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Squeezing Enhancement and Adiabatic Elimination in Quantum Feedback Networks

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Abstract

Classical feedback control and system theory are playing an important role in modelling, controlling and analysing complex devices in many branches of engineering. Recent developments like quantum computers and miniaturisation of existing applications and devices are increasing the importance of the ability to control systems with quantum effects. Efforts have been made recently to extent the simplicity and power of the language of classical control theory to quantum mechanical systems. Within this framework of "Quantum Feedback Networks" we are investigating two problems.

The first problem concerns the enhancement of squeezed states. It has been observed that the squeezing effect of squeezing devices can be enhancement by measurement based feedback techniques or use of optical cavities. We are investigating the possibility of feedback enhanced squeezing using coherent feedback control. Considered is a static ideal squeezing devices interacting with a single mode cavity undergoing coherent feedback using a beam splitter. We show that the overall squeezing of the output depends on the beam-splitter's reflectivity and that we are thus able to enhance the squeezing by choosing an appropriate configuration of the beam-splitter.

In the second part we investigate the question of compatibility of a rigorous approach to the adiabatic elimination of some degrees of freedom of a quantum mechanical systems and instantaneous feed-forward and feedback limits for quantum mechanical networks. The commutativity of both limits is not obvious but frequently assumed in quantum optics. We show that both limit procedures are instances of Schur complements and prove the commutativity of both limits by generalising a statement about successive Schur complements.

Acknowledgement

I first all want to thank my supervisor *Prof. John Gough*. It has been a long way from my arrival at Aberystwyth and the start of the PhD to the completion of this document. With a topic combining many disciplines and a personal background in only one of them (electrical engineering), I would not have been able to familiarise myself with all the knowledge required to work in the field nor would have been my research similarly fruitful without his help, guidance, encouragement, support and openness for questions and discussions.

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Contents

1 Introduction

This thesis will investigate two problems, both taking place within the framework of *quantum feedback networks*.

The framework or language of *quantum feedback networks* extends ideas of (classical) *control theory* to systems with quantum mechanical components. Classical control theory deals with the problem of designing a device that controls some dynamical system (usually referred to as the *plant*). In order to be able to design such a controller, tools and techniques are required to model and analyse the system. *Linear control theory*, which has emerged in the last century, offers here a rich and useful set of tools and results [11].

The notion of input/output systems and the language of block diagrams, which is used to represent networks of such systems, are very useful. They allow for reduction of bigger, complex systems to smaller, more easy to handle sub-components. In classical control theory however, these tools only extend scalar system variables who's dynamics are described by ordinary or partial differential equations. In quantum mechanics however, system variables (*observables*) are described by operators on Hilbert spaces.

One can establish a notion of quantum input/output systems similar to the classical case of linear systems and networks of such systems [22, 25, 28, 24]. Here we consider a specific class of open quantum systems. The dynamics of the joint system are described by quantum stochastic differential equations [49, 44]. Similarly to the classical case, all information about a single quantum input/output system can be encoded in some system matrices. One can build networks of systems by providing the parameter of the individual input/output systems (blocks) and the rules how the blocks are connected with each other. It has been shown that many problems in classical control theory can also be generalised or extended to the quantum case [67, 68, 47, 34, 71, 35, 12].

Due to the highly interdisciplinary background of the topic we will present an overview over the involved topics and introduce into the main concepts needed to understand the framework of quantum feedback networks. This will take place in Part I. In Part II we will deal with two problems. The first problem is an application of quantum coherent feedback which explores the possibilities of feedback enhanced squeezing.

A squeezed state is a minimum uncertainty state in which the variance in one quadrature is reduced on the cost of an increase in the other quadrature. There is a specific class of optical devices which are capable of creating these states. We explore how the squeezing of such an device can be enhanced by placing the device into a feedback arrangement.

The second problem concerns the compatibility of a procedure called adiabatic elimination with results about instantaneous feed-forward and feedback limits in quantum feedback net-

works. The adiabatic elimination limit eliminates some degrees of freedom of a specific class of quantum open systems. The feedback or feed-forward limit reduces internal degrees of freedom when building networks of quantum components. It is not obvious that both limit procedures commute, however, this assumption has been made frequently in quantum optics[1].

[1] [37]

Part I.

Background and Introductory Material

2

Principles of Quantum Mechanics

2.1. Introduction

In the first chapter we are going to introduce the basic notions and notations of quantum mechanics.

The theory of quantum mechanics has emerged from the discovery of the probabilistic behaviour of subatomic particles at the beginning of the 20^{th} century. Modelling this behaviour mathematically required a generalisation of the classical probabilistic set-up. This is due to the fact, that in the classical probability theory, distinct random variables are allowed to assume values of the sample space at the same time. However, one of the principles of quantum

mechanics is the fact that a measurement of a physical quantity of the system will perturb the system. When measuring certain pairs of physical quantities of quantum systems, represented by quantum random variables X and Y, the order of measurement will matter. Measuring the variable X before Y will lead to a different outcome than measuring Y first. A mathematical representation of this concepts requires a generalisation of Kolmogorov's probability theory.

In the following sections we are going to introduce the basic principles of the mathematical theory used to describe quantum mechanics. The content of this chapter is based on and following standard literature on the topic as for example [19], [56], [40] and [49].

2.2. The Quantum Probability Space

2.2.1. The finite dimensional case

We set the stage for a quantum probabilistic set-up with random variables, assuming a finite number of possible outcomes by choosing some Hilbert space $\mathcal{H}$ with $n = \dim \mathcal{H} < \infty$. We consider the set of projections $\mathcal{P}(\mathcal{H})$ on $\mathcal{H}$ and call the elements $E \in \mathcal{P}(\mathcal{H})$ events. Further we fix a positive operator with unit trace ρ and call ρ a state. The triple $(\mathcal{H}, \mathcal{P}(\mathcal{H}), \rho)$ is then called a quantum probability space. If ρ is a one dimensional projection, it is called a pure state, otherwise a mixed state. The set of all states ρ is convex, with the pure states being the extreme points. Each mixed state can therefore be expressed as a convex combination of pure states. Let $\mathcal{O}(\mathcal{H})$ be the set of self-adjoint operators in $\mathcal{H}$. The elements $X \in \mathcal{O}(\mathcal{H})$ are called random variables or *observables*. Note that the self-adjoint operator $\mathcal{O}(\mathcal{H})$ correspond to Hermitian matrices for $\dim \mathcal{H} < \infty$.

We now arrive at the following interpretation for the objects introduced so far. The number $\mathrm{tr}\{\rho E\}$ is interpreted as the probability that the event E, represented as an element of $\mathcal{P}(\mathcal{H})$, occurs for the state ρ. One can show that $0 \leq \mathrm{tr}\{\rho E\} \leq 1, \ \forall\, E \in \mathcal{P}(\mathcal{H})$.

Applying the spectral theorem one can show that any observable $X \in \mathcal{O}(\mathcal{H})$ assumes a spectral decomposition $X = \sum_j x_j E_j$, where the x_j are the eigenvalues of X and E_j is the projection into the eigenspace associated with x_j, which is interpreted as the event that X takes value x_j.

We can compute the quantity

$$\mathrm{tr}\left\{\rho X\right\} = \mathrm{tr}\left\{\rho \sum_j x_j E_j\right\} = \sum_j x_j \,\mathrm{tr}\left\{\rho E_j\right\}.$$

This is just the expectation of a random variable taking discrete values x_j with probability $p_j = \mathrm{tr}\left\{\rho E_j\right\}$, consistent with classical probability theory. We can state the k-th moment of X by $\mathrm{tr}\left\{\rho X^k\right\}$ and compute the characteristic function of X with $\mathrm{tr}\left\{\rho e^{itX}\right\}$.

2.2.2. The infinite dimensional case

In the previous section we considered random variables assuming a finite number of possible outcomes. However, simple examples of physical systems will give rise to the need of considering the general case, i.e. random variables taking an infinite number of outcomes. Following the set-up introduced in the previous section, this will lead to an infinite dimensional system space $\mathcal{H}$ and previous concepts will have to be generalised.

We will not go into the details and refer to [49] and [55] for further reading. We note that we will deal in the infinite case with *separable* Hilbert spaces. The state of the system will be described by those *trace class* operators, which are of unit trace and positive. Furthermore, the Hermitian matrices constituting the random variables of the set-up become the self-adjoint operators on $\mathcal{H}$.

2.2.3. The Bra-Ket Notation

We adopt the bra-ket notation introduced by Dirac. We denote a vector in $\mathcal{H}$ with $|\psi\rangle$ and the write for the linear map $\langle\phi, \cdot\rangle = \langle\phi|$ such that

$$\langle\phi, \psi\rangle = \langle\phi|\psi\rangle .$$

Given some observable $X \in \mathcal{O}(\mathcal{H})$ and some pure state $\rho = |u\rangle \langle u|$ and we see that we can write the expected value of X in state ρ as

$$\mathrm{tr}\{\rho X\} = \langle u, Xu\rangle = \langle u| X |u\rangle .$$

This notation is useful when labelling the vectors in a meaningful manner. Choose for example some operator $X \in \mathcal{O}(\mathcal{H})$ with eigenvalues x_j and eigenvectors $|\psi_k\rangle$. Using bra-ket notation we can label the eigenvectors with the associated eigenvalues, i.e. we can write $|x_j\rangle$ for the eigenvector of X associated with eigenvalue x_j such that $X|x_j\rangle = x_j |x_j\rangle$. Computing the expected value of X in the state $|x_j\rangle$ then yields $\mathrm{tr}\{\rho X\} = \langle x_j| X |x_j\rangle = x_j$. Given two state vectors $|\phi\rangle$ and $|\psi\rangle$, the quantity $\langle\psi|\phi\rangle$ is called the transition amplitude and its square $|\langle\psi|\phi\rangle|^2$ is interpreted as the transition probability from state $|\psi\rangle$ to state $|\phi\rangle$.

2.3. Dynamics

There are three equivalent ways to describe the dynamic evolution of a quantum mechanical system. The first, called the Schrödinger picture, describes the evolution by a time dependent state ρ_t whereas the observables remain time independent. In the Heisenberg picture, the observables are time dependent $X = X_t$ and the state of the system remains constant. In the interaction picture, we follow the Schrödinger evolution of the state for a given free dynamics, but have the Observables evolve according to a perturbation of this dynamics: this is useful if

we wish to examine the dynamics with respect to a fixed reference evolution.

Let $t \to U_t$ be a strongly continuous one-parameter group taking values in $\mathcal{U}(\mathcal{H})$, then by Stone's theorem there exists a unique observable H (called the Hamiltonian) such that $\lim_{t\to 0} t^{-1}(U_t - 1)\phi = -iH\phi$ for all ϕ in the domain of H. In the finite dimensional case, the domain will be all of $\mathcal{H}$, and we have $U_t = e^{-itH}$. We will also use this notation in the infinite dimensional case.

2.3.1. The Schrödinger picture

As mentioned before, in the Schrödinger, the state of the system is taken to be time dependent and the observables remain constant.

For a unitary dynamics $U_t = e^{-itH}$, the state evolves as $\rho_t = U_t \rho U_t^\dagger$, such that the expected value of the observable X in state ρ_t is given by $\mathrm{tr}\{\rho_t X\}$. The differential equation is the von Neumann equation

$$\frac{d}{dt}\rho_t = i[\rho_t, H].$$

If ρ_t is a pure state (a one dimensional projection), it can be expressed at $\rho_t = |U_t u(0)\rangle\langle U_t u(0)| = |u(t)\rangle\langle u(t)|$ and we obtain the Schrödinger equation

$$\frac{d}{dt}|u(t)\rangle = -iH|u(t)\rangle. \tag{2.3.1.1}$$

2.3.2. The Heisenberg picture

In the Heisenberg picture, the observables are taken to be time dependent and the state of the system is assumed to be fixed. We obtain the corresponding formulation by rearranging the equations obtained for the Schrödinger picture for some pure state $\rho_t = |u(t)\rangle\langle u(t)|$

$$\mathrm{tr}\{\rho_t X\} = \langle U_t u(0), X U_t u(0)\rangle = \left\langle u(0), \left(U_t^\dagger X U_t\right) u(0)\right\rangle = \langle u(0)| X_t |u(0)\rangle.$$

One can deduce the differential equation for $X_t = U_t^\dagger X U_t$

$$\frac{d}{dt} X_t = -i \left[X_t, H \right], \tag{2.3.2.1}$$

with $X_0 = X$.

2.4. Variance and Minimum Uncertainty

We defined the expected value of the observable X in state ρ in the previous section and denote it and the k-th moment of X with

$$\begin{aligned} \langle X \rangle_\rho &= \mathrm{tr} \left\{ \rho X \right\}, \\ \left\langle X^k \right\rangle_\rho &= \mathrm{tr} \left\{ \rho X^k \right\}. \end{aligned}$$

We can now define the mean-square deviation of X in state ρ by

$$(\Delta X)_\rho^2 = \left\langle X^2 \right\rangle_\rho - \langle X \rangle_\rho^2 .$$

Let $X, Y \in \mathcal{O}(\mathcal{H})$, then the variances of X and Y in state ρ obey the following inequality

$$(\Delta X)_u^2 (\Delta Y)_u^2 \geq \frac{1}{4} \mid \langle u | i \left[X, Y \right] u \rangle \mid^2 . \tag{2.4.0.2}$$

This inequality is know as Heisenberg's minimum uncertainty principle. If the observables X and Y do not commute we see that the the product of the variances can not be smaller than a certain minimum.

2.5. The Quantum Harmonic Oscillator

We introduce an important example for the quantum mechanical system, the quantum harmonic oscillator. We take take the system space $\mathcal{H}_{\text{osc}} = L^2(\mathbb{R})$. In Schrödinger representation we introduce the unbounded operators

$$(\hat{p}\psi)(x) = x \cdot \psi(x), \tag{2.5.0.3}$$

$$(\hat{q}\psi)(x) = -i\hbar \frac{\partial}{\partial x}\psi(x). \tag{2.5.0.4}$$

We call $\hat{p}$ the momentum and $\hat{q}$ the position operator. By choosing some test function $f(x) \in L^2(\mathbb{R})$, we can compute the cannonical commutation relation of $\hat{p}$ and $\hat{q}$, $[\hat{q}, \hat{p}]\, f(x) = i\hbar f(x)$ and obtain

$$[\hat{q}, \hat{p}] = (\hat{q}\hat{p} - \hat{p}\hat{q}) = i\hbar. \tag{2.5.0.5}$$

The Hamilton operator from Equation (2.3.1.1) and Equation (2.3.2.1) for a particle with mass m, interacting with some field can be specified by

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{q}^2. \tag{2.5.0.6}$$

This must be compared with the energy of a classical mass m, interacting with a spring with spring constant k, $E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2m}p^2 + V(q)$. The quantum mechanical equivalent by taking $p \to \hat{p}$ and $q \to \hat{q}$ yields

$$\hat{H} = \frac{1}{2m}\frac{\partial^2}{\partial^2 q} + V(q).$$

One can introduce new operators a and $a^\dagger$ by setting

$$\hat{a} = \frac{1}{2}(\hat{q} + i\hat{p}),\ \hat{a}^\dagger = \frac{1}{2}(\hat{q} - i\hat{p}), \tag{2.5.0.7}$$

such that

$$\hat{q} = \hat{a} + \hat{a}^\dagger,\ \hat{q} = \frac{1}{i}\left(\hat{a} - \hat{a}^\dagger\right).$$

The commutator $\left[\hat{a}, \hat{a}^\dagger\right]$ can be obtained from Equation (2.5.0.5) and Equation (2.5.0.7)

$$\left[\hat{a}, \hat{a}^\dagger\right] = 1.$$

The operators $\hat{a}$ and $\hat{a}^\dagger$ can be used to reformulate Equation (2.5.0.6) and hence we obtain the Hamilton operator:

$$H = \hbar\omega\left(\hat{a}\hat{a}^\dagger + \frac{1}{2}\right) = \hbar\omega\left(\hat{N} + \frac{1}{2}\right),$$

when defining the operator $\hat{N} = \hat{a}\hat{a}^\dagger$. We can compute the commutators $\left[\hat{H}, \hat{a}\right] = -\hbar\omega\hat{a}$ and $\left[\hat{H}, \hat{a}^\dagger\right] = \hbar\omega\hat{a}^\dagger$.

2.5.1. Number states

We are now going to investigate the eigenvectors and eigenvalues of the Hamiltonian $\hat{H}$. The operators $\hat{q}^2 = \hat{q}\hat{q}$ and $\hat{p}^2 = \hat{p}\hat{p}$ are positive, whence $\hat{H}$ is positive. As mentioned before, $\hat{H}$ can be interpreted as the energy of the system and that motivates us to call the eigenvectors $|E\rangle$ with eigenvalues E the energy vectors and energy eigenvalues. These eigenvectors and eigenvalues of $\hat{H}$ can be investigated by using above commutation relations and computing

$$\begin{aligned}
\hat{H}\,|E\rangle &= E\,|E\rangle\,, \\
\hat{H}\hat{a}^\dagger\,|E\rangle &= \hat{a}^\dagger\left(\hat{H} + \hbar\omega\right)|E\rangle = (E + \hbar\omega)\,\hat{a}^\dagger\,|E\rangle\,, \\
\hat{H}\hat{a}\,|E\rangle &= \hat{a}\left(\hat{H} - \hbar\omega\right)|E\rangle = (E - \hbar\omega)\,\hat{a}\,|E\rangle\,.
\end{aligned} \tag{2.5.1.1}$$

This motivates the following set-up. We call the energy eigenstates of $\hat{N} = \hat{a}\hat{a}^\dagger$ (recall that $\hat{H} = \hbar\omega\left(\hat{N} + \frac{1}{2}\right)$) the number states and label them with $|n\rangle$. If we denote the energy eigenstate associated with eigenvector $|n\rangle$ by $E_n = \hbar\omega\left(n + \frac{1}{2}\right)$ then we see that $\hat{H}\,|n\rangle =$

$E_n |n\rangle$ and $\hat{H}\hat{a}^\dagger |n\rangle = \hat{H} |n+1\rangle$ by above computation. Similarly we see that $\hat{H}\hat{a} |n\rangle = \hat{H} |n-1\rangle$. Since no eigenvalue may become negative ($\hat{H}$ is a positive operator), one can see that the lowest possible n must be $n = 0$ and that $\hat{a} |0\rangle = 0$, i.e. $\hat{a}$ annihilates the vacuum. We therefore see that $\hat{a}^\dagger |0\rangle = |1\rangle$ and the eigenvalues of the number operator are the whole numbers $n = 0, 1, 2, \ldots$. Similarly the energy eigenvalues E of $\hat{H}$ are given by $E = \frac{1}{2}\hbar\omega, \hbar\omega \left(1 + \frac{1}{2}\right), \hbar\omega \left(2 + \frac{1}{2}\right), \ldots$, and we therefore see that the energy of the system assumes discrete values.

This motivates us to call the operator $\hat{a}$ the annihilation operator, since it reduces the energy of the system by one quanta $\hbar\omega$ and to call $\hat{a}^\dagger$ the creation operator since it increases the systems energy by $\hbar\omega$. The vectors $|n\rangle$ form an orthonormal basis of $\mathcal{H}$. As seen above they can be deduced iteratively by application of operators $\hat{a}^\dagger$. To determine an appropriate normalisation we compute $\|\hat{a} |n\rangle\|^2 = \langle n| \hat{a}^\dagger a |n\rangle = n$ and $\left\|\hat{a}^\dagger |n\rangle\right\|^2 = \langle n| \hat{a}\hat{a}^\dagger |n\rangle = n + 1$, and thus we see that

$$\hat{a}^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle , \tag{2.5.1.2}$$

$$\hat{a} |n\rangle = \begin{cases} \sqrt{n} |n-1\rangle & \text{if} \quad n > 0 \\ 0 & \text{if} \quad n = 0 \end{cases} . \tag{2.5.1.3}$$

The normalised vectors $|n\rangle$ can be computed from the vacuum $|0\rangle$ by taking

$$|n\rangle = \frac{\hat{a}^{\dagger n}}{\sqrt{n!}} |0\rangle . \tag{2.5.1.4}$$

One can show that the number states are orthogonal and complete in $\mathcal{H}$. Let $|m\rangle$ and $|n\rangle$ be number states, then we have that

$$\langle m|n\rangle = \delta_{m,n},$$

and the resolution of identity

$$1 = \sum_{n\geq 0} |n\rangle \langle n| . \tag{2.5.1.5}$$

2.5.2. Coherent States

In the previous section we investigated the eigenstates of the operator $\hat{N} = \hat{a}^\dagger \hat{a}$. We are now investigating the eigenstates of the annihilation operator $\hat{a}$

$$\hat{a} |\alpha\rangle = \alpha |\alpha\rangle .$$

Using the resolution of identity for the number states Equation (2.5.1.5) one can expand the above equation in terms of the number states $|n\rangle$. Normalising the states $|\alpha\rangle$ then yields

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n\geq 0} \frac{\alpha^n}{\sqrt{n!}} |n\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n\geq 0} \frac{\alpha^n \hat{a}^{\dagger n}}{n!} |0\rangle .$$

These normalised states $|\alpha\rangle$ are know as coherent states. These states are not orthogonal, but complete and we find the resolution of identity in terms of $|\alpha\rangle$ with

$$\int \frac{\mathrm{d}^2\alpha}{\pi} |\alpha\rangle \langle\alpha| = 1.$$

One important property of coherent states is that they are those state for which equality holds in Equation (2.4.0.2), i.e.

$$\Delta\hat{q}_\rho \Delta\hat{p}_\rho = \sqrt{\tfrac{1}{2} \mid i [\hat{q}, \hat{p}] \mid^2} = \tfrac{1}{2}\hbar,$$

where $\rho = |\alpha\rangle \langle\alpha|$.

We can compute the characteristic functions of $\hat{q}, \hat{p}$ and $\hat{N}$ in state $|\alpha\rangle$ using $\hat{q} = \hat{a} + \hat{a}^\dagger$,

$\hat{p} = \frac{1}{i}\left(\hat{a} - \hat{a}^\dagger\right)$, $\langle\alpha|\alpha\rangle = 1$, $\langle n|m\rangle = \delta_{n,m}$ and the Baker-Campbell-Hausdorff theorem

$$\begin{aligned}
\langle\alpha| e^{it\hat{q}} |\alpha\rangle &= \langle\alpha| e^{it\left(\hat{a}+\hat{a}^\dagger\right)} |\alpha\rangle = \langle\alpha| e^{it\hat{a}^\dagger} e^{it\hat{a}} e^{-\frac{t^2}{2}} |\alpha\rangle , \\
&= e^{i(\alpha+\alpha^*)t - \frac{t^2}{2}}, \qquad (2.5.2.1) \\
\langle\alpha| e^{it\hat{p}} |\alpha\rangle &= e^{(\alpha-\alpha^*)t - \frac{t^2}{2}}, \qquad (2.5.2.2) \\
\langle\alpha| e^{it\hat{N}} |\alpha\rangle &= \langle n| \left(\sum_{n=0}^{\infty} \frac{(\alpha^*)^n}{\sqrt{n!}}\right) e^{it\hat{N}} \left(\sum_{m=0}^{\infty} \frac{(\alpha)^m}{\sqrt{m!}}\right) |m\rangle , \\
&= \langle n| \left(\sum_{n=0}^{\infty} \frac{(\alpha^*)^n}{\sqrt{n!}}\right) \left(\sum_{m=0}^{\infty} \frac{(\alpha e^{it})^m}{\sqrt{m!}}\right) |m\rangle \, e^{-|\alpha|^2}, \\
&= e^{|\alpha|^2\left(e^{it}-1\right)}. \qquad (2.5.2.3)
\end{aligned}$$

Equation (2.5.2.1) and (2.5.2.2) can be compared with the characteristic function of a normal distribution

$$\phi(t; \mu, \sigma^2) = e^{it\mu - \frac{1}{2}\sigma^2 t^2}$$

whereas Equation (2.5.2.3) can be compared with the characteristic function of the Poisson distribution

$$\phi(t; \lambda) = e^{\lambda\left(e^{it}-1\right)}.$$

We thus see that $\hat{p}$ and $\hat{q}$ in a coherent state $|\alpha\rangle$ are Gaussian with mean values $2\,\mathrm{Re}\,\alpha$ and $2\,\mathrm{Im}\,\alpha$ respectively and variance $\sigma^2 = 1$. The number operator $\hat{N}$ assumes a Poisson distribution with intensity $\lambda = \mid \alpha \mid^2$.

3

Quantum Open Systems

3.1. Introduction

We introduced the basic concepts of quantum mechanics in the last chapter and considered single, isolated systems. When modelling physical systems however, that system will never be isolated but always be interacting with its environment. If we focus on certain aspects of a system it might sometimes be sufficient to take the system to be isolated. For the more general case however, we consider quantum mechanical systems interacting with its environment, i.e. open systems. The approach presented in this chapter takes some general quantum mechanical system, interacting with an environment, modelled by a collection of harmonic oscillators.

In the following chapter we give some initial motivation and presents the approach by K. R. Parthasarathy [49].

Quantum stochastic calculus was introduced in 1984 by Hudson and Parthasarathy [31] as a generalization of Ito's theory of stochastic integration to processes based on Fock space. The motivation was to give explicit constructions of unitary dilations of quantum dynamical semigroups (semigroups of completely positive, identity preserving maps). In this way, concrete models of markovian open quantum systems could be built. An alternative formulation of quantum stochastic calculus was given by Gardiner and Collett [18] based on the scattering models from quantum field theory. Their approach used input and output processes. While the input processes gave an equivalent way of describing the noisy dynamics driving the system, the output processes allowed for a deeper interpretation and application of models. In particular, the outputs could be fed into a second system as input [17], or measured so as to perform an indirect measurement of the system [4].

Stochastic integrals of adapted processes with respect to the fundamental Fock space martingale processes of creation, annihilation and conservation, were defined by Hudson and Parthasarathy, and a quantum Ito formula established. The most relevant constructions for quantum physics are the unitary adapted processes. This was again originally investigated in [31] for the case of bounded coefficients, with subsequent extension to the unbounded case due to Chebotarev, Fagnola and Frigerio [8], and Fagnola [13]. For more information, see the monograph of Holevo [29].

3.2. Classical Itō Calculus

3.2.1. Stochastic Processes

Before introducing the quantum stochastic calculus, we have a look at the Itō calculus for classical stochastic processes.

Definition 3.1 (Stochastic Process) *A family of random variables taking values in* $(\mathbb{R}_t)_{t\in I}$ *for some index set* I*, on probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *is called a real stochastic process.*

Given a real-valued stochastic process $W(.)$, we can compute the finite dimensional distributions Prob $(W(t_1) \in A_1, W(t_2) \in A_2, \ldots, W_{t_n} \in A_n)$. An important example, the Wiener (Brownian motion) process, is given by $t_1, \ldots, t_n > 0$ and Borel subsets of $\mathbb{R}$, $A_1, \ldots, A_n$,

$$\begin{aligned} &\text{Prob}\left(W(t_1) \in A_1, W(t_2) \in A_2, \ldots, W_{t_n} \in A_n\right) \\ = &\int_{A_n} \ldots \int_{A_1} \mathrm{d}x_n \ldots \mathrm{d}x_1 \rho(x_n, t_n | x_{n-1}, t_{n-1}) \ldots \rho(x_2, t_2 | x_1, t_1) \rho(x_1, t_1 | 0, 0) \end{aligned}$$

where we take $0 < t_1 < t_2 < \ldots < t_n$, see Fig. 3.1 and we have

$$\rho\left(x_2, t_2 | x_1, t_1\right) = \frac{1}{\sqrt{2\pi\left(t_2 - t_1\right)}} e^{-\frac{1}{2}\frac{(x_2 - x_1)^2}{(t_2 - t_1)}}.$$

This is a Markov process, starting at the origin, with ρ as a transition mechanism.

The transition mechanism ρ is Gaussian. We note that the process enjoys the following properties [48]

- $W(0) = 0$, almost surely

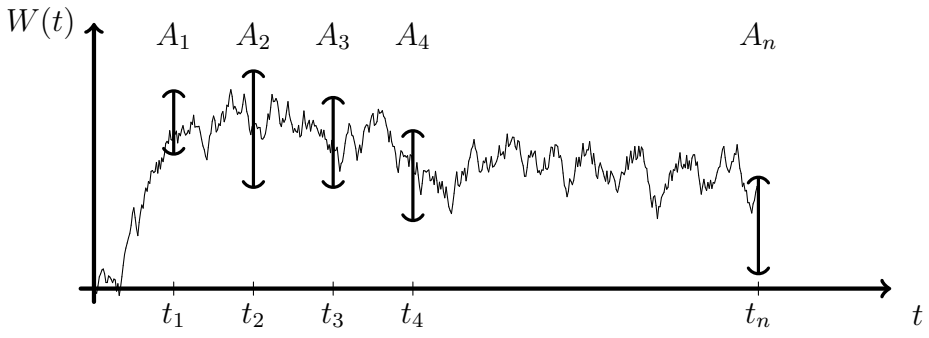

Figure 3.1.: A sample path of the Wiener process $W(t)$ over time t. What is the probability that the process at time t_i takes values in interval A_i, where $i = 1, ..., n$?

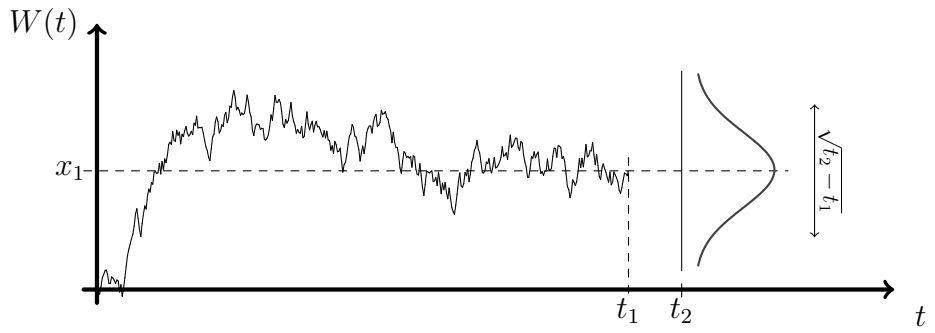

Figure 3.2.: Distribution of the increment $\Delta W(t)$ is a normal distribution with expectation zero and standard deviation $\sqrt{t_2 - t_1}$.

- Independent increments:

  For $t_1 < t_2 < t_3 < t_4$ the increments $W_{t_4} - W_{t_3}$ and $W_{t_2} - W_{t_1}$ are independent

- Increments are stationary and Gaussian

$$W_{t_2} - W_{t_1} \sim \mathcal{N}(0, t_2 - t_1)$$

- $W(.)$ almost surely continuous in time

Since $W(t) \sim \mathcal{N}(0, t_2 - t_1)$ we find that the first two moments of $\Delta W(t) = W(t + \Delta t) - W(t)$ are given by

$$\mathbb{E}\left[\Delta W(t)\right] = 0, \ \mathbb{E}\left[\left(\Delta W(t)\right)^2\right] = \Delta t.$$

We note that for $0 < s \leq t$, $\Delta W(t)$ is independent of $W(s)$. Thus, we have similarly for some function $g(W(t))$ due to independence of $W(t)$ and $\Delta W(t)$

$$\mathbb{E}\left[g(W(t))\Delta W(t)\right] = \mathbb{E}\left[g(W(t))\right]\mathbb{E}\left[\Delta W(t)\right] = 0,$$

and

$$\mathbb{E}\left[g(W(t))\left(\Delta W(t)\right)^2\right] = \mathbb{E}\left[g(W(t))\right]\mathbb{E}\left[\left(\Delta W(t)\right)^2\right] = \mathbb{E}\left[g(W(t))\right]\Delta t.$$

We recall the notion of a σ-algebra $\mathcal{F}(X)$ generated by a random variable X, this is the

smallest σ-algebra containing all the sets of the form $\{\omega \in \Omega \mid X(\omega) \in A\}$ for A a Borel subset of $\mathbb{R}$. More generally the σ-algebra generated by a collection of random variables is the smallest σ-algebra containing the σ-algebras generated by each of the random variables.

For the Wiener process $W(.)$, let $\mathcal{F}_{t]}$ be the σ-algebra generated by $\{W(s) \mid 0 \leq s \leq t\}$. We have the nested property for $t_0 < t_1 < t_2 < \cdots$,

$$\mathcal{F}_{t_0]} \subset \mathcal{F}_{t_1]} \subset \ldots \subset \mathcal{F}_{t_n]}.$$

The family $(\mathcal{F}_t])_{t\geq 0}$ is called a Wiener *filtration*.

We say that a process $X(.)$ is adapted to the filtration $\left(\mathcal{F}_{t]}\right)_{t\geq 0}$ if $X(t)$ is $F_{t]}$ measurable for each $t \geq 0$.

One can show that given some finite-mean process $X(.)$ adapted to the Wiener filtration, we have

$$\mathbb{E}\left[X_t \Delta W(t)\right] = \mathbb{E}\left[X_t\right] \mathbb{E}\left[\Delta W(t)\right] = 0$$

and

$$\mathbb{E}\left[X_t (\Delta W(t))^2\right] = \mathbb{E}\left[X_t\right] \Delta t.$$

3.2.2. The Wiener Itō Integral

We note $L^2(\Omega, \mathcal{F}, \mathbb{P})$ is a Hilbert space with inner product

$$\langle X, Y \rangle = \int_\Omega X^*(\omega) Y(\omega) \mathbb{P}(\omega) = \mathbb{E}\left[X^* Y\right].$$

We define an integral of the form

$$\int_S^T X(t) \mathrm{d}W(t)$$

which we take as being approximated by finite sums

$$I^n_{[S,T]} = \sum_n X_{t_n} \cdot (W(t_{n+1}) - W(t_n))$$

where $S < t_1 < t_2 < \ldots < t_n = T$. If $\int_S^T \mathbb{E}\left[X(t)^2\right] \mathrm{d}t < \infty$ and X is adapted then we can show that this sum converges as $n \to \infty$ in the L^2 sense, that is, there exists $I_{[s,T]}$ such that

$$\lim_{n\to\infty} \mathbb{E}\left[\left(I^n_{[S,T]} - I_{[S,T]}\right)^2\right] = 0.$$

One can show that due to arguments similarly to the ones presented before, we have that

$$\mathbb{E}\left[\int_S^T X_t \mathrm{d}W(t)\right] = 0,\ \mathbb{E}\left[\left(\int_S^T X_t \mathrm{d}W(t)\right)^2\right] = \int_S^T \mathbb{E}\left[X_t^2\right] \mathrm{d}t.$$

3.3. Quantum Stochastic Calculus

3.3.1. The Fock Space

Let $\mathcal{H}$ be a separable Hilbert space. We denote by

$$\mathcal{H}^{\otimes n} := \otimes_{i=1}^n \mathcal{H} = \underbrace{\mathcal{H} \otimes \mathcal{H} \otimes \ldots \otimes \mathcal{H}}_{n \text{ times}}$$

the n-fold tensor product of $\mathcal{H}$. Similarly for a sequence of vectors u_j, $j = 1, \ldots, n$ we define

$$\bigotimes_{j=0}^n u_j = u_1 \otimes \ldots \otimes u_n$$

and if $\forall j,\ u = u_j$ we denote the n-fold tensor product of the vector u by

$$u^{\otimes n} = \underbrace{u \otimes \ldots \otimes u}_{n \text{ times}}.$$

Let S_n be the group of all permutations of the set $\{1, \ldots, n\}$, and define for all $\sigma \in S_n$, U_σ by

$$U_\sigma \left(u_1 \otimes u_2 \otimes \ldots \otimes u_n\right) = u_{\sigma^{-1}(1)} \otimes u_{\sigma^{-1}(2)} \otimes \ldots \otimes u_{\sigma^{-1}(n)}.$$

We then introduce the symmetric and anti-symmetric tensor products of $\mathcal{H}$ by

$$\begin{aligned} \mathcal{H}^{\circledS n} &= \left\{u \in \mathcal{H}^{\otimes n} | U_\sigma u = u \, \forall \sigma \in S_n\right\} \\ \mathcal{H}^{\circledA n} &= \left\{u \in \mathcal{H}^{\otimes n} | U_\sigma u = \mathrm{sig}\,(\sigma)\, u \, \forall \sigma \in S_n\right\} \end{aligned}$$

with $\mathrm{sig}\,(\sigma)) = \pm 1$ depending on whether the permutation σ is even or odd.

We can then define the following Fock spaces [49]

$$\begin{aligned} \Gamma_{\text{free}}(\mathcal{H}) &= \bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes n}, \\ \Gamma_{\text{sym}}(\mathcal{H}) &= \bigoplus_{n=0}^{\infty} \mathcal{H}^{\circledS n}, \\ \Gamma_{\text{anti}}(\mathcal{H}) &= \bigoplus_{n=0}^{\infty} \mathcal{H}^{\circledA n}. \end{aligned}$$

Here $\mathcal{H}^{\otimes 0}$ is identified with the 1-dimensional Hilbert space $\mathbb{C}$. Physically, the symmetric case corresponds to boson systems and the anti-symmetric case to fermion systems. The n'th term in the direct sum corresponds to the n-particle space, i.e. the space describing a system with n-particles where the case $n = 0$ corresponds to the vacuum space, i.e. absence of any particles. We denote the vacuum vector with $\Omega = 1 \oplus 0 \oplus 0 \oplus \ldots$. The respective n-particle spaces are taken to be orthogonal.

We shall from now on deal only with the case of the symmetric Fock space for boson systems.

3.3.2. Exponential Vectors

We are now going to introduce the Fock space equivalent to coherent vectors as for the quantised harmonic oscillator, that is, exponential vectors. Let $u, v \in \mathcal{H}$ and $u^{\otimes 0} = 1$, then the exponential vector $\varepsilon(u) \in \Gamma_{\text{sym}}(\mathcal{H})$ with test function u is given by

$$\varepsilon(u) = \bigoplus_{n=0}^{\infty} \frac{1}{\sqrt{n!}} u^{\otimes n}.$$

The vacuum vector $\Omega = 1 \otimes 0 \otimes 0 \otimes \ldots$ is then given by $\varepsilon(0)$. Especially we note the identity

$$\langle \varepsilon(u), \varepsilon(v) \rangle = e^{\langle u, v \rangle}.$$

We note that the subspace generated by exponential vectors with test functions in a dense subset of $\mathcal{H}$ is dense in $\Gamma_{\text{sym}}(\mathcal{H})$.

3.3.3. The Weyl Operator

Let $\mathcal{H}$ be some separable Hilbert space. Consider an affine mapping of some element $v \in \mathcal{H}$ for $w \in \mathcal{H}$ and some unitary operator $U \in \mathcal{U}(\mathcal{H})$ of the form

$$v \mapsto Uv + w.$$

The map is parameterised by pairs $(w, U) \in \mathcal{H} \times \mathcal{U}(\mathcal{H})$. The space $\mathcal{H} \times \mathcal{U}(\mathcal{H})$ is an Euclidean group with group action

$$(v_2, U_2) \circ (v_1, U_1) = (U_2 v_1 + v_2, U_2 U_1).$$

We can define an operator $\mathcal{W}(v,U) : \mathcal{H} \times \mathcal{U}(\mathcal{H}) \mapsto \Gamma_{\text{sym}}(\mathcal{H})$ by the action

$$\mathcal{W}(v,U)\varepsilon(f) = e^{-\frac{1}{2}||a||^2 - \langle v, Uf\rangle}\varepsilon(Uf+v).$$

$\mathcal{W}(v,U)$ is called they Weyl operator associated with the pair $(v,U) \in \mathcal{H} \times \mathcal{U}(\mathcal{H})$. For $u_1, u_2 \in \mathcal{H}$ and $U_1, U_2 \in \mathcal{U}(\mathcal{H})$ we have

$$\mathcal{W}(u_1,U_1)\mathcal{W}(u_2,U_2) = e^{-i\,\text{Im}\,\langle u_1, U_1 u_2\rangle}\mathcal{W}((u_1,U_1)\circ(u_2,U_2)). \tag{3.3.3.1}$$

Note the special cases $U = 1$ and $v = 0$, i.e. $\mathcal{W}(v,1)$ and $\mathcal{W}(0,U)$ corresponding to pure translation and rotation respectively. We set

$$W(u) := \mathcal{W}(u,0),\ \Gamma(U) := \mathcal{W}(0,U),$$

where $\Gamma(U)$ is called the second quantisation of U. Using Eq. 3.3.3.1 we see that for $u \in \mathcal{H}$, $U, V \in \mathcal{U}(\mathcal{H})$ and $s,t \in \mathbb{R}$ we have

$$\begin{aligned}\Gamma(U)\Gamma(V) &= \Gamma(UV),\\ W(su)W(tu) &= W((t+s)\,u).\end{aligned}$$

The first relation shows that for any one parameter unitary semi-group $U_t = e^{-itH}$ in $\mathcal{H}$ there exists a corresponding one-parameter group $\{\Gamma(U_t) \mid t \in \mathbb{R}\}$ in $\Gamma_{\text{sym}}(\mathcal{H})$. The second relation shows that for every element $u \in \mathcal{H}$ we obtain a one-parameter group $\{W(tu) \mid t \in \mathbb{R}\}$ in $\Gamma_{\text{sym}}(\mathcal{H})$. In both cases, together with the fact that the Weyl operator is strongly continuous in its arguments, Stone's theorem shows that there exist unique self-adjoint operators $p(u)$ and

$\lambda(H)$ such that

$$\begin{aligned} W(tu) &= e^{-itp(u)}, \\ \Gamma(e^{-itH}) &= e^{-it\lambda(H)}. \end{aligned}$$

By setting $q(u) = -ip(u)$ we can define the operators

$$a(u) = \frac{1}{2}\left(q(u) + ip(u)\right),\ a^\dagger(u) = \frac{1}{2}\left(q(u) - ip(u)\right).$$

The properties of these operators will be presented in the next section.

Further one can write for any bounded operator $H \in \mathcal{B}(\mathcal{H})$

$$\lambda(H) = \lambda\left(\tfrac{1}{2}\left(H + H^\dagger\right)\right) + i\lambda\left(\tfrac{1}{2i}\left(H - H^\dagger\right)\right),\ \lambda(H^\dagger) = \lambda(H)^\dagger.$$

$\lambda(H)$ is called the differential second quantisation of H.

3.3.4. The Creation and Annihilation Operators

In the previous section we obtained operators $a(u), a^\dagger(u)$ and $\lambda(H)$ for $u \in \mathcal{H}$, $H \in \mathcal{B}(\mathcal{H})$. We show that $a(u)$ and $a^\dagger(u)$ admit properties that justify calling them annihilation and creation operators respectively.

Let $\mathcal{H}$ be some Hilbert space and $\Gamma_{\text{sym}}(\mathcal{H})$ the symmetric Fock space over $\mathcal{H}$. For vectors $u, v \in \Gamma_{\text{sym}}(\mathcal{H})$ one can show that $a^\dagger(u)$ maps $a^\dagger(u) : \mathcal{H}^{\circledS n} \to \mathcal{H}^{\circledS n+1}$ by

$$a^\dagger(u)v^{\otimes n} = \sqrt{n+1}\sum_{j=0}^{n} v^{\otimes r} \otimes u \otimes v^{\otimes n-r}.$$

The action of the creator is therefore taking some element from the n particle space $\mathcal{H}^{\otimes n}$ and mapping into the $n+1$ particle space where the sum produces again a completely symmetric

vector. Similarly we have for the annihilation operator $a(h) : \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes n-1}$

$$a(u)v^{\otimes n} = \sqrt{n}\,\langle u, v\rangle\, v^{\otimes n-1}.$$

The annihilation operator maps a vector of the n particle space into the $n-1$ particle space and produces again a completely symmetric vector.

As a direct consequence of this definition we see that

$$a(u)\varepsilon\,(0) = 0,$$

as in the case for the single quantised harmonic oscillator.

The action of the creation and annihilation operator on an exponential vector $\varepsilon\,(f)$ with $u, v \in \mathcal{H}$ are given by

$$\begin{aligned} a^{\dagger}(u)\varepsilon\,(v) &= \frac{d}{dt}\varepsilon\,(v+tu)\Big|_{t=0}, \\ a(u)\varepsilon\,(v) &= \langle u, v\rangle\,\varepsilon\,(v)\,. \end{aligned}$$

The canonical commutation relation reads as

$$\begin{aligned} \left[a(u), a^{\dagger}(v)\right] &= \langle u, v\rangle \cdot 1, \\ [a(u), a(v)] &= \left[a^{\dagger}(u), a^{\dagger}(v)\right] = 0. \end{aligned}$$

3.3.5. Filtrations & Adapted Processes

In the following section we are going to investigate the notion of filtrations and adapted processes on Fock spaces, as seen in the previous section for the classical Itō calculus .

Let in the following $h = L^2(0, \infty)$ the Hilbert space of square integrable functions $f(t)$ taking arguments in $t \in [0, \infty)$ and let $h^{\circledS n}$ be the n-fold symmetric tensor product of h.

We consider a decomposition of h of the form $L^2(0,\infty) = L^2(0,t') \oplus L^2(t',\infty)$. One way to visualise this decomposition is Fig. 3.3, i.e.

$$f(t) = f_{(0,t')}(t) \oplus 0 + 0 \oplus f_{(t',\infty)}(t)$$

with $f_{(a,b)}(t)$ being the restriction of f to the interval (a,b). We then have $f_{(0,t')}(t) \in L^2(0,t')$

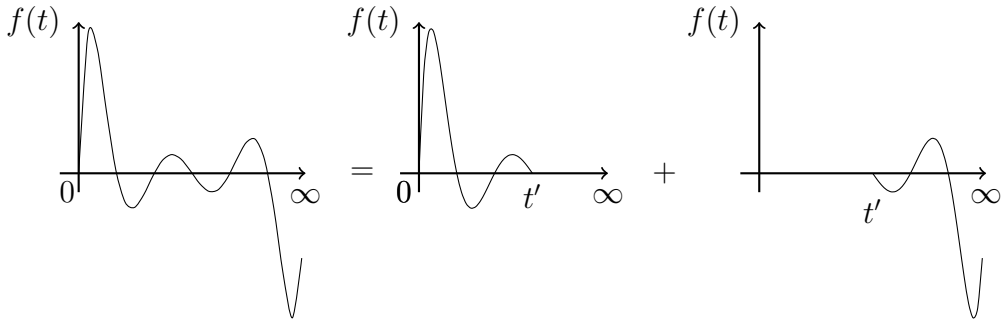

Figure 3.3.: Decomposition of a square-integrable function on $[0,\infty)$ into parts concentrated on $[0,t')$ and $[t',\infty)$.

and $f_{(t',\infty)}(t) \in L^2(t',\infty)$. The L^2-norm of $f(t)$ can then be expressed as

$$\int_0^\infty \mid f(t) \mid^2 \mathrm{d}t = \int_0^{t'} \mid f_{(0,t')}(t) \mid^2 \mathrm{d}t + \int_{t'}^\infty \mid f_{(t',\infty)}(t) \mid^2 \mathrm{d}t$$

since cross terms vanish, in other words we see directly that this also shows that every element of $L^2(0,t')$ is orthogonal to every element in $L^2(t',\infty)$.

This decomposition into past and future spaces $L^2(0,t')$ and $L^2(t',\infty)$ extends to Fock spaces by noting that for two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ we have that

$$\Gamma\left(\mathcal{H}_1 \oplus \mathcal{H}_2\right) = \Gamma\left(\mathcal{H}_1\right) \otimes \Gamma\left(\mathcal{H}_2\right).$$

Let $0 < t_1 < t_2 < \ldots < t_n < \infty$ and set $\mathcal{H}_{t]} = L^2(0,t)$, $\mathcal{H}_{[s,t]} = L^2(s,t)$ and $\mathcal{H}_{[t} = L^2(t,\infty)$.

Similarly as above we can yield a partition of $\mathcal{H} = \Gamma_{\text{sym}}\left(L^2(0,\infty)\right)$ of the form

$$\mathcal{H} = \mathcal{H}_{t_1]} \oplus \mathcal{H}_{[t_2,t_3]} \oplus \ldots \oplus \mathcal{H}_{[t_n}.$$

Let $h_0 = \mathcal{H}_{\text{sys}}$ be a complex separable Hilbert space and define

$$\mathcal{H} = h_0 \otimes \mathcal{F}.$$

We identify for $0 < s < t$

$$\mathcal{H}_{0]} = h_0,\ \mathcal{H}_{t]} = h_0 \otimes \mathcal{F}_{t]},\ \mathcal{H}_{[t} = \mathcal{F}_{[t},\ \mathcal{F}_{[s,t]} = \Gamma_{\text{sym}}\left(\mathcal{H}_{[s,t]}\right).$$

Due to the factorisation property of the Fock space we have

$$\mathcal{H} = \mathcal{H}_{t_1]} \otimes \mathcal{H}_{[t_2,t_3]} \otimes \ldots \otimes \mathcal{H}_{[t_n}.$$

Let $\mathcal{B}_{t]}$ be the set of all linear bounded operators that act trivially on the space $\mathcal{H}_{[t}$, i.e.

$$\mathcal{B}_{t]} = \left\{X \otimes \mathbb{1}_{[t} \mid X \in \mathcal{B}(\mathcal{H}_{t]}),\ \mathbb{1}_{[t} \text{ is the identity in } \mathcal{H}_{[t}\right\}.$$

h_0 is called the initial space and plays in the physical interpretation of the mathematical set-up the role of the Hilbert space representing the system which is driven by quantum noises living on the Fock space $\Gamma_{\text{sym}}(\mathcal{H})$. $\{\mathcal{B}_{t]}\}_{t\geq 0}$ is then an increasing family of non-commutative von Neumann algebras and plays the role of the filtration as encountered before.

Loosely speaking, a family of observables $\{X_t\}_{t\geq 0} \in \mathcal{B}(\mathcal{H})$ is then an adapted process if, for each $t \geq 0$, X_t acts trivially on the future space $\mathcal{H}_{[t}$.

Denote with $\left(f\mathbb{1}_{[0,t]}\right)(s)$, $f \in \mathcal{H}$ the function

$$\left(f\mathbb{1}_{[0,t]}\right)(s) = \begin{cases} f(s) & \text{if} \quad s \in [0,t] \\ 0 & \text{if} \quad s \notin [0,t] \end{cases},$$

and introduce the operator

$$\begin{aligned} \Pi_{[0,t]} &: [0,\infty) \to \mathcal{B}(\mathcal{H}) \\ \Pi_{[0,t]} &: f \mapsto f\mathbb{1}_{[0,t]}. \end{aligned}$$

Consider the creation, annihilation and differential quantisation operator $a^\dagger(u), a(u)$ and $\lambda(H)$ for $u \in \mathcal{H} = L^2(0,\infty)$, $H \in \mathcal{B}(\mathcal{H})$. We introduce as three important examples of adapted processes the creation-, annihilation and conservation process $B^\dagger(t), B(t)$ and $\Lambda(t)$ by setting $u(s) = \mathbb{1}_{[0,t]}$, $H = \Pi_{[0,t]}$ and obtain

$$\begin{aligned} B^\dagger(t) = \quad a^\dagger(\mathbb{1}_{[0,t]}) \quad &= \int_0^t b^\dagger(s)\mathrm{d}s, \\ B(t) = \quad a(\mathbb{1}_{[0,t]}) \quad &= \int_0^t b(s)\mathrm{d}s, \\ \Lambda(t) = \quad \lambda(\Pi_{[0,t]}) \quad &= \int_0^t b^\dagger(s)b(s)\mathrm{d}s = \Lambda^*(t). \end{aligned}$$

Here, $b^\dagger(s)$ and $b(s)$ are the formal derivatives of $B^\dagger(s)$ and $B(s)$ with singular canonical commutation relation

$$\left[b(t), b^\dagger(s)\right] = \delta(t-s).$$

The action of $B^\dagger(t)$ and $B(t)$ on exponential vectors $\varepsilon(u)$, $u \in \mathcal{H}$ are given by

$$\begin{aligned} B(t)\varepsilon(u) &= \left(\int_0^t u(s)\mathrm{d}s\right)\varepsilon(u), \\ B^\dagger(t)\varepsilon(u) &= \frac{\partial}{\partial x}\varepsilon\left(u + x\mathbb{1}_{[0,t]}\right)|_{x=0}, \end{aligned}$$

and we see that $\varepsilon(v)$ is an eigenvector of $B(t)$ with eigenvalue $\int_0^t v(s)\mathrm{d}s$.

We can compute the expectations with respect to the exponential vectors $\varepsilon(u)$ and $\varepsilon(v)$ and these are given by

$$\begin{aligned}
\langle \varepsilon(f)\,|B(t)\varepsilon(g)\rangle &= \left(\int_0^t g(s)\mathrm{d}s\right)\langle \varepsilon(f)\,|\varepsilon(g)\rangle\,,\\
\left\langle \varepsilon(f)\,|B^\dagger(t)\varepsilon(g)\right\rangle &= \left(\int_0^t f^*(s)\mathrm{d}s\right)\langle \varepsilon(f)\,|\varepsilon(g)\rangle\,,\\
\langle \varepsilon(f)\,|\Lambda(t)\varepsilon(g)\rangle &= \left(\int_0^t f^*(s)g(s)\mathrm{d}s\right)\langle \varepsilon(f)\,|\varepsilon(g)\rangle\,.
\end{aligned}$$

The canonical commutation relations for the integrated processes $B(t), B^\dagger(t)$ and $\Lambda(t)$ translate into

$$\begin{aligned}
[B(s),B(t)] &= \left[B^\dagger(s),B^\dagger(t)\right] = [\Lambda(s),\Lambda(t)] = 0,\ \forall\, s,t\\
\left[B(s),B^\dagger(t)\right] &= s\wedge t,\\
[B(s),\Lambda(t)] &= B(s\wedge t),\\
\left[\Lambda(s),B^\dagger(t)\right] &= B^\dagger(s\wedge t),
\end{aligned}$$

where $s\wedge t=\min(s,t)$.

3.3.6. Integration of Stochastic Processes

The stage is now set to introduce integrals of adapted quantum stochastic processes.

Let E,F,G,H be adapted locally square-integrable processes on $h_0\otimes\mathcal{F}$. Define finite future pointing differentials for a partition of time $0\le t_1<t_2<\ldots\le t_N=t$ of the form

$$\begin{aligned}
\Delta B(t_n) &= B(t_{n+1})-B(t_n),\\
\Delta B^\dagger(t_n) &= B^\dagger(t_{n+1})-B^\dagger(t_n),\\
\Delta\Lambda(t_n) &= \Lambda(t_{n+1})-\Lambda(t_n).
\end{aligned}$$

For a partition of $\mathcal{H} = h_0 \otimes \mathcal{F}$ of the form $\mathcal{H} = \mathcal{H}_{t_n]} \otimes \mathcal{H}_{[t_n,t_{n+1}]} \otimes \mathcal{H}_{[t_{n+1}}$ we obviously have due to adaptedness of $P(t_n) \in \{E(t_n), F(t_n), G(t_n), H(t_n)$ that $M(t_n)\}$ acts only non-trivially on $\mathcal{H}_{t_n]}$ whereas $\Delta D(t_n) \in \{\Delta B(t_n), \Delta B^\dagger(t_n), \Delta\Lambda(t_n)\}$ acts only non-trivially on $\mathcal{H}_{[t_n,t_{n+1}]}$. As a consequence we have

$$[P(t_n), \Delta D(t_n)] = 0$$

$\forall\, t_n$ and every of the processes and differentials $P(t_n)$ and $\Delta D(t_n)$. By similar arguments we can see that

$$\left[\Delta B^\dagger(t_n), \Delta B(t_m)\right] = [\Delta B(t_n), \Delta B(t_m)] = \left[\Delta B^\dagger(t_n), \Delta B^\dagger(t_m)\right] = 0$$

$\forall\, m \neq n$.

We now consider as an approximation for a stochastic integral the following finite sum

$$M_{(N)}(t) = \sum_{n=1}^{N} \left\{E(t_n)\Delta\Lambda(t_n) + F(t_n)\Delta B(t_n) + G(t_n)\Delta B^\dagger(t_n) + H(t_n)\Delta t_n\right\},$$

where $\Delta t_n = t_{n+1} - t_n$ and $E(t), F(t), G(t)$ and $H(t)$ are assumed to be simple processes, that is there exists an increasing sequence $t_n,\ t_0 = 0$ with $t_n \to \infty$ such that each of the processes are of the form

$$F = \sum_{n=0}^{\infty} F_n \mathbb{1}_{[t_n,t_{n+1}]}.$$

One can show that $M_N(t)$ converges as $N \to \infty$ (with $\min_n \mid t_{n+1} - t_n \mid \to 0$) to a quantum Itō integral

$$M(t) = \int_0^t \left\{E(s)\mathrm{d}\Lambda(s) + F(s)\mathrm{d}B(s) + G(s)\mathrm{d}B^\dagger(s) + H(s)\mathrm{d}s\right\},$$

that is

$$\lim_{N\to\infty} \left\| \{M_N(t) - M(t)\}\, u \otimes \varepsilon\left(f\right) \right\|^2 = 0.$$

We are now demonstrate the basic properties of the quantum Itō integral. Let

$$M_N(t) = \sum_n F(t_n)\Delta B(t_n),\ \tilde{M}_N(t) = \sum_n \tilde{G}(t_n)\Delta B^\dagger(t_n)$$

such that in the limit $N \to \infty$, $M_N(t) \to \int_0^t F(s)\mathrm{d}B(s)$ and $\tilde{M}_N(t) \to \int_0^t \tilde{G}(s)\mathrm{d}B^\dagger(s)$. We consider the product $M_N(t)\tilde{M}_N(t)$

$$M_N(t)\tilde{M}_N(t) = \sum_{n,m} F(t_n)\Delta B(t_n)\tilde{G}(t_m)\Delta B^\dagger(t_m).$$

By splitting up the sum into the cases $m > n$, $n > m$ and $m = n$ and evaluating the average in a state $u \otimes \varepsilon\left(f\right)$ one can show that

$$M_N(t)\tilde{M}_N(t) = \sum_n \left\{ F(t_n)\tilde{M}_N(t_n)\Delta B(t_n) + M_N(t_n)\tilde{G}(t_n)\Delta B^\dagger(t_n) + F(t_n)\tilde{G}(t_n)\Delta t_n \right\},$$

or, in the limit $N \to \infty$,

$$M(t)\tilde{M}(t) = \int_0^t F(s)\tilde{M}(s)\mathrm{d}B(s) + \int_0^t M(s)\tilde{G}(s)\mathrm{d}B^\dagger(s) + \int_0^t F(s)\tilde{G}(s)\mathrm{d}s.$$

Equivalently we can write

$$\mathrm{d}\left[M(t)\tilde{M}(t)\right] = \mathrm{d}M(t)\tilde{M}(t) + M(t)\mathrm{d}\tilde{M}(t) + \mathrm{d}M(t)\mathrm{d}\tilde{M}(t),$$

using $\mathrm{d}M(t) = F(t)\mathrm{d}B(t)$, $\mathrm{d}\tilde{M}(t) = \tilde{G}(t)\mathrm{d}B^\dagger(t)$ and obtaining the product rule for Itō differentials $\mathrm{d}B(t)\mathrm{d}B^\dagger(t) = \mathrm{d}t$. By similar means we can compute the rules for other pairs of differentials and obtain the Itō product table

	$\mathrm{d}B$	$\mathrm{d}\Lambda$	$\mathrm{d}B^\dagger$	$\mathrm{d}t$
$\mathrm{d}B$	0	$\mathrm{d}B$	$\mathrm{d}t$	0
$\mathrm{d}\Lambda$	0	$\mathrm{d}\Lambda$	$\mathrm{d}B^\dagger$	0
$\mathrm{d}B^\dagger$	0	0	0	0
$\mathrm{d}t$	0	0	0	0

Table 3.1.: Itō Table 1

3.3.7. The Hudson & Parthasarathy Quantum Stochastic Differential Equation

We obtain a special class of Quantum Stochastic Differential Equation by the following theorem:

Theorem 3.2 (Hudson & Parthasarathy QSDE) *Let L, S, H be bounded operators in h_0 where S is unitary and H is self-adjoint. Then there exists a unique unitary operator valued adapted regular process $U = \{U(t) \mid t \geq 0\}$ satisfying*

$$\mathrm{d}U = U\left\{L\mathrm{d}B^\dagger + (S-1)\,\mathrm{d}\Lambda - L^*S\mathrm{d}B - \left(\frac{1}{2}L^*L + iH\right)\mathrm{d}t\right\},\ U(0) = 1 \qquad (3.3.7.1)$$

Using the solution $U(t)$ to Eq. 3.3.7.1 we can describe the evolution of operators $X \in \mathcal{B}(h_0)$ in the Heisenberg picture by defining

$$j_t(X) := U(t)^*\,(X \otimes I)\,U(t),\ t \geq 0.$$

$\{j_t(X) \mid t \geq 0\}$ is then an adapted regular process satisfying the Heisenberg-Langevin equation

$$\begin{aligned}\mathrm{d}j_t(X) \;=\;& j_t(S^*\,[X,L])\mathrm{d}B^\dagger + j_t\,(S^*XS - X)\,\mathrm{d}\Lambda + j_t\,([L^*,X]\,S)\,\mathrm{d}B \\ &+ j_t(i\,[H,X])\mathrm{d}t - \tfrac{1}{2}\,\{L^*L + XL^*L - 2L^*XL\}\,\mathrm{d}t.\end{aligned}$$

3.3.8. Squeezing

Let us introduce the operators

$$\hat{\Sigma} = \hat{a}^2,\ \hat{\Sigma}^* = (\hat{a}^*)^2. \tag{3.3.8.1}$$

We find that

$$[\hat{a}, \hat{\Sigma}^*] = 2\hat{a}^*,\ [\hat{\Sigma}, \hat{a}^*] = 2\hat{a} \tag{3.3.8.2}$$

and

$$\begin{aligned} [\hat{\Sigma}, \hat{\Sigma}^*] &= 4\hat{N} + 2, \\ [\hat{N}, \hat{\Sigma}^*] &= 2\hat{\Sigma}^*, \\ [\hat{\Sigma}, \hat{N}] &= 2\hat{\Sigma}. \end{aligned} \tag{3.3.8.3}$$

In particular, the set consisting of linear combinations of $I, \hat{N}, \hat{\Sigma}$ and $\hat{\Sigma}^*$ is a Lie algebra with commutator as bracket.

Definition 3.3 *For complex ε, we define the squeezing operator by*

$$\hat{S}(\varepsilon) = \exp\left\{\frac{1}{2}\varepsilon\hat{\Sigma}^* - \frac{1}{2}\varepsilon^*\hat{\Sigma}\right\}. \tag{3.3.8.4}$$

This is a unitary family and we note that

$$\hat{S}(\varepsilon)^{-1} = \hat{S}(\varepsilon)^* = S(-\varepsilon).$$

Lemma 3.4 *Let ε have the polar form $re^{i\theta}$, then*

$$\hat{S}(\varepsilon)^* \hat{a} \hat{S}(\varepsilon) = \cosh(r)\ \hat{a} + \sinh(r)\ e^{i\theta}\hat{a}^*. \tag{3.3.8.5}$$

Proof:

Let $\hat{a}(u) = \hat{S}(u\varepsilon)^* \hat{a}\hat{S}(u\varepsilon)$ for real u, then

$$\frac{d}{du}\hat{a}(u) = \varepsilon\hat{a}(u)^*,$$

and so $\frac{d^2}{du^2}\hat{a}(u) = r^2\hat{a}(u)$. This is a simple 2nd order ODE with operator-valued initial conditions $\hat{a}(0) = \hat{a}$ and $\frac{d}{du}\hat{a}(u)|_{u=0} = \varepsilon\hat{a}^*$ yielding the solution $(3.3.8.5)$.

□

Definition 3.5 *The transformation* $\hat{a} \to \cosh(r)\ \hat{a} + \sinh(r)\ e^{i\theta}\hat{a}^*$ *preserves the canonical commutation relations and is referred to as a Bogoliubov transformation.*

Lemma 3.6 *Let* ε *have the polar form* $re^{i\theta}$*, then the squeezing operator may be placed in the following Wick ordered form*

$$\hat{S}(\varepsilon) = \zeta^{\hat{\Sigma}^*}(\cosh r)^{-\hat{N}+\frac{1}{2}}(\zeta^*)^{-\hat{\Sigma}}, \tag{3.3.8.6}$$

where $\zeta = \exp\left\{\frac{1}{2}e^{i\theta}\tanh r\right\}$.

3.4. Preliminary Results

3.4.1. A Trotter-Kato Theorem

The following results by Bouten et al. [6] are concerning convergence of QSDEs for a singular perturbation under certain conditions on the parametrisation of the QSDE. Their proof of the

result regarding singular perturbation makes use of a modified version of the Trotter-Kato theorem, proved as well in [6]. We shall quote both. The results presented in this section will be needed in the Chapter on Adiabatic Elimination.

The system under consideration is modelled by a sequence of QSDEs, parametrised by some scaling constant k

$$\mathrm{d}U_t^{(k)} = U_t^{(k)} \left\{ \sum_{jl=1}^{n} \left(N_{jl}^{(k)} - \delta_{jl} \right) \mathrm{d}\Lambda_t^{jl} + \sum_{j=1}^{n} M_j^{(k)} \mathrm{d}B_t^{j\dagger} + \sum_{j=1}^{n} L_j^{(k)} \mathrm{d}B_t^j + K^{(k)} \mathrm{d}t \right\}. \quad (3.4.1.1)$$

We are interested in the limit regime as $k \to \infty$ and ask if there exists a limit QSDE of the form

$$\mathrm{d}U_t = U_t \left\{ \sum_{jk=1}^{n} \left(N_{jk} - \delta_{jk} \right) \mathrm{d}\Lambda_t^{jk} + \sum_{j=1}^{n} M_j \mathrm{d}B_t^{j\dagger} + \sum_{j=1}^{n} L_j \mathrm{d}B_t^j + K \mathrm{d}t \right\}. \quad (3.4.1.2)$$

Lemma 3.7 *For $\alpha, \beta \in \mathbb{C}^n$ define $T_t^{(\alpha,\beta)} : \mathcal{H}_0 \to \mathcal{H}_0$ such that*

$$\left\langle u, T_t^{(\alpha,\beta)} v \right\rangle = e^{-\left(|\alpha|^2 + |\beta|^2\right)\frac{t}{2}} \left\langle u \otimes \varepsilon(\alpha \mathbb{1}_{[0,t]}), U(t) v \otimes \varepsilon(\beta \mathbb{1}_{[0,t]}) \right\rangle, \ \forall u, v \in \mathcal{H}_0, t \geq 0.$$

Then $T_t^{(\alpha,\beta)}$ is a strongly continuous contraction semigroup on $\mathcal{H}_0$, and the generator $\mathcal{L}^{(\alpha,\beta)}$ of this semigroup satisfies $\mathrm{Dom}(\mathcal{L}^{(\alpha,\beta)}) \supset \mathcal{D}_0$ such that for $u \in \mathcal{D}_0$

$$\mathcal{L}^{(\alpha,\beta)} u = \left(\alpha_j^* N_{jk} \beta_k + \alpha_j^* M_j + L_j \beta_j + K - \frac{|\ \alpha\ |^2 + |\ \beta\ |^2}{2} \right) u.$$

The same results holds for $T_t^{(k;\alpha,\beta)} : \mathcal{H} \to \mathcal{H}$ and $\mathcal{L}^{(k;\alpha,\beta)}$, defined by replacing U_t by $U_t^{(k)}$ and making the obvious modifications. In particular $\mathrm{Dom}(\mathcal{L}^{(k;\alpha,\beta)}) \supset \mathcal{D}$.

Theorem 3.8 (Trotter-Kato) *Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{H}_0 \subset \mathcal{H}$ be a closed subspace. For each $k \in \mathbb{N}$, let $T_t^{(k)}$ be a strongly continuous contraction semi-group on $\mathcal{H}$ with generator $\mathcal{L}^{(k)}$. Moreover, let T_t be strongly continuous contraction semi-group on $\mathcal{H}_0$ with generator $\mathcal{L}$. Let $\mathcal{D}_0$ be a core for $\mathcal{L}$. The following conditions are equivalent:*

i) For all $\Psi \in \mathcal{D}_0$ there exists $\Psi^{(k)} \in \mathrm{Dom}\,\mathcal{L}^{(k)}$ such that

$$\Psi^{(k)} \overset{k\to\infty}{\longrightarrow} \Psi,\ \mathcal{L}^{(k)}\Psi^{(k)} \overset{k\to\infty}{\longrightarrow} \mathcal{L}\Psi$$

ii) For all $T < \infty$ and $\psi \in \mathcal{H}_0$

$$\lim_{k\to\infty} \sup_{0\le t\le T} \left\| T_t^{(k)}\psi - T_t\psi \right\| = 0$$

The following quantum version of the Trotter-Kato theorem allows for making a link between convergence of the solution of the QSDE $U_t^{(k)}$ and convergence of the generator of the unitary evolution $\mathcal{L}^{(k;\alpha\beta)}$ which in turn is determined by the parametrisation of the QSDE.

Theorem 3.9 (QSDE Trotter-Kato, [6]) *The following conditions are equivalent*

a) For every $\alpha, \beta \in \mathbb{C}^n$ and $u \in \mathcal{D}_0$ there exists $u^{(k)} \in \mathrm{Dom}(\mathcal{L}^{(k;\alpha,\beta)})$ such that

$$u^{(k)} \overset{k\to\infty}{\longrightarrow} u,\ \ \mathcal{L}^{(k;\alpha,\beta)}u^{(k)} \overset{k\to\infty}{\longrightarrow} \mathcal{L}^{(\alpha,\beta)}u$$

b) For all $T < \infty$ and $\Psi \in \mathcal{H}_0$

$$\lim_{k\to\infty} \sup_{0\le t\le T} \left\| U_t^{*(k)}\Psi - U_t^*\Psi \right\|^2 = 0.$$

Moreover, if $\mathcal{D}$ is a core for all $\mathcal{L}^{(k;\alpha,\beta)}$, $k \in \mathbb{N}$, then we can always choose $\{u^{(k)}\} \subset \mathcal{D}$

In order to archive convergence for the singular perturbation problem further assumptions on the parameter of the QSDE parameter must be imposed:

Assumtions:

1) There exists operators $Y, Y^\dagger, A, A^\dagger, B, B^\dagger, F_i, F_i^\dagger, G_i, G_i^\dagger, W_{ij}, W_{ij}^\dagger$ with common invariant domain $\mathcal{D}$ such that

$$K^{(k)} = k^2 Y + kA + B, \;\; L_i^{(k)} = kF_i + G_i, \;\; N_{ij}^{(k)} = W_{ij}, \;\; \forall k \in \mathbb{N}, 1 \leq i, j \leq n. \quad (3.4.1.3)$$

2) There is a closed subspace $\mathcal{H}_0 \subset \mathcal{H}$ such that:

   a) $P_0 \mathcal{D} \subset \mathcal{D}$

   b) $Y P_0 = 0$ on $\mathcal{D}$

   c) There exist $\tilde{Y}, \tilde{Y}^\dagger$ with common invariant domain $\mathcal{D}$, such that $\tilde{Y} Y = Y \tilde{Y} = P_1$

   d) $F_j^\dagger P_0 = 0$ on $\mathcal{D}$, $\forall 1 \leq j \leq n$

   e) $P_0 A P_0 = 0$ on $\mathcal{D}$

   where P_0 and P_1 are the orthogonal projections onto $\mathcal{H}_0$ and $\mathcal{H}_0^\perp$ respectively and with choice of the dense domain $\mathcal{D}_0 = P_0 \mathcal{D}$ in $\mathcal{H}_0$.

3) (Limit coefficients) Define operators on $\mathcal{H}_0$

$$\begin{aligned} K &= P_0(B - A\tilde{Y}A)P_0, \\ L_i &= P_0(G_i - A\tilde{Y}F_i)P_0, \\ M_k &= -\sum_{l=1} P_0 W_{kl}(G_j^\dagger - F_j^\dagger \tilde{Y} A)P_0, \\ N_{kl} &= \sum_{j=1} P_0 W_{kj}(F_j^\dagger \tilde{Y} F_l + \delta_{jl})P_0. \end{aligned}$$

We are now ready to present the main result of [6]:

Theorem 3.10 *Under the assumptions as above, the singular perturbed equations (3.4.1.1) converge to the limit equations (Equation 3.4.1.2) on* $\mathcal{H}_0$

$$\lim_{k\to\infty} \sup_{0\leq t\leq T} \left\| U_t^{(k)*}\psi - U_t^*\psi \right\| = 0,\ \forall \psi \in \mathcal{H}_0 \otimes \mathcal{F}.$$

3.4.2. QSDEs with Unbounded Operators

The quantum stochastic calculus presented so far allowed for integration of operator processes with the condition that the operators have to be bounded. However, in practice and physical examples important operators such as the creation and annihilation operators are in fact unbounded.

For these cases the results of Fagnola [13, 14, 15, 16] are of help. We quote [6] for a version of the results :

Consider the initial space $\mathcal{H}$ and a fixed dense domain $\mathcal{D}_0 \subset \mathcal{H}$. Suppose that $\forall\, u \in \mathcal{D}_0,\ l \in \mathbb{N}$ there exists a constant $c(u,l)$ such that

1. $\forall u \in \mathcal{D}_0$ and for some $\epsilon > 0$ independent of u

$$\sum_{l=1}^{\infty} c(u,l)\epsilon^l < \infty$$

2. $\forall l \in \mathbb{N}$ and all choices of $X(1), \ldots, X(l)$, where $X(.)$ is one of $K, K^\dagger, L_i, L_i^\dagger, M_i, M_i^\dagger$, $N_{jk}, N_{jk}^\dagger$ (as encountered in the previous subsection), we have $\forall u \in \mathcal{D}_0$

$$\|X(1)\ldots X(l)\| \leq c(u,l)\sqrt{(l+m)!},$$

where m is the number of occurrences of K or $K^\dagger$ in the squence $X(1)\dots X(l)$.

Then there exists a unique operator valued cocycle $\{U_t \mid t \geq 0\}$ which satisfies the QSDE Eq. (3.3.7.1).

4

Classical Control Theory

4.1. Introduction

The topic of interest for this work, *Quantum Feedback Control*, combines elements of *Quantum Mechanics* with elements of *Control Theory*, a branch of engineering.

To prepare the ground for this interdisciplinary subject we provide a quick introduction into classical control theory in the following chapter.

Control theory is concerned with the control of some dynamical system, in literature usually called *the plant* or *the system*.

If we think of a dynamical system, for example a room equipped with a radiator and a thermometer (see Figure 4.1), we see that we can influence the system's state (room temperature) by adjusting the valve of the radiator and we get some output by reading off the current temperature of the thermometer.

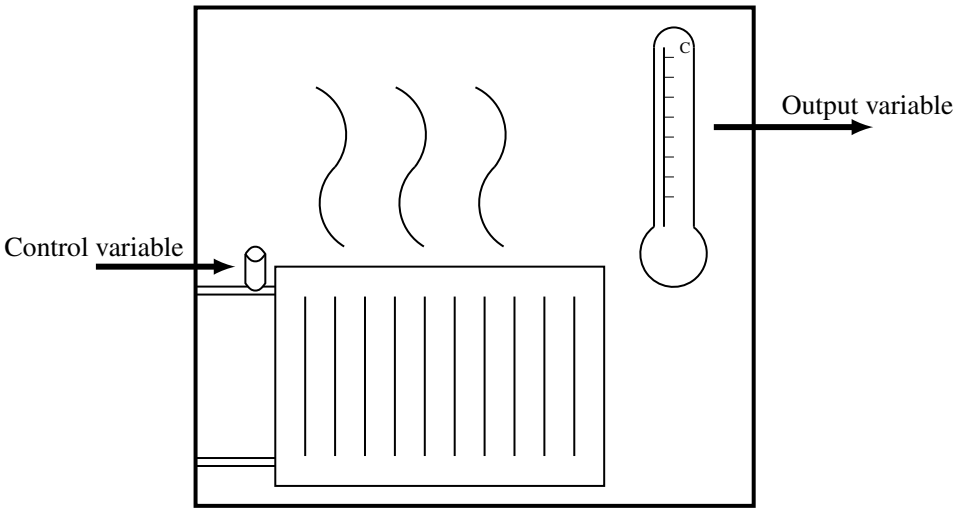

Figure 4.1.: Example for a dynamical system. A room heated by a radiator, with some control input adjusting the valve of the radiator and some output, i.e. the current temperature.

Clearly changing the input, i.e. adjusting the valve, changes the output, i.e. the temperature. But this will not happen instantaneously, there will be some dynamical behaviour.

If we want the output to exhibit a specific behaviour, for example we could want to room temperature to be always at exactly 20° Celsius, we will have to adjust the actuator, the valve, in a very specific manner.

One strategy to chose the specific valve setting might be to assemble a table, pairing a valve setting with a resulting room temperature.This table might be assembled using a mathematical model of the system. But the accuracy of this approach will be quite limited. On the one hand the mathematical model used to compute the pairs of the table will not be exact but only be an approximatio and errors have to be expected. On the other hand the system will be influenced by it's environment. In the given example, a specific valve setting will result in very different

room temperatures depending on how warm it is outside, i.e. the form of pertubation of the system by it's evrionment will heavily influence the accuracy of this kind of control.

Above described strategy is called *feedforward control*. In general we have some dynamical system with some inputs $u(t)$, some disturbance signal $w(t)$, some outputs $y(t)$ and a system state $x(t)$ with some law for the dynamical evolution of the state $\dot{x}(t) = f(u(t), x(t))$, see Fig. 4.2. The output will then be given by some law $y(t) = g(x(t), u(t))$.

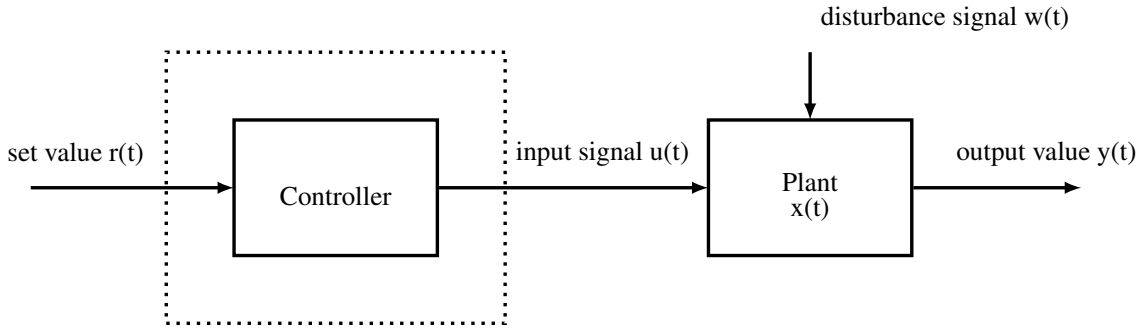

Figure 4.2.: Block diagram for a plant, controlled by a *feedforward controller*. We chose some *set value* or *reference value* $r(t)$, i.e. the value we want the output to assume. The controller computes a corresponding *input signal* or *actuating value* for the system. In reality the plant will be disturbed by it's environment such that there will be some final fixed error between set value and output value.

One could try to reduce the fixed error between reference value and output value for the set-up described above by providing the controller with knowledge about the current error! In this case we take the output value, compare it to the reference value and use result as an input for the controller. By this approach the controller will be able to react to a fixed error due to some disturbance or perturbation of the system by it's environment or due to some errors in the underlying mathematical model.

Under the assumption that the systems are linear, we can evaluate the difference between both control strategies quantitatively by computing the transfer functions (see Section 4.3) for both cases.

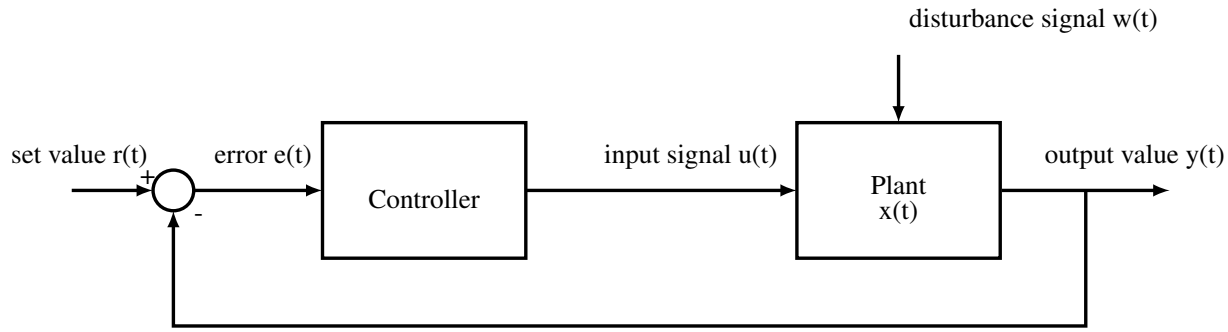

Figure 4.3.: Block diagram for a plant, controlled by a *feedback controller*. The output value is compared to the set value $e(t) = r(t) - y(t)$. The current error $e(t)$ is used as an input for the controller which computes an appropriate input signal $u(t)$ for the system.

4.1.1. Feedforward Control

We assume that the plant is described by a transfer function (obtained by taking the Laplace transform of ODE describing the model) $G(s)$ such that the ratio of the (Laplace transformed) output to input ratio is given by

$$\frac{Y_G(s)}{U_G(s)} = G(s).$$

We consider an overall output $Y(s)$ of the form signal + perturbation $W(s)$, $Y(s) = Y_G(s) + W(s)$. The input signal is computed by some controller with control law $U(s) = K(s)R(s)$ where $R(s)$ is the reference signal. We thus obtain

$$Y(s) = G(s)U(s) + W(s) = G(s)K(s)R(s) + W(s),$$

where the error between reference and output signal is given by

$$E(s) = (1 - G(s)K(s))\, R(s) + W(s).$$

The steady state error when taking $\lim_{t\to\infty}$ is given by

$$e(\infty) = \lim_{s\to 0} s \cdot \left[(1 - G(s)K(s))\, R(s) + W(s)\right].$$

We can readily see that we are able to minimize by an appropriate choice of $K(s)$ the error due to the reference signal $R(s)$ but not the error due to the disturbance $W(s)$.

4.1.2. Feedback Control

We repeat the same calculation as above, this time for an feedback arrangement as in Fig. 4.3. The plant is again described by the transfer function $G(s)$, the output is given by $Y(s) = G(s)U(s) + W(s)$ but this time the control law is given by

$$U(s) = K(s)\left(R(s) - Y(s)\right),$$

which corresponds to the set-up described in Fig. 4.3 such that

$$\begin{aligned} Y(s) &= G(s)K(s)R(s) - G(s)K(s)Y(s) + W(s), \\ Y(s) &= \frac{G(s)}{1 + G(s)K(s)}R(s) + \frac{1}{1 + G(s)K(s)}W(s). \end{aligned}$$

The output error will be given by

$$E(s) = \left(1 - \frac{G(s)}{1 + G(s)K(s)}\right)R(s) + \frac{1}{1 + G(s)K(s)}W(s),$$

and we see that the choice of controller $K(s)$ influences the error contribution by both the reference value and disturbance signal, i.e. we are able to minimize the contribution of both the reference signal and the disturbance signal to the output error by an appropriate controller.

4.1.3. Linear and Non-Linear Control Theory

The basis for many techniques of design of the controller is the mathematical model of the system being subject to the control. The derived model can be given, depending on the underlying physics, in different mathematical forms as for example as ordinary differential equations (ODEs), partial differential equations (PDEs) or stochastic differential equations (SDEs).

Furthermore these differential equations can be linear or non-linear.

Control problems involving non-linear differential equations are in general difficult to solve and a 'general theory' for such systems doesn't exist.

The most rich and general theory has been developed for linear, time invariant systems described by linear ODEs with constant coefficients and it is this class of systems we are going to discuss in this chapter.

4.2. The State Space Representation

There are two standard ways to represent and work with linear time-invariant (LTI) systems. The *state space* and the *transfer function*. In the following we are going to discuss the state space representation of a dynamical system.

If we consider a mathematical description of the system given by n linear, in general not homogeneous ODEs with constant coefficients

$$\begin{aligned}
\dot{x}_1(t) &= a_{11}x_1(t) + a_{12}x_2(t) + \ldots + a_{1n}x_n(t) + b_{11}u_1(t) + b_{12}u_2(t) + \ldots + b_{1n}u_n(t), \\
&\vdots \\
\dot{x}_n(t) &= a_{n1}x_1(t) + a_{n2}x_2(t) + \ldots + a_{nn}x_n(t) + b_{n1}u_1(t) + b_{n2}u_2(t) + \ldots + b_{nn}u_n(t),
\end{aligned}$$

we can collect the coefficients into matrices $\mathbf{A}$ and $\mathbf{B}$ and collect the variables $x_j(t)$, their

derivatives $\dot{x}_j(t)$ and the inputs $u_k(t)$ into vectors $\dot{\mathbf{x}}(t)$, $\mathbf{x}(t)$, $\mathbf{u}(t)$ and arrive at the following form

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t).$$

Note that it is sufficient to consider first order derivatives since higher order derivatives can be reduced by introducing new variables $x^{(n)} = y_n,\ x^{(n-1)} = y_{n-1}, \ldots$ such that

$$x^{(n)}(t) = b_1 x^{(n-1)}(t) + b_2 x^{(n-2)}(t) + b_3 x^{(n-3)}(t) + \ldots + b_{n-1} x^{(1)} b_n x(t)$$

turns into

$$\dot{y}(t) = b_1 y_{n-1}(t) + b_2 y_{n-2}(t) + \ldots b_n y_0(t).$$

If, in addition, we allow for some output $y(t)$ given by a linear combination of the variables $x_j(t)$ and inputs $u_k(t)$, we result in a model description

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t). \end{aligned} \tag{4.2.0.1}$$

The state of of system is now given as a vector in $\mathbb{R}^n$ (or more general in $\mathbb{C}^n$).

As we can see in Equation (4.2.0.1) the system is completely described by providing the matrices A, B, C and D. Since the entire information about the intrinsic dynamics of the is encoded in the matrix A this matrix is often called the *system matrix*.

The solution to the ODE Equation (4.2.0.1) is given by

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\mathrm{d}\tau, \tag{4.2.0.2}$$

where $e^{\mathbf{A}}$ is the matrix exponential

$$e^{\mathbf{A}} = \sum_{k=0}^{\infty} \frac{1}{k!} \mathbf{A}^k. \tag{4.2.0.3}$$

4.2.1. Stability of the System

We can see that the solution splits up into an autonomous part, describing the evolution of the initial state $\mathbf{x}(0)$ and a contribution due to the input signal $\mathbf{u}(t)$. We consider now the autonomous case where $\mathbf{u}(t) = 0$ such that we are left with

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0).$$

If $\mathbf{A} \in \mathbb{C}^{n\times n}$ is diagonalizable, the Jordan normal form of the matrix $\mathbf{A}$ will be a diagonal matrix with each element on the diagonal being an eigenvalue of $\mathbf{A}$,

$$\mathbf{A} = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}.$$

The matrix exponential $e^{\mathbf{A}t}$ will then be given by

$$e^{\mathbf{A}\mathbf{t}} = \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 & 0 \\ 0 & e^{\lambda_2 t} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & 0 & \dots & e^{\lambda_n t} \end{bmatrix}.$$

If we consider the limit $\lim_{t\to\infty} \mathbf{x}(t) = \lim_{t\to\infty} e^{\mathbf{A}t}\mathbf{x}(0)$ for the autonomous system we see that this will converge against 0 if all of the eigenvalues $\lambda_1, \dots, \lambda_n$ have a real part smaller

than 0, i.e.

$$\operatorname{Re}\lambda_j < 0 \,\forall j = 1,\ldots,n \Rightarrow \lim_{t\to\infty} \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 & 0 \\ 0 & e^{\lambda_2 t} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & 0 & \ldots & e^{\lambda_n t} \end{bmatrix} \mathbf{x}(0) = 0,\ \forall\, \mathbf{x}(0) \in \mathbb{C}^n.$$

If on the other side any of the eigenvalues has a positive real part, the corresponding entry of the matrix exponential will diverge and so will the systems state. This arguments extent to general matrices $\mathbf{A}$ and this kind of concept of stability is called *Hurwitz stability*.

Definition 4.1 (Hurwitz Matrix) *We call a matrix A a Hurwitz matrix if for all eigenvalues λ_j of A we have that*

$$\operatorname{Re}\lambda_j < 0.$$

If a system's system matrix is a Hurwitz matrix the system will be internally stable.

4.2.2. Controllability

Another important concept is the concept of *controllability*. The idea behind this concept is the question if for any given initial state $\mathbf{x}(0)$ and time $0 < t'$, there exists some integrable input signal $\mathbf{u}(t)$ such that

$$e^{\mathbf{A}t'}\mathbf{x}(0) + \int_0^{t'} e^{\mathbf{A}(t'-\tau)}\mathbf{B}\mathbf{u}(\tau)\mathrm{d}\tau = \mathbf{x}(t') = \mathbf{x}_{\text{target}}.$$

We define the set of reachable states by the set

$$\mathcal{R}_t = \{\xi \in \mathbb{R}^n \ :\ \exists\, \mathbf{u}(t) \text{ such that } \mathbf{x}(t) = \xi\}.$$

The answer to above question can be given in terms of the *controllability matrix*

$$R(A,B) = \left[A,\ A^2B,\ \ldots, A^{n-1}B\right].$$

Theorem 4.2 ([11]) *For each time $0 < t$ the set equality*

$$\mathcal{R}_t = \operatorname{Im} R(A,B)$$

holds.

We can see that the set of reachable states will be equal to $\mathbb{R}^n$ if and only if $\operatorname{Im} R(A,B)$ has dimension n which will be true if and only if $R(A,B)$ has full rank n. Furthermore we note that $R(A,B)$ is independent of time. We can therefore see that, if the conditions are meet, there will exist an input signal $\mathbf{u}(t)$ for any time $0 < t$ such that $\mathbf{x}_{\text{targ}} = \mathbf{x}(t)$.

Corollary 4.3 *The system $\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t)$, $\mathbf{x}(t) \in \mathbb{R}^n$ is controllable if and only if the matrix*

$$R(A,B) = \left[A,\ A^2B,\ \ldots, A^{n-1}B\right]$$

has full rank n.

Definition 4.4 *We call the pair (A,B) a controllable pair if the dynamical system $\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t)$ is controllable.*

In order to get some intuition for this relation, we assume the case were $\mathbf{x}(0)$ and have a look at

$$\mathbf{x}(t) = \int_0^{t'} e^{\mathbf{A}(t'-\tau)}\mathbf{B}\mathbf{u}(\tau)\mathrm{d}\tau, \tag{4.2.2.1}$$

were we require that $\mathbf{x}(t) \in \mathcal{R}_t$, i.e. $\mathbf{x}(t)$ is in the set of reachable elements.

Using the Cayley-Hamilton theorem (stating that every square matrix A satisfies it's own characteristic polynomial $\mathrm{char}_A(A) = 0$) it can be shown that every square matrix of dimension n can be written as

$$A^k = a_{n-1}A^{n-1} + \ldots + a_0 I, \tag{4.2.2.2}$$

for $k > n$. Using this with the definition of the matrix exponential Equation (4.2.0.3) one can show that there exist scalar functions $\phi_0(t), \ldots, \phi_{n-1}(t)$ for all $t > 0$ such that

$$e^{\mathbf{A}t} = \phi_0 I + \phi_1(t)A + \ldots + \phi_{n-1}(t)A^{n-1},$$

by expending every term $k > n$ of the series Equation (4.2.0.3) using Equation (4.2.2.2).

Applying this to Equation (4.2.2.1) we can write

$$\mathbf{x}(t) = \int_0^t \left[\phi_0(t-\tau)I + \ldots + \phi_{n-1}(t-\tau)A^{n-1}\right]\mathbf{B}\mathbf{u}(\tau)\mathrm{d}\tau \tag{4.2.2.3}$$

$$= \left[B,\ AB, \ldots, A^{n-1}B\right]\begin{bmatrix} \int_0^t \phi_0(t-\tau)\mathrm{d}\tau \\ \int_0^t \phi_1(t-\tau)\mathrm{d}\tau \\ \vdots \\ \int_0^t \phi_{n-1}(t-\tau)\mathrm{d}\tau \end{bmatrix}, \tag{4.2.2.4}$$

where we can readily see that $\mathbf{x}(t)$ will be in the image of the controllability matrix and for $\mathbf{x}(t) \in \mathbb{R}^n$ we require that $\mathbb{R}^n \subseteq \mathrm{Im}\, R(A, B)$ which in turn requires that the controllability matrix has full rank n.[1]

[1] [11]

4.2.3. Observability

In the previous section we have seen how the relation between the input signal and the set of reachable states is encoded in the matrices $\mathbf{A}$ and $\mathbf{B}$ and how one can determine if every state in the system space $\mathbb{R}^n$ is reachable by some appropriate input signal $\mathbf{u}(t)$.

Consider the autonomous system with $\mathbf{u}(t) = 0$,

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t), \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t), \end{aligned} \tag{4.2.3.1}$$

with solution

$$y(t) = Ce^{\mathbf{A}t}\mathbf{x}_0.$$

One could ask about the relation between the initial state $\mathbf{x}_0$ and the output $y(t)$, i.e. if we observe the output $y(t)$ over some finite time interval $0 \leq t \leq T$, will we be able to deduce the system's initial state $\mathbf{x}_0$?

Similarly to the previous section one can show how the information about this relation is encoded in the matrices $\mathbf{A}, \mathbf{C}$.

If one is able to deduce the initial state of the system by observing the output over some time interval, the system is called *observable* and this property is called *observability*. As before, the answer to this question is given in terms of a matrix, in this case the *observability matrix* given by

$$P(\mathbf{A}, \mathbf{C}) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}.$$

Theorem 4.5 ([11]) *The dynamical system with matrices* $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ *and* $\mathbf{D}$ *is observable if the matrix*

$$P(\mathbf{A}, \mathbf{C})$$

has full column rank.

4.3. Transfer Functions

4.3.1. The Transfer Function

The second standard representation of linear time-invariant linear systems is the *transfer function*. Consider again a system as before, with[2]

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t), \\ y(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}u(t). \end{aligned} \tag{4.3.1.1}$$

We chose an input of the form $u(t) = e^{st},\ s \in \mathbb{C}$. Substituting this in the general solution to the ODE will yield

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0 + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}e^{s\tau}\mathrm{d}\tau, \tag{4.3.1.2}$$

$$= e^{\mathbf{A}t}\mathbf{x}_0 + e^{\mathbf{A}t}\int_0^t e^{(s\mathbf{I}-A)\tau}\mathbf{B}\mathrm{d}\tau. \tag{4.3.1.3}$$

If we assume $s \notin \operatorname{spec}(A)$ we have that $(s\mathbf{I} - \mathbf{A})$ is invertible and we can solve the integral with

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0 + \left[(s\mathbf{I} - \mathbf{A})^{-1} e^{(s\mathbf{I}-A)\tau}\mathbf{B}\right]_0^t. \tag{4.3.1.4}$$

[2] [1], [11]

Substituting this into Equation (4.3.1.1), we obtain

$$y(t) = \mathbf{C}e^{\mathbf{A}t}\left(\mathbf{x}_0 - (s\mathbf{I} - \mathbf{A})^{-1} B\right) + \left(\mathbf{C}\,(s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B} + \mathbf{D}\right) e^{st}.$$

If the system is stable we will have that $e^{\mathbf{A}t} \to 0$ as $t \to \infty$ and so will the first term of above equation. The second term is proportional to the input $u(t) = e^{st}$ and the term

$$\mathbf{G}(s) = \mathbf{C}\,(s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B} + \mathbf{D} \tag{4.3.1.5}$$

can be interpreted as an input-output map, mapping the input $u(t)$ to the output $y(t)$. $\mathbf{G}(s)$ is called the *transfer function* of the system.

A second way to arrive at Equation (4.3.1.5) is given in terms of the Laplace transform

$$F(s) = \int_0^\infty e^{-st} f(t)\mathrm{d}t := \mathcal{L}\left[f(t)\right]$$

for some integrable function $f(t)$. One can show that taking the Laplace transform of the derivative of an function corresponds to multiplication by s, i.e.

$$\mathcal{L}\left[\frac{d}{dt}f(t)\right] = s \cdot F(s) + f(0).$$

Applying this relation to Equation (4.3.1.1), one can solve algebraically for $Y(s)$ and arrives under the condition that $x(0) = u(0) = 0$ at

$$Y(s) = \left(\mathbf{C}\,(s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B}\right) U(s), \tag{4.3.1.6}$$

this time for general input functions $u(t)$.

The *transfer function* is a very useful tool and many important properties of the system are encoded in it.

If we consider some rational transfer function of the form

$$G(s) = \frac{a(s)}{b(s)},$$

then we call all the roots of the polynomial $b(s)$ the *poles* of $G(s)$ and the roots of the polynomial $a(s)$ the *zeroes* of $G(s)$. One can show that the poles of $G(s)$ are equal to the eigenvalues of the corresponding system matrix $\mathbf{A}$.

As we can see, $G(s)$ will be unbounded if s is a pole of $G(s)$. If s is a zero of the system then, for example for an input as above with $u(t) = e^{st}$, we will obtain a zero output $y(t) = 0$ since $G(s) = 0$.

4.3.2. Networks of linear Systems

It is particularly convenient to compute networks of linear dynamical systems using the transfer function representation of the system. Here we consider a collection of linear systems, parametrized by matrices (A_j, B_j, C_j, D_j) with transfer functions $G_j(s)$ for $j = 1, \ldots n$.

The Series Product

The first basic network operation to consider is the series product, corresponding to taking systems $G_1(s)$ and $G_2(s)$ in series, see Fig. (4.4).

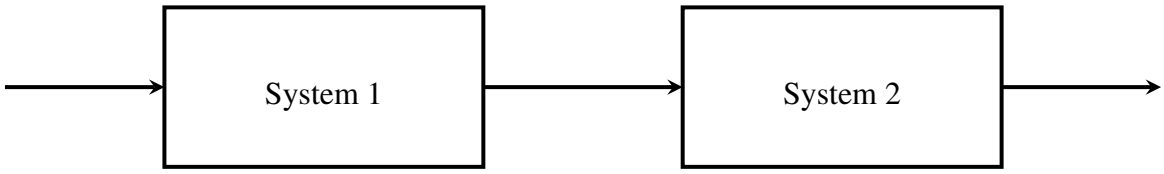

Figure 4.4.: Two systems in series, obtained by feeding the output of system 1 into the input of system 2.

The corresponding input-output function for the overall network is easily computed by writ-

ing

$$Y_1(s) = G_1(s)U_1(s),$$
$$Y_2(s) = G_2(s)U_2(s),$$

and setting $U_2(s) = Y_1(s)$. We obtain the overall transfer function

$$\frac{Y_2(s)}{U_1(s)} = G_2(s)G_1(s),$$

and more generally

$$\frac{Y_{\mathrm{n}}(s)}{U_1(s)} = \prod_{j=1}^{n} G_j(s)$$

for the series product of n-systems.

Parallel Systems

The second basic network operation is obtained by taking two systems parallel, see Fig. (4.5). The corresponding transfer function is obtained from the transfer functions of system 1 and 2 $G_1(s)$ and $G_2(s)$ by

$$\frac{Y(s)}{U(s)} = G_1(s) + G_2(s)$$

or in general by

$$\frac{Y(s)}{U(s)} = \sum_{j=1}^{n} G_j(s).$$

Systems in Loop

The third network operation is given by the system in loop, see Fig. (4.6). This is also the set-up considered in Section 4.1.2. The transfer function is given by

$$G_{\mathrm{cl}}(s) = \frac{Y(s)}{U(s)} = \frac{G_1(s)}{1 + G_1(s)G_2(s)}.$$

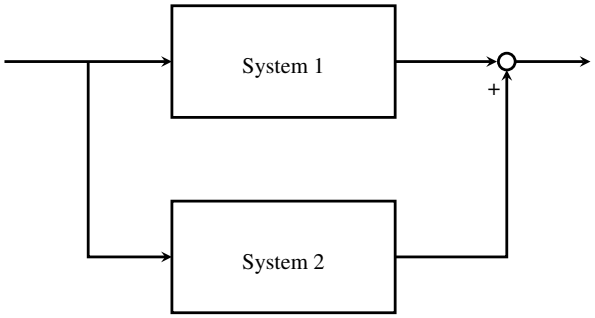

Figure 4.5.: Two parallel systems.

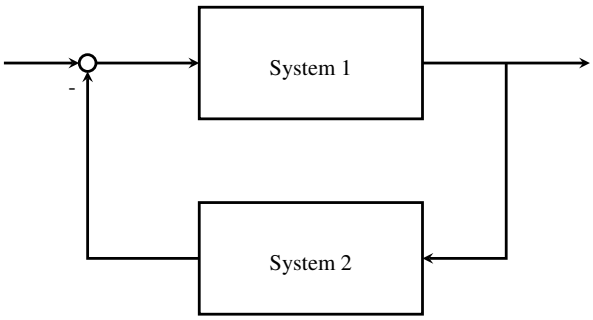

Figure 4.6.: Two systems in a feedback arrangement.

This can be stated explicitly in terms of the matrices A, B, C and D if the consider a closed-loop set-up as above with system 1 given by (A, B, C, D) and system 2 given by $(0, 0, 0, I)$. We can take the Laplace transform of Equation (4.3.1.1) and solve for $y(t)$. In this case one obtains

$$G_{\text{cl}}(s) = \frac{Y(s)}{U(s)} = C\left(sI - A\right)^{-1} B + D.$$

The transfer function associated with the system matrices A, B, C and D is sometimes denoted with

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right](s) = C\left(sI - A\right)^{-1} B + D.$$

The transfer function is well defined if $(sI - A)^{-1}$ exists.

5

Quantum Networks

5.1. Introduction

In classical control theory, the notion of an input-output system plays a key role. Bigger systems can often be thought of as a network of simpler input-output components. If one knows the rules to deduce the dynamics of the overall system from the dynamics of the individual blocks the network consists of, system design, simulation and control can become considerably easier. For classical systems this theory is well developed. We are interested in establishing a similar theory for quantum mechanical systems.

In the following we shall introduce a quantum counterpart of the notion of input-output

systems as developed in [19, Gardiner and Zoller], [22, Gough, Gohm and Yanagisawa], [25, Gough and James], [24, Gough and James] and present the rules deducible for networks of such general input-output systems.

5.2. Quantum Single-Input, Single-Output Systems

In section 3.3.7 we introduced the quantum stochastic calculus for the evolution of some quantum system under the influence of quantum noise. Theorem 3.3.7.1 guarantees existence of a solution for a quantum stochastic differential equation (QSDE) of the form

$$\mathrm{d}U = \left\{ L\mathrm{d}B^\dagger(t) + (S-1)\,\mathrm{d}\Lambda(t) - L^*S\mathrm{d}B(t) + \left(\frac{1}{2}L^*L + iH\right)\mathrm{d}t \right\} U(t) \equiv \mathrm{d}G(t)U(t),\ U(0) = 1,$$

or equivalently if we define the complex dampening K by

$$K = -\frac{1}{2}L^\dagger L - iH,$$

we can write

$$\mathrm{d}U(t) = \left\{ L\mathrm{d}B^\dagger(t) + (S-1)\,\mathrm{d}\Lambda(t) - L^*S\mathrm{d}B(t) + K\mathrm{d}t \right\} U \equiv \mathrm{d}G(t)U(t),\ U(0) = 1.$$

Note that we use here the right hand side Hudson & Parthasarathy equation, that is the solution $U(t)$ appearing on the right hand side, rather on the left hand side as before. In general we have the left hand QSDE of the form $\mathrm{d}V = V(\mathrm{d}H)$ and the right hand QSDE $\mathrm{d}U = (\mathrm{d}G)U$. If we let $V = U^\dagger$, then both QSDEs are equivalent if $\mathrm{d}H = (\mathrm{d}G)^\dagger$. When dealing with physical systems the right hand side QSDE is usually preferred. The left hand QSDE has it's advantages when dealing with unbounded operator.

The unitary evolution of the system under the influence of the quantum noises $\mathrm{d}B(t), \mathrm{d}B^\dagger(t)$

and $\mathrm{d}\Lambda(t)$ is given by the solution of the QSDE, $U(t)$, whereas this solution describes the evolution of both the field and the system. This allows us to perform measurements on the output field after interaction with the system, and leads to a quantum equivalent of the notion of an input-output system from classical control and system theory.

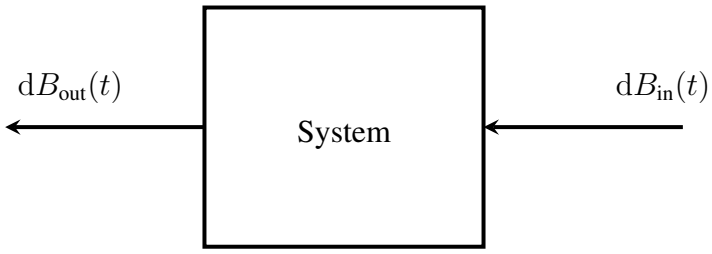

Figure 5.1.: Quantum input-output system under the influence of the input noise $\mathrm{d}B(t) = \mathrm{d}B_{\text{in}}(t)$ with output field $\mathrm{d}B_{\text{out}}(t)$.

Given some observable X on the initial space h_0, its evolution in interaction picture is given by

$$X(t) = U(t)^\dagger [X \otimes I] U(t)$$

where the systems outputs $\mathrm{d}B_{\text{out}}(t)$ are given by

$$\begin{aligned} B_{\text{out}}(t) &= U(t)^\dagger [I \otimes B_{\text{in}}(t)] U(t), \\ \Lambda_{\text{out}}(t) &= U(t)^\dagger [I \otimes \Lambda_{\text{in}}(t)] U(t), \end{aligned}$$

and obey the QSDEs

$$\begin{aligned} \mathrm{d}B_{\text{out}}(t) &= S(t)\mathrm{d}B_{\text{in}}(t) + L(t)\mathrm{d}t, \\ \mathrm{d}\Lambda_{\text{out}}(t) &= S^\dagger(t)\mathrm{d}\Lambda_{\text{in}}(t)S(t) + \mathrm{d}B_{\text{in}}^\dagger(t)S^\dagger(t)L(t) + L^\dagger(t)S(t)\mathrm{d}B_{\text{in}}(t) + L^\dagger(t)L(t)\mathrm{d}t. \end{aligned}$$

5.2.1. Multi-Channel Systems

So far we have considered a set-up allowing for system with one degree of freedom under the influence of some quantum noise due to some bosonic environment. We wish to extend this to allow for systems with multiple degrees of freedom under the influence of multiple, independent noise processes. We first of all deal with the generalisation of the model to multi-channel inputs. We extend the set-up by what is known as the multiplicity or colour space, $\mathfrak{K}$ which is taken to be a separable Hilbert space and is usually taken to be $\mathfrak{K} = \mathbb{C}^n$. We denote

$$L^2_{\mathfrak{K}}(0,\infty) := \mathfrak{K} \otimes L^2(0,\infty),$$

and obtain overall system Hilbert space $h_0 \otimes \Gamma(L^2_{\mathfrak{K}}(0,\infty))$. Let $\{e_i\}$ be a orthonormal basis for $\mathfrak{K}$. We define the for each j,k the independent processes

$$\begin{aligned} B_j(t) &:= a\left(e_j \otimes \mathbb{1}_{[0,t]}\right), \\ \Lambda_{jk}(t) &:= \lambda\left(|e_j\rangle\langle e_k| \otimes \Pi_{[0,t]}\right). \end{aligned}$$

We interpret $n = \dim \mathfrak{K}$ as the number of input and output channels of the system.

The coupling of the the n input processes to the system are now parameterised by n coupling operator $L_j, j = 1,\ldots,n$. The scattering between the field channel is described by n^2 operator $S_{jk} \in \mathcal{B}(h_0 \otimes k)$ where the matrix $S = (S_{jk})$ is to be taken unitary. We can collect these operators in column vector L and matrix S

$$L = \begin{bmatrix} L_1 \\ \vdots \\ L_n \end{bmatrix}, \quad S = \begin{bmatrix} S_{11} & \ldots & S_{1n} \\ \vdots & \ddots & \vdots \\ S_{n1} & \ldots & S_{nn} \end{bmatrix}.$$

In this case the general form of the QSDE leading to a unitary adapted solution is

$$\mathrm{d}U(t) = \Big\{ K \otimes \mathrm{d}t - L_j^\dagger S_{jk} \otimes \mathrm{d}B_k(t) + L_j \otimes \mathrm{d}B_j^\dagger(t) + (S_{jk} - \delta_{jk}) \otimes \mathrm{d}\Lambda_{jk}(t) \Big\} U(t)$$

with use of the Einstein sum convention, i.e. summation over repeated indices, and

$$K = -\frac{1}{2} L_i^\dagger L_i - iH.$$

The outputs are now given by

$$\begin{aligned} B_{\text{out},i} &= U^\dagger(t)[I \otimes B_i(t)]U(t), \\ \mathrm{d}B_{\text{out},j} &= S_{jk}(t)\mathrm{d}B_k(t) + L_j(t)\mathrm{d}t. \end{aligned}$$

5.3. Networks of Quantum Components

As we have seen, the class of physical models considered so far are parameterized by the triple (S, L, H) or by making use of the complex dampening $K = -\frac{1}{2}L^\dagger L - iH$ the triple (S, L, K). We refer to this triple as the Hudson and Parthasarathy system parameter. We remark that one can collect this parameter in a single operator called the *Itō matrix* or *Itō generator matrix* on $h_0 \otimes (\mathbb{C} \oplus k)$ by

$$G = \begin{bmatrix} K & -L^\dagger S \\ L & S - 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}L^\dagger L - iH & -L^\dagger S \\ L & S - 1 \end{bmatrix}.$$

Equivalently the model can be described by the *model matrix* V

$$V := G + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} K & -L^\dagger S \\ L & S \end{bmatrix}.$$

5.3.1. The Concatenation Product

Classical system and control theory allows for thinking of bigger and complex systems as being composed of multiple, simpler systems. These smaller systems are usually quit easily modelled. System theory then provides the tools to compute the model for the overall system given the simpler blocks and the way they are connected with each other.

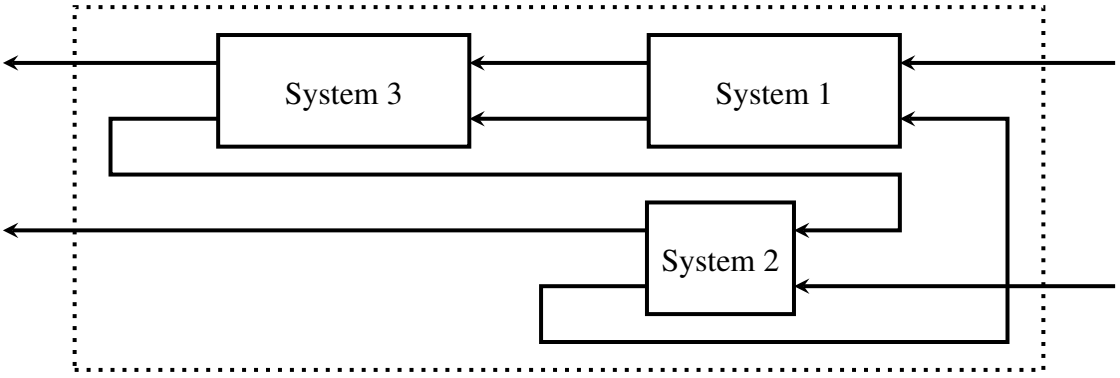

Figure 5.2.: An example of a general network involving multiple system components with feedback

The first operation we wish to introduce is the concatenation product. This product takes two system matrices V_1 and V_2, describing two blocks of our network with n and m inputs and outputs respectively and gives a new system matrix describing both systems at once with $n+m$ inputs and outputs. This concatenation of two blocks does not introduce any interconnection between the blocks and therefore does not change the input-output dynamics of the channels.

We follow [25] and define the concatenation of two model matrices in the following way:

Definition 5.1 *Let V_j be model matrices with Hilbert space h and colour space k_j for $j = 1, 2$. The concatenation of these model matrices is denoted by $V_1 \boxplus V_2$ describing a model with Hilbert space h and colour space $k_1 \oplus k_2$ and is defined by*

$$\begin{aligned} V_1 \boxplus V_2 &= \begin{bmatrix} -\frac{1}{2}L_1^\dagger L_1 - iH_1 & -L_1^\dagger S_1 \\ L_1 & S_1 \end{bmatrix} \boxplus \begin{bmatrix} -\frac{1}{2}L_2^\dagger L_2 - iH_2 & -L_2^\dagger S_2 \\ L_2 & S_2 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{1}{2}L_1^\dagger L_1 - \frac{1}{2}L_2^\dagger L_2 - i\,(H_1 + H_2) & -L_1^\dagger S_1 - L_2^\dagger S_2 \\ L_1 & S_1 \quad 0 \\ L_2 & 0 \quad S_2 \end{bmatrix}. \end{aligned}$$

As the authors remark, there is no further assumption on the decomposition of the system.

5.3.2. Network Models

A general network as for example Fig. 5.2 will be described by concatenations of its components together with a list of internal edges determining which ports are connected with each other.

Given a network with n components $V_j, j = 1, \ldots, n$ the networks model matrix V is given by $V = \boxplus_{j=1}^n V_j$ and takes for V_j associated with triple (S_j, L_j, H_j) the form

$$\begin{array}{c} \\ 0 \\ s_1 \\ s_2 \\ \vdots \\ s_n \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & r_1 & r_2 & \ldots & r_n \end{array} \\ \begin{pmatrix} -\sum_{j=1}^n \left(\frac{1}{2}L_j^\dagger L_j + iH_j\right) & -L_1^\dagger S_1 & -L_2^\dagger S_2 & \ldots & -L_n^\dagger S_n \\ L_1 & S_1 & 0 & \ldots & 0 \\ L_2 & 0 & S_2 & & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ L_n & 0 & \ldots & 0 & S_n \end{pmatrix} \end{array}$$

with respect to the labels $s_1, \ldots s_n$, $r_1, \ldots s_n$, labelling the rows and columns with respect to the input and output ports. Since each of the L_j is in general a vector with k_j entries, where k_j is the number of input and output ports of block V_j, each of the s_j, r_j is a k_j-tupel. We follow [25] and denote with $\mathcal{P}_{\text{in}}$ the set of labels of the input ports and with $\mathcal{P}_{\text{out}}$ the set of labels of the output ports.

We label blocks of the model matrix with respect to this labels by picking some $\alpha \in \mathcal{P}_{\text{in}} \cup \{0\}$ and $\beta \in \mathcal{P}_{\text{out}} \cup \{0\}$ and denoting $V_{\alpha,\beta}$

The network is then completely described by providing the network's model matrix and a list of all internal connections, that is edges $e = (s_n, r_m)$, where output s_n is fed into r_m.

5.3.3. Elimination of Internal Edges in the Zero Time Delay Limit

In general a network as shown in Fig. 5.2 will consist of a collection of blocks with a certain number of ports and a collection of edges of which some describe the inputs and outputs of the overall network and some which connect an output of a block with an input of another block and which are therefore internally. If we consider a limit, in which the time signals take to travel from the output of one block to the input of another block goes against zero, we can eliminate the additional degrees of freedom provided by the internal edges and result in a model that only knows external edges, see for example Fig. 5.3.

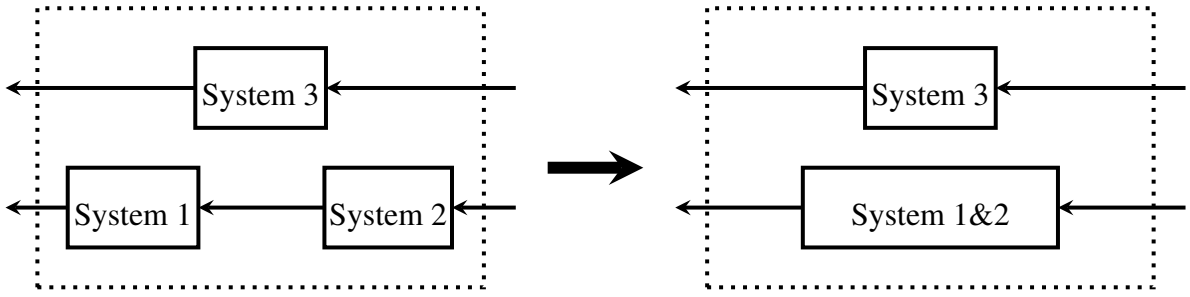

Figure 5.3.: Example for elimination of internal edges in a zero time delay limit. The reduced model consists only of external inputs and outputs.

Formulas for the model reduction such as the *series product* and the *feedback reduction*

formula have been derived in [22, 25, 24]. We are going to introduce as the most general case a theorem on the elimination of internal edges in a zero time delay limit as proven in [24], from which the series product and the feedback reduction formula can be deduced as special cases.

Theorem 5.2 (Gough, James [24]) *Let $e_0 = (r_0, s_0)$ be an internal channel with time delay $\tau_0 \geq 0$ in a quantum network $\mathcal{N}$ for which $1 - V_{s_0,r_0}$ is invertible. In the limit $\tau_0 \to 0^+$, the network reduces to $\mathcal{N}_{red}$ in which the input and output ports are $\mathcal{P}_{in}\backslash\{r_0\}$ and $\mathcal{P}_{out}\backslash\{s_0\}$ and the edge e_0 eliminated. (In the case where r_0 and s_0 are initially in different components then the components merge.) The reduced model matrix V^{red} then has the components*

$$V^{red}_{\alpha,\beta} = V_{\alpha,\beta} + V_{\alpha,r_0}\left(1 - V_{s_0,r_0}\right)^{-1} V_{s_0,\beta}, \tag{5.3.3.1}$$

for $\beta \in \{0\} \cup \mathcal{P}_{in}\backslash\{r_0\}$ and $\alpha \in \{0\} \cup \mathcal{P}_{out}\backslash\{s_0\}$.

The reduced model matrix can also be computed by reduction formulas for the parameter triple $(S, L, H) \to (S^{\text{red}}, L^{\text{red}}, H^{\text{red}})$.

Lemma 5.3 (Gough, James [24]) *Let V be the model matrix determined by the operators (S, L, H). Then the reduced model matrix V^{red} obtained by eliminating the edge $e_0 = (s_0, r_0)$ is determined by the operators $(S^{red}, L^{red}, H^{red})$ where*

$$\begin{aligned}
S^{red}_{sr} &= S_{sr} + S_{s,r_0}\left(1 - S_{s_0,r_0}\right)^{-1} S_{s_0,r}, \\
L^{red}_{s} &= L_s + S_{s,r_0}\left(1 - S_{s_0,r_0}\right)^{-1} L_{s_0}, \\
H^{red} &= H + \sum_{s\in\mathcal{P}_{out}} \operatorname{Im} L_s^\dagger S_{s,r_0}\left(1 - S_{s_0,r_0}\right)^{-1} L_{s_0},
\end{aligned}$$

where $r \in \mathcal{P}_{in}\backslash\{r_0\}$ and $s \in \mathcal{P}_{out}\backslash\{s_0\}$.

One can further show that the order in which internal channels are eliminated in the zero time delay limit doesn't matter, that is the reduction of two edges e_0 and e_1 commutes and the elimination of multiple edges can therefore be performed simultaneously [24, Lemma 17].

This can be archived by expressing S and L with respect to the decomposition of the colour space $k = k_{\text{int}} \oplus k_{\text{ext}}$

$$S = \begin{bmatrix} S_{ii} & S_{ie} \\ S_{ei} & S_{ee} \end{bmatrix}, \quad L = \begin{bmatrix} L_i \\ L_e \end{bmatrix}.$$

Define the unitary adjacency matrix η by

$$\eta_{sr} = \begin{cases} 1 & \text{if } (s,r) \text{ is an internal channel} \\ 0, & \text{otherwise} \end{cases}$$

The feedback reduced model can then be written as [24]

$$V^{\text{red}} = V_{\alpha,\beta} + V_{\alpha,i} \left(\eta - V_{ii}\right)^{-1} V_{i,\beta}$$

with $\alpha, \beta \in \{0, e\}$. Following Lemma 5.3, the reduced model matrix V^{red} can be determined from the operator triple $(S^{\text{red}}, L^{\text{red}}, H^{\text{red}})$ given by

$$\begin{aligned} S^{\text{red}} &= S_{ee} + S_{ei} \left(\eta - S_{ii}\right)^{-1} S_{ie}, \\ L^{\text{red}} &= L_e + S_{ei} \left(\eta - S_{ii}\right)^{-1} L_i, \\ H^{\text{red}} &= H + \sum_{j=i,e} \operatorname{Im} L_j^\dagger S_{ji} \left(\eta - S_{ii}\right)^{-1} L_i. \end{aligned}$$

5.4. The Series Product

As a first special case of the network reduction formulas presented in the previous section we deduce the rule for the *series product*. The series product describes the network reduction for two components where the output of the first component is feed into the input of the second

component, see for example Fig. 5.4. Obviously the series product only makes sense if we require that $\dim \mathfrak{K}_1 = \dim \mathfrak{K}_2$ where $\mathfrak{K}_j, j = 1, 2$ is the multiplicity space associated with model $V_j, j = 1, 2$, that is, the number of input channels accepted by block 2 matches the number of output channels of block 1.

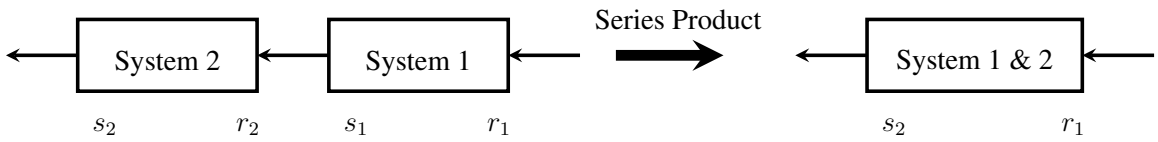

Figure 5.4.: The series product of two components. In the zero time delay limit both components merge into a single component after elimination of the internal edge.

The networks model matrix is given by

$$V = \begin{array}{c} \\ 0 \\ s_1 \\ s_2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & r_1 & r_2 \end{array} \\ \begin{pmatrix} -\sum_{j=1,2}\left(\frac{1}{2}L_j^\dagger L_j + iH_j\right) & -L_2^\dagger S_1 & -L_2^\dagger S_2 \\ L_1 & S_1 & 0 \\ L_2 & 0 & S_2 \end{pmatrix} \end{array}.$$

Elimination of the edge $e = (s_1, r_2)$ gives the reduced model matrix V_{series}

$$\begin{aligned} V_{\text{series}} &= \begin{bmatrix} -\sum_{j=1,2}\left(\frac{1}{2}L_j^\dagger L_j + iH_j\right) & -L^\dagger 1 S_1 \\ L_2 & 0 \end{bmatrix} + \begin{bmatrix} -L_2^\dagger S_2 \\ S_2 \end{bmatrix} (1-0)^{-1} [L_1, S_1] \\ &= \begin{bmatrix} -\sum_{j=1,2}\left(\frac{1}{2}L_j^\dagger L_j + iH_j\right) - L_2^\dagger S_2 L_1 & -L_1^\dagger S_1 + L_2^\dagger S_2 S_1 \\ L_2 + S_2 L_1 & S_2 S_1 \end{bmatrix}. \end{aligned}$$

We can readily read off the reduced model parameter

$$\begin{aligned} S_{\text{series}} &= S_2 S_1, \\ L_{\text{series}} &= L_2 + S_2 L_1, \\ H_{\text{series}} &= H_1 + H_2 + \text{Im}\left\{L_2^\dagger S_2 L_1\right\}. \end{aligned}$$

5.5. The Feedback Reduction

Another special case of the *elimination of internal edges in the zero time delay limit* we want to introduce is the feedback reduction formula, see Fig. 5.5. We have some 4-port system and some 2-port system where the 2-port system is located in some internal loop. This kind of set-up is a basic example for feedback and serves as a general case for common physical set-ups such as optical components in a feedback loop using beam splitter etc.

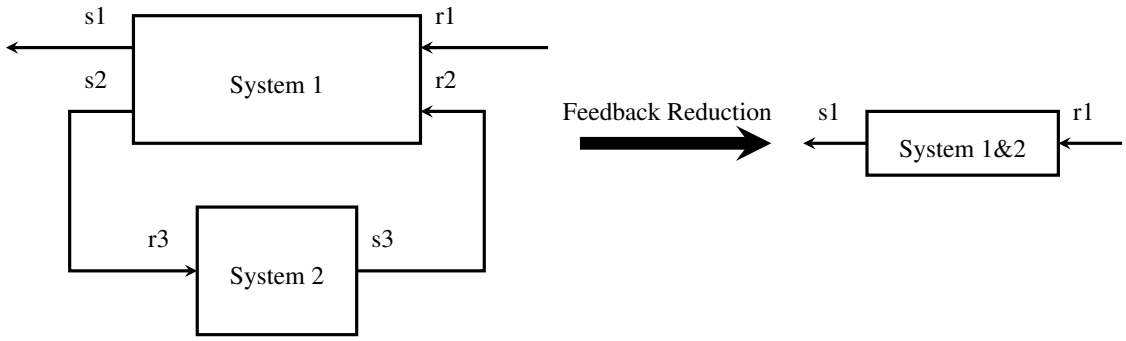

Figure 5.5.: Elimination of the internal edges: Feedback reduction.

We parametrise system 1, the 4-port component with triple (S_1, L_1, H_1) and colour space decomposition $\mathfrak{K}_1 = \mathfrak{K}_e \oplus \mathfrak{K}_i$ where

$$S_1 = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}, \quad L_1 = \begin{bmatrix} L_{1,e} \\ L_{1,i} \end{bmatrix}.$$

System 2 is given by the triple (S_2, L_2, H_2). We can compute the overall system model matrix V by taking the concatenation $V = V_1 \boxplus V_2$

$$V = \begin{array}{c} \\ 0 \\ s_1 \\ s_2 \\ s_3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & r_1 & r_2 & r_3 \end{array} \\ \begin{pmatrix} -\sum_{j=1,2}\left(\frac{1}{2}L_j^\dagger L_j + iH_j\right) & -L_{1,e}^\dagger S_{ie} & -L_{1,i}^\dagger S_{ei} & -L_2^\dagger S_2 \\ L_{1,e} & S_{11} & S_{12} & 0 \\ L_{1,i} & S_{21} & S_{22} & 0 \\ L_2 & 0 & 0 & S_2 \end{pmatrix} \end{array}.$$

Define the internal and extern sub-matrices by

$$S_{ii} = \begin{bmatrix} S_{22} & 0 \\ 0 & S_2 \end{bmatrix}, \; S_{ie} = \begin{bmatrix} T_{21} \\ 0 \end{bmatrix}, \; S_{ei} = (S_{12}, 0), \; S_{ee} = S_{11}, \; L_i = \begin{bmatrix} L_{1,i}, L_2 \end{bmatrix}, \; L_e = L_{1,e}.$$

The reduced model matrix is then determined by system parameter triple $(S_{\text{red}}, L_{\text{red}}, K_{\text{red}})$ given by

$$\begin{aligned} S_{\text{red}} &= S_{ee} + S_{ei} \left(\eta - S_{ii}\right)^{-1} S_{ie}, \\ L_{\text{red}} &= L_e + S_{ei} \left(\eta - S_{ii}\right)^{-1} L_i, \\ K_{\text{red}} &= H + \sum_{j=i,e} \text{Im} \left(L_j^{\dagger} S_{j,i} \left(\eta - S_{ii}\right)^{-1} L_i \right). \end{aligned} \tag{5.5.0.2}$$

6

Quantum Linear Systems

6.1. Introduction

Linear systems play a central role in classical control theory. It's only for this class of systems that a general theory of control of dynamical systems becomes available. In the following chapter we want to introduce a quantum mechanical equivalent for a linear system, that is the special case of a linear quantum open system as introduced in Chapter 3 and introduce techniques and methods becoming available for quantum linear systems and networks of quantum linear systems. This class of systems and the algebraic rules for networks of such systems in the previously introduced framework have been investigated in [22] and for the more general

case allowing for squeezing components in [26]. The following chapter will be based on this papers.

6.2. Linear Systems

As we have seen in Chapter 3, the dynamical evolution of a quantum open system is given by the solution $V(t)$ of the QSDE

$$\frac{d}{dt}V(t) = \Big\{(S_{ij} - \delta_{ij})\,\mathrm{d}\Lambda(t) + L_i \mathrm{d}B_i^\dagger(t) - L_i^\dagger S_{ij}\mathrm{d}B_j(t) - \Big(\tfrac{1}{2}L_i^\dagger L_i - iH\Big)\,\mathrm{d}t\Big\}\,V(t).$$

We obtain an unitary evolution of the system leading to linear dynamics by imposing the following structure of parameters

- S_{jk} are scalars
- The coupling operators are linear, i.e. of the form $L_j = \sum_k = c_{jk}a_k$ for some scalars c_{jk}
- The Hamiltonian H is quadratic, i.e. of the form $H = \sum_{j,k=1} \omega_{jk} a_j^\dagger a_k$ for some scalars ω_{jk}.

In this case the Heisenberg-Langevin equations for the annihilators $a(t) = V^\dagger(t)aV(t)$ are given by

$$\frac{d}{dt}a(t) = Aa(t) - C^\dagger S b_{\text{in}}(t), \tag{6.2.0.1}$$

$$b_{\text{out}}(t) = Sb_{\text{in}}(t) + Ca(t). \tag{6.2.0.2}$$

Set $B = -C^\dagger S$ and $D = S$, i.e.

$$\begin{aligned} \dot{a}(t) &= Aa(t) + Bb_{\text{in}}(t), \\ b_{\text{out}}(t) &= Ca(t) + Db_{\text{in}}(t), \end{aligned}$$

and we obtain a description for the system which looks very similar to the structure obtained in the classical case for the state space representation of linear time invariant systems, although the mathematical objects involved in both cases are very different in their nature (see Chapter 4).

6.2.1. Transfer Functions

Linear quantum systems allow for the introduction of Laplace transform techniques leading to input-output map descriptions of the system, that is a relation between incoming and outgoing fields of the form $b_{\text{out}}[s] = G[s]b_{\text{in}}[s]$ for some transfer function $G[s]$. We denote with $b[s]$ the Laplace transform of the field annihilator $b(t)$ defined by

$$b[s] = \int_0^\infty e^{-st} b(t) \mathrm{d}t \tag{6.2.1.1}$$

for $\mathrm{Re}\, s > 0$.

Applying the Laplace transform to the Heisenberg-Langevin equations Eq. (6.2.0.1,6.2.0.2) we obtain the input-output description

$$b_{\text{out}}[s] = \Xi[s] b_{\text{in}}[s] + \xi[s] a$$

with the transfer function

$$\Xi[s] = S - C(sI - A)^{-1} C^\dagger S$$

and the contribution of the initial operator $a = a(0)$

$$\xi[s] = C(sI - A)^{-1}.$$

We remark that, as a property of the Laplace transform, we have that

$$b[s^*]^* = \left(\int_0^\infty e^{-s^*t} b(t)\mathrm{d}t\right)^* = \int_0^\infty e^{-st} b^*(t)\mathrm{d}t.$$

One can adopt standard engineering notation from classical control theory and denote the transfer function by the matrix

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right](s) = D + C(sI - A)^{-1}B.$$

Similarly to the Laplace transform, define the transform of the past fields by

$$b[s] \triangleq \int_{-\infty}^0 e^{-st} b(t)\mathrm{d}t$$

such that the Fourier transform of the fields is given by

$$\hat{b}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{i\omega t} b(t)\mathrm{d}t = \frac{1}{\sqrt{2\pi}} b[0^+ - i\omega] + \frac{1}{\sqrt{2\pi}} b[0^- - i\omega].$$

The cannonical commutation relations translate into $[\hat{b}_{\text{in},k}(\omega), \hat{b}_{\text{in},l}(\omega')] = \delta_{kl}\delta_{\omega-\omega'}$.

6.2.2. Feedback Reduction

We are now ready to introduce quantum feedback networks. The first set-up under consideration is a form as seen in Fig. 6.1. We consider first the case without an inloop device, that is,

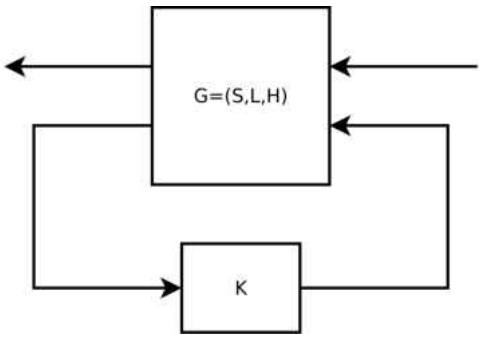

Figure 6.1.: Feedback arrangement of a 4-port system G and some in-loop component K

$K = (I, 0, 0)$. We obtain a partition of the transfer function matrix with respect to the choice of internal and external channel, that is

$$b_{\text{in}} = \begin{bmatrix} b_{\text{in},i} \\ b_{\text{in},e} \end{bmatrix},\ b_{\text{out}} = \begin{bmatrix} b_{\text{out},i} \\ b_{\text{out},e} \end{bmatrix},\ C = \begin{bmatrix} C_i \\ C_e \end{bmatrix},\ S = \begin{bmatrix} S_{ii} & S_{ei} \\ S_{ie} & S_{ee} \end{bmatrix}$$

with transfer function matrix

$$\Xi(s) = \left[\begin{array}{c|cc} A & -\sum_j C_j^\dagger S_{ij} & -\sum_j C_j^\dagger S_{je} \\ \hline C_i & S_{ii} & S_{ie} \\ C_e & S_{ie} & S_{ee} \end{array}\right](s) \tag{6.2.2.1}$$

Define the adjacency matrix η by

$$\eta_{sr} = \begin{cases} 1, \text{ if } (s,r) \text{ is an internal channel} \\ 0, \text{ otherwise} \end{cases}.$$

Theorem 6.1 *Let $(\eta - S_{ii})$ be invertible. The feedback system Eq. (6.2.2.1) has input-output relation $b_{out,e}[s] = \Xi_{red}(s) b_{in,e}[s] + \xi_{red}(s) a$ and the reduced transfer matrix function*

$$\Xi_{red}(s) = \left[\begin{array}{c|c} A_{red} & -C_{red}^\dagger S_{red} \\ \hline C_{red} & S_{red} \end{array}\right],\ \xi_{red} = C_{red} \frac{1}{s - A_{red}}$$

where

$$\begin{aligned} S_{red} &= S_{ee} + S_{ei}(\eta - S_{ii})^{-1} S_{ie}, \\ C_{red} &= S_{ei}(\eta - S_{ii})^{-1} C_i + C_e, \\ A_{red} &= A - \sum_{j=i,e} C_j^\dagger S_{ji}(\eta - S_{ii})^{-1} C_i \end{aligned}$$

See [22] for the proof.

6.2.3. The Series Product

Consider an arrangement as in Fig. 6.2, again in a zero time delay regime. We have the two systems $G_i = (S_i, L_i, H_i)$, $i = 1, 2$. Associated with this systems we have the transfer functions

$$\Xi_i(s) = \left[\begin{array}{c|c} A_i & -C_i^\dagger S_i \\ \hline C_i & S_i \end{array}\right].$$

We can reformulate this situation before making the connection, by describing the network components as one block with two inputs and two outputs by the concatenation

$$\Xi(s) = \left[\begin{array}{c|cc} A_1 + A_2 & -C_1^\dagger S_1, & C_2^\dagger S_2 \\ \hline C_1 & S_1 & 0 \\ C_2 & 0 & S_2 \end{array}\right].$$

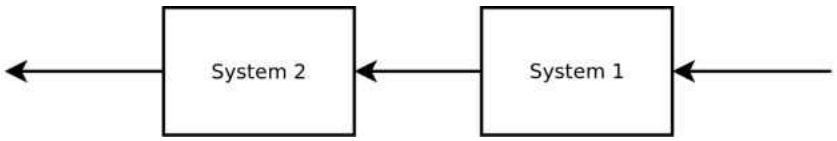

Figure 6.2.: The Series Product of two systems with the output of system 1 being feed into the input of system 2.

We again partition the parameter with respect to the internal and external channel

$$b_{\text{in}} = \begin{bmatrix} b_{\text{in},i} \\ b_{\text{in},e} \end{bmatrix} = \begin{bmatrix} b_{\text{in},2} \\ b_{\text{in},1} \end{bmatrix}, \qquad b_{\text{out}} = \begin{bmatrix} b_{\text{out},i} \\ b_{\text{out},e} \end{bmatrix} = \begin{bmatrix} b_{\text{out},1} \\ b_{\text{out},2} \end{bmatrix},$$
$$\begin{bmatrix} C_i \\ C_e \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, \qquad \begin{bmatrix} S_{ii} & S_{ei} \\ S_{ie} & S_{ee} \end{bmatrix} = \begin{bmatrix} 0 & S_1 \\ S_2 & 0 \end{bmatrix}.$$

The adjacency matrix for this case with one internal edge is trivial with $\eta = 1$ and we can compute the series product of the model $G_2 \triangleleft G_1$ using the formula for the reduced parameter given in Theorem 6.1 with

$$\Xi_{\text{series}}(s) = \left[\begin{array}{c|c} A_1 + A_2 - C_2^\dagger S_2 C_1 & -(C_2^\dagger S_2 + C_1^\dagger) S_1 \\ \hline C_2 + S_2 C_1 & S_2 S_1 \end{array}\right].$$

6.2.4. The Redheffer Star Product

One other important network arrangement is the one shown in Fig. 6.3, that is two systems with 2 inputs and outputs each in an feedback arrangement. This kind of set-up is used for example to model some system G driven by a noisy controller K.

The two system models can be parameterized by $G = (S_G, C_G, \Omega_G)$ and $K = (S_K, C_K, \Omega_K)$ with

$$S_G = \begin{bmatrix} S_{11}^G & S_{12}^G \\ S_{21}^G & S_{22}^G \end{bmatrix}, \; S_K = \begin{bmatrix} S_{33}^K & S_{34}^K \\ S_{43}^K & S_{44}^K \end{bmatrix}.$$

We establish the closed loop model by setting $b_{\text{out},2} = b_{\text{in},3}$ and $b_{\text{out},3} = b_{\text{in},2}$ We assume that

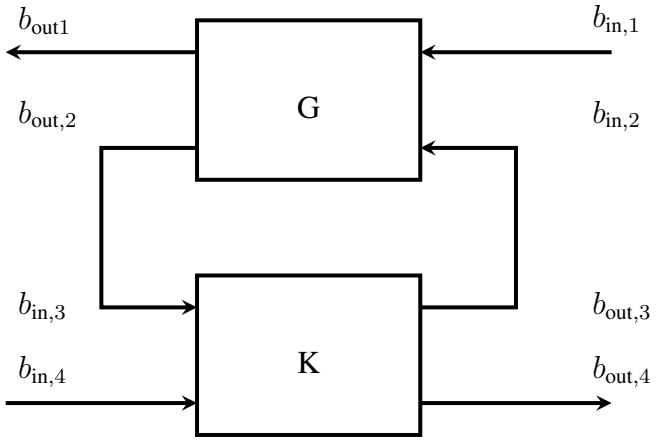

Figure 6.3.: The Set-up for the Redheffer Star Product. Two blocks with two inputs and outputs in an feedback arrangement.

the system observables of the two blocks live on distinct Hilbert spaces $\mathcal{H}_G$ and $\mathcal{H}_K$, that is, that the observables of block G and K commute.

One can again archive a partition of the system matrices with respect to the internal and external channels by

$$S = \begin{bmatrix} S_{ee} & S_{ei} \\ S_{ie} & S_{ii} \end{bmatrix}$$

with block components

$$\begin{aligned} S_{ee} &= \begin{bmatrix} S_{11}^G & 0 \\ 0 & S_{44}^K \end{bmatrix}, \; S_{ei} = \begin{bmatrix} S_{12}^G & 0 \\ 0 & S_{43}^K \end{bmatrix} \\ S_{ie} &= \begin{bmatrix} S_{21}^G & 0 \\ 0 & S_{34}^K \end{bmatrix}, \; S_{ee} = \begin{bmatrix} S_{22}^G & 0 \\ 0 & S_{33}^K \end{bmatrix}. \end{aligned}$$

The system model parameter after eliminating the internal channels can now be computed with,

$$S_\star = S_{ee} + S_{ei} \left(I - S_{ii}\right)^{-1} S_{ie} = \begin{bmatrix} S_{11}^\star & S_{14}^\star \\ S_{41}^\star & S_{44}^\star \end{bmatrix}$$

with block entries

$$\begin{aligned} S_{11}^{\star} &= S_{11}^{G} + S_{12}^{G} S_{33}^{K} \left(1 - S_{22}^{G} S_{33}^{K}\right)^{-1} S_{21}^{G}, \\ S_{14}^{\star} &= S_{12}^{G} \left(1 - S_{22}^{G} S_{33}^{K}\right)^{-1} S_{34}^{K}, \\ S_{41}^{\star} &= S_{43}^{K} \left(1 - S_{22}^{G} S_{33}^{K}\right)^{-1} S_{21}^{G}, \\ S_{41}^{\star} &= S_{44}^{K} + S_{43}^{K} \left(1 - S_{22}^{G} S_{33}^{K}\right)^{-1} S_{34}^{K}. \end{aligned}$$

The partitioned coupling vectors and the adjacency matrix are

$$L_e = \begin{bmatrix} L_1^G \\ L_4^K \end{bmatrix}, \; L_i = \begin{bmatrix} L_2^G \\ L_3^K \end{bmatrix}, \eta = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

We compute the coupling vectors after the elimination of the internal channels with

$$L_{\star} = L_e + S_{ei} \left(1 - S_{ii}\right)^{-1} L_i = \begin{bmatrix} L_i^{\star} \\ L_e^{\star} \end{bmatrix}$$

with entries

$$\begin{aligned} L_1^{\star} &= \left(C_1^G + S_{12}^G S_{33}^K \left(I - S_{22}^G S_{33}^K\right)^{-1} C_2^G\right) a_G + S_{12}^G \left(I - S_{22}^G S_{33}^K\right)^{-1} C_3^K a_K \\ L_2^{\star} &= S_{43}^K \left(I - S_{22}^G S_{33}^K\right)^{-1} a_G + \left(C_4^K + S_{43}^K S_{22}^G \left(I - S_{22}^G S_{33}^K\right)^{-1} C_3^K\right) a_K \end{aligned}$$

if we denote with a_G, and a_K denote the system mode annihilators of the systems G and K. The Hamiltonian of the system $H_{\star}$ is given by

$$H_{\star} = a_G^{\dagger} \left(\Omega_G + \Lambda_G\right) a_G + a_K^{\dagger} \left(\Omega_K + \Lambda_K\right) a_K + a_K^{\dagger} \Lambda_{KG} a_G,$$

with

$$\begin{aligned}
\Lambda_G &= \mathrm{Im}\begin{bmatrix} C_2^{G\dagger}\left(I - S_{22}^G S_{33}^K\right)^{-1} C_2^G \\ C_1^{G\dagger} S_{12}^G \left(I - S_{33}^K S_{22}^G\right)^{-1} S_{33}^K C_2^G \end{bmatrix}, \\
\Lambda_K &= \mathrm{Im}\begin{bmatrix} C_3^{K\dagger}\left(I - S_{33}^K S_{22}^G\right)^{-1} C_3^K \\ C_4^{K\dagger} S_{43}^K \left(I - S_{22}^G S_{33}^K\right)^{-1} S_{22}^G C_3^K \end{bmatrix}, \\
\Lambda_{GK} &= \mathrm{Im}\begin{bmatrix} C_3^{K\dagger}\left(I - S_{33}^K S_{22}^G\right) S_{33}^B C_2^G \\ C_4^{K\dagger} S_{43}^K \left(I - S_{22}^G S_{33}^K\right)^{-1} C_2^G \\ C_2^{G\dagger}\left(I - S_{22}^G S_{33}^K\right)^{-1} S_{22}^G C_3^K \\ -C_1^{A\dagger} S_{12}^G \left(I - S_{33}^K S_{22}^G\right)^{-1} C_3^K \end{bmatrix}.
\end{aligned}$$

6.3. Squeezing Components

So far we considered a class of linear system with coupling of the form $L_j = C_j a_j$. The framework of linear systems can be extended to the case of a coupling not only to the annihilators, but also to the creators, that is a coupling of the form $L_j = C_j^- a_j + C_j^+ a_j^\dagger$, i.e. the parameter for the stochastic Schrödinger equation are of the following structure for some scalars C^-, C^+, ω^- and ω^+

$$H = \sum_{\alpha,\beta=1}^{m} \left(a_\alpha^* \omega_{\alpha\beta}^- a_\beta + \tfrac{1}{2} a_\alpha^* a_\beta^* \omega_{\alpha\beta}^+ + \tfrac{1}{2} a_\alpha a_\beta \omega_{\alpha\beta}^{+*}\right) \tag{6.3.0.1}$$

$$L_j = \sum_{\alpha=1}^{m} \left(C_{j\alpha}^- a_\alpha + C_{j\alpha}^+ a_\alpha^*\right) \tag{6.3.0.2}$$

and S is taken to be unitary where the S_{ij}'s are scalars. At this point it is convenient to introduced the doubled up notation. Let $x = [x_1^{\mathrm{T}}, \ldots, x_n^{\mathrm{T}}]^{\mathrm{T}}$ be a vector with operator entries of length n. Denote for some matrix X, $X^\sharp = (X_{jk}^*)$, where $*$ is the complex conjugate or

Hilbert space adjoint. We define the doubled up vector of size $2n$, $\breve{x}$ by setting

$$\breve{x} = [x_1^{\mathrm{T}}, \ldots, x_n^{\mathrm{T}}, x_1^{\sharp\mathrm{T}}, \ldots, x_n^{\sharp\mathrm{T}}]^{\mathrm{T}},$$

and the doubled up matrix

$$\tilde{X} \quad = \quad \Delta(X^-, X^+) = \begin{bmatrix} X^- & X^+ \\ X^{+\sharp} & X^{-\sharp} \end{bmatrix}, \tag{6.3.0.3}$$

such that a relation of the form $b = C^- a + C^- a^\dagger$ can be described by $\breve{b} = \Delta(C^-, C^+)\breve{a} = \tilde{C}\breve{a}$. We define the involution $\flat$ for a $2n \times 2m$ sized doubled up matrices by

$$\tilde{X}^\flat \triangleq J_m X^\dagger J_n$$

where

$$J_n \triangleq \begin{bmatrix} I_n & 0 \\ 0 & -I_n \end{bmatrix}$$

with the $n \times n$ identity matrix I_n.

We can see that we have an effective change in parameterisation from the triple $G = (S, L, H)$ to the triple $G = (\tilde{S}, \tilde{C}, \tilde{\Omega})$ with doubled up matrices

$$\tilde{C} \quad = \quad \Delta(C^-, C^+) = \begin{bmatrix} C^- & C^+ \\ C^{+\sharp} & C^{-\sharp} \end{bmatrix}, \tag{6.3.0.4}$$

$$\tilde{\Omega} \quad = \quad \Delta(\Omega^-, \Omega^+) = \begin{bmatrix} \Omega^- & \Omega^+ \\ \Omega^{+\sharp} & \Omega^{-\sharp} \end{bmatrix}, \tag{6.3.0.5}$$

where $\Omega = (\omega_{ij})$. We can now establish the equations of motion for the system oscillators

$\frac{d}{dt}a = V^*(t)a_jU(t)$ and the input-output relation $b_{\text{out}}(t) = U^*(t)b(t)U(t)$ in doubled up form

$$\frac{d}{dt}\breve{a} = \tilde{A}\breve{a}(t) - \tilde{C}^\flat\tilde{S}\breve{b}(t) \tag{6.3.0.6}$$

$$\breve{b}_{\text{out}} = \tilde{C}\breve{a}(t) + \tilde{S}\breve{b}(t) \tag{6.3.0.7}$$

with $\tilde{A} = \Delta(A_-, A_+)$, $\tilde{C} = \Delta(C_-, C_+)$, $\tilde{S} = \Delta(S)$ and

$$A_\pm = -\tfrac{1}{2}\left(C_-^\dagger C_\mp - C_+^\top C_\pm^\sharp\right) - i\Omega_\pm.$$

We can establish the relations

$$\tilde{A} + \tilde{A}^\dagger = -\tfrac{1}{2}\tilde{C}^\dagger\tilde{C},\ \tfrac{1}{2i}(A_- - A_-^\dagger) = \Omega_-,\ \tfrac{1}{2}(A_+ + A_+^\top) = \Omega_+.$$

6.3.1. Transfer Functions

The transfer function in case of coupling to both the annihilators and creators takes the form

$$b_{\text{out},i}[s] = \Xi_{ij}^-(s)b_{\text{in}}[s] + \Xi_{ij}^+(s)b_{\text{in},j}^*[s].$$

6.3.2. The Series Product

We consider an arrangement of a type as shown in Fig. (6.2), that is two systems with the output of one system feed into the input of a second system. We consider a zero time delay regime, that is, we take the time τ the signal needs to travel from output 1 to input 2 to be $\tau = 0$.

The systems 1 and 2 are parameterised by $G_1 = (S_1, C_1, A_1)$ and $G_2 = (S_2, C_2, A_2)$. We denote the series product of the two systems by $G_2 \triangleleft G_1$. The model after making the

connection is given by $G_{\text{series}} = (\tilde{S}_{\text{series}}, \tilde{L}_{\text{series}}, \tilde{H}_{\text{series}})$ and parameter

$$\begin{aligned}\tilde{S}_{\text{series}} &= \tilde{S}_2\tilde{S}_1,\\ \tilde{L}_{\text{series}} &= \tilde{C}_2 + \tilde{S}_2\tilde{C}_1,\\ \tilde{H}_{\text{series}} &= \tilde{\Omega} + \mathrm{Im}_\flat\, \tilde{C}_2^\flat \tilde{S}_2\tilde{C}_1.\end{aligned}$$

As one can show, the series product leads to an factorisation of the form $\tilde{G}_2 \triangleleft \tilde{G}_1 = \tilde{G}_2 \cdot \tilde{G}_1$, which is the kind of structure expected from classical control theory, only for the case, where both systems have no system modes in common. Otherwise the series product for doubled up systems is non-trivial and given as above.

6.3.3. Feedback Reduction

Consider a set-up of the form Fig. (6.1). We have a 4-port system with an internal loop and some component K within this loop. We are again interested in a zero time delay regime and want to deduce a model for the system after the internal channels have been eliminated, i.e. deduce the input output map for the single input single output system after the internal connections have been made.

We partition the two port system with respect to the internal and external input and output channels, that is,

$$b_{\text{in}} = \begin{bmatrix} b_{\text{in},i} \\ b_{\text{in},e} \end{bmatrix},\ b_{\text{out}} = \begin{bmatrix} b_{\text{out},i} \\ b_{\text{out},e} \end{bmatrix},\ S_\pm = \begin{bmatrix} S_{ii}^\pm & S_{ie}^\pm \\ S_{ei}^\pm & S_{ee}^\pm \end{bmatrix}, C_\pm = \begin{bmatrix} C_e^\pm \\ C_i^\pm \end{bmatrix}.$$

The input output maps for the field channels are given by

$$\breve{b}_{\text{out},j}[s] = \sum_{k=e,i} \hat{G}_{jk}[s]\breve{b}_k[s],$$

where the transfer function of the system is given by

$$\hat{G}(s) = \left[\begin{array}{c|c} \tilde{A} & -[\tilde{C}_e^\flat, \tilde{C}_i^\flat]\tilde{S} \\ \hline \begin{bmatrix} \tilde{C}_e \\ \tilde{C}_i \end{bmatrix} & \tilde{S} \end{array}\right].$$

The closed loop model after making the connections in the instantaneous feedback limit, that is as the time the signal needs to travel along the edge $\tau \to 0$, under the assumption that $(I - \tilde{S}_{ii})$ is invertible is given by:

$$\begin{aligned} \tilde{S}_{\text{red}} &= \tilde{S}_{ee} + \tilde{S}_{ei}(I - \tilde{S}_{ii})^{-1}\tilde{S}_{ie}, \\ \tilde{C}_{\text{red}} &= \tilde{C}_e + \tilde{S}_{ei}(I - \tilde{S}_{ii})^{-1}\tilde{C}_i, \\ \tilde{A}_{\text{red}} &= \tilde{A} - \sum_{j=e,i} \tilde{C}_J^\flat \tilde{S}_{ij}(I - \tilde{S}_{ii})^{-1}\tilde{C}_i. \end{aligned}$$

Part II.

Results

7

Systems in Loop: Squeezing

7.1. Introduction

An early application of feedback to enhance the squeezing of an (infrared) cavity mode was given by Wiseman et al. [64]. Here the mode is coupled to a second harmonic (green) mode which is subjected to a quantum nondemolition measurement. In contrast, we wish to examine the squeezing of the input noise field by a cavity mode acting as an idealized squeezing device. Here the feedback is coherent, rather than measurement-based, and we consider a set up involving a simple beam splitter to introduce the feedback loop. We shall work in the limit of instantaneous feedback throughout. We shall be interested in the class of linear dynamical

systems [67], [22],[45], and indeed will study static components wherein the internal degrees of freedom have been eliminated. The degenerate parametric amplifier (DPA) is a well known non-linear device capable of squeezing input fields [41], [65], [9]. We follow the treatment of Gardiner [19]. For a single quantum input field coupled to a single cavity mode a with coupling strength $\sqrt{\kappa}$ and Hamiltonian

$$H_{\text{DPA}} = \frac{i\epsilon}{2}\left(a^{\dagger 2} - a^2\right).$$

there is an approximate squeezing parameter given by, [19, Section 7.2.9],

$$r_{\text{DPA}} = \ln \frac{\kappa + \epsilon}{\kappa - \epsilon}. \tag{7.1.0.1}$$

Here the amplification is due to the specific choice of the Hamiltonian H_{DPA}. Without feedback, the method of obtaining maximal squeezing for a degenerate parametric amplifier is to try and realize the Hamiltonian for the internal mode with parameter coefficient ε as close to the threshold value $(\varepsilon = \kappa)$ as possible, see [19, Section 10.2]. As originally noted by Yanagisawa and Kimura [67], the value of the reflective damping for an in-loop mode will depend on the reflectivity value α:

$$\kappa(\alpha) = \frac{1-\alpha}{1+\alpha}\kappa$$

Our strategy is to use coherent feedback for a fixed degenerate parametric amplifier (below threshold, and therefore internally stable [45]) and tune the reflectivity of the beam splitter so as to select the degree of squeezing. The degenerate parametric amplifier is an idealized device in which one assumes that κ and ε are large but with fixed ratio. We shall investigate the situation where both these parameters are finite. Also, we introduce additional quantum damping into the model to see the effect of loss.

7.2. The Degenerate Parametric Amplifier

7.2.1. The System

A Degenerate Parametric Amplifier (DPA) is a device with a system Hamiltonian [19] of the form

$$H_{\text{DPA}} = \frac{i\epsilon}{2}\left(a^{\dagger 2} - a^2\right).$$

Let us consider a DPA model with two inputs and outputs, one serving as the signals in- and output, one playing the role of a loss mechanism. The resulting system is given by the triple $G = (S, L, H_{\text{DPA}})$ with

$$S = I_{2\times 2},\ C_- = \begin{bmatrix} \sqrt{\gamma} \\ \sqrt{\kappa} \end{bmatrix}, C_+ = 0, \omega_- = 0, \omega_+ = \frac{\epsilon}{2}$$

with resulting doubled-up matrices (see Equation 6.3.0.3)

$$\tilde{S} = I_{4\times 4}, \tilde{C} = \Delta(C_-, 0), \Omega = \Delta(0, \omega_+).$$

We can compute the transfer function $\Xi_{\text{DPA}}(s)$ of the system using

$$\tilde{A} = -\frac{1}{2}\tilde{C}^\flat\tilde{C} - i\tilde{\Omega} = -\frac{1}{2}\begin{bmatrix} \kappa+\gamma & \epsilon \\ -\epsilon & \kappa+\gamma \end{bmatrix}.$$

such that

$$\begin{aligned} \tilde{\Xi}_{\text{DPA}}[s] &= \left[\begin{array}{c|c} \tilde{A} & \tilde{C} \\ \hline \tilde{S} & -\tilde{C}^\flat\tilde{S} \end{array}\right](s) = \tilde{S} - \tilde{C}\left(sI - \tilde{A}\right)^{-1}\tilde{C}^\flat\tilde{S} \\ &= \begin{bmatrix} \Xi^-[s] & \Xi^+[s] \\ \Xi^+[s] & \Xi^-[s] \end{bmatrix} \end{aligned}$$

with components

$$\Xi^{-}[s] = \frac{1}{P(s)} \begin{bmatrix} \frac{\kappa^2+\gamma^2-\epsilon^2}{4} + \gamma s + s^2 & -\sqrt{\kappa\gamma}\left(s + \frac{\kappa+\gamma}{2}\right) \\ -\sqrt{\kappa\gamma}\left(s + \frac{\kappa+\gamma}{2}\right) & \frac{\kappa^2+\gamma^2-\epsilon^2}{4} + \gamma s + s^2 \end{bmatrix}, \tag{7.2.1.1}$$

$$\Xi^{+}[s] = -\frac{\epsilon}{2P(s)} \begin{bmatrix} \kappa & \sqrt{\kappa\gamma} \\ \sqrt{\kappa\gamma} & \gamma \end{bmatrix}, \tag{7.2.1.2}$$

where the denominator $P(s)$ is given by

$$P(s) = \left(s^2 + \left(\frac{\kappa+\gamma+\epsilon}{2}\right)^2\right)\left(s^2 + \left(\frac{\kappa+\gamma-\epsilon}{2}\right)^2\right).$$

The output of the system is now given by

$$\breve{b}_{\text{out}}[s] = \tilde{\Xi}_{\text{DPA}}[s]\breve{b}_{\text{in}}[s].$$

We note that the resulting system will be Hurwitz stable, i.e. all eigenvalues of $\tilde{A}$ have real part value strictly smaller then 1, if and only if $\kappa + \gamma < \epsilon$.

7.2.2. The Spectrum of the Output

The DPA is a device that is capable of squeezing. It is able to transform a minimum uncertainty state with equal variances in both quadratures into a minimum uncertainty state with an increased variance in one and a decreased variance in the other quadrature. In order to analyse this behaviour in detail we are interested in the spectrum of the output quadratures.

Recall the definition of the transfer function defined in terms of the Laplace transform Eq. (6.2.1.1)

$$b[s] = \int_0^\infty e^{-st} b(t) \mathrm{d}t. \tag{7.2.2.1}$$

The Fourier transform of the fields on the other hand is given by

$$\hat{b}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega t} b(t) \mathrm{d}t. \tag{7.2.2.2}$$

Comparing this with the Laplace transform, we can see that the Laplace transform corresponds to the transform of the future fields, whereas the Fourier transform realises the transform of the past and future fields. The transfer function (Laplace transform) is for linear networks explicitly given in terms system matrices or system parameter, see Chapter 6.2.1. We are therefore interested in formulating the Fourier transform in terms of the transfer function of the system of interest.

We complete the Laplace transform by added the transformed past fields, i.e.

$$\hat{b}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega t} b(t) \mathrm{d} = \frac{1}{\sqrt{2\pi}} b[0^+ i\omega] + \frac{1}{\sqrt{2\pi}} c[0^- - i\omega], \tag{7.2.2.3}$$

where the transform of the past fields is defined by

$$c[s] \triangleq \int_{-\infty}^{0} e^{-st} b(t) \mathrm{d}t. \tag{7.2.2.4}$$

The canonical commutation relations in terms of the transformed fields read as

$$\left[\hat{b}_j^*(\omega), \hat{b}_k(\omega'\right] = \delta_{jk} \delta(\omega - \omega'). \tag{7.2.2.5}$$

One can evaluate the expectation of $b^*[0^+ - i\omega] b^*[0^+ - i\omega]$ and $c^*[0^+ - i\omega] c^*[0^+ - i\omega]$ in the vaccum state using Eq. (7.2.2.1), Eq. (7.2.2.4) and the canonical commutation relation of the input fields to obtain

$$\begin{aligned} \langle b_{\text{in},j}[0^+ - i\omega] b^*_{\text{in},k}[0^+ - i\omega'] \rangle &= \delta_{jk} \zeta_+(\omega + \omega'), \\ \langle c_{\text{in},j}[0^- - i\omega] c^*_{\text{in},k}[0^- - i\omega'] \rangle &= \delta_{jk} \zeta_-(\omega + \omega'). \end{aligned} \tag{7.2.2.6}$$

Here $\zeta_{\pm}$ are the Heitler functions defined by

$$\zeta_{+}(\omega) = \int_{0}^{\infty} e^{i\omega t}\mathrm{d}t,\ \zeta_{-}(\omega) = \int_{-\infty} e^{i\omega t}\mathrm{d}t,$$

or

$$\zeta_{\pm}(\omega) = \pi\delta \pm \mathrm{PV}\frac{1}{\omega}.$$

Since in practical applications we deal witht he combination of past plus future fields, we only accounter combinations of the form $\zeta_{-} + \zeta_{+} = 2\pi\delta$. As a direct result we see that

$$\langle \hat{b}_{\mathrm{in},j}(\omega)\hat{b}^{*}_{\mathrm{in},k}(\omega')\rangle = \delta_{jk}\delta(\omega - \omega'),$$

since from Eq. (7.2.2.3) and Eq. (7.2.2.6) we see that we have the sum

$$\frac{1}{2\pi}\langle b_{\mathrm{in},j}[0^{+} - i\omega]b^{*}_{\mathrm{in},k}[0^{+} - i\omega']\rangle + \frac{1}{2\pi}\langle c_{\mathrm{in},j}[0^{-} - i\omega]c^{*}_{\mathrm{in},k}[0^{-} - i\omega']\rangle.$$

Following the notation for the transfer function of the (future) fields, we can denote an input-output-map for the past fields

$$c_{\mathrm{out},j}[s] = \Xi^{-}_{jk}(s)c_{\mathrm{in},k}[s] + \Xi^{+}_{jk}(s)c^{*}_{\mathrm{in},k}[s].$$

We can clean up the sign change by introducing the following matrices

$$\mathcal{S}^{-}_{jk}(\omega) = \Xi^{-}_{jk}(-i\omega),\ \mathcal{S}^{+}_{jk}(\omega) = \Xi^{+}_{jk}(-i\omega),$$

such that we result with an input-output description of the form

$$\hat{b}_{\mathrm{out},j}(\omega) = \mathcal{S}^{-}_{jk}(\omega)\hat{b}_{\mathrm{in},k}(\omega) + \mathcal{S}^{+}_{jk}(\omega)\hat{b}_{\mathrm{in},k}(-\omega)^{*}.$$

We may now determine the correlation functions of the output from the transfer functions as defined above (in the vacuum state),

$$\begin{aligned}\langle \hat{b}^*_{\text{out},j}(\omega)\hat{b}_{\text{out},j}(\omega')\rangle &= \mathcal{N}_{jk}(\omega)\delta(\omega-\omega'),\\ \langle \hat{b}_{\text{out},j}(\omega)\hat{b}_{\text{out},j}(\omega')\rangle &= \mathcal{M}_{jk}(\omega)\delta(\omega+\omega').\end{aligned} \tag{7.2.2.7}$$

where

$$\mathcal{N}_{jk}(\omega)=\mathcal{S}^+_{jk}(\omega)^*\mathcal{S}^+_{jk}(\omega),\ \mathcal{M}_{jk}(\omega)=\mathcal{S}^-_{jk}(\omega)\mathcal{S}^+_{jk}(-\omega).$$

We also remark the following identities

$$\mathcal{N}_{jk}(\omega)^*=\mathcal{N}_{jk}(\omega),\ \mathcal{M}_{jk}(\omega)^*=\mathcal{M}_{jk}(-\omega).$$

Following the introduction of the doubled-up notation for squeezing components in quantum feedback networks we denote

$$\tilde{\Xi}(\omega)=\Delta(S_-(\omega),S_+(\omega)).$$

This defined a Bogoliubov matrix for each real ω where it is well defined. It particularly ensures that the transformation from the inputs to the outputs preserves the canonical commutation relation.

7.2.3. Power spectrum density

We consider a generalised quadrature of the form

$$q_{\text{out},j}(t,\theta)=e^{i\theta}b_{\text{out},j}(t)+e^{-i\theta}b^\dagger_{\text{out},j}(t)$$

for a fixed phase $\theta \in [0, 2\pi)$. If we choose $\theta = 0$ and $\theta = \frac{\pi}{2}$ we get the usual quadratures

$$\begin{aligned} q^x_{\text{out},j}(t,0) &= b_{\text{out},j}(t) + b^\dagger_{\text{out},j}(t), \\ q^y_{\text{out},j}(t,\frac{\pi}{2}) &= -\frac{1}{i}\left(b_{\text{out},j}(t) - b^\dagger_{\text{out},j}(t)\right). \end{aligned}$$

The integrated processes are given by

$$\mathrm{d}Q_{\text{out},j}(t,\theta) = \int_0^t q_{\text{out},j}(t',\theta)\mathrm{d}t',$$

and the Itō increments read as

$$\mathrm{d}Q_{\text{out},j}(t,\theta)\mathrm{d}Q_{\text{out},k}(t,\theta) = \delta_{jk}\mathrm{d}t. \tag{7.2.3.1}$$

Following Barchielli and Gregoratti [2], we set

$$\mathcal{P}_{jk}(\omega,\theta,T) = \frac{1}{T}\left\langle \int_0^T e^{i\omega t_1} q_{\text{out},j}(t_1,\theta)\mathrm{d}t_1 \int_0^T e^{-i\omega t_2} q_{\text{out},k}(t_2,\theta)\mathrm{d}t_2 \right\rangle,$$

$$\mathcal{P}^{\text{el}}_{jk}(\omega,\theta,T) = \frac{1}{T}\left\langle \int_0^T e^{i\omega t_1} q_{\text{out},j}(t_1,\theta)\mathrm{d}t_1 \right\rangle \left\langle \int_0^T e^{-i\omega t_2} q_{\text{out},k}(t_2,\theta)\mathrm{d}t_2 \right\rangle,$$

and

$$\mathcal{P}^{\text{inel}}_{jk}(\omega,\theta,T) = \mathcal{P}_{jk}(\omega,\theta,T) - \mathcal{P}^{\text{el}}_{jk}(\omega,\theta,T).$$

This corresponds to the spectrum of the output over a finite time horizon. Whenever the limit exists, we define the power spectral density matrix by

$$\lim_{t\to\infty} \mathcal{P}_{jk}(\omega,\theta,T) = \mathcal{P}_{kl}(\omega,\theta).$$

Similarly we define $\mathcal{P}^{\text{el}}_{jk}(\omega,\theta)$ and $\mathcal{P}^{\text{inel}}_{jk}(\omega,\theta)$.

Now, Eq. (7.2.3.1) implies

$$\mathcal{P}^{\text{inel}}_{jk}(\omega,\theta)=\delta_{jk}.$$

The Fourier transform of the generalised output quadratures are given by

$$\hat{q}_{\text{out},j}=e^{i\theta}\hat{b}_{\text{out},j}(\omega)+e^{-i\theta}\hat{b}_{\text{out},j}(-\omega)^*,$$

and it is readily verified that

$$\langle\hat{q}_{\text{out},j}(\omega,\theta)\hat{q}_{\text{out},k}(\omega',\theta)\rangle=\mathcal{P}_{jk}(\omega,\theta)\delta(\omega+\omega').$$

We can state the explicit expression by using Eq. (7.2.2.7) and obtain

$$\mathcal{P}_{11}(\omega,\theta)=1+\mathcal{N}_{11}(-\omega)+\mathcal{N}_{11}(\omega)+e^{2i\theta}\mathcal{M}_{11}(\omega)+e^{-2i\theta}\mathcal{M}_{11}(-\omega)^*$$

where, after summing over repeated indices and using the system's transfer functions from the previous section we have that

$$\begin{aligned}\mathcal{N}_{jk}(\omega) &= \Xi^+_{jl}[-i\omega]^*\Xi^+_{kl}[-i\omega]=\frac{\epsilon^2\left(\kappa^2+\kappa\gamma\right)}{4|P(-i\omega)|^2},\\ \mathcal{M}_{jk}(\omega)^* &= \Xi^-_{jl}(-i\omega)\Xi^+_{kl}(i\omega)=\frac{\kappa\gamma\left(\omega^2+\left(\frac{\kappa+\gamma}{2}\right)^2+\left(\frac{\epsilon}{2}\right)^2\right)}{2|P(-i\omega)|^2}.\end{aligned}$$

In the lossless case, i.e. $\gamma=0$ we reduce the DPA to a single input single output component and we have identities $\left|\Xi^-_{11}(-\omega)\right|^2-\left|\Xi^+_{11}(-\omega)\right|^2=1$ and $|\mathcal{M}_{11}(\omega)|^2=(\mathcal{N}_{11}(\omega)+1)\mathcal{N}_{11}(\omega)$ and compute spectrum of the output with

$$\mathcal{P}_{11}(\omega,\theta)=\frac{\left(\omega^2+\left(\frac{\kappa^2+\epsilon^2}{4}\right)\right)^2+\frac{\kappa^2\epsilon^2}{4}+\epsilon\kappa\left(\omega^2+\left(\frac{\kappa^2+\epsilon^2}{4}\right)\right)\cos 2\theta}{|P(-i\omega)|^2}. \tag{7.2.3.2}$$

We see that the spectral power of the output as a function of the phase θ takes a maximum for

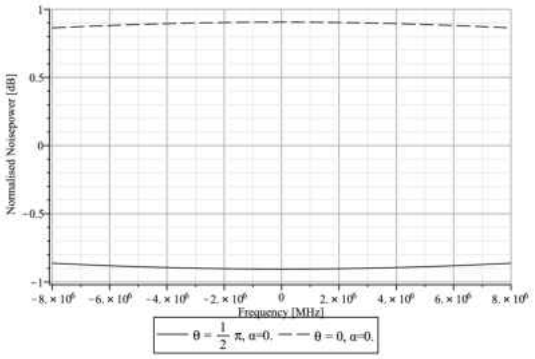

Figure 7.1.: Spectrum of the degenerate parametric amplifier, lossless case, open loop with $\epsilon = 7.35e6$ and $\kappa = 7.2e7$. The solid and dashed line correspond to the squeezing and anti-squeezing respectively.

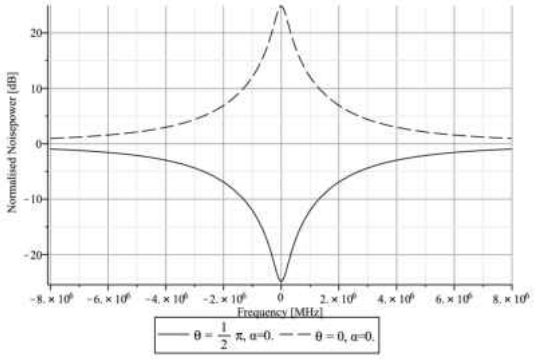

Figure 7.2.: Spectrum of the degenerate parametric amplifier, lossless case, open loop with $\epsilon = 7.35e6$ and $\kappa = 4.2e6$. The solid and dashed line correspond to the squeezing and anti-squeezing respectively.

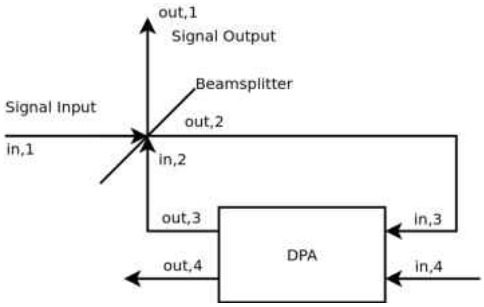

Figure 7.3.: Degenerate Parametric Amplifier in a closed loop arrangement. The additional input and output for the DPA serves as a loss mechanism.

$\theta = 0$ and a minimum for $\theta = \frac{\pi}{2}$. In this two cases we are measureing the noise power of the anti-squeezed and squeezed quadrature respectively.

Fig. (7.1) and Fig. (7.2) shows the power spectrum for the open loop, lossless DPA. The higher coupling of the field to the DPA via coupling constant κ results in an effective squeezing over a wider frequency bandwidth.

7.3. The Degenerate Parametric Amplifier in Loop

Next we consider a set-up as in Fig. (7.3) where the DPA is in a feedback arrangement. The beamsplitter can be modelled as a four port input output system with parameter $G_{\text{BS}} = (H = 0, L = 0, S_{\text{BS}})$ where

$$S_{\text{BS}} = \begin{bmatrix} \alpha & \beta \\ \beta & -\alpha \end{bmatrix}$$

with the beamsplitters reflectivity α and reflectivity β. Since S_{BS} has to be unitary, i.e. $\det S_{\text{BS}} = \pm 1$ we have constrained $|\alpha|^2 + |\beta|^2 = 1$. The input-output relation for the beamsplitter is given by

$$\begin{bmatrix} b_{\text{out},1}(t) \\ b_{\text{out},2}(t) \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \beta & -\alpha \end{bmatrix} \begin{bmatrix} b_{\text{in},1}(t) \\ b_{\text{in},2}(t) \end{bmatrix}.$$

We can use the feedback reduction formula provided in Chapter 6.2.2, i.e.

$$\begin{aligned} S_{\text{red}} &= S_{\text{e,e}} + S_{\text{e,i}} \left(\eta - S_{\text{i,i}}\right)^{-1} S_{\text{i,e}} \\ L_{\text{red}} &= L_{\text{e,e}} + S_{\text{e,i}} \left(\eta - S_{\text{i,i}}\right)^{-1} L_{\text{i,e}} \\ H_{\text{red}} &= H + \sum_{j=i,e} \Im[L_j^\dagger S_{\text{j,i}} (\eta - S_{\text{i,i}})^{-1} L_i] \end{aligned}$$

to deduce the closed loop model. With respect to the lables (in1,in2,in3,in4) and (out1,out2,out3,out4) we have the scattering matrix of the concatenated system

$$S = \begin{bmatrix} \alpha & \beta & 0 & 0 \\ \beta & -\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and the adjacency and submatrices

$$\eta = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ S_{\text{ii}} = \begin{bmatrix} -\alpha & 0 \\ 0 & 1 \end{bmatrix},\ S_{\text{ie}} = S_{\text{ie}} = \begin{bmatrix} \beta & 0 \\ 0 & 0 \end{bmatrix},$$
$$S_{\text{ee}} = \begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix},\ L_{\text{i}} = \begin{bmatrix} 0 \\ \sqrt{\kappa} \end{bmatrix} a,\ L_{\text{e}} = \begin{bmatrix} \sqrt{\gamma} \\ 0 \end{bmatrix} a.$$

The closed loop system in the instantaneous feedback limit is given by

$$L_{\text{red}} = \begin{bmatrix} \sqrt{\frac{1-\alpha}{1+\alpha}\kappa} \\ \sqrt{\gamma} \end{bmatrix} a,\ H_{\text{red}} = H,\ S_{\text{red}} = I$$

with the new closed loop transfer functions as above but with the coupling constant being replaced by $\kappa \to \frac{1-\alpha}{1+\alpha}\kappa$. We can compute the frequency dependend squeezing parameter Eq. (7.1.0.1) for the closed loop case by defining $|S_-(\omega)| = \cosh r_{\text{DPA}}(\omega, \alpha)$ such that $r_{\text{DPA}}(\omega, \alpha) = \frac{1}{2} \ln |S_-(\omega)| - |S_+(\omega)|$, i.e.

$$r_{\text{DPA}}(\omega, \alpha) = \frac{1}{2} \ln \frac{\omega^2 + \left(\frac{\kappa(\alpha)+\epsilon}{2}\right)^2}{\omega^2 + \left(\frac{\kappa(\alpha)-\epsilon}{2}\right)^2}.$$

7.3.1. Spectrum of the Output

We can now analyse the closed loop set-up and compare the spectral squeezing of the outputs for the open loop and closed loop case.

The power spectrum of the output is similar as for the open loop case as above but with κ being substituted by $\kappa(\alpha)$, i.e.

$$\mathcal{P}_{11}(\omega, \theta) = \frac{\left(\omega^2 + \left(\frac{\kappa(\alpha)^2+\epsilon^2}{4}\right)\right)^2 + \frac{\kappa(\alpha)^2\epsilon^2}{4} + \epsilon\kappa(\alpha)\left(\omega^2 + \left(\frac{\kappa(\alpha)^2+\epsilon^2}{4}\right)\right)\cos 2\theta}{|P(-i\omega, \alpha)|^2}, \qquad (7.3.1.1)$$

with $\kappa(\alpha) = \frac{1-\alpha}{1+\alpha}\kappa$. As we can see in Fig. 7.4 we find that, by adjusting the beamsplitters

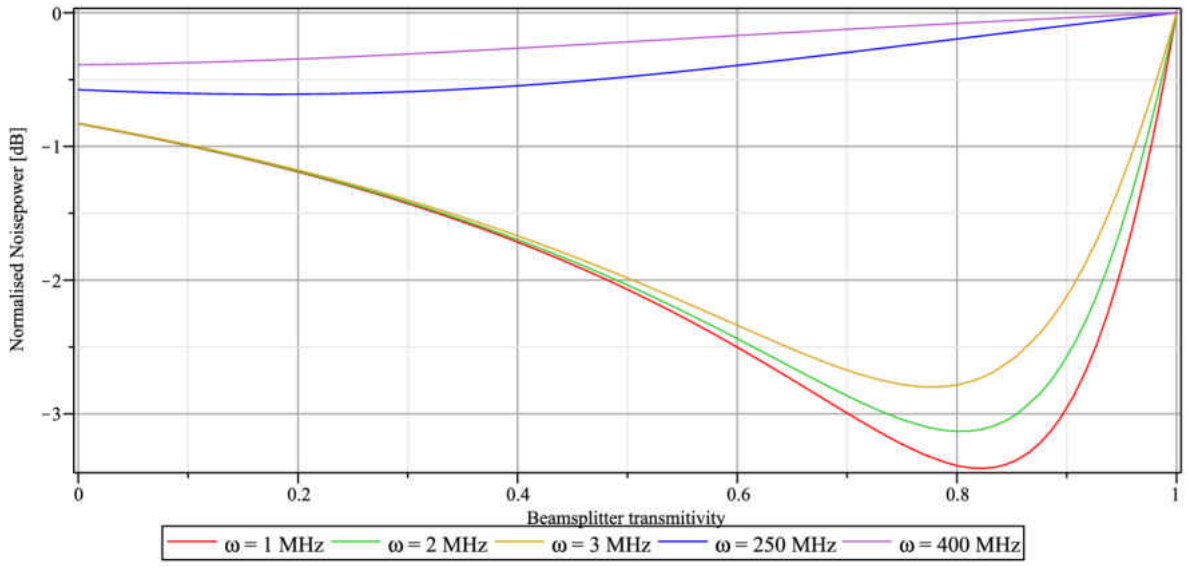

Figure 7.4.: Normalised Noisepower of the output for the closed loop feedback arrangement Fig. 7.3 over the reflectivity α of the beamsplitter for various frequencies.

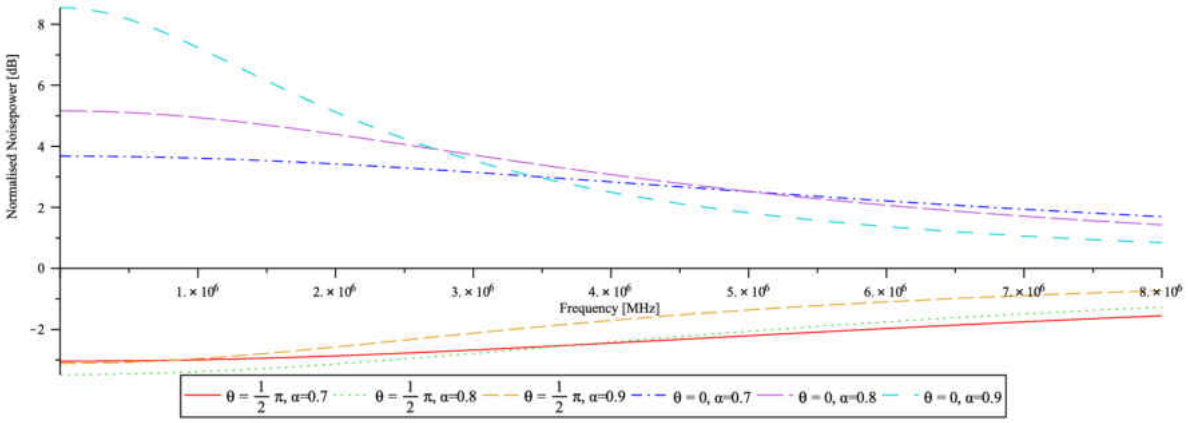

Figure 7.5.: Squeezing and anti-squeezing normalised noisepower for the closed loop set-up Eq. (7.3.1.1) as in [33] for different values for the beamsplitters reflectivity.

reflectivity, the squeezing can be significantly enhanced for lower frequencies compared to the open loop case $\alpha = 0$ by choosing an appropriate reflectivity. For high frequencies (lines for 250Mhz and 400MHz) the output noisepower ceases to have a minimum within the intervall $0 < \alpha < 1$. Hence we will not be able to enhance the squeezing of the output by adjusting the beamsplitters reflectivity.

The squeezing parameter for the closed loop case (see above) is given by

$$r_{\text{DPA}}(\omega, \alpha) = \frac{1}{2} \ln \frac{\omega^2 + \left(\frac{\kappa(\alpha)+\epsilon}{2}\right)^2}{\omega^2 + \left(\frac{\kappa(\alpha)-\epsilon}{2}\right)^2}.$$

In order to determine the value for α for which we archive maximum squeezing we solve $\frac{\mathrm{d}r_{\text{DPA}}(\omega,\alpha)}{\mathrm{d}\alpha} = 0$ for α and find that the maximum of $r_{\text{DPA}}(\omega, \alpha)$ is achieved at

$$\alpha_{\text{max}}(\omega) = \frac{\kappa^2 + 4\omega^2 + \epsilon^2 - 2\kappa\sqrt{4\omega^2 + \epsilon^2}}{\kappa^2 - 4\omega^2 - \epsilon^2}.$$

As we have seen above $r_{\text{DPA}}(\omega, \alpha)$ will not always have an maximum for $0 < \alpha < 1$ at high frequencies. We are therefore interested in the maximum frequency for which such a maximum for $0 < \alpha < 1$ exists, i.e. for up to which frequency the closed loop setup will be able to enhance the squeezing of the output by selection of a suitable reflectivity α. We notice that $\alpha_{\text{max}} < 0$ for large frequencies and want to find the maximal frequency ω_{max} for which $\alpha_{\text{max}}(\omega), \omega < \omega_{\text{max}}$ is restricted to $0 \leq \alpha_{\text{max}} \leq 1$. This will be satisfied for

$$\omega_{\text{max}} = \frac{1}{2}\sqrt{\kappa^2 - \epsilon^2}.$$

7.4. A static model

Next we consider the DPA in a limit regime where we rescale the system parameter κ, ϵ, γ by some parameter k, i.e. we result in a sequence of models parameterised by the triple

$(\kappa_k, \epsilon_k, \gamma_k)$ and let $k \to \infty$.

Recall, the entrys of the doubled up transfer function were given by

$$\begin{aligned}
\Xi^{-}[s] &= \frac{1}{P(s)} \begin{bmatrix} \frac{\kappa^2+\gamma^2-\epsilon^2}{4} + \gamma s + s^2 & -\sqrt{\kappa\gamma}\left(s + \frac{\kappa+\gamma}{2}\right) \\ -\sqrt{\kappa\gamma}\left(s + \frac{\kappa+\gamma}{2}\right) & \frac{\kappa^2+\gamma^2-\epsilon^2}{4} + \gamma s + s^2 \end{bmatrix}, \\
\Xi^{+}[s] &= -\frac{\epsilon}{2P(s)} \begin{bmatrix} \kappa & \sqrt{\kappa\gamma} \\ \sqrt{\kappa\gamma} & \kappa \end{bmatrix}
\end{aligned}$$

and as we can see we have that $\Xi^{\pm}_{(k)}[s] = \Xi^{\pm}[\frac{s}{k}]$ such that the strong coupling limit is equivalent to a low frequency limit.

After the limit $\lim_{k\to\infty}$ we have the following limit model

$$\begin{aligned}
S_{-} &= \lim_{k\to\infty} \Xi^{-}_{(k)}[s] = \frac{1}{(\kappa+\gamma)^2 - \epsilon^2} \begin{bmatrix} \gamma^2 - \kappa^2 - \epsilon^2 & -2\sqrt{\kappa\gamma}\,(\kappa+\gamma) \\ -2\sqrt{\kappa\gamma}\,(\kappa+\gamma) & \kappa^2 - \gamma^2 - \epsilon^2 \end{bmatrix} \\
S_{+} &= \lim_{k\to\infty} \Xi^{+}_{(k)}[s] = \frac{-2\epsilon}{(\kappa+\gamma)^2 - \epsilon^2} \begin{bmatrix} \kappa & \sqrt{\kappa\gamma} \\ \sqrt{\kappa\gamma} & \gamma \end{bmatrix}.
\end{aligned}$$

This limit is again a squeezing device since

$$(\mathcal{M}_{jk})(\omega) = (\Xi^{-}_{jl}(-i\omega)\Xi^{+}_{kl}(i\omega)) = \frac{2\epsilon((\kappa+\gamma)^2 + \epsilon^2)}{(\kappa+\gamma)^2 - \epsilon^2} \begin{bmatrix} \kappa & \sqrt{\kappa\gamma} \\ \sqrt{\kappa\gamma} & \gamma \end{bmatrix}$$

For the single input single output case ($\gamma = 0$) we can again compute the squeezing function r for the static limit model

$$r_{\text{DPA}} = \frac{1}{2} \ln \left[\frac{\kappa + \epsilon}{\kappa - \epsilon} \right].$$

This coincides with the squeezing parameter given in [19] and is equivalent to the squeezing parameter as before for the case $\omega = 0$.

7.5. Squeezing Enhancement in the Static Limit

In the previous section we found that in the closed loop situation the spectral squeezing of the output is a function of the beam splitter's reflectivity α. For the lossless DPA in loop we find that the modified squeezing parameter is given by

$$r_{\text{DPA}}(\alpha) = \frac{1}{2} \ln \left[\frac{\kappa(\alpha) + \epsilon}{\kappa(\alpha) - \epsilon} \right],$$

where $\kappa(\alpha) = \frac{1-\alpha}{1+\alpha}\kappa$ as before. We observe that the squeezing parameter diverges if $\kappa(\alpha) = \epsilon$ which is the case for $\alpha_{\text{crit}} = \frac{\kappa-\epsilon}{\kappa+\epsilon}$. The dynamic model takes for this choice of α the value

$$r(\omega, \alpha_{\text{crit}}) = \frac{1}{2} \left[\frac{\omega^2 + \epsilon^2}{\omega^2} \right]$$

which again posseses a singularity at $\omega = 0$ for $\alpha = \alpha_{\text{crit}}$. Note that the open loop system is stable for $\kappa > \epsilon$, which is in the closed loop set up modified to $\kappa(\alpha) > \epsilon$. For α_{crit} we have $\kappa(\alpha_{\text{crit}}) = \epsilon$ and therefore see that for the critial squeezing the system becomes unstable.

So far we investigated the lossless case $\gamma = 0$ which allowed for the definition of the squeezing function. In the lossy case the stability condition modifies to $\kappa(\alpha) + \gamma > \epsilon$. Under the assumation of Hurwitz stability of the system and that the dissipation is smaller then the pumping ($\gamma < \epsilon$) we find that for $\alpha \in (\alpha_{\text{crit}})$ the system is again stable and that α_{crit} is given by

$$\alpha_{\text{crit}} = \frac{\kappa - \epsilon + \gamma}{\kappa + \epsilon - \gamma}.$$

The critical value of $\kappa(\alpha)$ changes to $\kappa(\alpha_{\text{crit}}) = \epsilon - \gamma$ and the 1,1 entries of the matrices

$\mathcal{N}(\omega,\alpha)$ and $\mathcal{M}(\omega,\alpha)$ are given by

$$\begin{aligned}\mathcal{N}_{11}(\omega,\alpha) &= \frac{\epsilon^3[\epsilon-\gamma]}{4[\omega^2+\epsilon^2]\omega^2}\\ \mathcal{M}_{11}(\omega,\alpha) &= \frac{\epsilon[\epsilon-\gamma][\omega^2-(\frac{\epsilon}{2})^2]}{2[\omega^2+\epsilon^2]\omega^2}.\end{aligned}$$

Both expressions diverge in the limit $k\to\infty$ for a similar limiting procedure as before when replacing the triple (κ,γ,ϵ) by $(k\kappa,k\gamma,k\epsilon)$.

7.6. A Model with Time Delays

We are now going to present an approach for a model for an in-loop DPA with time delays, that is an arrangement of the form Fig. 7.6. The beam splitter has the input-output relation

$$\begin{aligned}\breve{u} &= \alpha\breve{d}+\beta\breve{b}_{\text{in}}\\ \breve{b}_{\text{out}} &= \beta\breve{d}-\alpha\breve{b}_{\text{in}}.\end{aligned}$$

We denote with the transfer functions $\tilde{G}(s)$ and $\tilde{H}(s)$ the input output maps $\tilde{G}:\breve{u}\to\breve{d}$ and $\tilde{H}:\breve{c}_{\text{in}}\to\breve{d}$. The effective time delay in frequency domain is modelled by multiplication with $e^{s\tau}$. The input $\breve{d}$ is given in terms of the time delays and the transfer functions $\tilde{G}$, $\tilde{H}$ by

$$\begin{aligned}\breve{d} &= e^{s\tau_a}\tilde{G}(s)e^{s\tau_b}\breve{u}+e^{s\tau_a}\tilde{H}(s)\tilde{c}_{\text{in}}\\ &= e^{s(\tau_a+\tau_b)}\tilde{G}(s)\left(\alpha\breve{d}+\beta\breve{b}_{\text{in}}\right)+e^{s\tau_a}\tilde{H}(s)\breve{c}_{\text{in}}\\ &= (1-\alpha e^{s(\tau_a+\tau_b)}\tilde{G})^{-1}\left[\beta e^{s(\tau_a+\tau_b)}\tilde{G}(s)\breve{b}_{\text{in}}+e^{s\tau_b}\tilde{H}(s)\breve{c}_{\text{in}}\right]\end{aligned}$$

such that

$$\breve{b}_{\text{out}}=\left[-\alpha+\beta^2\left(1-\alpha e^{s(\tau_a+\tau_b)}\tilde{G}(s)\right)^{-1}e^{s(\tau_a+\tau_b)}\tilde{G}(s)\right]\breve{b}_{\text{in}}+\beta\left(1-\alpha e^{s(\tau_a+\tau_b)}\right)^{-1}e^{s\tau_b}\tilde{H}(s)\breve{c}_{\text{in}}.$$

With the transfer functions $\tilde{G}(s)$ and $\tilde{H}(s)$ as before

$$\begin{aligned}
\tilde{G}(s) &= \frac{1}{P(s)}\begin{bmatrix} s^2+\gamma s+\frac{\gamma^2-\kappa^2-\epsilon^2}{4} & -\frac{\epsilon\kappa}{2} \\ -\frac{\epsilon\kappa}{2} & s^2+\gamma s+\frac{\gamma^2-\kappa^2-\epsilon^2}{4} \end{bmatrix}, \\
\tilde{H}(s) &= \frac{\sqrt{\gamma\kappa}}{P(s)}\begin{bmatrix} s+\frac{\kappa+\gamma}{2} & \frac{\epsilon}{2} \\ \frac{\epsilon}{2} & s+\frac{\kappa+\gamma}{2} \end{bmatrix}, \\
P(s) &= \left(s+\frac{\kappa+\gamma-\epsilon}{2}\right)\left(s+\frac{\kappa+\gamma+\epsilon}{2}\right).
\end{aligned}$$

The new transfer functions mapping $\tilde{G}'(s) : \breve{b}_{\text{in}} \to \breve{b}_{\text{out}}$ and $\tilde{H}'(s) : \breve{c}_{\text{in}} \to \breve{b}_{\text{out}}$ are then given by

$$\begin{aligned}
\tilde{G}'(s) &= \frac{1}{P'(s)}\begin{bmatrix} A(s) & B(s) \\ B(s) & A(s) \end{bmatrix}, \\
\tilde{H}'(s) &= e^{s\tau_b}\frac{\sqrt{\gamma\kappa}}{P'(s)}\begin{bmatrix} \frac{\sqrt{1-\alpha^2}}{1-\alpha\chi}s+\frac{\gamma}{2}\frac{\sqrt{1-\alpha^2}}{1-\alpha\chi}+\frac{\kappa}{2}\frac{\sqrt{1-\alpha^2}(1+\alpha\chi)}{(1-\alpha\chi)^2} & \frac{\epsilon}{2}\frac{\sqrt{1-\alpha^2}}{1-\alpha\chi} \\ \frac{\epsilon}{2}\frac{\sqrt{1-\alpha^2}}{1-\alpha\chi} & \frac{\sqrt{1-\alpha^2}}{1-\alpha\chi}s+\frac{\gamma}{2}\frac{\sqrt{1-\alpha^2}}{1-\alpha\chi}+\frac{\kappa}{2}\frac{\sqrt{1-\alpha^2}(1+\alpha\chi)}{(1-\alpha\chi)^2} \end{bmatrix}, \\
&= \frac{1}{P'(s)}\begin{bmatrix} C(s) & D(s) \\ D(s) & C(s) \end{bmatrix},
\end{aligned}$$

with

$$\begin{aligned}
A(s) &= \frac{\chi-\alpha}{1-\alpha\chi(s)}s^2+\left[\frac{1-\alpha}{1-\alpha\chi}\gamma-\frac{\alpha(1-\chi^2)}{(1-\alpha\chi)^2}\kappa\right]s+\frac{\gamma^2}{4}\frac{(1-\alpha)}{(1-\alpha\chi)}-\frac{\kappa^2}{4}\frac{(\chi+\alpha)(1+\alpha\chi)}{(1-\alpha\chi)^2} \\
&- \frac{\epsilon^2}{4}\frac{(\chi-\alpha)}{(1-\alpha\chi)}+\frac{1}{2}\frac{\alpha(1-\chi^2)}{(1-\alpha\chi)^2}\gamma\kappa, \\
B(s) &= -\frac{\epsilon}{2}\frac{(1-\alpha^2)\chi}{(1-\alpha\chi)^2}\kappa, \\
P'(s) &= (s+\frac{\gamma}{2}+\frac{1}{2}\frac{1+\alpha\chi}{1-\alpha\chi}\kappa+\frac{\epsilon}{2})(s+\frac{\gamma}{2}+\frac{1}{2}\frac{1+\alpha\chi}{1-\alpha\chi}\kappa-\frac{\epsilon}{2}).
\end{aligned}$$

and $\chi = \chi(s) = e^{s(\tau_a+\tau_b)}$.

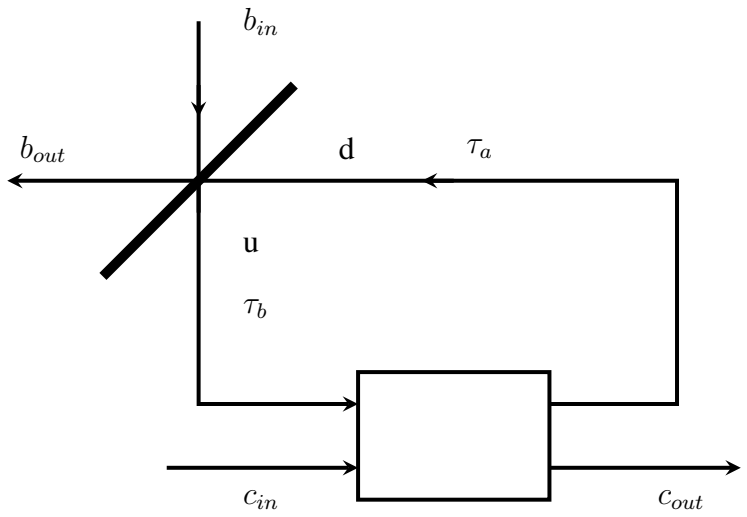

Figure 7.6.: Network with a DPA in loop and time delays

7.7. Further Research and Citations

The paper[1] on the results presented in this chapter, has been cited in various other works since its publication. These are [72, G. Zhang, H. Joseph Lee, B. Huang and H. Zhang], [33, Iida, Yukawa, Yonezawa, Yamamoto and Furusawa] and [66, N. Yamamoto].

7.7.1. Experimental Demonstration of Coherent Feedback Control on Optical Field Squeezing

In [33] the authors present an experimental demonstration of the approach to feedback enhanced squeezing as presented in this chapter and [28]. In order to allow for the specific details of their experimental set up they generalised the models used in [28, 68] by allowing for an additional loss mechanism and finite time delays within the inner loop of the arrangement. The system's equation of motion is given by

[1] J. Gough and S. Wildfeuer [28]

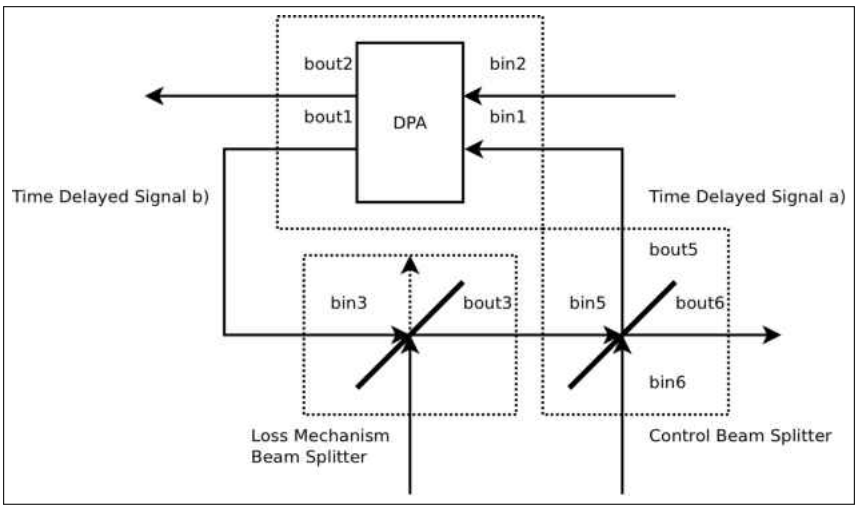

Figure 7.7.: Generalised Model for the coherent feedback arrangement with an additional loss mechanism (modelled by the beam splitter) and finite time delays within the loop.

$$\begin{aligned}\frac{d}{dt}a(t) &= -i\omega_0 a(t) + \epsilon e^{-2i\omega_0 t}a^\dagger(t) \\ &- \frac{\kappa+\gamma}{2}a(t) + \sqrt{\kappa}b_{in,1}(t) + \sqrt{\gamma}b_{in,2}(t)\end{aligned}$$

with input-output relations

$$b_{out,1}(t) = \sqrt{\kappa}a(t) - b_{in,1}(t).$$

A time delay of the form $b_\tau(t) = b(t-\tau)$ is modelled in frequency domain by multiplication by $e^{-s\tau}$ such that the inputs $b_{in,1}$ and $b_{in,3}$ with respect to time delays τ_a and τ_b are given by

$$\begin{aligned}b_{in,1}(t) &= b_{out,5}(t-\tau_a)e^{i\omega_0\tau_a} \\ b_{in,3}(t) &= b_{out,1}(t-\tau_b)e^{i\omega_0\tau_b}.\end{aligned}$$

The acquired open loop transfer functions in [33] coincide with Equations (7.2.1.1, 7.2.1.2).

The authors come to the conclusion that the experimental results agree well with the theoretical results.

8

Adiabatic Elimination

8.1. Introduction

In 2008 Luc Bouten, Ramon van Handel and Andrew Silberfarb [6] presented a rigorous approach to the adiabatic elimination problem by using a version of the Trotter-Kato theorem. The result extents to the language of quantum feedback networks as discussed in this thesis. The methods and procedures providing the rules for building quantum networks however utilizes other limits to obtain named networks. The instantaneous feed-forward limit for examples rises from taking the limit $\tau \to 0$ where τ is the time needed by the signals to travel from the output of some system A to the input of another system B. After taking this limit we

obtain a single component describing the joint behaviour of both systems in series. Here the question of commutativity of both limiting procedures arises. The assumption that both limits commute is frequently used in quantum optics when modelling quantum optical set-ups[1]. We therefore study the problem of commutativity of both limits in the following chapter.

We can provide an intuitive naïve equivalent to the adiabatic elimination of components of a system with the following example. Consider Fig. 8.1.a, i.e. a system driven by some input u_{in}. The system itself is coupled to the input by some (damped) mechanical spring Y and the spring is coupled to some system X. The set-up can be described by the equations of motion

$$\begin{aligned}\dot{X} &= f(X, Y, u_{\text{in}}),\\ \dot{Y} &= g(X, Y, u_{\text{in}}),\\ u_{\text{out}} &= h(X, Y, u_{\text{in}}).\end{aligned}$$

We assume that the spring has a spring-constant k and we rescale the spring-constant by some scaling factor γ, i.e. $k \to \gamma \cdot k$. We consider the limit $\gamma \to \infty$ and will observe that the spring relaxes infinitely fast to the input state. We will be left with a set-up where the spring follows the input infinitely fast and we can think of the system X as being driven by the input directly. Mathematically this can be described by setting $\dot{Y} = 0$ and solving $g(X, Y, u_{\text{in}}) = 0$. We will obtain $Y = j(X, u_{\text{in}})$ which can be substituted into the equations for X and u_{out}. We thus obtain a set of equations of motion which are independent of the spring component Y, hence where the spring has been eliminated.

[1] [37]

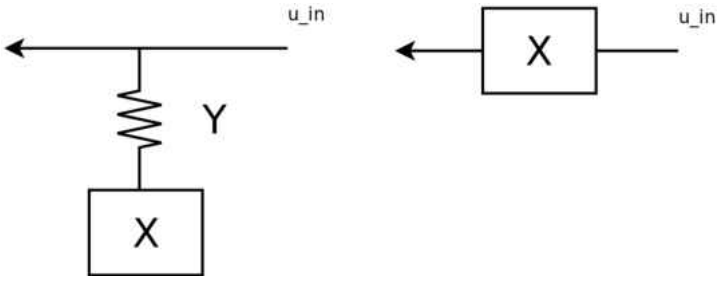

Figure 8.1.: a) A system with a slow and an oscillatory component. b) The same system after elimination of the oscillatory component.

8.2. Adiabatic Elimination in Quantum Feedback Networks

In order to set the stage for the Adiabatic Elimination, we assume that the system consists of two distinct parts. One collection of oscillators living on Hilbert space $\mathfrak{h}_{\text{osc}}$ and the remaining degrees of freedom of the system living on the auxiliary space $\mathfrak{h}_{\text{aux}}$. The overall system Hilbert space is then given by the tensor product $\mathfrak{h}_{\text{aux}} \otimes \mathfrak{h}_{\text{osc}}$.

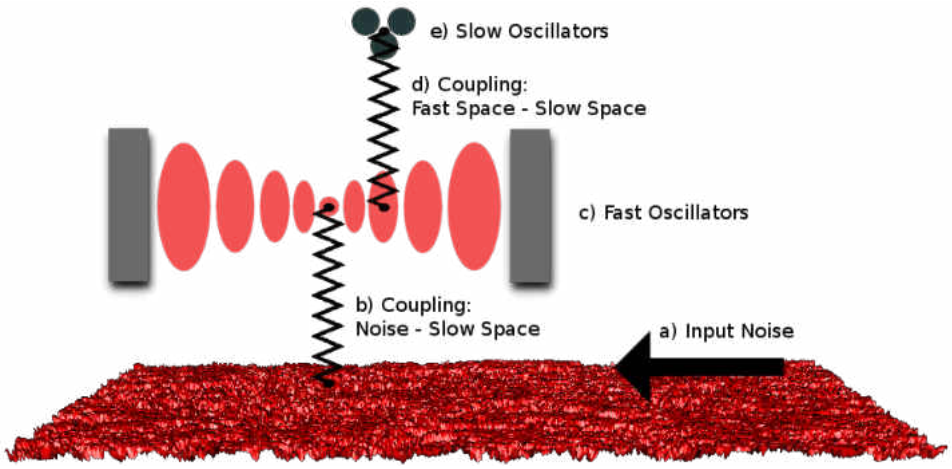

Figure 8.2.: Adiabatic Elimination of the cavity. The set-up consists of a cavity (fast oscillator) being driven by the input field and some slow degrees of freedom (for example: slow oscillator). In the Adiabatic Elimination limit the fast oscillator become increasingly strong coupled to the input field until they become 'enslaved' to the input and we are left with the slow components being driven by the input directly.

The Adiabatic Elimination describes the limit where the oscillators get increasingly strong

coupled to the input field such that, in the limit $k \to \infty$, we are left with the degrees of freedom of the system living on the auxiliary space $\mathfrak{h}_{\text{aux}}$. The pre- and post-limit models are described by the QSDEs

$$\mathrm{d}U_t(k) = \Big\{(S_{jk}(k) - \delta_{jk})\,\mathrm{d}\Lambda_{jk}(t) + L_j(k)\mathrm{d}B_j(t)^\dagger + L_j^{(k)\dagger} S_{jk}\mathrm{d}B_k(t) + K(k)\mathrm{d}t\Big\}\,U_t(k),$$

and

$$\mathrm{d}U_t = \Big\{(S_{jk} - \delta_{jk})\,\mathrm{d}\Lambda_{jk}(t) + L_j\mathrm{d}B_j(t)^\dagger + L_j^\dagger S_{jk}\mathrm{d}B_k(t) + K\mathrm{d}t\Big\}\,U_t,$$

respectively.

In order to ensure convergence of the solution of the QSDE in the strong coupling limit we can invoke the results by Bouten et al. [6] as quoted in Section 3.4.1. The convergence is ensured when the coefficients of the pre-limit QSDE assume a specific structure (Equation (3.4.1.3)) and satisfy some conditions (see assumptions of Theorem 3.10).

In order to satisfy this assumption on the structure of the pre-limit coefficients we assume that the QSDE parameter triple $(S(k), L(k), K(k))$ for the quantum open model takes the following form:

$$\begin{aligned} S(k) &= S \otimes I, \\ L(k) &= k\sum_j C_j \otimes a_j + G \otimes I, \\ K(k) &= k^2 \sum_{jl} A_{jl} \otimes a_j^* a_l + k\sum_j Z_j \otimes a_j^* + k\sum_j X_j \otimes a_j + R \otimes I. \end{aligned} \tag{8.2.1.1}$$

Here a_j and a_j^* are the annihilators and creators for the j's mode of the oscillators. $k > 0$ is the scaling parameter, scaling the coupling strength of the oscillators the input field. The operators $S, G, R, C_j, X_j, Z_j, A_{jl}$ are living on the auxiliary space $\mathfrak{h}_{\text{aux}}$ where A_{jl} is assumed to have a bounded inverse.

This parametrisation describes a model with the cavity having m oscillator modes. We may collect the operators C_j, X_j into the row vectors $C = [C_1, \ldots, C_m]$, $X = [X_1, \ldots, X_m]$, the operators Z_j into the column vector $Z = [Z_1, \ldots, Z_m]^{\mathrm{T}}$ and the operators A_{jl} into the matrix $A = (A_{jl})$.

In the limit $k \to \infty$, the oscillators will become increasingly strong coupled to the input field. In this strong coupling limit we consider the oscillator modes to be permanently relaxed to the input noise fields state, i.e. the ground state. We denote the ground state of the oscillators with $|0\rangle_{\text{osc}}$. In this case, as the degrees of freedom of the oscillators will become eliminated, we are left with the degrees of freedom of the auxiliary system $\mathfrak{h}_{\text{aux}}$.

We obtain as a special case of Theorem 3.10 the following statement:

Theorem 8.1 *Let $U(t,k)$ be the unitary adapted evolution associated with the triple $(S(k), L(k), K(k))$ and define the slow space as $\mathfrak{h}_s = \mathfrak{h}_{aux} \otimes \{\mathbb{C}\,|0\rangle_{osc}\}$. If the operator $Y = \sum_{jl} A_{jl} \otimes a_j^* a_l$ has kernel space equal to the slow space, then we have the limit*

$$\lim_{k\to\infty} \sup_{0\le t\le T} \left\| U(t,k)\Phi - \hat{U}(t)\Phi \right\| = 0,$$

for all $T > 0$ and $\Phi \in \mathfrak{h}_s \otimes \mathfrak{F}$, where $\hat{U}(t)$ is the unitary adapted evolution associated with the triple $(\hat{S} \otimes |0\rangle\langle 0|_{osc}, \hat{L} \otimes |0\rangle\langle 0|_{osc}, \hat{K} \otimes |0\rangle\langle 0|_{osc})$ and

$$\begin{aligned} \hat{S} &= (I + CA^{-1}C^*)S, \\ \hat{L} &= G - CA^{-1}Z, \\ \hat{K} &= R - XA^{-1}Z. \end{aligned}$$

In favour of an easier notation we will drop the factor of $\otimes\,|0\rangle\langle 0|_{\text{osc}}$ for the remainder of the chapter.

To prove the theorem we have show that Assumptions 1) to 3) of Theorem 3.10 are satisfied. The structural requirements of Assumption 1) are fulfilled by construction of the pre-limit coefficients Equation (8.2.1.1).

The operators appearing in Equation (8.2.1.3) and their adjoints are assumed to have a common invariant domain $\mathcal{D} \subset \mathfrak{h}_{\text{aux}} \otimes \mathfrak{h}_{\text{osc}}$. On $\mathfrak{h}_{\text{osc}}$ this domain is given by [13]

$$\text{span}\left\{|n\rangle : \sum n_j = N | N \in \mathbb{N}\right\}. \tag{8.2.1.2}$$

Assumptions 2) were given by:

i) $P_0 \mathcal{D} \subset \mathcal{D}$

ii) $\alpha_2 P_0 = 0$ on $\mathcal{D}$

iii) There exist $\tilde{\alpha}_2, \tilde{\alpha}_2^\dagger$ with common invariant domain $\mathcal{D}$, such that $\tilde{\alpha}_2 \alpha_2 = \alpha_2 \tilde{\alpha}_2 = P_1$

iv) $\beta_{1,j}^\dagger P_0 = 0$ on $\mathcal{D}$, $\forall 1 \leq j \leq n$

v) $P_0 \alpha_1 P_0 = 0$ on $\mathcal{D}$

The operators appearing in these assumptions are the operators of the parameter triple

$$\begin{aligned} \gamma_{jk}(k) &= \epsilon_{jk}, \\ \beta_j(k) &= k\beta_{1,j} + \beta_{0,j}, \\ \alpha(k) &= k^2\alpha_2 + k\alpha_1 + \alpha_0, \end{aligned} \tag{8.2.1.3}$$

as given in the result of Bouten et al. in [6].

This must be compared with the assumed structure of the coefficients according to Theorem

8.1, i.e.

$$
\begin{aligned}
S(k) &= S \otimes I, \\
L(k) &= k \sum_j C_j \otimes a_j + G \otimes I, \\
K(k) &= k^2 \sum_{jl} A_{jl} \otimes a_j^* a_l + k \sum_j Z_j \otimes a_j^* + k \sum_j X_j \otimes a_j + R \otimes I.
\end{aligned} \tag{8.2.1.4}
$$

We have to be careful with the left and right hand formulation of the QSDE, i.e. $\mathrm{d}V(t) = V(t)\left(\mathrm{d}\tilde{G}\right)$ as used in [6] versus the right hand formulation $\mathrm{d}U(t) = (\mathrm{d}G(t))\,U(t)$ as used in this chapter.

Since Equation (8.2.1.3) and Assumptions 2) are formulated with respect to the left-hand QSDE and Equation (8.2.1.4) is formulated with respect to the right-hand QSDE we see that the operators appearing in Assumptions 2) translate therefore to

$$
\begin{aligned}
\alpha_2 &= \left(\sum_{jk} A_{kl} \otimes a_j^* a_k \right)^*, \\
\beta_{1,j} &= \left(\sum_j C_j \otimes a_j \right)^*, \\
\alpha_1 &= \left(\sum_j Z_j \otimes a_j^* + X_j \otimes a_j \right)^*.
\end{aligned}
$$

The slow space was defined as $\mathfrak{h}_s = \mathfrak{h}_{\text{aux}} \otimes \{\mathbb{C}\,|0\rangle_{\text{osc}}\}$ and we can readily see that ii) and iv) of Assumptions 2) will be satisfied since in each case we encounter $a_j\,|0\rangle_{\text{osc}}$. Assumption v) will be satisfied for the same reason and the fact that $P_0 a^*\,|0\rangle_{\text{osc}} = 0$. i) will be satisfied by the specific form of the invariant domain in $\mathfrak{h}_{\text{osc}}$, Equation (8.2.1.2).

We can now continue with the proof of Theorem 8.1:

Proof:

By Assumption 2), we are left with the verification of the existence of $\tilde{\alpha}_2$ and $\tilde{\alpha_2}^\dagger$ such that $\tilde{\alpha_2}\alpha_2 = \alpha_2\tilde{\alpha_2} = P_f$.

Let us set

$$M_N = \mathfrak{h}_{\text{aux}} \otimes \text{span}\left\{ |n\rangle : \sum_j n_j = N \right\},$$

for $N = 0, 1, 2, \ldots$. In particular, we have the direct sum of orthogonal subspaces $\mathfrak{h}_{\text{aux}} \otimes \mathfrak{h}_{\text{osc}} = \bigoplus_{N\geq 0} M_N$. Let $P_0 = P_s$ be the orthogonal projection onto the "slow space"

$$\mathfrak{h}_s = \mathfrak{h}_{\text{aux}} \otimes |0\rangle_{\text{osc}} = M_0$$

and $P_1 = P_f = I - P_s$ the projection onto the "fast space"

$$h_f = \bigoplus_{n=1}^{\infty} M_n.$$

The overall space $\mathfrak{h}_{\text{aux}} \otimes \mathfrak{h}_{\text{osc}}$ decomposes then into $\mathfrak{h}_{\text{aux}} \otimes \mathfrak{h}_{\text{osc}} = h_s \oplus h_f$

Recall the hypothesis that $\mathrm{Ker}(Y) = \mathfrak{h}_{\text{s}}$. We first have the following:

Lemma 8.2 *Under the hypothesis* $\mathrm{Ker}(Y) = \mathfrak{h}_s$*, the subspaces* M_N *are stable under* $Y_N = Y\Big|_{M_N}$*, and we have*

$$(P_f Y P_f)^{-1} = \bigotimes_{N\geq 0} Y_N^{-1}.$$

Moreover, let $|\delta_j\rangle$ *be the state where the* j*th mode is in the first excited state and all others are in the ground state, then* $(Y_1)^{-1}\sum_j \phi_j \otimes |\delta_j\rangle = \sum_{jl}(A^{-1})_{jl}\phi_l \otimes |\delta_j\rangle$.

Proof:

Consider the decomposition of the system space $\mathfrak{h}_{\text{aux}} \otimes \mathfrak{h}_{\text{osc}} = h_s \oplus h_f$ as presented above. Y is an operator on $\mathfrak{h}_{\text{aux}} \otimes \mathfrak{h}_{\text{osc}}$ and assumes with respect to the decomposition into "fast" and

"slow" space the following form

$$Y = \begin{bmatrix} Y_{ss} & Y_{sf} \\ Y_{fs} & Y_{ff} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & Y_{ff} \end{bmatrix}.$$

The right-hand-side follows since $Y = \sum_{jk} A^*_{jk} \otimes a^*_k a_j$ acting on some element of $\phi \in \bigoplus_{n=0}^{\infty} M_n$ doesn't change the number of photons and each subspace M_n is therefore stable under Y whence every off-diagonal element of Y in this representation will be zero, i.e. $Y_{fs} = Y_{sf} = 0$ and by direct sum decomposition we see that

$$Y_{ff}^{-1} = (P_f Y_N P_f)^{-1} = \bigoplus_{N \geq 1} Y_N^{-1}.$$

We also have that $\mathrm{Ker}\, Y = h_s$ implies $Y_{ss} = 0$ and invertibility of $Y_{ff} = P_f Y P_f$.

The remaining identity is easily checked from $Y \sum_j \phi_j \otimes |\delta_j\rangle = \sum_{jl} A_{jl} \phi_l \otimes |\delta_j\rangle$ and setting this equal to $\sum_j \tilde{\phi}_j \otimes |\delta_j\rangle$ we deduce that $\phi_l = (A^{-1})_{jl} \tilde{\phi}_j$.

□

Lemma 8.2 shows the specific form of $\alpha_2^\dagger = Y$ and we can easily see that if we choose for example the Moore-Penrose pseudo inverse $\tilde{\alpha_2}^\dagger = Y^-$ with

$$Y^- = \begin{bmatrix} 0 & 0 \\ 0 & Y_{ff}^{-1} \end{bmatrix}$$

we have indeed that $Y^- Y = Y Y^- = P_f$. We obtain an equivalent statement for $\alpha_2 = Y^*$ by showing that $\mathrm{Ker}\, Y = \mathrm{Ker}\, Y^* = M_0$ such that above arguments hold mutatis mutandis for Y^*.

Corrollary 8.3 $\mathrm{Ker}(Y^*) = M_0$.

Proof:

By the preceding lemma we have that $\mathfrak{h}_f = P_f\mathfrak{h}_{\text{aux}} \otimes \mathfrak{h}_{\text{osc}}$ is stable under Y. Therefore, for any $\phi \in M_0$ and $\psi \in \mathfrak{h}_{\text{aux}} \otimes \mathfrak{h}_{\text{osc}}$ we have that $\langle \phi | Y\psi \rangle = \langle \phi | YP_f\psi \rangle = 0$. It follows that $\langle Y^*\phi | \psi \rangle = \langle \phi | Y\psi \rangle = 0 \; \forall \psi \in \mathfrak{h}_{\text{aux}} \otimes \mathfrak{h}_{\text{osc}}$, thus $Y^*\psi = 0$ for any $\psi \in M_0$ and we conclude that $M_0 \subseteq \mathrm{Ker}(Y^*)$. We show the converse, i.e. $\mathrm{Ker}(Y^*) \subseteq M_0$ by contradiction. To do this, suppose that $\exists\, \varphi \in P_f\mathfrak{h}_{\text{aux}} \otimes \mathfrak{h}_{\text{osc}}$ with $\varphi \neq 0$ such that $\langle Y^*\varphi | \psi \rangle = 0 \; \forall \; \psi \in \mathfrak{h}_{\text{aux}} \otimes \mathfrak{h}_{\text{osc}}$. It follows that $\langle \varphi | Y\psi \rangle = 0$ and therefore $\langle \varphi | YP_f\psi \rangle = 0 \; \forall \; \psi \in \mathfrak{h}_{\text{aux}} \otimes \mathfrak{h}_{\text{osc}}$. But since $\mathfrak{h}_f$ is stable under Y and $Y\Big|_{\mathfrak{h}_f}$ is invertiblem, it follows that $\varphi \in \mathfrak{h}_{\text{s}}$. But this contradicts the hypothesis that φ is a nonzero element of $\mathfrak{h}_{\text{f}}$ and therefore we conclude that $\mathrm{Ker}(Y^*) \subseteq M_0$. This concludes the proof.

□

We now state a sufficient condition for $\mathrm{Ker}(Y) = M_0 = \mathrm{Ker}(Y^*)$. Let us first recall the following definition:

Definition 8.4 (See Chapter 6) *A bounded Hilbert space operator A is strictly Hurwitz stable if*

$$\mathrm{Re}\,\langle \psi | A\psi \rangle < 0, \; \forall \; \psi \neq 0.$$

Lemma 8.5 *Let $A_{jl} \in \mathcal{B}(\mathfrak{h}_{aux})$ such that $A = (A_{jl}) \in \mathcal{B}(\mathfrak{h}_{aux} \otimes \mathbb{C}^m)$ is strictly Hurwitz stable. The the operator*

$$Y = \sum_{jl} A_{jl} \otimes a_j^* a_l$$

on $\mathfrak{h}_{aux} \otimes \mathfrak{h}_{osc}$ has kernel consisting of vectors of the form $\phi \otimes |0\rangle_{osc}$, where $\phi \in \mathfrak{h}_{aux}$.

Proof:

We see that for $\psi \in \mathfrak{h}_{\text{aux}} \otimes \mathfrak{h}_{\text{osc}}$

$$\langle \psi | Y \psi \rangle = \sum_{jl} \langle \psi | \left(I \otimes a_j \right)^* \left(I \otimes a_l \right) \psi \rangle = \sum_{jl} \langle \psi_j | A_{jl} \otimes I \psi_l \rangle$$

where $\psi_j = (I \otimes b_j)\, \psi$. We may decompose $\psi_j = \sum_n \psi_j(n) \otimes |n\rangle$, where $|n\rangle$ is the orthonormal basis of number states for the oscillators and $\psi_j(n) \in \mathfrak{h}_{\text{aux}}$. Then

$$\langle \psi | Y \psi \rangle = \sum_n \sum_{jl} \langle \psi_j(n) | A_{jl} \psi_l(n) \rangle$$

and, for each fixed n, we have $\sum_{jl} \langle \psi_j(n) | A_{jl} \psi_l(n) \rangle \leq 0$ with equality if and only if the $\psi_j(n) = 0$ since (A_{jl}) is assumed to be strictly Hurwitz. In particular, if we assume that ψ is in the kernel of Y then we deduce that $\psi_j(n) = 0$ for each n and $j = 1, \ldots, m$. It follows that $\psi_j = (I \otimes a_j)\psi = 0$ for each $j = 1, \ldots, m$, and this implies that $\psi = \phi \otimes |0\rangle_{\text{osc}}$ for some $\phi \in \mathfrak{h}_{\text{aux}}$ as required.

□

Note, however, that as we shall see below, for Theorem 8.1 to hold it is enough that $\mathrm{Ker}(Y) = M_0$.

Next we have to ensure that the post-limit parameter given in Assumption 3) satisfy the Hudson-Parthasarathy relations, i.e. that $\hat{S}$ is unitary and that $\hat{K} + \hat{K}^* = -\hat{L}^* \hat{L}$.

> **Lemma 8.6** *The operator $\hat{S}$ is unitary and $\hat{K} + \hat{K}^* + \hat{L}^* \hat{L} = 0$.*

Proof:

We first show that $I + CA^{-1}C^*$ is invertible. Suppose that $u \in \mathrm{Ker}(I + CA^{-1}C^*)$

$$\begin{aligned} u &= -CA^{-1}C^*u \Rightarrow C^*u = -C^*CA^{-1}C^*u \Rightarrow (I + C^*CA^{-1})C^*u = 0 \\ &\Rightarrow (A + C^*C)A^{-1}C^*u = 0 \Rightarrow -A^*AC^*u = 0 \Rightarrow C^*u = 0 \end{aligned}$$

so substitute $C^*u = 0$ into $u = -CA^{-1}C^*u$ we see that $u = 0$, therefore $\mathrm{Ker}(\hat{S}) = 0$. As S is unitary, we have that

$$\begin{aligned} \hat{S}\hat{S}^* &= (I + C^*A^{-1}C)(I + CA^{*-1}C^*) \\ &= I + CA^{-1}\left(A + A^* + C^*C\right)A^{*-1} = I \end{aligned}$$

using $A + A^* = -C^*C$. Similarly $\hat{S}^*\hat{S}$. Likewise we use Eq. (8.2.1.10) to show that

$$\begin{aligned} \hat{K} + \hat{K}^* + \hat{L}^*\hat{L} &= R - XA^{-1}Z + R^* - Z^*A^{*-1}X^* + \left(G^* - Z^*A^{*-1}C^*\right)\left(G - CA^{-1}Z\right) \\ &= -\left(X - G^*C\right)A^{-1}Z - Z^*A^{*-1}\left(X^* - C^*G - C^*CA^{-1}Z\right) \\ &= Z^*A^{-1}Z + Z^*A^{*-1}\left(Z + C^*CA^{-1}Z\right) \\ &= Z^*A^{*-1}\left(A + A^* + C^*C\right)A^{-1}Z \\ &= 0. \end{aligned}$$

□

We now verified all assumptions for Theorem 3.10 and are ready to complete the prove of Theorem 8.1. We first recall both forms of the QSDEs as found in this Chapter and [6].

Let $V(t,k) = U(t,k)^*$, then V satisfies the left QSED (using summation convention)

$$\mathrm{d}V_t^{(k)} = V_t^{(k)}\left\{\alpha^{(k)}\mathrm{d}t + \beta_l^{(k)}\mathrm{d}B_l(t) + \gamma_j^{(k)}\mathrm{d}B_j^*(t) + \left(\epsilon_{jl}^{(k)} - \delta_{jl}\right)\mathrm{d}\Lambda_{jl}(t)\right\} \tag{8.2.1.5}$$

where $\alpha^{(k)} = k^2\alpha_2 + k\alpha_1 + \alpha_0 = K(k)^*$, $\beta_j^{(k)} = k\beta_{0,j} + \beta_{0,j} = L_j(t)^*$, $\gamma_j^{(k)} = -S_{jl}^*L_l$, and

$\epsilon_{jl} = S^*_{jl}$. As discussed before, the results in [6] are stated for the left QSDE whereas the our results are stated for the right QSDE

$$\mathrm{d}U^{(k)}_t = \left\{\left(S^{(k)}_{jl} - \delta_{jl}\right)\mathrm{d}\Lambda^{jl}_t + L^{(k)\dagger}_j S^{(k)}_{jl}\mathrm{d}B^l_t + L^{(k)}_j \mathrm{d}B^{j\dagger}_t + K^{(k)}\mathrm{d}t\right\} U^{(k)}_t, \tag{8.2.1.6}$$

where the operator appearing in Eq. (8.2.1.6) are given by Equation (8.2.1.4).

The limit coefficients in Assumption 3 are then given by

$$\begin{aligned}
\hat{\alpha} &= P_s\left(\alpha_0 - \alpha_1\tilde{\alpha}_2\alpha_1\right)P_s = \left(R^* - Z^*A^{*-1}X^*\right)\otimes|0\rangle\langle 0|_{\text{osc}} \equiv \hat{K}^*\otimes|0\rangle\langle 0|_{\text{osc}} \\
\hat{\beta} &= P_s\left(\beta_0 - \alpha_1\tilde{\alpha}_2\beta_1\right)P_s = \left(G^* - Z^*A^{*-1}C^*\right)\otimes|0\rangle\langle 0|_{\text{osc}} \equiv \hat{L}^*\otimes|0\rangle\langle 0|_{\text{osc}} \\
\hat{\epsilon} &= P_s\epsilon\left(I + \beta^*_1\tilde{\alpha}_2\beta^*_1\right)P_s = S^*\left(I + C^*A^{*-1}C^*\right)\otimes|0\rangle\langle 0|_{\text{osc}} \equiv \hat{S}^*\otimes|0\rangle\langle 0|_{\text{osc}} \\
\hat{\gamma} &= -\hat{\epsilon}\hat{\beta}^* \equiv -\hat{S}^*\hat{L}\otimes|0\rangle\langle 0|_{\text{osc}}
\end{aligned}$$

with $(\hat{S}, \hat{L}, \hat{K})$ as given in the statement of Theorem 8.1. The action of the Moore-Penrose inverse $\tilde{\alpha}_2$ on some element $\alpha_1 P_s\phi\otimes\psi = X^*\phi\otimes|\delta_j\rangle_{\text{osc}}$, where $|\delta_j\rangle_{\text{osc}}$ is the state where all oscillators are in the ground state and the j'th oscillator is in the first excited state, is given by Lemma 8.2. These coefficients evidently satisfy the requirements of Assumption 3, namely, to generate a unitary adapted Hudson-Parthasarathy equation on a common invariant domain in M_0, as we established in Lemma 8.6 .

□

> **Remark 8.2.1.1** *We drop for the remainder of the chapter the* $\otimes|0\rangle\langle 0|_{osc}$*, as it is clear that in the limit the oscillators will be relaxed to the ground state* $|0\rangle_{osc}$.

After dropping the $\otimes|0\rangle\langle 0|_{\text{osc}}$ and collecting the oscillator creators and annihilators in a

similar manner as in the previous sections, i.e.

$$a^* = [a_1^*, \ldots, a_m^*], \; a = \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix}$$

we can simply write

$$S(k) = S, \; L(k) = kCa + G, \; K(k) = k^2 a^* A a + k a^* Z + k X a + R. \tag{8.2.1.7}$$

Consider a self-adjoint Hamiltonian of the from by

$$H(k) = k^2 a^* \Omega a + k a^* \Gamma + k \Gamma^* a + \Theta,$$

and remember the definition of the complex dampening $K(k) = -\frac{1}{2} L^*(k) L(k) - iH(k)$. Using the parameter as given in Equation (8.2.1.7) and comparing both sides of the equation we see that

$$\begin{aligned} A &= -\tfrac{1}{2} C^* C - i\Omega, \\ Z &= -\tfrac{1}{2} C^* G - i\Gamma, \\ X &= -\tfrac{1}{2} G^* C - i\Gamma^*, \\ R &= -\tfrac{1}{2} G^* G - i\Theta. \end{aligned}$$

We can deduce the following identities when computing $K(k) + K^*(k)$:

$$A + A^* = -C^* C, \tag{8.2.1.8}$$

$$X + Z^* = -G^* C, \tag{8.2.1.9}$$

$$R + R^* = -G^* G. \tag{8.2.1.10}$$

8.3. Adiabatic Elimination and Systems in Series

We now wish to verify commutativity of the adiabatic elimination with the instantaneous feedforward limit as encountered in Chapter 5.5.4 and Chapter 6.6.2.3.

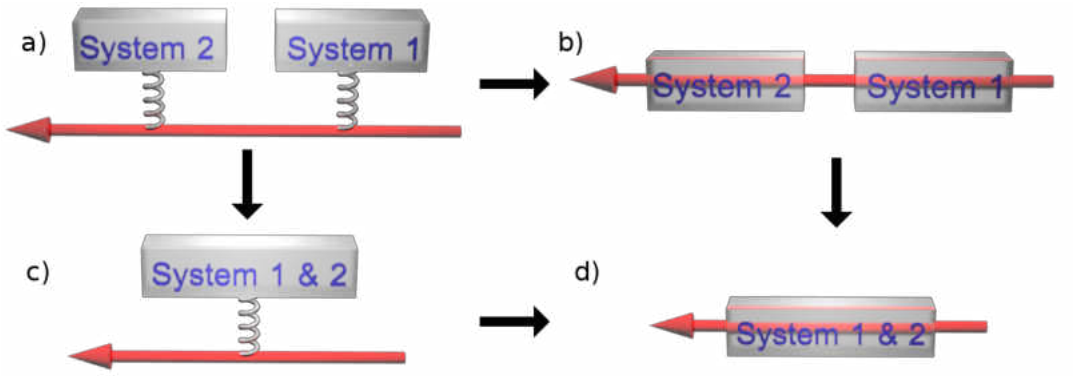

Figure 8.3.: Illustration of the commutativity of the adiabatic elimination with the series product. The question arising is if the order we take the limits changes the resulting model (lower right corner). a) -> b) Adiabatic Elimination of the oscillator of both system 1 and 2; a) -> c) Taking the series product of system 1 and 2; b) -> d) Taking the series product after the adiabatic elimination; c) -> d) Adiabatic elimination after the series product.

We remind the reader that the series product or instantaneous feedforward limit of models $G_j, j = 1, 2$ was denoted by $G_2 \triangleleft G_1$ and that the adiabatic elimination of model G is denoted by $\mathcal{A}(G)$.

We therefore want to verify that

$$\mathcal{A}(G_2 \triangleleft G_1) = (\mathcal{A}(G_2)) \triangleleft (\mathcal{A}(G_2)).$$

The model under consideration consists of two oscillator modes $a_j, j = 1, 2$, one for each system 1 and 2, see Fig. 8.3. We collect these oscillators in vector $a = [a_1^*, a_2^*]^\dagger$ and obtain for system $j = 1, 2$ the parameter triples $(S_j(k), L_j(k), K_j(k))$

$$
\begin{aligned}
S_1(k) &= S_1, \\
L_1(k) &= k[C_1, 0]a + G_1, \\
K_1(k) &= k^2 a^* \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} + k a^* \begin{bmatrix} Z_1 \\ 0 \end{bmatrix} + k[X_1, 0]a + R_1, \qquad (8.3.0.11)
\end{aligned}
$$

and

$$
\begin{aligned}
S_2(k) &= S_2, \\
L_2(k) &= k[0, C_2]a + G_2, \\
K_2(k) &= k^2 a^* \begin{bmatrix} 0 & 0 \\ 0 & A_2 \end{bmatrix} + k a^* \begin{bmatrix} 0 \\ Z_2 \end{bmatrix} + k[0, X_2]a + R_2. \qquad (8.3.0.12)
\end{aligned}
$$

Lemma 8.7 *Let $G_j, j = 1, 2$ be models with parameter triple (S_j, L_j, K_j). If the Adiabatic Elimination limits $\mathcal{A}(G_j)$ exist, then*

$$\mathcal{A}(G_2 \triangleleft G_1) = (\mathcal{A}(G_2)) \triangleleft (\mathcal{A}(G_2))$$

Proof:

We compute the resulting models by followings the paths a) -> b) -> d) or a) -> c) -> d) in Fig. 8.3 and compare the results. Recall the formula for the series product

$$S_{\text{Ser}} = S_2 S_1, \;\; L_{\text{Ser}} = L_2 + S_2 L_1, \;\; K_{\text{Ser}} = K_1 + K_2 - L_2^\dagger S_2 L_1.$$

Adiabatic elimination followed by the series product: $(\mathcal{A}(G_2)) \triangleleft (\mathcal{A}(G_1))$

We can now compute the path a) -> c) -> d), that is adiabatic elimination followed by the instantaneous feedforward limit. The adiabatic elimination of reduces model $j = 1, 2$ to

$$\begin{aligned}\hat{S}_j &= \left(I + C_j A_j^{-1} C_j^*\right) S_j, \\ \hat{L}_j &= G_j - C_j A_j^{-1} Z_j, \\ \hat{K}_j &= R_j - X_j A_j^{-1} Z_j.\end{aligned}$$

If we compute now the series product, we obtain

$$\begin{aligned}S_{\text{Ser}} &= \hat{S}_2 \hat{S}_1, \\ L_{\text{Ser}} &= \hat{L}_2 + \hat{S}_2 \hat{L}_1, \\ K_{\text{Ser}} &= \hat{K}_1 + \hat{K}_2 - \hat{L}_2^\dagger \hat{S}_2 \hat{L}_1. \end{aligned} \tag{8.3.0.13}$$

The series product followed by adiabatic elimination: $\mathcal{A}\left(G_2 \triangleleft G_1\right)$

We can compute the model parameter by the converse direction by following path path a) -> b) -> d) in Fig. 8.3. The series product of models Eq. 8.3.0.11 and 8.3.0.12 gives

$$\begin{aligned}S_{\text{Ser}}(k) &= S_2 S_1 \\ L_{\text{Ser}}(k) &= L_2(k) + S_2(k) L_1(k) = k[S_2 C_1, C_2]a + G_1 + S_2 G_1 \\ K_{\text{Ser}}(k) &= K_1(k) + K_2(k) - L_2^\dagger(k) S_2(k) L_1(k) \\ &= K^2 a^* \begin{bmatrix} A_1 & 0 \\ -C_2^* S_2 C_1 & A_2 \end{bmatrix} a + k a^* \begin{bmatrix} Z_1 \\ Z_2 - C_2^* S_2 G_1 \end{bmatrix} + k[X_1 - G_2^* S_2 C_1, X_2]a \\ &+ R_1 + R_2 - G_2^* S_2 G_1\end{aligned}$$

Performing now the adiabatic elimination results in

$$\hat{S}_{\text{Ser}} = \left(I + [S_2 C_1, C_2] \begin{bmatrix} A_1 & 0 \\ -C_2^* S - 2C_1 & A - 2 \end{bmatrix}^{-1} \begin{bmatrix} C_1^* S_2^* \\ C_2^* \end{bmatrix} \right) S_2 S_1,$$

$$\hat{L}_{\text{Ser}} = (G_1 + S_2 G_1) - [S_2 C_1, C_2] \begin{bmatrix} A_1 & 0 \\ -C_2^* S - 2C_1 & A - 2 \end{bmatrix}^{-1} \begin{bmatrix} Z_1 \\ Z_2 - C_2^* S - 2G_1 \end{bmatrix},$$

$$\begin{aligned} \hat{K}_{\text{Ser}} = {} & (R_1 + R_2 - G_2^* S_2 G_1) \\ & - [X_1 - G_2^* S_2 C_1, X_2] \begin{bmatrix} A_1 & 0 \\ -C_2^* S - 2C_1 & A - 2 \end{bmatrix}^{-1} \begin{bmatrix} Z_1 \\ Z_2 - C_2^* S - 2G_1 \end{bmatrix}. \end{aligned} \quad (8.3.0.14)$$

We are now left with verifying that Eq. 8.3.0.13 and Eq. 8.3.0.14 coincide. In order to archive this we have to compute the matrix inverse appearing in Eq. 8.3.0.14. A useful tool to accomplish this is the Banachiewicz inversion formula [70], see Equation A.1.0.2. We therefore obtain

$$\begin{bmatrix} A_1 & 0 \\ -C_2^* S - 2C_1 & A_2 \end{bmatrix}^{-1} = \begin{bmatrix} A_1^{-1} & 0 \\ A_2^{-1} C_2^* S_2 C_1 A_1^{-1} & A_2^{-1} \end{bmatrix}.$$

We are now able to write out the expressions in Eq. 8.3.0.14 explicitly and obtain for $\hat{S}_{\text{Ser}}$

$$\begin{aligned} \hat{S}_{\text{Ser}} &= \left(I + S_2 C_1 A_1^{-1} C_1^* S_2^* + C_2 A_2^{-1} C_2^* S_2 C_1 A_1^{-1} C_1^* S_2^* + C_2 A_2^{-1} C_2^* \right) S_2 S_1, \\ &= \left(I + C_2 A - 2^{-1} C_2^* \right) S_2 \left(I + C_1 A_1^{-1} C_1^* \right) S_1, \\ &= \hat{S}_2 \hat{S}_1. \end{aligned}$$

Similarly for the coupling operator $\hat{L}_{\text{Ser}}$

$$\begin{aligned}\hat{L}_{\text{Ser}} &= (G_1 + S_2 G_1) - S_2 C_1 A_1^{-1} Z_1 - C_2 A_2^{-1} C_2^* S_2 A_1 Z_1 - C_2 A_2^{-1} Z_2 + C_2 A_2^{-1} C_2^* S_2 G_1, \\ &= \left(G_2 - C_2 A_2^{-1} Z_2\right) + \left(I + C_2 A_2^{-1} C_2^*\right) S_2 \left(G_1 - C_1 A_1^{-1} Z_1\right), \\ &= \hat{L}_2 + \hat{S}_2 \hat{L}_1.\end{aligned}$$

If we expand the complex dampening $\hat{K}_{\text{Ser}}$ we obtain

$$\hat{K}_{\text{Ser}} = R_1 + R_2 - X_1 A_1^{-1} Z_1 + G_2^* S_2 C_1 A_1^{-1} Z_1 - X_2 A_2^{-1} Z_2 + X_2 A_2^{-1} C_2^* S_2 \left(G_1 - C_1 Z_1\right).$$

We want to show that $\hat{K}_{\text{Ser}}$ as above equals

$$\begin{aligned}\hat{K}_1 + \hat{K}_2 - \hat{L}_2^* \hat{S}_2 \hat{L}_1 &= R_1 + R_2 - X_1 A_1^{-1} Z_1 - X_2 A_2^{-1} Z_2 \\ &- \left(G_2^* - Z_2^* A_2^{-1*} C_2^*\right) \left(I + C_2 A_2^{-1} C_2^*\right) S_2 \left(G_1 - C_1 A_1^{-1} Z_1\right).\end{aligned}$$

We recall that $A_2 = -\frac{1}{2} C_2^* C_2 - i\Omega_2$ such that

$$A_2 + C_2^* C_2 = +\frac{1}{2} C_2^* C_2 - i\Omega_2 = -A_2^* \tag{8.3.0.15}$$

and

$$\begin{aligned}A_2^{-1*} C_2^* \left(I + C_2 A_2^{-1} C_2\right) &= A_2^{-1*} \left(I + C_2^* C_2 A_2^{-1}\right) C_2^*, \\ &= A - 2^{-1*} \left(A_2 + C_2^* C_2\right) A - 2^{-1} C_2^*, \\ &\overset{(8.3.0.15)}{=} A_2^{-1*} \left(-A_2^*\right) A_2^{-1*} C_2^*, \\ &= -A_2^{-1} C_2^*.\end{aligned} \tag{8.3.0.16}$$

Using this we get

$$\hat{K}_{\text{Ser}} - \left(\hat{K}_1 + \hat{K}_2 - \hat{L}_2^* \hat{S}_2 \hat{L}_1\right) = \left(X_2 + G_2^* C_2 + Z_2^*\right) A_2^{-1} C_2^* S_2 \left(G_1 - C_1 Z_1\right).$$

But now recall relations Eq. (8.2.1.10), i.e. $X_2 + Z_2^* = -G_2^* C_2$ from which we can see that the right hand side of this equation vanishes.

We therefore can see that model parameter obtained by either performing the adiabatic elimination or series product first don't differ and both limits therefore commute

□

8.4. Adiabatic Elimination and Systems in Loop: Introduction

We now want to verify that the same result as in the previous section also holds for the feedback reduction limit, see Fig. 8.4. One could now try to attempt a similar way to verify

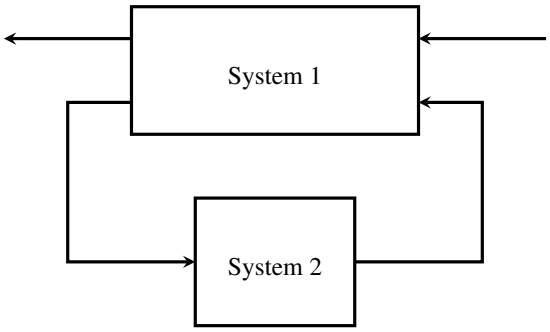

Figure 8.4.: Adiabatic elimination for systems in loop. System 2 has a fast oscillator component and some remaining degrees of freedom.

commutativity with the adiabatic elimination as in the previous chapter, that is, computing the models obtained by performing both, the adiabatic elimination limit or the feedback reduction first and compare the results. However, if one tries this procedure with the most general system models, the algebra becomes quite challenging.

As a motivating example we compute the resulting models for both orders of taking the limits for a simpler model.

Here we take the four port model 'System 1' in Fig. 8.4 to be a beam splitter, that is a model with parametrisation

$$S_1 = \begin{bmatrix} S_{ii} & S_{ei} \\ S_{ie} & S_{ee} \end{bmatrix} = \begin{bmatrix} \alpha & \sqrt{1-\alpha^2} \\ \sqrt{1-\alpha^2} & -\alpha \end{bmatrix}, \;\; L = 0, \;\; K = 0,$$

with transmissivity $\alpha \in [0, 1]$. We take the in-loop device to have a single oscillator mode coupled to the field and no remaining degrees of freedom

$$\begin{aligned} S_2(k) &= S_2, \\ L_2(k) &= k\sqrt{\gamma}a_2, \\ K_2(k) &= -\frac{1}{2}k^2 a_2^* \gamma a_2. \end{aligned}$$

In terms of the operators appearing in Theorem 8.1 we get

$$S_2 = S_2, \; A_2 = -\tfrac{1}{2}\gamma, \; C_2 = \sqrt{\gamma}, \; Z_2 = X_2 = R_2 = G_2 = 0.$$

The reduced model after taking the feedback reduction in the zero time delay limit is given in Section 5.5.5:

$$\begin{aligned} S_{\text{red}} &= S_{ee} + S_{ei} S_2 \left(I - S_{ii} S_2\right)^{-1} S_{ie}, \\ L_{\text{red}} &= S_{ei} \left(I - S_{ii} S_2\right)^{-1} L_2, \\ H_{\text{red}} &= K_2 - L_2^* S_2 \left(I - S_{ii} S_2\right)^{-1} L_2 \end{aligned} \tag{8.4.0.17}$$

We recall the parameter after the adiabatic elimination as given in Theorem 8.1:

$$\begin{aligned}\hat{S} &= (I + C_2 A_2^{-1} C_2^*) S_2 = -S_2, \\ \hat{L} &= G_2 - C_2 A_2^{-1} Z_2 = 0, \\ \hat{K} &= R_2 - X_2 A_2^{-1} Z_2 = 0. \end{aligned} \tag{8.4.0.18}$$

The parameter taking the adiabatic elimination first and taking the feedback reduction limit are given by

$$\begin{aligned}\hat{S} &= \alpha + \sqrt{1-\alpha^2}\,(-S_2)\,\frac{1}{1-(-\alpha)\,(1-\alpha^2)} = \frac{\alpha - S_2}{1-\alpha S_2}, \\ \hat{L} &= 0, \\ \hat{K} &= 0. \end{aligned} \tag{8.4.0.19}$$

Taking the feedback reduction first results in

$$\begin{aligned}\tilde{S}(k) &= \alpha + \left(1-\alpha^2\right) S_2 \frac{1}{1+\alpha S_2}, \\ \tilde{L}(k) &= k\sqrt{1-\alpha^2}\,\frac{1}{1+\alpha S_2}\sqrt{\gamma}\,a_0, \\ \tilde{K}(k) &= K_2(k) - L_2(k)^* \frac{S_2 T_{22}}{1 - S_2 T_{22}} L_2(k) = k^2 a_2^* \left(-\tfrac{1}{2}\gamma + \gamma \frac{\alpha S_2}{1+\alpha S_2}\right) a_2. \end{aligned}$$

We have to compare this with the structure of pre-limit parameter as given with Theorem 8.1

$$\begin{aligned}S(k) &= S \otimes I, \\ L(k) &= k \sum_j C_j \otimes a_j + G \otimes I, \\ K(k) &= k^2 \sum_{jl} A_{jl} \otimes a_J^* a_l + k \sum_j Z_j \otimes a_j^* + k \sum_j X_j \otimes a_j + R \otimes I. \end{aligned} \tag{8.4.0.20}$$

and obtain the following parameter for the adiabatic elimination:

$$\begin{aligned} A &= -\frac{\gamma}{2}\frac{1-\alpha S_2}{1+\alpha S_2},\ C = \frac{\sqrt{1-\alpha^2}\sqrt{\gamma}}{1+\alpha S_2}, \\ S &= \alpha + \frac{(1-\alpha^2)\,S_2}{1+\alpha S_2},\ G = X = Z = R = 0. \end{aligned}$$

After the adiabatic elimination we therefore obtain $\hat{L} = 0, \hat{K} = 0$ and

$$\begin{aligned} \hat{S} &= \left(1 - \frac{(1-\alpha^2)\,\gamma}{|\,1+\alpha S_2\,|}\frac{2}{\gamma}\frac{1+\alpha S_2}{1-\alpha S_2}\right)\left(\alpha + \frac{(1-\alpha^2)\,S_2}{1+\alpha S_2}\right) \\ &= \frac{(\alpha S_2^* - 1)\,(1+\alpha S_2)}{(1+\alpha S_2^*)\,(1-\alpha S_2)}\left(\frac{\alpha + S_2}{1+\alpha S_2}\right) \\ &= \frac{\alpha - S_2}{1-\alpha S_2}. \end{aligned} \tag{8.4.0.21}$$

And we have indeed that Eq. 8.4.0.19 equals Eq. 8.4.0.21 which shows that for this example both limits commute.

8.5. The Generalised Schur Complement

The key to understand the relation between Eq. (8.4.0.21) and Eq. (8.4.0.19) is to encode both, the adiabatic elimination and the feedback reduction as instances of Schur complements in the model Itō matrix. The properties of the Schur complement as presented in Appendix A are stated for matrices with complex scalar entries. In order to apply these results in the set-up of this chapter these statements have to be generalised to operator valued matrices.

Let h be a Hilbert space and consider a decomposition of h of the form $h = \oplus_{j\in J} h_j$ for some finite index set J. Let M be a bounded invertible operator on h and pick some non-trivial

subsets $A, B \subset J$ and write for $a_k \in A, k = 1, \ldots, n$, $b_l \in B, l = 1, \ldots, m$

$$M_{A,B} = \begin{bmatrix} M_{a_1,b_1} & M_{a_1,b_2} & \ldots & M_{a_1,b_m} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ M_{a_n,b_1} & M_{a_n,b_2} & \ldots & M_{a_n,b_m} \end{bmatrix}$$

Let A and B be two sets. We denote the relative complement of B in A by A/B.

If we drop the requirement of M being invertible we obtain in the spirit of Definition A.2 the following:

Definition 8.8 (Generalised Schur Complement) *Let A and B be non-trivial subsets of the finite index set J and choose non-trivial subsets $C \subset A$ and $D \subset B$. Furthermore take $\mid A \mid = \mid B \mid$ and $\mid C \mid = \mid D \mid$. Suppose the sub-block $M_{C,D}$ possesses a generalised inverse $(M_{C,D})^-$, then the Schur complement of $M_{A,B}$ relative to $M_{C,D}$ is defined to be*

$$M_{A,B}/M_{C,D} = M_{A/C,B/D} - M_{A/C,D} \, (M_{C,D})^- \, M_{C,B/D}.$$

The generalised Schur complement is well-defined and independent of the choice of generalised inverse if and only if $\operatorname{Im} M_{C,B/D} \subset \operatorname{Im} M_{C,D}$ *and* $\operatorname{Ker} M_{C,D} \subset \operatorname{Ker} M_{A/C,D}$.

The next result is a generalisation of the quotient rule Eq. (A.1.0.4), [70, Theorem 1.4],[62, Eq. (4.116)] to operator valued matrices with extension to a statement about commutativity of the order of taking the Schur complement with respect to indices B and C.

This result is the main technical result needed to prove commutativity of the adiabatic elimination and the feedback reduction.

Lemma 8.9 *Suppose that A, B, C is a partition of the finite index set J then, whenever the generalised Schur complements are well-defined, we have the rule*

$$\begin{aligned} M/M_{B\cup C,B\cup C} &= (M/M_{C,C})\,/\,(M/M_{C,C})_{B,B}\,, \\ &= (M/M_{B,B})\,/\,(M/M_{B,B})_{C,C}\,. \end{aligned} \tag{8.5.0.22}$$

Proof (Lemma 8.9):

The proof follows by comparison of $M/M_{B\cup C,B\cup C}$, $(M/M_{C,C})\,/\,(M/M_{C,C})_{B,B}$ and $(M/M_{B,B})\,/\,(M/M_{B,B})_{C,C}$.

We compute the first part of Lemma 8.9 $M/M_{B\cup C,B\cup C}$ with

$$\begin{aligned} M/M_{B\cup C,B\cup C} &= \begin{bmatrix} M_{A,A} & M_{A,B} & M_{A,C} \\ M_{B,A} & M_{B,B} & M_{B,C} \\ M_{C,A} & M_{C,B} & M_{C,C} \end{bmatrix} / \begin{bmatrix} M_{B,B} & M_{B,C} \\ M_{C,B} & M_{C,C} \end{bmatrix}, \\ &= M_{A,A} - [M_{A,B},\ M_{A,C}] \begin{bmatrix} M_{B,B} & M_{B,C} \\ M_{C,B} & M_{C,C} \end{bmatrix}^{-} \begin{bmatrix} M_{B,A} \\ M_{C,A} \end{bmatrix}. \end{aligned} \tag{8.5.0.23}$$

The inverse in the last equation can be computed explicitly using Banachiewicz inversion

formula in Lemma A.4:

$$
\begin{aligned}
&\begin{bmatrix} M_{B,B} & M_{B,C} \\ M_{C,B} & M_{C,C} \end{bmatrix}^{-}, \\
&\overset{Eq.(A.1.0.10)}{=} \begin{bmatrix} M_{B,B}^{-} + M_{B,B}^{-} M_{B,C} X^{-} M_{C,B} M_{B,B}^{-} & -M_{B,B}^{-} M_{B,C} X^{-} \\ -X^{-} M_{C,B} M_{B,B}^{-} & X^{-} \end{bmatrix}, \qquad (8.5.0.24) \\
&\overset{Eq.(A.1.0.11)}{=} \begin{bmatrix} Y^{-} & -Y^{-} M_{B,C} M_{C,C}^{-} \\ -M_{C,C}^{-} M_{C,B} Y^{-} & M_{C,C}^{-} + M_{C,C}^{-} M_{C,B} Y^{-} M_{B,C} M_{C,C}^{-} \end{bmatrix}, \qquad (8.5.0.25)
\end{aligned}
$$

where $X = M_{B\cup C,B\cup C}/M_{B,B}$, $Y = M_{B\cup C,B\cup C}/M_{C,C}$. Using this and multiplying out Eq. (8.5.0.23) yields

$$
\begin{aligned}
& M/M_{B\cup C,B\cup C}, \\
\overset{Eq.(8.5.0.24)}{=}\ & M_{A,A} - M_{A,B} X M_{B,A} + M_{A,B} X M_{B,C} M_{C,C}^{-} M_{C,A} + M_{A,C} M_{C,C}^{-} M_{C,B} X M_{B,A} \\
& - M_{A,C} M_{C,C}^{-} M_{C,A} - M_{A,C} M_{C,C}^{-} M_{C,B} X M_{B,C} M_{C,C}^{-} M_{C,A}, \\
\overset{Eq.(8.5.0.24)}{=}\ & M_{A\cup C,A\cup C}/M_{C,C} - M_{A\cup C,B\cup C}/M_{C,C} \left(M_{B\cup C,B\cup C}/M_{C,C}\right)^{-} M_{B\cup C,A\cup C}/M_{C,C}, \qquad (8.5.0.26) \\
\overset{Eq.(8.5.0.25)}{=}\ & M_{A\cup B,A\cup B}/M_{B,B} - M_{A\cup B,B\cup C}/M_{B,B} \left(M_{B\cup C,B\cup C}/M_{B,B}\right)^{-} M_{B\cup C,A\cup B}/M_{B,B}. \qquad (8.5.0.27)
\end{aligned}
$$

Next we compute the right hand side of the first equation in Lemma 8.9, that is

$(M/M_{C,C})\,/\,(M/M_{C,C})_{B,B}$:

$$\begin{aligned} N = M/M_{C,C} &= \begin{bmatrix} M_{A,A} & M_{A,B} & M_{A,C} \\ M_{B,A} & M_{B,B} & M_{B,C} \\ M_{C,A} & M_{C,B} & M_{C,C} \end{bmatrix} / M_{C,C}, \\ &= \begin{bmatrix} M_{A,A} & M_{A,B} \\ M_{B,A} & M_{B,B} \end{bmatrix} - \begin{bmatrix} M_{A,C} \\ M_{B,C} \end{bmatrix} (M_{C,C})^{-} \, [M_{C,A},\ M_{C,B}], \\ &= \begin{bmatrix} M_{A,A} - M_{A,C} M_{C,C}^{-} M_{C,A} & M_{A,B} - M_{A,C} M_{C,C}^{-} M_{C,B} \\ M_{B,A} - M_{B,C} M_{C,C}^{-} M_{C,A} & M_{B,B} - M_{B,C} M_{C,C}^{-} M_{C,B} \end{bmatrix}, \\ &= \begin{bmatrix} M_{A\cup C,A\cup C}/M_{C,C} & M_{A\cup C,B\cup C}/M_{C,C} \\ M_{B\cup C,A\cup C}/M_{C,C} & M_{B\cup C,B\cup C}/M_{C,C} \end{bmatrix}. \end{aligned}$$

Performing the second Schur complement $N/N_{B,B} = (M/M_{C,C})/(M/M_{C,C})_{B,B}$ results in:

$$N/N_{B,B} = M_{A\cup C,A\cup C}/M_{C,C} - M_{A\cup C,B\cup C}/M_{C,C}\,(M_{B\cup C,B\cup C}/M_{C,C})^{-}\, M_{B\cup C,A\cup C}/M_{C,C},$$

which is identical to Equation (8.5.0.26) as required. The second equality of Lemma 8.9 follows mutatis mutandis from the last calculation when interchanging B and C and comparing with Eq. (8.5.0.27).

□

Next we establish the conditions we have to impose in order for all Schur complements appearing in Lemma 8.9 to be well-defined and independent of the choices of generalised inverses.

Lemma 8.10 *If*

$$\operatorname{Ker}\begin{bmatrix} M_{B,B} & M_{B,C} \\ M_{C,B} & M_{C,C} \end{bmatrix} \subseteq \operatorname{Ker}[M_{A,B}, M_{A,C}] \tag{8.5.0.28}$$

$$\operatorname{Im}\begin{bmatrix} M_{B,A} \\ M_{C,A} \end{bmatrix} \subseteq \operatorname{Im}\begin{bmatrix} M_{B,B} & M_{B,C} \\ M_{C,B} & M_{C,C} \end{bmatrix} \tag{8.5.0.29}$$

$$\operatorname{Ker} M_{C,C} \subseteq \operatorname{Ker} M_{B,C} \tag{8.5.0.30}$$

$$\operatorname{Im} M_{C,B} \subseteq \operatorname{Im} M_{C,C} \tag{8.5.0.31}$$

$$\operatorname{Ker} M_{B,B} \subseteq \operatorname{Ker} M_{C,B} \tag{8.5.0.32}$$

$$\operatorname{Im} M_{B,C} \subseteq \operatorname{Im} M_{B,B} \tag{8.5.0.33}$$

then the Schur complements $(M/M_{C,C})/(M/M_{C,C})_{B,B}$ *and* $(M/M_{B,B})/(M/M_{B,B})_{C,C}$ *are well-defined and independent of the choice of generalised inverses.*

Proof:

Collecting all Schur complements appearing directly in Lemma 8.9, we see that we require

i) $M/M_{B\cup C,B\cup C}$,

ii) $M/M_{C,C}, M/M_{B,B}$,

iii) $(M/M_{C,C})/(M/M_{C,C})_{B,B}$ and $(M/M_{B,B})/(M/M_{B,B})_{C,C}$,

to be well-defined. Additionally, when computing $M/M_{B\cup C,B\cup C}$ (see proof of Lemma 8.9, Eq. (8.5.0.26)-(8.5.0.27)) we obtain

$$\begin{aligned} &M/M_{B\cup C,B\cup C}, \\ =&M_{A\cup C,A\cup C}/M_{C,C} - M_{A\cup C,B\cup C}/M_{C,C}\left(M_{B\cup C,B\cup C}/M_{C,C}\right)^{-} M_{B\cup C,A\cup C}/M_{C,C}, \\ =&M_{A\cup B,A\cup B}/M_{B,B} - M_{A\cup B,B\cup C}/M_{B,B}\left(M_{B\cup C,B\cup C}/M_{B,B}\right)^{-} M_{B\cup C,A\cup B}/M_{B,B}, \end{aligned}$$

such that we have to add the following Schur complements to the list of complements we require to be well-defined:

iv) $M_{A\cup C,A\cup C}/M_{C,C},\ M_{A\cup C,B\cup C}/M_{C,C},\ M_{B\cup C,A\cup C}/M_{C,C},\ M_{B\cup C,B\cup C}/M_{C,C}$

v) $M_{A\cup B,A\cup B}/M_{B,B},\ M_{A\cup B,B\cup C}/M_{B,B},\ M_{B\cup C,B\cup C}/M_{B,B},\ M_{B\cup C,A\cup B}/M_{B,B}$

vi) $M_{A\cup C,A\cup C}/M_{C,C} - M_{A\cup C,B\cup C}/M_{C,C}\left(M_{B\cup C,B\cup C}/M_{C,C}\right)^{-} M_{B\cup C,A\cup C}/M_{C,C}$

Recall that the requirements on the kernel and image space inclusion for the Schur complement to be well-defined, Eq. (A.1.0.9). We can think of M assuming the partition

$$M = \begin{bmatrix} M_{A,A} & M_{A,B} & M_{A,C} \\ M_{B,A} & M_{B,B} & M_{B,C} \\ M_{C,A} & M_{C,B} & M_{C,C} \end{bmatrix}.$$

Now,

$$M/M_{B\cup C,B\cup C} = M_{A,A} - [M_{A,B}, M_{A,C}] \begin{bmatrix} M_{B,B} & M_{B,C} \\ M_{C,B} & M_{C,C} \end{bmatrix}^{-} \begin{bmatrix} M_{B,A} \\ M_{C,A} \end{bmatrix}$$

is well-defined due to Relation (8.5.0.28) and Relation (8.5.0.29). Similarly

$$M_{B\cup C,B\cup C}/M_{C,C} = \begin{bmatrix} M_{B,B} & M_{B,C} \\ M_{C,B} & M_{C,C} \end{bmatrix} /M_{B,B} = M_{C,C} - M_{C,B}M_{B,B}^{-}M_{B,C}$$

is well-defined because of Relation (8.5.0.32) and Relation (8.5.0.33) and

$$M_{B\cup C,B\cup C}/M_{B,B} = \begin{bmatrix} M_{B,B} & M_{B,C} \\ M_{C,B} & M_{C,C} \end{bmatrix} /M_{C,C} = M_{B,B} - M_{B,C}M_{C,C}^{-}M_{C,B}$$

is well-defined by of Relation (8.5.0.30) and Relation (8.5.0.31).

To ensure that $M/M_{B,B}, M/M_{C,C}, M_{A\cup C,A\cup C}/M_{C,C}, M_{A\cup C,B\cup C}/M_{C,C}$ and

$M_{B\cup C,A\cup C}/M_{C,C}$ are well-defined we have to require that

$$\begin{aligned}
\mathrm{Ker}\, M_{B,B} &\subseteq \mathrm{Ker} \begin{bmatrix} M_{A,B} \\ M_{C,B} \end{bmatrix}, && (8.5.0.34)\\
\mathrm{Im}[M_{B,A}, M_{B,C}] &\subseteq \mathrm{Im}\, M_{B,B}, && (8.5.0.35)\\
\mathrm{Ker}\, M_{C,C} &\subseteq \mathrm{Ker} \begin{bmatrix} M_{A,C} \\ M_{B,C} \end{bmatrix}, && (8.5.0.36)\\
\mathrm{Im}[M_{C,A}, M_{C,B}] &\subseteq \mathrm{Im}\, M_{C,C}. && (8.5.0.37)
\end{aligned}$$

One can show that relations Eq. (8.5.0.28) - (8.5.0.33) imply Eq. (8.5.0.34) - (8.5.0.37) by the following:

By Relation (8.5.0.32) we see that $M_{B,B}x = 0 \implies M_{C,B}x = 0$ such that together with Relation (8.5.0.28) $M_{B,B}x = 0 \implies \begin{bmatrix} M_{B,B} \\ M_{C,B} \end{bmatrix} x = 0 \implies M_{A,B}x = 0$ and therefore

$$\mathrm{Ker}\, M_{B,B} \subseteq \mathrm{Ker} \begin{bmatrix} M_{A,B} \\ M_{C,B} \end{bmatrix},$$

i.e. Relation (8.5.0.34) holds. Now, by Relation (8.5.0.29) we see that $\forall x \exists y, z$ such that

$$\begin{bmatrix} M_{B,A} \\ M_{C,A} \end{bmatrix} x = \begin{bmatrix} M_{B,B}y + M_{B,C}z \\ M_{C,B}x + M_{C,C}z \end{bmatrix}. \qquad (8.5.0.38)$$

Similarly by Relations (8.5.0.31) and (8.5.0.33), $\exists v, w$ such that

$$\begin{aligned}
M_{C,B}y &= M_{C,C}v, && (8.5.0.39)\\
M_{B,C}z &= M_{B,B}w, && (8.5.0.40)
\end{aligned}$$

and therefore by combining Relations (8.5.0.38) - (8.5.0.40) we see that $\mathrm{Im}\, M_{B,A} \subseteq M_{B,B}$

and $\mathrm{Im}\, M_{C,A} \subseteq M_{C,C}$. This together with Relation (8.5.0.33) gives

$$\mathrm{Im}[M_{B,A}, M_{B,C}] \subseteq \mathrm{Im}\, M_{B,B},$$

that is Relation (8.5.0.35). By similar arguments we can see that

- Relations (8.5.0.28) and (8.5.0.30) $\implies$ (8.5.0.36)
- Relations (8.5.0.29) and (8.5.0.31) $\implies$ (8.5.0.37)
- Relations (8.5.0.28), (8.5.0.29), (8.5.0.32) & (8.5.0.33) $\implies$ $\mathrm{Im}\, M_{B,A} \subseteq \mathrm{Im}\, M_{B,B}$, $\mathrm{Ker}\, M_{B,B} \subseteq \mathrm{Ker}\, M_{A,B}$ $\implies$ $M_{A\cup B,A\cup B}/M_{B,B}$ is well-defined
- Relations (8.5.0.28), (8.5.0.32) & (8.5.0.33) $\implies$ $\mathrm{Im}\, M_{B,C} \subseteq M_{B,B}$, $\mathrm{Ker}\, M_{B,B} \subseteq \mathrm{Ker}\, M_{A,B}$ $\implies$ $M_{A\cup B,B\cup C}/M_{B,B}$ is well-defined
- Relations (8.5.0.29), (8.5.0.32) & (8.5.0.33) $\implies$ $\mathrm{Im}\, M_{B,A} \subseteq M_{B,B}$, $\mathrm{Ker}\, M_{B,B} \subseteq \mathrm{Ker}\, M_{C,B}$ $\implies$ $M_{B\cup C,A\cup B}/M_{B,B}$ is well-defined

Next we have to show that Relations (8.5.0.28) - (8.5.0.37) ensure that $(M/M_{C,C})/(M/M_{C,C})_{B,B}$ is well-defined. See Eq. (8.5)

$$\begin{aligned}
&(M/M_{C,C})/(M/M_{C,C})_{B,B} \\
=&M_{A,A} - M_{A,C}(M_{C,C})^{-}M_{C,A} - (M_{A,B} - M_{A,C}(M_{C,C})^{-} \\
&\times M_{C,B})(M_{B,B} - M_{B,C}(M_{C,C})^{-}M_{C,B})^{-}(M_{B,A} - M_{B,C}(M_{C,C})^{-}M_{C,A}),
\end{aligned}$$

and we therefore have to require that

$$\mathrm{Ker}\left(M_{B,B} - M_{B,C}(M_{C,C})^{-}M_{C,B}\right) \subseteq \mathrm{Ker}\left(M_{A,B} - M_{A,C}(M_{C,C})^{-}M_{C,B}\right) \quad (8.5.0.41)$$

$$\mathrm{Im}\left(M_{B,A} - M_{B,C}(M_{C,C})^{-}M_{C,A}\right) \subseteq \mathrm{Im}\left(M_{B,B} - M_{B,C}(M_{C,C})^{-}M_{C,B}\right) \quad (8.5.0.42)$$

Now, take $v \in \mathrm{Im}\,(M_{B,A} - M_{B,C}(M_{C,C})^{-}M_{C,A})$ such that

$$\begin{aligned} v &= \left(M_{B,A} - M_{B,C}(M_{C,C})^{-}M_{C,A}\right) w, \\ &= [1,\ -M_{B,C}M_{C,C}^{-}] \begin{bmatrix} M_{B,A}w \\ M_{C,A}w \end{bmatrix}. \end{aligned}$$

Relation (8.5.0.29) implies that $\forall w\ \exists\ x, y$ such that

$$\begin{bmatrix} M_{B,A}w \\ M_{C,A}w \end{bmatrix} = \begin{bmatrix} M_{B,B}x + M_{B,C}y \\ M_{C,B}x + M_{C,C}y \end{bmatrix}$$

and therefore

$$\begin{aligned} v &= [1,\ -M_{B,C}M_{C,C}^{-}] \begin{bmatrix} M_{B,A}w \\ M_{C,A}w \end{bmatrix}, \\ &= [1,\ -M_{B,C}M_{C,C}^{-}] \begin{bmatrix} M_{B,B}x + M_{B,C}y \\ M_{C,B}x + M_{C,C}y \end{bmatrix}, \\ &= M_{B,B}x + M_{B,C}y - M_{B,C}M_{C,C}^{-}M_{C,C}x - M_{B,C}M_{C,C}^{-}M_{C,C}y, \\ &= \left(M_{B,B} - M_{B,C}M_{C,C}^{-}M_{C,B}\right) x, \end{aligned}$$

where the last line follows since $M_{B,C}M_{C,C}^{-}M_{C,C} = M_{B,C}$ by Relation (8.5.0.30) and Lemma A.1 and hence we verified Relation (8.5.0.41).

To show Relation (8.5.0.42) we pick some $x \in \mathrm{Ker}\left(M_{B,B} - M_{B,C}M_{C,C}^{-}M_{C,B}\right)$

$$M_{B,B} - M_{B,C}M_{C,C}^{-}M_{C,B}x = [M_{B,B},\ M_{B,C}] \begin{bmatrix} x \\ -M_{C,C}^{-}M_{C,B}x \end{bmatrix} = 0$$

and therefore

$$\begin{bmatrix} x \\ -M_{C,C}^{-}M_{C,B}x \end{bmatrix} \in \mathrm{Ker}[M_{B,B},\ M_{B,C}].$$

By Relation (8.5.0.37) and Lemma A.1 we have $M_{C,C}M_{C,C}^{-}M_{C,B} = M_{C,B}$ and thus

$$\left(M_{C,B} - M_{C,C}M_{C,C}^{-}M_{C,B}\right)x = 0$$

and therefore

$$\begin{bmatrix} x \\ -M_{C,C}^{-}M_{C,B}x \end{bmatrix} \in \begin{bmatrix} M_{B,B} & M_{B,C} \\ M_{C,B} & M_{C,C} \end{bmatrix}.$$

Combine this with Relation (8.5.0.28) and we see that

$$\begin{bmatrix} x \\ -M_{C,C}^{-}M_{C,B}x \end{bmatrix} \in \mathrm{Ker}[M_{A,B},\ M_{A,C}]$$

which implies that $\left(M_{A,B} - M_{A,C}M_{C,C}^{-}M_{C,B}\right)x = 0$ and therefore verifies Relation (8.5.0.42).

□

8.6. Adiabatic Elimination and Systems in Loop: The Main Result

8.6.1. Adiabatic Elimination as a Schur complement

We are now ready to formulate the Adiabatic Elimination limit and the Feedback Reduction limit as instances of Schur complements. In this section we will formulate the adiabatic elimination limit as a Schur complement. Recall the post-limit system parameter for the Adiabatic

Elimination limit given in Theorem 8.1,

$$\hat{S} = (I + CA^{-1}C^*)S,$$
$$\hat{L} = G - CA^{-1}Z,$$
$$\hat{K} = R - XA^{-1}Z,$$

and observe that these parameter have the structure of Schur complements $M/A = D - CA^{-1}B$. Consider the pre-limit Itō matrix (as introduced in Chapter 5.3),

$$\mathbf{G}(k) = \begin{bmatrix} K(k) & -L(k)^*S \\ L(k) & S - I \end{bmatrix},$$

such that the post-limit system parameter are given by

$$S(k) = [I, ka^*] \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I \\ ka \end{bmatrix}, \tag{8.6.1.1}$$
$$L(k) = [I, ka^*] \begin{bmatrix} G & C \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I \\ ka \end{bmatrix}, \tag{8.6.1.2}$$
$$K(k) = [I, ka^*] \begin{bmatrix} R & X \\ Z & A \end{bmatrix} \begin{bmatrix} I \\ ka \end{bmatrix}. \tag{8.6.1.3}$$

The system Itō matrix is given as an operator on $\mathfrak{h} \otimes (\mathbb{C} \oplus \mathfrak{K})$ with initial space $\mathfrak{h}$. Now, remember that the set-up of the Adiabatic Elimination limit assumed a decomposition of the Hilbert space $\mathfrak{h}$ of the form $\mathfrak{h} = \mathfrak{h}_{\text{aux}} \oplus \mathfrak{h}_{\text{fast}}$ such that the system Itō matrix can be decomposed with respect to

$$\mathfrak{h} \otimes (\mathbb{C} \oplus \mathfrak{K}) = [\mathfrak{h}_{\text{aux}} \otimes (\mathbb{C} \oplus \mathfrak{K})] \oplus [h_{\text{osc}} \otimes (\mathbb{C} \oplus \mathfrak{K})]$$

with

$$\mathbf{G}(k) = k^2 a^* g_{ff} a + k a^* g_{fs} + k g_{sf} a + g_{ss} = [I,\ ka^*] \begin{bmatrix} g_{ss} & g_{sf} \\ g_{fs} & g_{ff} \end{bmatrix} \begin{bmatrix} I \\ ka \end{bmatrix}. \tag{8.6.1.4}$$

One can see that the entries of the Itō system matrix are given by

$$\begin{aligned} g_{ss} &= \begin{bmatrix} R & -G^* S \\ G & S - I \end{bmatrix}, & g_{sf} &= \begin{bmatrix} X & 0 \\ C & 0 \end{bmatrix}, \\ g_{fs} &= \begin{bmatrix} Z & -C^* S \\ 0 & 0 \end{bmatrix}, & g_{ff} &= \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned} \tag{8.6.1.5}$$

We can now observe that

$$\begin{aligned} g/g_{ff} = g_{ss} - g_{sf} \bar{g_{ff}} g_{fs} &= \begin{bmatrix} R & -G^* S \\ G & S - I \end{bmatrix} - \begin{bmatrix} X & 0 \\ C & 0 \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}^- \begin{bmatrix} Z & -C^* S \\ 0 & 0 \end{bmatrix}, \\ &= \begin{bmatrix} R - XA^{-1}Z & -G^* S + XA^{-1}C^* S \\ G - CA^{-1}Z & (S + CA^{-1}C^* S) - I \end{bmatrix}, \end{aligned} \tag{8.6.1.6}$$

where the choice of generalised inverse is the Moore-Penrose inverse

$$\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}^- = \begin{bmatrix} A^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

This must be compared to the Itō system matrix $\hat{G}$ obtained after taking the adiabatic elimination limit

$$\hat{G} = \begin{bmatrix} \hat{K} & -\hat{L}^* \hat{S} \\ \hat{L} & \hat{S} - I \end{bmatrix} = \begin{bmatrix} R - XA^{-1}Z & -G^* S + XA^{-1}C^* S \\ G - CA^{-1}Z & (I + CA^{-1}C^*)S - I \end{bmatrix} \tag{8.6.1.7}$$

where the right hand side follows from the post-limit parameter given in Theorem 8.1 and the top-right corner follows from the identity

$$\begin{aligned}
-\hat{L}^*\hat{S} &= -\left(G - CA^{-1}Z\right)^*\left(I + CA^{-1}C^*\right)S, \\
&= -G^*\left(I + CA^{-1}C^*\right)S + Z^*\left(A^{-1}\right)^* C^*\left(CA^{-1}C^*\right)S, \\
&\overset{\text{Eq. (8.3.0.16)}}{=} -G^*\left(I + CA^{-1}C^*\right)S + Z^*A^{-1}C^*S, \\
&= -G^*S - \left(G^*C + Z^*\right)A^{-1}C^*S, \\
&\overset{\text{Eq. (8.2.1.9)}}{=} -G^*S + XA^{-1}C^*S.
\end{aligned}$$

We see that Eq. (8.6.1.6) and Eq. (8.6.1.7) coincide and that the adiabatic elimination limit is given as an Schur complement in the matrix $g = \begin{bmatrix} g_{ss} & g_{sf} \\ g_{fs} & g_{ff} \end{bmatrix}$ *by*

$$\hat{G} = g/g_{ff} = g_{ss} - g_{sf}\bar{g_{ff}}g_{fs}.$$

8.6.2. Feedback reduction as a Schur complement

Recall the representation of networks of quantum components in Chapter 5.3 and the feedback reduction formula given in Section 5.5. As the most general case of the set-up introduced in Section 8.4 we assume a set-up of the following form:

We consider a collection $j = 1, 2, \ldots n$ components with parameter triple (S_j, L_j, K_j) and we may as shown in Section 5.5 collect this parameter into a single model (S, L, K) with

$$S = \begin{bmatrix} S_1 & 0 & \ldots & 0 \\ 0 & S_2 & \ldots & 0 \\ \vdots & & \ddots & 0 \\ 0 & \ldots & 0 & S_n \end{bmatrix}, \; L = \begin{bmatrix} L_1 \\ L_2 \\ \vdots \\ L_n \end{bmatrix}, \; K = \sum_{j=1}^{n} K_j.$$

To complete the description of of the network we have to provide a list of internal edges (see Section 5.3). With this list we can obtain a decomposition of the networks colour space $\mathfrak{K}$ with respect to the networks internal and external channels, that is a decomposition $\mathfrak{K} = \mathfrak{K}_e \oplus \mathfrak{K}_i$. The Itō system matrix will then be given as an operator on $\mathfrak{h} \otimes (\mathbb{C} \oplus \mathfrak{K})$ with $\mathfrak{h} = \bigotimes_{j=1}^{n} \mathfrak{h}_j$ where $\mathfrak{h}_j$ is the Hilbert space corresponding to component (S_j, L_j, K_j).

We can now partition the models Itō matrix in a similar way to the previous section with respect to the internal and external channels

$$\mathfrak{h} \otimes (\mathbb{C} \oplus \mathfrak{K}) = [\mathfrak{h} \otimes (\mathbb{C} \oplus \mathfrak{K}_e)] \oplus [\mathfrak{h} \otimes \mathfrak{K}_i],$$

and obtain

$$G = \begin{bmatrix} G_{ee} & G_{ei} \\ G_{ie} & G_{ii} \end{bmatrix}.$$

Now the feedback reduction formula as given in Section 5.5 reads as

$$\begin{aligned} S^{\text{red}} &= S_{ee} + S_{ei} (\eta - S_{ii})^{-1} S_{ie}, \\ L^{\text{red}} &= L_e + S_{ei} (\eta - S_{ii})^{-1} L_i, \\ H^{\text{red}} &= H + \sum_{j=i,e} \operatorname{Im} L_j^{\dagger} S_{ji} (\eta - S_{ii})^{-1} L_i. \end{aligned}$$

This reduction formula is formulated with respect to the model matrix V (see Section 5.3) and the reduced model matrix after eliminating the internal edges in the zero time delay limit is given by Theorem 5.2 with

$$V_{\alpha\beta}^{\text{red}} = V_{\alpha\beta} + V_{\alpha r_0} (1 - V_{s_0 r_0})^{-1} V_{s_0 \beta}, \tag{8.6.2.1}$$

given that $(1 - V_{s_0 r_0})^{-1}$ exists and with $\alpha \in \{0\} \cup \mathcal{P}_{\text{out}}/\{s_0\}$, $\beta \in \{0\} \cup \mathcal{P}_{\text{in}}/\{r_0\}$.

Denote with $\mathcal{F}$ the operation of eliminating the internal edges (in the feedback reduction

limit) of model G.

Equation (8.6.2.1) has here the structure of a linear fractional transformation which as already been encountered when computing the transfer functions of linear quantum components, see Chapter 6. The linear fractional transformation and the Schur complement are closely related and by formulating the feedback reduction formula in terms of the Itō system matrix G, that is by substituting S with $S - I$, the feedback reduction formula assumes the structure of a Schur complement as desired and we see that,

$$\mathcal{F}G = g_{ee} - g_{ei}\left(g_{ii}\right)^{-1} g_{ie}.$$

8.6.3. Commutativity of the Limits

Since we encoded both objects of interest, the adiabatic elimination limit and the feedback reduction in the zero time delay limit as instances of Schur complements in the networks Itō matrix we are now ready to establish the main statement, i.e. that both limits commute. In order to archive this we have to partition the networks Itō matrix with respect to the internal and external channels and the fast and slow part as seen before. That given, we can establish the commutativity by invoking Lemma 8.9 to ensure that the order of the Schur complements commutes and thus the order of limits doesn't change the result.

We quote the structure imposed on the system parameter given in Equation (8.2.1.1)

$$\begin{aligned} S(k) &= S \otimes I, \\ L(k) &= k\sum_j C_j \otimes a_j + G \otimes I, \\ K(k) &= k^2 \sum_{jl} A_{jl} \otimes a_j^* a_l + k\sum_j Z_j \otimes a_j^* + k\sum_j X_j \otimes a_j + R \otimes I. \end{aligned} \tag{8.6.3.1}$$

We are setting the stage by assuming that the system parameter triple $(S(k), L(k), K(k))$

assume the following partition with respect to the internal and external channels

$$S = \begin{bmatrix} S_{ee} & S_{ei} \\ S_{ie} & S_{ii} \end{bmatrix},\ C = \begin{bmatrix} C_e \\ C_i \end{bmatrix},\ G = \begin{bmatrix} G_e \\ G_i \end{bmatrix}. \tag{8.6.3.2}$$

We are now ready to construct the Itō matrix, partitioned with respect to slow, fast, internal and external components.

Combining the partitioned Itō matrix Equation (8.6.1.4)-(8.6.1.5) with the partitioned operator entries Equation (8.6.3.2) yields

$$g(k) = \begin{bmatrix} g_{ss} & g_{sf} \\ g_{fs} & g_{ff} \end{bmatrix} = \left[\begin{array}{ccc|ccc} R & -G^*S_e & -G^*S_i & X & 0 & 0 \\ G_e & S_{ee}-I & S_{ei} & C_e & 0 & 0 \\ G_i & S_{ie} & S_{ii}-I & C_i & 0 & 0 \\ \hline Z & -C^*S_e & -C^*S_i & A & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \} \text{ slow, external} \\ \} \text{ slow, internal} \\ \\ \} \text{ fast, external} \\ \} \text{ fast, internal} \end{array}, \tag{8.6.3.3}$$

$$G(k) = [I, a^*] \begin{bmatrix} g_{ss} & g_{sf} \\ g_{fs} & g_{ff} \end{bmatrix} \begin{bmatrix} I \\ a \end{bmatrix}$$

which corresponds to $G(k)$ partitioned with respect to the decomposition

$$(\mathfrak{h}_{\text{aux}} \oplus \mathfrak{h}_{\text{osc}}) \otimes (\mathbb{C} \oplus \mathfrak{K}_e \oplus \mathfrak{K}_i) = \underbrace{[\mathfrak{h}_{\text{aux}} \otimes (\mathbb{C} \oplus \mathfrak{K}_e)]}_{\text{slow,ext.}} \oplus \underbrace{[\mathfrak{h}_{\text{aux}} \otimes \mathfrak{K}_i]}_{\text{slow, int.}} \oplus \underbrace{[\mathfrak{h}_{\text{osc}} \otimes (\mathbb{C} \oplus \mathfrak{K}_e)]}_{\text{fast, ext.}} \oplus \underbrace{[\mathfrak{h}_{\text{osc}} \otimes \mathfrak{K}_i]}_{\text{fast, int.}}.$$

The fast components are then assembled in the Itō matrix

$$g_{ff} = \begin{bmatrix} A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and the internal components are given by

$$g_{ii} = \begin{bmatrix} S_{ii} - I & 0 \\ 0 & 0 \end{bmatrix}.$$

The adiabatic elimination limit corresponds now to

$$\mathcal{A}G(k) = g(k)/g_{ff},$$

and the feedback reduction limit corresponds to

$$\mathcal{F}G(k) = [I, a^*]\,(g(k)/g_{ii}) \begin{bmatrix} I \\ a \end{bmatrix}.$$

We see that the adiabatic elimination limit and feedback reduction limit are both given as Schur complements in one specific Itō matrix, Equation (8.6.3.3). The commutativity can now be established by invoking Lemma 8.9:

Theorem 8.11 *Let $G(k)$ and $\mathcal{F}G(k)$ correspond to strictly Hurwitz stable open quantum systems (i.e., the A matrix of each system is strictly Hurwitz stable), and suppose that $S_{ii} + C_i A^{-1} C^* S_i - I$ and $S_{ii} - I$ are invertible. Then in the notation established above we have*

$$\mathcal{A}\mathcal{F}G(k) = \mathcal{F}\mathcal{A}G(k).$$

Proof:

The proof follows from Lemma 8.9. In order to ensure that all Schur complements appearing in Lemma 8.9 are well-defined and independent of the choice of generalised inverse we can invoke Lemma 8.10. We are now left with checking that for the Itō matrix Equation (8.6.3.3) the conditions Equation (8.5.0.28) - (8.5.0.33) of Lemma 8.10 are satisfied.

We can collect the entries of the Itō matrix Equation (8.6.3.3) in the following way

$$g(k) = \begin{bmatrix} g_{ss} & g_{sf} \\ g_{fs} & g_{ff} \end{bmatrix} = \left[\begin{array}{cc|cc} R_1 & X_1 & M_1 & 0 \\ G_1 & S_{ii} - I & C_1 & 0 \\ \hline Z_1 & -C^* S_i & A & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] = \begin{bmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & g_{44} \end{bmatrix},$$

where we have the following correspondence of labels: 1 = slow external, 2 = slow internal, 3 = fast external and 4 = fast internal. We further have the matrices

$$R_1 = \begin{bmatrix} R & -G^* S_e \\ G_e & S_{ee} - I \end{bmatrix}, \quad X_1 = \begin{bmatrix} X \\ C_e \end{bmatrix}, \; M_1 = \begin{bmatrix} -G^* S_i \\ S_{ei} \end{bmatrix}, \; S_e = \begin{bmatrix} S_{ee} \\ S_{ie} \end{bmatrix}, \; S_i = \begin{bmatrix} S_{ei} \\ S_{ii} \end{bmatrix}$$
$$G_1 = \begin{bmatrix} G_i, \; S_{ie} \end{bmatrix}, \qquad Z_1 = \begin{bmatrix} Z, \; -C^* S_e \end{bmatrix}.$$

We identify the index sets A, B, C appearing in Lemma 8.10 with $A = \{1\}$, $B = \{2\}$ and $C = \{3, 4\}$. The first relation in Lemma 8.10 reads therefore as

$$\mathrm{Ker} \begin{bmatrix} S_{ii} - I & C_i & 0 \\ -C^* S_i & A & 0 \\ 0 & 0 & 0 \end{bmatrix} \subseteq \mathrm{Ker} \begin{bmatrix} M_1, \; X_1, \; 0 \end{bmatrix}. \tag{8.6.3.4}$$

Let $(x, y, z)^{\mathrm{T}}$ be an element of

$$\mathrm{Ker} \begin{bmatrix} S_{ii} - I & C_i & 0 \\ -C^* S_i & A & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We now obviously have

$$\begin{aligned} -C^* S_i x + A y = 0 \qquad &\Longrightarrow \qquad y = A^{-1} C^* S_i x, \\ (S_{ii} - I)\, x + C_i y = 0 \qquad &\Longrightarrow \qquad 0 = \left(S_{ii} + C_i A^{-1} C^* S_i - I\right) x, \end{aligned}$$

but $(S_{ii} + C_i A^{-1} C^* S_i - I)$ is invertible by hypothesis whence x has to be zero which implies that $y = 0$ and the kernel space of above matrix consists of vectors of the form $(0, 0, z)^{\mathrm{T}}$, z arbitrary, which are clearly contained in

$$\mathrm{Ker} \left[\begin{array}{ccc} M_1, & X_1, & 0 \end{array} \right].$$

We thus verified Relation (8.6.3.4).

In order to verify the second relation (8.5.0.28) we have to show that $\forall\, x\ \exists\, y, z$ such that

$$\left[\begin{array}{c} G_1 x \\ Z_1 x \end{array} \right] = \left[\begin{array}{cc} S_{ii} - I & C_i \\ -C^* S_i & A \end{array} \right] \left[\begin{array}{c} y \\ z \end{array} \right]. \tag{8.6.3.5}$$

This will be true if the matrix

$$\left[\begin{array}{cc} S_{ii} - I & C_i \\ -C^* S_i & A \end{array} \right] \tag{8.6.3.6}$$

is invertible. Recall the Banachiewicz inversion formula for block matrices, i.e.

$$M = \left[\begin{array}{cc} A & B \\ C & D \end{array} \right], \; M^{-1} = \left[\begin{array}{cc} A^{-1} + A^{-1} B \left(M/A\right)^{-1} C A^{-1} & -A^{-1} B \left(M/A\right)^{-1} \\ -\left(M/A\right)^{-1} C A^{-1} & \left(M/A\right)^{-1} \end{array} \right].$$

which shows that M is invertible if (M/A) and A are invertible. The matrix Equation (8.6.3.6) is therefore invertible if $S_{ii} - I + C_i A^{-1} C^* S_i$ and A are invertible which is true by hypothesis

whence Relation (8.6.3.5) holds and we have

$$\mathrm{Im}\begin{bmatrix} G_1 x \\ Z_1 x \\ 0 \end{bmatrix} \subseteq \mathrm{Im}\begin{bmatrix} S_{ii} - I & C_i & 0 \\ -C^* S_i & A & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Now, since A is invertible we see directly that

$$\mathrm{Ker}\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \subseteq \mathrm{Ker}\begin{bmatrix} C_i, & 0 \end{bmatrix},$$
$$\mathrm{Im}\begin{bmatrix} -C^* S_i \\ 0 \end{bmatrix} \subseteq \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix},$$

which verifies Relations (8.5.0.30) and (8.5.0.31). The remaining Relations (8.5.0.32) and (8.5.0.33) follow in a similar manner from the invertibility of $S_{ii} - I$

$$\mathrm{Ker}\,(S_{ii} - I) \subseteq \mathrm{Ker}\begin{bmatrix} -C^* S_i \\ 0 \end{bmatrix},$$
$$\mathrm{Im}\begin{bmatrix} C_i, & 0 \end{bmatrix} \subseteq \mathrm{Im}\,(S_{ii} - I).$$

We thus verified the conditions of Lemma 8.10 and hence Lemma 8.9 holds which proves the theorem.

□

We thus showed that the adiabatic elimination limit and feedback reduction limit commute given that $S_{ii} - I + C_i A^{-1} C^* S_i$ and $S_{ii} - I$ are invertible. This conditions are the conditions for the existence of the feedback reduction limit, where $S_{ii} - I + C_i A^{-1} C^* S_i$ is the internal component of $\hat{S}$, the scattering matrix after taking the adiabatic elimination limit.

We note that the Hurwitz-Stability is only a sufficient but not a necessary condition. In

general we must only ensure that the kernel space condition for the operator Y (see Theorem 8.1) is satisfied.

8.7. Conclusion

In the previous sections of this chapter we studied the question of whether the operations of taking the adiabatic elimination of oscillatory degrees of freedom of a system and taking the instantaneous feed-forward limit of two such systems commute.

The question of commutativity is interesting from a practical and methodical point of view. Part of the elegance of the system theory, among other things, is the possibility to reduce the problem of studying some bigger system to studying its smaller parts. Instead of handling one big system one can start by modelling easy to handle components of the system and obtain the overall system at a later stage by using the network rules to connect these components. It is therefore preferred to model single components and adiabatically eliminate the oscillatory components on the component level. This enables one to study the behaviour of the reduced, isolated components. The opposite approach, i.e. building the network first and then eliminating the oscillatory components would mask the behaviour of the reduced system.

What we found in the course of this chapter is, that under some not too limiting conditions, both limits do commute. The result has been established by showing that both, the adiabatic elimination limit and the instantaneous feed-forward limit are instances of Schur complements. By generalising statements about successive Schur complements we were able to establish the result on an algebraic level.

Part III.

Appendix

A

The Schur Complement

A.1. Introduction

In this Chapter we introduce the Schur complement and present some of its properties. We follow [70] and [62]. There is a wide variety of results in in linear algebra making use of the Schur complement. We are going to introduce a couple of results needed for later use in Chapter 8.

Let M be a block matrix of the form

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}. \tag{A.1.0.1}$$

The Schur complement of A in M, denoted by M/A is defined by

$$M/A = D - CA^{-1}B.$$

The first one result is a very useful inversion formula for block matrices of the form Eq. A.1.0.1, credited to Banachiewicz.

Let M be a block matrix partitioned as in Eq. A.1.0.1 and let A be invertible. The inverse of M is then given by

$$M^{-1} = \begin{bmatrix} A^{-1} + A^{-1}B\left(M/A\right)^{-1}CA^{-1} & -A^{-1}B\left(M/A\right)^{-1} \\ -\left(M/A\right)^{-1}CA^{-1} & \left(M/A\right)^{-1} \end{bmatrix}. \tag{A.1.0.2}$$

If we assume instead that D is non-singular then we have parallel to Eq. (A.1.0.2) [70, Page 13] that

$$M^{-1} = \begin{bmatrix} (M/D)^{-1} & -(M/D)^{-1}BD^{-1} \\ -D^{-1}C(M/D)^{-1} & D^{-1} + D^{-1}C(M/D)^{-1}BD^{-1} \end{bmatrix}. \tag{A.1.0.3}$$

If both A and D are invertible both Eq. (A.1.0.2) and Eq. (A.1.0.3) hold.

Another useful property of the Schur complement is given by the *quotient rule* . If we

consider a partition of M of the form

$$M = \left(\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right)$$

and assume that both, A and A_{11} are invertible, then

$$M/A = (M/A_{11}) / (A/A_{11}) . \tag{A.1.0.4}$$

A useful generalisation of the Schur complement is obtained by dropping the requirement that the matrix A is invertible. This in especially in the application of the Schur complement in Chapter 8 not given. The solution to this problem is given by generalised inverses.

The generalised inverse of a matrix $M \in M_{n\times m}$ is the non-unique matrix $M^- \in M_{m\times n}$ such that

$$MM^-M = M. \tag{A.1.0.5}$$

Lemma A.1 *Let N be some matrix, MM^- acts like the identity matrix in*

$$MM^-N = N, \quad \textit{if and only if} \quad \operatorname{Im} N \subset \operatorname{Im} M \tag{A.1.0.6}$$

$$NM^-M = N, \quad \textit{if and only if} \quad \operatorname{Ker} M \subset \operatorname{Ker} N \tag{A.1.0.7}$$

Similary, for some matrices P and Q, the matrix

$$PM^-Q \tag{A.1.0.8}$$

is independent of the choice of generalised inverse M^- if and only if $\operatorname{Im} Q \subset \operatorname{Im} M$ *and* $\operatorname{Ker} M \subset \operatorname{Ker} P$.

Proof:

See [70, Chapter 1].

□

In the case the conditions in Lemma A.1 are satisfied, we might choose the well known Moore-Penrose generalised inverse X with properties

$$NXN = N,\ XNX = X,\ (NX)^* = NX,\ (XN)^* = XN. \tag{A.1.0.9}$$

Definition A.2 *Define for a block matrix M of the form*

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

the generalised Schur complement of A in M by by $M/A = D - CA^{-}B$.

Lemma A.3 *The generalised Schur complement M/A is well defined and independent of the choice of generalised inverse A^{-} if*

$$\operatorname{Im} B \subset \operatorname{Im} A \text{ and } \operatorname{Ker} A \subset \operatorname{Ker} C.$$

The next result we wish to generalise is the Banachiewicz inversion formula, see Eq. (A.1.0.2) or [70, Eq. (0.7.2)],[62, Theorem 4.6]

Lemma A.4 (Generalised Banachiewicz Formula) *Let M be partitioned according to*

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

Assume that $\operatorname{Im} B \subseteq \operatorname{Im} A$ *and* $\operatorname{Ker} A \subseteq \operatorname{Ker} C$ *whence the Schur complement M/A is well defined and independent of the choice of generalised inverse A^- of A. Then the generalised inverse of M is given by*

$$M^- = \begin{bmatrix} A^- + A^- B\,(M/A)^-\, CA^- & -A^- B\,(M/A)^- \\ -\,(M/A)^-\, CA^- & (M/A)^- \end{bmatrix}. \tag{A.1.0.10}$$

Similarly, if we assume that $\operatorname{Im} C \subseteq \operatorname{Im} D$ *and* $\operatorname{Ker} C \subseteq \operatorname{Ker} B$ *such that M/D is well defined and independent of the generalized inverse D^- of D, then*

$$M^- = \begin{bmatrix} (M/D)^- & -(M/D)^- BD^- \\ -D^- C(M/D)^- & D^- + D^- C(M/D)^- BD^- \end{bmatrix}. \tag{A.1.0.11}$$

Proof:

To prove the first part of the statement we compute the matrix MM^-M and verify that this equals M. Set $X = M/A$ and obtain for the block $(MM^-M)_{11}$:

$$\begin{aligned} \left(MM^-M\right)_{11} &= AA^-A + AA^-BX^-CA^-A - BX^-CA^-A - AA^-BX^-C + BX^-C, \\ &= AA^-A + AA^-BX^-C\left(AA^- - 1\right) - BX^-C\left(AA^- - 1\right), \\ &= AA^-A + \left(AA^- - 1\right)BX^-C\left(AA^- - 1\right), \\ &= A, \end{aligned}$$

where $AA^{-}A = A$ by the definition of the generalised inverse Eq. (A.1.0.5) and $(AA^{-} - 1)\, B = 0$ by the assumption $\mathrm{Im}\, B \subseteq \mathrm{Im}\, A$. We next evaluate the block $(MM^{-}M)_{12}$

$$\begin{aligned}\left(MM^{-}M\right)_{12} &= B + \left(A^{-}A - 1\right) B + \left(A^{-}A - 1\right) BX^{-}CA^{-}B \left(A^{-}A - 1\right) BX^{-}D, \\ &= B,\end{aligned}$$

where $AA^{-}B = B$ and therefore $(AA^{-} - 1)\, B = 0$ since $\mathrm{Im}\, B \subseteq \mathrm{Im}\, A$ see Eq. (A.1.0.6).

$$\begin{aligned}\left(MM^{-}M\right)_{21} &= C + C\left(A^{-}A - 1\right) + \left(CA^{-}B - D\right) X^{-}C \left(A^{-}A - 1\right), \\ &= C.\end{aligned}$$

This follows from $\mathrm{Ker}\, A \subseteq \mathrm{Ker}\, C$ which implies $CA^{-}A = C$ and thus $C\,(A^{-}A - 1) = 0$. Similarly

$$\begin{aligned}\left(MM^{-}M\right)_{22} &= D - \left(D - CA^{-}B\right) - \left(D - CA^{-}B\right) X^{-} \left(D - CA^{-}B\right), \\ &= D - X + XX^{-}X, \\ &= D.\end{aligned}$$

because $X = M/A = D - CA^{-}B$ and thus we can see that $MM^{-}M = M$ holds.

For the second part of the statement we compute again $MM^{-}M$ with M^{-} as in Eq. (A.1.0.11). We obtain:

$$\begin{aligned}(MM^{-}M)_{11} &= A(M/D)^{-}(A - BD^{-}C) - BD^{-}C(M/D)^{-}(A - BD^{-}C) + BD^{-}C, \\ &= A - BD^{-}C + BD^{-}C, \\ &= A,\end{aligned}$$

since $(M/D) = A - BD^{-}C$ and $BD^{-}C(M/D)^{-}(M/D) = BD^{-}C$.

$$\begin{aligned}(MM^{-}M)_{12} &= A(M/D)^{-}B(1 - D^{-}D) + BD^{-}C(M/D)^{-}B(1 - D^{-}D) + BD^{-}D, \\ &= B,\end{aligned}$$

because of $BD^{-}D = B$ and thus $B(1 - D^{-}D) = 0$.

$$\begin{aligned}(MM^{-}M)_{21} &= C(M/D)^{-}(A - BD^{-}C) + D^{-}DC(M/D)^{-}(A - BD^{-}C) + D^{-}DC, \\ &= C,\end{aligned}$$

by $D^{-}DC = C$ and $C(M/D)^{-}(M/D) = C$. The last element computes as

$$\begin{aligned}(MM^{-}M)_{21} &= C(M/D)^{-}B(1 - D^{-}D) - D^{-}DC(M/D)^{-}B(1 - D^{-}D) + D^{-}DD, \\ &= D\end{aligned}$$

since $D^{-}DD = D$ and $B(1 - D^{-}D) = 0$ as before. We see that $MM^{-}M = M$ holds for M^{-} as in Eq. (A.1.0.11) which proves the statement.

□

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