\rangle \propto \exp(\Gamma t)$. The decay rate $\Gamma = \gamma D_0 n$ consists of the density D_0 of the macroscopic body and the number of particles n in the center of mass position Q' that do not lie in the volume occupied by the body in the center of mass position Q''. The important observation here is that the bigger the system, the faster the collapse. The free parameter is still γ, for which Adler [29] gave an estimate of $\gamma = 10^{-8}\, s^{-1}$, and which describes the spontaneous collapse of a single nuclei within the CSL model. Clearly the non-linearity in (2.1.44) enters in a very specific way and one may ask what the evolution looks like if the Schrödinger equation is modified in a different non-linear way. Recently Bassi et al. [38] showed that:

> the only collapse models that are non-linear Markovian extensions of the Schrödinger equation, and which do not allow faster-than- light-signaling, are those for which the evolution of the density operator $\rho = \mathbb{E}[|\psi\rangle\langle\psi|]$ is governed by a master equation in Lindblad form (2.1.25).

Hence they are mathematically similar to decoherence models which describe an exponential decay through a semigroup dynamics, as for example the amplitude

damping channel (2.1.36) would do. This is the reason why we will later in Sec. 4.2 refer to such a dynamics as intrinsic decoherence. Then the following question arises: imagine one observes an exponential decay in an experiment, can one distinguish between an exponential decay that is caused by some system-environment interaction or by some collapse model? Despite decoherence, how can we verify collapse models if they predict similar features to decoherence? Or more generally, how can we find out whether intrinsic non-unitary dynamics, which is not caused by some interaction with another system, does exist? Before we propose a solution to this problem in Sec. 4.2, we are going to introduce some basic concepts of quantum control theory.

2.2 Quantum control theory

Control theory in general has a long history and it became explicit in engineering with the demands of the industrial revolution in the 19th century. From describing the natural evolution of a mechanical system it became important to influence the dynamics in a systematic way in order to get desired outcomes. One has to distinguish between closed loop and open loop control schemes. In a closed loop control scheme the outputs of the system are routed back to the inputs, which is also called a feedback loop. Open loop schemes, which will be the focus of this thesis, compute the inputs into the systems only based on the current state of the system and its describing model. We consider control systems that can be formulated in a bilinear way [10], i.e. they are described by a differential equation of the form

$$\dot{x}(t) = \left(A + \sum_i u_i(t) B_i \right) x(t), \qquad x(t_0) = x_0, \tag{2.2.1}$$

where x can be a vector or a matrix describing the state or the evolution of the system, A and B_i are some $d \times d$ matrices and u_i are the control functions. Given a control system of the form (2.2.1) one may ask the following questions:

1. Which states or transformations can we reach from the initial condition x_0 by modulating the control functions?

2. How do we have to modulate the control functions u_i to steer the system from x_0 to some goal-state or transformation x_G at some time T?

In a quantum mechanical setting the electromagnetic field plays the role of the control, and the atom, nucleus or electron, or any other quantum mechanical system, is the object of the control [9]. The bilinear control equation is given by the Schrödinger equation for the state vector or the time evolution operator. Here we are mainly interested in the implementation of quantum gates described by unitary transformations, rather than in state to state transfer. Therefore the equation we are going to study is the Schrödinger equation for the time evolution operator

$$\dot{U}(t) = -iH(t)U(t), \quad U(0) = \mathbb{1}, \tag{2.2.2}$$

with

$$H(t) = H_0 + \sum_{i=1}^{n} u_i(t)H_i, \tag{2.2.3}$$

where we refer to H_0 as the *drift Hamiltonian* and to H_i as the *control Hamiltonians* with $u_i(t)$ being the corresponding control fields. Note that one possibility to study the control of an open quantum system is to study the control properties of the master equation (2.1.22) where the control enters in the Hamiltonian part of (2.1.25) and dissipation is described by $\mathcal{D}$. Before we will come back to this approach in Sec. 2.3, we first address in the following two sections the questions (1) and (2) for a closed quantum system. Given a control system (2.2.3), which unitary transformation can we implement and how can we implement them? The first question is treated with Lie algebraic and group theoretical methods, whereas in Sec. 2.2.2 a gradient based algorithm is presented that allows us to numerically calculate the control fields for a given target transformation.

2.2.1 The reachable set and controllability

To introduce some important concepts from control theory we start again with the bilinear control system (2.2.1). We consider the bilinear control system on a Lie

subgroup G of $Gl(d,\mathbb{C})$ with $Gl(d,\mathbb{C})$ being the general linear group consisting of all invertible $d \times d$ matrices with complex entries. The solutions to the bilinear control system (2.2.1) are hence given by elements of $Gl(d,\mathbb{C})$. Moreover, let $\mathfrak{g}$ be the corresponding Lie algebra in the sense that the tangent space of G at the identity coincides with $\mathfrak{g}$ [39]. For instance, the Lie algebra $\mathfrak{gl}(d,\mathbb{C})$ corresponding to $Gl(d,\mathbb{C})$ is the algebra of all complex $d \times d$ matrices. Denote by $\mathcal{R}_T(x_0)$ the set of all $x \in G$ that are for $T > 0$ solutions to (2.2.1), i.e. there exists some modulation of the control functions such that every $x \in \mathcal{R}_T(x_0)$ can be reached from $x_0 \in G$ for a time $T > 0$. The *reachable set* $\mathcal{R}$ is then defined as

$$\mathcal{R}(x_0) = \bigcup_{T \geq 0} \mathcal{R}_T(x_0). \tag{2.2.4}$$

We call as system *accessible* if for all $x_0 \in G$ the reachable set $\mathcal{R}(x_0)$ has non-empty interior in G and *fully controllable* if for all $x_0 \in G$ the closure of the reachable set is equal to G. More details on control theoretic terminology can be found in [40].
Consider now as a control system the Schrödinger equation for the time evolution operator (2.2.2) with the Hamiltonian (2.2.3). Clearly, in this case we have $G = U(d)$ with $U(d)$ being the unitary group, consisting of all $d \times d$ unitary matrices, and $\mathfrak{g} = \mathfrak{u}(d)$ being the algebra of skew-hermetian $d \times d$ matrices. The reachable set $\mathcal{R}(\mathbb{1}) \equiv \mathcal{R}$ consists of all unitary operations that can be reached as a solution to (2.2.2) by varying the control functions $u_i(t)$, which we assume are piecewise constant. It can be shown [11, 41] that for finite dimensional quantum system

$$\bar{\mathcal{R}} = e^{\mathfrak{L}}, \tag{2.2.5}$$

with $\bar{\mathcal{R}}$ being the closure of the reachable set and $e^{\mathfrak{L}}$ being the Lie group that corresponds to the *dynamical Lie algebra*

$$\mathfrak{L} = \mathfrak{Lie}(iH_0, iH_1, \ldots, iH_n), \tag{2.2.6}$$

which is spanned by real linear combinations and iterated commutators of the drift and the control Hamiltonians. The unitary control system (2.2.2) is fully controllable if $\mathfrak{L} = \mathfrak{u}(d)$, so that $\bar{\mathcal{R}}$ is equal to the unitary group. This is known as the as the *Lie*

rank criterion [11, 41]. In others words, if the system is fully controllable, every unitary matrix can be implemented with arbitrary high precision by choosing a proper control, sometimes called universal control. For traceless matrices, such as the Pauli matrices for spin systems, we have $\mathfrak{L} = \mathfrak{s}u(d)$ if the system is fully controllable where $\mathfrak{su}(d)$ is the special unitary algebra of skew-hermitian $d \times d$ matrices with trace 1. The corresponding Lie group is the special unitary group $SU(d)$ that consists of all unitary $d \times d$ matrices with determinant 1, so that $SU(d) \subset U(d) \subset Gl(d, \mathbb{C})$. For future considerations we further introduce the special linear group $Sl(d, \mathbb{R})$, consisting of all real $d \times d$ matrices with determinant 1, the orthogonal- and the special orthogonal group $O(d)$ and $SO(d)$, consisting of all orthogonal $d \times d$ matrices with determinant ± 1 and 1 respectively, and the symplectic group $Sp(2d, \mathbb{R})$ consisting of all $2d \times 2d$ matrices S satisfying $S\Omega S^T = \Omega$ where

$$\Omega = \bigoplus_{j=1}^{n} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{2.2.7}$$

is the symplectic form. Their corresponding Lie algebras are given by $\mathfrak{sl}(d, \mathbb{R})$, $\mathfrak{o}(d)$, $\mathfrak{so}(d)$ and $\mathfrak{sp}(2d, \mathbb{R})$ respectively. Note that if the Lie group G is compact and connected, such as the unitary and the special unitary group, full controllability is equivalent to accessibility [11, 41]. That means that if a finite dimensional closed quantum system is fully controllable it is also accessible and vice versa. Thus for closed finite dimensional quantum system we do not have to distinguish between accessibility and controllability, whereas for open quantum systems described by a Lindblad master equation this distinction will become important later. The dynamical Lie algebra will play an important role in this thesis and the Lie rank criterion is a powerful tool to determine whether a system is fully controllable. A full proof of the Lie rank criterion can for example be found in [9]. Here we outline a proof using the Trotter formulas

$$\lim_{n\to\infty} \left(e^{B\frac{t}{n}} e^{C\frac{t}{n}} \right)^n = e^{(B+C)t}, \tag{2.2.8}$$

$$\lim_{n\to\infty} \left(e^{-B\frac{t}{n}} e^{-C\frac{t}{n}} e^{B\frac{t}{n}} e^{C\frac{t}{n}} \right)^{n^2} = e^{[B,C]t^2}, \tag{2.2.9}$$

valid for two arbitrary matrices B, C, and writing the reachable set as

$$\mathcal{R} = \left\{ \prod_{j=1}^{n} e^{-iA_j\alpha_j} \quad | A_j \in \mathcal{A}, \alpha_j \geq 0, n \in \mathbb{N} \right\}, \tag{2.2.10}$$

with $\mathcal{A}$ the set of all Hamiltonians between we can switch at will. We will show in the following that the elements that are contained in the closure of the reachable set are given by unitaries which are generated by all possible linear combinations and iterated commutators of the drift and the control Hamiltonians. Therefore $\bar{\mathcal{R}} = e^{\mathfrak{L}}$. We begin with noting that each $A_j \in \mathcal{A}$ may be written as $A_j = H_0 + \sum_i u_i^{(j)} H_i$ with some constant control field amplitudes $u_1^{(j)}, \ldots, u_n^{(j)} \in \mathbb{R}$. Notice first of all that if all control fields are zero, we have $e^{-ia H_0} \in \mathcal{R}$ for all positive constants α. Due to the quantum recurrence theorem [25],

$$\forall \epsilon \text{ and } \beta > 0, \ \exists \gamma > \beta \text{ such that } \left\| e^{-i\gamma H_0} - \mathbb{1} \right\| < \epsilon, \tag{2.2.11}$$

which implies that the evolution $e^{-i(\gamma-\alpha)H_0}$ is effectively given by $e^{i\alpha H_0}$, we have $e^{\pm i\alpha H_0} \in \bar{\mathcal{R}}$. By considering successively the operations $e^{i\alpha/n H_0} e^{-i\alpha/n A_1}$, with $A_1 = H_0 + u_i^{(1)} H_i$ and using (2.2.8), we can create every unitary evolution that is generated by the drift and the control Hamiltonians alone. Any unitary operation that is generated by some real linear combination of $iH_0, \ldots, iH_n$ can be created in a similar way. It remains to show that also any unitary operation that is generated by nested commutators and their real linear combinations can be implemented. Consider n^2 times the successive operations $e^{-i\sqrt{\alpha}/n H_j} e^{-i\sqrt{\alpha}/n H_m} e^{i\sqrt{\alpha}/n H_j} e^{i\sqrt{\alpha}/n H_m}$ and with (2.2.9) and (2.2.11) we find

$$e^{-i\alpha[H_j, H_m]} \in \bar{\mathcal{R}}, \quad \forall\, \alpha \in \mathbb{R}, \quad \forall\, m, j = 0, \ldots, n. \tag{2.2.12}$$

Unitaries containing higher order commutators and their real linear combinations can be created analogously. Hence every $U = e^{\theta}$ with $\theta \in \mathfrak{L}$ and $\mathfrak{L}$ being the dynamical Lie algebra (2.2.6) can be implemented with arbitrary high precision.
The Lie rank criterion is a very powerful method to determine the unitary operations that can be implemented with the resources one has in an experimental setting by calculating the corresponding dynamical Lie algebra. It works well for finite

dimensional unitary gates since they are subgroups of the compact group $SU(d)$. A crucial point in the outline of the above proof was the use of the recurrence theorem (2.2.11), which allows us to reverse the sign in front of the drift Hamiltonian. In fact the compactness of a Lie group is a sufficient criterion to go from negative to positive times.

But what about infinite dimensional quantum systems such as quantum harmonic oscillators? Since we are going to use the framework of *symplectic transformations* in Sec. 4.3, we already give here a brief introduction into this subject and connect it to recent results [42] in quantum control. We consider a system described and controlled by Hamiltonians that are quadratic in the quadrature operators, $\hat{x}, \hat{p}$, i.e

$$H = \frac{1}{2} \sum_{i,j} A_{i,j} \boldsymbol{R}_i \boldsymbol{R}_j, \tag{2.2.13}$$

where the vector $\boldsymbol{R}$ is defined as

$$\boldsymbol{R} = (\hat{x}_1, \hat{p}_1, \ldots, \hat{x}_n, \hat{p}_n)^T, \tag{2.2.14}$$

and A is a real and symmetric $2n \times 2n$ matrix. The canonical commutation relation can be written as $[\boldsymbol{R}_i, \boldsymbol{R}_j] = i\Omega_{i,j}$ with Ω being the symplectic form introduced in (2.2.7). Note that if the quadrature operators are collected in the way $\boldsymbol{R} = (\hat{x}_1, \ldots, \hat{x}_n, \hat{p}_1, \ldots, \hat{p}_n)^T$ the symplectic form J is given by

$$J = \begin{pmatrix} 0 & \mathbb{1}_{n\times n} \\ -\mathbb{1}_{n\times n} & 0 \end{pmatrix}. \tag{2.2.15}$$

In the following we work in the representation defined through (2.2.14) using Ω, whereas in Sec. 4.3 we will also work in the other representation using J. The time evolution operator $U = \exp(-iHt)$, that corresponds to the quadratic Hamiltonian (2.2.13), acts on an infinite dimensional Hilbert space and it is also called a Gaussian operation since it preserves the properties of Gaussian states. The time evolution of the quadrature operators collected in the vector $\boldsymbol{R}$ can be written as $U^\dagger \boldsymbol{R} U = S\boldsymbol{R}$ where

$$S = e^{-tA\Omega}, \tag{2.2.16}$$

belongs to the symplectic group $Sp(2n, \mathbb{R})$. This shows that for quadratic Hamiltonians there is a one-to-one correspondence between the time evolution on an infinite dimensional Hilbert space and a time evolution given by a finite dimensional symplectic matrix (2.2.16). Hence for quadratic Hamiltonians we can recast the Schrödinger equation for the time evolution operator as

$$\dot{S}(t) = \mathcal{G}(t)S(t), \qquad S(0) = \mathbb{1}, \tag{2.2.17}$$

with

$$\mathcal{G}(t) = \left(A_0 + \sum_{j=1}^{n} u_j(t) A_j \right) \Omega, \tag{2.2.18}$$

so that the control properties of (2.2.17) can now be studied within the symplectic group. The symplectic group is not compact and thus from a first perspective the Lie rank criterion does not apply. The key result of [42] was that, if the matrix A is positive definite, the recurrence theorem can be applied and hence the Lie rank criterion remains a necessary and sufficient condition to asses controllability. Therefore the system (2.2.17) is fully controllable if $\mathfrak{L} = \mathfrak{sp}(2n, \mathbb{R})$ with $\mathfrak{sp}(2n, \mathbb{R})$ being the symplectic algebra containing all elements s that satisfy $\Omega s = -s^T \Omega$.

Clearly, in order to identify the operations that can be implemented in a quantum system through the controls, we need to calculate the dynamical Lie algebra (2.2.6). Especially for large quantum system this can still be a challenging task since it involves the computation of many iterated commutators and their real linear combinations. However, there exists an algorithm that iteratively creates from the drift and the control Hamiltonians a basis of the dynamical Lie algebra [10]. To do so the drift and the control Hamiltonians are written as column vectors arranged in a matrix M. Commutators are calculated iteratively and a linearly independent element of the dynamical Lie algebra is obtained if the rank of M increases. If the rank of M does not increase anymore, a basis for the dynamical Lie algebra was found. For further details we refer to [43]. Unfortunately the algorithm works well only for small system sizes. The vectors, which are created in that way, can involve constants that differ many orders of magnitude from each other, so that the calculation of the

rank becomes unstable. Intuitively it is hard to numerically decide if two vectors are linearly independent if they are almost parallel. Nevertheless, this method will be used for a small system size in order to visualize in Sec. 3.2.4 through a tree structure the dynamical Lie algebra of a specific system.

So far we have discussed how to determine the unitary operations which can be implemented in a given experimental setting. In the next section we are going to present a gradient based algorithm that calculates the control pulses to implement some specific target operation.

2.2.2 GRAPE algorithm

Here we describe how the gradient ascent pulse shape engineering (GRAPE) algorithm works, allowing us to numerically calculate the control fields in order to reach a specific target. It was originally developed in [44] and further improved in [45, 46]. To be as general as possible we consider again the bilinear control system (2.2.1) and we denote by $x(t, t_0)$ its solution from t_0 up to time t. Clearly, for closed finite dimensional quantum systems, $x(t, t_0)$ is given by the solution $U(t, t_0)$ of the unitary control system (2.2.2), for quadratic Hamiltonian control systems (2.2.16) by $S(t, t_0)$, and for open quantum system described by a Lindblad master equation (2.1.22) by the CPTP map $\Lambda_{t,t_0} = \hat{T} \exp(\int_{t_0}^{t} dt' \mathcal{L}_{t'})$, where $\mathcal{L}_t$ is the Lindbladian that includes the controls.

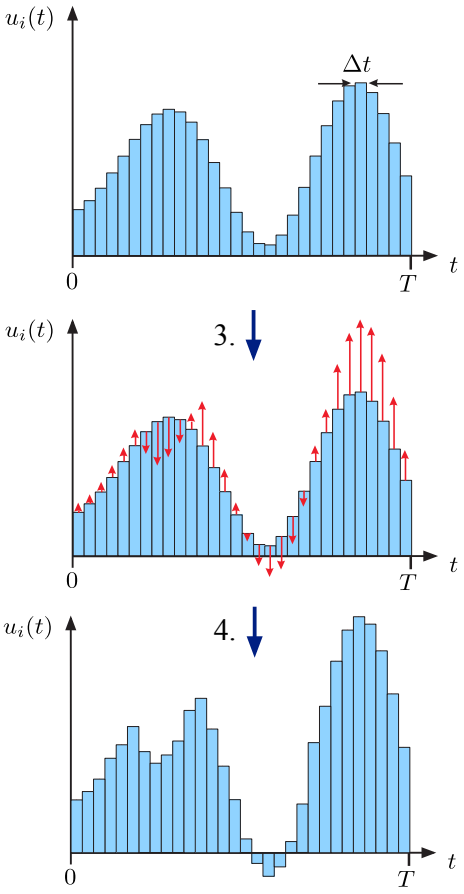
We discretize the control fields $u_i(t)$ using piecewise constant functions and dividing the total time T into M time slices $[t_{p-1}, t_p]$ of duration $\Delta t = T/M$. We set $u_i(t) = u_{ip}$ for $t \in [t_{p-1}, t_p]$. The idea of the algorithm is to minimize an error functional $\varepsilon(\boldsymbol{u})$ with respect to the piecewise constant control field amplitudes that are collected in the vector $\boldsymbol{u}$. The goal is to find the vector $\boldsymbol{u}^*$, such that $\varepsilon(\boldsymbol{u}^*)$ attains its global minimum. This is done iteratively by updating the control fields according to a second order quasi Newton method

$$\boldsymbol{u}^{(k+1)} = \boldsymbol{u}^{(k)} - \alpha_k (\boldsymbol{H}_k)^{-1} \nabla \varepsilon(\boldsymbol{u}^{(k)}), \tag{2.2.19}$$

starting with an initial guess $\boldsymbol{u}^{(0)}$ of the control fields, which is taken to be random throughout the thesis. The initial Hessian $\boldsymbol{H}_0$ is taken to be the identity, the Hessian

Figure 2.2: GRAPE:

1. Initial guess pulse $\boldsymbol{u}^{(0)}$.
2. Calculate the time evolution $x(T)$ and the corresponding error functional $\varepsilon(\boldsymbol{u}^{(0)})$.
3. Compute the gradient $\nabla\varepsilon(\boldsymbol{u}^{(0)})$ using (2.2.20).
4. Update the control fields according to (2.2.19).
5. Start again from 2. and iterate until ε has reached its minimum.



$\boldsymbol{H}_k$ is constructed from the past gradient history according to the Broyden-Fletcher-Goldfarb-Shanno (BFGS) method and the parameter α_k, which characterizes the length of the search, is set to one by default. A summary of the mains steps of the algorithm can be found in Fig. 2.2.2, which was adopted from [44]. From (2.2.19) we see that the update of the control fields requires the calculation of derivatives $\frac{\partial}{\partial u_{ip}}\epsilon(u_{ip})$, which can be done by approximating the gradient by the differential quotient. However, this method is very slow and the step size parameter has to be chosen carefully. Typically the functional $\epsilon(\boldsymbol{u})$ is a simple function of $x(t)$, and it can be shown [47] that for piecewise control functions its gradient with respect to

the control fields can be written as

$$\frac{x(T,t_0)}{\partial u_{ip}} = x(T,t_p)\left[\int_{t_{p-1}}^{t_p} d\tau\, x(t_p,\tau)B_i x(\tau,t_{p-1})\right] x(t_{p-1},t_0), \tag{2.2.20}$$

with

$$x(t,t_{p-1}) = \exp((t-t_{p-1})\mathcal{G}(u_{ip})), \tag{2.2.21}$$

and

$$\mathcal{G}(u_{ip}) = A + \sum_i u_{ip}B_i, \tag{2.2.22}$$

being the time independent generator of the dynamics evaluated at the piecewise constant values of the control field amplitudes. The integral in (2.2.20) can be evaluated exactly using the augmented matrix exponential formula

$$\exp\begin{pmatrix} D & E \\ 0 & F \end{pmatrix} = \begin{pmatrix} e^D & \int_0^1 e^{D(1-s)}Ee^{Fs}ds \\ 0 & e^F \end{pmatrix}, \tag{2.2.23}$$

and identifying $D = F = \mathcal{G}(u_{ip})\Delta t$ and $E = B_i\Delta t$.

So far we have discussed how the control fields can be calculated in an iterative way, without specifying the functional ε that needs to be minimized. Since we are dealing throughout the thesis with the implementation of quantum operations, which are described by matrices, a convenient choice of ε would be the minimization of

$$\varepsilon = \lambda \left\| x(T) - x_G \right\|_{\mathrm{HS}}^2, \tag{2.2.24}$$

where x_G is the goal operation, $\|\cdot\|_{\mathrm{HS}}^2$ is the Hilbert-Schmidt norm and λ is some normalization constant. Taking $\lambda = 1/(2d)$, with d being the dimension of the quantum system, then for unitary control systems (2.2.2) ε takes the form

$$\varepsilon = 1 - \frac{2}{d}\Re\{\mathrm{tr}\{U(T)U_G^\dagger\}\}, \tag{2.2.25}$$

such that we need to maximize the normalized Hilbert-Schmidt scalar product $g = \frac{1}{d}\mathrm{tr}\{U(T)U_G^\dagger\}$ between the goal unitary operation U_G and the actual time evolution $U(T)$. Note that a maximization of g would respect the global phase of the evolution,

which is not relevant. Hence a more common choice, that does not respect the global phase, would be the maximization of $f_1 = |g|^2$ [45]. For open quantum systems, which are described by a Lindblad master equation (2.1.22), the numerical calculation of the control fields can be carried out choosing $\lambda = 1$ and minimizing

$$\varepsilon = \|\Lambda_T - \mathcal{E}_G\|_{\mathrm{HS}}^2 \,, \tag{2.2.26}$$

with $\Lambda_T = \hat{T} \exp(\int_0^T dt\, \mathcal{L}_t)$ and $\mathcal{E}_G$ some goal CPTP map, both treated as $d^2 \times d^2$ matrices obtained from row vectorization (2.1.18) of the density operator. Note that compared to (2.2.25) the Hilbert Schmidt norm (2.2.26) is not upper bounded by one, but it provides an upper bound for the diamond norm

$$\|\Lambda_T - \mathcal{E}_G\|_\diamond \leq d\sqrt{\varepsilon}, \tag{2.2.27}$$

which measures how much two quantum channels can be distinguished operationally. It takes its maximum value of 2 if the two quantum channels are perfectly distinguishable [48].

Clearly, we cannot be sure that the algorithm approaches the global minimum of ε since the control landscape can exhibit local minima as well [49, 50]. Depending on the initial guess pulse $\boldsymbol{u}^{(0)}$, the gradient based search can stop in a local minimum with $\varepsilon \neq 0$. This effect can be reduced by taking the minimum value of ε over a sample of randomly chosen initial pulses.

2.3 Quantum control and open quantum systems

Having introduced the relevant concepts of open quantum systems and quantum control theory we want to characterize now the difficulties in combining the two fields. To study the control of an open quantum system there are two approaches one can follow.

1st approach: study the control properties of the Lindblad master equation

$$\dot{\rho}(t) = -i[H(t), \rho(t)] + \mathcal{D}(\rho), \tag{2.3.1}$$

where the dissipator $\mathcal{D}$ is given by (2.1.26) and the control enters in a coherent way in the Hamiltonian $H(t)$ given by (2.2.3). We first note that the controlled Lindblad master equation (2.3.1) can be written as an affine control system using the vector of coherence representation introduced in Sec. 2.1.3. Using numerical gate optimization within the controlled master equation formalism, fidelity limits for implementing specific gates were studied in [46, 51, 52, 53]. Usually if the strength of the dissipation gets stronger, the fidelity for implementing unitary gates through the controls drops down. This behavior is related to the fact that one can relatively easy show that [40]:

the Lindblad master equation is never fully controllable with unitary controls.

Roughly speaking the irreversible nature of the Lindblad master equation does not allow us to go backwards by applying the controls coherently, unless we wait for infinitely long time for the steady state to be reached [54]. The dissipative part drives the system towards some fixed point, while changing the entropy of the system. It is not possible to counteract this process by coherently applying control fields since they correspond to unitary dynamics leaving the entropy invariant. One might think that, analogous to the closed system case, the reachable set can be characterized by calculating the corresponding dynamical Lie algebra. Unfortunately this is not the case because the group structure is lost. Without the controls the solutions Λ_t to (2.3.1) form a semigroup. The inverse element, given by the inverse of Λ_t, is

in general not a physical map as mentioned in Sec. 2.1.1. Therefore the reachable set takes the form of a Lie subsemigroup characterized in terms of Lie wedges [55], for which currently no efficient procedure is known to compute them in general. Although substantial progress has recently been made [56], by solving very simple models, the efficient determination of the reachable set of the controlled Lindblad master equation remains an open problem. However, accessibility of the Lindblad master equation is well studied [57] using the vector of coherence formalism and we will come back to it in Chap. 6. Regarding the control of open quantum systems it is furthermore not clear whether (2.3.1) is a valid description of an open system that is subject to coherent control. Unless we steer the system adiabatically [58], the approximations that lead to the Lindblad master equation have to be checked carefully. This brings us to the second approach.

2nd approach: one can tackle the problem already *before the derivation of a reduced dynamics*. That is we study the control system before the infamous "bath trace" is performed and then we make conclusions on the control properties of the open system. Within this approach one starts from a Hamiltonian description of system and environment

$$H(t) = H_S(t) \otimes \mathbb{1} + \mathbb{1} \otimes H_B + H_{S,B}, \tag{2.3.2}$$

where the controls enter in the system Hamiltonian $H_S(t)$. One can then apply the whole Lie algebraic framework to determine the unitary operations that can be implemented on the system, while interacting with the environment. Previous work focused on the numerical optimization of specific single-qubit transformations in the presence of a non-Markovian bosonic environment [59, 60]. The problem with a complete characterization of the control properties of such a system–environment dynamics is that it requires the investigation of infinite-dimensional systems, which is almost equally challenging as that of open systems described by a Lindblad master equation. For infinite dimensional systems the mathematics that is needed in order to characterize the reachable operations is much more intricate and the few results that exist are confined to systems with discrete spectra [61, 62, 63]. To tackle

this issue one can begin by studying the control properties of systems that interact with a finite set of environmental degrees of freedom. The studies [46, 47, 64, 65] investigated the numerical optimization of a specific control target on a spin coupled to a finite set of environmental spins. In particular in [64, 65] it was pointed out that decoherence induced by the environmental spins can be suppressed through control. However, the full characterization of the control properties of such open systems is missing and therefore still an open question. Furthermore, due to its apparent relevance for realistic applications, the scaling of control timescales with the size of the environment, when only the central system can be accessed by control fields, deserves extensive studies.

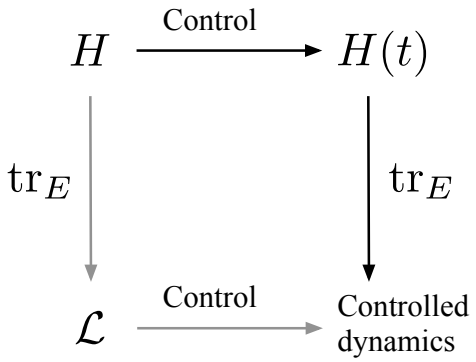


Figure 2.3.: Schematic representation of the two approaches for describing the control of an open quantum system. In the first approach (gray arrows) one first traces over the environment in order to obtain a description for the reduced dynamics in terms of a Lindblad master equation. Afterwards one adds the control fields coherently and studies the control properties of the controlled master equation. In the second approach (black arrows) one considers the full Hamiltonian dynamics (system+environment) and applies the time dependent control fields on the system, so that the total Hamiltonian becomes time dependent. Then one studies the control properties of the total system using methods from closed quantum systems. Afterwards one makes conclusions on the control properties of the system alone by considering the reduced dynamics.

In Fig. 2.3 we schematically summarized the two approaches. Either we study the control properties of system and environment on a Hamiltonian level and make afterwards conclusions on the control properties of the system by taking the partial

trace, or we investigate the control properties of the coherently controlled master equation. Since it is not clear whether the approximations that lead to a Lindblad master equation are valid in the presence of time dependent and unconstrained control fields, we begin in the next chapter by characterizing the control properties of a specific system-environment Hamiltonian. We determine the dynamical Lie algebra of the total system, so that afterwards we can conclude on the control properties of the system alone. In order to avoid the difficulties arising for infinite dimensional environments we consider an environment with finitely many degrees of freedom which is scalable. Moreover we study the minimum time needed in order to implement a target unitary gate on the system as a function of the size of the environment.

3 Control of open quantum systems: case study of the central spin model

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Following the second approach for describing the control of an open quantum system we have similarly to [64, 65] decided to examine the central-spin model [66, 67, 68, 69, 70, 86, 87, 88], where a central spin interacts with a finite set of surrounding environmental spins. Control is exerted through a classical field applied on the central spin. The central-spin model represents a finite but scalable system, whose thermodynamic limit is well defined and it is of experimental relevance even at small environmental sizes because it can be used to describe the main sources of decoherence in NV centers [74, 75, 76, 77] and quantum dots [78, 79, 80]. In addition the central-spin model has been subject to a series of studies concerning its reduced dynamics, so there is hope to bring the more theoretical and the more application-oriented research lines together on a practically relevant system. In this spirit here we consider a spin bath controlled through the central system. The novel aspect of this work lies in the complete characterization of the control properties of both the central system and the bath for two different cases of system-bath coupling. Known Lie algebraic methods that were introduced in Sec. 2.2.1 and successfully applied for other spin systems (see for example [81, 82, 83, 84]) are used to identify the

unitary operations that can be implemented, by acting on the central spin alone, both on the central spin and on the bath. In addition we employ extensive numerics, using an open source package developed by S. Manches et al. [45], to estimate the minimum time required to implement a unitary transformation on the central spin as a function of the number of environmental spins. This allows us to draw some relevant conclusions on the scaling of control timescales in a dissipative set-up.

3.1 The model

We consider a system consisting of a central spin surrounded by N spins as shown in Fig. 3.1, hereafter denoted as the spin star [1]. The spins surrounding the central spin

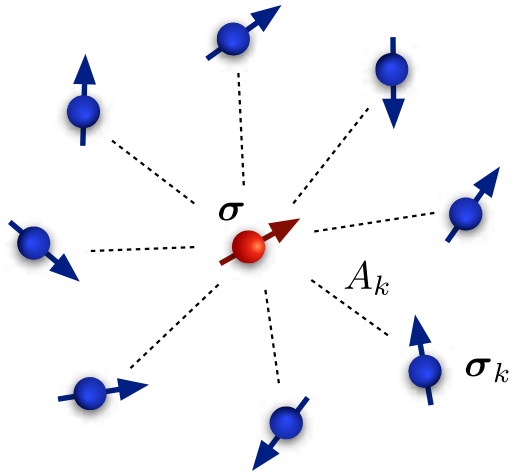

Figure 3.1.: The model described by Hamiltonian (3.1.1): a central spin (red) described by $\boldsymbol{\sigma}$ interacts via an isotropic Heisenberg interaction with N surrounding spins (blue) each described by $\boldsymbol{\sigma}^{(k)}$. The coupling between the system and the kth bath spin is given by A_k. The central spin interacts additionally with a classical control field as described by the Hamiltonian (3.1.4).

will be hereafter referred to as the *bath* spins keeping in mind that, strictly speaking, they represent a true spin bath only in the thermodynamic limit. We assume that the central spin interacts with the bath spins via an isotropic Heisenberg interaction

[1] The term spin star was first used by A. Hutton and S. Bose in [67] where a similar model involving only XY coupling was considered.

and that it is additionally subject to a constant magnetic field. The model is thus described by the following Hamiltonian

$$H_0 = \sigma_y + \sum_{k=1}^{N} A_k \boldsymbol{\sigma} \cdot \boldsymbol{\sigma}^{(k)}, \tag{3.1.1}$$

where A_k is the coupling between the central and the kth bath spin, and $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ and $\boldsymbol{\sigma}^{(k)} = (\sigma_x^{(k)}, \sigma_y^{(k)}, \sigma_z^{(k)})^T$ are the Pauli matrices acting on the central and the kth bath spin respectively. Due to the isotropy of the Heisenberg interaction, the specific choice of σ_y as the central spin Hamiltonian does not represent a loss of generality. Under the assumption of equal system-bath couplings, i.e. $A_k = A$ for each k, the dynamics of the central spin and the entanglement properties of this and similar models have been studied analytically in [67, 68, 69] by means of a non-Markovian master equation. The exact solution for the reduced dynamics in the different coupling case can be found in [86, 87, 88]. If all couplings are equal, in fact, the Hamiltonian (3.1.1) can be rewritten as a two-particle Hamiltonian

$$H_0 = i(\sigma_- - \sigma_+) + 2A(\sigma_- J_+ + \sigma_+ J_- + \sigma_z J_z), \tag{3.1.2}$$

where $\sigma_\pm = (\sigma_x \pm i\sigma_y)/2$ are the lowering and raising operators acting on the central spin and the bath is regarded as a single effective particle with angular momentum operator

$$\boldsymbol{J} = \frac{1}{2} \sum_{k=1}^{N} \boldsymbol{\sigma}^{(k)}, \tag{3.1.3}$$

and corresponding raising and lowering operators given by $J_\pm = J_x \pm i J_y$. The Hamiltonian (3.1.2) conserves the square of the bath angular momentum, i.e. $[\boldsymbol{J}^2, H_0] = 0$. Hence, noting $[J_z, \boldsymbol{J}^2] = 0$, simultaneous eigenstates of $\boldsymbol{J}^2$ and J_z represent a convenient basis for the bath. However, since the operators $\boldsymbol{J}^2$ and J_z alone do not form a complete set of commuting observables, the subspaces defined by their eigenvalues, denoted by j and m respectively, are not in general one-dimensional. We therefore introduce an additional quantum number ν corresponding to the eigenvalues of certain permutation operators acting on the bath spins and commuting with H_0. The

permutation operators do not need to be specified as the controllability analysis is independent of them. Due to the conservation of j and ν, the bath Hilbert space can be written as a direct sum of the subspaces $\mathcal{H}_{j,\nu}$ and the total Hilbert space can be written as $\mathcal{H} = \mathcal{H}_S \otimes (\bigoplus_{j,\nu} \mathcal{H}_{j,\nu})$ where $\mathcal{H}_S$ is the Hilbert space of the central spin. This Hilbert space structure, as detailed in the following section, lies at the heart of the spin-star controllability properties in the equal coupling scenario.

Having defined the model Hamiltonian H_0, we now move on to introduce controls. As discussed in the beginning of this chapter, we assume that only the central spin can be accessed to and controlled. In order to obtain non-trivial dynamics, the control field acting on the central spin must not commute with H_0. A convenient choice is therefore represented by a classical magnetic field $B(t)$ along the z direction as described by the control Hamiltonian

$$H_c(t) = B(t)\sigma_z. \tag{3.1.4}$$

The full Hamiltonian is thus

$$H(t) = H_0 + H_c(t). \tag{3.1.5}$$

Despite representing quite an extreme simplification, still the spin-star model described by Eq. (3.1.1) already captures some relevant features of the spin-bath decoherence processes occurring in solid-state systems used for the implementation of quantum technologies such as nitrogen vacancy centers [74, 75] and quantum dots [78, 79, 80], although the interactions are here highly anisotropic [2]. The spin-star model therefore represents an interesting and challenging playground for an investigation of controllability of open systems which can also be of practical relevance.

[2] For an anisotropic Heisenberg interaction the proof A.2.2 for full controllability of the central spin still holds for almost all choices of the anisotropy parameters. The proof is based on the non-vanishing determinant of the Vandermonde matrix which is an analytic function and hence for most of the choices of the parameters different from zero.

3.2 Controllability considerations

We now focus on the investigation of which unitary transformations can be implemented on the spin star, in particular on the central spin, using the control field $B(t)$. The dynamics is governed by the Schrödinger equation for the time evolution operator

$$\dot{U}(t) = -i(H_0 + H_c(t))U(t), \qquad U(0) = \mathbb{1}, \tag{3.2.1}$$

where the drift Hamiltonian H_0 and the control Hamiltonian H_c are those given in Eqs. (3.1.1) and (3.1.4). All unitary operations which can be implemented on the system constitute the reachable set $\mathcal{R}$, which was introduced in (2.2.4). Remember that the the closure $\bar{\mathcal{R}}$ of the reachable set consists of the unitaries which can be achieved with arbitrary high precision. It is equal [9] to the Lie group $e^{\mathfrak{L}}$, where $\mathfrak{L} = \mathfrak{Lie}(iH_0, iH_c)$ is the dynamical Lie algebra spanned by real linear combinations and nested commutators of iH_0 and iH_c. We recall that the system is said to be fully controllable if the Lie group is equal to the unitary group or, in our case of traceless Hamiltonians, to the special unitary group [11, 41]. To analyze the controllability of the spin star we thus need to calculate the associated dynamical Lie algebra.

Without bath spins, i.e. for $N = 0$, the central spin is fully controllable because $[i\sigma_y, i\sigma_z] = 2i\sigma_x$ and $\mathfrak{L} = \mathfrak{su}(2)$. When $N > 0$ it is no longer obvious whether the central spin is fully controllable or not: on the one hand H_0 is necessary to achieve rotations around the x axis, on the other the interaction with the bath spins introduces noise on the central spin. We will therefore study how the bath influences the controllability of the central spin. The controllability of similar spin star models that consists of an anisotropic interaction of the central spin with the bath spins was studied in [83, 84]. Two classical fields were used to control the central spin and it was shown, by using the graph criterion [85], that then the whole system becomes controllable. However this method is based on finding the eigenstates of the system and therefore it is not applicable for the general Hamiltonian (3.1.1) when all coupling constants are different from each other. Moreover, in this case

the controllability of the central spin is trivial, and can in principle be achieved arbitrarily quickly through strong control fields, which means that such models are not relevant in the context of the present study.
In the following we will first consider the case when the central spin couples with the same coupling strength to each bath spin and then the case when the couplings are all different from each other.

3.2.1 Equal couplings

As discussed in Sec. 3.1, when the central spin couples to each bath spin with the same strength, the bath spins behave like a collective spin described by the angular momentum operator (3.1.3) whose square is conserved. Since the control Hamiltonian (3.1.4) acts only on the central spin, this symmetry is conserved also in presence of the control field, thus implying that the spin star is not fully controllable (see also [89]). However, by performing repeated commutators of iH_0 and iH_c and taking their real linear combinations, we can obtain the operators $i\sigma_\alpha$, iJ_α and $i\sigma_\alpha J_\beta$ with $\alpha, \beta = x, y, z$ (see App. A.1 for details). This implies that the full $\mathfrak{su}(2)$ algebra acting on the Hilbert space of the central spin is contained in the dynamical Lie algebra regardless of the number of bath spins. The central spin is thus fully controllable even in presence of decoherence or, in other words, the noise induced on the central spin as a result of the interaction with the bath can be effectively switched off. More generally, the dynamical Lie algebra for equal couplings contains all elements of the form (see App. A.1)

$$i\sigma_\alpha(J_+^l J_-^k J_z^s + h.c.), \quad \alpha = x, y, z \;\; l, k, s \in \mathbb{N}_0. \tag{3.2.2}$$

Equation (3.2.2) implies full controllability of the spin star within each subspace $\mathcal{H}_S \otimes (\bigoplus_\nu \mathcal{H}_{j,\nu})$ which can be achieved by properly combining the operators J_+^l and J_-^k in such a way to act only on a given j-subspace. Even without full controllability it is still possible to perform many interesting and practically relevant operations on the spin star such as entangling the central spin with the bath or using the bath

as a data bus. Such protocols were recently experimentally demonstrated in [90]. The dimension of the dynamical Lie algebra can be obtained by determining the size of the subspaces of fixed ν [67] as $\dim(\mathfrak{L}) = \sum_j((2(2j+1))^2 - 1)$. For a given N, j can only take the values $j = 1/2, 3/2, \ldots, N/2$ when N is odd and $j = 0, 1, \ldots, N/2$ when N is even, we obtain

$$\dim(\mathfrak{L}) = \begin{cases} \frac{1}{6}(2+N)(9+4N(4+N)), & \text{for } N \text{ even}, \\ \frac{1}{6}(1+N)(3+2N)(7+2N), & \text{for } N \text{ odd}, \end{cases} \tag{3.2.3}$$

which shows that the dimension of the dynamical Lie algebra scales polynomially, $\propto N^3$, with the size of the bath.

3.2.2 Different couplings

In the previous section we learned that in the equal-coupling case the central spin is fully controllable but, due to the symmetries of the system, the whole spin star is not. The situation changes if all coupling constants A_k are different from one another. In this case the system has no more symmetries and the bath spins do not behave like a collective spin anymore. Full controllability of the central spin still holds for almost all choices of the coupling constants and is independent of both the size and the initial state of the bath, (see App. A.2.1). In addition each single bath spin is fully controllable, (see App. A.2.2), thus allowing us to write

$$\sigma_\alpha^{(k)} \in \mathfrak{L}, \quad \forall k = 1, \ldots N, \quad \alpha = x, y, z. \tag{3.2.4}$$

Hence, due to the Heisenberg interaction between the central and the bath spins, full controllability of the spin star $\mathfrak{L} = \mathfrak{su}\left(2^{N+1}\right)$ is achieved [92]. As a consequence, the dimension of the dynamical Lie algebra scales exponentially with the bath size. By acting with a control field on the central spin alone all degrees of freedom, even the unaccessible ones, can be used for quantum information tasks.

3.2.3 Implementing CPTP maps

An interesting generalization of the above is to consider the ability to implement completely positive trace preserving (CPTP) maps, which were introduced in Sec. 2.1.2, on the central system. This is especially relevant in view of the growing interest towards open quantum system simulators [93, 94, 95, 96] and quantum reservoir engineering [4, 97]. We find that arbitrary CPTP maps $\mathcal{E}(\rho_S)$ can be implemented: first, let us consider the unequal coupling case with $N \geq 2$. We initialize two spins of the bath in a pure state ϕ_B through consecutive unitary operations and measurements on the central spin. Using controls we then implement the unitary U of the Stinespring representation, see Sec. 2.1.2, $\mathcal{E}(\rho_S) = \mathrm{tr}_{12}\left\{U(\rho_S \otimes \phi_B)U^\dagger\right\}$ of $\mathcal{E}$, and thus $\mathcal{E}$. Second, for equal couplings even though the whole system is not fully controllable it is still possible to implement every unitary operation within the subspaces $\mathcal{H}_S \otimes (\bigoplus_\nu \mathcal{H}_{j,\nu})$. Provided they are large enough ($j > 3/2$, implying $N > 3$) and provided the bath can be initialized appropriately, we can again implement a Stinespring dilation of $\mathcal{E}$.

3.2.4 Numerical calculation of the dynamical Lie algebra

In this section we will examine more in detail the structure of the dynamical Lie algebra, $\mathfrak{L}$, using a numerical algorithm similar to those discussed in [10] and [43]. In order to obtain a complete operator basis for $\mathfrak{L}$ it is enough to repeatedly compute the commutators with iH_0 and iH_c, until the rank of $\mathfrak{L}$ does not increase any further [10]. Such a procedure can be visualized as a tree, the so-called Lie tree. Indeed in Fig. 3.2 we show the Lie tree of a spin star with $N = 2$ bath spins for both equal, a), and different, b), couplings. The numbers inside the circles label the elements of $\mathfrak{L}$ starting with iH_c and iH_0 which correspond to 1 and 2. The blue/red branches indicate that the new element was obtained by commutation with iH_c/iH_0 respectively. The number k denotes the depth of the tree nodes starting with $k = 1$ for $[iH_c, iH_0]$. More generally, we define the depth of an element of $\mathfrak{L}$ as the

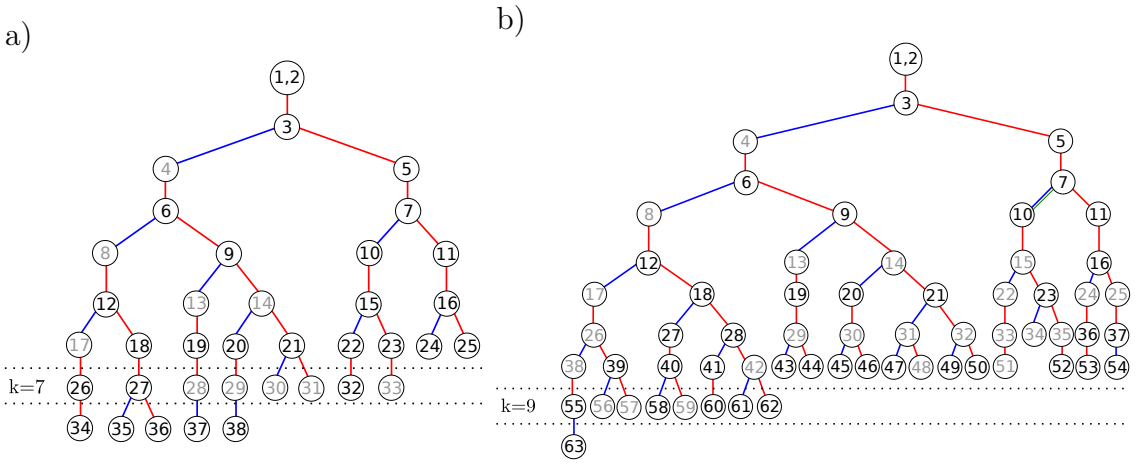

Figure 3.2.: Tree structure of the dynamical Lie algebra for $N = 2$ bath spins and either equal (a)) or different (b)) couplings. The numbers in the circles represent the elements of the dynamical Lie algebra and the branches indicate whether the new linearly independent term was obtained by commutation with iH_c (blue) or iH_0 (red). The index k indicates the depth of the commutator. Numbers in gray denote the elements which have non-zero overlap with $i\sigma_x$ on the central spin.

maximal depth of nodes required to express it via linear combinations. Although the tree structure is not unique, because it depends on the order according to which commutators are performed, using the Jacobi identity it can be shown that the depth of an element is independent of the specific tree structure. To achieve full controllability of the central spin the crucial element to be obtained is $i\sigma_x$. In order to determine its depth, we highlight nodes in gray corresponding to the basis elements that have non-zero overlap with $i\sigma_x$.

By comparing panels a) and b) of Fig. 3.2, where the depth of σ_x is $k = 7$ and $k = 9$ respectively, we can conclude that the value of the couplings, i.e. the presence of symmetries of the drift Hamiltonian, affects the depth at which full controllability of the central spin is achieved. In both cases the tree structure is rather rich which is reflected by the complex proof of the central-spin full controllability presented in the appendix.

In the equal coupling case the depth of $i\sigma_x$ is upper bounded by 24 since it can be obtained in fashion independent of the bath size (see App. A.1). In contrast,

for different couplings, the depth of $i\sigma_x$ in the proof App. A.2.1 indicates a linear scaling with the bath size. However, this only represents an upper bound on the scaling because a different proof might exist yielding a lower depth. By considering a perturbation expansion of the time-evolution operator, it is tempting to conjecture that the depth of an element of the dynamical Lie algebra is related to the minimum time required to achieve its unitary companion. Unfortunately we do not have enough numerical data to decide this conjecture and leave it as an open problem for future studies.

3.3 Influence of the bath on the minimum gate time

So far we have discussed which unitary transformations can be implemented *in principle* on the spin star by a generic control field $B(t)$. By this we mean that no explicit statement is made about the time required to achieve the desired unitary. In practice, we not only need to reach the desired unitary but we need to do so in a reasonable time. Therefore we now to turn to the question of how the minimum time, $T^*(U_G)$, required to implement a target unitary transformation, U_G, on the central spin (hereafter minimum gate time) scales with the number of bath spins. To do this we need to identify the control pulse allowing to implement U_G in the shortest time possible for different numbers of bath spins. To this end we need to resort to numerical gate optimization.

We used the Grape algorithm [44] as outlined in Sec. 2.2.2 and implemented in the open source optimal control package DYNAMO [98]. The algorithm uses a gradient based method that maximizes the following gate fidelity

$$f_1(\tau) = \left| \frac{1}{2^{N+1}} \mathrm{tr}\{U_G^\dagger U(\tau)\} \right|^2 , \tag{3.3.1}$$

given by the modulus square of the normalized overlap, at a given time τ, between the target transformation, U_G, and the actual evolution, U. The fidelity (3.3.1)

involves choosing a target unitary operation acting on the whole spin star. However, since we are interested in implementing unitary transformations on the central spin alone (and in general we cannot access the bath degrees of freedom), such a choice is somewhat arbitrary and limiting. In an open system set up a better and more motivated fidelity measure is therefore given by [99]

$$f_2(\tau) = 1 - \lambda \min_V ||U_G \otimes V - U(\tau)||^2, \tag{3.3.2}$$

where U_G is the target unitary on the central system, V a generic unitary on the bath, U is the actual evolution at time τ of the full system and λ is a normalization constant. Due to the minimization over all the unitaries acting on the bath, as opposed to the fidelity in Eq. (3.3.1), the fidelity defined in Eq. (3.3.2), reaches its maximum if the goal transformation has been implemented on the central system, regardless of the bath evolution. Using the Frobenius norm and choosing $\lambda = 1/(2 * 2^{N+1})$, the minimization can be carried out explicitly yielding [46]

$$f_2(\tau) = \frac{1}{2^{N+1}} \mathrm{tr}\{\sqrt{Q^\dagger Q}\}, \tag{3.3.3}$$

with $Q = \mathrm{tr}_S\{(U_G \otimes \mathbb{1}_{\mathrm{bath}})^\dagger U(\tau)\}$, and $\mathrm{tr}_S\{\cdot\}$ the partial trace over the central spin degrees of freedom.

After having included the gate fidelity f_2 into the DYNAMO package, we have performed the optimization of f_1 and f_2 by using the exact gradient formula developed in [46, 47]. The time τ, from now on called the driving time, has been divided into M equidistant time intervals $\Delta t = 0,05$ chosen to be smaller than the inverse of the highest eigenvalue of the Hamiltonian (3.1.1) to ensure a proper resolution of the dynamics. For a given bath size, in order to estimate the minimum gate time T^*, we have optimized both figures of merit $f_1(\tau)$ and $f_2(\tau)$ for different values of τ. We additionally optimized over randomly chosen initial pulses meaning that at each τ the maximum value of the fidelity over the different realizations is taken. An additional optimization over many initial pulses is performed to minimize the effect of local minima in the numerical routine. Finally we have rescaled the coupling constants A_k by a factor $1/\sqrt{N}$. Such choice follows the considerations made in [68, 69]

about the thermodynamical limit of the central-spin and similar models in the equal coupling case. Different choices of renormalizations are possible depending on the specific process being modeled [3]. However, since we expect the minimum time to be related to the Lie algebraic properties of the model, hence to be independent of parameter details, we believe it is reasonable to assume that the different renormalization choices do not affect the scaling of the minimum time in the different and equal coupling case.

We emphasize that our numerical calculations can only provide upper bounds to the minimum gate time, because the choice of initial control field can affect the time at which the given fidelity reaches a predetermined threshold value.

3.3.1 Optimizing f_1

We begin with the optimization of the fidelity $f_1(\tau)$ defined in Eq. (3.3.1) and choose the identity as the target unitary on the bath. As a target transformation U_G on the central system we consider both the Hadamard gate and the $\pi/8$ gate since these one-qubit gates form a universal set [100]. We begin with equal couplings, set $A = 1$, and investigate the minimum time required for the implementation of the Hadamard gate (Fig. 3.3 panel a)) and the $\pi/8$ gate (Fig. 3.3 panel b)) on the central spin. In Fig. 3.3 we plot the maximum value of $f_1(\tau)$ as a function of the driving time τ for different number of bath spins N and maximized over 200 randomly chosen initial pulses. Points that seem to break the continuity of the curves are statistical fluctuations and have no physical relevance as confirmed by optimizations over a higher number of initial pulses. The computational effort required by the optimizations is intensive which is the reason why, when optimizing f_1, we restricted ourselves to a statistical sample of 200 random initial pulses for each time and, when optimizing f_2, to 500. The black curve corresponds to $N = 0$ whereas the other curves to increasing values of N: in panel a) $N = 1, \ldots, 7$ and in

[3] For hyperfine coupling the coupling constants A_k are rescaled by a factor $1/N$ because A_k is proportional to the absolute square of the electron wave function that is normalized over a volume containing N nuclei.

panel *b*) $N = 1, \ldots, 5$. We observe the following:

1) Short time behavior in the equal coupling case: for $\tau = 0$ we have obtained $f_1(0) = 0$ for the Hadamard gate and $f_1(0) = (2 + \sqrt{2})/4$ for the $\pi/8$ gate. The plots show the bath detrimental effect on gate optimization on short time scales. Indeed after an initial extremely short time window where all curves exhibit the same increasing behaviour, reflecting the fact that correlations between the central system and the bath have not been established yet, the maximum value of the fidelities in presence of the spin bath then drops compared to the $N = 0$ case. Note that for short times the $\pi/8$ gate can be reached with fidelities above 0.99 independently of the number of bath spins. This reflects the fact that the $\pi/8$ gate is up to a global phase identical to a rotation around the z axis which can always be achieved at short times with a sufficiently large control-field amplitude.

2) Long time behavior in the equal coupling case: after a region of decreasing slope, all dissipative fidelities increase again until, for all N, a maximum value above 0.995 is reached. The increasing bath size results in a time shift of the maximum value. The achieved maximum values are the same for both the Hadamard and the $\pi/8$ gate.

3) Different coupling case: to study the effect of the bath spins in more detail we will from now on focus only on the optimization of the Hadamard gate on the central spin.

Fig. 3.4 shows the maximum value of the fidelity as a function of the driving time in the different coupling case. The couplings are randomly chosen from a uniform distribution between 1 and 2. The curves have been obtained for $N = 1, \ldots, 3$ bath spins. As before, a maximum value above 0.995 is reached for all bath sizes but the driving time needed to reach it is much longer with respect to the equal coupling case.

4) Estimation of T^*: in Fig. 3.5 we plot the estimated minimum gate time T^* against the number of bath spins for different and equal couplings. Our estimate has been obtained by setting a threshold value for the fidelity $f_1 = 0.995$ and extracting the corresponding T^* from the data plotted in Fig. 3.3 and 3.4. The

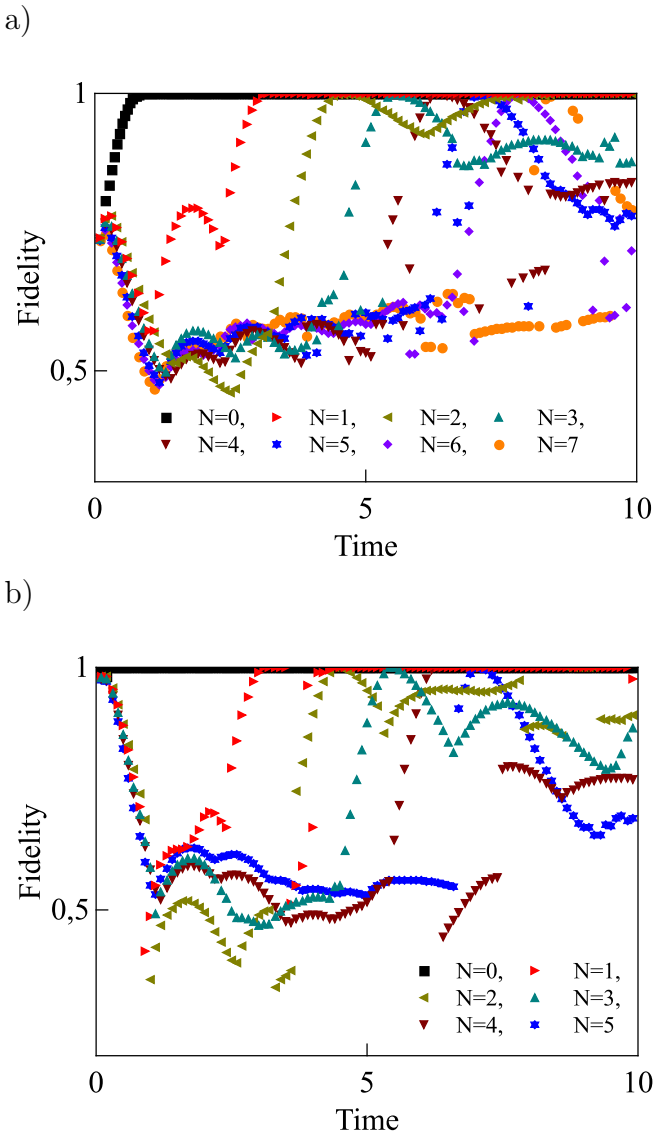

Figure 3.3.: Maximum value of the fidelity $f_1(\tau)$ as a function of the driving time τ extracted from 200 random initial pulses for equal couplings and target unitary a) Hadamard on the central spin and identity on the bath; b) $\pi/8$ on the central spin and identity on the bath. Both plots have been obtained for different numbers N of bath spins as indicated on the figure.

inset shows the minimum gate time versus the number of bath spins for different couplings on a logarithmic scale. It should be mentioned here that the point that belongs to $N = 4$ for different couplings was obtained by searching only in the expected time window for a fidelity above the mentioned threshold. Furthermore, as already mentioned, our results can only provide an upper bound on T^*. Nevertheless

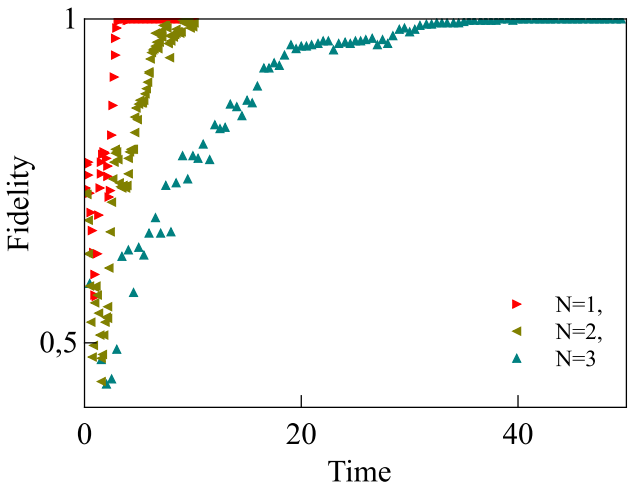

Figure 3.4.: Maximum value of the fidelity $f_1(\tau)$ as a function of the driving time τ extracted from 200 random initial pulses for different couplings and bath size $N = 1, \ldots, 3$.

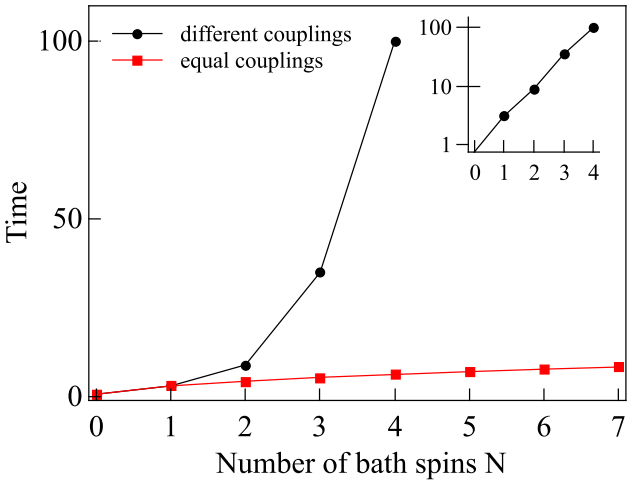

Figure 3.5.: Minimum gate time T^* needed to reach a value of the fidelity of at least $f_1(\tau) = 0.995$ as a function of the number of bath spins N for both different and equal couplings. The inset shows the curve for different couplings on a logarithmic scale.

Fig. 3.5 clearly suggests a significantly different scaling behaviour of the minimum gate time in the two different coupling regimes. In the equal coupling case, when the whole system is not fully controllable, the gate time seems to depend weakly on the number of bath spins (red curve) in strong contrast with the fully controllable case

(black curve) where the dependence on the bath size is at least polynomial (black curve). Consistently with our controllability analysis, the scaling of the minimum gate time suggests that in the equal coupling case the decoherence affecting the central spin can be suppressed in reasonable time regardless of the size of the bath. On the other hand, in the fully controllable case, for higher number of bath spins (dramatically) longer gate times can be expected. This seems consistent with the intuition that if the dimension of the Lie Algebra grows exponentially with N, then the implementation of a generic element of the corresponding Lie group requires an exponentially increasing time.

3.3.2 Optimizing f_2

Until now we have investigated the scaling of the minimum gate time by optimizing f_1 and choosing the identity as a target operation on the bath. We now want to see whether the optimization of f_2, Eq. (3.3.3) exhibits significant deviations from this behaviour. Unfortunately the optimization of f_2 with the GRAPE algorithm resulted extremely sensitive to local minima, especially for increasing number of bath spins, consistently also with the results presented in [46]. In order to minimize this effect a much higher number of random initial pulses was required. Due to computational restrictions, we had therefore to limit our investigation to $N = 2$ and 500 random initial pulses.

In Fig. 3.6 the maximum value of the fidelities $f_1(\tau)$ (dark yellow curve) and $f_2(\tau)$ (orange curve) is plotted as a function of the driving time τ for both equal (panel a)) and different couplings (panel b)). Intuitively we would expect a shorter minimum gate time when the target transformation is specified only on the central spin because in this case the constraint on the bath evolution is weaker. Each control pulse maximizing $f_1(\tau)$ is a specific solution for $f_2(\tau)$ as well, hence the fidelity $f_2(\tau)$ should at least attain the same maximum values as $f_1(\tau)$. However, from Fig. 3.6, we see that values of the maxima reached by f_2 around T^* are slightly below those reached by f_1, thus witnessing an increased sensitivity of the optimization to local

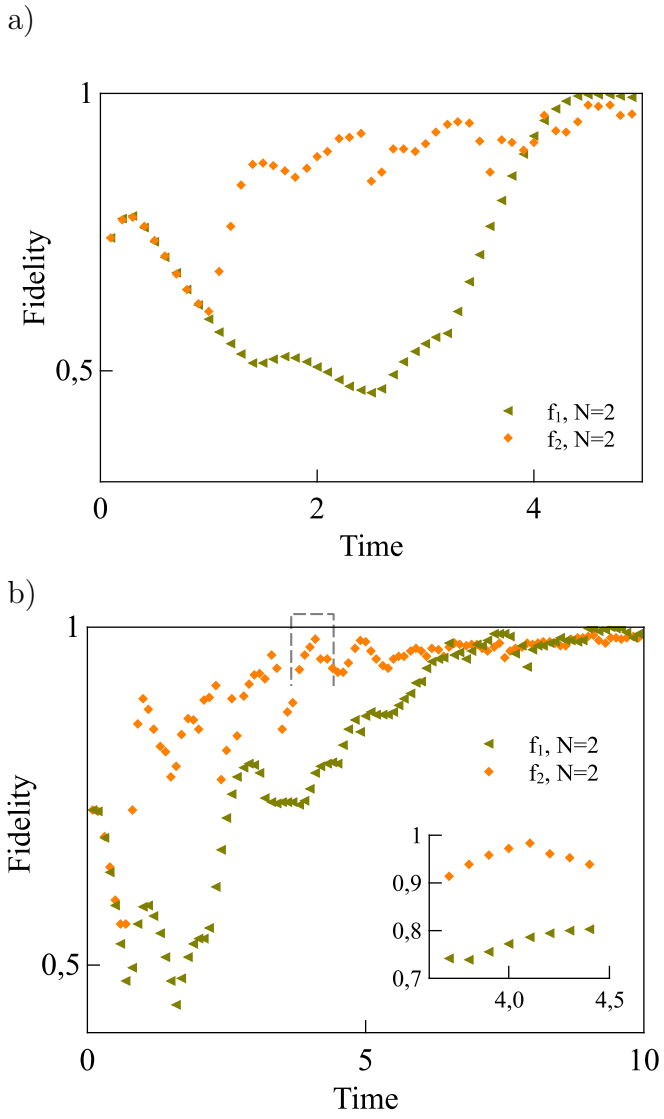

Figure 3.6.: Maximum values of the fidelities $f_1(\tau)$ and $f_2(\tau)$ as a function of the driving time τ extracted from 500 randomly chosen initial pulses for $N = 2$: *a*) equal couplings, *b*) different couplings where the inset shows the time window $\tau \in [3.7, 4.4]$ and the maximum was extracted from 10^4 random initial pulses.

minima within this time window. From Fig. 3.6 we also note that up to a certain time the curves relative to $f_2(\tau)$ and $f_1(\tau)$ are identical, thus implying that within this time window there is no difference between setting the target on the full system or on the central spin only. This behavior however changes at increasing times since higher fidelities can be achieved on shorter timescales if the target is only set on

the central spin. Only at the end of the time window the curves seems to become similar again. However, for equal couplings, even though f_2 reaches higher values at shorter times, it never crosses the threshold of $f_2 = 0.995$ before T^* thus leading us to conclude that in this case, for sufficiently high threshold values, the fidelity used does not significantly affect the estimate of the upper bound on the minimum gate time. For different couplings, instead, values that are close to the threshold can be reached at short times: the inset plot shows a time window in which $f_2(\tau)$ reaches a maximal value of 0.98.

In conclusion, for equal couplings, the numerical results suggest that the minimum gate time depends weakly on the size of the bath and perhaps reaches a saturation value. This is consistent to the theoretical prediction that the depth of an element of the dynamical Lie algebra is related to the minimum gate time to achieve its unitary companion (see Sec. 3.2.4). This behavior appears to be the same for both fidelities. For different couplings, instead, the numerical results hint at a different behaviour of the minimum gate time according to whether the target is defined on the whole system or on the central spin only. In the latter case in fact not only the minimum gate time seems to be shorter but also we can not even rule out the possibility that it scales as in the equal coupling case. More conclusive statements require much bigger computational resources and more sophisticated analytical techniques as for example those suggested in [101, 102]. However these methods are not easily generalized to high dimensional systems considered here.

3.4 Conclusions

By analytical calculation of the dynamical Lie algebra, we showed that a central spin interacting with a surrounding spin bath is fully controllable for almost all choices of the coupling constants and any bath size. If the central spin couples to the bath with unequal couplings, this property extends to the whole spin star, environmental spins included. To our knowledge, this is the first explicit example of a system that is universal for quantum computation using only a single control

field. We can therefore conclude that quite remarkably, by controlling the central system, the bath can be i) effectively switched off; ii) arbitrarily engineered. The possibility of controlling the environment via the central spin can be exploited to implement, on the central spin itself, not only arbitrary unitaries but, more generally, arbitrary (completely positive trace preserving) dynamical maps. This result can be of practical relevance both for quantum simulations of open system and for quantum reservoir engineering.

Alongside these purely analytical findings we also performed an extensive numerical investigation of control timescales and how these are affected by both the bath size and the symmetries of the system. In the maximally symmetric scenario, when all the bath spins can be regarded as a single collective particle, our estimate for the minimum time required to perform a gate under dissipative dynamics shows that it scales relatively slow, perhaps reaching a saturation value, as a function of the bath size. On the contrary, in absence of symmetries, i.e. when each environmental spin interacts differently with the central system, the scaling of the minimum gate time appears to be much faster (we conjecture exponentially faster).

Our results might have interesting applications in NV centers, which are essentially electron spins in a finite nuclear spin bath. One recently demonstrated method to overcome the short coherence time of the electron spin is to store its state in the nuclear spins, which have longer decoherence times. Our control results then suggest that this might not be the best strategy, and that instead one might apply a more complex shaped pulse to the electron spin to keep it fresh for longer.

4 Dynamical decoupling and decoherence

In the previous section we saw that the modulation of a classical control field allows us to implement quantum gates in an open system setting in a noiseless manner. Decoherence caused by the interaction with the environment can be circumvented by shaping the control pulses according to the underlying system-environment model and the target gate one wants to implement. The possibility to fully control the central system through one control field in the presence of an environment relies on the system-environment interaction that is present. To numerically calculate the control pules, in order to implement a target operation, we need full knowledge of the form of the open system Hamiltonian and the coupling constants that are contained in it. Additionally there might be other environmental degrees of freedom that couple to the system of interest, yielding an additional noise source for the implementation of quantum gates through the calculated pulses. Sometimes only the protection of a quantum system from decoherence, rather than the additional implementation of gates, is of main interest in order to keep quantum features like entanglement and coherent superpositions alive. In other words, one is interested in a strategy that suppresses the interactions with environment, regardless of how the interactions appear, so that the quantum system effectively evolves unitarily or does not evolve at all. Such a strategy is dynamical decoupling.

Dynamical decoupling is a highly successful strategy to protect quantum systems from decoherence [3]. Its particular strength is that it is applicable even if the de-

tails of the system-environment coupling are unknown. The idea is to rapidly rotate the quantum system by means of classical fields to average the system-environment coupling to zero. In this respect dynamical decoupling can be regarded as a specific instance of quantum control. Historically dynamical decoupling dates back to pioneering work in nuclear magnetic resonance (NMR) by U. Haeberlen and J. S. Waugh [103]. They developed pulse sequences in order to increase the resolution in NMR spectroscopy by coherently averaging out interactions [104]. Prominent examples are spin-echo techniques, such as the famous Hahn-echo [105], allowing us to measure relaxation times through applying a sequence of rotations on a spin and measuring the echo signal. In the context of suppressing decoherence and quantum information theory the theoretical framework was developed by L. Viola and S. Lloyd [106, 107] in the late 90's. Over the years the efficiency of different decoupling schemes was studied and improved for several environmental models in [108, 109, 110, 111, 112, 113]. Many experiments, such as [114, 115, 116], demonstrate the applicability of dynamical decoupling in an impressive way by prolonging coherence times a few orders of magnitude. Additionally dynamical decoupling can be combined with the implementation of quantum gates, which makes it a viable option to error correction [117, 118].

The chapter is organized as follows. We start by introducing the concept and the terminology of dynamical decoupling, embedding it afterwards into the framework of control theory. This will lead to the question of how fast we have to apply dynamical decoupling to effectively suppress decoherence, which will be the subject of Sec. 4.1.1. Since we want to analyze dynamical decoupling also for infinite dimensional environments in the next section, we give in Sec. 4.1.2 a brief introduction into the mathematical-terminology and the difficulties concerning unbounded operators. Based on dynamical decoupling we present afterwards a method to distinguish decoherence from intrinsic noise terms, for instance arising from collapse models that were introduced in Sec. 2.1.5. Usually dynamical decoupling is introduced and formulated for finite dimensional quantum systems interacting with an environment. We finish the chapter by developing in Sec. 4.3 a general framework for continuous

variable systems that are described by quadratic Hamiltonians (2.2.13).

4.1 The concept

To begin with we consider again the simple one qubit decoherence model from Sec. 2.1.4, given by the system-environment Hamiltonian

$$H_{S,B} = \sigma_z \otimes B. \tag{4.1.1}$$

Imagine we are able to perform instantaneously $\pi/2$ rotations of the qubit around the x-axis described by the unitary *decoupling operation* $\sigma_x \otimes \mathbb{1} \equiv \sigma_x$. Note that with such a decoupling operation we can reverse the sign in front of the system-environment Hamiltonian, i.e. $\sigma_x H_{S,B} \sigma_x = -H_{S,B}$. Then, by applying the decoupling operation, letting the system evolve under $H_{S,B}$ for a time Δt and applying the decoupling operation again, we can switch off the system-environment interaction

$$e^{-iH_{S,B}\Delta t}\sigma_x e^{-iH_{S,B}\Delta t}\sigma_x = \mathbb{1}, \tag{4.1.2}$$

for times $2\Delta t$, that is, the total system does not evolve anymore. We have chosen the decoupling operation in such a way that the sign in front of the system-environment Hamiltonian can be reversed. This requires some knowledge about the form of the system-environment Hamiltonian. Clearly a system-environment interaction which is described by the central spin model (3.1.1) from the last section cannot be suppressed in this way. Moreover the rotated Hamiltonian $\sigma_x H_{S,B} \sigma_x$ commutes with the original one, so that decoherence can be suppressed for all times. For a general system-environment Hamiltonian

$$H = H_S \otimes \mathbb{1} + \mathbb{1} \otimes H_B + \sum_\alpha S_\alpha \otimes B_\alpha, \tag{4.1.3}$$

this is generally not the case since higher order terms, including commutators of the rotated Hamiltonians, enter in the dynamics. As we will see below, one particular strength of dynamical decoupling is the fact that generic system-environment interactions can always be suppressed by choosing the decoupling operations properly.

Before we start formulating dynamical decoupling in the framework of quantum control theory we first want to discuss the essence why dynamical decoupling works for generic interactions. In the following we consider a system that consists of N qubits with Hilbert space $\mathcal{H}_S = \mathbb{C}^{2^N}$ and for the moment we assume that the environment is finite dimensional. We take unitary decoupling operations v from a finite set V of unitaries with $|V|$ being the number of elements in the set. The dynamics is modified by applying the decoupling operations instantaneously in time intervals $\Delta t = \frac{t}{n}$, such that within one decoupling cycle, consisting of transversing through the elements of V, the evolution is modified according to

$$U_{\text{cyc}} = \prod_{v \in V} v^\dagger e^{-iHt/n} v. \tag{4.1.4}$$

Throughout this chapter we consider perfect decoupling operations, while bounds for the non-perfect case can be found for example in [119, 120, 121]. If we repeat this cycle many times we obtain, using a generalized Trotter formula [122],

$$U(t) = \lim_{n\to\infty} U_{\text{cyc}}^n = \exp\left[-it \sum_{v\in V} \left(v^\dagger H_S v \otimes \mathbb{1} + \mathbb{1} \otimes H_B + \sum_\alpha v^\dagger S_\alpha v \otimes B_\alpha \right)\right], \tag{4.1.5}$$

where the right hand side involves a map of the form

$$\Pi(x) := \frac{1}{|V|} \sum_{v\in V} v^\dagger x v. \tag{4.1.6}$$

We remark here that throughout this chapter we are going to study the properties of such maps arising in different contexts. We proceed and notice that under the assumption that V is a group, Π maps all $x \in B(\mathcal{H}_S)$ onto the commutant $V' = \{A \mid [A, v] = 0,\ \forall v \in V\}$. Now, if we define a proper *decoupling set* V as an irreducible representation of a finite group of unitary operations v, then we have, by Schurs Lemma (see e.g [39]), that the commutant is trivial, $V' = \mathbb{C}\mathbb{1}_S$, and consequently

$$\Pi(x) = \frac{\text{tr}\{x\}}{\dim(\mathcal{H}_S)} \mathbb{1}_S, \quad \forall x \in B(\mathcal{H}_S), \tag{4.1.7}$$

so that

$$\rho_S(t) = \mathrm{tr}_B\{U(t)(\rho_S(0) \otimes \rho_B(0))U^\dagger\} = \rho_S(0). \tag{4.1.8}$$

This shows that decoherence can always be suppressed by choosing the decoupling set V to be a finite irreducible representation of the unitary group. Note that for spin systems the Hamiltonian is traceless and hence the righthand side of (4.1.7) is zero. For a single qubit such a decoupling set is given by the Pauli group $V = \{\mathbb{1}, \sigma_x, \sigma_y, \sigma_z\}$, whereas for N qubit systems it consists of 4^N combinations of the Pauli spin operators on the tensor factors.

Up to this point we have considered a decoupling sequence of the form U_{cyc}, yielding in the limit of infinitely fast decoupling ($\Delta t \to 0$) a dynamics governed by $\Pi(H)$. Therefore, if the map Π acts irreducibly, the resulting dynamics is given by a global phase or, in the case where the system operators are all traceless, by the identity. This is the essence of dynamical decoupling. In the following we will see that a decoupling sequence of a different form, $v_1 \exp(-iH\Delta t)v_2 \exp(-iH\Delta t)v_3$, can always be brought into the form U_{cyc}. This can be achieved by transforming the dynamics into a frame in which the action of the decoupling operations alone is separated from the modified dynamics.

Usually dynamical decoupling is introduced within the framework of control theory. In order to link it to quantum control we will outline the connections below, following the presentation in [3]. In spirit of the second approach for controlling open quantum systems (see Sec. 2.3) we consider a total Hamiltonian of the form

$$H(t) = H_0 + H_c(t) \otimes \mathbb{1}, \tag{4.1.9}$$

where H_0 is the Hamiltonian of the open quantum system (4.1.3), the controller $H_c(t) = \sum_i u_i(t)H_i$ contains the control Hamiltonians only acting on the system, and the control propagator is defined as

$$U_c(t) = \hat{T} \exp\left(\int_0^t dt'\, H_c(t')\right). \tag{4.1.10}$$

It is now natural to seek for a representation in which the intended control action is isolated from the rest. This can be done by transforming the total system into the

toggling frame. In this frame the total system rotates with the control propagator, so that the evolution in this frame is given by

$$\tilde{U}(t) = \hat{T} \exp\left(\int_0^t dt' \tilde{H}(t')\right), \tag{4.1.11}$$

with $\tilde{H}(t) = U_c^\dagger(t) H_0 U_c(t)$ being the Hamiltonian modified through the controller. The evolution in the Schrödinger picture is thus obtained by $U(t) = U_c(t)\tilde{U}(t)$. If we assume that the controller is periodic, i.e.

$$U_c(t + NT_c) = U_c(t), \qquad N \in \mathbb{N}, \tag{4.1.12}$$

both frames coincide for times $t = NT_c$. For periodic controllers the framework of average Hamiltonian theory [103] can be applied, yielding

$$U(NT_c) = \tilde{U}(NT_c) = e^{-iNT_c\bar{H}}, \tag{4.1.13}$$

with $\bar{H}$ being an averaged Hamiltonian given by the Magnus expansion $\bar{H} = \sum_{l=0}^{\infty} \bar{H}^{(l)}$ for which the lowest order is given by

$$\bar{H}^{(0)} = \frac{1}{T_c} \int_0^{T_c} dt' U_c^\dagger(t') H_0 U_c(t'). \tag{4.1.14}$$

Higher orders contain nested commutators and they scale as

$$\left\|\bar{H}^{(l)}\right\| = \mathcal{O}(\|H_0\| \left(\|H_0\| \, T_c\right)^l), \tag{4.1.15}$$

with $\|\cdot\|$ any unitarily invariant norm. Here we already note that for unbounded operators B_α, as they would appear for infinite dimensional environments, special care has to be taken by invoking such an expansion. In particular the error estimation (4.1.15) becomes meaningless since the norm of H_0 becomes infinite.

It is now easy to introduce *bang-bang* dynamical decoupling. We assume that each periodic decoupling cycle consists of a sequence $v_1, \ldots, v_M$ of M unitary decoupling operations that are separated by Δt, such that $M\Delta t = T_c$. The lowest order (4.1.14) of the averaged Hamiltonian then becomes

$$\bar{H}^{(0)} = \frac{1}{T_c} \sum_{k=1}^{M} \Delta t U_k^\dagger H_0 U_k, \tag{4.1.16}$$

with $U_k = v_{k-1}, \ldots, v_0$ and $v_0 = \mathbb{1}$. Clearly, if the decoupling operations are taken from a set V that forms a group, every U_k is an element of the group, and consequently the sum over k in (4.1.16) can be taken over all group elements. We thus get the expression (4.1.5) that was obtained with a generalized Trotter formula. To summarize, we saw that for periodic decoupling schemes the toggling frame coincides with the physical frame at times NT_c, i.e. $U_c(NT_c) = \mathbb{1}$. This shows that periodic, group-based decoupling schemes formulated in the framework of control theory (4.1.9) are equivalent to the scheme (4.1.4) in which the decoupling operations are applied in the order v, free evolution, $v^\dagger$. In other words, it is always possible by transforming into the toggling frame to obtain a decoupling procedure that is described by (4.1.4). We can conclude that decoupling is possible if the *decoupling condition*

$$\Pi(S) = \frac{1}{|V|} \sum_{v \in V} v^\dagger S v = 0, \qquad \forall S \in B(\mathcal{H}_S), \tag{4.1.17}$$

is satisfied, which ensures that the first order term in Δt vanishes. This is always possible by choosing V to be a finite group of irreducible unitary representations, where we used that the system operators are all traceless.

So far we have restricted the discussion to deterministic decoupling schemes, which has in the bang-bang case the disadvantage that we have to steer the controller in each decoupling cycle through all elements of V. Since the length of each cycle is proportional to the size $|V|$ of the decoupling set, dynamical decoupling becomes inefficient for large quantum systems. To overcome these limitations the framework of random dynamical decoupling was introduced [107]. Instead of steering the controller deterministically through V, the decoupling operations are taken uniformly random from the decoupling set, so that a random walk on the unitary group is induced. The dynamics is then governed by the expectation [107],

$$\mathbb{E}[U_c^\dagger(t) H_0 U_c(t)] = \int_V d\nu_V \, v^\dagger H_0 v, \qquad t \in (0, \Delta t), \tag{4.1.18}$$

with ν_V being the Haar measure on V, normalized to $\nu_V(V) = 1$. Clearly, if V is a irreducible representation of the unitary group, (4.1.18) is just given by a multiple of

the identity. Moreover, if we take the decoupling operations uniformly random from a finite discrete decoupling set, the random walk of $U_c(t)$ may be enforced through a sequence of equally spaced bang-bang operations, such that the integral in (4.1.18) is replaced by a finite sum over all elements in V. Note that within a deterministic decoupling scheme a similar average is obtained through the time average over a decoupling cycle (4.1.16) by invoking the Magnus expansion. By contrast, this definition of random decoupling is intrinsically acyclic, and the control path almost never returns the system to the physical frame. However, the available information about the past control trajectory may be exploited to bring the state of the system back to the physical frame if desired [107].

4.1.1 Bounds

Up to now we have discussed how in principle decoherence can be suppressed through dynamical decoupling. The instantaneous application of unitary operations satisfying the decoupling condition (4.1.17) makes it possible to suppress the interactions with the environment up to first order in Δt, so that in the limit $\Delta t \to 0$ decoherence can be fully suppressed. Obviously in an experimental situation this limit is not attainable since it corresponds to an infinite amount of energy. Hence we want to investigate now how fast the decoupling operations have to be applied to effectively suppress the interactions with environment. Typically, error estimates are given in terms of the higher orders of the Magnus expansion (4.1.15) or the Dyson series [107, 123]. The obtained error bounds are proportional to the total time T and the norm of the system-environment Hamiltonian characterizing the strength of the interaction with the environment. Here we develop a bound for the trace distance between the initial state of the system $\rho_S(0)$ and the evolved system state $\rho_S(T)$ after some time T. We consider a decoupling sequence of the form (4.1.4)

and develop an error estimate similar to the one obtained by Suzuki [122],

$$\left\| \exp\left(\sum_{j=1}^{p} A_j\right) - \left(\prod_{j=1}^{p} e^{\frac{1}{n}A_j}\right)^n \right\| \leq \frac{1}{2n}\left(\sum_{j>k} \|[A_j, A_k]\|\right) \exp\left(\sum_{j=1}^{p} \|A_j\|\right), \tag{4.1.19}$$

for a generalized Trotter formula, which is valid for a normed space.
In the following we first show that for unitary operations the exponential factor in (4.1.19) disappears such that the bound becomes more tight. Let g and h be unitary operations and $\|\cdot\|$ be any unitarily invariant norm, i.e $\|UAV\| = \|A\|$ with U, V unitary. Using the triangle inequality we then have

$$\begin{aligned} \|g^n - h^n\| &= \left\|g^{n-1}(g-h) + g^{n-2}(g-h)h + \ldots + (g-h)h^{n-1}\right\|, \\ &\leq \left\|g^{n-1}(g-h)\right\| + \left\|g^{n-2}(g-h)h\right\| + \ldots + \left\|(g-h)h^{n-1}\right\|, \\ &= n\,\|g-h\|\,. \end{aligned} \tag{4.1.20}$$

Now, if we take $g = \exp\left(\frac{T}{n}\sum_{j=1}^{p}\mathcal{K}_j\right)$, $h = \prod_{j=1}^{p} e^{\frac{T}{n}\mathcal{K}_j}$ with $\mathcal{K}_j(\cdot) = -i[v_j^\dagger H v_j, \cdot]$ being the generator of a unitary map rotated by the decoupling operations v_j, we further obtain, using identity 2 in [122],

$$\|g-h\| \leq \frac{1}{2}\frac{T^2}{n^2}\sum_{j>k}\|[\mathcal{K}_j, \mathcal{K}_k]\|. \tag{4.1.21}$$

With (4.1.20) we arrive at

$$\left\| \exp\left(T\sum_{j=1}^{p}\mathcal{K}_j\right) - \left(\prod_{j=1}^{p}\exp\left(\frac{T}{n}\mathcal{K}_j\right)\right)^n \right\| \leq \frac{1}{2}T\Delta t\sum_{j>k}\|[\mathcal{K}_j, \mathcal{K}_k]\|, \tag{4.1.22}$$

where the temporal spacing of the decoupling operations $\Delta t = \frac{T}{n}$ was introduced. We note that (4.1.22) is up an exponential factor similar to (4.1.19). We proceed by using the decoupling condition (4.1.17) and noticing that then the first term of the left hand side of (4.1.22) becomes the identity. The trace distance $\|\rho_S(0) - \rho_S(T)\|_1$ between the initial state and the state at time T of the system is therefore given by

$$\|\rho_S(0) - \rho_S(T)\|_1 = \left\| \mathrm{tr}_B\left\{\left(\exp\left(T\sum_{j=1}^{p}\mathcal{K}_j\right) - \left(\prod_{j=1}^{p}\exp\left(\frac{T}{n}\mathcal{K}_j\right)\right)^n\right)\rho(0)\right\}\right\|_1, \tag{4.1.23}$$

with $\rho(0)$ being the initial state of system and environment and $\mathrm{tr}_B\{\cdot\}$ denotes the partial trace over the environmental degrees of freedom. Now we can make use of the properties of the Schatten p-norms $\|A\|_p = (\mathrm{tr}\{(A^\dagger A)^{p/2}\})^{1/p}$ under the partial trace [124], i.e. $\|\mathrm{tr}_B\{A\}\|_p \leq d_B^{(p-1)/p} \|A\|_p$, where d_B is the dimension of the environmental Hilbert space. For the 1-norm this leads with (4.1.22) to

$$\|\rho_S(0) - \rho_S(T)\|_1 \leq \frac{1}{2} T\Delta t \sum_{j>k} \|[\mathcal{K}_j, \mathcal{K}_k]\|_\infty, \tag{4.1.24}$$

where we used $\|\mathcal{K}(\rho)\|_1 \leq \|\mathcal{K}\|_\infty \|\rho\|_1$ with $\|\cdot\|_\infty$ being the operator norm. Using the matrix representation of $\mathcal{K}$, obtained from row vectorization of the density operator (2.1.27), we finally find

$$\begin{aligned}\|\rho_S(0) - \rho_S(T)\|_1 &\leq T\Delta t \sum_{j>k} \left\| [v_j^\dagger H v_j, v_k^\dagger H v_k] \right\|_\infty , \\ &\leq T\Delta t \frac{|V|(|V|-1)}{2} \max_{j\neq k} \left\| [v_j^\dagger H v_j, v_k^\dagger H v_k] \right\|_\infty , \end{aligned} \tag{4.1.25}$$

with $|V|$ being the number of elements in the decoupling set. Clearly, for a spacing $\Delta t \to 0$ between the decoupling operations, dynamical decoupling suppresses the interactions with the environment completely. This is reflected in the bound (4.1.25) in such a way that for $\Delta t \to 0$ the initial state of the system does not change. Moreover we see that the suppression of decoherence depends on the evolution time T. The longer we want to preserve a quantum state from decoherence, the faster we have to apply the decoupling operations. Using the triangle inequality, unitary invariance, and submultiplicativity we further find $\left\| [v_j^\dagger H v_j, v_k^\dagger H v_k] \right\|_\infty \leq 2 \|H\|_\infty^2$, such that the bound (4.1.25) becomes

$$\|\rho_S(0) - \rho_S(T)\|_1 \leq T\Delta t(|V|(|V|-1)) \|H\|_\infty^2 , \tag{4.1.26}$$

noting that this bound is similar to one of the bounds in [123]. Because the operator norm of H is given by the highest eigenvalue, one concludes that the decoupling operations have to be applied faster than the highest frequency of the open system Hamiltonian in order to effectively suppress decoherence. We want to emphasize here two points. The first one is that the bound (4.1.26) does not capture the fact

that if the rotated Hamiltonians commute with each other, dynamical decoupling works perfectly for arbitrary spacings of the decoupling operations. This is reflected in the bound (4.1.25). Second, both bounds become meaningless for infinite dimensional environments described by unbounded operators. We pointed out in Sec. 2.1.4 that a system dynamics which is Markovian, in the sense that it is described by a semigroup dynamics yielding an exponential decay, can only be obtained if the system-environment Hamiltonian is unbounded in both directions. The typical conclusion that dynamical decoupling only works for non-Markovian environments [3], because one has to be faster than the fastest timescale of the overall dynamics [106], which is $\propto \|H\|_\infty = \infty$ for unbounded operators, is not based on rigorous mathematical grounds. In particular one has to check carefully if a pertubative expansion like the Magnus expansion, the Trotter series or the Dyson series is applicable to find error bounds in an infinite dimensional setting. In Sec. 4.2 we give an example of a system-environment Hamiltonian that leads to an exponential decay and that can be decoupled perfectly. Before, we first give a short introduction into the mathematical terminology we have to face in order to rigorously proof that dynamical decoupling also works for unbounded operators acting on the environment.

4.1.2 Facing unbounded operators

We begin with the definition of a bounded operator. For a bounded operator A there always exists a constant $c \geq 0$ so that

$$\|A\psi\| \leq c\,\|\psi\|\,, \qquad \forall \psi \in \mathcal{H}. \tag{4.1.27}$$

Clearly in a finite dimensional setting this is always satisfied. To introduce the notion of an unbounded operator we start with an example, taken from [125], that considers the position operator $\hat{x}$ on the space of square integrable function $L^2(\mathbb{R})$,

i.e. functions $f : \mathbb{R} \to \mathbb{C}$ for which $\int_{-\infty}^{\infty} dx\, |f(x)|^2 < \infty$. Take $\phi \in L^2(\mathbb{R})$ defined by

$$\phi(x) = \begin{cases} 0 & \text{for } x < 0, \\ \frac{\sqrt{3}}{2} x & \text{for } 0 \leq x < 1, \\ \frac{\sqrt{3}}{2x} & \text{for } x \geq 1, \end{cases} \tag{4.1.28}$$

and note that $\int_{\infty}^{\infty} dx\, |\phi(x)|^2 = 1$. The position operator is defined as $\hat{x}\psi(x) = x\psi(x)$ for all $\psi \in L^2(\mathbb{R})$. Setting $\varphi \equiv \hat{x}\phi$ and taking the definition of ϕ from a above one can easily check that $\int_{-\infty}^{\infty} dx\, |\varphi(x)|^2 = \infty$ so that $\varphi \notin L^2(\mathbb{R})$. The position operator acts on a normalized state and moves it out of the Hilbert space, here the space of square integrable functions. Obviously (4.1.27) is not satisfied for all $\psi \in L^2(\mathbb{R})$. In other words the action of $\hat{x}$ cannot be defined on the entire Hilbert space. In fact the Hellinger-Töplitz theorem (see for example [126]) states that if a linear and self-adjoint operator is defined on the entire Hilbert space it is necessarily bounded. This suggest that an unbounded operator B can only be defined on a dense linear subspace $D(B)$ of the Hilbert space $\mathcal{H}$, which is called the *domain* of B. To make sense of unbounded operators we first have to specify the domain so that B is a linear map from its domain into $\mathcal{H}$. This is of particular importance if we deal with different unbounded operators. For instance, consider A and B to be unbounded operators with $D(A)$ and $D(B)$ the domains respectively. Then the sum $A + B$ only exists on the domain $D(A) \cap D(B)$.

Regarding dynamical decoupling for infinite dimensional environments, we have to deal with unbounded Hamiltonians that have a sum like structure

$$\Pi(H) = \frac{1}{|V|} \sum_{v \in V} v^\dagger H v, \tag{4.1.29}$$

where

$$H = H_S \otimes \mathbb{1} + \mathbb{1} \otimes H_B + \sum_{\alpha} S_\alpha \otimes B_\alpha, \tag{4.1.30}$$

is the system-environment Hamiltonian and v are the unitary decoupling operations only acting on the system. Consider for example the single qubit case in which the

Hamiltonian only consist of a system-environment interaction of the form

$$H_{SB} = \sigma_+\sigma_- \otimes B_1 + \sigma_-\sigma_+ \otimes B_2, \tag{4.1.31}$$

where B_1 and B_2 are unbounded operators. If we take $D(H_{SB}) = (D(B_1), D(B_2))^T$ to be the domain for H_{SB}, written in the σ_z basis, and for the rotated version $\sigma_x H_{SB}\sigma_x$ the domain $D(\sigma_x H_{SB}\sigma_x) = (D(B_2), D(B_1))^T$, it is clear that they are not necessarily the same unless we take $D(B_1) = D(B_2)$. This example illustrates that we have to choose the domain carefully in order to formulate dynamical decoupling for unbounded environmental operators. In particular, in order to make sense of $\Pi(H)\psi = (\mathbb{1} \otimes H_B)\psi$, obtained from the decoupling condition (4.1.17) and arising in the limit of infinitely fast decoupling, we have to make sure that ψ is in the domain of H_B and in the domain of the rotated Hamiltonians $v^\dagger H v$. This can be achieved by choosing for example

$$D(H) = \mathbb{C}^d \otimes D(H_B), \tag{4.1.32}$$

to be the domain, assuming that $D(H_B)$ is contained in the domains of the environmental operators B_α. Note that if we chose the domain in this way, the domain stays the same when we apply the decoupling operations since they only act on $\mathbb{C}^d$. However, this choice might be too restrictive and we will see later that the most general setting is obtained through the notion of a *core*. Before we introduce the concept of a core we will first introduce the definition of a self-adjoint operator in an infinite dimensional setting and an important theorem that establishes the connection between self-adjoint operators and one parameter unitary groups. Followed by further definitions from functional calculus and operator theory we give afterwards the definition of a core.

Let B be a densely defined operator on a Hilbert space $\mathcal{H}$ and denote by $B^\dagger$ its adjoint. For a definition of the adjoint of an unbounded operator and its domain $D(B^\dagger)$ see [126]. Then B is called *hermitian* if $D(B) \subset D(B^\dagger)$ and $B\psi = B^\dagger\psi$ for all $\psi \in D(B)$. Equivalently, B is hermitian if and only if

$$\langle B\phi|\psi\rangle = \langle\phi|B\psi\rangle, \quad \forall\, \phi, \psi \in D(B). \tag{4.1.33}$$

A hermitian operator B is *self-adjoint* if and only if the domain of B coincides with the domain of its adjoint, i.e. $D(B) = D(B^\dagger)$ and we write $B = B^\dagger$. Note that one can show, using the definition of $D(H^\dagger)$, that if H is self-adjoint on $D(H)$ given by (4.1.32), also $v^\dagger H v$ is self-adjoint on $D(H)$ since the decoupling operations only act on the system part, which is assumed to be finite dimensional.

For finite dimensional quantum systems the exponential of an operator is defined through its power series that converges in norm. For an unbounded operator the series expansion can not be used directly and the question arises how the time evolution operator can be defined. First of all note that in the framework of functional calculus it can be shown [126] that for a self-adjoint operator H on a Hilbert space $\mathcal{H}$ with domain $D(H)$ we have for $U(t) = e^{itH}$ that:

i) $U(t)$ is a strongly continuous one-parameter unitary group,

ii) for $\psi \in D(H)$, $\frac{(U(t)-\mathbb{1})\psi}{t} \to iH\psi$ as $t \to 0$,

iii) if $\lim_{t\to 0} \frac{(U(t)-\mathbb{1})\psi}{t}$ exists, then $\psi \in D(H)$.

This shows that, if we specify the domain, also for self-adjoint unbounded operators we get by exponentiation a one parameter unitary group describing the time evolution in a quantum mechanical setting. In fact the connection is much deeper. Stones theorem [126] states that:

> if $U(t)$ is a one parameter unitary group on a Hilbert space $\mathcal{H}$, then there exist a self-adjoint operator H on $\mathcal{H}$ such that $U = e^{itH}$.

The crucial point here is that the operator generating the one parameter unitary group has to be self-adjoint, which requires for unbounded operators the specification of its domain. Stones theorem tells us that not only we can create a one parameter unitary group by exponentiating iH, it is also the only way to obtain a one parameter unitary group.

To define a core of an unbounded operator we need a few further definitions [126]. Let A and B be densely defined operators on a Hilbert space $\mathcal{H}$ with domains $D(A)$ and $D(B)$ respectively, then:

> *Extension of an operator:* B is said to be an *extension* of A if $D(A) \subseteq D(B)$ and if $B\psi = A\psi$ for all $\psi \in D(A)$. We write $A \subseteq B$.
>
> *Closure of an operator:* A is *closed* if for every sequence $\phi_n \subset D(A)$ such that $\phi_n \to \phi$ and $A\phi_n \to \psi$ as $n \to \infty$, it follows that $\phi \in D(A)$ and $A\phi = \psi$. If A has a closed extension we say it is *closable*. For each closable operator A there is a unique smallest closed extension, which we denote by $\bar{A}$, and which is called the *closure* of A.
>
> *Essentially self-adjoint:* if A is hermitian it is called essentially self-adjoint if its closure $\bar{A}$ is self-adjoint.

Finally we can now come to the definition of a core. If a hermitian operator A is closed, a subset $\mathcal{C} \subset D(A)$ is called a core for A if the closure of A over $\mathcal{C}$ gives us A. We write $\overline{A|_{\mathcal{C}}} = A$. The importance of a core becomes clear with the following fact. If an operator A is essentially self-adjoint, then it has one and only one self-adjoint extension. Conversely if A is a self-adjoint operator it is enough to give a core for A for which $\overline{A|_{\mathcal{C}}}$ is self-adjoint. In words, instead of finding the domain of A precisely, we can find the domain over which the smallest closed extension of A is self-adjoint, i.e. over which A is essentially self-adjoint.

Now we want to come back to the relevance of the introduced terminology for dynamical decoupling. Instead of characterizing the domain $D(H)$ of the self-adjoint operator H precisely we can just give a suitable core $\mathcal{C}$ for it. We then assume that $\mathcal{C}$ is also core of H_B and each B_α, all assumed to be self-adjoint on a certain dense domain $D(H_B)$ and $D(B_\alpha)$ [1], noting that $D(H_B)$ and $D(B_\alpha)$ might be different from each other. Then the sum $\Pi(H)$ is well defined on $\mathbb{C}^d \otimes \mathcal{C}$.

Having established the mathematical framework for dynamical decoupling of unbounded environmental operators, we will prove in the following section that also

[1] or we assume that they are essentially self-adjoint on $\mathcal{C}$

finite dimensional quantum systems interacting with an infinite dimensional environment can be decoupled and therefore protected from decoherence. Moreover we will show that noise described by some Lindbladian can never be suppressed through dynamical decoupling. This will lead to a strategy allowing us to distinguish between decoherence, induced by the interaction with environment, from intrinsic noise terms that are described by Lindblad operators, arising for instance from collapse models.

4.2 Distinguishing decoherence from collapse models by dynamical decoupling

Based on the published work:

C. Arenz, R. Hillier, M. Fraas and D. Burgarth, Phys. Rev. A **92**, 022102 (2015)

R. Hillier, C. Arenz and D. Burgarth, J. Phys. A: Math. Theor. **48**, 155301 (2015)

Despite of its puzzling nature and persistent foundational problems, such as the infamous measurement problem, quantum mechanics remains one of the most precise and successful physical theories to date. This makes it hard to develop alternative theories [29, 32, 127], which are either bound to agree with quantum mechanics on all measurable aspects – and therefore being indistinguishable from it – or must disagree with it only at the most subtle level, which means that such theories are hard to falsify experimentally. While in our daily life quantum effects do not appear to play a role, this does not imply that it is an incomplete theory, as the onset of classicality can – at least up to a certain degree [21, 22] – be explained from *within* quantum theory, using the concept of decoherence, which was introduced and discussed in Sec. 2.1.4 regarding the emergence of classicality.

Decoherence arises from the coupling of a quantum object with other degrees of freedom, which washes out quantum mechanical features. Besides being a major obstacle to quantum computing, decoherence is also an obstacle to the tests of theories alternative to quantum mechanics, since it tends to obscure the – already minimal – deviations they predict from the usual Schrödinger dynamics. Even worse, since most alternative theories aim to explain the onset of classicality, they predict features identical in their mathematical nature to decoherence [38]. For instance we saw in Sec. 2.1.5 that collapse models (CM) lead to an exponential decay that would

similarly occur from an unbounded interaction with an environment. The main aim of this article is to demonstrate that while these models might be *mathematically* identical, they are *physically* distinguishable, irrespectively of decoherence. At first, this seems impossible. Especially in quantum information theory, the Church of the Larger Hilbert Space – the idea that any noisy dynamics or state might equally well be represented by a noiseless one on a dilated space – is so deeply rooted that such a distinction seems heretic.

A method to distinguish decoherence from CM which is obvious but impractical is to derive ab initio predictions of decoherence and compare these with experiments. Unfortunately, the predictive power of decoherence models till date is low, as they contain many free parameters to fit. We therefore aim to develop methods which are independent of the details of the decoherence involved, as well as of the specific CM considered.

Our work is based on a very simple idea, namely that dynamical decoupling only works for systems which are *truly* coupled to environments, but not for systems which have intrinsic noise terms, as arriving from axiomatic modifications of Schrödinger's equation, such as the CM that were introduced in Sec. 2.1.5. This seems to leave us with an amazingly simple strategy to distinguish decoherence from CM: apply decoupling, and if it works, then the noise was due to standard quantum theory; if it does not work, it can provide evidence for CM. Is this therefore the most successful "failed" experiment ever? Of course not: we need to be convinced that the experiment did not work *despite good effort*, in other words, we need to know quantitatively how much the experiment can fail while still being in the realms of standard decoherence; and how much it can succeed despite being in the realms of CM. This poses an additional problem. As already pointed out in the previous section, it is a common view that dynamical decoupling only works for environments inducing non-exponential decay (sometimes referred to as 'non-Markovian', although this term is used ambiguously in the literature). This means that if the observed quantum dynamics shows exponential behaviour, we would not be able to distinguish it from CM. On the other hand, most CMs predict exponential

decay [38].

The reason for this common view is that exponential decay can only be obtained from an unbounded interaction with the environment [6] (see additionally Sec. 2.1.4 for further details), for which standard error analysis of dynamical decoupling fails [107]. Perhaps surprisingly, we will prove in Sec. 4.2.3 that in general even unbounded Hamiltonians can be decoupled and hence distinguished from intrinsic decoherence. This general proof is illustrated by an analytically solvable example 4.2.3. We can conclude that non-exponential dynamics is in general not the underlying mechanism of dynamical decoupling. This result extends the applicability of decoupling to a vast class of system-environment interactions and has applications in quantum engineering beyond the scope of this work.

Finally, dynamical decoupling arises in the limit of infinitely fast quantum gates, so in practice it is never perfect. How fast should these operations be so that decoherence and CM can be distinguished? Below, we provide numerical simulations of two common models and asymptotical bounds. As we will see below, the convergence speed can depend strongly on the initial bath state, which implies that *Missing think*, e.g., depending only on the observed decay rates of the system, cannot be provided. Nevertheless, experimental evidence can be provided if a saturation of fidelity is observed under increasingly fast operations. For the parameter range explorable by our scheme, we can do the following rough estimate. The strongest intrinsinc decay rates for qubits predicted by CM are of the order of $10^{-8}\,s^{-1}$ corresponding to a half-life time of several *years* [29]. Precision measurements of qubits on the other hand are very well developed meaning that coherence decay of the order of percent can be detected. This means that if one aims to keep a qubit from detectable decay for several days, the first CM models could be detected or excluded. At present qubit coherence times can be prolonged by dynamical decoupling up to six hours [128]. This is still a few orders of magnitude off the theoretical predictions, which is comparable to the usual CM tests in the macroscopic superposition regime.

Our results pave the way to test CM in low-dimensional systems, including qubits, where CM predicts very weak effects [29], but where dynamical decoupling is very

efficient, and where accurate tomography can be performed [129]. This is a different parameter regime compared to tests using macroscopic superpositions [130, 131, 132, 133], where CM predict stronger effects but dynamical decoupling is challenging (see, however, [134]).

4.2.1 Dynamical decoupling of bounded Hamiltonians and Lindbladians

We recall that the idea of dynamical decoupling is to rapidly rotate the quantum system by means of classical fields to average the system-environment coupling to zero. First we want to review the key aspects of dynamical decoupling that were introduced in the beginning of this chapter. We consider unitary decoupling operations v taken from the set V of $|V|$ unitary $d\times d$ matrices satisfying $\frac{1}{|V|}\sum_{v\in V} v^\dagger x v = \frac{1}{d}\mathrm{tr}(x)\mathbb{1}$ for any matrix x. We remember that for a single qubit such set are the Pauli matrices $V = \{\mathbb{1}, \sigma_x, \sigma_y, \sigma_z\}$. While we saw that usually dynamical decoupling is discussed in the realm of a unitary time evolution, we already allow here a noisy dynamics generated by a Lindbladian $\mathcal{L}$ because we later want to see what happens for CM. This dynamics is now modified by decoupling operations $v_i \in V$ with $i = 1, \ldots, n$ applied instantaneously in time steps Δt. After time $t = n\Delta t$ the system has evolved according to

$$\mathcal{E}_t(\cdot) = \prod_{i=1}^{n} \mathrm{Ad}(v_i^\dagger)\exp(\Delta t\mathcal{L})\mathrm{Ad}(v_i)(\cdot), \tag{4.2.1}$$

where $\mathrm{Ad}(v_i)(\cdot) = v_i(\cdot)v_i^\dagger$ and the product is time-ordered. The generalization to time-dependent generators is straight forward and will be used later in the examples. The decoupling operations are chosen uniformly random from V, which has some advantage over deterministic schemes [107, 110]. Notice that our definition of random dynamical decoupling differs slightly from [107]. The time evolution (4.2.1) becomes a stochastic process with expected dynamics determined by

$$\bar{\mathcal{L}} := \frac{1}{|V|}\sum_{v\in V} \mathrm{Ad}(v^\dagger)\mathcal{L}\mathrm{Ad}(v). \tag{4.2.2}$$

This leads to the *decoupling condition* $\bar{\mathcal{L}} = 0$, which one requires in order to successfully suppress decoherence. Remember that this condition is independent of whether we use a deterministic or random decoupling scheme [106]. The idea behind this condition is that it ensures the cancellation of $\mathcal{L}$ in first order in $\Delta t||\mathcal{L}||$. For $\Delta t \to 0$, keeping the total time t fixed, the time evolution (4.2.1) becomes therefore effectively the identity.

We saw in the beginning of this chapter that for finite dimensional systems a Hamiltonian dynamics $\mathcal{L}(\cdot) = i[H, \cdot]$ can always be suppressed through dynamical decoupling using a proper decoupling set V. In Sec. 4.2.3 we prove that this is even true for finite dimensional system that interact with an environment through unbounded Hamiltonians. But what happens for CM? Note first of all that we mentioned in Sec. 2.1.5 that for CM models that modify the Schrödinger equation in a nonlinear way, it was argued in [38] that under the assumption of the no-signalling principle the resulting dynamics is described by a time independent Lindblad operator

$$\mathcal{L}(\cdot) = \sum_{j=1}^{d^2-1} \gamma_j (2L_j(\cdot)L_j^\dagger - (L_j^\dagger L_j(\cdot) + (\cdot)L_j^\dagger L_j)), \tag{4.2.3}$$

yielding the averaged Lindbladian

$$\bar{\mathcal{L}}(\cdot) = \sum_{j=1}^{d^2-1} 2\gamma_j \left(\frac{1}{|V|} \sum_{v \in V} v^\dagger L_j v (\cdot) v^\dagger L_j^\dagger v - \frac{1}{d} \mathrm{tr}(L_j^\dagger L_j)(\cdot) \right). \tag{4.2.4}$$

We will henceforth refer such AQT dynamics as *intrinsic decoherence*. In order to avoid confusion, we will write *extrinsic decoherence* for decoherence arising in standard quantum theory. Surprisingly if the dynamics includes intrinsic decoherence, the decoupling condition can *never* be fulfilled. We prove in the following that $\bar{\mathcal{L}}$ is always different from zero for intrinsic decoherence. We note that this allows us to mathematically define intrinsic decoherence as a dynamics that is generated by some $\mathcal{L}$ for which the decoupling condition can never be satisfied.

Theorem 4.1 *A decoupling set V satisfies the decoupling condition $\bar{\mathcal{L}} = 0$ if and only if the dynamics is purely unitary.*

Proof. We write the generator $\mathcal{L}$ of the dynamics in Christensen-Evans form [16],

$$\mathcal{L}(\rho) = \Phi(\rho) + a\rho + \rho a^\dagger, \tag{4.2.5}$$

with a certain $a \in B(\mathcal{H})$ and Φ a completely positive map, which is not a multiple of id. Note that adding $2\lambda\mathrm{id}$ to Φ has the same effect on $\mathcal{L}$ as adding $\lambda\mathbb{1}$ to a. Suppose first that the dynamics $\Lambda_t = \exp(\mathcal{L}t)$ is purely unitary, i.e. $\Phi = 0$ and $a^\dagger = -a$. We thus obtain

$$\frac{1}{|V|}\sum_{v\in V}\mathrm{Ad}(v^\dagger)\mathcal{L}\mathrm{Ad}(v)(\rho) = \frac{1}{d}\mathrm{tr}\{a + a^\dagger\} = 0, \qquad \forall\rho \in S(\mathcal{H}), \tag{4.2.6}$$

so V satisfies the decoupling condition. If instead Λ_t is not purely unitary we have

$$\bar{\Phi} := \frac{1}{|V|}\sum_{v\in V}\mathrm{Ad}(v^\dagger)\Phi\mathrm{Ad}(v), \tag{4.2.7}$$

is completely positive and non-zero. Therefore to satisfy the decoupling condition we need that $\bar{\Phi}(\rho) = -1/d\,\mathrm{tr}\{a + a^\dagger\}\rho$, $\forall\rho \in S(\mathcal{H})$. Suppose this is true. Then for every pure state $|\phi\rangle\langle\phi| \in S(\mathcal{H})$ we have

$$\bar{\Phi}(|\phi\rangle\langle\phi|) = -\frac{1}{d}\mathrm{tr}\{a + a^\dagger\}|\phi\rangle\langle\phi| \tag{4.2.8}$$

But for every $|\xi\rangle \in \mathcal{H}$ with $\langle\xi|\phi\rangle = 0$ we need

$$0 = \langle\xi|\bar{\Phi}(|\phi\rangle\langle\phi|)|\xi\rangle \tag{4.2.9}$$

$$= \frac{1}{|V|}\sum_{v\in V}\langle\xi|\mathrm{Ad}(v^\dagger)\Phi\mathrm{Ad}(v)(|\phi\rangle\langle\phi|)|\xi\rangle, \tag{4.2.10}$$

with each single term equal to zero, due to the positivity of Φ. Since V includes the identity we hence have

$$\langle\xi|\Phi(|\phi\rangle\langle\phi|)|\xi\rangle = 0, \tag{4.2.11}$$

for all orthogonal vectors $|\xi\rangle$ and $|\phi\rangle$. Let us write the CP map Φ in Kraus form with Kraus operators M_j, then

$$\sum_j |\langle\xi|M_j|\phi\rangle|^2 = 0. \tag{4.2.12}$$

For every j we see that $|\phi\rangle$ must be an eigenvector of M_j. This hold for every $|\phi\rangle \in \mathcal{H}$, so $M_j \in \mathbb{C}\mathbb{1}$ and thus Φ is a multiple of id, which contradicts our initial assumption. Therefore

$$\Phi(\rho) \neq -\frac{1}{d}\mathrm{tr}\{a + a^\dagger\}\rho, \tag{4.2.13}$$

and $\frac{1}{|V|}\sum_{v\in V} \mathrm{Ad}(v^\dagger)\mathcal{L}\mathrm{Ad}(v) \neq 0$. □

Intuitively the irreversible nature of the non-unitary dynamics, i.e. the increase of entropy, makes it impossible to counteract the loss of coherence with unitary decoupling pulses. Although intuitively clear, this is a remarkable result since it enables us to distinguish two different seemingly equal decoherence mechanisms. We remark that the generalization to time-dependent Lindbladians is straightforward allowing our technique also to discriminate non-exponential collapse models from extrinsic decoherence.

In the limit of arbitrarily fast decoupling operations ($\Delta t \to 0$) dynamical decoupling works perfectly for extrinsic decoherence. However, in practice even dynamical decoupling of extrinsic decoherence can never be perfect meaning that higher orders in $\Delta t||\mathcal{L}||$ enter the resulting dynamics. To detect the presence of intrinsic decoherence we therefore need to develop an extrapolation for $\Delta t \to 0$. Furthermore to distinguish extrinsic and intrinsic decoherence we need bounds. Using a central limit theorem, such bounds are developed in [135] for the expectation of the decoupling error $\bar{\epsilon}$, while here we will focus on specific examples. The decoupling error $\epsilon = \mathrm{tr}\{(\mathbb{1} - \mathcal{E}_t)^\dagger(\mathbb{1} - \mathcal{E}_t)\}/d^2$ compares the free evolution under random dynamical decoupling with the identity operation. In the limit $\Delta t \to 0$, keeping the total time t fixed, the decoupling error becomes [135],

$$\epsilon = \frac{1}{d^2}\mathrm{tr}\left((\mathbb{1} - \exp(\bar{\mathcal{L}}t))^\dagger(\mathbb{1} - \exp(\bar{\mathcal{L}}t))\right), \tag{4.2.14}$$

where for extrinsic decoherence the time evolution of the total system is followed by the partial trace over the environment yielding $\epsilon = 0$ for $\Delta t \to 0$. Note that the decoupling error can be estimated in an experiment by performing process tomography [136]. Simpler "fingerprints" to distinguish CM which do not require process

tomography can easily be derived for specific systems. In the following we emphasize the physics calculating bounds for two common models.

4.2.2 Models and bounds

To demonstrate our method we consider two different types of decoherence of a single qubit, namely amplitude damping and pure dephasing introduced in Sec. 2.1.3.

Two qubit model

To begin with suppose that one observes a dynamics described by an amplitude damping (AD) channel, given by the Lindblad operator

$$\mathcal{L}_{\mathrm{AD}}(\cdot) = -\gamma(\sigma_+\sigma_-(\cdot) + (\cdot)\sigma_+\sigma_- - 2\sigma_-(\cdot)\sigma_+), \tag{4.2.15}$$

with $\sigma_\pm$ the raising and lowering Pauli operators. Within the extrinsic decoherence model such amplitude damping dynamics can be obtained by a time dependent interaction with an ancilla qubit (A) initialized in its ground state. The total Hamiltonian reads

$$H(t) = g(t)(\sigma_+ \otimes \sigma_-^{(A)} + \sigma_- \otimes \sigma_+^{(A)}), \tag{4.2.16}$$

with the time dependent coupling constant $g(t) = \gamma/\sqrt{\exp(2\gamma t) - 1}$. The Hamiltonian $H(t)$ commutes with itself at all times such that the time evolution of the composite system can easily be integrated. After tracing over the ancilla qubit one obtains precisely the two Kraus operators which describe the amplitude damping channel generated by (4.2.15). Note that at $t = 0$ the interaction strength $g(t)$ diverges while the time evolution operator remains well defined. Clearly there are other possible choices of the system-bath Hamiltonian that lead to the same dynamics. For example within the Born-Markov approximation the same Lindblad operator (4.2.15) is obtained by a time independent interaction of the qubit with a bath of harmonic oscillators at zero temperature. However as a toy model, (4.2.16) has the advantage of being simpler. Such time-dependent dilations may also find

applications in other context.

Now we turn to the question how well dynamical decoupling can distinguish between extrinsic decoherence, given by the Hamiltonian (4.2.16), and pure intrinsic decoherence given by the Lindbladian (4.2.15). Using (4.2.4) one finds for the intrinsic decoherence case the averaged Lindblad operator $\bar{\mathcal{L}}_{\mathrm{AD}}(\cdot) = -\gamma(\mathbb{1}(\cdot) - \sigma_-(\cdot)\sigma_+ - \sigma_+(\cdot)\sigma_-)$, which determines the dynamics in the limit of infinitely fast decoupling operations. The first observation is that $\bar{\mathcal{L}}_{\mathrm{AD}}$ does not vanish. With (4.2.14) we can furthermore derive the following asymptotic behaviour for the decoupling error in the intrinsic decoherence case

$$\epsilon_{\mathrm{AD}}^{\mathrm{int}} \to \frac{1}{4}\left(3 - e^{-\gamma t}\left(4 - e^{-3\gamma t}\right)\right), \quad \Delta t \to 0, \tag{4.2.17}$$

and for $\gamma t \gg 1$ it approaches a value of 3/4. In Fig. 4.1 we evaluated the averaged

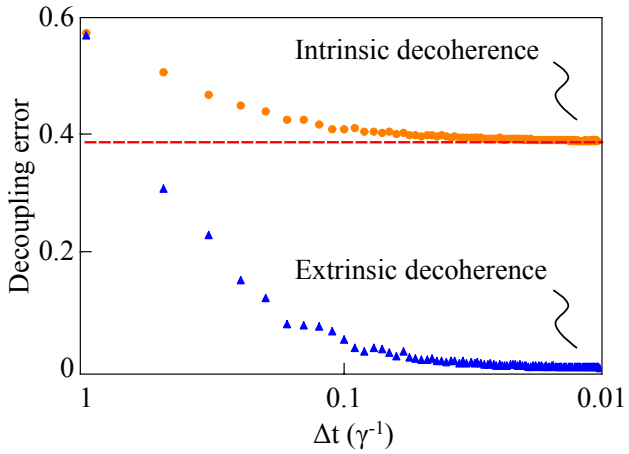

Figure 4.1.: Averaged decoupling error under random dynamical decoupling as a function of Δt on an inverse logarithmic scale for the total time $t = \gamma^{-1}$. The circles correspond to pure intrinsic decoherence described by (4.2.15), the triangles to extrinsic decoherence given by (4.2.16) and the dashed line shows the asymptotic behavior (4.2.17) for the intrinsic decoherence case for $\Delta t \to 0$. The average was taken over 100 trajectories.

decoupling error for intrinsic and extrinsic decoherence as a function of Δt for a fixed total time $t = \gamma^{-1}$. We see that for the Hamiltonian model (4.2.16) the decoupling error tends to zero. The asymptotic behaviour of the averaged trajectories allows us

to distinguish intrinsic from extrinsic decoherence: for purely intrinsic decoherence we have (4.2.17), while for purely extrinsic it is 0, and everything in-between must correspond to a mixture of the two. The actual speed of convergence to the limit in the extrinsic case depends on the chosen dilation [107], so that we cannot say how small Δt has to be chosen in order to distinguish with certainty.

Spin-boson model

Next, we consider a more realistic and experimentally relevant model describing pure dephasing (PD) in the σ_z basis of the qubit. The Lindbladian reads

$$\mathcal{L}_{\mathrm{PD}}(t)(\cdot) = -\frac{\gamma(t)}{4}[\sigma_z, [\sigma_z, \cdot\,]], \tag{4.2.18}$$

where the time dependent damping rate $\gamma(t)$ will be specified later. As extrinsic decoherence such PD would arise from an interaction with a bosonic heat bath given by

$$H = \sum_k \omega_k a_k^\dagger a_k + \sigma_z \sum_k (g_k a_k^\dagger + g_k^* a_k), \tag{4.2.19}$$

where $a_k^\dagger, a_k$ are the bosonic creation and annihilation operators of the kth field mode and g_k are coupling constants quantifying the interaction strength to each harmonic oscillator. After tracing over the bath degrees of freedom [1, 137, 138] one finds for the time dependent damping rate $\gamma(t) = 4\int_0^t ds \int_0^\infty d\omega I(\omega) \coth\left(\frac{\omega}{2T}\right)\cos(\omega s)$ where the continuum limit was performed and the spectral density $I(\omega)$, which contains the statistical properties of the bath, and the temperature T of the bath were introduced.

For an intrinsic dephasing mechanism given by (4.2.18) the decoupling operations V do not affect the dynamics $v^\dagger \sigma_z v = \pm\sigma_z$ for all $v \in V$ such that $\mathcal{L}_{\mathrm{PD}} = \bar{\mathcal{L}}_{\mathrm{PD}}$. Therefore the decoupling error in the intrinsic decoherence case is governed by the dynamics generated by $\mathcal{L}_{\mathrm{PD}}$ and one finds independently of Δt,

$$\epsilon_{\mathrm{PD}}^{\mathrm{int}} = \frac{1}{2}\left[1 - \exp\left(-\int_0^t \gamma(t')dt'\right)\right]^2, \tag{4.2.20}$$

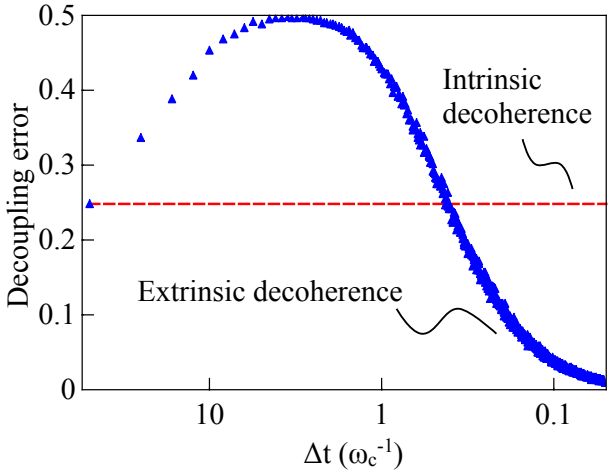

Figure 4.2.: Averaged decoupling error under random dynamical decoupling as a function of Δt on an inverse logarithmic scale evaluated for $t = 50\,\omega_c^{-1}$. The triangles correspond to extrinsic decoherence given by the spin boson model (4.2.19) where the dashed line corresponds to intrinsic decoherence (4.2.18) which is independent of Δt here (4.2.20). The average was taken over 100 trajectories.

showing that the asymptotic decoupling error is given by 1/2. Based on the spin-boson Hamiltonian (4.2.19) it was shown in [110] that under random dynamical decoupling the spectral density gets renormalized by a factor that ensures for $\Delta t \to 0$ the suppression of decoherence.

Because the decoupling operations V give the same spectral density as in [110] we can easily evaluate the averaged decoupling error for extrinsic and intrinsic decoherence (Fig. 4.2). We chose an ohmic spectral density with a sharp cut off $I(\omega) = 1/4\kappa\omega\theta(\omega - \omega_c)$ with $\kappa = 0.25$ a measure of the coupling strength to the environment and $\omega_c = 100$ the cut off frequency. We calculated the averaged decoupling error in the low temperature limit $\omega_c/T = 10^2$.

Note that for $\Delta t \gtrsim 0.5\,\omega_c^{-1}$ decoherence gets accelerated as reported in [110] in the extrinsic case since the decoupling error is higher than the decoupling error that is obtained for the dynamics generated by $\mathcal{L}_{\mathrm{PD}}$.

4.2.3 Dynamical decoupling for unbounded Hamiltonians

Many physical environments are modeled as infinite dimensional system, often with *unbounded* interactions. In order to discuss dynamical decoupling of such systems, we find it enlightening to start with a specific, analytically solvable model, before providing a general proof that generally even unbounded time-independent Hamiltonians can be decoupled.

Shallow pocket model

We now provide an analytically solvable model of an unbounded, time-independent Hamiltonian which, without approximations, leads to a time-independent dephasing Lindbladian, but can be decoupled arbitrarily well. It is an example of an exact time-independent dilation describing a small system coupled to a fictitious particle on a line. After tracing over the decrees of freedom of the particle we obtain a time independent Lindblad generator for the reduced dynamics of the system. The particle cannot store energy internally – hence the name – and the dynamics is governed by an interaction Hamiltonian

$$H = \frac{g}{2}\sigma_z \otimes \hat{x} = \frac{g}{2}\begin{pmatrix} \hat{x} & 0 \\ 0 & -\hat{x} \end{pmatrix}, \tag{4.2.21}$$

where $\hat{x}$ is the position operator, the small system is a qubit for simplicity and g a coupling constant. The Hamiltonian is diagonal and the evolution of a joint density matrix in the σ_z basis is

$$\rho(t,x) = \begin{pmatrix} \rho_{11}(0,x) & \rho_{10}(0,x)e^{igxt} \\ c.c. & \rho_{00}(0,x) \end{pmatrix}. \tag{4.2.22}$$

A reduced dynamic displaying exponential decay is achieved by choosing an initial state $\rho \otimes |\psi\rangle\langle\psi|$ where

$$\langle x|\psi\rangle = \sqrt{\frac{\gamma}{\pi}}\frac{1}{x+i\gamma}. \tag{4.2.23}$$

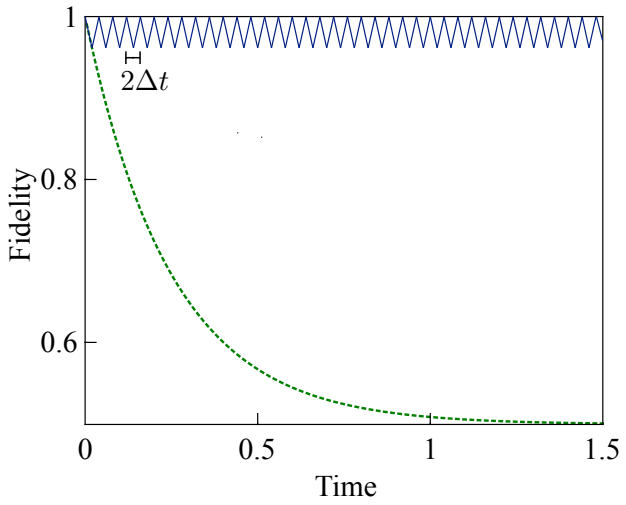

Figure 4.3.: Schematic representation of the fidelity for exponential dephasing (dotted green line) to stay in a coherent superposition of ground and excited state. The solid blue line shows the dynamics of the qubit under dynamical decoupling. Note that after the time Δt, when the sign in front of H was reversed, the fidelity changes according to $\mathcal{F}(t) = \frac{1}{2}\left(e^{-g\gamma|t-\Delta t|}+1\right)$ with $0 \leq t \leq \Delta t$.

After integrating out the particle degree of freedom we obtain, through the Fourier transform of a Lorentzian, a purely exponential decay of the off diagonal terms,

$$\rho(t) = \begin{pmatrix} \rho_{11}(0) & \rho_{10}(0)e^{-g\gamma t} \\ c.c. & \rho_{00}(0) \end{pmatrix}, \tag{4.2.24}$$

which corresponds to a time-independent dephasing Lindbladian

$$\mathcal{L}(\cdot) = -g\frac{\gamma}{4}[\sigma_z,[\sigma_z,\cdot]]. \tag{4.2.25}$$

Note that for an initial bath state with a Gaussian shape a decay $\propto e^{-t^2}$ is obtained. The model can be perfectly decoupled using $\mathbb{Z}_2$ controls $v_0 = \mathbb{1}$, $v_1 = \sigma_x$. In fact $v_1^\dagger H v_1 = -H$ and hence

$$v_0 \exp(-i\Delta t H) v_0^\dagger v_1 \exp(-i\Delta t H) v_1^\dagger = \mathbb{1}. \tag{4.2.26}$$

This model displays similar effects as the above ones, which means that the explicit time-dependence of the Hamiltonian/Lindbladian of the first two examples is not relevant to the discussion. In Fig. 4.3 we show the fidelity $\mathcal{F}(t) = \frac{1}{2}\left(e^{-g\gamma t}+1\right)$

(dotted green line) of being in a coherent superposition of ground and excited state obtained from the dynamics generated by the Lindbladian (4.2.25). The solid blue line shows the reduced dynamics of the shallow pocket model under dynamical decoupling (4.2.26).

The shallow pocket model is a counterexample to dynamical decoupling working for non-exponential decay only. For a fixed decoupling time τ the fidelity never drops below $\mathcal{F}(\tau)$. The model also highlights some of the unpleasant mathematical properties required for modeling strict exponential decay: the initial state of the system is not in the domain of the interaction[2], which in turn is unbounded below and above [6]. Such properties indicate that the general proof below requires a certain degree of mathematical precision. In order to proof that unbounded interactions can be decoupled we need the mathematical terminology provided in the previous section (Sec. 4.1.2).

General proof

It is a fact of nature and an ubiquitous challenge in the mathematical treatment of quantum mechanics that unbounded Hamiltonians cannot be defined everywhere [126, Chapter VIII]. We saw in Sec. 4.1.2 that a domain $D(H)$ has to be specified in order to make a clear sense of an unbounded Hamiltonian H. Remember that for example the notion of self-adjointness, properties of a sum $H_1 + H_2$, etc has to take the domain into account. Starting with a pioneering work of von Neumann a machinery has been developed with a purpose to circumvent these problems when dealing with a derived quantum mechanical phenomena. This is precisely our case, we show that whenever a Hamiltonian which couples a finite-dimensional system of size d to an infinite-dimensional bath can be reasonably defined then it can be decoupled perfectly.

All Hamiltonians under our consideration have a sum-like structure consisting of the system\bath free Hamiltonians and the interactions. As we pointed out a core

[2]Nevertheless, its time evolution is well defined due to the interaction being self-adjoint

$\mathcal{C}$ of an operator [126] is then a natural notion to make sense of this sum in the most general setting. We postpone this technical discussion by few paragraphs and start with a natural – albeit less general – setting where this notion is not needed. It includes for example the case when the interaction Hamiltonian is relatively bounded with respect to the free Hamiltonian.

We assume that a Hamiltonian describing the system is a densely defined self-adjoint operator of the form $H = H_S \otimes \mathbb{1} + \mathbb{1} \otimes H_B + \sum_\alpha S_\alpha \otimes R_\alpha$ on the tensor product Hilbert space $\mathcal{H}_{SB} = \mathcal{H}_S \otimes \mathcal{H}_B$, with H_B itself self-adjoint on a dense domain $D(H_B)$ and $D(H) = \mathbb{C}^d \otimes D(H_B)$. For simplicity we only consider deterministic decoupling schemes here, while the random case can be proved using [139, Th.2.2] (c.f. forthcoming work for details). The announced perfect decoupling of such a Hamiltonian might be surprising given that the usual derivation of dynamical decoupling hinges on a perturbative expansion $\exp(-i\Delta t A) \sim \mathbb{1} + -i\Delta t A + O(\Delta t^2)$ and a limit formula

$$\left(1 + \frac{A}{n} + O(n^{-2})\right)^n \to \exp(A). \tag{4.2.27}$$

In particular all standard error bounds [107] become infinite for unbounded Hamiltonians. These apparent problems can be circumvented by means of a deep generalization of the above limit formula due to Chernoff [140], c.f. also [141, Chapter 8.]: *Let $F(t), ||F(t)|| \leq 1$ be a family of operators on a Hilbert space $\mathcal{H}$ with $F(0) = \mathbb{1}$ and suppose that $(F(t) - \mathbb{1})(\psi)/t \to A\psi$ as $t \to 0$, for every $\psi \in \mathcal{H}$ in a core of A. Then we have*

$$\lim_{n\to\infty} F\left(\frac{t}{n}\right)^n (\psi) = \exp(tA)\psi, \quad \psi \in \mathcal{H}. \tag{4.2.28}$$

We apply Chernoffs theorem with $F(t) = \Pi_{v\in V}\, v^\dagger \exp(-iHt/|V|)v$ and H as above. Then for $\psi \in D(H)$,

$$\frac{(F(t) - \mathbb{1})(\psi)}{t} \to -i\left(\frac{1}{|V|}\sum_{v\in V} v^\dagger H v\right)\psi = -i(\mathbb{1} \otimes H_B)\psi, \tag{4.2.29}$$

due to the decoupling property of V, as $t \to 0$ and for every ψ in the domain of all $v^\dagger H v$'s. Note that the convergence in (4.2.29) is not obvious since the use of the

Taylor series is not well defined for unbounded operators. Along the lines of [142] it can be proven instead on the group level, by rearranging the exponentials in such a way that Stone's theorem (cf. Sec. 4.1.2) can be used. Consider for example as a system a qubit with V the Pauli group. We can evaluate the limit (4.2.29) using

$$\begin{aligned}\frac{(F(t)-\mathbb{1})(\psi)}{t} &= \frac{1}{t}\left(e^{-i\sigma_z H\sigma_z t}-\mathbb{1}\right)\psi + \frac{1}{t}e^{-i\sigma_z H\sigma_z t}\left(e^{-i\sigma_y H\sigma_y t}-\mathbb{1}\right)\psi \\ &+ \frac{1}{t}e^{-i\sigma_z H\sigma_z t}e^{-i\sigma_y H\sigma_y t}\left(e^{-i\sigma_x H\sigma_x t}-\mathbb{1}\right)\psi, \\ &+ \frac{1}{t}e^{-i\sigma_z H\sigma_z t}e^{-i\sigma_y H\sigma_y t}e^{-i\sigma_x H\sigma_x t}\left(e^{-iHt}-\mathbb{1}\right)\psi, \end{aligned} \tag{4.2.30}$$

with $\psi \in \mathbb{C}^2 \otimes D(H_B)$. By assumption all $v^\dagger H v$ are self-adjoint on this domain, so we can apply Stone's theorem for each summand of (4.2.30) yielding the desired result (4.2.29) as t goes to zero. We conclude that perfect dynamical decoupling

$$\lim_{n\to\infty} \mathrm{tr}_B\{\mathcal{E}_t(\rho)\} = \mathrm{tr}_B\left\{e^{-it\mathbb{1}\otimes H_B}\rho e^{it\mathbb{1}\otimes H_B}\right\} = \mathrm{tr}_B\{\rho\}, \tag{4.2.31}$$

is possible where ρ is the density operator of the system and the bath.

Notice that many examples including the shallow pocket model verify the above assumptions of self-adjointness. Nevertheless, we aim for even bigger generality and to achieve this we introduce the notion of a core into our discussion. We review that a core of an operator is a subspace of its domain such that restriction of the operator to the core and subsequent closure gives back the original operator. Clearly the domain itself is a core, but it might be too big in certain applications like the present one.

We may assume that H is formally given as above with some unknown dense domain $D(H)$, with H_B and each R_α self-adjoint on certain dense domains $D(H_B)$ and $D(R_\alpha)$, which might be different, but with all H_B and R_α having a common core $\mathcal{C}$. This is the minimal assumption to make in order to have the sum definition of H well-defined at all. Under this assumption the sum $\sum_{v\in V} v^\dagger H v$ is then also well-defined on $\mathbb{C}^d \otimes \mathcal{C}$ and its closure is exactly (an extension of) $\mathbb{1}\otimes H_B$. For any $\psi \in \mathbb{C}^d \otimes \mathcal{C}$ the conditions of Chernoff's theorem, and in particular $(F(t)-\mathbb{1})(\psi)/t \to -i(\mathbb{1}\otimes H_B)\psi$, are then satisfied, so (4.2.29) follows again.

Clearly if H is self-adjoint with domain $\mathbb{C}^d \otimes D(H_B)$ then all $v^\dagger H v$ are also self-adjoint on that domain, but there are cases of H with different domains, and that is when the above criterion with cores is needed.

We now discuss the question of how small Δt needs to be to efficiently decouple. For bounded operators, *the motion induced by the decoupling field needs to be faster than the fastest time-scale characterizing the unwanted interactions* [106]. In the unbounded case, such a simple time-scale defined only by the interaction cannot be provided, as the convergence speed also crucially depends on the state, given by the speed of convergence of Chernoffs Theorem (4.2.28). Clearly there exist a $\tau(\psi, \epsilon) = \frac{t}{n}$ larger than zero for which $F(\tau)^n \psi$ is up to an error ϵ given by $\exp(tA)\psi$. Assuming that system and bath are initially uncorrelated, we may (through purification) without loss of generality assume that the initial bath state ψ_B is pure. We can then define $\tau(\epsilon) = \inf_{\psi_S} \tau(\psi_S \otimes \psi_B) > 0$ as the critical time-scale for dynamical decoupling, where we used that the system space is finite-dimensional. This time-scale is harder to calculate than the finite-dimensional one, but we see a priori reasons why it should be much smaller than the latter.

4.2.4 Conclusion

So far we have considered the two extreme cases in which either extrinsic or intrinsic decoherence is present assuming the two mechanisms take place with the same decay rate. Clearly in an experimental situation both, a mixture $\mathcal{L} = \mathcal{L}_{\text{int}} + \mathcal{L}_{\text{ext}}$ of extrinsic and intrinsic decoherence could be present. In this case, the asymptotic behavior of the gate error would be between those two extremal cases. It seems difficult to determine a general precise value, but estimates for the amount of intrinsic decoherence can be obtained based on the bounds $\|\overline{\mathcal{L}}_{\text{int}}\| \leq \|\mathcal{L}_{\text{int}}\|$. The effective Lindbladian $\overline{\mathcal{L}}_{\text{int}}$ can be determined using process tomography. For intrinsic decoherence decay rates predicted by collapse models we are at present a few orders of magnitude away from the regime in which this becomes feasible. But with current advances in qubit design and a world-wide effort to increase the number of clean

qubits this could come within reach soon.

Our results pave the way towards the experimental verification of collapse models (CM) – despite the presence of (extrinsic) decoherence. Even if the quantum noise is due to some unbounded coupling to an infinite dimensional environment we proved that the system evolution can be decoupled and hence distinguished from CM. Furthermore, this decoupling of unbounded Hamiltonians has applications in quantum engineering beyond the scope of this paper. It is fascinating to contemplate that in the vast experimental evidence for dynamical decoupling such CMs have already been discovered.

4.2.5 Review and implications

Here we want to review the results from the last section and discuss briefly some of its implications. So what have we shown? We have proven that any non-unitary time evolution, described by some Lindblad operator, cannot be decoupled with unitary decoupling operations. Hence, if there exists intrinsic non-unitary dynamics, we can verify it by applying dynamical decoupling infinitely fast. Such non-unitary dynamics can for example arise from non-linear stochastic extensions of the Schrödinger equation, such as collapse models introduced in Sec. 2.1.5. Clearly we have not studied dynamical decoupling on the level of non-linear stochastic Schrödinger equations and hence we did not prove that such modifications of the Schrödinger equation cannot be decoupled. Intuitively we expect that this is not possible since any non-linear extension changes the unitary character of the Schrödinger equation. Nevertheless, our results suggest that if we apply dynamical decoupling, and if we observe a saturation of the decoupling error above zero, we can conclude that this behavior is a strong hint for the presence of some intrinsic non-unitary noise terms.

Regarding the control of open quantum systems our results also show the difference between the two approaches introduced in Sec. 2.3. Consider dephasing in σ_z direction and unitary control in σ_x direction within the first approach. The Lindbladian

of the controlled master equation then reads

$$\mathcal{L}_t(\rho) = -iu(t)[\sigma_x, \rho] + \mathcal{L}(\rho), \tag{4.2.32}$$

with $u(t)$ being the control field and $\mathcal{L}(\cdot)$ is given by (4.2.25). If we now consider a bang-bang dynamical decoupling sequence we cannot stop the system from dephasing since $\sigma_x \mathcal{L}(\sigma_x \rho \sigma_x)\sigma_x = \mathcal{L}(\rho)$. On the other hand, regarding the second approach, such a dephasing process can be obtained from the shallow pocket model for which the controlled total Hamiltonian is given by

$$H(t) = u(t)\sigma_x \otimes \mathbb{1} + \sigma_z \otimes \hat{x}. \tag{4.2.33}$$

We have shown that the shallow pocket model can be decoupled perfectly, meaning that here a bang-bang decoupling sequence can stop the system from dephasing. In

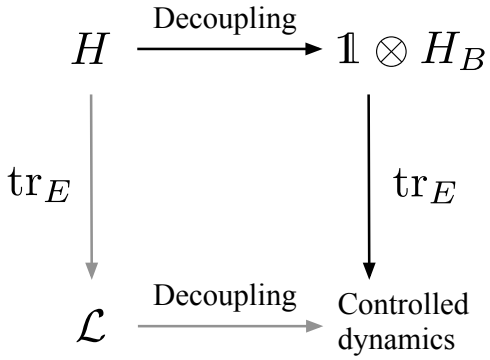

Figure 4.4.: Schematic illustration of the two approaches for describing the control of an open quantum from Sec. 2.3 for dynamical decoupling. The resulting controlled dynamics depends on which approach was taken. Following the black arrows, decoherence can always be suppressed through dynamical decoupling, i.e. the controlled dynamics is given by the identity map. For the grey arrows dynamical decoupling will never succeed. Here the controlled dynamics is given by $\exp(\bar{\mathcal{L}}t)$.

other words, within the system-environment Hamiltonian approach we can suppress decoherence completely, whereas in the controlled master equation description for controlling an open quantum system this is not possible. The example shows that the two descriptions lead to extremely different predictions for the controlled system dynamics. See Fig. 4.4 for an illustration of the two different approaches for

describing the control of an open quantum system regarding dynamical decoupling.

Another interesting consequence of our results is related to the definition of Markovian dynamics. In the literature an open quantum system is typically referred to be Markovian if the system dynamics is generated by a time-independent Lindblad operator, that is, if it is described by a one parameter semigroup, in particular if the resulting CPTP map is divisible [143]. This definition is usually considered to be related to the properties of the system-environment interaction and the memory timescales of the environment alone, rather than being dependent on what we do with the quantum system [144]. The shallow pocket model shows that, if we define Markovian dynamics in the above sense, the characterization of an open quantum system in terms of Markovian and non-Markovian is questionable if only properties of the environment and its interactions with the system are taken into account. Without acting on the system, the shallow pocket model leads, without approximations, to an exponential decay and therefore we can refer to it as a Markovian open system. On the other hand, if we apply dynamical decoupling, the resulting dynamics is given by the identity map and thus "highly" non-Markovian since we reverse the flow of information through the decoupling operations. We conclude that if we take a semigroup dynamics to be the definition of a Markovian evolution, Markovianity not only depends on the properties of the environment, but also on how we interact with the system. Looking at it from a different angle, the derivation of a Markovian master equation, in the case where the system is subject to some time dependent control fields, is not always possible if we do not constrain the control fields. The relevant timescales of the system depend on the controls. A comparison with the timescales of the environment in order to apply the Born-Markov approximation becomes problematic. To conclude, the statement that a non-Markovian environment is needed in order to make dynamical decoupling work [3] becomes meaningless since the notion of Markovianity depends on what we do with the system. Quite the opposite, it was recently shown that non-Markovianity is sometimes detrimental for the efficiency of dynamical decoupling [145].

4.3 Dynamical decoupling of quadratic Hamiltonians

In preparation:

C. Arenz, R. Hillier, and D. Burgarth (2016).

In the last sections we saw that any finite dimensional quantum system can always be decoupled from the environment using unitary operations acting only on the system. This is true even if the environment is modeled as an infinite dimensional system described by some unbounded operators. Hence, for every finite dimensional quantum system, decoherence that is induced by interactions with the environment can be perfectly suppressed by applying the decoupling operations infinitely fast. Mathematically speaking, the decoupling condition (4.1.17) can always be satisfied, independently of the system-environment interaction that is present. This can be achieved by using an irreducible representation of a finite subgroup of the unitary group as a decoupling set. The question arises if the results for finite dimensional quantum systems can be one-to-one translated into an infinite dimensional setting? Is it possible to suppress decoherence for systems that are, for example, described by quantum harmonic oscillators even if the details of the interaction with the environment are unknown? Before we treat this problem we begin with an example from [134] which is, up to best of our knowledge, the only study that can be found in the literature where dynamical decoupling is investigated for a specific infinite dimensional system. Afterwards we prove that not every infinite dimensional system can be decoupled from the environment and thus protected from decoherence using dynamical decoupling. Then we focus on an important class of Hamiltonians, i.e Hamiltonians which are quadratic in the quadrature operators $\hat{x}, \hat{p}$ so that the dynamics can be represented by a symplectic transformation. Considering such Hamiltonians, which were already introduced in Sec. 2.2.1, we start by formulating

dynamical decoupling within the framework of control theory, in order to adopt the tools that were described in Sec. 4.1. This will lead to the definition of a decoupling condition, the characterization of a decoupling set and the formulation of random dynamical decoupling in a symplectic setting. Afterwards we investigate dynamical decoupling in order to suppress decoherence that is induced by an interaction described through a quadratic Hamiltonian.

We start by considering a quantum harmonic oscillator described by the bosonic annihilation and creation operators a and $a^\dagger$. The harmonic oscillator interacts with a bosonic environment through

$$H_{S,E} = \sum_k g_k (a b_k^\dagger + a^\dagger b_k), \tag{4.3.1}$$

where $b_k, b_k^\dagger$ are the bosonic annihilation and creation operators of the environmental modes and g_k are real coupling constants describing the strength of the interaction with each mode. The total Hamiltonian then reads

$$H = \omega a^\dagger a + \sum_k \omega_k b_k^\dagger b_k + H_{S,E}, \tag{4.3.2}$$

where ω, ω_k are the frequencies of the system and the environmental oscillators respectively. This model has been investigated a lot in the literature (see e.g. [194] for an overview) and it describes, for example, decoherence and photon losses of a quantized cavity mode which interacts with the free electromagnetic field outside the cavity. Now suppose we are able to instantaneously perform the decoupling operation

$$v = e^{-i\pi a^\dagger a}, \tag{4.3.3}$$

which corresponds to a π-rotation of the system oscillator in phase space. Note that $v^\dagger H_{S,E} v = -H_{S,E}$ such that, analogous to the simple qubit example from Sec. 4.1 (see equation (4.1.1) and (4.1.2)), the system-environment interaction can be suppressed in the limit of infinitely fast decoupling. In other words, decoherence that is caused by an interaction (4.3.1) can always be suppressed if we are able to rotate the system oscillator quickly around π. In [134] the efficiency of this scheme was

studied in detail in terms of spectral properties of the environment. For our purposes what is important is that a system-environment interaction of the form (4.3.1) can always be suppressed by applying the decoupling operation (4.3.3). For generic system-environment interactions we cannot always expect this to be possible with the decoupling operation v, since the scheme relies on reversing the sign in front of the interaction Hamiltonian. Hence the question arises if, analogous to the finite dimensional case, a decoupling condition can be formulated and a set of unitary operations can be found satisfying this condition. First we show that such a set cannot exist for generic interactions in an infinite dimensional setting by considering a specific class of system operators H.

Theorem 4.2 *Let H be a self-adjoint operator with a positive unbounded spectrum. Under the usual domain assumptions there does not exist a decoupling set V with $|V| < \infty$ satisfying*

$$\frac{1}{|V|}\sum_{v\in V} v^\dagger H v = \lambda \mathbb{1}, \qquad \lambda \in \mathbb{R}. \tag{4.3.4}$$

Proof. We first notice that the left-hand side of (4.3.4) is given by a positive map. Hence we need $\lambda \geq 0$. In the next step we want to show that λ must be strictly positive. We show this by contradiction. Assume there exist a non-empty decoupling set V of unitary operations such that

$$\sum_{v\in V} v^\dagger H v = 0, \tag{4.3.5}$$

is satisfied. Then equivalently

$$\sum_{v\in V} \left\| H^{1/2} v\phi \right\|^2 = 0 \tag{4.3.6}$$

$$\Leftrightarrow \left\| H^{1/2} v\phi \right\|^2 = 0, \qquad \forall v \in V \tag{4.3.7}$$

$$\Leftrightarrow \left\| v\phi \right\|^2 = 0, \qquad \forall v \in V \tag{4.3.8}$$

has to hold for all states ϕ in a common core $\mathcal{C}$ of all $v^\dagger H v$. Since $\|v\phi\| = \|\phi\| = 1$, we need $v = 0$ for all $v \in V$ which contradicts our initial assumption. Hence λ must

be strictly positive. Next we show that there cannot exist a decoupling set satisfying (4.3.4) with $\lambda > 0$. Again we show this by contradiction. We assume that (4.3.4) with $\lambda > 0$ holds. Then we can multiply (4.3.4) from the left with some $v \in V$ and from the right with its adjoint such that

$$H + \sum_{\tilde{v}\in\tilde{V}} v^\dagger H v = \tilde{\lambda}\mathbb{1}, \tag{4.3.9}$$

has to hold where $\tilde{\lambda} = |V|\lambda$. Now, multiplying both sides by a density operator ρ and taking the trace we find equivalently

$$\langle H\rangle_\rho + \sum_{\tilde{v}\in\tilde{V}} \mathrm{tr}\{v^\dagger H v \rho\} = \tilde{\lambda}\mathbb{1}. \tag{4.3.10}$$

Since H has a positive unbounded spectrum we can always pick a state ρ such that $\langle H\rangle_\rho = \tilde{\lambda}$. Thus we need

$$\sum_{\tilde{v}\in\tilde{V}} \mathrm{tr}\{\tilde{v}^\dagger H \tilde{v} \rho\} = 0, \tag{4.3.11}$$

$$\Leftrightarrow \sum_{\tilde{v}\in\tilde{V}} \left\| H^{1/2}\tilde{v}\phi_j \right\|^2 = 0, \tag{4.3.12}$$

which cannot be satisfied as shown before in steps (4.3.5)-(4.3.8). Therefore there does not exist a decoupling set V satisfying the condition (4.3.4). □

Theorem 4.2 implies that interactions described by system operators with a positive unbounded spectrum can never be suppressed with a finite number of unitary decoupling operations. Thus it is not possible to protect such systems from decoherence using dynamical decoupling. The theorem also shows that, for example, a harmonic oscillator can never be stopped oscillating by applying unitary decoupling operations infinitely fast. An alternative proof that the condition (4.3.4) cannot hold for all operators acting on an infinite dimensional space can easily be obtained using an argument based on the commutant. In an infinite dimensional space the commutant of a finite dimensional decoupling set always contains infinitely many elements. Therefore not all operator acting on an infinite dimensional space can be mapped onto the identity and a scalar factor using finitely many unitary operations. In fact

there are infinitely many operators that commute with the decoupling operations and thus they are unaffected by dynamical decoupling. However, the statement of theorem 4.2 is stronger, since it characterizes a class of self-adjoint operators that cannot be decoupled.

We saw that we cannot decouple every infinite dimensional system from the environment. In the following we study in more detail an important class of infinite dimensional system, i.e. those that can be described by quadratic Hamiltonians. This has the advantage that we can avoid the mathematical difficulties arising in infinite dimensional systems by representing the dynamics through a symplectic transformation on a finite dimensional space. We start by formulating dynamical decoupling of quadratic Hamiltonians in the context of control theory using the framework of symplectic transformations that was established in Sec. 2.2.1.

4.3.1 Quantum control theory framework

Following the control theory framework introduced in Sec. 4.1 we divide the total Hamiltonian $H(t) = H_0 + H_c(t) \otimes \mathbb{1}$ into the system-environment Hamiltonian H_0 and the controller $H_c(t)$, both assumed to be quadratic in the quadrature operators $\hat{x}, \hat{p}$ such that they are given in the form (2.2.13). Along the lines of the presentation of control theory of quadratic Hamiltonians in Sec. 2.2.1 the total evolution is then given by a symplectic transformation $S(t) \in Sp(2n, \mathbb{R})$, which obeys the differential equation

$$\frac{d}{dt}S(t) = (\mathcal{G}_0 + \mathcal{G}_c(t))S(t), \tag{4.3.13}$$

where $\mathcal{G}_0 = -A_0\Omega$ encompasses the system-environment interaction and $G_c(t) = -A_c(t)\Omega$ is the description of the controller in the symplectic picture. Remember that $A \in \mathbb{R}^{2n\times 2n},\ A = A^T$ is a real and symmetric matrix with n being the number of oscillators of the total system and Ω is the symplectic form given by (2.2.7). Analogous to the unitary case the first step is to separate the action of the controller from the rest of the dynamics. This can easily be done by noting that

$S(t) = S_c(t)\tilde{S}(t)$ solves (4.3.13) with

$$S_c(t) = \hat{T} \exp\left(\int_0^t dt' \, \mathcal{G}_c(t') \right), \tag{4.3.14}$$

being the action of the controller alone and

$$\tilde{S}(t) = \hat{T} \exp\left(\int_0^t dt' \, S_c^{-1}(t') \mathcal{G}_0 S_c(t') \right). \tag{4.3.15}$$

We assume that the controller is periodic and cyclic, i.e. $\mathcal{G}_c(t + NT_c) = \mathcal{G}_c(t)$ with $N \in \mathbb{N}$ and $S_c(NT_c) = \mathbb{1}$, such that

$$S(NT_c) = \tilde{S}(NT_c) = (\tilde{S}(T_c))^N. \tag{4.3.16}$$

Using the Magnus expansion $\bar{\mathcal{G}} = \sum_{l=0}^{\infty} \bar{\mathcal{G}}^{(l)}$ (cf. Sec. 4.1) we find

$$S(NT_c) = e^{NT_c \bar{\mathcal{G}}}, \tag{4.3.17}$$

where the first order of the Magnus expansion is given by

$$\bar{\mathcal{G}}^{(0)} = \frac{1}{T_c} \int_0^{T_c} dt \, S_c^{-1} \mathcal{G}_0 S_c(t). \tag{4.3.18}$$

If we assume that each cycle consists of a bang-bang decoupling sequence, enforced by symplectic decoupling operations $g_1, \ldots, g_M \in G \subset Sp(2n, \mathbb{R})$ separated by Δt such that $T_c = M\Delta t$, the first order of the Magnus expansion becomes

$$\bar{\mathcal{G}}^{(0)} = \frac{1}{T_c} \sum_{k=1}^{M} \Delta t S_k^{-1} \mathcal{G}_0 S_k, \tag{4.3.19}$$

with $S_k = g_{k-1}^{-1} \ldots g_0^{-1}$ and $g_0 = \mathbb{1}$. If the decoupling operations are taken from a decoupling set G which forms a group, steering each cycle through the group elements, $M = |G|$, leads to

$$\bar{\mathcal{G}}^{(0)} = \frac{1}{|G|} \sum_{g \in G} g \mathcal{G}_0 g^{-1} = -\frac{1}{|G|} \sum_{g \in G} g A_0 \Omega g^{-1}. \tag{4.3.20}$$

We thus obtain an averaged dynamics that is similar to the one obtained in the unitary framework. One might wonder if, analogous to the unitary case, there does exist a decoupling set G, such that for all A_0 the first order of the Magnus expansion is just given by the identity and a scalar factor. In the limit of infinitely fast decoupling ($\Delta t \to 0$), keeping the total time fixed, this would allow us to "freeze" the evolution.

4.3.2 Decoupling condition, decoupling set and random decoupling

To begin with we want to analyze if there exists a finite decoupling set $G \subset Sp(2n, \mathbb{R})$ satisfying the condition

$$\frac{1}{|G|} \sum_{g \in G} g A \Omega g^{-1} = \lambda \mathbb{1}_{2n \times 2n}, \quad \forall A \in \mathbb{R}^{2n \times 2n},\ A = A^T, \tag{4.3.21}$$

for some $\lambda \in \mathbb{R}$. We define

$$\Pi(A) := \frac{1}{|G|} \sum_{g \in G} g A g^T, \tag{4.3.22}$$

and we use $g^T \Omega = \Omega g^{-1}$ yielding with $\Omega^{-1} = -\Omega$ that equivalently

$$\Pi(A) = -\lambda \Omega, \tag{4.3.23}$$

has to hold for all real and symmetric matrices A. Since $\Pi(A)$ is symmetric and $\Omega^T = -\Omega$ we first notice that $\lambda = 0$, such that condition (4.3.21) becomes

$$\Pi(A) = 0, \quad \forall A \in \mathbb{R}^{2n \times 2n},\ A = A^T. \tag{4.3.24}$$

Note that if there would exist a set of symplectic decoupling operations G that satisfies the above condition for all real and symmetric matrices A, infinitely fast decoupling would lead to a modified evolution given by the identity. Like in the unitary case the whole system does not evolve anymore. We already saw that this is in general not possible for infinite dimensional systems, which is reflected in the fact that for non-interacting harmonic oscillators ($A = \mathbb{1}$) the condition (4.3.24) can never be satisfied. However maybe we are demanding too much. Therefore we will now study in more detail the properties of the map Π for specific decoupling sets G. In particular we will modify condition (4.3.24) and identify decoupling sets satisfying the new conditions.

Modified decoupling condition, homogenization and decoupling sets

The first case we consider is a modification of condition (4.3.21) of the kind

$$\frac{1}{|G|}\sum_{g\in G} gA\Omega g^{-1} = \left(\bigoplus_{j=1}^{n}\lambda_j \mathbb{1}_{2\times 2}\right)\Omega, \qquad \forall A\in\mathbb{R}^{2n\times 2n},\ A=A^T. \tag{4.3.25}$$

Such a modification has the physical meaning that, for instance, a network of interacting harmonic oscillators become decoupled from each other. Instead of requiring that the total dynamics is given by the identity, we now require that the harmonic oscillators do not interact and rotate with frequency λ_j. Equivalently we find a *modified decoupling condition*

$$\Pi(A) = \bigoplus_{j=1}^{n}\lambda_j \mathbb{1}_{2\times 2}, \tag{4.3.26}$$

which has to hold for all real and symmetric matrices A. If we can find a decoupling set G, satisfying the modified decoupling condition (4.3.26), we are able to map a network of interacting harmonic oscillators to a set of non-interacting harmonic oscillators, each rotating with a frequency λ_j.

Theorem 4.3 *The modified decoupling condition* (4.3.26) *can always be satisfied by taking a decoupling set*

$$G = \{R(\phi_1,\ldots,\phi_n)\,|\,\phi_1,\ldots,\phi_n\in\{\pi/2,\pi,3\pi/2,2\pi\}\}, \tag{4.3.27}$$

that consists of local rotations $R\in Sp(2n,\mathbb{R})\cap\bigoplus_{j=1}^{n}SO(2)$ *given by*

$$R(\phi_1,\ldots,\phi_n) = \bigoplus_{j=1}^{n}R(\phi_j) = \bigoplus_{j=1}^{n}\begin{pmatrix}\cos(\phi_j) & -\sin(\phi_j)\\ \sin(\phi_j) & \cos(\phi_j)\end{pmatrix}. \tag{4.3.28}$$

Proof. We partition the symmetric matrix $A\in\mathbb{R}^{2n\times 2n}$ into 2×2 blocks $A^{(i,j)}\in\mathbb{R}^{2\times 2},\ i,j=1,\ldots,n$ according to

$$A = \begin{pmatrix} A^{(1,1)} & A^{(1,2)} & \ldots & A^{(1,n)}\\ A^{(2,1)} & A^{(2,2)} & \ldots & A^{(2,n)}\\ \vdots & \vdots & \ddots & \vdots\\ A^{(n,1)} & A^{(n,2)} & \ldots & A^{(n,n)}\end{pmatrix}, \tag{4.3.29}$$

so that a summand $F(\phi_1,\ldots,\phi_n)(A)$ of $\Pi(A)$, given by (4.3.22), becomes

$$F(\phi_1,\ldots,\phi_n) = \begin{pmatrix} R(\phi_1)A^{(1,1)}R^T(\phi_1) & R(\phi_1)A^{(1,2)}R^T(\phi_2) & \cdots & R(\phi_1)A^{(1,n)}R^T(\phi_n) \\ R(\phi_2)A^{(2,1)}R^T(\phi_1) & R(\phi_2)A^{(2,2)}R^T(\phi_2) & \cdots & R(\phi_2)A^{(2,n)}R^T(\phi_n) \\ \vdots & \vdots & \ddots & \vdots \\ R(\phi_n)A^{(n,1)}R^T(\phi_1) & R(\phi_n)A^{(n,2)}R^T(\phi_n) & \cdots & R(\phi_n)A^{(n,n)}R^T(\phi_n) \end{pmatrix}. \quad (4.3.30)$$

Since the blocks on the diagonal are symmetric we have that

$$\sum_{\phi_i\in\{\pi/2,\pi,3\pi/2,2\pi\}} R(\phi_i)A^{(i,i)}R^T(\phi_i) = 2\mathrm{tr}\{A^{(i,i)}\}, \quad (4.3.31)$$

$$\sum_{\phi_i,\phi_j\in\{\pi/2,\pi,3\pi/2,2\pi\}} R(\phi_i)A^{(i,j)}R^T(\phi_j) = 0 \quad \text{for } i\neq j, \quad (4.3.32)$$

and thus we arrive at

$$\begin{aligned}\Pi(A) &= \frac{1}{|G|}\sum_{\phi_1,\ldots,\phi_n\in\{\pi/2,\pi,3\pi/2,2\pi\}} F(\phi_1\ldots,\phi_n)(A), \\ &= \frac{1}{2}\bigoplus_{j=1}^{n}(A_{2j-1,2j-1}+A_{2j,2j})\mathbb{1}_{2\times 2}, \end{aligned} \quad (4.3.33)$$

which is the desired result. We identify $\lambda_j = \frac{1}{2}(A_{2j-1,2j-1}+A_{2j,2j})$. □

In other words, we can always map a network of interacting harmonic oscillators to a system of non-interacting harmonic oscillators if we are able to rotate each oscillator rapidly around $\pi/2,\pi,3\pi/2,2\pi$. Note that the decoupling set (4.3.27) forms a group (mod 2π) with $|G|=4^n$, and that the cyclic condition $S_c(T_c)=\prod_{g\in G} g = \mathbb{1}_{2n\times 2n}$ is fulfilled. Hence the toggling frame, which is obtained through a transformation with $S_c(T_c)$, coincides with the physical frame at times $t = NT_c$ with $N\in\mathbb{N}$. At this stage one might wonder why we haven chosen the framework of control theory in order to formulate dynamical decoupling for quadratic Hamiltonians. We could have simply looked at a decoupling sequence of the form (4.1.5), yielding with the help of the generalized Trotter formula (4.1.19), the same results. The reason lies in the observation that if we separate the controller from the system dynamics in such

a way that we are able to control the frequencies λ_j of the harmonic oscillators,

$$A_c(t) = \bigoplus_{j=1}^{n} \lambda_j(t) \mathbb{1}_{2\times 2}, \tag{4.3.34}$$

and

$$(A_0)_{j,j} = 0, \quad \forall j = 1, \ldots, 2n, \tag{4.3.35}$$

we can achieve $\bar{\mathcal{G}}^{(0)} = 0$ by taking the decoupling operations from the decoupling set (4.3.27). If we separate the controller and the system dynamics in this way the original decoupling condition (4.3.24) can be satisfied. This is not really surprising, because roughly speaking we use the operations that cannot be decoupled to decouple the rest.

In the preceding considerations we required that the harmonic oscillators do not interact, and that they rotate with frequencies λ_j if we apply the decoupling operations infinitely fast. We found a set of operations (4.3.27) which allows us to obtain the desired dynamics. Instead of demanding that the harmonic oscillators rotate with different frequencies, we can ask whether there exists a set of symplectic operations so that, in the limit of infinitely fast decoupling, the harmonic oscillators are decoupled from each other and rotate with the same frequency λ. We call this process *homogenization*. Mathematically speaking we need a set of symplectic operations G satisfying the *homogenization condition*

$$\Pi(A) = \lambda \mathbb{1}_{2n\times 2n}, \tag{4.3.36}$$

for all real and symmetric matrices A. In order to find such a decoupling set we will now work under the assumption that the symplectic form J is given by

$$J = \begin{pmatrix} 0 & \mathbb{1}_{n\times n} \\ -\mathbb{1}_{n\times n} & 0 \end{pmatrix}, \tag{4.3.37}$$

referring to the preliminary section for further details.

Theorem 4.4 *The homogenization condition* (4.3.36) *can always be satisfied with a decoupling set*

$$G = \langle O(n,\mathbb{Z}) \oplus O(n,\mathbb{Z}), J \rangle \,, \tag{4.3.38}$$

forming a group generated by $O(n,\mathbb{Z}) \oplus O(n,\mathbb{Z})$ *with* $O(n,\mathbb{Z})$ *being the orthogonal group over* $\mathbb{Z}$ *and* J *is the symplectic form* (4.3.37).

Proof. We first partition A into four $n \times n$ blocks $A^{(i,j)},\ i,j = 1,2$ where $(A^{(1,2)})^T = A^{(2,1)}$ and $(A^{(i,i)})^T = A^{(i,i)}$. We further note that we can write

$$\Pi(A) = \frac{1}{|G|} \sum_{g \in O(n,\mathbb{Z})} \left[\begin{pmatrix} gA^{(1,1)}g^T & gA^{(1,2)}g^T \\ gA^{(1,2)}g^T & gA^{(2,2)}g^T \end{pmatrix} + J \begin{pmatrix} gA^{(1,1)}g^T & gA^{(1,2)}g^T \\ gA^{(2,1)}g^T & gA^{(2,2)}g^T \end{pmatrix} J^T \right]. \tag{4.3.39}$$

Next we show that there exists a finite subset $D \subset O(n,\mathbb{Z})$ acting irreducibly

$$\frac{1}{|D|} \sum_{g \in D} gBg^T = \frac{\mathrm{tr}\{B\}}{n} \mathbb{1}_{n\times n}, \qquad \forall B \in \mathbb{R}^{n\times n}, \tag{4.3.40}$$

i.e the commutant D' is given by multiples of the identity. We can construct such a subset by noticing that a $n \times n$ permutation matrix P_π with entries $\pm 1, 0$ is an orthogonal matrix and hence the set of permutation matrices is a subset of $O(n,\mathbb{Z})$. Such a permutation matrix permutes the elements of a vector and changes the signs of its entries according to

$$P_\pi (v_1, ..., v_n)^T = (\pm v_{\pi(1)}, \ldots, \pm v_{\pi(n)})^T. \tag{4.3.41}$$

Note that through linear combinations we can for instance create the vector $(v_{\pi(1)}, 0, \ldots, 0)^T$. Hence the vectors that are obtained in this way span the whole space. Since there is no invariant subspace we have found a finite irreducible representation of $O(n,\mathbb{Z})$ given by the permutation matrices with entries $\pm 1, 0$. Then (4.3.39) becomes

$$\Pi(A) = \frac{1}{2} \begin{pmatrix} \frac{\mathrm{tr}\{A^{(1,1)}\}}{n} \mathbb{1}_{n\times n} & \frac{\mathrm{tr}\{A^{(1,2)}\}}{n} \mathbb{1}_{n\times n} \\ \frac{\mathrm{tr}\{A^{(2,1)}\}}{n} \mathbb{1}_{n\times n} & \frac{\mathrm{tr}\{A^{(2,2)}\}}{n} \mathbb{1}_{n\times n} \end{pmatrix} + \frac{1}{2} J \begin{pmatrix} \frac{\mathrm{tr}\{A^{(1,1)}\}}{n} \mathbb{1}_{n\times n} & \frac{\mathrm{tr}\{A^{(1,2)}\}}{n} \mathbb{1}_{n\times n} \\ \frac{\mathrm{tr}\{A^{(2,1)}\}}{n} \mathbb{1}_{n\times n} & \frac{\mathrm{tr}\{A^{(2,2)}\}}{n} \mathbb{1}_{n\times n} \end{pmatrix} J^T, \tag{4.3.42}$$

and with

$$J\begin{pmatrix} a_1\mathbb{1}_{n\times n} & a_2\mathbb{1}_{n\times n} \\ a_3\mathbb{1}_{n\times n} & a_4\mathbb{1}_{n\times n}\end{pmatrix}J^T = \begin{pmatrix} a_4\mathbb{1}_{n\times n} & -a_3\mathbb{1}_{n\times n} \\ -a_2\mathbb{1}_{n\times n} & a_1\mathbb{1}_{n\times n}\end{pmatrix}, \tag{4.3.43}$$

we arrive at

$$\Pi(A) = \frac{1}{2n}\begin{pmatrix} (\mathrm{tr}\{A^{(1,1)}\}+\mathrm{tr}\{A^{(2,2)}\})\mathbb{1}_{n\times n} & (\mathrm{tr}\{A^{(1,2)}\}-\mathrm{tr}\{A^{(2,1)}\})\mathbb{1}_{n\times n} \\ (\mathrm{tr}\{A^{(2,1)}\}-\mathrm{tr}\{A^{(1,2)}\})\mathbb{1}_{n\times n} & (\mathrm{tr}\{A^{(2,2)}\}+\mathrm{tr}\{A^{(2,2)}\}\mathbb{1}_{n\times n}\end{pmatrix}, \tag{4.3.44}$$

$$= \frac{\mathrm{tr}\{A\}}{2n}\mathbb{1}_{2n\times 2n}, \tag{4.3.45}$$

which completes the proof with $\lambda = \frac{\mathrm{tr}\{A\}}{2n}$ being an averaged frequency. □

We note that $O(n,\mathbb{Z}) \oplus O(n,\mathbb{Z})JO^T(n,\mathbb{Z}) \oplus O^T(n,\mathbb{Z}) = J$ and hence $O(n,\mathbb{Z}) \oplus O(n,\mathbb{Z})$ are symplectic operations with determinant 1, and therefore (global) rotations. An irreducible representation is given by the permutation matrices (4.3.41) swapping the coordinates of the oscillators. We conclude that through quickly applied rotations, i.e quickly swapping the coordinates of the oscillators, we can map a network of n harmonic oscillators into one "big" oscillator rotating with an averaged frequency $\lambda = \frac{\mathrm{tr}\{A\}}{2n}$.

Random dynamical decoupling

Instead of taking the decoupling operations deterministically from a decoupling set G, we now consider the case in which the decoupling operations $g_j \in G$ with $j = 1,\ldots,N$ are taken uniformly random from G. Moving away from the control theory based approach we study a dynamics that is modified according to

$$S(\Delta tN) = \prod_{j=1}^{N} g_j e^{-A\Omega\Delta t} g_j^{-1}, \tag{4.3.46}$$

describing a random walk on the symplectic group. Keeping the total time $T = N\Delta t$ fixed, in the limit of infinitely fast decoupling operations the expected dynamics is governed by [135]

$$\bar{\mathcal{G}} := -\frac{1}{|G|}\sum_{g\in G} gA\Omega g^{-1}. \tag{4.3.47}$$

We saw in the last section that if the decoupling set is given by local rotations R, then the modified decoupling condition (4.3.26) is satisfied, such that the expected dynamics is, in first order of Δt, described by the evolution of uncoupled harmonic oscillators, i.e. $\bar{\mathcal{G}} = -\frac{1}{2}\left(\bigoplus_{j=1}^{n}(A_{2j-1,2j-1} + A_{2j,2j})\mathbb{1}_{2\times 2}\right)\Omega \equiv -A_G\Omega$. Instead of sampling from a decoupling set (4.3.27) with finitely many elements we can also sample from a set containing infinitely many rotations

$$G_\infty = \{R(\phi_1, \ldots, \phi_n) \,|\, \phi_1, \ldots, \phi_n \in [0, 2\pi]\}. \tag{4.3.48}$$

The lowest order in Δt is then given by the Haar average

$$\begin{aligned}\bar{\mathcal{G}} &= -\frac{1}{(2\pi)^n}\int_0^{2\pi} d\phi_1 \ldots \int_0^{2\pi} d\phi_n \, R(\phi_1, \ldots, \phi_n) A\Omega R^{-1}(\phi_1, \ldots, \phi_n), \\ &= -A_G\Omega,\end{aligned} \tag{4.3.49}$$

yielding the same result. In the limit of infinitely fast decoupling operations it does not matter whether we sample from a finite decoupling set (4.3.27) or an infinite decoupling set (4.3.48). In Fig. 4.5 we numerically studied the performance of the two different random decoupling schemes by investigating the gate error

$$\epsilon = \|S_G - S(T)\|_{\mathrm{HS}}^2 \,, \tag{4.3.50}$$

for different values of the temporal spacing Δt of the decoupling operations and a goal evolution $S_G = \exp(-A_G\Omega)$. As a system we took two interacting harmonic oscillators ($n = 2$), described by some randomly chosen real and symmetric matrix A, $T = 1$ and the gate error in Fig. 4.5 was evaluated by taking the average over 20 trajectories. The numerical analysis suggests that even for a finite temporal spacing Δt the performance of random dynamical decoupling is independent of whether we sample from a finite or an infinite set of local rotations. We can conclude that using a random decoupling scheme we can turn a system of interacting harmonic oscillators into non-interacting ones by either rapidly rotating the oscillators uniformly random around $\pi/2,\ \pi,\ 3\pi/2,\ 2\pi$, or uniformly random around angles between 0 and 2π. This might have the advantage that the controller does not need to implement the rotations around fixed angles precisely. Comparing the two random decoupling schemes the performance as a function of Δt is the same.

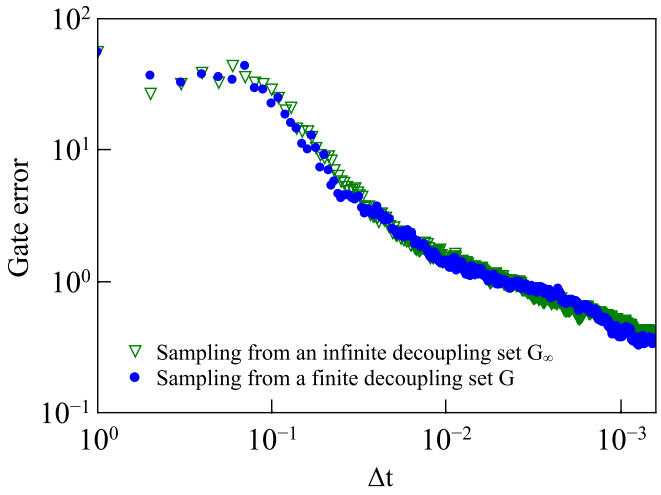

Figure 4.5.: Evaluation of the gate error (4.3.50), on a double logarithmic scale, for different values of Δt for a system of two interacting harmonic oscillators described by a randomly chosen real symmetric matrix A and a total time $T = 1$. The decoupling operations were taken uniformly random from a finite decoupling set (blue dots), given by (4.3.27), and from an infinite decoupling set (green triangles), given by (4.3.48). The gate error was evaluated by taking the average over 20 trajectories.

4.3.3 Suppression of decoherence for quadratic Hamiltonians

Now we want to come back to our initial motivation, the suppression of decoherence induced by generic quadratic system-environment interactions. First of all remember that within the framework of unitary dynamics, in order to suppress decoherence, we need to suppress terms of the form $S_\alpha \otimes B_\alpha$ by acting on the system alone. This led to the decoupling condition $\sum_{v\in V} v^\dagger S_\alpha v = \lambda \mathbb{1}$, which needs to be satisfied for all S_α. We already proved in theorem 4.2 that for infinite dimensional systems this condition can never be satisfied for all S_α's, whereas for finite dimensional systems this is always possible by choosing the decoupling set V to be an irreducible representation of a finite subgroup of the unitary group. In the previous section, using the formalism of symplectic transformations, we constructed two modified "decoupling conditions" for systems that are described by quadratic Hamiltonians. These conditions describe the mapping of a network of harmonic oscillators to oscillators that do not interact, and either rotate with different frequencies or with the same

frequency (homogenization). But note that in this case the desired dynamics is achieved by frequently applying the decoupling operations on the total system. In general we have no access to the environment, and therefore we now want to formulate a condition that allows us to suppress the system-environment interactions if we can only act on the system. We first partition the total system into the system of interest (S) and the environment (E), noting that for symplectic dynamics we have a direct sum structure of the underlying space. We thus write

$$A = \left(\begin{array}{c|c} A_S & I \\ \hline I^T & A_E \end{array}\right), \tag{4.3.51}$$

where $A_S \in \mathbb{R}^{2n_S \times 2n_S}$, $A_E \in \mathbb{R}^{2n_E \times 2n_E}$ are symmetric matrices describing the uncoupled dynamics of S and E, and $I \in \mathbb{R}^{2n_S \times 2n_E}$ describes the interactions between system and environment. Now, if we apply the decoupling operations only on the system, $\tilde{g} = g \oplus \mathbb{1}_{2n_E \times 2n_E}$, we obtain in the limit of infinitely fast decoupling a dynamics governed by

$$\tilde{\Pi}(A) = \left(\begin{array}{c|c} \Pi(A_S) & \frac{1}{|G|}\sum_{g\in G} gI \\ \hline \frac{1}{|G|}\left(\sum_{g\in G} gI\right)^T & A_E \end{array}\right), \tag{4.3.52}$$

with $\tilde{\Pi}(A) = \frac{1}{|\tilde{G}|}\sum_{\tilde{g}\in\tilde{G}} \tilde{g}A\tilde{g}^T$ and $\Pi(\cdot)$ given by (4.3.22). Obviously, in order to suppress the interactions with the environment, we can see that we need a decoupling set G satisfying

$$\sum_{g\in G} g = 0. \tag{4.3.53}$$

Clearly, both decoupling sets (4.3.27) and (4.3.38) from the previous section satisfy the above condition, while leading to a different system dynamics which is uncoupled from the environment. The simplest decoupling set one can imagine satisfying

(4.3.53) is just given by $G = \{\mathbb{1}_{2n_S \times 2n_S},\ -\mathbb{1}_{2n_S \times 2n_S}\}$, noting that it leaves the system invariant, i.e.

$$\tilde{\Pi}(A) = \left(\begin{array}{c|c} A_S & 0 \\ \hline 0 & A_E \end{array}\right). \tag{4.3.54}$$

The two operations correspond to "no-rotation" and a global rotation around π of the system oscillators. It shows that the operation from [134], introduced in the beginning, allows us to decouple arbitrary quadratic system-environment interactions too. This is not really surprising, since in the unitary picture we can always reverse the sign in front of interaction parts of the form $\hat{x} \otimes B_1$ and $\hat{p} \otimes B_2$ by applying $\exp(i\pi a^\dagger a)$. Here however we want to emphasize two things. First of all, in contrast to finite dimensional systems, the system can always be decoupled from the environment using two operations, independent of how big the system or the environment is. For qubit systems we saw that the size of the decoupling set scales exponentially with the number of qubits. Hence the bigger the system, the more operations we need in order to protect the system from decoherence. There is another interesting difference to the finite dimensional case. For finite dimensional systems, on the one hand the irreducible action of the decoupling set suppresses all interactions with the environment, while on the other it destroys the system dynamics in the sense that it is given by the identity. For quadratic Hamiltonians we can suppress the interactions with the environment without disturbing the system dynamics at the same time. To push it a bit further, suppose we can build a network of harmonic oscillators whose interactions and frequencies can be engineered arbitrarily, so that we can create every A we like. Since we are able to decouple every block from the rest without disturbing it, using two frequently applied operations, the considered block undergoes a dynamics determined by some A_S, which we assumed can be engineered arbitrarily. Hence by applying this procedure to different blocks we can implement every Gaussian operation we encoded before in A. In this way a sequence of Gaussian operations can be implemented by consecutively decouple

different blocks. Alternatively, if we use the decoupling set (4.3.38) we can force a chosen part of the network to behave like one big oscillator that is decoupled from the rest of the network.

Before we get too lost in speculations we should come back to the suppression of decoherence. A rough estimate of how fast the decoupling operations $-\mathbb{1}_{2n_S\times 2n_S}$ and $\mathbb{1}_{2n_S\times 2n_S}$ have to be applied in steps Δt, in order to decouple the system from the environment for a fixed total time T, can be given using the Trotter formula (4.1.19). If we assume that the entries of A are bounded from above by K we find for the gate error (4.3.50) between the full and the uncoupled dynamics

$$\epsilon \leq 8\Delta t T(n_S + n_E)^2 K^2 e^{2T(n_S+n_E)K}. \tag{4.3.55}$$

Clearly, because of the exponential factor the bound becomes useless for big system and environmental sizes. Hence we will now study numerically how efficient dynamical decoupling, as a function of the number N of environmental oscillators, is. We study the system that was introduced in the beginning of this chapter (4.3.2), noting that we could have taken any other system-environment interaction that is described by a quadratic Hamiltonian. In terms of the quadrature operators the total Hamiltonian reads

$$H = \frac{\omega}{2}(\hat{x}^2 + \hat{p}^2) + \sum_{k=1}^{N} \frac{\omega_k}{2}(\hat{x}_k^2 + \hat{p}_k^2) + \sum_{k=1}^{N} g_k(\hat{x}\hat{x}_k + \hat{p}\hat{p}_k). \tag{4.3.56}$$

We investigate a random dynamical decoupling scheme by randomly rotating the system oscillator through

$$R(\phi) = \begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix} \oplus \mathbb{1}_{2N\times 2N}, \tag{4.3.57}$$

noting that the decoupling condition (4.3.53) is satisfied since $\int_0^{2\pi} d\phi\, R(\phi) = 0$. In Fig. 4.6 we evaluated the gate error (4.3.50) for different values of Δt and different numbers N of environmental modes for a total time $T = 1$. As a goal operation we took $S_G = \exp(-A_G\Omega)$ with $A_G = \mathrm{diag}(\omega, \omega, \omega_1, \omega_1, \ldots, \omega_N, \omega_N)$, while the symplectic dynamics is modified by applying instantaneously and uniformly random

rotations (4.3.57) so that the dynamics is given by (4.3.46). The frequency of the system oscillator ω was set to 1 and the coupling constants g_k and the frequencies ω_k of the environmental oscillators were chosen uniformly random between 0 and 1. The gate error was evaluated by taking the average over 20 trajectories. The inset plot shows the gate error as a function of N for a fixed spacing $\Delta t = 10^{-3}$ between the decoupling operations.

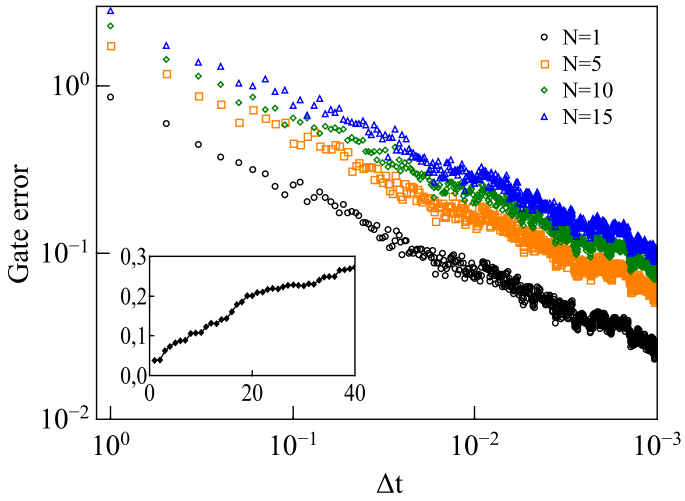

Figure 4.6.: Evaluation of the gate error (4.3.50), on a double logarithmic scale, for different values of Δt between random rotations (4.3.57) of the system oscillator and different numbers N of environmental oscillators for the decoherence model given by (4.3.56). The total time was chosen to be $T = 1$, the frequency of the system oscillator ω was set to 1, the coupling constants g_k and the frequencies ω_k of the environmental oscillators were chosen uniformly random between 0 and 1. The goal operation was $S_G = \exp(-A_G\Omega)$, while the gate error was evaluated by taking the average over 20 trajectories. The inset plot shows the gate error as a function of the number N of environmental oscillators for fixed $\Delta t = 10^{-3}$.

The numerical analysis indicates that, in order to suppress decoherence through random dynamical decoupling for this particular model (4.3.2), the size of the environment and the strength of the system-environment interactions are important for the efficiency of the scheme. Instead of increasing exponentially, the inset plot suggest that the gate error increases linearly when the number of environmental oscillators increases. This behavior indicates that there is room for improvement of

the bound (4.3.55). Based on a central limit theorem from [135], the development of more accurate bounds for random dynamical decoupling of quadratic Hamiltonians will be the subject of future work.

So far we have only discussed the suppression of decoherence through random dynamical decoupling for continuous variable systems that are described by quadratic Hamiltonians. What about other types of interactions that contain powers of the quadrature operators of the system? Consider an interaction of the form $x^n \otimes B$, where n is any natural number and B some hermitian environmental operator describing the interaction with a finite or an infinite dimensional environment. For $n = 1$ and B given by $\hat{x}$ or $\hat{p}$ we have seen that such interaction terms can be suppressed by rotating the system. For n odd we expect that this is still true since in the limit of infinitely fast rotations we have that the dynamics is governed by the average

$$\frac{1}{2\pi}\int_0^{2\pi} d\phi\, e^{i\frac{\phi}{2}(\hat{x}^2+\hat{p}^2)}\,(\hat{x}^n \otimes B)\, e^{-i\frac{\phi}{2}(\hat{x}^2+\hat{p}^2)} = \begin{cases} 0 & \text{for } n \text{ odd}, \\ \propto (\hat{x}^2+\hat{p}^2)^{n/2} \otimes B & \text{for } n \text{ even}. \end{cases} \tag{4.3.58}$$

Through randomly rotating the system we cannot suppress interaction terms that contain even powers of the position operator of the system. The same holds for the momentum operator $\hat{p}$. In fact for n even $\hat{x}$ has an unbounded spectrum over the positive part of the real line and we showed in the beginning that there does not exist a finite set of unitaries allowing us to suppress such interactions.

4.3.4 Conclusions

We showed that not every infinite dimensional quantum system can be decoupled from the environment and therefore protected from decoherence using dynamical decoupling. Afterwards we studied, in more detail, a particular class of such systems, namely dynamical decoupling of Hamiltonians that are quadratic in the quadrature operators. We first showed that through dynamical decoupling a network of interacting harmonic oscillators can be mapped to non-interacting ones, either rotating

with different frequencies or with the same frequency. Because, in the second case, the whole system behaves like one "big" oscillator, we called this process homogenization. In both cases the effective dynamics can be obtained by applying rapid rotations to the total system. A random dynamical decoupling scheme has the same effect. Afterwards we focussed on the suppression of decoherence for interactions that are described by quadratic Hamiltonians. Remarkably, we showed that, in contrast to finite dimensional systems, we can always suppress the interactions with the environment without averaging the system Hamiltonian to zero. This can always be achieved with two simple operations, independent of the system or the environmental size. We demonstrated random dynamical decoupling on a common decoherence model and we found that the performance of dynamical decoupling depends linearly on the number of environmental oscillators that couple to the system. The determination of bounds in terms of the strength of the system-environment interaction and the total time we want to protect the system from decoherence will be the subject of future work.

Our results pave the way for protecting an infinite dimensional quantum system from decoherence towards the reduction of noise in systems that are described by continuous variables. For instance, dynamical decoupling has the potential to decrease the environmentally induced errors in optical quantum computing [146] and quantum metrology [147]. Moreover, dynamical decoupling for continuous variables might assist in verifying collapse models in the macroscopic superposition regime [130, 131, 132, 133] by reducing extrinsic decoherence so that the small derivations from the usual Schrödinger dynamics become more visible.

In the last chapter we were concerned with the suppression of decoherence through dynamical decoupling. We learned that decoherence described by a Lindblad-type evolution cannot be suppressed. Instead of fighting against such noise processes we will see in the next chapter that sometimes they can have a beneficial effect on the dynamics that is accompanied by some controls. In the realm of the second approach for controlling open quantum systems we study how a Lindblad-type evolution can

increase the set of unitary operations that can be implemented with classical control fields. It will turn out that such noise process can turn parts of a quantum system into a system capable of universal quantum computations, whereas without the noise process this would not be possible.

5 Universal control induced by noise

Based on the submitted work:

C. Arenz, D. Burgarth, P. Facchi, V. Giovannetti, H. Nakazato, S. Pascazio and K. Yuasa, submitted to Phys. Rev. A (2016), arXiv: quant-ph/:1601.01212.

The interaction of a quantum system with its environment is usually considered to be detrimental for quantum information processing. Quantum features one wants to use for quantum information tasks are washed out quickly so that the implementation of quantum gates becomes noisy. In the last decades, however, it has been observed that sometimes noise can be *beneficial*. Rather than fighting against the environment, dissipative state preparation [97, 148, 149, 150] and dissipative quantum computing [4, 151, 152] turned out to be valuable alternatives to unitary gate designs. In the context of quantum control theory state preparation and the implementation of unitary gates through the modulation of classical control fields in the presence of a dissipative environment has been studied [46, 51, 52, 153] and the set of reachable operations has been analyzed [56, 154]. The environment can be used as a resource to increase the set of operations that can be implemented through the controls [155, 156]. If the dissipative process admits some set of states robust against the environmental perturbations, the fidelity for the implementation of a gate within the subspaces spanned is not influenced by the noise and the dynamics there is free from decoherence. The existence of the decoherence-free subspaces (DFS's) [157, 158, 159, 160, 161, 162, 163, 164, 165, 166] and the

interplay between weak coherent processes and fast relaxation processes make it possible to implement unitary gates over the steady state manifold in a noiseless manner [167, 168, 169, 170]. Here we show that such a noise process can even raise the fidelity for implementing a desired gate. The action of the strong dissipation allows the implementation of gate operations which cannot be realized without the help of the dissipation. The complexity of the dynamics is enhanced by the noise. To show this we build upon the recent results obtained in Ref. [155]. On the basis of the quantum Zeno effect [171] it was shown that frequent projective measurements can enrich the dynamics steered by a set of control Hamiltonians. Consider two control Hamiltonians H_1 and H_2 which are commutative with each other,

$$[H_1, H_2] = 0. \tag{5.0.1}$$

One is allowed to switch them on and off at will, but can induce only trivial dynamics on the system due to the commutativity. If one additionally performs frequent projective measurements described by a hermitian projection P during the control, the system is confined to the subspace specified by the projection P due to the quantum Zeno effect (quantum Zeno subspace [171, 172]), where the system evolves unitarily (quantum Zeno dynamics [171, 173]) according to the projected counterparts of the control Hamiltonians, PH_1P and PH_2P. These projected Hamiltonians do not necessarily commute any more,

$$[PH_1P, PH_2P] \neq 0. \tag{5.0.2}$$

The measurement forces the system to evolve within the Zeno subspace, in which more complex operations can be realized thanks to the noncommutativity. The same effect can be induced by an infinitely strong dissipative process [168, 169]. It was shown in Ref. [155] that a strong amplitude damping channel acting only locally on one out of many qubits in a chain typically turns a pair of commuting Hamiltonians into a pair of projected Hamiltonians that allow us to perform universal quantum computation over the whole chain of qubits apart from the projected one. The amplitude damping acting locally on one qubit out of many, however, is a very

special type of noise, and the assumption that it acts only locally seems unrealistic. On the other hand, this effect, *noise-induced universal quantum computation*, should arise in more general settings.

Here we show that the universal controllability over the system can be achieved with the help of more general noise models, including the ones widely studied in the context of DFS's [157, 158, 159, 160, 161, 162, 164, 165, 166, 168, 169]. DFS's will be identified as the equivalent to the quantum Zeno subspaces. Even if we are originally able to perform only trivial controls by commuting control Hamiltonians, a strong amplitude damping process projects the system onto DFS's, where we achieve universal controllability over the system. We characterize the set of reachable operations within DFS's and provide examples for which universal sets of gates can be implemented. Moreover, we perform numerical gate optimization to study how strong the dissipative process needs to be to implement such gates with high precision. As a byproduct a new fidelity function which can be applied in other optimization problems for open quantum systems is developed.

5.1 Basic Concepts

5.1.1 DFS's

DFS's can be exploited as a passive strategy for protecting quantum information against noise [3]. The theory has been developed in terms of interaction Hamiltonians [157, 158, 159, 162, 163] as well as of quantum dynamical semigroups [160, 161, 166, 165]. Many experiments, such as [174, 175, 176, 177], demonstrate the importance of DFS's for noiseless quantum computation. An experimental setup in waveguide QED has also been discussed recently [178]. Moreover the combinations with error correcting schemes [161] and dynamical strategies for decoherence control [106, 108, 179, 180, 181, 182] are promising possibilities for robust quantum information processing [183].

A DFS can be seen as a degenerate pointer basis, which is invariant against the dissi-

pative process. Consider a purely dissipative dynamics described by the Lindbladian generator

$$\mathcal{D}(\rho) = \sum_{j=1}^{d^2-1} \gamma_j (2L_j \rho L_j^\dagger - (L_j^\dagger L_j \rho + \rho L_j^\dagger L_j)), \tag{5.1.1}$$

with ρ the density operator of the system, L_j the Lindblad operators acting on the system, and γ_j non-negative constants. Here we restrict ourselves to finite-dimensional quantum system with Hilbert space $\mathcal{H}$ of dimension d and write $S(\mathcal{H})$ for the state space of $\mathcal{H}$. A DFS $\mathcal{H}_{\rm DFS}^{(i)} \subset \mathcal{H}$ is spanned by $\{|\psi_1^{(i)}\rangle, \ldots, |\psi_{d_i}^{(i)}\rangle\}$ characterized by

$$\begin{aligned} &L_j|\psi_k^{(i)}\rangle = \lambda_j^{(i)}|\psi_k^{(i)}\rangle, \quad G|\psi_k^{(i)}\rangle = b^{(i)}|\psi_k^{(i)}\rangle, \\ &j = 1, \ldots, d^2-1;\ k = 1, \ldots, d_i, \end{aligned} \tag{5.1.2}$$

with $G = \sum_{j=1}^{d^2-1} \gamma_j L_j^\dagger L_j$, $\lambda_j^{(i)}$ complex, and $b^{(i)} = \sum_{j=1}^{d^2-1} \gamma_j |\lambda_j^{(i)}|^2$ [184]. Clearly if we prepare the system in an initial state $\rho_0 \in S(\mathcal{H}_{\rm DFS}^{(i)})$, this state is protected from dissipation driven by the dissipator $\mathcal{D}$ in (5.1.1). We denote by $\mathcal{P}$ the (super)projection (which is not necessarily self-dual) onto the steady state manifold which consists of all quantum states ρ satisfying $\mathcal{D}(\rho) = 0$. We assume that the steady states are attractive, i.e.,

$$\lim_{t\to\infty} e^{\mathcal{D}t} = \mathcal{P}, \tag{5.1.3}$$

to which we refer as the long-time/strong-damping limit. In practice, the strong dissipative process quickly destroys the quantum coherence along a given set of directions.

5.1.2 Quantum Control

Having introduced the concept of DFS's we briefly review some results from quantum control theory introduced in the preliminary section (see Sec. 2.2.1). Consider a quantum system described by a Hamiltonian H_0, which suffers from dissipation

described by the dissipator $\mathcal{D}$ in (5.1.1). We try to steer the system by modulating external fields $\{u_1(t), \ldots, u_m(t)\}$ to switch on and off control Hamiltonians $\{H_1, \ldots, H_m\}$. The evolution of the system is generated by

$$\mathcal{L}_t(\rho) = -i[H(t), \rho] + \mathcal{D}(\rho), \tag{5.1.4}$$

with

$$H(t) = H_0 + \sum_{\ell=1}^{m} u_\ell(t) H_\ell. \tag{5.1.5}$$

H_0 is a drift Hamiltonian, and we do not have access to it. As discussed in Sec. 2.2.1 it is known [9] that in the absence of the dissipator $\mathcal{D}$, every unitary operation in the closure of the *dynamical Lie group* $e^{\mathfrak{L}}$ can be implemented with arbitrarily high precision, with

$$\mathfrak{L} = \mathfrak{Lie}(iH_0, iH_1, \ldots, iH_m), \tag{5.1.6}$$

being the real dynamical Lie algebra formed by real linear combinations of the operators $iH_0, iH_1, \ldots, iH_m$ and of their iterated commutators. If $\mathfrak{L} \supseteq \mathfrak{su}(d)$ (for traceless operators), where $\mathfrak{su}(d)$ is the special unitary algebra, the system is said to be fully controllable, that is, every unitary can be implemented up to a global phase.

5.2 Noise-Induced Universal Quantum Computation

Our question is the following. Suppose that the dynamical Lie algebra $\mathfrak{L}$ generated by our Hamiltonians $\{H_0, H_1, \ldots, H_m\}$ is strictly smaller than $\mathfrak{su}(d)$ and only limited unitaries are realizable by our control in the absence of the dissipation $\mathcal{D}$. How is the set of reachable operations enlarged by the action of a strong dissipation $\mathcal{D}$ on the system?

To this end we need to know how the system evolves under the influence of the strong dissipation $\mathcal{D}$ [168, 169]. To begin with we consider the situation in which no

drift term H_0 is present and the dissipator $\mathcal{D}$ can be switched on and off arbitrarily as well as the control Hamiltonians $\{H_1, \ldots, H_m\}$. Afterwards we discuss the case in which we have no control over the dissipative part $\mathcal{D}$ and the drift Hamiltonian H_0, assuming that the control fields are all constant. Finally this leads to the general case (5.1.4).

If we are allowed to control $\mathcal{D}$ arbitrarily, we can switch rapidly between $\mathcal{P}$ and a unitary evolution that is generated by $\mathcal{K}_c(\cdot) = -i[H_c, \cdot]$ with some $H_c \in \{H_0, H_1, \ldots, H_m\}$ and in the limit of infinitely frequent switching

$$\lim_{n\to\infty} (\mathcal{P} e^{\mathcal{K}_c t/n} \mathcal{P})^n = e^{\mathcal{P}\mathcal{K}_c\mathcal{P}t}\mathcal{P}. \tag{5.2.1}$$

It can be shown [168, 185] that

$$(\mathcal{P}\mathcal{K}_c\mathcal{P})(\rho) = -i[P_i H_c P_i, \rho], \quad \forall \rho \in S(\mathcal{H}^{(i)}_{\mathrm{DFS}}), \tag{5.2.2}$$

where $P_i = \sum_{k=1}^{d_i} |\psi_k^{(i)}\rangle\langle\psi_k^{(i)}|$ is the hermitian projection onto the ith DFS. Clearly this implies that if we prepare the system in a DFS, say in the ith DFS, it remains there evolving unitarily with the projected Hamiltonian $P_i H_c P_i$. Furthermore if the evolution generated by $\mathcal{D}$ is unital, i.e., $\mathcal{D}(\mathbb{1}) = 0$, the system evolves over the steady state manifold according to $\mathcal{P}\mathcal{K}_c\mathcal{P}(\cdot) = -i[\mathcal{P}(H_c), \cdot]$, and for an abelian interaction algebra [186], generated by the L_j's in (5.1.1) and their conjugates, we have $\mathcal{P}(H_c) = \sum_i P_i H_c P_i$ [168]. The mechanism is similar to that of the quantum Zeno subspaces induced by other means, such as frequent measurements, strong continuous couplings, and frequent unitary kicks [171, 172, 182]. The projective measurement is effectively performed by the dissipative process. The measurement is nonselective [187]: the transitions among different subspaces are hindered and the dynamics within each subspace is governed by the projected Hamiltonian $P_i H_c P_i$.

So far we have discussed the case in which the dissipator $\mathcal{D}$ as well as the control Hamiltonians $\{H_1, \ldots, H_m\}$ can be controlled arbitrarily, in the absence of the drift Hamiltonian H_0. Typically one has no access to the dissipative part $\mathcal{D}$ in (5.1.4) that arises for example from an interaction with the environment. If we assume that the control fields are all constant, the generator (5.1.4) including the drift Hamiltonian

H_0 reads

$$\mathcal{L} = g\mathcal{K} + \mathcal{D}, \tag{5.2.3}$$

where we have introduced the constant g that measures the strength of the coherent part $\mathcal{K}(\cdot) = -i[H, \cdot]$ in comparison with the dissipative part $\mathcal{D}$. Based on a pertubative expansion it has been shown [168, 169] that

$$\|(e^{t\mathcal{L}} - e^{gt\mathcal{P}\mathcal{K}\mathcal{P}})\mathcal{P}\| \leq O(g\tau_R), \tag{5.2.4}$$

where $\tau_R^{-1} = \min_{h>0} |\Re\{\lambda_h\}|$, with λ_h the nonvanishing eigenvalues of $\mathcal{D}$, defines the longest relaxation timescale τ_R. The norm is the usual operator norm and $gt = O(1)$. Thanks to this, we notice that on a timescale on which the dissipative dynamics is much faster than the coherent dynamics, the dynamics is effectively governed by (5.2.1). Similarly to (5.2.2), if the system is initially prepared in a DFS, say in the ith DFS, the system evolves unitarily within the same DFS in the limit $g\tau_R \to 0$ with $gt = O(1)$, driven by the projected Hamiltonian $P_i H P_i$. Again, this is intuitively clear: if the dynamics is dominated by the fast dissipative process, the latter defines the subspaces within which the system can evolve. The presence of the coherent component $\mathcal{K}$ only modifies the motion within each subspace.

It is now easy to treat the general case (5.1.4). In spirit of the Trotter formula (cf. Sec. 2.2.1), by switching among the control Hamiltonians under $g\tau_R \to 0$ and $gt = O(1)$, we can implement with arbitrarily high precision every $U_i = e^{\mathfrak{L}_{\rm DFS}^{(i)}}$ in the relevant DFS, with

$$\mathfrak{L}_{\rm DFS}^{(i)} = \mathfrak{Lie}(iP_iH_0P_i, iP_iH_1P_i, \ldots, iP_iH_mP_i), \tag{5.2.5}$$

being the real Lie algebra generated by the drift Hamiltonian H_0 and the control Hamiltonians $\{H_1, \ldots, H_m\}$ projected by the projection P_i. Note that for a unital evolution $e^{\mathcal{D}t}$ the Lie algebra over the DFS's reads

$$\mathfrak{L}_{\rm DFS} = \mathfrak{Lie}(i\mathcal{P}(H_0), i\mathcal{P}(H_1), \ldots, i\mathcal{P}(H_m)). \tag{5.2.6}$$

The projection P_i can now be identified as the equivalent of the frequent projective measurement that projects the system onto the quantum Zeno subspace specified

by P_i: the strong dissipation does the same job as the Zeno measurement. In the strong-damping limit the system is confined in the DFS's, evolving unitarily and steered by the projected Hamiltonians.

Although the dimensions of the DFS's are smaller than the dimension of the original Hilbert space, the dynamics induced by the projected control Hamiltonians within the DFS's can be much more complex than the one induced by the original control Hamiltonians in the absence of the dissipation, since $\dim(\mathfrak{L}_{\rm DFS})$ is in general larger than $\dim(\mathfrak{L})$ [155]. One can even achieve the universal controllability over the DFS's, with the help of the strong dissipation.

5.3 Universal control in DFS's: Examples

On the basis of the observation that the projected drift and control Hamiltonians do not necessarily commute any more, we saw in the last section that the Lie algebra over the DFS's might be larger than the Lie algebra over the original Hilbert space. In the following we present three different examples, for which the universal controllability over the DFS's is achieved, even though only "simple" operations can be implemented over the original Hilbert space in the absence of dissipation.

5.3.1 Two Qubits

We first provide a simplest example with only two qubits, which is essentially the same as that presented in Ref. [155]: one of the two qubits, say qubit 2, is subject to a strong amplitude-damping process. We also discuss the same model but with a pure dephasing process on qubit 2, instead of the amplitude-damping process.

The drift Hamiltonian reads

$$H_0 = \sigma_x \otimes (\sigma_x + \sigma_z), \tag{5.3.1a}$$

while we have a control Hamiltonian

$$H_1 = \sigma_y \otimes (\sigma_x - \sigma_z), \tag{5.3.1b}$$

where σ_α with $\alpha = x, y, z$ are the Pauli operators. Note that these Hamiltonians commute with each other, $[H_0, H_1] = 0$. Therefore in the absence of noise the Lie algebra $\mathfrak{L} = \mathfrak{Lie}(iH_0, iH_1)$ is spanned just by $\{iH_0, iH_1\}$ and hence is only two-dimensional, $\dim(\mathfrak{L}) = 2$. We now add amplitude-damping on qubit 2, generated by

$$\mathcal{D}(\rho) = \gamma(2\sigma_-^{(2)}\rho\sigma_+^{(2)} - (\sigma_+^{(2)}\sigma_-^{(2)}\rho + \rho\sigma_+^{(2)}\sigma_-^{(2)})), \tag{5.3.2}$$

with $\sigma_\pm^{(2)} = \mathbb{1} \otimes (\sigma_x \pm i\sigma_y)/2$ the raising and lowering operators acting nontrivially only on qubit 2. It projects the system as [188]

$$\begin{aligned} e^{\mathcal{D}t}\rho &= (P + Qe^{-\gamma t})\rho(P + Qe^{-\gamma t}) + (1 - e^{-2\gamma t})L\rho L^\dagger \\ &\xrightarrow{\gamma t\to\infty} \mathcal{P}(\rho) = P\rho P + L\rho L^\dagger, \end{aligned} \tag{5.3.3}$$

where $P = \mathbb{1} \otimes |0\rangle\langle 0|$, $Q = \mathbb{1} \otimes |1\rangle\langle 1|$, and $L = \sigma_-^{(2)} = \mathbb{1} \otimes |0\rangle\langle 1|$ with $|0\rangle$ and $|1\rangle$ being the eigenstates of σ_z belonging to the eigenvalues -1 and $+1$, respectively. The dissipator (5.3.2) admits a single DFS identified by the hermitian projection P onto

$$\mathcal{H}_{\text{DFS}} = \text{span}\{|0\rangle \otimes |0\rangle, |1\rangle \otimes |0\rangle\}. \tag{5.3.4}$$

In the strong-damping limit our Hamiltonians are projected to

$$PH_0P = -\sigma_x \otimes |0\rangle\langle 0|, \tag{5.3.5a}$$

$$PH_1P = \sigma_y \otimes |0\rangle\langle 0|, \tag{5.3.5b}$$

and the Lie algebra over the DFS is given by

$$\mathfrak{L}_{\text{DFS}} = \mathfrak{Lie}(iPH_0P, iPH_1P) = \mathfrak{su}(2) \otimes |0\rangle\langle 0|. \tag{5.3.6}$$

That is, in the strong-damping limit qubit 1 becomes fully controllable, i.e., every $U \in \text{SU}(2)$ can be implemented on qubit 1.

Now let us replace the amplitude-damping process on qubit 2 by a pure dephasing process generated by

$$\mathcal{D}(\rho) = -\gamma[\sigma_z^{(2)}, [\sigma_z^{(2)}, \rho]], \tag{5.3.7}$$

where $\sigma_z^{(2)} = \mathbb{1} \otimes \sigma_z$. In this case the system is projected as [188]

$$\begin{aligned} e^{\mathcal{D}t}\rho &= P_0\rho P_0 + P_1\rho P_1 + P_0\rho P_1 e^{-4\gamma t} + P_1\rho P_0 e^{-4\gamma t}, \\ &\xrightarrow{\gamma t\to\infty} \mathcal{P}(\rho) = P_0\rho P_0 + P_1\rho P_1, \end{aligned} \tag{5.3.8}$$

where $P_i = \mathbb{1} \otimes |i\rangle\langle i|$ with $i = 0, 1$. This dephasing process admits two orthogonal DFS's identified by the hermitian projections P_0 and P_1,

$$\mathcal{H}_{\rm DSF}^{(0)} = \text{span}\{|0\rangle \otimes |0\rangle, |1\rangle \otimes |0\rangle\}, \tag{5.3.9a}$$

$$\mathcal{H}_{\rm DFS}^{(1)} = \text{span}\{|0\rangle \otimes |1\rangle, |1\rangle \otimes |1\rangle\}. \tag{5.3.9b}$$

Since the evolution generated by (5.3.7) is unital, in the strong-dephasing limit our Hamiltonians are projected to

$$\mathcal{P}(H_0) = \sigma_x \otimes \sigma_z, \tag{5.3.10a}$$

$$\mathcal{P}(H_1) = -\sigma_y \otimes \sigma_z, \tag{5.3.10b}$$

and the Lie algebra over the DFS's $\mathfrak{L}_{\rm DFS} = \mathfrak{Lie}(i\mathcal{P}(H_0), i\mathcal{P}(H_1))$ is spanned by $\{\sigma_x \otimes \sigma_z, \sigma_y \otimes \sigma_z, \sigma_z \otimes \mathbb{1}\}$: its dimension is $\dim(\mathfrak{L}_{\rm DFS}) = 3$ and is increased from $\dim(\mathfrak{L}) = 2$ by the action of the strong pure dephasing on qubit 2. In particular, if qubit 2 starts from the state $|i\rangle$ with $i = 0$ or 1 the Lie algebra over the ith DFS reads

$$\mathfrak{L}_{\rm DFS}^{(i)} = \mathfrak{Lie}(iP_iH_0P_i, iP_iH_1P_i) = \mathfrak{su}(2) \otimes |i\rangle\langle i|, \tag{5.3.11}$$

and qubit 1 is fully controllable. Although in this case we do not have the full controllability over all DFS's, universal quantum computation is possible on qubit 1 within one of the two DFS's. We see that using the framework of DFS's the previous results on amplitude damping channels extend naturally to other types of noise.

5.3.2 N-Level Atom with an Unstable Level

The next example involves an atom with energy eigenstates $|1\rangle, \ldots, |N\rangle$ plus a higher lying unstable state $|e\rangle$ that decays to the lower lying states with rates $\gamma_1, \ldots, \gamma_N$,

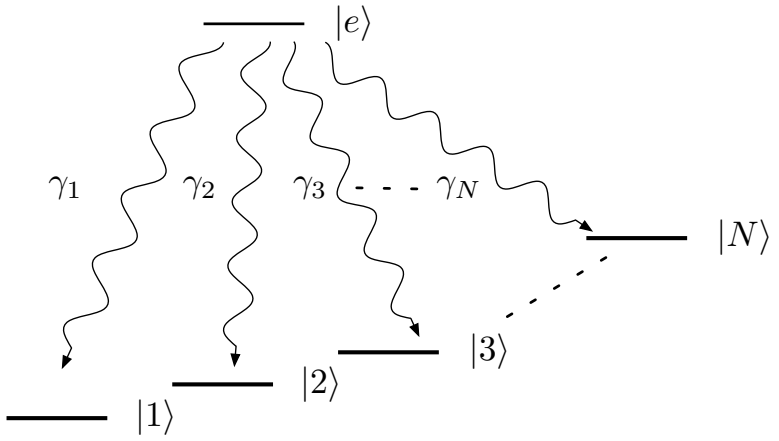

Figure 5.1.: Schematic representation of an N-level atom with a higher lying unstable level $|e\rangle$ that decays with rates $\gamma_1, \ldots, \gamma_N$ to the lower lying levels $|1\rangle, \ldots, |N\rangle$ spanning a DFS.

as schematically represented in Fig. 5.1. We assume that $N \geq 2$. A similar level structure manifests for example in a Rydberg atom, for which the quantum Zeno dynamics has recently been demonstrated in an impressive way [189].

We will consider a decay process described by

$$\mathcal{D}(\rho) = \sum_{j=1}^{N} \gamma_j (2L_j \rho L_j^\dagger - (L_j^\dagger L_j \rho + \rho L_j^\dagger L_j)) \tag{5.3.12}$$

with $L_j = |j\rangle\langle e|$ where $j = 1, \ldots, N$. The system is projected as [188]

$$\begin{aligned} e^{\mathcal{D}t}\rho &= (P + Qe^{-\Gamma t})\rho(P + Qe^{-\Gamma t}) + \frac{1}{\Gamma}(1 - e^{-2\Gamma t}) \sum_{j=1}^{N} \gamma_j L_j \rho L_j^\dagger \\ &\xrightarrow{\Gamma t \to \infty} \mathcal{P}(\rho) = P\rho P + \frac{1}{\Gamma} \sum_{j=1}^{N} \gamma_j L_j \rho L_j^\dagger, \end{aligned} \tag{5.3.13}$$

where $P = \mathbb{1} - |e\rangle\langle e|$, $Q = |e\rangle\langle e|$, and $\Gamma = \sum_{j=1}^{N} \gamma_j$. The dissipator (5.3.12) admits a DFS identified by the hermitian projection P, namely, spanned by the lower lying levels

$$\mathcal{H}_{\mathrm{DFS}} = \mathrm{span}\{|1\rangle, \ldots, |N\rangle\}. \tag{5.3.14}$$

Now we are going to introduce a drift Hamiltonian and a control Hamiltonian. We take an example from Ref. [190], for which the universal control is achieved through

frequent projective measurements described by a hermitian projection P. Note that here P is realized through the strong-damping limit of the CPTP map that is generated by the dissipator (5.3.12). The drift Hamiltonian

$$H_0 = |e\rangle\langle 2| + |2\rangle\langle e| + \sum_{j=1}^{N-1}(|j\rangle\langle j+1| + |j+1\rangle\langle j|), \tag{5.3.15a}$$

consists of the interactions among the lower lying levels $\{|1\rangle, \ldots, |N\rangle\}$ and additional driving terms stimulating the transitions between $|e\rangle$ and $|2\rangle$. The control Hamiltonian, on the other hand, reads

$$H_1 = |e\rangle\langle e| + |1\rangle\langle 1| - (|e\rangle\langle 1| + |1\rangle\langle e|). \tag{5.3.15b}$$

Again, these Hamiltonians commute with each other, $[H_0, H_1] = 0$. Therefore in the absence of the noise $\mathcal{D}$ the Lie algebra $\mathfrak{L} = \mathfrak{Lie}(iH_0, iH_1)$ is spanned just by $\{iH_0, iH_1\}$ and hence is only two-dimensional, $\dim(\mathfrak{L}) = 2$, as in the previous example. These Hamiltonians are projected by the strong dissipation (5.3.13) to

$$PH_0P = \sum_{j=1}^{N-1}(|j\rangle\langle j+1| + |j+1\rangle\langle j|), \tag{5.3.16a}$$

$$PH_1P = |1\rangle\langle 1|. \tag{5.3.16b}$$

This pair of Hamiltonians is known to generate the full unitary algebra $\mathfrak{u}(N)$ (see e.g. [191]). We get

$$\mathfrak{L}_{\mathrm{DFS}} = \mathfrak{Lie}(iPH_0P, iPH_1P) = \mathfrak{u}(N)P. \tag{5.3.17}$$

Its dimension is $\dim(\mathfrak{L}_{\mathrm{DFS}}) = N^2$, while $\dim(\mathfrak{L}) = 2$ in the absence of the dissipation. Compared to the previous two-qubit example we observe here a more dramatic increase of the complexity in the dynamics over the DFS through projection.

5.3.3 Ising Chain of N Qubits under Collective Decoherence

The third example is a chain of N qubits interacting with each other via nearest-neighbor Ising-type couplings,

$$H_0 = \sum_{n=1}^{N-1} \sigma_z^{(n)}\sigma_z^{(n+1)}, \tag{5.3.18a}$$

where $\sigma_\alpha^{(n)} = \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes \sigma_\alpha \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}$ with $\alpha = x, y, z$ are the Pauli operators acting on the nth qubit. We assume that $N \geq 3$. In addition we are allowed to switch on and off the coupling between the first two qubits,

$$H_1 = \sigma_z^{(1)} \sigma_z^{(2)}. \tag{5.3.18b}$$

These Hamiltonians trivially commute with each other, $[H_0, H_1] = 0$, and our control over the chain of qubits is very poor. Suppose then that this system undergoes a strong collective decoherence described by the Lindbladian generator

$$\mathcal{D}(\rho) = \sum_{\alpha=x,y,z} \gamma_\alpha (2 S_\alpha \rho S_\alpha - (S_\alpha^2 \rho + \rho S_\alpha^2)), \tag{5.3.19}$$

that preserves the identity, where

$$S_\alpha = \frac{1}{2} \sum_{n=1}^{N} \sigma_\alpha^{(n)} \quad , \alpha = x, y, z, \tag{5.3.20}$$

are the collective spin operators. This noise model is well studied in the context of DFS's, and is known to admit multiple DFS's labeled by the total spin J of the whole chain (i.e., J gives the total spin angular momentum of the chain by $S^2 = \sum_{\alpha=x,y,z} S_\alpha^2 = J(J+1)$ [163, 165, 168]). The dimensions of the DFS's are given by [192],

$$d_{J,N} = \frac{(2J+1)N!}{(N/2+J+1)!(N/2-J)!}, \tag{5.3.21}$$

and are listed in Table 5.1 for small numbers of qubits N.

To see how our Hamiltonians H_0 and H_1 are projected by the collective decoherence $\Lambda_t = e^{\mathcal{D}t}$ in the strong-damping limit, let us look at its dual channel $\Lambda_t^\star = e^{\mathcal{D}^\star t}$ defined by

$$\mathrm{tr}\{A \Lambda_t(\rho)\} = \mathrm{tr}\{\Lambda_t^\star(A) \rho\}, \tag{5.3.22}$$

for an arbitrary observable A and state ρ, and note that $\mathcal{D}^\star = \mathcal{D}$ in this case, since S_α in the generator $\mathcal{D}$ in (5.3.19) are hermitian. By this channel, each component

of our Hamiltonians $\sigma_z^{(n)}\sigma_z^{(n+1)}$ evolves according to

$$\mathcal{D}\begin{pmatrix}\sigma_x^{(n)}\sigma_x^{(n+1)}\\ \sigma_y^{(n)}\sigma_y^{(n+1)}\\ \sigma_z^{(n)}\sigma_z^{(n+1)}\end{pmatrix} = -2\begin{pmatrix}\gamma_y+\gamma_z & -\gamma_z & -\gamma_y\\ -\gamma_z & \gamma_z+\gamma_x & -\gamma_x\\ -\gamma_y & -\gamma_x & \gamma_x+\gamma_y\end{pmatrix}\begin{pmatrix}\sigma_x^{(n)}\sigma_x^{(n+1)}\\ \sigma_y^{(n)}\sigma_y^{(n+1)}\\ \sigma_z^{(n)}\sigma_z^{(n+1)}\end{pmatrix}, \tag{5.3.23}$$

and in the strong-damping limit the operators $\sigma_\alpha^{(n)}\sigma_\alpha^{(n+1)}$, $\alpha = x, y, z$ are projected to

$$\begin{aligned}\Lambda_t\begin{pmatrix}\sigma_x^{(n)}\sigma_x^{(n+1)}\\ \sigma_y^{(n)}\sigma_y^{(n+1)}\\ \sigma_z^{(n)}\sigma_z^{(n+1)}\end{pmatrix} &\xrightarrow{\bar{\gamma}t\to\infty} \mathcal{P}\begin{pmatrix}\sigma_x^{(n)}\sigma_x^{(n+1)}\\ \sigma_y^{(n)}\sigma_y^{(n+1)}\\ \sigma_z^{(n)}\sigma_z^{(n+1)}\end{pmatrix}\\ &= \frac{1}{3}\begin{pmatrix}\boldsymbol{\sigma}^{(n)}\cdot\boldsymbol{\sigma}^{(n+1)}\\ \boldsymbol{\sigma}^{(n)}\cdot\boldsymbol{\sigma}^{(n+1)}\\ \boldsymbol{\sigma}^{(n)}\cdot\boldsymbol{\sigma}^{(n+1)}\end{pmatrix},\end{aligned} \tag{5.3.24}$$

where $\bar{\gamma}$ is a characteristic timescale of the decoherence, e.g., the smaller nonvanishing eigenvalue of the matrix in (5.3.23). The operators become rotationally symmetric by the projection. In particular, our Hamiltonians H_0 and H_1 are projected to

$$\mathcal{P}(H_0) = \frac{1}{3}\sum_{n=1}^{N-1}\boldsymbol{\sigma}^{(n)}\cdot\boldsymbol{\sigma}^{(n+1)}, \tag{5.3.25a}$$

$$\mathcal{P}(H_1) = \frac{1}{3}\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)}. \tag{5.3.25b}$$

The Ising chain (5.3.18) thus becomes the Heisenberg chain (5.3.25) by the projection $\mathcal{P}$. The projected Hamiltonians are not commutative anymore with each other. Now we look at the Lie algebra

$$\mathfrak{L}_{\mathrm{DFS}} = \mathfrak{Lie}(i\mathcal{P}(H_0), i\mathcal{P}(H_1)), \tag{5.3.26}$$

generated by the projected Hamiltonians $\mathcal{P}(H_0)$ and $\mathcal{P}(H_1)$. Recall that the projected Hamiltonians in (5.3.25) are rotationally symmetric, reflecting the character of the decoherence model (5.3.19). Commutators preserve this rotational symmetry,

as we will see below. Then, all the elements of the Lie algebra $\mathfrak{L}_{\rm DFS}$ are rotationally symmetric, and are given in terms of the two- and three-body operators (see App. B.1 for details)

$$H_{mn} = \boldsymbol{\sigma}^{(m)} \cdot \boldsymbol{\sigma}^{(n)}, \quad H_{ijk} = \boldsymbol{\sigma}^{(i)} \cdot (\boldsymbol{\sigma}^{(j)} \times \boldsymbol{\sigma}^{(k)}),$$
$$m < n;\ i < j < k;\ m, n, i, j, k = 1, \ldots, N. \tag{5.3.27}$$

In Ref. [165], it is proved that any SU transformations on the DFS's induced by the strong collective decoherence (5.3.19) can be realized if we are able to apply SWAP interactions between any pair of qubits. Note that the SWAP Hamiltonians can be constructed from the rotationally symmetric two-body operators $H_{mn} = \boldsymbol{\sigma}^{(m)} \cdot \boldsymbol{\sigma}^{(n)}$: the SWAP operator S_{mn} swapping the states of qubits m and n is given by $S_{mn} = (1 + \boldsymbol{\sigma}^{(m)} \cdot \boldsymbol{\sigma}^{(n)})/2$. Since we have proven in App. B.1 that all the rotationally symmetric two-body operators $H_{mn} = \boldsymbol{\sigma}^{(m)} \cdot \boldsymbol{\sigma}^{(n)}$ can be generated by the projected Hamiltonians $\mathcal{P}(H_0)$ and $\mathcal{P}(H_1)$, the SWAP Hamiltonians S_{mn} between any pair of qubits can be applied, and by the theorem proved in Ref. [165] all the generators of $\bigoplus_J \mathfrak{su}(d_{J,N})$ can be constructed. Namely,

$$\mathfrak{L}_{\rm DFS} = \mathfrak{Lie}(i\mathcal{P}(H_0), i\mathcal{P}(H_1)) \supset \bigoplus_J \mathfrak{su}(d_{J,N}). \tag{5.3.28}$$

This means that we are able to perform universal quantum computation over all DFS's by the projected Hamiltonians $\mathcal{P}(H_0)$ and $\mathcal{P}(H_1)$. Notice, however, that the full unitary algebra $\bigoplus_J \mathfrak{u}(d_{J,N})$ over the DFS's is not attainable. For instance, not all the rotationally symmetric four-body operators $(\boldsymbol{\sigma}^{(i)} \cdot \boldsymbol{\sigma}^{(j)})(\boldsymbol{\sigma}^{(k)} \cdot \boldsymbol{\sigma}^{(\ell)}) = H_{ij}H_{k\ell}$ can be generated. Combinations of them can be generated by the rotationally symmetric two- and three-body operators through

$$i[H_{ij}, H_{jk\ell}] = 2(H_{ik}H_{j\ell} - H_{i\ell}H_{jk}), \tag{5.3.29}$$

but we realize that we can generate only differences of four-body operators. The other commutators such as

$$i[H_{ijk}, H_{ij}H_{k\ell}] = 4(H_{j\ell} - H_{i\ell}) + 2(H_{i\ell}H_{jk} - H_{ik}H_{j\ell}), \tag{5.3.30}$$

	$N=1$	$N=2$	$N=3$	$N=4$	$N=5$	$N=6$
$J=0$		1		2		5
$J=\frac{1}{2}$	1		2		5	
$J=1$		1		3		9
$J=\frac{3}{2}$			1		4	
$J=2$				1		5
$J=\frac{5}{2}$					1	
$J=3$						1
$\dim \mathfrak{L}_{\mathrm{DFS}}$	0	1	4	12	40	129
$\sum_J \dim(\mathfrak{su}(d_{J,N}))$	0	0	3	11	39	128
$\sum_J \dim(\mathfrak{u}(d_{J,N}))$	1	2	5	14	42	132

Table 5.1.: The dimensions $d_{J,N}$ of the DFS's, and the dimension of the Lie algebra $\mathfrak{L}_{\mathrm{DFS}} = \mathfrak{Lie}(i\mathcal{P}(H_0), i\mathcal{P}(H_1))$ compared with the dimensions of the $\mathfrak{u}$ and $\mathfrak{su}$ algebras over the DFS's, for small numbers of qubits N.

do not help to break the differences to get a single piece of four-body operator. This is because commutators yield something antisymmetric with respect to some of the qubits involved in the operators. In order to single out each piece of four-body operator from the differences, we need a sum of four-body operators, but it is not available or provided through commutators. We thus cannot generate the full algebra over the DFS's.

See Table 5.1, where the dimension of the Lie algebra $\dim(\mathfrak{L}_{\mathrm{DFS}})$ is compared with the dimension of the $\mathfrak{su}$ algebra $\sum_J \dim(\mathfrak{su}(d_{J,N}))$ and that of the full unitary algebra $\sum_J \dim(\mathfrak{u}(d_{J,N}))$ over the DFS's. The dimension of the Lie algebra $\dim(\mathfrak{L}_{\mathrm{DFS}})$ is indeed larger than $\sum_J \dim(\mathfrak{su}(d_{J,N}))$, but is smaller than $\sum_J \dim(\mathfrak{u}(d_{J,N}))$. Anyway, the dimension of the Lie algebra is greatly enhanced from $\dim(\mathfrak{L}) = 2$, as $\dim(\mathfrak{L}_{\mathrm{DFS}}) \simeq 4^N N^{-3/2}/\sqrt{\pi}$ for large N, as estimated in App. B.2.

In summary, we started with two commuting Hamiltonians H_0 and H_1 in (5.3.18), which are projected to $\mathcal{P}(H_0)$ and $\mathcal{P}(H_1)$ in (5.3.25), respectively, by the strong collective decoherence (5.3.19). As a consequence, the Ising chain (5.3.18) is changed

into the Heisenberg chain (5.3.25), and our projected Hamiltonians $\mathcal{P}(H_0)$ and $\mathcal{P}(H_1)$ are not commutative anymore with each other. They generate the full algebra of $\bigoplus_J \mathfrak{su}(d_{J,N})$ on the DFS's. Remarkably the noise is turning the Ising chain (classical) into the Heisenberg chain (quantum), and we are able to perform a universal quantum computation over the DFS's.

5.4 Gate optimization and subsystem fidelity

In this section we analyze how the process fidelity scales with the noise strength. To this end we resort to the numerical gate optimization using the quantum control package implemented in QuTip [193]. We study the two-qubit example discussed in Sec. 5.3.1, with the amplitude damping (5.3.2) for different values of γ. For the sake of simplicity the drift Hamiltonian (5.3.1a) is treated as a control Hamiltonian as well.

We wish to optimize the control fields $u_\ell(t)$ (recall (5.1.5)) to implement some goal operation $\mathcal{E}_G$. Denote by $\Lambda_T = \hat{T}\exp(\int_0^T dt\,\mathcal{L}_t)$ the CPTP map at time T, where $\mathcal{L}_t$ is the Lindbladian given in (5.1.4) and $\hat{T}$ indicates time-ordered product. The optimization is performed to minimize the gate error

$$\varepsilon_1 = \|\Lambda_T - \mathcal{E}_G\|_{\mathrm{HS}}^2, \tag{5.4.1}$$

where $\|\cdot\|_{\mathrm{HS}}$ is the Hilbert-Schmidt norm with $\mathcal{E}_G$ and Λ_T being treated as $d^2 \times d^2$ matrices obtained by the row-vectorization of the density operator of a d-dimensional system (see Eq. (2.1.18) from Sec. 2.1.2). In general, for two CPTP maps Φ_1 and Φ_2, the Hilbert-Schmidt norm of the difference between their corresponding matrices provides an upper bound $\|\Phi_1 - \Phi_2\|_\diamond \leq d\|\Phi_1 - \Phi_2\|_{\mathrm{HS}}$ on the diamond norm $\|\cdot\|_\diamond$. The diamond norm [48] takes its maximal value 2 when the two quantum channels Φ_1 and Φ_2 are perfectly distinguishable. The minimization of (5.4.1) is done by a gradient-based algorithm [44] dividing the total time T into equidistant time intervals, on which the control fields are piecewise constant. For further details of the algorithm we refer to Sec. 2.2.2.

We are actually interested in the reduced dynamics of system 1, i.e., in the map $\mathcal{E}_T^{(1)}(\rho_1) = \mathrm{tr}_2\{\Lambda_T(\rho_1 \otimes \rho_2)\}$ with ρ_1 and ρ_2 the initial states of systems 1 and 2, respectively, and $\mathrm{tr}_2\{\cdot\}$ the partial trace over system 2. We wish to optimize Λ_T such that $\mathcal{E}_T^{(1)}$ becomes some goal unitary map $\mathcal{E}_G^{(1)} = \mathcal{U}_G$ with $\mathcal{U}_G(\rho) = U_G \rho U_G^\dagger$ and $U_G \in \mathrm{SU}(d)$. Our measure of error ε_1 in (5.4.1), however, depends also on how the channels Λ_T and $\mathcal{E}_G$ act on system 2: even if $\mathcal{E}_T^{(1)}$ coincides with the goal unitary $\mathcal{E}_G^{(1)} = \mathcal{U}_G$, the total maps Λ_T and $\mathcal{E}_G$ can be different and our measure of error ε_1 can be nonvanishing. In addition, the reduced map $\mathcal{E}_T^{(1)}$ depends on the initial state of system 2. We notice, on the other hand, that since the goal operation on system 1 is unitary $\mathcal{U}_G$ the total goal operation must factorize $\mathcal{E}_G = \mathcal{U}_G \otimes \tilde{\mathcal{E}}$ with $\tilde{\mathcal{E}}$ an arbitrary CPTP map acting on system 2. What is more relevant is how close the reduced channel $\mathcal{E}_T^{(1)}$ is to the goal unitary $\mathcal{U}_G$. Therefore it would be more appropriate to perform an additional minimization of ε_1 in (5.4.1) over $\tilde{\mathcal{E}}$. To obtain the subsystem fidelity for purely unitary channels this minimization can be carried out analytically [64, 99] (see also Eq. (3.3.2) and Eq. (3.3.3) from Sec. 3.3) but unfortunately for arbitrary CPTP channels this is a challenging task. Instead we use the normalized Choi representation $J(\mathcal{E})$ of a quantum channel $\mathcal{E}$ [7] (see Sec. 2.1.2) to derive a lower bound of ε_1,

$$\begin{aligned} \varepsilon_1/d^2 &= \|J(\Lambda_T) - S(J(\mathcal{U}_G) \otimes J(\tilde{\mathcal{E}}))S\|_{\mathrm{HS}}^2 \\ &\geq \mathrm{tr}\{J^2(\Lambda_T)(\mathbb{1} - S(J(\mathcal{U}_G) \otimes \mathbb{1}_2)S)\} \equiv \varepsilon_2, \end{aligned} \tag{5.4.2}$$

where the SWAP operator S between systems 1 and 2 is introduced because in general for two CPTP maps Φ_1 and Φ_2, $J(\Phi_1 \otimes \Phi_2) = S(J(\Phi_1) \otimes J(\Phi_2))S$. For details of the derivation of the lower bound (5.4.2) we refer to App. B.3. Clearly the minimization over $\tilde{\mathcal{E}}$ on the left-hand side of (5.4.2) is now lower bounded by ε_2, which is independent of $\tilde{\mathcal{E}}$ and is zero if and only if the goal unitary operation on system 1 is reached. Thus the lower bound becomes tighter and tighter when Λ_T factorizes into the goal unitary $\mathcal{U}_G$ on system 1 and some arbitrary $\tilde{\mathcal{E}}$ on system 2. The strategy to study the convergence of the map to the goal operation as γ is increased can now be summarized as follows. We implement ε_2 and its gradient with

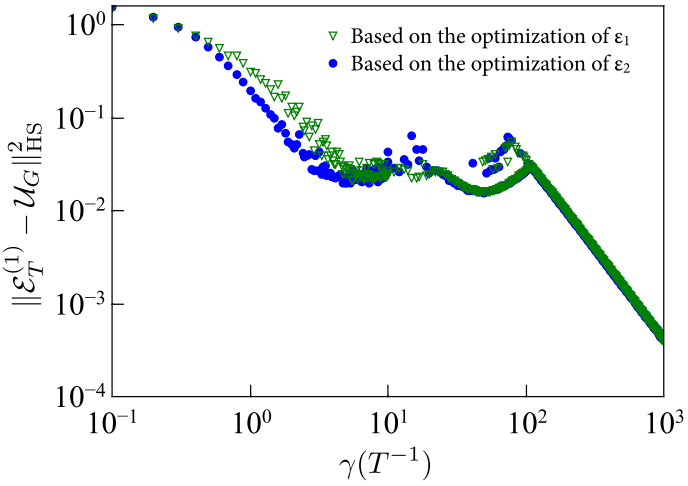

Figure 5.2.: Numerical gate optimization for the two-qubit model in Sec. 5.3.1 with the amplitude damping (5.3.2) for different values of γ. The gate error between the reduced dynamics $\mathcal{E}_T^{(1)}$ and the Hadamard gate on qubit 1 obtained from the numerical minimizations of ε_1 (green triangles) and ε_2 (blue points) for different values of γ with gate time $T = 1$. Qubit 2 is initially prepared in the totally mixed state, and for ε_1, $\tilde{\mathcal{E}}$ is chosen to be the superprojection $\mathcal{P}$ that brings qubit 2 into the ground state $|0\rangle$. To reduce the effect of local minima in the minimum value 100 randomly chosen initial pulses are taken.

respect to the control fields on QuTip, and minimize ε_1 and ε_2 for different values of γ. For ε_1, $\tilde{\mathcal{E}}$ is chosen to be the superprojection $\mathcal{P}$ in (5.3.3) that brings qubit 2 into the ground state $|0\rangle$. On the basis of the minimizations of ε_1 and ε_2 we evaluated in Fig. 5.2 the gate error $\|\mathcal{E}_T^{(1)} - \mathcal{U}_G\|^2_{\mathrm{HS}}$ by specifying the initial state of qubit 2 in the totally mixed state and tracing out the auxiliary degrees of freedom. The target unitary operation U_G on qubit 1 was chosen to be the Hadamard gate. We observe that despite the enhanced freedom in ε_2 the curves based on the minimizations of ε_1 and ε_2 are similar to each other. For noise strengths above $\gamma \approx 10\,T^{-1}$ gate errors below 10^{-1} can be reached, corresponding to the upper bound 0.2 for the diamond norm. It demonstrates that with intermediate noise strengths reasonable fidelity can be reached.

5.5 Conclusions

We showed that every dissipative process exhibiting a DFS can enlarge the set of unitary operations that can be implemented by means of classical control fields. We provided three examples for which a universal set of gates can be implemented over a DFS whereas over the original Hilbert space only "simple" operations are possible. In particular we showed that a realistic noise model can map a commutative classical system into a universal quantum one. Numerical gate optimization was performed to study how strong the dissipative process needs to be to implement some unitary gate over the DFS with high precision. As a result a subsystem fidelity for open quantum systems was developed. Our results pave the way to experimental feasibility studies in noisy systems.

6 Hamiltonian and Lindbladian purification

In preparation:

C. Arenz, D. Burgarth, P. Facchi, V. Giovannetti, H. Nakazato, S. Pascazio and K. Yuasa (2016).

The results of the last chapter were based on the observation that the projected counterparts PH_1P and PH_1P of two commuting Hamiltonians H_1 and H_2 do not necessarily commute anymore. The hermitian projection P can either be enforced by a frequent projective measurement or by a strong dissipative process exhibiting a decoherence free subspace. Both processes can increase the dimension of the dynamical Lie algebra such that we are able to implement new unitary operations. The crucial point here is that we started with two commuting Hamiltonians and thus the dynamical Lie algebra is two dimensional, allowing us only to implement simple operations through controlling H_1 and H_2. We could also consider the other way around. Given two Hamiltonians, can we make them commutative? Or in general, given a set of density operators, a set of Hamiltonians and a set of Lindblad operators, can we make these objects commutative through a general framework? From now on we call this procedure *purification*, because similarly to the purification of mixed quantum states [100] we will add an auxiliary system to make the considered object commutative on an extended space. We note here that the term purification regarding Lindbladians was already used by Lindblad [15] to indicate that a Lindbladian consists only of one Lindblad operator L. Here we use the term in a slightly

different way, though it has the same meaning when only the evolution of observables acting only on the original system is considered, since then, within our purification scheme, the Lindbladian can be rewritten with a single L. The second question that goes hand in hand with the purification scheme is the question of how we can come back? How do we get back the old dynamics of the original system on which, for instance, the two Hamiltonians do not commute anymore? We have already seen in the last chapter that, based on the Zeno effect, the old Hamiltonian dynamics on the original system is recovered if we frequently observe the auxiliary system. Before we establish a similar framework for Lindblad operators we will review the main results from [190], in which Hamiltonian purification was discussed in detail. This motivated us to develop another scheme for Lindbladians, which is presented afterwards and which is also applicable for Hamiltonians and density operators. At the end of this chapter we discuss some applications regarding accessibility of an open quantum system that is described by a Lindblad master equation.

To begin with we will first properly define Hamiltonian purification [190]. Let $\mathcal{S} = \{H_1, \ldots, H_n\}$ be a set of Hamiltonians acting on a Hilbert space $\mathcal{H}_d$ of dimension d and $\tilde{\mathcal{S}} = \{\tilde{H}_1, \ldots, \tilde{H}_n\}$ be a set Hamiltonians acting on an extended Hilbert space $\mathcal{H}_{d_E}$, which includes $\mathcal{H}_d$ as a proper subspace. We call $\tilde{\mathcal{S}}$ a purifying set of $\mathcal{S}$ if all elements of $\tilde{\mathcal{S}}$ commute with each other, i.e.

$$[\tilde{H}_i, \tilde{H}_j] = 0, \quad \forall\, i, j = 1, \ldots, n. \tag{6.0.1}$$

They are related to those from $\mathcal{S}$ through

$$H_j = P\tilde{H}_j P, \quad \forall\, j = 1, \ldots, n, \tag{6.0.2}$$

with P being the orthogonal projection onto $\mathcal{H}_d$. For the sake of simplicity we consider now two Hamiltonians H_1 and H_2. Then proposition 1 in [190] states that a purifying set can be constructed on $\mathcal{H}_{d_E} = \mathcal{H}_d \otimes \mathcal{H}_{d_A}$ with an auxiliary Hilbert space $\mathcal{H}_{d_A}$ given by the space of a single qubit. The purification reads

$$\begin{aligned} \tilde{H}_1 &= H_1 \otimes \mathbb{1}_2 + H_2 \otimes \sigma_x, \\ \tilde{H}_2 &= H_2 \otimes \mathbb{1}_2 + H_1 \otimes \sigma_x \end{aligned} \tag{6.0.3}$$

where the orthogonal projection that leads to the old Hamiltonians is given by

$$P = \mathbb{1}_d \otimes \frac{\mathbb{1}_2 + \sigma_z}{2}. \tag{6.0.4}$$

For a generic set $\mathcal{S}$ that consist of n linearly independent Hamiltonians it can be shown [190] that there exist always a purifying set $\tilde{\mathcal{S}}$, where the minimal dimension $d_E^{(min)}$ of the extended Hilbert space is bounded above by $d_E^{(\min)} \leq nd$.
The question arises if a purifying set can be constructed for Lindblad operators in a similar way. In the following we provide a very easy purification scheme that allows us to explicitly construct a purifying set for an arbitrary number of Lindblad operators.

6.1 Lindbladian purification

We consider a set of n Lindbladians $\mathcal{S} = \{\mathcal{D}_1, \ldots, \mathcal{D}_n\}$ of the form

$$\mathcal{D}_i(\cdot) = 2L_i(\cdot)L_i^\dagger - (L_i^\dagger L_i(\cdot) + (\cdot)L_i^\dagger L_i), \tag{6.1.1}$$

that is we consider only the dissipative part where we refer to L_i as the Lindblad operator acting on a Hilbert space $\mathcal{H}_d$. First of all note that two Lindbladians $\mathcal{D}_1$ and $\mathcal{D}_2$ are commutative if their corresponding Lindblad operators commute, i.e. $[L_1, L_2] = 0$. If we denote by $\tilde{\mathcal{D}}_j$ the elements of the purifying set $\tilde{\mathcal{S}}$, then we have the following:
a purifying set can always be constructed using an auxiliary Hilbert space $\mathcal{H}_{d_A}$ of dimension $d_A = n$ through the construction

$$\tilde{L}_i = L_i \otimes |i\rangle\langle i|, \quad i = 1, \ldots, n, \tag{6.1.2}$$

with $\{|i\rangle\}_{i=1}^n$ being an orthonormal basis of $\mathcal{H}_{d_A}$. Obviously through such a construction all purified Lindblad operators $\tilde{L}_i$ commute with each other, and as such we have found a general way to construct a purifying set of Lindbladians. Note that any coherent term $\mathcal{K}_i(\cdot) = -i[H_i, \cdot]$ can be purified in the same way by purifying the Hamiltonian according to $\tilde{H}_i = H_i \otimes |i\rangle\langle i|$.

For the Hamiltonian purification scheme (6.0.3) the old dynamics can be obtained through frequently measuring the auxiliary system in a projective way. We find in the Zeno limit

$$\lim_{n\to\infty}\left(Pe^{-i(\tilde{H}_1+\tilde{H}_2)t/n)}P\right)^n = e^{-i(H_1+H_2)t}P, \tag{6.1.3}$$

that the dynamics is governed by the old Hamiltonians H_1 and H_2, where P is the hermitian projection given by (6.0.4). For semigroup dynamics generated by $\mathcal{D}$ we can alternate the action of $\Lambda_{t/n} = \exp(\mathcal{D}t/n)$ and the action of a CPTP map $\mathcal{P}$, which is assumed to be a super projection $\mathcal{P}^2 = \mathcal{P}$. Analogously we find

$$\Phi_t \equiv \lim_{n\to\infty}(\mathcal{P}\Lambda_{t/n}\mathcal{P})^n = e^{\mathcal{J}t}\mathcal{P}, \tag{6.1.4}$$

where $\mathcal{J} = \mathcal{P}\mathcal{D}\mathcal{P}$, remembering from the previous chapter that such a dynamics can be obtained through a strong dissipative process, with $\mathcal{P}$ being the super projection onto the steady state manifold. If we define $\mathcal{P}(\rho_0) \equiv \rho(0)$ we see that (6.1.4) solves the master equation

$$\dot{\rho}(t) = \mathcal{J}(\rho(t)), \tag{6.1.5}$$

where it is worth mentioning that $\mathcal{J}$ is in general not in Lindblad form. However, the solution to the above master equation defines a proper semigroup as long as we consider initial states that belong to the image of $\mathcal{P}$. Having established the Zeno limit for semigroup dynamics (6.1.4) we now want to come back to the question of how we get from the purified version of the Lindbladians $\tilde{\mathcal{D}}_i$ back the old dynamics generated by $\mathcal{D}_i$. Instead of measuring the auxiliary system in a projective way we consider a non-selective measurement [187], described by the CPTP map

$$\mathcal{P}(\cdot) = \sum_n P_n(\cdot)P_n, \tag{6.1.6}$$

where $\{P_n\}_n$ is a set of complete and orthogonal hermitian projectors acting only non-trivially on the auxiliary system. A non-selective measurement has the physical meaning that we do not select the outcomes of the measurement. If we frequently

measure the auxiliary system in a non-selective way described by $\mathcal{P}$ from above, the purified version of the Lindbladian in the Zeno limit (6.1.4) becomes

$$\mathcal{J}_i(\cdot) = \sum_{n,m} (2\hat{L}_{i,m,n}(\cdot)\hat{L}^\dagger_{i,m,n} - (\hat{L}^\dagger_{i,m,n}\hat{L}_{i,m,n}(\cdot)P_n + P_n(\cdot)\hat{L}^\dagger_{i,m,n}\hat{L}_{i,m,n})), \tag{6.1.7}$$

where $\hat{L}_{i,m,n} = P_m\tilde{L}_iP_n$ with $\tilde{L}_i$ being the purified Lindblad operator given by (6.1.2). Now we take $P_n = \mathbb{1} \otimes |\phi_n\rangle\langle\phi_n|$ with $\{|\phi_n\rangle\}_{n=1}^{d_A}$ as an orthonormal basis for the auxiliary Hilbert space. Then it can be shown (for further details see App. C.1) that, if we trace over the auxiliary Hilbert space the solution to the reduced master equation $\dot{\rho}_S(t) = \mathrm{tr}_A\{\mathcal{J}_i(\rho(t))\}$ is given by

$$\rho_S(t) = \sum_{n=1}^{d_A} e^{\mathcal{L}_{i,n}}(\langle\phi_n|\rho(0)|\phi_n\rangle), \tag{6.1.8}$$

with $\rho(0)$ being the initial state of the composite system and

$$\mathcal{L}_{i,n}(\cdot) = |\langle\phi_n|i\rangle|^2(2L_i(\cdot)L_i^\dagger - (L_i^\dagger L_i(\cdot) + (\cdot)L_i^\dagger L_i)). \tag{6.1.9}$$

Note that $\langle\phi_n|\rho(0)|\phi_n\rangle = \sum_{k,n;k',n}\rho_{k,n;k',n}|k\rangle\langle k'|$, where $\{|k\rangle\}_{k=1}^{d}$ is an orthonormal basis of the original Hilbert space, is not necessarily a density operator. Now, if we assume that the basis in which the non-selective measurement is performed and the basis that appears in the purification scheme (6.1.2) are two mutually unbiased bases [195], we have that $|\langle\phi_n|i\rangle|^2 = \frac{1}{d_A}$ for all $n, i \in \{1, \ldots, d_A\}$ such that the solution (6.1.8) becomes

$$\rho_S(t) = e^{\mathcal{D}_i\frac{t}{d_A}}\rho_S(0). \tag{6.1.10}$$

Therefore, up to a modified decay rate, we obtain the original dynamics that is generated by the unpurified Lindbladian $\mathcal{D}_i$. Note that if we consider a purification of two Lindbladians with $\tilde{L}_1 = L_1 \otimes |1\rangle\langle 1|$ and $\tilde{L_2} = L_2 \otimes |2\rangle\langle 2|$, the basis in which the non-selective measurement is done is just given by the symmetric and anti-symmetric superpositions $|\pm\rangle = \frac{1}{\sqrt{2}}(|1\rangle \pm |2\rangle)$. To summarize, we saw that by using a frequent non-selective measurement of the auxiliary system we can turn a set of commuting Lindbladians into a set of non-commuting Lindbladians acting on the

original system.

Before we discuss some application of this observation in the next section, we first want to analyze the long time behavior of the reduced dynamics that is obtained from an overall dynamics generated by a purified Lindbladian $\tilde{\mathcal{D}}$. We consider $\tilde{L} = L \otimes A$ with some hermitian $A \in \mathbb{C}^{d_A \times d_A}$. If we denote by $\{|\phi_i\rangle\}_{i=1}^{d_A}$ the eigenbasis of A with eigenvalues λ_i respectively and if assume that the original system and the auxiliary system are initially uncorrelated, the density operator of the system $\rho_S(t)$ evolves according to

$$\rho_S(t) = \sum_{j=1}^{d_A} p_j e^{\mathcal{L}_j t} \rho_S(0) \equiv \mathcal{E}_t(\rho_S(0)), \tag{6.1.11}$$

where $p_j = \langle \phi_j | \rho_A(0) | \phi_j \rangle$, $\rho_A(0)$ is the initial state of the auxiliary system and

$$\mathcal{L}_j(\cdot) = \lambda_j^2 (2L(\cdot)L - (L^\dagger L(\cdot) + (\cdot)L^\dagger L)). \tag{6.1.12}$$

We are interested in the long time behavior of the map $\mathcal{E}_t$ where we assume that the original Lindbladian $\mathcal{D}$ has a unique fixed point, i.e. $\lim_{t\to\infty} e^{\mathcal{D}t}\rho_S(0) = \rho_{ss}$ for all $\rho_S(0) \in S(\mathcal{H}_d)$. If A has full support in the auxiliary Hilbert space, $\mathrm{supp}(A) = d_A$, we find

$$\begin{aligned} \lim_{t\to\infty} \mathcal{E}_t(\rho_S(0)) &= \sum_{j=1}^{d_A} p_j \lim_{t\to\infty} e^{\mathcal{L}_j t} \rho_S(0), \\ &= \rho_{ss}, \end{aligned} \tag{6.1.13}$$

that the asymptotic behavior of the reduced dynamics does not change. On the contrary, if A does not have full support, $\mathrm{supp}(A) = k < d_A$, the asymptotic behavior reads

$$\lim_{t\to\infty} \mathcal{E}_t(\rho_S(0)) = \left(\sum_{j=1}^{k} p_j \right) \rho_{ss} + \left(1 - \sum_{j=1}^{k} p_j \right) \rho_S(0). \tag{6.1.14}$$

The asymptotic state is given by a convex combination of the initial state of the original system and the fixed point of the unpurified Lindbladian. If we now frequently measure the auxiliary system in a non-selective way, we change the long

time behavior of the original system in such a way that it always converges through

$$\rho_S(t) = e^{\gamma \mathcal{D} t} \rho_S(0), \tag{6.1.15}$$

to the unique fixed point ρ_{ss}, where the modified decay rate is given by $\gamma = \frac{1}{d_A} \sum_{j=1}^{k} \lambda_j^2$. In other words, if the unique fixed point of the original system is given by the ground state, we can "cool" the original system by frequently measuring the auxiliary system in a non-selective way. In terms of quantum computing, if the original Lindbladian $\mathcal{D}$ is given by the Lindbladian that can be used for universal quantum computation [4], we can turn the original system into a system that is capable of universal quantum computational tasks.

6.2 Turning a non-accessible open system into an accessible one

Now we want to come back to the question of what we can gain by measuring the auxiliary system in a non-selective way in order to make the Lindbladians acting on the reduced system non-commutative. We consider the controlled master equation

$$\dot{\rho}(t) = -i \mathrm{ad}_{H(t)}(\rho(t)) + \mathcal{D}(\rho(t)), \tag{6.2.1}$$

where we use the notation $\mathrm{ad}_{H(t)}(\cdot) = [H(t), \cdot]$ with $H(t)$, given by (2.2.3), containing the drift and m control Hamiltonians. For the sake of simplicity we assume that the dissipative part generates unital dynamics, such that in the vector of coherence representation of the master equation (cf. Sec. 2.1.3) the system (6.2.1) is equivalent to the bilinear control system

$$\dot{\boldsymbol{v}} = \left(A_0 + \sum_{k=1}^{m} u_k(t) A_k \right) \boldsymbol{v}, \tag{6.2.2}$$

for the vector of coherence $\boldsymbol{v} = (v_1, \ldots, v_d)^T$ where $d = N^2 - 1$, with N being the dimension of the quantum system. The drift term $A_0 \cong -i\mathrm{ad}_{H_0} + \mathcal{D}$ and the control part $A_k \cong -i\mathrm{ad}_{H_k}$ are isomorphic to the old representation (6.2.1), where $A_k \in \mathfrak{so}(d)$

and $A_0 \in \mathfrak{gl}(d,\mathbb{R})$ such that the solutions to (6.2.1) are given by transformation $x(t) \in Gl(d,\mathbb{R})$.

Instead of studying the control system (6.2.2) we study the control system

$$\dot{x}(t) = \left(A_0 + \sum_{k=1}^{m} u_k(t) A_k \right) x(t), \qquad x(0) = \mathbb{1}, \tag{6.2.3}$$

on the Lie group $Gl(d,\mathbb{R})$. Remember from Sec. 2.2.1 that the reachable set $\mathcal{R}(\mathbb{1})$ consists of all transformations that can be reached from the identity with the given controls. Note that the Lie Group $Gl(d,\mathbb{R})$ is not connected nor compact such that we can not apply the Lie rank criterion to analyze which transformations can be implemented with the controls. Moreover we pointed out in Sec. 2.3 that the Lindblad master equation is never fully controllable with unitary controls. Hence we decided to study in the following the accessibility of the control system (6.2.3), that is we investigate the case in which the reachable set has a non-empty interior in $Gl(d,\mathbb{R})$. It is known [11, 196] that the system (6.2.3) is accessible if its corresponding dynamical Lie algebra

$$\mathfrak{L} = \mathfrak{Lie}(A_0, \ldots, A_m), \tag{6.2.4}$$

is given by $\mathfrak{L} = \mathfrak{gl}(d,\mathbb{R})$. In particular we want to show in the following, on a specific example, that a frequent non-selective measurement can turn a non-accessible open system into an accessible one. We consider two qubits with a drift part $\tilde{\mathcal{L}}(\cdot) = -i\mathrm{ad}_{\tilde{H}_0}(\cdot) + \tilde{\mathcal{D}}(\cdot)$ where

$$\tilde{H}_0 = \sigma_z \otimes |1\rangle\langle 1|, \qquad \tilde{L} = \frac{\gamma}{2}\sigma_z \otimes |1\rangle\langle 1|, \tag{6.2.5}$$

and a control Hamiltonian $\tilde{H}_1 = \sigma_y \otimes |2\rangle\langle 2|$ such that $[\tilde{\mathcal{L}}, \mathrm{ad}_{\tilde{H}_1}] = 0$. Hence the Lie algebra (6.2.4) is two dimensional and the system is not accessible. Now, if we frequently measure the second qubit in the basis $|\pm\rangle = \frac{1}{\sqrt{2}}(|1\rangle \pm |2\rangle)$ in a non-selective way, we find with (6.1.8) and (6.1.10) that the reduced dynamics of qubit one is governed by

$$H_0 = \sigma_z, \qquad L = \frac{\gamma}{2}\sigma_z, \qquad H_1 = \sigma_y, \tag{6.2.6}$$

which leads in the vector of coherence representation to

$$A_0 = \begin{pmatrix} -\gamma & -1 & 0 \\ 1 & -\gamma & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}. \tag{6.2.7}$$

One can easily check that A_0 and A_1 generate the full algebra, i.e $\mathfrak{Lie}(A_0, A_1) = \mathfrak{gl}(3, \mathbb{R})$, such that the reduced system is accessible. A frequent non-selective measurement has turned a non-accessible system into an accessible one, noting that the dimension of the Lie algebra (6.2.4) has increased from 2 to 9.

From Kuranishi (see e.g. [11]) we know that if $\mathfrak{g}$ is a semi simple Lie algebra, there exists two elements $A, B \in \mathfrak{g}$ generating the full algebra $\mathfrak{Lie}(A, B) = \mathfrak{g}$. Moreover, the set of all $(A, B) \in \mathfrak{sl}(d, \mathbb{R}) \times \mathfrak{sl}(d, \mathbb{R})$ such that A, B generates the full algebra is open and dense in $\mathfrak{sl}(d, \mathbb{R}) \times \mathfrak{sl}(d, \mathbb{R})$. Hence almost all bilinear control systems with a single control are accessible, which we refer to as generically accessible. Motivated by this result the question arises whether the controlled master equation is generically accessible. The main difficulty in answering this question is the fact that, due to the constrains of the Lindblad generator, the set of all possible drift terms is a restricted subset $\mathfrak{Lind}$ of $\mathfrak{gl}(d, \mathbb{R})$, while the controls are restricted to be in $\mathfrak{l} \subseteq \mathfrak{so}(d)$. However, one of the key results in [57] states that the set of all pairs $(A_0, A_1) \in \mathfrak{Lind} \times \mathfrak{l}$, such that $\mathfrak{Lie}(A_0, A_1) = \mathfrak{gl}(d, \mathbb{R})$, is open and dense in $\mathfrak{Lind} \times \mathfrak{l}$. Therefore the controlled master equation is generically accessible with a single unitary control [57]. Clearly, finding a pair that generates the full algebra for arbitrary system sizes is not trivial and still a work in progress. Furthermore note that the results for the unital case can be generalized to arbitrary Lindbladians by introducing the semi direct product [57].

6.3 Summary and conclusion

We extended the work that considers Hamiltonian purification [190] by establishing a new purification scheme for Lindbladians which is also applicable for Hamiltonian

dynamics. Every set of m Lindblad generators can be made commutative on an m dimensional auxiliary space by extending the Lindblad operators through hermetian projectors that form an orthonormal basis in the auxiliary space. Using the quantum Zeno effect the old, non-commutative dynamics on the original system can be obtained by frequently measuring the auxiliary system in a non-selective way. We saw that, without measuring the auxiliary system, the asymptotic state of the reduced dynamics of the original system is given by a convex combination of the asymptotic state of the unpurified Lindbladian and the initial state of the original system. If the original Lindbladian exhibits a unique fixed point, the reduced dynamics always converges to it when we perform a frequent non-selective measurement on the auxiliary system. The ability to select a unique fixed point through non-selective measurements might find some applications in reservoir engineering schemes and dissipative quantum computing. Moreover we saw that we can turn a non-accessible open quantum system into an accessible one by performing frequent non-selective measurements. We considered two qubits which are subject to a dynamics generated by a Lindbladian and a control Hamiltonian that commute with each other. The corresponding Lie algebra is two dimensional and hence the system is not accessible. If we now frequently measure the second qubit, qubit one becomes accessible.
Clearly, the presented purification scheme also works for observables and density operators, although, except for the partial trace, an operational way that allows us to obtain the original observable or state is not known to us. Nevertheless, we can conclude that every quantum object can be made commutative on a larger space. Since non-commutativity is a unique feature of quantum mechanics, and in fact it was argued in [197] that the only difference between quantum and classical is commutativity, it is tempting to say that every quantum system can be made classical on a larger space.

7 Summary and outlook

Our investigations into control of an open quantum system brought us from a Hamiltonian approach of formulating the control of an open system to the study of the coherently controlled Lindblad master equation. Unsurprisingly, the overall message of the thesis is that the two approaches can lead to extremely different predictions of the open system dynamics which is subject to coherent control. Clearly, we cannot change the non-unitary character of the Lindblad master equation by applying unitary controls. Dynamical decoupling demonstrates the difference between the two approaches in a dramatic way. For finite dimensional quantum systems decoherence can always be suppressed through dynamical decoupling within the Hamiltonian approach, whereas in the description of decoherence through a Linbdladian dynamical decoupling will never succeed.

Based on a system-environment Hamiltonian, we began in Chap. 3 with characterizing the unitary operations that can be implemented on a single spin interacting through a Heisenberg interaction with a spin environment. We showed that independent of the number of environmental spins the system spin is always fully controllable by acting with a single control field on the system spin. Hence decoherence that is induced by the spin environment can be circumvented by the possibility of implementing any unitary transformation on the system spin in a noiseless manner. Remarkably, if the couplings to each environmental spin are different from each other, the whole system becomes fully controllable by acting on the system spin alone. In this respect we can even control the environment through the system such that the total system becomes universal for quantum information tasks. Based on

the GRAPE algorithm we did extensive numerical studies in order to find the minimum time needed to implement a target gate on the system spin with high fidelity. Unfortunately, we found that in the case where the total system is fully controllable, the minimum time increases exponentially with the number of environmental spins. Instead of implementing a specific gate through the controls, we studied in Chap. 4 the suppression of decoherence for a generic system-environment interaction through frequently applying instantaneous control pulses, known as dynamical decoupling. We first developed bounds, characterizing how fast we have to apply dynamical decoupling, in order to effectively suppress decoherence. Afterwards we proved that non-unitary dynamics described by a Lindblad operator can never be suppressed through unitary decoupling operations, whereas every system-environment interaction, even if it is described by some unbounded bath operator, can. This observation led to a new method to distinguish intrinsic non-unitary dynamics, arising for example from collapse models, from decoherence. Taking dynamical decoupling as an example of controlling an open quantum system, the obtained results imply a dramatic difference in describing the coherent control of an open quantum system through the two afore mentioned approaches. Within the master equation approach decoherence can never be suppressed, whereas on the Hamiltonian level this is always possible. Up to this point dynamical decoupling was studied for finite dimensional quantum system because a general framework for infinite dimensional systems is missing in the literature. We first showed that not every infinite dimensional system can be decoupled from the environment and hence protected from decoherence using dynamical decoupling. Afterwards we developed a general framework for Hamiltonians that are quadratic in the quadrature operators, allowing us to represent the dynamics by a symplectic transformation. We showed that, independent of environment, or the system size, decoherence that is induced by a quadratic interaction can always be suppressed using two operations applied on the system. Moreover, in contrast to the finite dimensional case, these operations do not disturb the system by leaving the system Hamiltonian invariant. Afterwards we studied the performance of random dynamical decoupling for a harmonic oscillator that is subject to decoher-

ence described through a common system-environment model. We found that the efficiency to suppress decoherence scales linearly with the number of environmental oscillators coupled to the system.

In Chap. 5 we moved on to investigate the control properties of the coherently controlled Lindblad master equation. Though the Lindblad master equation is never fully controllable with unitary controls, we saw that if the Lindbladian exhibits a decoherence free subspace, a strong noise process can turn parts of a quantum system into a system universal for quantum information tasks. Based on the Zeno effect the strong noise process forces the dynamics, accompanied by controls, to evolve in a protected subspace in which more complex operations can be implemented. Performing numerical gate optimization we investigated how strong the noise process has to be in order to implement high fidelity quantum operations that could not be implemented without the noise process. As a byproduct a new fidelity function was developed that might find applications in other open system optimization problems. Finally in Chap. 6 we developed and discussed a scheme that allows us to make Hamiltonians and Lindbladians commutative on an extended Hilbert space. Through a frequent non-selective measurement on the auxiliary system we can recover the old, possibly non-commutative, dynamics of the original system. This effect has two implications. Using a non-selective measurement we can engineer the fixed points of the reduced dynamics and we can turn a non-accessible open quantum system into an accessible one.

So what have we learned? To summarize we follow the sailing boat analogy from the introduction. The sail serves here as our control and currents underneath the boat represent non-unitary dynamics. In the presence of the wind it is possible to fully control the boat, but it is hopeless to fight against (intrinsic) currents that carry the boat away. Even if we rotate the sail rapidly, we will never stop the boat, since we can only suppress the wind through such a procedure. Nevertheless, sometimes such currents are beneficial, driving the boat into some direction so that together with controlling the sail new directions are explored. However, in order to steer the boat in the presence of noise, we first aim for an appropriate description of its tra-

jectory subject to the controls. In this respect the description of controlling an open quantum system needs further investigations. Besides the points already raised at the end or within each chapter, the results of the thesis suggest several directions for future research.

For example, a detailed analyzes of where the master equation description fails, how a controlled system-environment Hamiltonian is related to the reduced dynamics, and in particular a characterization of the CPTP maps arising from controlled Hamiltonians is needed. The derivation of a master equation based on a controlled system-environment Hamiltonian using Floquet theory [198] and recently obtained error estimates for truncating the environmental Hilbert space [199] could provide deeper insights. Regarding dynamical decoupling, are the mathematical subtleties, for instance concerning domain questions of unbounded operators, technical issues or are they relevant for the physics, possibly yielding new effects that can be observed? What are the relevant timescales to effectively decouple a quantum system from an infinite dimensional environment? Here the development of bounds that capture the dependence on the initial state of the environment would be highly desirable. Furthermore, the combination of open loop control with closed loop control schemes, such as feedback, is less investigated in the literature. Can we gain something from a noisy dynamics using feedback and open loop control? For example, can a feedback loop help to project onto a decoherence free subspace, even if the noise is not sufficiently strong, so that together with properly shaping the control fields universal control can be achieved?

To summarize further, the development of effective descriptions for the control of complex, possibly noisy, quantum systems with many degrees of freedom, in order to steer its dynamic towards desired outcomes, should be the subject of future studies. Especially the use of noise through control and the control of systems for which not all parameters are known, or no complete model exist, should be explored more. Estimation [200, 201], and adaptive control schemes [202], may prove to be valuable tools for developing a deeper understanding of such systems, and allowing us to move towards real life applications that make use of quantum mechanical phenomena.

A Supplements chapter 3

A.1 Lie algebra for equal couplings

First we want to show that $i\sigma_x \in \mathfrak{L}$ where we define $A_1 \equiv iH_0$ and $A_2 \equiv iH_c$. Building the double commutator $[A_2, [A_1, A_2]]$ we get up to a constant the element

$$A_3 = i(\sigma_y + \sigma_+ J_- + \sigma_- J_+), \tag{A.1.1}$$

which leads with $A_1 - A_3$ to

$$A_4 = i\sigma_z J_z. \tag{A.1.2}$$

After calculating $[[A_1, A_2], A_3]$ and using the properties of J_- and , J_+ we find up to a constant the element

$$A_5 = \sigma_z(J_- - J_+) - 2i\sigma_z(\vec{J}^2 - J_z^2 - J_z) + i\sigma_-\sigma_+ J_z. \tag{A.1.3}$$

The last two terms of A_5 commute with A_4 and therefore $[A_4, A_5]$ yields, up to a constant

$$A_6 = iJ_x. \tag{A.1.4}$$

By commuting A_6 with $[A_1, A_2]$ we find $i\sigma_x J_z$ and by commuting with A_2, A_4, A_6 we obtain the following elements

$$\begin{aligned} &i\sigma_z,\ i\sigma_x J_z,\ i\sigma_y J_z,\ i\sigma_z J_z, \\ &iJ_x,\ i\sigma_x J_y,\ i\sigma_y J_y,\ i\sigma_z J_y, \end{aligned} \tag{A.1.5}$$

which can be used to isolate

$$A_7 = i(\sigma_y + \sigma_x J_x), \tag{A.1.6}$$

from A_1. By commuting A_7 with $i\sigma_x J_z$ we obtain iJ_y and the commutator $[i\sigma_x J_z, iJ_y]$ yields up to a constant $i\sigma_x J_x$ which can be used, together with A_7, to reach $i\sigma_y$. We then also have $i\sigma_x$ by using A_2. In fact we showed that

$$i\sigma_\alpha,\ iJ_\beta,\ i\sigma_\alpha J_\beta \in \mathfrak{L}, \quad \forall \alpha, \beta = x, y, z. \tag{A.1.7}$$

Due to the fact that the ladder operators $\sigma_\pm$ and $J_\pm$ define another representation it is easy to verify that $i(\sigma_\alpha J_\beta + h.c.) \in \mathfrak{L}$ holds also for $\alpha, \beta = \pm, z$.
With the elements we found so far we can find other elements by building their commutators and creating real linear combinations. Next we show that

$$i\sigma_\alpha(J_+^l J_-^k J_z^s + h.c.) \in \mathfrak{L}, \quad \forall l, k, s \in \mathbb{N}, \quad \alpha = x, y, z. \tag{A.1.8}$$

Essentially, this characterizes the dynamical Lie algebra up to normal ordering of operators. We will proceed by induction and define

$$A(K) = \text{span}\{i\sigma_\alpha(J_+^l J_-^k J_z^s + h.c.) \mid l + k + s \leq K,\ \alpha = x, y, z\}, \tag{A.1.9}$$

where hereafter Greek indices describe some x, y, z for the Pauli spin operators and some $\pm, z$ for the angular momentum operators.
The initial step is to prove that $A(1) \subset \mathfrak{L}$. This is trivial because we already have proven with (A.1.7) that $\{i\sigma_\alpha(J_\beta + h.c.)\}$ is a subset of $\mathcal{L}$. We can therefore go to the inductive step and show that if $A(K) \subset \mathfrak{L}$ then $A(K+1) \subset \mathfrak{L}$.
Take any $a = i\sigma_\alpha(J_+^l J_-^k J_z^s + h.c.) \in A(K+1)$ with $l + k + s = K + 1$ and calculate for $s > 0$ the commutator

$$\begin{aligned}&[i\sigma_\alpha(J_+^l J_-^k J_z^{s-1} + h.c.), i\sigma_\beta J_z] = \\ &\sigma_\beta\sigma_\alpha(J_z J_+^l J_-^k J_z^{s-1} + J_z^s J_+^k J_-^l) - \sigma_\alpha\sigma_\beta(J_+^l J_-^k J_z^s + J_z^{s-1} J_+^k J_-^l J_z),\end{aligned} \tag{A.1.10}$$

keeping in mind that if $i\sigma_\alpha(J_+^l J_-^k J_z^{s-1} + h.c.) \in A(K)$ then the above commutator is by construction an element of $\mathfrak{L}$. Due to the anticommutation rules of the Pauli

spin operators, we can always choose a σ_β so to obtain from Eq. (A.1.10) up to a constant the following

$$[i\sigma_\alpha(J_+^l J_-^k J_z^{s-1} + h.c.), i\sigma_\beta J_z] = a + O, \tag{A.1.11}$$

with $O \in A(K)$. The cases $l > 0$ and $k > 0$ can be treated analogously and therefore we showed that $a \in \mathfrak{L},\ \forall l, k, s \in \mathbb{N}$. $\square$

A.2 Controllability proofs

A.2.1 Controllability of the central spin

In this section we will prove controllability of the central spin by using the determinant of a Vandermonde matrix along the lines of [91]. We want to prove that $\mathfrak{su}(2) \subset \mathfrak{L}$, $\forall N \in \mathbb{N}$ for almost all values of the couplings constants A_k. By $\mathfrak{su}(2)$ we denote the special unitary algebra acting on the central spin. To be as general as possible we rewrite the system Hamiltonian (3.1.1) as

$$H_0 = \sigma_y + \sum_{n=1}^{\tilde{N}} h_n(\sigma_x J_x^{(n)} + \sigma_y J_y^{(n)} + \sigma_z J_z^{(n)}), \tag{A.2.1}$$

where each set n of bath spins with identical A_k's are combined as collective particles, coupled to the central system with strength h_n and with corresponding angular momentum operators $J_\alpha^{(n)}$ with $\alpha = x, y, z$. We assume that $|h_n| \neq |h_m|$ and $|h_n - h_m| \neq |h_i - h_j|$ with $(n, m) \neq (i, j) \neq (j, i)$. In general these assumptions are only instrumental to the analytical proof and have neither physical meaning nor are necessary in practice as witnessed by numerical calculations of the dimension of the dynamical Lie algebra. An exception occurs for the full controllability of the whole spin star. In this instance, which will be discussed later in A.2.2, both the analytical proof and the numerical calculations show that the assumption $|h_n| \neq |h_m|$ is necessary.

In order to prove full controllability of the central spin, we need to prove that the operator $i\sigma_x$ acting on the central spin belongs to the dynamical Lie algebra $\mathfrak{L}$. To

this end we begin by commuting iH_0 with the control Hamiltonian (3.1.4) and get by real linear combinations the elements

$$B_1 = i(\sigma_y + \sum_{n=1}^{\tilde{N}} h_n(\sigma_x J_x^{(n)} + \sigma_y J_y^{(n)})), \tag{A.2.2}$$

$$B_2 = i\sigma_z \sum_{n=1}^{\tilde{N}} h_n J_z^{(n)}, \tag{A.2.3}$$

$$B_3 = i(\sigma_x + \sum_{n=1}^{\tilde{N}} h_n(\sigma_x J_y^{(n)} - \sigma_y J_x^{(n)})). \tag{A.2.4}$$

We can now observe that proving $i\sigma_x \in \mathfrak{L}$ amounts to prove that $iJ_x^{(i)} \in \mathfrak{L}$. Indeed $i\sigma_x$ is obtained by performing commutators of $iJ_x^{(i)}$ and $B1$, B_2, B_3 and real linear combinations of the resulting elements. The double commutator $[[B_1, B_3], B_2]$ yields up to a constant the element

$$\begin{aligned} B_4 = i(&\sum_{n=1}^{\tilde{N}} h_n^2 J_x^{(n)} \\ &+ \sum_{n>m=1}^{\tilde{N}} (h_n - h_m) h_n h_m (J_x^{(n)} J_y^{(m)} - J_x^{(m)} J_y^{(n)})). \end{aligned} \tag{A.2.5}$$

At this point the key observation is that up to a constant

$$\begin{aligned} &\sum_{n>m=1}^{\tilde{N}} c_{n,m}[[(J_x^{(n)} J_y^{(m)} - J_x^{(m)} J_y^{(n)}), B_2], B_2] \\ &= \sum_{n>m=1}^{\tilde{N}} (h_n - h_m)^2 c_{n,m} (J_x^{(n)} J_y^{(m)} - J_x^{(m)} J_y^{(n)}), \end{aligned} \tag{A.2.6}$$

and

$$\sum_{n=1}^{\tilde{N}} d_n [[J_x^{(n)}, B_2], B_2] = \sum_{n=1}^{\tilde{N}} h_n^2 d_n J_x^{(n)}, \tag{A.2.7}$$

with $c_{n,m}$ and d_n some coefficients. Using the operator B_4 and Eqs. (A.2.6) and (A.2.7), we can create operators of the form

$$\begin{aligned} B^{(s)} = i(&\sum_{n=1}^{\tilde{N}} h_n^{2(s+1)} J_x^{(n)} \\ &+ \sum_{n>m=1}^{\tilde{N}} (h_n - h_m)^{2s+1} h_n h_m (J_x^{(n)} J_y^{(m)} - J_x^{(m)} J_y^{(n)})), \end{aligned} \tag{A.2.8}$$

with $B^{(0)} = B_4$, $[[B^{(s)}, B_2], B_2] = B^{(s+1)}$ and $s = 0, \ldots, \tilde{N} - 1$.
We now need to show that the operators of the kind $X^{(s)} \equiv \sum_{n=1}^{\tilde{N}} h_n^{2(s+1)} J_x^{(n)}$ contained in each $B^{(s)}$, Eq. (A.2.8), are all linearly independent. In fact, if all $X^{(s)}$ are linearly independent then the determinant of the matrix corresponding to the linear set of equations

$$\begin{pmatrix} h_1^2 & h_2^2 & \cdots & h_{\tilde{N}}^2 \\ h_1^4 & h_2^4 & \cdots & h_{\tilde{N}}^4 \\ \vdots & \vdots & \ddots & \vdots \\ h_1^{2\tilde{N}} & h_2^{2\tilde{N}} & \cdots & h_{\tilde{N}}^{2\tilde{N}} \end{pmatrix} \begin{pmatrix} J_x^{(1)} \\ J_x^{(2)} \\ \vdots \\ J_x^{(\tilde{N})} \end{pmatrix} = \begin{pmatrix} X^{(0)} \\ X^{(1)} \\ \vdots \\ X^{(\tilde{N}-1)} \end{pmatrix} \tag{A.2.9}$$

is non-vanishing. We now define $\tilde{h}_n = h_n^2$, divide the columns of the matrix (A.2.9) by $\tilde{h}_n$ and then transpose. In this way we obtain a Vandermonde matrix whose determinant $\prod_{1 \le i < j \le \tilde{N}} (\tilde{h}_j - \tilde{h}_i)$ is non-vanishing if $|h_j| \neq |h_i|$, $\forall i \neq j$ as assumed in the beginning.

By real linear combination of the operators $B^{(s)}$ the operator

$$\tilde{B}_i = i(J_x^{(i)} + \sum_{n>m=1}^{\tilde{N}} \xi_{n,m}^{(i)} (J_x^{(n)} J_y^{(m)} - J_x^{(m)} J_y^{(n)})), \tag{A.2.10}$$

can be selected. If all $\xi_{n,m}^{(i)}$ in Eq. (A.2.10) are zero then we immediately obtain the operator $iJ_x^{(i)}$ as an element of $\mathfrak{L}$. If this is not the case, using Eqs. (A.2.6) and (A.2.7) we can again construct s operators of the form

$$\begin{aligned} \chi^{(s)} = i(&h_i^{2s} J_x^{(i)} \\ &+ \sum_{n>m=1}^{\tilde{N}} (h_n - h_m)^{2s} \xi_{n,m}^{(i)} (J_x^{(n)} J_y^{(m)} - J_x^{(m)} J_y^{(n)})) \end{aligned} \tag{A.2.11}$$

with $s = 1, \ldots, (\tilde{N}^2 - \tilde{N})/2$ assuming that all coefficients $\xi_{n,m}^{(i)}$ are different from zero. As before we can associate them to a Vandermonde matrix with non-vanishing determinant provided that $|h_n - h_m| \neq |h_i - h_j|$, $\forall (n,m) \neq (i,j) \neq (j,i)$. By real linear combinations of the $\chi^{(s)}$'s we can then select the operator

$$\tilde{\chi}_{n,m} = i(\lambda J_x^{(i)} + \omega_{n,m} (J_x^{(n)} J_y^{(m)} - J_x^{(m)} J_y^{(n)})). \tag{A.2.12}$$

If the coefficient λ is zero we can obtain $iJ_x^{(i)}$ by real linear combinations of $\omega_{n,m}(J_x^{(n)}J_y^{(m)} - J_x^{(m)}J_y^{(n)})$ and the B_i's (A.2.10). Instead, if $\lambda \neq 0$, using Eqs. (A.2.6) and (A.2.7), we can obtain from $\tilde{\chi}_{n,m}$ a second linearly independent operator with the same structure and then, by real linear combination of the two operators, the operator $iJ_x^{(i)}$. Since $iJ_x^{(i)} \in \mathfrak{L}$ we have $i\sigma_x \otimes \mathbb{1}_{bath} \in \mathfrak{L}$ and hence the central spin is fully controllable.

A.2.2 Full controllability

By commuting $iJ_x^{(i)}$ with B_1 and B_2 and using the full controllability of the central spins we obtain by real linear combinations $iJ_y^{(i)} \in \mathfrak{L}$ and hence $iJ_z^{(i)} \in \mathfrak{L}$. This implies that each collective particle contained in Eq. (A.2.1) is fully controllable. If all system-bath coupling constants are different from each other this implies full controllability of each bath spin and due to the Heisenberg interaction with the central spin the Lie algebra is given by $\mathfrak{su}(2^{N+1})$ [92] meaning that the whole system is fully controllable. We emphasize that controllability of the whole spin star can only be achieved if all coupling constants are different from each other, because in this case the existence of symmetric manifolds is prevented. The numerical calculation of the dimension of the dynamical Lie algebra shows that even the absolute value of the coupling constants has to be different from each other.

B Supplements chapter 5

B.1 Characterization of $\mathfrak{L}_{\mathrm{DFS}}$ for the qubit chain model

We prove the following lemma: the Lie algebra $\mathfrak{L}_{\mathrm{DFS}}$ generated by the two projected Hamiltonians $\mathcal{P}(H_0)$ and $\mathcal{P}(H_1)$ in (5.3.25) includes all the rotationally symmetric two- and three-body operators, $H_{mn} = \boldsymbol{\sigma}^{(m)} \cdot \boldsymbol{\sigma}^{(n)}$ and $H_{ijk} = \boldsymbol{\sigma}^{(i)} \cdot (\boldsymbol{\sigma}^{(j)} \times \boldsymbol{\sigma}^{(k)})$, $m < n$; $i < j < k$; $m, n, i, j, k = 1, \ldots, N$ defined in (5.3.27), for any number of qubits $N \geq 3$.

Proof. Let us introduce

$$\tilde{H}_0 = \mathcal{P}(H_0), \qquad \tilde{H}_1 = \mathcal{P}(H_1). \tag{B.1.1}$$

The first commutator reads

$$i[\tilde{H}_0, \tilde{H}_1] = 2H_{123}. \tag{B.1.2}$$

Then, by commuting $\tilde{H}_1 = H_{12}$ with the newly generated H_{123} twice, we have

$$i[H_{12}, H_{123}] = 4(H_{13} - H_{23}), \tag{B.1.3a}$$

$$i[i[H_{12}, H_{123}], H_{123}] = 16(H_{13} + H_{23} - 2H_{12}), \tag{B.1.3b}$$

from which we gain H_{13} and H_{23}. All the rotationally symmetric operators up to the third qubit (three two-body operators H_{12}, H_{23}, H_{13} and a three-body operator H_{123}) are in our hands.

For $N \geq 4$, we proceed by induction. Suppose that all the rotationally symmetric two- and three-body operators for the first n qubits are at our disposal. It is actually the case for $n = 3$, as we saw above. Then, we are able to extend one qubit further, generating all the two- and three-body operators involving the $(n+1)$th qubit by the following procedure.

1. Commute $H_{(n-1)n}$ with $\tilde{H}_0$ to extend to the $(n+1)$th qubit,

$$i[H_{(n-1)n}, \tilde{H}_0] = -2(H_{(n-2)(n-1)n} - H_{(n-1)n(n+1)}). \tag{B.1.4}$$

We acquire $H_{(n-1)n(n+1)}$.

2. By commuting $H_{(n-1)n}$ with the newly generated $H_{(n-1)n(n+1)}$ twice, we have

$$i[H_{(n-1)n}, H_{(n-1)n(n+1)}] = 4(H_{(n-1)(n+1)} - H_{n(n+1)}), \tag{B.1.5}$$

$$i[i[H_{(n-1)n}, H_{(n-1)n(n+1)}], H_{(n-1)n(n+1)}] = 16(H_{(n-1)(n+1)} + H_{n(n+1)} - 2H_{(n-1)n}), \tag{B.1.6}$$

from which we gain $H_{(n-1)(n+1)}$ and $H_{n(n+1)}$.

3. Then, iterate the following steps for $m = n-2, n-3, \ldots, 1$,

$$i[H_{m(m+1)}, H_{(m+1)(n+1)}] = 2H_{m(m+1)(n+1)}, \tag{B.1.7a}$$

$$i[H_{m(m+1)}, H_{m(m+1)(n+1)}] = 4(H_{m(n+1)} - H_{(m+1)(n+1)}), \tag{B.1.7b}$$

to get $H_{m(n+1)}$, $m = 1, \ldots, n-2$. All the two-body operators involving the $(n+1)$th qubit are thus in our hands.

4. Combining the two-body operators, we can generate any three-body operators involving the $(n+1)$th qubit,

$$\begin{aligned} &i[H_{m_1 m_2}, H_{m_2(n+1)}] = 2H_{m_1 m_2(n+1)} \\ &m_1, m_2 = 1, \ldots, n;\ m_1 < m_2 \leq n. \end{aligned} \tag{B.1.8}$$

In this way, all the rotationally symmetric two- and three-body operators for the first $n+1$ qubits are generated. Then, by induction, we can generate all the rotationally symmetric two- and three-body operators for any number of qubits N. □

B.2 Asymptotic dimension of the Lie algebra $\mathfrak{L}_{\rm DFS}$ for the qubit chain model

Let us estimate the asymptotic dimension for a large N of the Lie algebra $\mathfrak{L}_{\rm DFS}$ in (5.3.28) generated by the projected Hamiltonians $\mathcal{P}(H_0)$ and $\mathcal{P}(H_1)$ for the chain of N qubits discussed in Sec. 5.3.3. As commented in Sec. 5.3.3, the dimension of $\mathfrak{L}_{\rm DFS}$ is bounded by the dimension of $\bigoplus_J \mathfrak{su}(d_{J,N})$ and the dimension of $\bigoplus_J \mathfrak{u}(d_{J,N})$, i.e.,

$$\sum_J (d_{J,N}^2 - 1) < \dim \mathfrak{L}_{\rm DFS} < \sum_J d_{J,N}^2. \tag{B.2.1}$$

As we will see, the lower bound is dominated by the first contribution $\sum_J d_{J,N}^2$ for large N, and the difference between the lower and upper bounds becomes relatively negligible in the asymptotic regime. Observe also that the dimensions $d_{J,N}$ of the DFS's given in (5.3.21) can be cast as

$$d_{J,N} = \left(1 - \frac{2K}{N+1}\right) \binom{N+1}{K}, \qquad K = N/2 - J = 0, 1, \ldots, \lfloor \tfrac{N}{2} \rfloor, \tag{B.2.2}$$

where $\lfloor x \rfloor$ denotes the largest integer not greater than x. Approximating the binomial coefficient by

$$\binom{n}{k} = \frac{2^n}{\sqrt{\pi n/2}} e^{-2n(k/n-1/2)^2} [1 + O(1/\sqrt{n})], \tag{B.2.3}$$

the dimension of the Lie algebra is estimated as

$$\begin{aligned} \dim \mathfrak{L}_{\rm DFS} &\sim \sum_J d_{J,N}^2 \\ &= \sum_{K=0}^{\lfloor \frac{N}{2} \rfloor} \left(1 - \frac{2K}{N+1}\right)^2 \binom{N+1}{K}^2 \\ &\sim \frac{N+1}{2} \int_0^1 dx\, x^2 \frac{4^{N+1}}{\pi(N+1)/2} e^{-(N+1)x^2} \\ &\sim \frac{4^N}{\sqrt{\pi} N^{3/2}}, \end{aligned} \tag{B.2.4}$$

where the continuum limit is taken through $x = 1 - 2K/(N+1)$.

B.3 Derivation of the lower bound ε_2

Here we derive the lower bound (5.4.2). Using the definition of the Hilbert-Schmidt norm $\|A\|_{\mathrm{HS}}^2 = \mathrm{tr}\{A^\dagger A\}$ for a matrix A we can rewrite the left-hand side of (5.4.2),

$$\begin{aligned}&\left\|J(\Lambda_T) - S(J(\mathcal{U}_G)\otimes J(\tilde{\mathcal{E}}))S\right\|_{\mathrm{HS}}^2\\ &= \mathrm{tr}\{J^2(\Lambda_T)\} + \mathrm{tr}_2\{J^2(\tilde{\mathcal{E}})\} - 2\mathrm{tr}\{SJ(\Lambda_T)S(J(\mathcal{U}_G)\otimes J(\tilde{\mathcal{E}}))\},\end{aligned} \tag{B.3.1}$$

where $\mathrm{tr}_2\{\cdot\}$ denotes the partial trace over the second system and the properties of the normalized Choi state J were used, i.e., $J^\dagger = J$, $\mathrm{tr}\{J\} = 1$ and $J^2 = J$ for a unitary map. The third term of the right-hand side of (B.3.1) can be rewritten as

$$\begin{aligned}&\mathrm{tr}\{SJ(\Lambda_T)S(J(\mathcal{U}_G)\otimes J(\tilde{\mathcal{E}}))\}\\ &= \mathrm{tr}\{SJ(\Lambda_T)S(J(\mathcal{U}_G)\otimes \mathbb{1}_2)(J(\mathcal{U}_G)\otimes J(\tilde{\mathcal{E}}))\}\\ &\leq \mathrm{tr}\{SJ^2(\Lambda_T)S(J(\mathcal{U}_G\otimes\mathbb{1}_2))\}^{1/2}\mathrm{tr}_2\{J^2(\tilde{\mathcal{E}})\}^{1/2}\\ &\leq \frac{1}{2}\left(\mathrm{tr}\{J^2(\Lambda_T)S(J(\mathcal{U}_G)\otimes\mathbb{1}_2)S\} + \mathrm{tr}_2\{J^2(\tilde{\mathcal{E}})\}\right),\end{aligned} \tag{B.3.2}$$

where from the second line to the third the Cauchy-Schwarz inequality and from the third line to the forth the inequality between the arithmetic and the geometric means have been used. Combining (B.3.1) and (B.3.2) we arrive at

$$\begin{aligned}&\left\|J(\Lambda_T) - S(J(\mathcal{U}_G)\otimes J(\tilde{\mathcal{E}}))S\right\|_{\mathrm{HS}}^2\\ &\geq \mathrm{tr}\{J^2(\Lambda_T)\} + \mathrm{tr}_2\{J^2(\tilde{\mathcal{E}})\} - \mathrm{tr}\{J^2(\Lambda_T)S(J(\mathcal{U}_G)\otimes\mathbb{1}_2)S\} - \mathrm{tr}_2\{J^2(\tilde{\mathcal{E}})\}\\ &= \mathrm{tr}\{J^2(\Lambda_T)(\mathbb{1} - S(J(\mathcal{U}_G)\otimes\mathbb{1}_2)S)\},\end{aligned} \tag{B.3.3}$$

which is the desired result. Note that for pure unitary maps $\Lambda_T = \mathcal{U}_T$ the lower bound simplifies further

$$\varepsilon_2 = 1 - \mathrm{tr}\{J(\mathcal{U}_T)S(J(\mathcal{U}_G)\otimes\mathbb{1}_2)S\}. \tag{B.3.4}$$

C Supplements chapter 6

C.1 Derivation of the reduced dynamics

We consider a purification $\tilde{L} = L \otimes A$ with $A \in \mathbb{C}^{d_A \times d_A}$, $A = A^\dagger$, and a non-selective measurement $\mathcal{P}$ with $P_n = \mathbb{1} \otimes |\phi_n\rangle\langle\phi_n|$ where $B = \{|\phi_n\rangle\}_{n=1}^{d_A}$ is an orthonormal basis for the auxiliary Hilbert space. Here we want to show that if we take B to be mutually orthogonal to the eigenbasis of A, the reduced dynamics of the original system is given by

$$\rho_S(t) = e^{\frac{\gamma}{d_A}\mathcal{D}t}\rho_S(0), \tag{C.1.1}$$

where $\gamma = \sum_{k=1}^{d_A} \lambda_k^2$ with λ_k being the eigenvalues of A and

$$\mathcal{D}(\cdot) = 2L(\cdot)L^\dagger - (L^\dagger L(\cdot) + (\cdot)L^\dagger L), \tag{C.1.2}$$

the Lindbladian of the original system. Clearly, for a purification of the form $A = |i\rangle\langle i|,\ i \in \{1, \ldots, d_A\}$ and $\{|i\rangle\}$ an orthonormal basis for $\mathcal{H}_{d_A}$ we obtain (6.1.10) from Sec. 6.1.

Using (6.1.7) we find that in the Zeno limit (6.1.4) the overall dynamics is governed by $\Phi_t = e^{\mathcal{J}t}\mathcal{P}$ where

$$\begin{aligned}\mathcal{J}(\cdot) &= \mathcal{P}\tilde{\mathcal{D}}\mathcal{P}(\cdot) \\ &= \sum_{n,m} \langle\phi_m|A|\phi_n\rangle\langle\phi_n|A|\phi_m\rangle (2L \otimes |\phi_m\rangle\langle\phi_n|(\cdot)L^\dagger \otimes |\phi_n\rangle\langle\phi_m| \\ &\quad - (L^\dagger L \otimes |\phi_n\rangle\langle\phi_n|(\cdot)\mathbb{1} \otimes |\phi_n\rangle\langle\phi_n| + \mathbb{1} \otimes |\phi_n\rangle\langle\phi_n|(\cdot)L^\dagger L \otimes |\phi_n\rangle\langle\phi_n|)).\end{aligned} \tag{C.1.3}$$

With $\mathcal{J}(\mathcal{P}(\rho)) = \mathcal{J}(\rho)$ and $\mathrm{tr}_A\{\mathcal{P}(\rho)\} = \mathrm{tr}_A\{\rho\} = \rho_A$ we find

$$\rho_S(t) = \mathrm{tr}_A\left\{e^{\mathcal{J}t}\mathcal{P}(\rho(0))\right\} = \mathrm{tr}_A\left\{e^{\mathcal{J}t}\rho(0)\right\}, \tag{C.1.4}$$

such that $\rho_S(t)$ obeys the differential equation

$$\begin{aligned}\dot{\rho}_S(t) &= \mathrm{tr}_A\{\mathcal{J}(\rho(t))\},\\ &= \sum_{n=1}^{d_A}\mathcal{L}_n(\langle\phi_n|\rho(t)|\phi_n\rangle),\end{aligned} \tag{C.1.5}$$

where

$$\mathcal{L}_n(\cdot) = \langle\phi_n|A^2|\phi_n\rangle(2L(\cdot)L^\dagger - (L^\dagger L(\cdot) + (\cdot)L^\dagger L)). \tag{C.1.6}$$

The formal solution to (C.1.5) reads

$$\rho_S(t) = \sum_{n=1}^{d_A} e^{\mathcal{L}_n t}(\langle\phi_n|\rho(0)|\phi_n\rangle), \tag{C.1.7}$$

and if we take $\{|\phi_n\rangle\}_{n=1}^{d_A}$ to be a basis that is mutually orthogonal to the eigenbasis $\{|\psi_k\rangle\}_{k=1}^{d_A}$ of A with eigenvalues λ_k we have that $\langle\phi_n|A^2|\phi_n\rangle = \frac{1}{d_A}\sum_{k=1}^{d_A}\lambda_k = \frac{\gamma}{d_A}$ for all $n \in \{1,\ldots,d_A\}$. Hence the solution becomes

$$\rho_S(t) = e^{\frac{\gamma}{d_A}\mathcal{D}t}\sum_{n=1}^{d_A}(\langle\phi_n|\rho(0)|\phi_n\rangle) = e^{\frac{\gamma}{d_A}\mathcal{D}t}\mathrm{tr}_A\{\rho(0)\} = e^{\frac{\gamma}{d_E}\mathcal{D}t}\rho_S(0), \tag{C.1.8}$$

which is the desired result.

8 Bibliography

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