On a Construction for Menon Designs using Affine Designs

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Contents

Chapter 1

Introduction

In this chapter we state some basic definitions and results that are used throughout this Thesis.

Definition 1.1. A ***structure*** (or an incidence structure) is two finite sets of objects, called *points* and *blocks*, with an incidence relation I between them, i.e. a structure is a triple $(\mathcal{P}, \mathcal{B}, I)$,where $\mathcal{P}$ is the set of points, $\mathcal{B}$ is the set of blocks, with $\mathcal{P} \cap \mathcal{B} = \emptyset$ and incidence $I \subseteq \mathcal{P} \times \mathcal{B}$.

Note.
- The set of points is denoted by v, i.e. $|\mathcal{P}| = v$ and the set of blocks is denoted by b, i.e. $|\mathcal{B}| = b$.
- If the ordered pair $(p, \mathcal{B})$ is in I we say that p is ***incident*** with $\mathcal{B}$, or that $\mathcal{B}$ contains the point p, or that p is on $\mathcal{B}$.

Definition 1.2. Let S be a structure with $v > 0$ points. If there exist integers λ, t with $\lambda > 0$ and $0 \leq t \leq v$ such that every subset of t points of S is incident with exactly λ common blocks then we say that S is a ***t-structure*** for λ.

Definition 1.3. An incidence structure is called ***uniform*** if its block set is non-empty and if each of its blocks contains exactly $k > 0$ points.

Definition 1.4. A uniform $t - structure$ with block size k is called a $t - (v, k, \lambda)$ structure.

Theorem 1.5. *If S is a $t - (v, k, \lambda)$ structure then for any integer s satisfying $0 \leq s \leq t$, S is also an $s - (v, k, \lambda_s)$ structure where* $\lambda_s = \lambda \dfrac{(v-s)(v-s-1)\ldots(v-t+1)}{(k-s)(k-s-1)\ldots(k-t+1)}$.

Definition 1.6. A $t - (v, k, \lambda)$ ***design*** $\mathcal{D}$ is a uniform t-structure for λ for which, any point is on a constant number of blocks.

Note.
- The number of blocks containing a point is called ***replication number*** and is denoted by the letter $\mathbf{r}$.

- The number $r - \lambda$ is the ***order*** of a $t-(v, k, \lambda)$ design. Any t-design is also an s-design for any s, $1 \leq s < t$.

- A $2-(v, k, \lambda)$ design with $b = v$, is called a ***symmetric*** design.

- A ***subdesign*** of a $t-(v, k, \lambda)$ design $(P, \mathcal{B}, I)$ is a triple (S, C, J), where $S \subseteq P$, $C \subseteq \mathcal{B}$ and $J \subseteq I$ such that if $(p, E) \in J$ then $p \in S$ and $E \in C$.

Definition 1.7. If $\mathcal{D}$ is a $2-(v, k, \lambda)$ design, then the binary $b \times v$ matrix $A = (a_{ij})$, where

$$a_{ij} = \begin{cases} 1, & \text{if the i-th block contains j} \\ 0, & \text{otherwise.} \end{cases}$$

is called ***the incidence matrix*** of the design.

Note. The incidence matrix of a design is not unique but depends on the order in which we write the blocks.

Theorem 1.8. *Suppose that S is a $2-$ structure for λ. Let $P_1, \ldots, P_v$ be the points of S, let $x_1, \ldots, x_b$ be the blocks and let $r_i = |P_i|$ (the number of blocks on P_i). If A is the incidence matrix of S with this labeling then: The relation $A^T A = N + \lambda J_v$ is true , where N is a diagonal v by v matrix with $n_i = r_i - \lambda$ in the i_{th} position on the diagonal and J is the all one matrix.*

To prove this theorem, consider that the $(i, j)^{th}$ entry of $A^T A$ is the product of the i^{th} and j^{th} rows of A. If $i \neq j$, this inner product counts a 1 whenever there is a block on both P_i and P_j and 0 otherwise; so the entry is λ. When $i = j$, the inner product or row i with itself counts the number of blocks on P_i, so the entry is $|P_i| = r_i = n_i + \lambda$.

Definition 1.9. The ***dual*** design $\mathcal{D}^*$ of $\mathcal{D}$ is obtained by interchanging the roles of points and blocks in $\mathcal{D}$.

In other words, if $\mathcal{D} = (X, \mathcal{B}, I)$ is an incidence structure, then the dual incidence structure is $\mathcal{D}^* = (\mathcal{B}, X, I^*)$ where $(\mathcal{B}, X) \in I^*$ if and only if $(X, \mathcal{B}) \in I$.

Theorem 1.10. *The dual $\mathcal{D}^*$ of a (v, k, λ) design $\mathcal{D}$, is also a (v, k, λ) design.*

Proof. Let A be an incident matrix for $\mathcal{D}$, so that $B = A^T$ is an incidence matrix for $\mathcal{D}^*$. By Theorem 1.8, $A^T A = N + \lambda J_v$, so that $A(A^T A) = nA + \lambda A J_v = nA + \lambda k A = nA + \lambda J_v A = (nI_v + \lambda J_v)A$. By Theorem 1.4.1 in [19], A is non-singular and we get $AA^T = N + \lambda J_v = B^T B$. Thus $\mathcal{D}^*$ is a (v, k, λ) design as well. $\square$

Definition 1.11. The ***complement*** $\bar{\mathcal{D}}$ of $\mathcal{D}$ is defined by $\bar{\mathcal{D}} = (\mathcal{B}, \mathcal{P}, \bar{\mathcal{I}})$, where $\bar{\mathcal{I}} = \mathcal{P} \times \mathcal{B} - \mathcal{I}$.

If $\mathcal{D}$ is a $2-(v,k,\lambda)$ design then $\bar{\mathcal{D}}$ is a $2-(v, v-k, b-2r+\lambda)$ design with the same order as $\mathcal{D}$.

Definition 1.12. The ***derived*** design $\mathcal{D}_p$ has point set $\{\mathcal{P}\backslash\{p\}\}$ and block set $\{B\backslash\{p\} : p \in B \in \mathcal{B}\}$.

Definition 1.13. A symmetric design $\mathcal{D}$ is said to be ***quasi-3*** for blocks if there exist integers α and β called *triple intersection numbers* such that:
$|A \cap B \cap C| \in \{\alpha, \beta\}$ for any three distinct blocks A, B and C of $\mathcal{D}$.

Definition 1.14. A symmetric $2-(4n-1, 2n-1, n-1)$ design, where n is a positive integer, is called a ***Hadamard 2-design*** or order n.

Another way of writing the parameters of a Hadamard 2-design, is $2-(4\lambda+3, 2\lambda+1, \lambda)$.

Definition 1.15. A design with parameters $3-(4n, 2n, n-1)$ is called a ***Hadamard 3-design***.

Theorem 1.16.

1. *any two distinct blocks in a Hadamard 3 design, meet in* 0 *or* n *points*
2. *for any block of the Hadamard 3-design, the complement of this block is also a block of the design.*

Proof. The following proof can be found in [15].

1. Let $\mathcal{D} = (X, \mathcal{B})$ be a $3-(4n, 2n, n-1)$ design and let $B_1, B_2 \in \mathcal{B}$. Suppose that $B_1 \neq B_2$ and $B_1 \cap B_2 \neq \emptyset$. Let $p \in B_1 \cap B_2$. Then $\mathcal{D}_p$ is a symmetric $(4n-1, 2n-1, n-1)$ design. Therefore, $|(B_1\backslash\{p\}) \cap (B_2\backslash\{p\})| = n-1$, which implies that $|B_1 \cap B_2| = n$.
2. Let $B \in \mathcal{B}$ and let m be the number of blocks $A \in \mathcal{B}$ such that $|A \cap B| = n$. Counting in two ways pairs (p,A) where $A \in \mathcal{B}$, $A \neq B$, and $p \in A \cap B$ yields $2n(4n-2) = mn$, so $m = 8n-4$. Since the design $\mathcal{D}$ has exactly $8n-2$ blocks (from the relation vr=bk on designs), there is a unique block in $\mathcal{D}$ that is disjoint from B. This block is the complement of B.

□

Definition 1.17. A $t-(v,k,\lambda)$ design is ***resolvable*** if its block set, $\mathcal{B}$, has a partition, called a *parallelism*, into ***parallel classes*** of blocks such that two distinct blocks in the same parallel class are always disjoint and every point belongs to exactly one block from each parallel class.

Definition 1.18. A resolvable design is called *affine resolvable* or simply ***affine***, if any two non-parallel blocks (i.e blocks from different parallel classes) meet in a constant number $\mu > 0$ of points.

Note 1.19. *Each parallel class of an affine design consists of* $m = \frac{v}{k}$ *blocks, where we call* m *the* ***class number*** *of the affine design, and* $\mu = \frac{k}{m}$.

Definition 1.20. A symmetric $2-(q^2+q+1, q+1, 1)$ design is called a ***projective plane*** of order q.

The smallest projective plane has order 2. It is known as the ***Fano*** plane.

Definition 1.21. A $2-(q^2, q, 1)$ design is called an ***affine plane*** of order q.

In particular, there exists a projective plane of order q if and only if there exists an affine plane of order q.

Theorem 1.22. *(Bose 1942) In a resolvable* $2-(v, k, \lambda)$ *design* $\mathcal{D}$ *we have*

$$\lambda \geq \frac{(k-1)}{(m-1)}$$

with equality if and only if $\mathcal{D}$ *is affine.*

By Theorem 1.22 and Note 1.19, the parameters of $\mathcal{D}$ in the affine case can be expressed entirely in terms of μ and m as follows: $v = \mu m^2$, $k = \mu m$, $\lambda = \dfrac{(\mu m - 1)}{(m-1)}$, $r = \dfrac{(\mu m^2 - 1)}{(m-1)}$ and $b = rm$.

Note. A parallel class in a Hadamard $3-design$ consists of a block and its complement. Thus Hadamard $3-designs$ admit resolutions, and by Theorem 1.16, these resolutions are also affine.
In other words any design with parameters $3-(4n, 2n, n-1)$ is necessarily affine!

Definition 1.23. A ***Latin square*** or order n is an $n \times n$ array with entries $1, 2, \ldots, n$ having the property that each entry occurs exactly once in each row and in each column.

Chapter 2

On Hadamard Matrices

2.1 Introduction

In this chapter we study Regular Hadamard Matrices, since they give rise to a class of designs which we will study thoroughly in the following chapters.
In section 2.2 we state some basic results on Hadamard Matrices.
In section 2.3 we study Regular Hadamard Matrices and the Kronecker product of such matrices.

2.2 Basic Results on Hadamard Matrices

Definition 2.1. A matrix H of order n with every entry equal to 1 or -1 is called a ***Hadamard Matrix*** (HM) if $HH^T = nI$.

Definition 2.2. A Hadamard Matrix with constant row sum is called a ***Regular Hadamard Matrix*** (RHM).

Put another way, a $(+1, -1) - matrix$ is a HM if the inner product of two distinct rows is 0. It is apparent that if the rows and columns of a HM are permuted, the matrix remains Hadamard.
It is also true that if any row or column is multiplied by -1, the Hadamard property is retained. Thus, it is always possible to arrange the first row and first column of a HM to contain only $+1$ entries. A Hadamard Matrix in this form is said to be ***normalized***.

Note. We say that two Hadamard Matrices H_1 and H_2 are equivalent if and only if one can be obtained from the other by permutation of the rows and columns and a series of addition of elements of group independently of the rows and columns.

The classification of HM has been done up to order 28. More precisely, there is a unique equivalence class of HM of each order 1;2;4;8 and 12. The number of classes for orders 16;20;24 and 28 are 5, 3, 60 and 487, respectively.

Below, Example 2.3 shows a RHM H and and its normalized form H' which is no longer a RHM. Example 2.4 shows a method of constructing a HM of order 36 by using any Latin square L of order 6.

Example 2.3.

$$H = \begin{pmatrix} 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{pmatrix}$$

$$H' = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix}.$$

Example 2.4. Let $L = (l_{ij})$ be a Latin square of order 6. Now let H be a matrix with rows and columns indexed by the 36 cells of the array: its entry in the position corresponding to a pair (c, c') of distinct cells is defined to be $+1$ if c and c' lie in the same row, or in the same column, or have the same entry. All other entries (including the diagonal ones) are -1. Then H is a HM. Using the Latin square

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 0 & 3 & 2 & 5 & 4 \\ 2 & 3 & 4 & 5 & 0 & 1 \\ 3 & 2 & 5 & 4 & 1 & 0 \\ 4 & 5 & 0 & 1 & 2 & 3 \\ 5 & 4 & 1 & 0 & 3 & 2 \end{pmatrix}$$

we get the 36×36 HM that we describe in Appendix A using MATLAB. Note that this HM in not regular.

Theorem 2.5. *The order of a HM is* 1, 2 *or* $4n$ *where* n *is an integer.*

Proof. *[1]*

Theorem 2.6. *The row sum of a RHM of order* $n \geq 4$ *is even and not equal to 0. If it is equal to* s*, then* $n = s^2$ *and hence* $s = \pm\sqrt{n}$*.*

Proof. *[15, Proposition 4.4.2]*

Theorem 2.7. *If H is a HM with constant row sums, then H has also constant column sums.*

Proof. Let H be an $n \times n$ HM with constant row sum, say s. Since $HH^T = nI$, we have that H is non-singular and thus the inverse of H exists and is equal to $\frac{1}{n}H^T$.
Given that H has constant row sum equal to s, we have that:
$HJ = sJ$, where J is the $n \times n$ all one matrix
$\Leftrightarrow J = sH^{-1}J$
$\Leftrightarrow J = s\frac{1}{n}H^T J$. Since $H^{-1} = \frac{1}{n}H^T$ and $s \neq 0$, we have that
$H^T J = \frac{n}{s}J$.
Hence, H^T has constant row sum equal to s, since by Theorem 2.6, $n = s^2$, and $\frac{n}{s} = \frac{s^2}{s} = s$.
Hence H has also constant column sum equal to s. □

Corollary 2.8. *If H is a RHM of order n and row sum equal to s, then H^T is also a RHM with row sum equal to s.*

2.3 Kronecker Product of Regular Hadamard Matrices

Definition 2.9. The Kronecker product of an $m \times n$ matrix $A = [a_{ij}]$ and an $m' \times n'$ matrix B over a commutative ring, is the $(mm') \times (nn')$ block matrix

$$A \bigotimes B = \begin{pmatrix} a_{11}B & a_{12}B & ... & a_{1n}B \\ a_{21}B & a_{22}B & ... & a_{2n}B \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1}B & a_{m2}B & ... & a_{mn}B \end{pmatrix}$$

Theorem 2.10. *The Kronecker product of RH Matrices is a RHM. Particularly, if A and B have row sums $\pm 2u$ and $\pm 2v$ respectively then the Kronecker product $A \bigotimes B$ will have row sum $\pm 4uv$ accordingly.*

Proof. *[15, Proposition 4.4.9]*

The converse of Theorem 2.10 is also true:

Theorem 2.11. *Let A and B be two Hadamard Matrices. If the Kronecker product of A and B is regular, then A and B are also regular.*

Proof. Let A and B be two Hadamard matrices of order a and b respectively, and let J_{ab}, J_a, J_b be the $ab \times ab$, $a \times a$, $b \times b$, all one matrices . Assuming that $A \bigotimes B$ is regular with row sum $2w$, for $w \geq 2$, we have that:

$$(A \bigotimes B)J_{ab} = 2wJ_{ab} \iff (A \bigotimes B)(J_a \bigotimes J_b) = 2wJ_{ab} \iff (AJ_a) \bigotimes (BJ_b) = 2wJ_{ab}$$

$$\iff (AJ_a) \bigotimes (BJ_b) = \begin{pmatrix} 2w & 2w & \dots & 2w \\ 2w & 2w & \dots & 2w \\ \vdots & \vdots & \vdots & \vdots \\ 2w & 2w & \dots & 2w \end{pmatrix} ab.$$

This final equivalence gives us:

$$AJ_a = 2wJ_a$$

and

$$BJ_b = 2wJ_b$$

and thus A and B have constant row sums.

□

After studying the Kronecker product of RH Matrices, we will now deal with their product.

Note. If A is a RHM of order $n = 4u^2$, then by Theorem 2.6 , the row sum s, of A, is equal to $\pm\sqrt{n}$, i.e. $s = \pm 2u$. In addition, by Theorem 2.7 , A has also constant column sum equal to $\dfrac{n}{s} = \dfrac{4u^2}{\pm 2u} = \pm 2u$.

Definition 2.12. Let A be a RHM of order $4u^2$. If A has constant row/column sum equal to $(+2u)$, $u \geq 0$, then A is said to be of type $(+)$, and if A has constant row/column sum equal to $(-2u)$, it is said to be of type $(-)$.

Theorem 2.13. *If $A = [a_{ij}]$ and $B = [b_{jk}]$ are two Regular Hadamard Matrices of order $n = 4u^2$, then the product AB of A and B is a matrix with constant row/column sum but is not Hadamard.*

Proof. First we will show that AB has constant row/column sum in the case where one of the matrices is of type (+) and the other is of type (-). W.l.o.g. we assume that the matrix A is of type (+) and the matrix B is of type (-).

If J denotes the $n \times n$ matrix with $J_{ij} = 1$ for all i, j, then:
$AJ = JA = 2uJ$ and $BJ = JB = -2uJ$. Therefore,
$ABJ = A(-2uJ) = -2uAJ = -2u(2uJ) = -4u^2J$ and similarly $JAB = -4u^2$.
Hence the product AB of A and B is a matrix with constant row/column sum equal to $-4u^2$.

Now if matrices A and B are both of type (+) or (-), then the same argument shows that $ABJ = JAB = 4u^2$.

Now, we will show that AB is not Hadamard , by showing that $(AB)(AB)^T \neq nI_n$.

Since A and B are Hadamard Matrices of order n, we have that $AA^T = nI_n$ and $BB^T = nI_n$. Hence:

$(AB)(AB)^T = (AB)(B^T A^T)$

$=ABB^T A^T$

$=A(nI_n)A^T$

$=nAI_nA^T$

$=nAA^T$

$=nnI_n$.

Thus $(AB)(AB)^T = n^2 I_n$ and hence AB is not Hadamard.

□

The following simple example shows that the product of two Regular Hadamard Matrices A and B, not only doesn't satisfy the condition $(AB)(AB)^T = n^2 I_n$, but it also has entries that are not equal to ± 1.

Example 2.14. Let

$$A = B = \begin{pmatrix} 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{pmatrix}$$

Taking the product of the 1st row with 3rd column of the array $A(=B)$, we see that the element $(AB)_{13}$ of the matrix AB is equal to 4.

Chapter 3

Menon Designs

3.1 Introduction

Regular Hadamard Matrices yield a family of symmetric designs, called Menon designs.
In Section 3.2 we state some basic definitions and results on Menon designs and incidence structures in general.
In Section 3.3 we define special triples in Menon designs and study whether these designs may have special triples or not. Then we study whether designs associated with Regular Hadamard Matrices that are the Kronecker product of other RH Matrices, may have special triples or not.

3.2 Basic Definitions and Results

Definition 3.1. Let u be a nonzero integer. A symmetric $2-\big(4u^2, 2u^2-u, u^2-u\big)$ design and the complementary $2-\big(4u^2, 2u^2+u, u^2+u\big)$ design are called ***Menon designs*** of order u^2.

Theorem 3.2. *Let $\mathcal{D}$ be a Menon design with parameters $\big(4u^2, 2u^2-u, u^2-u\big)$. Given any pair of blocks X and Y in $\mathcal{D}$, $X \neq Y$, there exist at most one block $Z \neq X, Y$ containing $X \cap Y$.*

Proof. Let X and Y be two blocks of the Menon design $\mathcal{D}$, $X \neq Y$, and let S be the intersection of these blocks, i.e $S = X \cap Y$. Then $X - S$ and $Y - S$ are disjoint because X and Y cannot intersect in any points outside S. Recall by Theorem 1.10 that the dual of a symmetric design has the same parameters as the original design. Hence, any two blocks of a $2-(4u^2, 2u^2-u, u^2-u)$ Menon design also meet in u^2-u points. Thus if Z is a third block containing S then $X-S$, $Y-S$ and $Z-S$ are disjoint subsets, each containing u^2 points. Since there are only $4u^2-(4u^2-u) = u$ points outside $S \cup Y \cup Z$, there cannot exist another block Z' different than Z that also contains S. Thus, given any two blocks X and Y in $\mathcal{D}$, $X \neq Y$, there exist at most one block $Z \neq X, Y$ containing their intersection. $\square$

It should be noted that we will use Theorem 3.2 in Section 3.3.

The following theorem which can be found in [15] is a well-known theorem that associates Regular Hadamard Matrices with Menon designs, but we include the proof for completeness.

Theorem 3.3. *Let H be a $(\pm 1)-$ matrix of order $n \geq 4$ and let $N = \frac{1}{2}(J - H)$. Then H is a RHM with row sum $2u$ if and only if N is the incidence matrix of a Menon design with parameters $\left(4u^2, 2u^2 - u, u^2 - u\right)$.*
(Respectively, H is a RH matrix with row sum $-2u$ if and only if N is the incidence matrix of a Menon design with parameters $\left(4u^2, 2u^2 + u, u^2 + u\right)$)

Proof. If H is a RHM with row sum $2u$, then $n = 4u^2$, $HH^T = 4u^2 I$ and $HJ = 2uJ$. Therefore, $NN^T = \frac{1}{4}(J - H)(J - H)^T = u^2 I + (u^2 - u)J$. According to Theorem 1.8 this implies that N is the incidence matrix of a $2 - (4u^2, 2u^2 + u, u^2 + u)$ design.
Conversely, if N is an incidence matrix of a symmetric $2 - (4u^2, 2u^2 + u, u^2 + u)$ design, then $HH^T = (J - 2N)(J - 2N)^T = 4u^2 I$ and $HJ^T = 2uJ$. Thus H is a RH matrix. □

Note 3.4. *To obtain the matrix $N = \dfrac{1}{2}(J - H)$ from the RH Matrix H, first we subtract all entries from 1. By doing this, the entries that were equal to 1 become equal to 0 and the entries that were equal to -1 become equal to 2. Then by multiply the matrix with $\dfrac{1}{2}$ we change the entries that became equal to 2 with 1. This procedure leads to a $(0,1)$-incidence matrix for the corresponding Menon design. During the rest of this Thesis we will be using this technique to obtain incidence matrices of designs from Regular Hadamard Matrices by changing the -1's with $+1$'s and $+1$'s with 0's.*

The following is an example of a HM that when the procedure mentioned in Note 3.4 is applied, it produces the incidence matrix of a Menon design.

Example 3.5. Let

$$H = \begin{pmatrix} 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{pmatrix}$$

be the RHM used in Example 2.3. Then

$$N = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

where $N = \dfrac{1}{2}(J - H)$ mentioned in Theorem 3.3. According to Definition 1.7 matrix N is the incidence matrix of a design with 4 points and 4 blocks. Since every row of N has only one entry equal to 1, we have that every block of the design is incident with exactly 1 point i.e the design has parameters $b = v = 4$ and $k = 1$. Furthermore every two columns of matrix N have 0 entries equal to 1 in common, i.e the design has $\lambda = 0$ and finally N has constant row and column sum equal to 1. Thus matrix N is the incidence matrix of a Menon design with parameters $2-(4u^2, 2u^2-u, u^2-u)$ where $u = 1$.

Theorem 3.6. *For any integer $u \neq 0$, a $(0,1)-$ matrix M is the incidence matrix of a symmetric $2-\big(4u^2, 2u^2-u, u^2-u\big)$ Menon design if and only if $J-2M$ is a RHM with row sum $2u$.*

Proof. "$\Longleftarrow$"

Assume that $J-2M$ is a RHM with row sum $2u$.

By Theorem 2.6, since $J-2M$ has row sum $2u$, it will be of order $4u^2$. Since matrix $J-2M$ has entries equal to $+1$ or -1, matrix M has entries equal to 1 or 0. Thus M is a $(0,1)$-matrix of order $4u^2$ and hence is the incidence matrix of a structure with $4u^2$ points. Now, since $J-2M$ is a HM, of order $4u^2$:

$(J-2M)(J-2M)^T = 4u^2 I$

$\Longrightarrow (J-2M)(J^T-2M^T) = 4u^2 I$

$\Longrightarrow JJ^T - 2JM^T - 2MJ^T + 4MM^T = 4u^2 I$

But,$(J-2M)J = 2uJ = J^2 - 2MJ = 4u^2 J - 2MJ$.

Thus $MJ = (2u^2-u)J$. So we have:

$4u^2 J - 2J(2u^2-u) - 2(2u^2-u)J + 4MM^T = 4u^2 I$

$\Longrightarrow 4u^2 J - 4(2u^2-u)J + 4MM^T = 4u^2 I$

$\Longrightarrow 4u^2 J - 8u^2 J + 4uJ + 4MM^T = 4u^2 I$

$\Longrightarrow 4MM^T = 4u^2 I + 4u^2 J - 4uJ$

$\Longrightarrow MM^T = u^2 I + (u^2-u)J$.

Using Theorem 1.8, this proves that M is the incidence matrix of a 2-structure with λ equal to u^2-u having replication number $r = 2u^2-u$. Also, since $J-2M$ is a RHM, so is its transpose, by Corollary 2.8, i.e. $J-2M^T$ is a RHM. Applying this argument to $J-2M^T$ it follows that M^T is the incidence matrix of a 2-structure and thus M is the incidence matrix of a symmetric 2-design having $b = v$ and $r = k$.

"$\Longrightarrow$":

Suppose now that M is the incidence matrix of a symmetric $\big(4u^2, 2u^2-u, u^2-u\big)$ design.

Since M is the incidence matrix of a $2-\big(4u^2, 2u^2-u, u^2-u\big)$ Menon design, every column of M has $(2u^2-u\ )$ 1's and $[4u^2-(2u^2-u)]$=$2u^2+u$ 0's. Thus, $J-2M$ has $(2u^2-u)$ -1's and $(2u^2+u)$ 1's. So the sum of each column of $J-2M$ is equal to $(2u^2+u)\cdot 1+(2u^2-u)\cdot(-1)$ =$2u^2+u$-$2u^2+u$=$2u$. Similarly, the sum of every row is equal to $2u$.

We have shown that $J-2M$ is a ± 1 matrix with constant row and column sum. We still need to show that $(J-2M)(J-2M)^T)$=$4u^2 I_n$. By performing calculations we have:

$(J-2M)(J-2M)^T) = (J-2M)(J^T-2M^T) = JJ^T - 2JM^T - 2MJ^T + 4MM^T$.
Since M is the the incidence matrix of a $2-\big(4u^2, 2u^2-u, u^2-u\big)$ design, we know that $MM^T = u^2I + (u^2-u)J$, so
$(J-2M)(J-2M)^T = 4u^2J - 2(2u^2-u)J - 2(2u^2-u)J + 4(u^2I + (u^2-u)J)$
$= 4u^2J - 4u^2J + 2uJ - 4u^2J + 2uJ + 4u^2I + 4u^2J - 4uJ = 4u^2I$. □

3.3 Special Triples in Menon Designs

Definition 3.7. Let $\mathcal{D}$ be a $2-\big(4u^2, 2u^2-u, u^2-u\big)$ Menon design. A ***special triple*** is a set of three distinct blocks with the property that each of these blocks contains the intersection of the other two blocks.

In Chapter 2, Definition 2.12 states that a RHM of order $4u^2$ with row/column sum equal to $(+2u)$ is said to be of type $(+)$, and of type $(-)$ in the case where it has row/column sum equal to $(-2u)$. In Theorem 3.3 we've seen that if H is a RHM of order n with row sum $(+2u)$, i.e of type $(+)$, then the matrix $N = \dfrac{1}{2}(J-H)$,where J is the $n \times n$ matrix with all entries equal to ones, is the incidence matrix of a Menon design $\mathcal{D}$ with parameters $\big(4u^2, 2u^2-u, u^2-u\big)$.

The following definition provides a notation for Menon design arising from a RHM of type $(+)$ or type $(-)$.

Definition 3.8. A Menon design $\mathcal{D}$ with parameters $2-\big(4u^2, 2u^2-u, u^2-u\big)$ for some $u > 0$ arising from a RHM of type $(+)$ is called type $\mathcal{D}^+$ Menon design, and a Menon design $\mathcal{D}$ with parameters $2-\big(4u^2, 2u^2+u, u^2+u\big)$ arising from a RHM of type $(-)$ is called type $\mathcal{D}^-$ Menon design.

In Theorem 3.2 we have seen that for any pair of distinct blocks X and Y in a Menon design, there exist at most one block Z different than X and Y that contains their intersection. The following theorem shows that in fact there exist pairs of blocks that don't belong to special triples.

Theorem 3.9. *In a Menon design of type $\mathcal{D}^+$, there exist pairs of blocks that are not in special triples.*

Proof. Let $X \in \mathcal{B}$. Assume that if Y is a block different than X, there exist a block, say Z, with $Z \neq X, Y$, such that the intersection of X and Y is contained in Z.

Using Theorem 3.2 we know that given a block Y different than X, a unique Z is determined. Thus, if X is a block then the blocks different than X can be paired as $\{Y, Z\}$ and therefore there must be an even number of them. But this is impossible since the number of blocks different than X is $4u^2-1$ which is an odd number. □

We now consider the question of whether Menon designs may have special triples or not .

Using Note 3.4, recall that in a RHM of type $(+)$, i.e with row sum equal to $2u$, if we replace the -1's with 1's and the 1's with 0's we get the incidence matrix of a Menon $2-\big(4u^2, 2u^2-u, u^2-u\big)$ design.

Remark 1.
For a Menon $2-\big(4u^2, 2u^2-u, u^2-u\big)$ design $\mathcal{D}^+$, it is possible to have special triples. Simple counting gives us the reason:
Let X and Y be two blocks of the design , $X \neq Y$, and let S be the intersection of these blocks, i.e $S = X \cap Y$. Then $X-S$ and $Y-S$ are disjoint because X and Y cannot intersect in any points outside S. Thus if Z is a third block containing S then $X-S$, $Y-S$ and $Z-S$ are disjoint subsets, each containing u^2 points. Since there are $4u^2-(u^2-u+u^2+u^2) = u^2+u$ points that don't belong to either $X-S$ or $Y-S$, we conclude that since this number is greater that the block size of $\mathcal{D}$, such a block Z can exist, and therefore the design could have special triples.

Remark 2.
Using the same counting technique, we see that in the case of a $2-\big(4u^2, 2u^2+u, u^2+u\big)$ design D^+, there are $4u^2-(u^2+u-u^2-u^2) = u^2-u$ points that don't belong to either $X-S$ or $Y-S$ and therefore since this number is less that u^2 there cannot exist a block Z that forms a special triple with X and Y. Thus a $2-(4u^2, 2u^2+u, u^2+u)$ Menon design $\mathcal{D}^-$ cannot have special triples.

Note. In the case where $u=2$, the $2-(16,6,2)$ Menon design cannot have special triples because: if B_1, B_2, B_3 form a special triple, then their intersection is a set of two points $\{p_1, p_2\}$. But this implies that the pair of points $\{p_1, p_2\}$ is incident with 3 common blocks, contradicting the fact that for a $2-(v,k,\lambda)$ design, every subset of 2 points is incident with exactly λ common blocks.

Remark 3.
If $M = M_1 \otimes M_2$ is a RHM, where M_1 and M_2 are Hadamard Matrices, then the design corresponding to M does not have special triples. (The proof of this result is described in Appendix E.)
In Theorem 2.11 we've shown that if the Kronecker product of two Hadamard Matrices is a RHM, then the two Matrices must also be Regular.
Combining Theorem 2.11 with this result, we conclude that Menon designs corresponding to RHM that were constructed by the Kronecker product of RH matrices, cannot have special triples.

Note. It is possible for a Kronecker product of two Hadamard Matrices to have three rows for which the common positions of the -1's in two of these rows are positions in which the third also has -1's, i.e. it is possible to have a special triple for the corresponding design. For example, let $M^{(1)} = \begin{bmatrix} -1 & -1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{bmatrix}$ and $M^{(2)} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$. Then rows $(2,2)$, $(3,3)$ and $(4,4)$ of $M^{(1)} \otimes M^{(2)}$ are the following $\begin{bmatrix} 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix}$ and the common positions of the -1's in any two rows are $\{(1,4),(2,3),(3,2),(4,1)\}$.

Chapter 4

Alpha arcs

4.1 Introduction

In Section 4.2 we state some basic definitions and results on α-arcs in a $2-(v,k,\lambda)$ design. In Section 4.3 we study the possible values of the cardinality of an α-arc in a $2-(4u^2, 2u^2 \pm u, u^2 \pm u)$ Menon design. As it turns out, the size of an α-arc is related to u and the value of α depends on the size of the arc.

4.2 Basic Definitions and Results

Definition 4.1. A non-empty subset A of s points of a design $\mathcal{D}$ is called an α-arc , where $\alpha \neq 0$, if $|B \cap A| \in \{0, \alpha\}$ for every block B of $\mathcal{D}$.

A block is said to be a ***secant*** or ***passant*** of an α-arc according as it meets the arc in α or 0 points.

Definition 4.2. If A is an α-arc of a design $\mathcal{D}$, then $\mathcal{D}_A$ denotes the ***induced design*** defined on the points of A, whose blocks are the secants of A, with induced incidence.

Thus a secant B induces a block of $\mathcal{D}_A$ whose points are those of $A \cap B$. Clearly $\mathcal{D}_A$ is a $1-(s,\alpha,r)$ design, where $|A| = s$ and r is the replication number of $\mathcal{D}$.

Note. The ***nth multiple design*** of a design is obtained by repeating each of its blocks m times. If $\mathcal{D}$ is a $2-(v,k,\lambda)$ design with an α-arc A, then the induced design on A, $\mathcal{D}_A$, is a $2-(s,\alpha,\lambda)$ design, with replication number r, the same as in $\mathcal{D}$. Hence if $\mathcal{D}_A$ is a multiple of a symmetric design, then it is the $(\dfrac{r}{\alpha})$th multiple of a symmetric $2-(s,\alpha,\lambda')$ design denoted by $[\mathcal{D}_A]$, where $\lambda' = \dfrac{\lambda\alpha}{r}$. In this case, A is called a ***symmetric*** α-arc.

Lemma 4.3. *Let A be an α-arc in a 2-(v, k, λ) design $\mathcal{D}$. Then:*
(a) $\mathcal{D}_A$ is a $2-(s,\alpha,\lambda)$ design, where $s = |A| = 1 + r(\alpha - 1)/\lambda$.
(b) A has exactly rs/α secants and b - rs/α passants.
(c) Any point not in A is on exactly $\lambda s/\alpha$ secants and $r - \lambda s/\alpha$ passants.
(d) The passants of A form an $(r-\lambda)/\alpha$-arc in the dual design of $\mathcal{D}$, $\mathcal{D}^*$.

Proof. *The proof of this lemma can be found in [18].*

Note. The $(\dfrac{r-\lambda}{\alpha})$-arc formed by the passants of the α-arc A, mentioned in Lemma 4.3, will be called ***the dual arc*** of A.

The following remarks refer to an α-arc A of a $2-(v,k,\lambda)$ design $\mathcal{D}$.

Remarks:

1. If x is a point not in A and γ, δ represent the number of blocks on x not meeting A and meeting A respectively, then $\gamma + \delta = k$. If x, y are points on B and $y \in A$, counting the number of flags (y, B), gives $\alpha\delta = \alpha s$. Thus, $\delta = s$ and $\gamma = k - s$.

2. Let $\mathcal{D}_1$ be a subdesign of $\mathcal{D}$ whose points are the points of $\mathcal{D}$ not in A and whose blocks are the blocks of $\mathcal{D}$ not meeting A, i.e the passants of $\mathcal{D}$. Then, $\mathcal{D}_1$ has $v - s$ points, and every block contains k points. By using Lemma 4.3, we know that any point not in A is on exactly $r - \dfrac{\lambda s}{\alpha}$ passants. Hence, $\mathcal{D}_1$ is a $1-(v-s,k,r-\dfrac{\lambda s}{\alpha})$ design.

3. Let $\mathcal{D}_2$ be a subdesign of $\mathcal{D}$ whose points are the points of $\mathcal{D}$ not in A and whose blocks are the blocks of $\mathcal{D}$ meeting A, i.e the secants of $\mathcal{D}$. Then again, $\mathcal{D}_2$ has $v - s$ points, but every block contains $k - \alpha$ points. By using Lemma 4.3, we know that any point not in A is on exactly $\dfrac{\lambda s}{\alpha}$ secants. Hence, $\mathcal{D}_2$ is a $1-(v-s,k-\alpha,\dfrac{\lambda s}{\alpha})$ design.

4. Given that the subdesign $\mathcal{D}_1$ is a $1-(v-s,k,r-\dfrac{\lambda s}{\alpha})$ design, then $\mathcal{D}_1$ is also a $2-(v-s,k,\lambda)$ design. Then, the dual $\mathcal{D}_1^*$ of $\mathcal{D}_1$, is a $2-(b-\dfrac{rs}{\alpha},r-\dfrac{\lambda s}{\alpha},\lambda)$ design, since $\mathcal{D}_1$ has $b-\dfrac{rs}{\alpha}$ blocks and every point in $\mathcal{D}_1$ is on $r-\dfrac{\lambda s}{\alpha}$ blocks.

The following lemma considering the number of common secants and passants of two disjoint arcs, can be found in [18]. In their paper, A.N.Al Kenani and V.C.Mavron, give an outline of the proof so we give a detailed proof of this lemma for completeness.

Lemma 4.4. *Let $\mathcal{D}$ be a $2-(v,k,\lambda)$ design and let A_i be an α_i-arc of $\mathcal{D}$ with $|A_i| = s_i$ for $i = 1,2$, and $A_1 \cap A_2{=}\emptyset$. Then the number of secants common to A_1 and A_2 is*

$$\frac{\lambda s_1 s_2}{\alpha_1 \alpha_2}$$

and the number of passants common to A_1 and A_2 is

$$b - (\frac{r a_1 \alpha_2 + r a_2 \alpha_1 - \lambda s_1 s_2}{\alpha_1 \alpha_2}).$$

Proof. Let x be the number of common secants. Counting in two ways the number of ordered triples (p_1, p_2, B), where $p_i \in A_i$ and B is a block containing p_i, for $i = 1,2$, gives $s_1 s_2 \lambda = x\alpha_1\alpha_2$. So the number of secants common to A_1 and A_2 is $x = \dfrac{s_1 s_2 \lambda}{\alpha_1 \alpha_2}$.
For the number of common passants,
given that $A_1 \cap A_2{=}\emptyset$, then a block of the design will either

1. meet A_1 and not A_2 or
2. meet A_2 and not A_1 or
3. meet both A_1 and A_2, or
4. meet neither of A_1 and A_2 (and these are the passants common to A_1 and A_2).

Thus to find the number of common passants to A_1 and A_2, we perform the following calculation:
b-(secants of A_1 + secants of A_2 - secants common to both A_1 and A_2).

By Lemma 4.3, the number of secants of A_1 is

$$\frac{r s_1}{\alpha_1}$$

and the number of secants of A_2 is

$$\frac{r s_2}{\alpha_2}.$$

In addition, as we have shown, the number of secants common to A_1 and A_2 is

$$\frac{\lambda s_1 s_2}{\alpha_1 \alpha_2},$$

so simple calculations give that the number of common passants to A_1 and A_2 is:

$$b - (\frac{rs_1\alpha_2 + rs_2\alpha_1 - \lambda s_1 s_2}{\alpha_1\alpha_2}).$$

□

4.3 Menon Designs and α-arcs

According to Lemma 4.3(a) the cardinality of an α-arc A in a $2-(v,k,\lambda)$ design is $s = 1 + \frac{r(\alpha-1)}{\lambda}$.
For a $2-\big(4u^2, 2u^2-u, u^2-u\big)$ Menon design, if there exist an α-arc, then $s = 1 + \frac{(2u-1)(\alpha-1)}{u-1}$.
In this section, Theorem 4.6 proves that the only possible values of s are $2u$ and $2u^2+u$.

Lemma 4.5. *Let $\mathcal{D}$ be a Menon design with parameters $2-\big(4u^2, 2u^2-u, u^2-u\big)$ and let A be an α-arc of $\mathcal{D}$ with $|A| = s$, where $s = 1 + \frac{(2u-1)(\alpha-1)}{u-1}$. Then $(u-1)$ must divide $(\alpha-1)$.*

Proof. Since s is an integer, $\frac{(2u-1)(\alpha-1)}{u-1}$ must be an integer as well, i.e $(u-1)$ must divide $(2u-1)(\alpha-1)$. But, since $(2u-1)$ and $(u-1)$ are coprime, i.e. $(2u-1, u-1) = 1$, we get that $(u-1)$ must divide $(\alpha-1)$. □

Note.
1. In the proof of Lemma 4.5 we claim that $(2u-1)$ and $(u-1)$ are coprime. This is true since, $(2u-1) - 2(u-1) = 1$ implies that any number dividing $2u-1$ and $u-1$ must also divide 1, hence $2u-1$ and $u-1$ are coprime.
2. In the case where $u = 2$, $\alpha = \frac{s-1}{3} + 1$ and therefore 3 must divide $s-1$.

Theorem 4.6. *Let $\mathcal{D}$ be a $2-(4u^2, 2u^2-u, u^2-u)$ design and A an α-arc of $\mathcal{D}$ with $|A| = s$. Then s can take only the values $2u$ or $2u^2+u$. In the case where $s = 2u$, α is equal to u and in the case where $s = 2u^2+u$, α is equal to u^2.*

Proof. Let y be the number of blocks meeting A in α points, i.e. y is the number of secants. By Lemma 4.3(b), $y = \frac{rs}{\alpha}$ and therefore $y\alpha = rs$. Since for the design $\mathcal{D}$, $r = k$, we get the following equation:

$$y\alpha = sk = s(2u^2-u). \tag{4.1}$$

Moreover, by Lemma 4.3(a), the cardinality of A is s, where $s = 1 + \frac{r(\alpha-1)}{\lambda}$. Performing calculations gives $(s-1)\lambda = r(\alpha-1)$. Thus:

$$\begin{aligned} y\alpha(\alpha-1) &= \frac{rs}{\alpha}\cdot\alpha(\alpha-1) \\ &= rs(\alpha-1) \\ &= s(s-1)\lambda \\ &= s(s-1)(u^2-u). \end{aligned}$$

The preceding calculations give the following equation:

$$y\alpha(\alpha-1) = s(s-1)(u^2-u). \tag{4.2}$$

By combining equations (4.1) and (4.2) we get:

$$\alpha - 1 = \frac{s(s-1)(u^2-u)}{y\alpha} = \frac{s(s-1)u(u-1)}{s(2u^2-u)} = \frac{(s-1)u(u-1)}{u(2u-1)}$$

Hence,

$$\alpha - 1 = \frac{(s-1)(u-1)}{(2u-1)}. \tag{4.3}$$

Equation (4.3) gives:

$$s = 1 + \frac{(\alpha-1)(2u-1)}{u-1}. \tag{4.4}$$

Working with equation (4.4), since s must be an integer, $(u-1)$ must divide $(\alpha-1)(2u-1)$. Recall that by lemma 4.5, $(u-1)$ must divide $(\alpha-1)$, i.e:

$$\alpha - 1 = l(u-1), l \in Z. \tag{4.5}$$

Thus, equation 4.4 becomes:

$$s = 1 + l(2u-1), l \in Z. \tag{4.6}$$

Note that:

- If $s = 0$, then A is the empty set and $y = 4u^2$.
- If $s = 1$, then A is a one-point set and $y = 2u^2 - u$ since there are $(2u^2 - u)$ blocks containing one point.
- If $s = 4u^2$, then A is the trivial α-arc consisting of all points and therefore $y = 4u^2$ since every block meets this arc in $2u^2 - u$ points.

So by excluding the trivial cases where $s = 0, 1$ or $4u^2$, assume now that $2 \leq s \leq 4u^2 - 1$.

Since $s < 4u^2$, then:

$1 + l(2u - 1) < 4u^2$, by equation (4.6)

$\Rightarrow l(2u - 1) < 4u^2 - 1$

$\Rightarrow l(2u - 1) < (2u - 1)(2u + 1)$

$\Rightarrow l < 2u + 1$

Since, l is an integer, we get that $l \leq 2u$.

Now, by equation (4.1):

$$y = \frac{s(2u^2 - u)}{\alpha}.$$

Since y is the number of blocks meeting A in α points, $y \leq 4u^2$. We will work on the difference between y and $4u^2 - 1$:

$$\begin{aligned} y - (4u^2 - 1) &= \frac{s(2u^2 - u)}{\alpha} - (4u^2 - 1) \\ &= \frac{su(2u - 1) - \alpha(2u - 1)(2u + 1)}{\alpha} \\ &= \frac{(2u - 1)[su - \alpha(2u + 1)]}{\alpha}. \qquad (*) \end{aligned}$$

Simplifying the term $su - \alpha(2u + 1)$ by using equations (4.5) and (4.6) gives:

$$\begin{aligned} [su - \alpha(2u + 1)] &= [1 + l(2u - 1)]u - [1 + l(u - 1)](2u + 1) \\ &= [(1 + 2lu - l)u] - (1 + lu - l)(2u + 1) \\ &= (u + 2lu^2 - lu) - (2u + 1 + 2lu^2 + lu - 2lu - l) \\ &= u - 2u - 1 + l \\ &= l - (u + 1). \end{aligned}$$

Hence, by substituting the term $su - \alpha(2u + 1)$ with $l - (u + 1)$ in $(*)$ we get:

$$y - (4u^2 - 1) = \frac{[l - (u + 1)](2u - 1)}{1 + l(u - 1)}. \qquad (4.7)$$

We will now show that $l \leq u+1$.

Assume that $l > u+1$. Then:

$$\frac{[l-(u+1)](2u-1)}{1+l(u-1)} > 0,$$

thus by equation (4.7):

$$y > 4u^2 - 1.$$

But $y \leq 4u^2$ as well. Hence,

$$4u^2 - 1 < y \leq 4u^2$$

and since y is an integer, we get that $y = 4u^2$. Having $y = 4u^2$ implies that $y - (4u^2 - 1) = 1$. Setting equation (4.7) equal to 1, we get:

$$\frac{[l-(u+1)](2u-1)}{1+l(u-1)} = 1$$

$$\Rightarrow [l-(u+1)](2u-1) = 1 + l(u-1)$$

$$\Rightarrow (l-u-1)(2u-1) = 1 + lu - l$$

$$\Rightarrow 2lu - l - 2u^2 + u - 2u + 1 = 1 + lu - l$$

$$\Rightarrow lu - u - 2u^2 = 0$$

$$\Rightarrow l - 1 - 2u = 0$$

$$\Rightarrow l = 1 + 2u.$$

But this is a contradiction since we have already proven that $l \leq 2u$.

Therefore, from the contradiction, we conclude that

$$l \leq u + 1. \tag{4.8}$$

We will use equation (4.8) to study the possible values of l that will lead us to the possible values of s and α which is the objective of this proof.

CASE 1: $l = u+1$
In this case, by equation (4.6), $s = 1 + (u+1)(2u-1)$, giving us $s = 2u^2 + u$ and by equation (4.5), $\alpha = 1 + (u+1)(u-1)$, giving us $\alpha = u^2$.
CASE 2: $0 < l \leq u$

1. l=1

   By equation (4.6), $s = 1+(2u-1)$, giving us $s = 2u$, and by equation (4.5), $\alpha = 1+(u-1)$, giving us $\alpha = u$.

2. l=2

   By equation (4.6), $s = 1 + 2(2u - 1)$, giving us $s = 4u - 1$, and by equation (4.5), $\alpha = 1 + 2(u - 1)$, giving us $\alpha = 2u - 1$.

   We will show that in fact l cannot take the value 2.

   Referring to Lemma 4.3(d), if A is an α-arc in a $2-(v, k, \lambda)$ design $\mathcal{D}$, then the passants of A form an $(\dfrac{r-\lambda}{\alpha})$-arc in $\mathcal{D}^*$. Recall that $\mathcal{D}^*$ is the dual of $\mathcal{D}$, r is the replication number of $\mathcal{D}$ and a passant is a block meeting the arc in 0 points.
   In a $2-\left(4u^2, 2u^2-u, u^2-u\right)$ design,

$$\frac{r-\lambda}{\alpha} = \frac{(2u^2-u)-(u^2-u)}{\alpha} = \frac{2u^2-u-u^2+u}{\alpha} = \frac{u^2}{\alpha}.$$

   By this, we conclude that α must divide u^2. But in this case where $l = 2$, we have shown that $\alpha = 2u - 1$, i.e. $2u - 1$ must divide u^2. This cannot happen because that would imply that $2u - 1 \mid u$. But $(2u - 1, u) = 1$, therefore $l \neq 2$.

3. $3 \leq l \leq u$

To study this general case, we use equation (4.7) that involves the integer number:

$$\frac{[l-(u+1)](2u-1)}{1+l(u-1)}.$$

For this number to be an integer, the following must be true:

$$1 + l(u-1) \mid [l-(u+1)](2u-1). \tag{4.9}$$

We will show, by means of contradiction that $1+l(u-1)$ does not divide $[l-(u+1)](2u-1)$ and therefore the case where $3 \leq l \leq u$ cannot occur.

We need the following lemma:

Lemma 4.7. *If a, b, c are integers and $a \mid bc$ then $a \leq (a, b)(a, c)$.*

Proof. Since $a \mid bc$, $bc = ak$ for some $k \in Z$ and therefore:

$$\frac{bc}{(a,b)} = \frac{ak}{(a,b)}.$$

Hence $\frac{a}{(a,b)}$ divides $\frac{b}{(a,b)}c$. Since $(\frac{a}{(a,b)}, \frac{b}{(a,b)}) = 1$,we conclude that $\frac{a}{(a,b)}$ must divide c. Since also $\frac{a}{(a,b)}$ divides a, it follows that $\frac{a}{(a,b)}$ divides (a,c). So,

$$\frac{a}{(a,b)} \leq (a,c)$$

and hence

$$a \leq (a,b)(a,c).$$

$\square$

The following result is well known and will be used in the rest of the proof:
Result:

1. Let a, b, c be integers. Then $(a + bc, b) = (a, b)$.

Continuing with our proof, we will now show that $(1 + l(u-1), 2u-1) = (l-2, 2u-1)$.

We observe that $1 + l(u-1) = (l-2)(u-1) + 1(2u-1)$. By Result 1, $((l-2)(u-1) + (2u-1) \cdot 1, 2u-1) = ((l-2)(u-1), 2u-1)$. Hence,

$$(1 + l(u-1), 2u-1) = (l-2, 2u-1). \qquad (**)$$

Using Lemma 4.7 and assuming that equation (4.9) is true, we get that:

$$1 + l(u-1) \leq (1 + l(u-1), l-u-1)(1 + l(u-1), 2u-1).$$

But since for any two numbers a and b it is true that $(a,b) \leq b$, we have that: $1 + l(u-1) \leq (l-u-1) \cdot (1 + l(u-1), 2u-1)$.
Moreover, since $(1 + l(u-1), 2u-1) = (l-2, 2u-1)$ by $(**)$, we have that:

$$1 + l(u-1) \leq (l-u-1)(l-2, 2u-1) \Longleftrightarrow$$

$$1 + l(u-1) \leq (2-u)(l-2). \tag{4.10}$$

Equation (4.10) is true $\forall l \geq 3$, since:
$l \geq 3 \Rightarrow l-1 \geq 2$

$$\Rightarrow l-1-u \geq 2-u.$$

Also,
$l-2 \geq (l-2, 2u-1)$.

By observing that $l > l-2$ and $1-u > u-1$ $(u \geq 3)$, equation (4.10) becomes:

$$1 + l(u-1) < l(1-u)$$

which is a contradiction.
So we conclude that $1 + l(u-1)$ does not divide $[l-(u+1)](2u-1)$ and therefore l cannot take values in the range $[3, u]$.

We have shown that the only possible values for l are 1 or $u+1$. If $l = 1$, using equation (4.5)

$$\alpha - 1 = l(u-1)$$

and equation (4.6)

$$s = 1 + l(2u-1)$$

we get:

i) if $l = 1$ then $\alpha = u$ and $s = 2u$, and

ii) if $l = u+1$ then $\alpha = u^2$ and $s = 2u^2 + u$.

These final observations complete the proof of the theorem.

$\square$

Remarks:

1. In the case where $s = 2u^2 + u$ and $\alpha = u^2$, the complement of any block of a $2-\left(4u^2, 2u^2-u, u^2-u\right)$ design $\mathcal{D}$ is an α-arc for the design. This is true because in $\mathcal{D}$, every two blocks contain $\lambda = u^2 - u$ common points and thus there are u^2 remaining points for each block that don't belong to any other block. So by taking the α-arc A to be the complement of a block, then every other block will contain u^2 points of A that doesn't belong to any other set, i.e $|B \cap A| \in \{0, \alpha\}$ for every block B of $\mathcal{D}$.

   Particularly, all u^2-arcs A with $|A| = s = 2u^2 + u$ are complements of blocks! To verify this, recall that by Lemma 4.3, the number of blocks meeting A, i.e. the secants, is $\dfrac{rs}{\alpha} = \dfrac{(2u^2-u)(2u^2+u)}{u^2} = 4u^2 - 1$. So there exist a unique passant block, say B. Since $|B| = 2u^2 - u$ and the complement of B has $2u^2 + u = |A|$ points, then A is the complement of B.

2. A similar argument holds in the case of the complement design of $\mathcal{D}$, i.e. the $2-\left(4u^2, 2u^2+u, u^2+u\right)$ design $\bar{\mathcal{D}}$. For every block of $\bar{\mathcal{D}}$, its complement is a u^2-arc of cardinality $s = 2u^2 - u$ having $4u^2 - 1$ secants and a unique passant.

Chapter 5

The ADLS Method

5.1 Introduction

In Section 5.2 we state some basic definitions and results that will be used in Section 5.3 where we describe a new method called ADLS, for constructing Menon designs using an Affine design and a Latin square.

In Section 5.4 we study α-arcs in the designs arising from the ADLS method and also study the induced designs on these α-arcs.

In Section 5.5 we study whether Menon designs constructed via the ADLS method may have special triples.

Section 5.6 studies the resulting designs when in the ADLS method we change the Latin square by permuting its rows or columns or its sybmols. At the end of this section we show that in the case of n=1, all designs coming from the ADLS method are isomorphic and in Section 5.7 we prove that the unique design arising in this case, is the design from the $L_2(4)$ association scheme.

In Section 5.8 we study alpha arcs even further.

5.2 Basic Definitions and Results

Definition 5.1. Two designs $\mathcal{D} = (X, \mathcal{B})$ and $\mathcal{D}' = (X', \mathcal{B}')$ are isomorphic if and only if there exist a bijection $\tau : X \cup \mathcal{B} \to X' \cup \mathcal{B}'$ such that:

1. $\tau(X) = X'$, $\tau(\mathcal{B}) = \mathcal{B}'$
2. pIB if and only if $\tau(p)I\tau(B)$.

Bijection τ is called an ***isomorphism***.

If $\mathcal{D} = \mathcal{D}'$ then τ is called an ***automorphism***.

The set of all automorphisms of a design $\mathcal{D}$ form the ***full automorphism group*** of $\mathcal{D}$.

Definition 5.2. A correlation of a symmetric design $\mathcal{D} = (X, \mathcal{B})$ is an isomorphism $\tau : \mathcal{D} \to \mathcal{D}^*$, where $\mathcal{D}^*$ denotes the dual of $\mathcal{D}$.

Definition 5.3. An incidence structure $\mathcal{D}$ is called ***self-dual*** if $\mathcal{D}$ and $\mathcal{D}^*$ are isomorphic incident structures.

Definition 5.4. A correlation τ of order 2, i.e $\tau^2 = 1$, is called a ***polarity*** .

For a polarity τ, the conditions of Definition 5.1 may be written as: $pI\tau(q)$ if and only if $qI\tau(p)$, $\forall p, q \in X$.

5.3 The ADLS Method(Affine Design-Latin Square)

The ADLS method uses an affine design and a Latin square to construct Menon designs. This method is described below.

Let Π be a 3-$(4n, 2n, n-1)$ affine design. Denote the parallel classes of Π as C_i, $i \in S \backslash \{0\}$, where $S = \{0, 1, \ldots, 4n-1\}$ and the points of Π as P_i, where $i \in S$.

Let $\varphi : S^2 \to S$ be a function satisfying:

i) $\varphi(i, i) = 0$ for all $i \in S$

ii) for each $j \in S$, the mapping $i \mapsto \varphi(i, j)$ is a permutation of S

iii) for each $i \in S$, the mapping $j \mapsto \varphi(i, j)$ is a permutation of S.

Such a mapping φ determines a unique Latin square L with entries from S and all diagonal entries 0, where $L_{ij} = \varphi(i, j)$. Conversely, given such a Latin square L we can define a mapping φ with the required properties.

The ADLS method uses the affine design Π and the Latin square L to define the incidence structure $\mathcal{D}$ with points (i, j), $i, j \in S$, and blocks $[x, y]$, $x, y \in S$. The structure is defined as follows:
The point (i, j) is incident with the block $[x, y]$ if and only if $y \neq j$ and the points P_i and P_x are on the same block of $C_{\varphi(j,y)}$.

Theorem 5.5. *$\mathcal{D}$ is a 2-$\big(16n^2, 8n^2 - 2n, 4n^2 - 2n\big)$ Menon design of order $4n^2$.*

Proof. By definition, $\mathcal{D}$ has $16n^2$ points.

Let $[x, y]$ be a block. We may choose an element $j \in S \backslash \{y\}$ in $4n - 1$ ways. For each such j, there are $2n$ points P_i on the unique block of the parallel class $C_{\varphi(j,y)}$ containing P_x. Hence, there are $2n(4n - 1) = 8n^2 - 2n$ points (i, j) incident with $[x, y]$.

Now let (i_1, j_1) and (i_2, j_2) be distinct points. We consider separately the cases $j_1 = j_2$ and $j_1 \neq j_2$.

If $j_1 = j_2$ then $i_1 \neq i_2$. There are $2n-1$ blocks of Π containing both P_{i_1} and P_{i_2}. Each of these blocks contains $2n$ points P_x and determines a unique class C_z containing it, and hence a unique $y \in S$ with $z = \varphi(j_1, y)$. Hence, there are $2n(2n-1) = 4n^2 - 2n$ blocks $[x, y]$ incident with both (i_1, j_1) and (i_2, j_2) in this case.

If $j_1 \neq j_2$, there are $4n-2$ choices for $y \in S \backslash \{j_1, j_2\}$. Each such choice determines a unique class $C_{\varphi(j_1,y)}$ and a unique block B in this class containing P_{i_1}. It also determines a unique class $C_{\varphi(j_2,y)}$, which is different from $C_{\varphi(j_1,y)}$, and a unique block B' in this class containing P_{i_2}. The point P_x may be chosen arbitrarily in $B \cap B'$. There are n such choices. Hence, there are $n(4n-2) = 4n^2 - 2n$ blocks $[x, y]$ incident with both (i_1, j_1) and (i_2, j_2) in this case. □

Note. Appendix D contains the implementation of the ADLS method, for producing Menon designs of order 16 using MATLAB.

Lemma 5.6. *If the Latin square L used in the ADLS method is symmetric, then the function $\tau : (i, j) \leftrightarrow [i, j]$ is a polarity of the constructed design $\mathcal{D}$.*

Proof. Given that L is symmetric, i.e $\varphi(i, j) = \varphi(j, i)$ we have that if (i, j) and (i', j') are two points of $\mathcal{D}$ then:
$(i, j) \in \tau((i', j')) \Leftrightarrow$
$(i, j) \in [i', j'] \Leftrightarrow$
$j \neq j'$ and P_i, $P_{i'}$ belong to the same block of $C_{\varphi(j,j')} \Leftrightarrow$
$j' \neq j$ and $P_{i'}$, P_i belong to the same block of $C_{\varphi(j',j)}$, since L is symmetric $\Leftrightarrow$
$(i', j') \in [i, j] \Leftrightarrow$
$(i', j') \in \tau((i, j))$.

Therefore, τ is a polarity of $\mathcal{D}$.

□

Note 5.7. *A point x is called an* ***absolute point*** *of a polarity τ, if $x \in \tau(x)$. If (i, j) is a point of $\mathcal{D}$, for this point to be absolute, we must have that $(i, j) \in \tau(i, j)$ and thus $(i, j) \in [i, j]$. But this cannot happen since j is equal to itself. So we conclude that for this function τ, no point is absolute.*

By using GRAPE, [24], we have found an example where the design has polarities, 576 particularly, without the Latin square being symmetric. See Appendix B for the complete example.

5.4 Alpha arcs in the Designs constructed via the ADLS Method

Recall that Theorem 4.6 shows that if $\mathcal{D}$ is a $2-\big(4u^2, 2u^2-u, u^2-u\big)$ Menon design, then $\mathcal{D}$ can only have u-arcs of cardinality $2u$ or u^2-arcs of cardinality $2u^2+u$. Since in this chapter we use the parameters $2-\big(16n^2, 8n^2-2n, 4n^2-2n\big)$ for the Menon designs arising via the ADLS method, by replacing u with $2n$ we can express the results of this theorem in terms of n. That is, the $2-\big(16n^2, 8n^2-2n, 4n^2-2n\big)$ Menon design $\mathcal{D}$ constructed via the ADLS method can only have $2n$-arcs of cardinality $4n$ or $4n^2$-arcs of cardinality $8n^2+2n$.
The following theorem, studies the structure of some of the arcs of cardinality $4n$.

Theorem 5.8. *The set Δ_k=$\{(i,k) : 0 \leq i \leq 4n-1\}$, with $0 \leq k \leq 4n-1$, is an α-arc with $\alpha = 2n$ for the $2-\big(16n^2, 8n^2-2n, 4n^2-2n\big)$ Menon design $\mathcal{D}$, constructed via the ADLS method.*

Proof. A point (i,k) of $\mathcal{D}$ is incident with a block $[x,y]$ if and only if $y \neq k$ and the points P_i and P_x are on the same block of $C_{\varphi(k,y)}$.

In the case where $y \neq k$, there are $2n$ points P_i on the unique block of $C_{\varphi(k,y)}$ containing P_x. So in this case, the block $[x,y]$ meets Δ_k in $2n$ points.

In the case where $y = k$, the blocks $[x,k]$ meet Δ_k in zero points.

Hence, Δ_k is a $2n$-arc for the design $\mathcal{D}$. Particularly $\mathcal{D}$ has only one passant, the block $[x,k]$ and $4n-1$ secants, the blocks $[x,y]$ with $y \neq k$. □

Note. Since the set Δ_k=$\{(i,k) : 0 \leq i \leq 4n-1\}$ is an α-arc for every $k \in \{0,1,\dots 4n-1\}$, we have that the point set of the 2-$\big(16n^2, 8n^2-2n, 4n^2-2n\big)$ Menon design $\mathcal{D}$, can be partitioned into α-arcs of the form Δ_0, Δ_1, ..., Δ_{4n-1}.

After we studied the set Δ_k, and showed that it is a $2n$-arc for the design $\mathcal{D}$, we defined the set $S_i = \{(i,j) : 0 \leq j \leq 4n-1\}$ and studied its intersection sizes with blocks of $\mathcal{D}$ in order to investigate whether S_i could be an α-arc as well. As it turned out, S_i is not an α-arc for $\mathcal{D}$, but we present our work below, as a lemma, for completeness.

Lemma 5.9. *Let $S_i = \{(i,j) : 0 \leq j \leq 4n-1\}$ be a subset of points of the design $\mathcal{D}$ coming from the ADLS method. Then, the intersection sizes of S_i with blocks of $\mathcal{D}$ are $4n-1$ and $2n-1$.*

Proof. Let $[x,y]$ be a block of $\mathcal{D}$.

1. If $x = i$, then $|S_i \cap [i,y]| = 4n-1$, since:
   $(i,j) \in [i,y] \Leftrightarrow j \neq y$ and P_i belongs in the same block of the parallel class $C_{\varphi(j,y)}$. But every point of Π belongs in $4n-1$ blocks, hence there are $4n-1$ different choices for j, different than y such that $P_i \in C_{\varphi(j,y)}$.

2. If $x \neq i$, then $|S_i \cap [x,y]| = 2n-1$, since:
   $(i,j) \in [x,y] \Leftrightarrow j \neq y$ and $\{P_i, P_x\}$ belong in the same block of the parallel class $C_{\varphi(j,y)}$. But any two points of Π meet in $2n-1$ blocks. Each of these blocks determine a unique class containing them and a unique j such that $\{P_i, P_x\} \in C_{\varphi(j,y)}$.

□

Theorem 5.10. *The induced design $\mathcal{D}_{\Delta_k}$ is a 3-design, where Δ_k=$\{(i,k) : 0 \leq i \leq 4n-1\}$ is a $2n$-arc of the 2-$\big(16n^2, 8n^2-2n, 4n^2-2n\big)$ Menon design $\mathcal{D}$.*

Proof. Recall that by Definition 4.2, $\mathcal{D}_{\Delta_k}$ has $4n$ points, since $|\Delta_k| = 4n$. In addition, every block of $\mathcal{D}_{\Delta_k}$ has $2n$ points, since every secant meets Δ_k in $2n$ points.

We will now work on the number of blocks of $\mathcal{D}_{\Delta_k}$.

First we observe that (i,k) belongs to the block $[x,y]$ if and only if $y \neq k$ and the points P_i and P_x are on the same block of $C_{\varphi(k,y)}$ containing P_x. Since there are $2n$ P_x's on the specific block of $C_{\varphi(k,y)}$ in the $3-(4n,2n,n-1)$ affine design Π used in the ADLS method, for every such x the secants $[x,y]$ will induce the same blocks in $\mathcal{D}_{\Delta_k}$. Since there are

$$\frac{rs}{\alpha} = \frac{(8n^2-2n)4n}{2n} = 4n(4n-1)$$

secants, where s is the cardinality of Δ_k, we get that $\mathcal{D}_A$ has

$$\frac{4n(4n-1)}{2n} = 2(4n-1) = 8n-2$$

blocks.

Recall that a Hadamard $3-(4n,2n,n-1)$ design has $8n-2$ blocks. So, the design Π also has $8n-2$ blocks. Because of this, we can define a function f from the points and blocks of $\mathcal{D}_{\Delta_k}$ to the points and blocks of Π respectively.
First we note that the blocks of $\mathcal{D}_{\Delta_k}$ are the point subsets $\Delta_k \cap [x,y]$, $y \neq 0$.
Furthermore:
$\Delta_k \cap [x,y]$=$\Delta_k \cap [x',y] \Leftrightarrow$ the points P_x and $P_{x'}$ are on the same block of $C_{\varphi(k,y)}$.
So we define f as follows:
f:$\mathcal{D}_{\Delta_k} \to \Pi$,
$f((i,k)) \to P_i$ on points, and
$\Delta_k \cap [x,y] \to B_{x,y}$ on blocks, where $B_{x,y}$ denotes the block of $C_{\varphi(k,y)}$ on P_x.

We will now show that f is an isomorphism.
Since the designs $\mathcal{D}_{\Delta_k}$ and Π have the same number of points and blocks, the function f is a bijection.
Now let the point (i,k) be incident with the block $\Delta_k \cap [x,y]$, in $\mathcal{D}_{\Delta_k}$. Then: $(i,k) \in \Delta_k \cap [x,y]$ $\Leftrightarrow y \neq k$ and the points P_i and P_x are on the same block of $C_{\varphi(k,y)}$,

$\Leftrightarrow$ the point P_i belongs in the block of $C_{\varphi(k,y)}$ containing P_x,
$\Leftrightarrow f((i,k)) \in f(\Delta_k \cap [x,y])$.
Hence, $\mathcal{D}_{\Delta_k}$ and Π are isomorphic, and thus $\mathcal{D}_{\Delta_k}$ is a 3-design.

$\square$

5.5 Special Triples in the Designs constructed via the ADLS Method

Theorem 5.11. *The 2-$\big(16n^2, 8n^2-2n, 4n^2-2n\big)$ Menon design $\mathcal{D}$ of order $4n^2$ constructed via the ADLS method using the 3-$(4n, 2n, n-1)$ affine design Π and the Latin square L of order $4n$ on the alphabet $\{0, 1, \ldots, 4n-1\}$ with zero diagonal , cannot have special triples.*

To prove this theorem, recall that:

- The parallel classes of the affine design Π are C_i where $i \in S\backslash\{0\}$, $S = \{0, 1, \ldots, 4n-1\}$ and the points of Π are P_i where $i \in S$.

- Representing the points and blocks of $\mathcal{D}$ as (i,j) and $[x,y]$ respectively, incidence is defined as:
  $(i,j) \in [x,y] \Leftrightarrow j \neq y$ and the points P_i and P_x of Π are on the same block of the parallel class $C_{\phi(j,y)}$, where $\phi(j,y) = L_{jy}$.

- A special triple in a design, as given by Definition 3.7, is a set of three distinct blocks with the property that each of these blocks contains the intersection of the other two blocks. In the case of the Menon design $\mathcal{D}$ with parameters 2-$\big(16n^2, 8n^2-2n, 4n^2-2n\big)$, a special triple would imply three blocks meeting in exactly $4n^2-2n$ points.

Proof. In $\mathcal{D}$, for any three distinct blocks $[x,y]$, $[x',y']$, $[x'',y'']$ to form a special triple we must have that $\big|[x,y] \cap [x',y'] \cap [x'',y'']\big| = 4n^2 - 2n$. The set $\{[x,y] \cap [x',y'] \cap [x'',y'']\}$ contains the points (i,j) such that $j \neq \{y, y', y''\}$ and :

- the points P_i and P_x are on the same block A of the parallel class $C_{\phi(j,y)}$ and

- the points P_i and P_x' are on the same block A' of the parallel class $C_{\phi(j,y')}$ and

- the points P_i and P_x'' are on the same block A'' of the parallel class $C_{\phi(j,y'')}$.

We will count the number of possible points (i,j) in the intersection of the three blocks. Consider the following cases:

1. Let y, y', y'' be distinct. Then the parallel classes $C_{\phi(j,y)}$, $C_{\phi(j,y')}$, $C_{\phi(j,y'')}$ are also distinct and thus the blocks A, A' and A'' are mutually non-parallel. This implies that the number of possible j's is equal to $4n-3$, since $j \neq y, y', y''$.
   Having chosen j, the number of possible i's is equal to a, where $a = |A \cap A' \cap A''|$, since the point P_i must belong in blocks A, A' and A''. So the number I of possible points (i, j) is equal to $a(4n-3)$.
   We know that in the $3-(4n, 2n, n-1)$ affine design Π, any two blocks meet in 0 or n points, thus:
   $a = |A \cap A' \cap A''| \leq |A \cap A'| = n$, i.e. $a \leq n$. So:
   $I = a(4n-3) \leq n(4n-3) = 4n^2 - 3n < 4n^2 - 2n$ and thus the design cannot have special triples in this case.

2. Suppose that exactly two of the y's are equal, say $y = y' \neq y''$.
   Given that $y = y'$ then $x \neq x'$, otherwise the blocks $[x, y]$ and $[x', y']$ would be equal. Since $y = y'$ we have that $C_{\phi(j,y)} = C_{\phi(j,y')}$. Furthermore, the point P_i belongs to block A of the parallel class $C_{\phi(j,y)}$ and to block A' of the parallel class $C_{\phi(j,y')} = C_{\phi(j,y)}$. Since each point is on a unique block from each parallel class, then $A = A'$.

   By Theorem 1.5 we know that since Π is a $3-(4n, 2n, n-1)$ design, it is also a $2-(4n, 2n, 2n-1)$ design. Thus the two distinct points P_x and $P_{x'}$ of Π are contained in exactly $2n-1$ blocks. So there exist $2n-1$ or $2n-2$ possible choices for j according as y'' isn't or is a possible j. Again, having chosen j, the number of possible i's is equal to $a = |A \cap A''| = n$.

   Note that since $y \neq y''$, the blocks A and A'' are non-parallel. So the number I of possible points (i, j) in the intersection of the three blocks $[x, y]$, $[x', y']$ and $[x'', y'']$ can take the following two values:

   i) $I = n(2n-1) = 2n^2 - n < 4n^2 - 2n$

   ii) $I = n(2n-2) = 2n^2 - 2n < 4n^2 - 2n$

   For both cases, I is clearly less than $4n^2 - 2n$ showing that again in this case, the design cannot have special triples.

3. Suppose, $y = y' = y''$.
   Given that all y's are the same, then $\{x, x', x''\}$ must be distinct otherwise the blocks $[x, y], [x', y'], [x'', y'']$ would not be distinct. The condition $j \neq y, y', y''$ is reduced to $j \neq y$ and the points P_i, P_x, P_x' and P_x'' must belong to the same block A of the parallel class $C_{\phi(j,y)} = C_{\phi(j,y)} = C_{\phi(j,y')}$ and therefore $A = A' = A''$. So the number of possible j's is equal to the number of blocks containing the distinct points $P_x, P_{x'}, P_{x''}$, that is $n-1$, since Π is a $3-(4n, 2n, n-1)$ design. Moreover the number of possible i's is equal to $A \cap A' \cap A'' = |A| = 2n$, since $A = A' = A''$.

So the number I of possible points (i, j) in the intersection of the three blocks is: $I = 2n(n-1) = 2n^2 - 2n < 4n^2 - 2n$ and thus the design cannot have special triples in this case.

□

Result:

Theorem 5.11 characterizes the Menon designs arising from the ADLS method as not having special triples, thus we now know that if a design with the parameters $2-(16n^2, 8n^2 - 2n, 4n^2 - 2n)$ has special triples, it could not have been constructed via the ADLS method!

Theorem 5.12. *The Menon design $\mathcal{D}$ with parameters 2-$\big(16n^2, 8n^2 - 2n, 4n^2 - 2n\big)$ arising from the ADLS method is quasi-3 for blocks only in the case where $n = 1$.*

Proof. As Definition 1.13 states, a design is said to be quasi-3 for blocks if there exist integers α and β called intersection numbers such that $|A \cap B \cap C| \in \{\alpha, \beta\}$ for any three distinct blocks A, B and C of the design. Looking back at the proof of Theorem 5.11 we see that any three distinct blocks of the Menon design $\mathcal{D}$ can have only the following possible intersection numbers:

1) $a(4n-3)$
where a is the intersection number of three blocks of the affine design Π.

2) **i)** $n(2n-1)$

ii) $n(2n-2) = 2n(n-1)$.

3) $2n(n-1)$.

For the design $\mathcal{D}$ to be quasi-3 for blocks, either $a(4n-3) = n(2n-1)$ or $a(4n-3) = 2n(n-1)$.

i) Case: $a(4n-3) = n(2n-1)$.
This implies that $(4n-3)$ divides $n(2n-1)$. Since $(4n-3)$ is odd, then: $(4n-3, 2n-1) = (4n-3, 2(2n-1)) = (4n-3, 4n-2) = 1$. So $(4n-3)$ and $(2n-1)$ are coprime and therefore $(4n-3)$ must divide n. But this happens only when $n = 1$.

ii) Case: $a(4n-3) = 2n(n-1)$.
Similar argument shows that:
$(4n-3, n-1) = (4n-3, 4n-4) = 1$, since $(4n-3)$ is odd. Thus the numbers $(4n-3)$ and $(n-1)$ are coprime. Since the numbers $(4n-3)$ and 2 are also coprime, $(4n-3)$ must divide n. But this happens only when $n = 1$.

□

Now, by replacing $n = 1$ to the equalities in **i)** and **ii)** above, we get $a = 1$ and $a = 0$ respectively. So, we have the following result:

Corollary 5.13. *In the case where $n = 1$, the $2-(16, 6, 2)$ Menon design arising from the ADLS method is quasi-3 for blocks with intersection sizes 0 or 1.*

Note. It can be seen from the arguments presented in Theorem 5.12, that the possible intersection numbers of three blocks of the design $\mathcal{D}$ are the numbers:

1) $a(4n-3)$

where a is the intersection number of three blocks of the affine design Π.

2) **i)** $n(2n-1)$

ii) $n(2n-2) = 2n(n-1)$.

3) $2n(n-1)$

Since these numbers are independent from our choice of the Latin square used in the ADLS method, we can conclude that in the case where $n = 1$ the dual design of the constructed design $\mathcal{D}$, $\mathcal{D}^*$ is also quasi-3 for blocks and thus the design $\mathcal{D}$ is quasi-3 for points, i.e any three points of $\mathcal{D}$ are contained in $n(2n-1) = 1$ or $2n(n-1) = 0$ common blocks.

5.6 Isomorphism of Designs constructed via the ADLS Method

Let $\mathcal{D}$ be the Menon design constructed via the ADLS method using the $3-(4n, 2n, n-1)$ affine design Π and the Latin square L.

Theorem 5.14. *The dual $\mathcal{D}^*$ of $\mathcal{D}$ is obtained using the same Π but replacing L by its dual Latin square L^*, i.e the transpose of L.*

Proof. Recall that if $\mathcal{D} = (X, B, I)$ is an incidence structure, then the dual incidence structure is $\mathcal{D}^* = (B, X, I^*)$ where $(B, X) \in I^*$ if and only if $(X, B) \in I$.

Also, recall that $\varphi(i, j) = L_{ij}$ so from now on we will use the notation $C_{L_{ij}}$ instead of $C_{\varphi(i,j)}$.

Let $(i, j) \in [x, y]$ in $\mathcal{D}$. Then:

$j \neq y$ and the points P_i and P_x belong in the same block of the parallel class $C_{L_{jy}}$. But since L^* is the transpose of L, $L_{jy} = L^*_{yj}$, and therefore:

$y \neq j$ and P_x and P_i belong in the same block of the parallel class $C_{L^*_{(y,j)}}$. But this is equivalent to:

$(x, y) \in [i, j]$ in $\mathcal{D}^*$.

So, we have shown that the dual $\mathcal{D}^*$ of $\mathcal{D}$ is obtained using the same Π but replacing L by its dual Latin square L^*. □

Definition 5.15. If we can change one Latin square into another, by means of any or all of the operations below, we say that the two squares are ***isotopic***.
The operations are:

1. permute the rows
2. permute the columns
3. permute the symbols. (i.e. rename the symbols without changing their relative positions)

Given a Latin square whose entries consist of the symbols $1, 2, ..., n$, we can permute the columns so that the first row consists of $1, 2, ..., n$ in their natural order. After doing this we could permute the rows so that the first column is also in the natural order. The resulting square is of course isotopic to the original square and is a convenient representative of the isotopy class of this square. Such a square is said to be in ***standard form*** or ***reduced***.

The equivalence classes of the Latin squares under the isotopy relation are called ***isotopy classes***.

Observation:
In the case of Latin squares of order 4, on the symbols $\{0, 1, 2, 3\}$ with zero diagonal, there are only two isotopic classes. The representatives of these classes, with zero diagonal, are:

$$L_1 = \begin{matrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{matrix}$$

and

$$L_2 = \begin{matrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 3 & 2 & 0 & 1 \\ 2 & 3 & 1 & 0 \end{matrix}$$

Since the Latin squares used in our construction have zero diagonal, if we permute the rows of such a square, we must permute the columns with the same permutation to get an isotopic Latin square with zero diagonal!

The following theorem shows that under these conditions, the Menon designs obtained by permuting the rows and columns of a Latin square are isomorphic.

Theorem 5.16. *Let $\sigma \in S_4n$. Given a Latin square L of order $4n$ on the symbols $\{0, 1, 2, \ldots, 4n-1\}$ with zero diagonal, let L' be the Latin square obtained by applying σ to the rows and columns of L. Then the Menon designs $\mathcal{D}$ and $\mathcal{D}'$ arising by the ADLS method using the Latin squares L and L' respectively, and the affine $3-(4n, 2n, n-1)$ design Π, are isomorphic.*

Proof. By the way we defined the Latin square L', the following relation holds:
$L_{ij} = L'_{\sigma(i)\sigma(j)}$.
We will show that the mapping $f \colon \mathcal{D} \to \mathcal{D}'$ defined by:
$(i, j) \mapsto (i, \sigma(j))$, on the points and
$[x, y] \mapsto [x, \sigma(y)]$, on the blocks,
is an isomorphism.

Let the point (i, j) belong in the block $[x, y]$. Then:
$(i, j) \in [x, y] \Leftrightarrow j \neq y$ and the points P_i, P_x belong in the same block of the parallel class $C_{L_{jy}}$.
But since $j \neq y$ and $\sigma \in S_n$, we have that $\sigma(j) \neq \sigma(y)$, and in addition $L_{ij} = L'_{\sigma(i)\sigma(j)}$, from the previous observation. So:
$(i, j) \in [x, y] \Leftrightarrow \sigma(j) \neq \sigma(y)$ and the points P_i, P_x belong in the same block of the parallel class $C_{L'_{\sigma(j)\sigma(y)}}$,
$\Leftrightarrow (i, \sigma(j)) \in [x, \sigma(y)]$.

Thus, the designs $\mathcal{D}$ and $\mathcal{D}'$ are isomorphic. $\square$

The following theorem studies the effect on the designs arising via the ADLS method when the non-zero symbols of the Latin square are permuted. Before we state the theorem, we need the following:

1. The function $\chi : Aut\Pi \to S_{4n-1}$ defined by $\Phi \to \Phi_0$, where Φ_0 is the resulting permutation on the parallel classes when Φ is applied to the elements of Π, is a homomorphism whose kernel is the normal subgroup Δ of all dilatations in $Aut\Pi$.

2. The subgroup Δ of all dilatations in $Aut\ \Pi$ contains , by the definition of the Kernel, all automorphisms of Π that fix its parallel classes. Moreover the image G of χ is a subgroup of S_{4n-1} and by the Fundamental Homomorphism Theorem, $G \cong \dfrac{Aut\Pi}{\Delta}$.

3. G is indeed a subgroup of S_{4n-1}, since any two automorphisms of Π may induce the same permutation of S_{4n-1}. This is true because for any automorphisms σ, τ with $\sigma \in \Delta\tau$

we have that:
$\sigma \in \Delta\tau$
$\Leftrightarrow \sigma\tau^{-1} \in \Delta$ (i.e $\sigma\tau^{-1}$ fixes the parallel classes)
$\Leftrightarrow \sigma\tau^{-1}C_i = C_i$ (for every parallel class C_i, where $i = 1, 2, \ldots 4n$)
$\Leftrightarrow \sigma C_i = \tau C_i$
$\Leftrightarrow \chi(\sigma) = \chi(\tau)$, i.e.
σ and τ induce the same permutation on the parallel classes.

Theorem 5.17. *Let Π be the $3-(4n, 2n, n-1)$ affine design and L the Latin square used in the ADLS method to obtain design $\mathcal{D}$. Consider the function $\chi : Aut\Pi \to S_{4n-1}$ defined by $\Phi \to \Phi_0$ where Φ_0 is the permutation induced by Φ, on the parallel classes of Π.*
Let G be the image of χ, and $\alpha \in G$ to be a permutation of S_{4n} such that $\alpha(0) = 0$.
If L^α is the Latin square obtained by applying α to the symbols of L and $\mathcal{D}^\alpha$ is the Menon design constructed via the ADLS method using Π and L^α, then the designs $\mathcal{D}$ and $\mathcal{D}^\alpha$ are isomorphic.

Moreover, an isomorphism between $\mathcal{D}$ and $\mathcal{D}^\alpha$ fixes the $2n$-arc
$\Delta_j = \{(i, j) : 0 \le i \le 4n-1\}$.

Proof. Given that $\alpha \in G$, and $\Phi \in Aut\Pi$, then the mapping $\Psi^\alpha : \mathcal{D} \to \mathcal{D}^\alpha$: $(i, j) \to (\Phi(i), j)$ on points, and $[x, y] \to [\Phi(x), y]$ on blocks, determines an isomorphism since ,
if the point (i, j) belongs in the block $[x, y]$ in $\mathcal{D}$, then:
$(i, j) \in [x, y] \Leftrightarrow j \neq y$ and the points P_i, P_x belong in the same block of the parallel class $C_{L_{jy}}$
$\Leftrightarrow j \neq y$ and the points $\Phi(P_i)$ and $\Phi(P_x)$ belong in the same block of the parallel class of $\Phi(C_{L_{jy}})$.
(These two statements are equivalent since Φ is an automorphism of Π).

But, $\Phi(C_{L_{jy}})$=$C_{\alpha(L_{jy})} \to C_{L^\alpha_{jy}}$ (since Φ induces the permutation α on the $4n-1$ parallel classes, and maps L_{ij} to L^α_{ij}).

Thus:
$(i, j) \in [x, y] \Leftrightarrow j \neq y$ and the points $\Phi(P_i)$ and $\Phi(P_x)$ belong in the same block of the parallel class $C_{L^\alpha_{jy}}$ in Π
$\Leftrightarrow (\Phi(i), j) \in [\Phi(x), y]$ in $\mathcal{D}^\alpha$.
So we have shown that Ψ^α is an isomorphism.

In addition this isomorphism, fixes every arc $\Delta_j = \{(i, j) : 0 \le i \le 4n-1\}$, where j can be considered as a point of Π, since:
$\Phi(\Delta_j) = \{(\Phi(i), j) : 0 \le i \le 4n-1\}$, and since Φ is a permutation on the points of Π, we have that $\Phi(\Delta_j) = \Delta_j$, $\forall j$. □

Note 5.18. *In the case where $n = 1$, Π is the affine plane of order 2. Its automorphism group is S_4 and this group acts on the 3 parallel classes like S_3. Thus any permutation in S_3 will be induced by an automorphism of Π.*
It follows by Theorem 5.17 that in the case, where $n = 1$, permuting the elements 1,2,3 of L gives isomorphic Menon designs.

Since all Menon designs arising via the ADLS method in the case where $n = 1$ are isomorphic, we can conclude that this unique Menon design $\mathcal{D}$ is self-dual, since in Section 5.14 we've shown that the dual design $\mathcal{D}^*$ of $\mathcal{D}$, is obtained by changing the Latin square L with its transpose L^T, and by keeping the affine design Π the same.

Computer search showed that in the case where $n = 2$ the designs are not self dual and thus we can conclude that for $n = 2$, not all designs arising via the ADLS method are isomorphic!

In Theorem 5.16 we've proven that if we permute the rows and columns of the Latin square used in the ADLS method, we obtain isomorphic designs.
In Theorem 5.17 we've proven that if there exist a permutation that is induced on the parallel classes of the affine design Π, then when this permutation is applied to the Latin square, the corresponding designs are isomorphic.
An interesting question one may ask is whether the converse of Theorem 5.17 is true. Even though we have not been able to prove that this is actually true, the following thoughts lead towards this belief:

Heuristic Argument:

Let $\alpha \in S_{4n}$, where $\alpha(0) = 0$. If there exist an isomorphism $\mathcal{D} \to \mathcal{D}^\alpha$ fixing each arc Δ_j, then $\alpha \in G$.

Proof:

Suppose there exist an isomorphism $\Psi^\alpha : \mathcal{D} \to \mathcal{D}^\alpha$, that fixes every arc Δ_j. Then the action of Ψ on points of $\mathcal{D}$ is given by: $\Psi(i, j) = (i', j)$, and on blocks by: $\Psi[x, y] = [x'', y]$, where the mappings $i \to i'$ and $x \to x''$ are permutations of the point set of Π.

Recall that incidence in $\mathcal{D}$ is given by:
$(i, j) \in [x, y] \Leftrightarrow j \neq y$ and the points P_i and P_x are on the same block of the class $C_{L_{jy}}$,

and incidence in $\mathcal{D}^\alpha$ is given by:

$(i', j) \in [x'', y] \Leftrightarrow j \neq y$ and the points $P_{i'}$ and $P_{x''}$ are on the same block of $C_{L^\alpha_{jy}}$.
Since Ψ is an isomorphism from $\mathcal{D}$ to $\mathcal{D}^\alpha$,

1. the statement: P_i and P_x are on the same block of the class $C_{L_{jy}}$ is equivalent with the

statement: $P_{i'}$ and $P_{x''}$ are on the same block of $C_{L^{\alpha}_{jy}}$, for all i, x.
But, since the statement: $P_{i'}$ and $P_{x''}$ are on the same block of $C_{L^{\alpha}_{jy}}$, is true for all i, x, if we choose $i = i''$, then this statement is equivalent with:

2. $P_{i'}$ and $P_{x'}$ are on the same block of $C_{L^{\alpha}_{jy}}$.

Combining (1) and (2), we have that:

3. P_i and P_x are on the same block of the class $C_{L_{jy}}$
$\Leftrightarrow$
$P_{i'}$ and $P_{x'}$ are on the same block of $C_{L^{\alpha}_{jy}}$.

Now, since Ψ fixes all arcs Δ_j, then if a block B in $\mathcal{D}$ meets Δ_j, then the image of B, $\Psi(B)$ will also meet Δ_j in $\mathcal{D}^{\alpha}$. From this we can conclude that Ψ also permutes the blocks of $\mathcal{D}$, and thus, the mapping $\Omega : \Pi \to \Pi$ is an automorphism of Π.

What we need to show now, is that the action of Ω on the parallel classes is α.
So, if we take a fixed point P_i in a block $B \in C_{L_{jy}}$, then its image $P_{i'}$ will be in another block $B' \in C_{L^{\alpha}_{jy}}$. Equivalence (3), tells as that for any other point $P_x \in B$, its image $P_{x'}$ must also be in $B' \in C_{L^{\alpha}_{jy}}$. Hence every point of B has its image in B' and thus every block of the parallel class is induced by α. From this, can we suggest that every parallel class is induced by α as well.

5.7 The ADLS Method and the $L_2(4)$ Association Scheme

It is known that there exist only 3 non-isomorphic designs with the parameters $2-(16,6,2)$. As we've shown, only one arises from our construction. We will show that this is actually the design obtained by the $L_2(4)$ association scheme.

Theorem 5.19. *The $2-(16n^2, 8n^2-2n, 4n^2-2n)$ Menon design constructed via the ADLS method using the $3-(4n, 2n, n-1)$ affine design Π and Latin square L, is isomorphic to the $2-(16,6,2)$ design arising from the $L_2(4)$ association scheme, in the case where $n = 1$.*

Proof. Let $\mathcal{D}$ be the Menon design constructed from the affine design Π and the Latin square

$$L = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{bmatrix}$$

Now, let $\mathcal{D}_L$ be the $2-(16,6,2)$ design from the $L_2(4)$ association scheme. The points of $\mathcal{D}_L$ are (i,j) with $i, j \in \{0,1,2,3\}$, the blocks of $\mathcal{D}_L$ are $[x,y]$ with $x, y \in \{0,1,2,3\}$ and incidence is defined by $(i,j) \in [x,y]$ if, and only if, $i = x$ and $j \neq y$ or $i \neq x$ and $j = y$.

Also, let τ_0 be the identity permutation on $\{0,1,2,3\}$, $\tau_1 = (0,1)(2,3)$, $\tau_2 = (0,2)(1,3)$ and $\tau_3 = (0,3)(1,2)$.
We will show that the following mappings of the points and blocks of Π to the points and blocks of $\mathcal{D}_L$, respectively,
$(i,j) \mapsto (i, \tau_i(j))$
$[x,y] \mapsto [x, \tau_x(y)]$,
determine an isomorphism.

In the design $\mathcal{D}$, the statement $(i,j) \in [x,y]$ is equivalent with:
$j \neq y$, and the points P_i and P_x are in the same block of the parallel class $C_{L_{jy}}$.

In the design $\mathcal{D}_L$, the statement $(i,\tau_i(j)) \in [x,\tau_x(y)]$ is equivalent with:
$i = x$ and $\tau_i(j) \neq \tau_x(y)$ or,
$i \neq x$ and $\tau_i(j) = \tau_x(y)$.

In order to show that these mappings determine an isomorphism, we need to show that these two statements are equivalent.

Assume that the point (i,j) is on the block $[x,y]$ in $\mathcal{D}$, i.e. the first statement holds. Now, in the case where $i = x$, the permutations τ_i and τ_x are the same, and $\tau_i(j) \neq \tau_x(y)$, since $j \neq y$. Therefore the point $(i,\tau_i(j))$ is on the block $[x,\tau_x(y)]$ in $\mathcal{D}_L$.

We will now study the case where $i \neq x$.

Let A=$\{\tau_0,\tau_1,\tau_2,\tau_3\}$. Then, A is the abelian Klein 4-group, with group action the composition of two permutations.

If in the Latin square L, each entry i is replaced by τ_i, the resulting square

*	τ_0	τ_1	τ_2	τ_3
τ_0	τ_0	τ_1	τ_2	τ_3
τ_1	τ_1	τ_0	τ_3	τ_2
τ_2	τ_2	τ_3	τ_0	τ_1
τ_3	τ_3	τ_2	τ_1	τ_0

is the Cayley table (the multiplication table) of the group A.

In particular,

1. $\tau_{L_{i,j}} = \tau_i\tau_j = \tau_j\tau_i$, for each $i, j \in \{0, 1, 2, 3\}$.

   In addition, we observe that:

2. $\tau_f(0) = f$, $\forall f \in \{0, 1, 2, 3\}$.

   Thus if $i, j \in \{0, 1, 2, 3\}$ then:
   $\tau_i(j) = \tau_i(\tau_j(0))$, since $\tau_j(0) = j$
   $\Leftrightarrow$
   $\tau_i(j) = \tau_j(\tau_i(0))$, since A is abelian,
   $\Leftrightarrow$
   $\tau_i(j) = \tau_j(i)$, since $\tau_i(0) = i$.
   Moreover, since $\tau_{L_{i,j}} = \tau_i\tau_j$, by observation 1, we have:
   $\tau_{L_{ij}}(0) = \tau_i\tau_j(0)$ which is equivalent with:
   $L_{ij} = \tau_i(j)$, (since $L_{ij}(0) = L_{ij}$ and $\tau_i\tau_j(0) = \tau_i(j)$, by observation 2.)

   Using the same argument, we also get:
   $L_{ij} = \tau_j(i)$. Hence:

3. $\tau_i(j) = \tau_j(i) = L_{i,j}$, $\forall i, j \in \{0, 1, 2, 3\}$.

Now, since the parallel classes of Π are $C_1 = \{\{P_0, P_1\}, \{P_2, P_3\}\}$, $C_2 = \{\{P_0, P_2\}, \{P_1, P_3\}\}$, and $C_3 = \{\{P_0, P_3\}, \{P_1, P_2\}\}$, then the statement "$i \neq x$ and P_i and P_x are in the same block of class C_w" is equivalent to "$i \neq x$ and (i, x) is a cycle in τ_w".

Thus, the following statements are equivalent:

- $j \neq y$, $i \neq x$ and P_i and P_x are in the same block of the parallel class $C_{L_{j,y}}$
- $j \neq y$, $i \neq x$ and (i, x) is a cycle of $\tau_y\tau_j$,(by observation 1)
- $j \neq y$ and $\tau_y\tau_j(i) = x$, (since every cycle contains only two elements)
- $j \neq y$ and $\tau_j(i) = \tau_y(x)$, (since $\tau_y\tau_j(i) = x \Leftrightarrow \tau_y(\tau_y\tau_j(i)) = \tau_y(x) \Leftrightarrow \tau_j(i) = \tau_y(x)$, by the properties of the group A)

Lastly, since $\tau_j(i) = \tau_i(j)$ and $\tau_x(y) = \tau_y(x)$, we conclude that:
$j \neq y$, and the points P_i and P_x are in the same block of the parallel class $C_{L_{jy}}$ is equivalent with:
$i \neq x$ and $\tau_i(j) = \tau_x(y)$, and therefore the two designs $\mathcal{D}$ and $\mathcal{D}_L$ are isomorphic. $\square$

5.8 More on alpha Arcs of the Designs constructed via the ADLS Method

Recall that in Theorem 4.6 we've seen that the design $\mathcal{D}$ constructed via the ADLS method can only have $2n$-arcs of cardinality $4n$ or $4n^2$-arcs of cardinality $8n^2+2n$. In the latter case, we've shown that the $4n^2$-arcs of cardinality $8n^2+2n$ are all complements of the blocks of $\mathcal{D}$, thus $\mathcal{D}$ has $16n^2$ $4n^2$-arcs of cardinality $8n^2+2n$.

In this section we will show that in the case where $n=1$, the $2-(16,6,2)$ design $\mathcal{D}$ arising via the ADLS method, has a total of 60 2-arcs of cardinality 4.

Let Π be a 3-$(4n, 2n, n-1)$ affine design with parallel classes C_i where $i \in S\backslash\{0\}$ and $S=\{0,1,\ldots,4n-1\}$ and let L be the Latin square on the alphabet $\{0,1,\ldots,4n-1\}$ having zero diagonal. If $\mathcal{D}$ is the $2-(16n^2, 8n^2-2n, 4n^2-2n)$ design from the ADLS method, we've shown in Theorem 5.19 that for n=1, the $2-(16,6,2)$ design $\mathcal{D}$ is isomorphic to the $2-(16,6,2)$ design $\mathcal{D}_L$ constructed via the $L_2(4)$ association scheme.

Recall that the points of D_L are (i,j) with $i,j \in \{0,1,2,3\}$, and the blocks of D_L are $[x,y]$ with $x,y \in \{0,1,2,3\}$.
Incidence is defined by:
$(i,j) \in [x,y]$ if, and only if, $i=x$ and $j \neq y$ or $i \neq x$ and $j=y$.

Theorem 5.20. *If $\mathcal{D}$ is the $2-\left(16n^2, 8n^2-2n, 4n^2-2n\right)$ design from the ADLS method, then in the case where n=1, the $2-(16,6,2)$ design $\mathcal{D}$ has a total of 60 2-arcs of cardinality 4.*

Proof. The proof of Theorem 5.20 lies in the following lemmas.

Lemma 5.21. *The set Y, containing 4-element sets of the form $\{(i,j),(i,l),(k,j),(k,l)\}$ where $\{i,k\}$ and $\{j,l\}$ are arbitrary 2-element subsets of $\{0,1,2,3\}$, contains 36 2-arcs for the design constructed via the ADLS method.*

Proof. Since $\{i,k\}$ and $\{j,l\}$ are arbitrary 2-element subsets of $\{0,1,2,3\}$, and there are 6 ways of choosing a set of 2 elements out of a set of 4 elements, we get that the set Y contains $6 \cdot 6 = 36$ 4-element sets of the form $\{(i,j),(i,l),(k,j),(k,l)\}$.

Now, since the $2-(16,6,2)$ design $\mathcal{D}$ from the ADLS method is isomorphic to the $2-(16,6,2)$ design $\mathcal{D}_L$ arising from the $L_2(4)$ association scheme, we will prove the lemma for the design $\mathcal{D}_L$.
Remember that the points of $\mathcal{D}_L$ are (i,j) with $i,j \in \{0,1,2,3\}$.

Let Y_1=$\{(i,j),(i,l),(k,j),(k,l)\}$ be a 4-element subset of Y, with i,j,k,l fixed, and let $[x,y]$ be a block of $\mathcal{D}_L$. Then we have that:

1. if $x=i$ then $[x,y]=[i,y]$ contains the points (i,n) where $n\neq y$ and (m,y) where $m\neq i$, thus:

   - if $y\neq\{j,l\}\Rightarrow$ n can take the values of j or $l\Rightarrow[i,y]\cap Y_1=\{(i,j),(i,l)\}$, i.e $|[i,y]\cap Y_1|=2$.

   - if $y=j\Rightarrow n$ cannot be equal to $j\Rightarrow[i,y]\cap Y_1=\{(i,l),(k,j)\}$, for $k\neq i$, i.e $|[i,y]\cap Y_1|=2$.

   - if $y=l\Rightarrow$ n cannot be equal to $l\Rightarrow[i,y]\cap Y_1=\{(i,j),(k,l)\}$, for $k\neq i$, i.e $|[i,y]\cap Y_1|=2$.

   The same argument holds in the case where $x=k$.

2. if $x\neq i$, then:

   - if $x\neq k$:
     since the block $[x,y]$ contains the points (x,n) where $n\neq y$ and (m,y) where $m\neq x$ and Y_1 contains the points $\{(i,j),(i,l),(k,j),(k,l)\}$ for fixed $i,j,k,l\in\{0,1,2,3\}$, we have that:

     i) if $y=j\Rightarrow[x,y]\cap Y_1=\{(i,j),(k,j)\}$, i.e. $|[x,y]\cap Y_1|=2$.
     ii) if $y=l\Rightarrow[x,y]\cap Y_1=\{(i,l),(k,l)\}$, i.e. $|[x,y]\cap Y_1|=2$.
     iii) if $y\neq\{j,l\}\Rightarrow[x,y]\cap Y_1=\emptyset$.

   - if $x=k$ we have the same case as 1) and thus $|[x,y]\cap Y_1|=2$.

Cases 1. and 2. show that every block $[x,y]$ of $\mathcal{D}_L$ meets Y_1 in 0 or 2 points and since the choice of Y_1 was arbitrary, we conclude that every 4-element subset of Y is a 2-arc for the design $\mathcal{D}_L$. □

Lemma 5.22. *If σ is an arbitrary element of the symmetric group S_4 then the set X containing 4-element sets of the form $\{(i,\sigma(i)) : i=0,1,2,3\}$ contains 24 2-arcs for the design $\mathcal{D}$ constructed via the ADLS method.*

Proof. As in Lemma 5.21, we will prove this lemma for the $2-(16,6,2)$ design $\mathcal{D}_L$ arising from the $L_2(4)$ association scheme.

Since σ is a permutation of the S_4 group, we can assume that X contains 4-element sets of the form $\{(a,b),(c,d),(e,f),(g,h)\}$, with a,b,c,$d \in \{0,1,2,3\}$, $a \neq c$, $c \neq e$,$e \neq g$, $g \neq a$ and $b \neq d$, $d \neq f$, $f \neq h$, $h \neq b$.

Let $X_1 = \{(a,b),(c,d),(e,f),(g,h)\}$ be a fixed 4-element subset of X.

1. Let $B = [a,b]$ be a block of $\mathcal{D}_L$. The incidence relation for the design $\mathcal{D}_L$ tells us that the point (a,b) does not belong in B. Thus B contains points of the form (a,n) where $n \neq b$. Since X_1 contains exactly one point with x-coordinate equal to a, and the point (a,b) belongs in X_1, we conclude that X_1 does not contain any points of the form (a,n) where $n \neq b$. The block B also contains points of the form (n,b) where $n \neq a$. But for a point (n,b) to belong in X_1, n must be equal to a. Hence, $B \cap X_1 = \emptyset$.
   The same argument holds for blocks $[c,d]$, $[e,f]$ and $[g,h]$.

2. Let $B = [a,p]$ be a block of $\mathcal{D}_L$, with $p \neq b$. This block contains points of the form (a,m) where $m \neq p$. Since $p \neq b$, B contains the point (a,b) which is also contained in X_1.
   In addition, block B contains points of the form (n,p) where $n \neq a$. Since X_1 contains 4 points with different coordinates, it will contain a point of the form (q,p) for some $q \in \{0,1,2,3\}$, where $q \neq a$. Hence $|B \cap X_1| = 2$.

So the set X contains 4-element sets that meet every block of $\mathcal{D}_L$ in 0 or 2 points, and thus the 24 4-elements sets of X are 2-arcs for the design $\mathcal{D}_L$. □

Lemma 5.23. *The design $\mathcal{D}$ constructed via the ADLS method has a total of 60 2-arcs of cardinality 4.*

Proof. As in Lemma 5.21 and lemma 5.22, we will prove this lemma for the design $\mathcal{D}_L$.
Since the points of $\mathcal{D}_L$ are (i,j) where $i,j \in \{0,1,2,3\}$, all 4-element subsets of the point set of $\mathcal{D}_L$ can be distinguished into the following categories:

1. 4-element subsets containing points with different coordinates

2. 4-element subsets containing points where two of them have the same first or second coordinate

3. 4-element subsets containing points where 3 of them have the same first or second coordinate

4. 4-element subsets containing points where all points have the same first or second coordinate.

Clearly, categories 3 and 4 contain 4-element subsets that are not 2-arcs. For example, in category 3, if the subset is $\{(i,a),(i,b),(i,c),(j,d)\}$ where $i,a,b,c,d,j \in \{0,1,2,3\}$ then the block $[i,k]$, for some $k \in \{0,1,2,3\}$ meets this set in more than 2 points and hence cannot be a 2-arc.
So a 2-arc of cardinality 4 will fall either in category 1 or 2.

We have already shown that the 4-element subsets of the set X mentioned in Lemma 5.22, containing points with different coordinates, i.e.subsets of category 1, are 2-arcs for $\mathcal{D}_L$.
So it is enough to show that any 4-element subset with two points having the same first or second coordinate, i.e 4-element subsets of category 2, must belong in the set Y mentioned in Lemma 5.21.

Let Y_2 be a 4-element subset of category 2. Then two of its points are of the form (i,m) and (i,n) for some $i,m,n \in \{0,1,2,3\}$. Then, the other two points of Y_2 will be either:

i) (j,a) and (j,b) where $j \neq i$, $a \neq m$ and $n \neq b$, or

ii) (k,n) and (k,l) where $l \neq n,m$ and $k \neq i$.

If we have case i), then Y_2 falls in category 1.
But if we have case ii), then the block $[i,y]$ for some y depending on the choice of i, will only meet $Y_2 = \{(i,m),(i,n),(k,n),(k,l)\}$ in point (i,n) and thus Y_2 cannot be a 2-arc.
So, for Y_2 to be a 2-arc, l must be equal to m and thus Y_2 must belong in Y.

□

□

Theorem 5.24. *If $\mathcal{D}'$ is the design with point set the points of the design $\mathcal{D}$ constructed by the ADLS method,in the case where $n=1$, and block set the 60 2-arcs of $\mathcal{D}$, then $\mathcal{D}'$ is a resolvable $2-(16,4,3)$ design.*

Proof. Obviously, the design $\mathcal{D}'$ has 16 points and every block of $\mathcal{D}'$ contains 4 points.

We will now show that any point of $\mathcal{D}'$ is on 15 of its blocks.
We've shown that the 2-arcs of the design $\mathcal{D}$ constructed by the ADLS method are either of the form $\{(i,j),(i,l),(k,j),(k,l)\}$ where $\{i,k\}$ and $\{j,l\}$ are arbitrary 2-element subsets of $\{0,1,2,3\}$ (i.e belonging in the set Y), or of the form $\{(a,b),(c,d),(e,f),(g,h)\}$, with a,b,c,d $\in \{0,1,2,3\}$, $a \neq c$, $c \neq e$,$e \neq g$, $g \neq a$ and $b \neq d$, $d \neq f$, $f \neq h$, $h \neq b$ (i.e belonging in the set X).
Let (i,j) be a point of $\mathcal{D}'$. An element of the set X contains (i,j) if the other 3 of its points have their x-coordinate different than i. There are $3 \times 2 = 6$ such choices and thus every point (i,j) is on 6 2-arcs belonging in X.

Now, an element of Y contains the point (i,j) if $l \neq j$ and $k \neq i$. There are 3 choices for l and 3 choices for k, thus a point (i,j) is on $3 \times 3 = 9$ 2-arcs belonging in Y.
Hence, every point of the design $\mathcal{D}'$ is on 15 of its blocks.

We will now show that any pair of points of $\mathcal{D}'$ is on 3 of its blocks.

Any pair of points of $\mathcal{D}'$ will either have:

1. the same x-coordinate, or
2. the same y-coordinate, or
3. different x and y coordinates.

In case 1, if we have the points (i, j) and (i, l), then these two points belong only in elements of the set $Y = \{(i, j), (i, l), (k, j), (k, l)\}$ and there are 3 choices for the k.
A similar argument works for case 2, i.e for two points of the form (i, j) and (k, j).

In case 3, if we have two points (i, j) and (k, l), then this pair belongs only in one element of Y. In addition an element of the set X that contains the points (i, j) and (k, l), by definition, will contain two other points (m, n) and (p, q) for which $m \neq i, k$ and $p \neq i, j, m$. There are 2 choices for m and 1 choice for o. In the same way, n must not be equal to j and l and q must not be equal to j, l and q. Again, there are 2 choices for n and 1 choice for q.
Therefore , a pair of points (i, j), (k, l) of $\mathcal{D}'$ is on 1 element of Y and $2 \times 1 = 2$ elements of X and thus any pair of points (i, j) and (k, l) of $\mathcal{D}'$ if on 3 blocks.

Last, since we've seen that any point (i, j) of $\mathcal{D}'$ is on 6 elements of X and 9 elements of Y, we conclude that $\mathcal{D}'$ can be partitioned into 15 parallel classes, six of them containing 4 distinct elements of X each, and 9 of them containing 4 elements of Y each. Thus, $\mathcal{D}'$ is resolvable. □

Note. As we've shown, in Theorem 5.20 the $2-(16, 6, 2)$ Menon design constructed by the ADLS method has a total of 60 2-arcs. Computer search showed that the other two non-isomorphic $2-(16, 6, 2)$ Menon designs have a total of 28 and 12 2-arcs respectively.
We've also shown that the $2-(16, 6, 2)$ Menon design constructed by the ADLS method is self dual. This is also true for the other two $2-(16, 6, 2)$ Menon designs.

Note. Sharad S.Sane and Natwar N. Roghelia, in their paper *Classification of (16,6,2) designs by Ovals*, [25],prove the same result using "good triples".

Definition 5.25. Given a $t-(v, k, f)$ design $\mathcal{D}$, the m-th multiple $m\mathcal{D}$ of $\mathcal{D}$ is obtained by repeating each of the blocks of $\mathcal{D}$ m times.

Obviously, mD is $t-(v, k, mf)$ design.

Note. If the design $\mathcal{D}'$ is the multiple of 3 affine planes of order 4, i.e. each line of the affine plane is counted 3 times as a block, then any two blocks in $\mathcal{D}'$ would meet in either 1 or 4 points. In the latter case the blocks are multiples of the same line. Since, we've shown by computer search that any two blocks of $\mathcal{D}'$ meet in 0, 1 or 2 points, we conclude that $\mathcal{D}'$ is not a multiple of the affine plane of order 4. However we have found a subdesign of $\mathcal{D}'$ that is an affine plane of order 4, i.e a $2-(16, 4, 1)$ design. We give this subdesign in Appendix C.

Appendix A: An example of a 36x36 RHM

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% given a Latin square L of order 6, create a Regular 6x6 HM indexed by the 6^2 cells
% of the Latin square and whose entries [(i,j),(x,y)] are equal to 1 if.f i=x and j~=y
% or i~=x and j=y or Lij=Lxy
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

L=[0 1 2 3 4 5;
   1 0 3 2 5 4;
   2 3 4 5 0 1;
   3 2 5 4 1 0;
   4 5 0 1 2 3;
   5 4 1 0 3 2];

for k=1:36,
    for l=1:36,
        H(k,l)=-1;
    end
end
%disp(H)

counter_row=1; for i=1:6,
    for j=1:6,
        counter_column=1;
        for x=1:6,
            for y=1:6,
if(i==x)&&(j~=y) ||(i~=x)&&(j==y)||((i~=x)&& (j~=y)&&
(L(i,j)==L(x,y)))
                         H(counter_row, counter_column)=1;
                        counter_column=counter_column+1;
else
             counter_column=counter_column+1;
end


           end %end y
        end %end x
```

```
          counter_row=counter_row+1;
    end %end j

end %end i


disp('this is the hadamard matrix')
 disp(H)


%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

this is the hadamard matrix

  Columns 1 through 18
    -1   1   1   1   1   1   1   1  -1  -1  -1  -1   1  -1  -1  -1   1  -1
     1  -1   1   1   1   1   1   1  -1  -1  -1  -1  -1   1  -1  -1  -1   1
     1   1  -1   1   1   1  -1  -1   1   1  -1  -1   1  -1   1  -1  -1  -1
     1   1   1  -1   1   1  -1  -1   1   1  -1  -1  -1   1  -1   1  -1  -1
     1   1   1   1  -1   1  -1  -1  -1  -1   1   1  -1  -1   1  -1   1  -1
     1   1   1   1   1  -1  -1  -1  -1  -1   1   1  -1  -1  -1   1  -1   1
     1   1  -1  -1  -1  -1  -1   1   1   1   1   1   1  -1  -1  -1  -1   1
     1   1  -1  -1  -1  -1   1  -1   1   1   1   1  -1   1  -1  -1   1  -1
    -1  -1   1   1  -1  -1   1   1  -1   1   1   1  -1   1   1  -1  -1  -1
    -1  -1   1   1  -1  -1   1   1   1  -1   1   1   1  -1  -1   1  -1  -1
    -1  -1  -1  -1   1   1   1   1   1   1  -1   1  -1  -1  -1   1   1  -1
    -1  -1  -1  -1   1   1   1   1   1   1   1  -1  -1  -1   1  -1  -1   1
     1  -1   1  -1  -1  -1   1  -1  -1   1  -1  -1  -1   1   1   1   1   1
    -1   1  -1   1  -1  -1  -1   1   1  -1  -1  -1   1  -1   1   1   1   1
    -1  -1   1  -1   1  -1  -1  -1   1  -1  -1   1   1   1  -1   1   1   1
    -1  -1  -1   1  -1   1  -1  -1  -1   1   1  -1   1   1   1  -1   1   1
     1  -1  -1  -1   1  -1  -1   1  -1  -1   1  -1   1   1   1   1  -1   1
    -1   1  -1  -1  -1   1   1  -1  -1  -1  -1   1   1   1   1   1   1  -1
     1  -1  -1   1  -1  -1   1  -1   1  -1  -1  -1   1   1  -1  -1  -1  -1
    -1   1   1  -1  -1  -1  -1   1  -1   1  -1  -1   1   1  -1  -1  -1  -1
    -1  -1   1  -1  -1   1  -1  -1   1  -1   1  -1  -1  -1   1   1  -1  -1
    -1  -1  -1   1   1  -1  -1  -1  -1   1  -1   1  -1  -1   1   1  -1  -1
    -1   1  -1  -1   1  -1   1  -1  -1  -1   1  -1  -1  -1  -1  -1   1   1
     1  -1  -1  -1  -1   1  -1   1  -1  -1  -1   1  -1  -1  -1  -1   1   1
     1  -1  -1  -1   1  -1   1  -1  -1  -1  -1   1   1  -1   1  -1  -1  -1
    -1   1  -1  -1  -1   1  -1   1  -1  -1   1  -1  -1   1  -1   1  -1  -1
     1  -1   1  -1  -1  -1  -1   1   1  -1  -1  -1  -1  -1   1  -1   1  -1
    -1   1  -1   1  -1  -1   1  -1  -1   1  -1  -1  -1  -1  -1   1  -1   1
    -1  -1   1  -1   1  -1  -1  -1  -1   1   1  -1   1  -1  -1  -1   1  -1
    -1  -1  -1   1  -1   1  -1  -1   1  -1  -1   1  -1   1  -1  -1  -1   1
     1  -1  -1  -1  -1   1   1  -1  -1  -1   1  -1   1  -1  -1   1  -1  -1
    -1   1  -1  -1   1  -1  -1   1  -1  -1  -1   1  -1   1   1  -1  -1  -1
    -1   1   1  -1  -1  -1   1  -1   1  -1  -1  -1  -1  -1   1  -1  -1   1
```

```
   1  -1  -1   1  -1  -1  -1   1  -1   1  -1  -1  -1  -1  -1   1   1  -1
  -1  -1  -1   1   1  -1  -1  -1   1  -1   1  -1  -1   1  -1  -1   1  -1
  -1  -1   1  -1  -1   1  -1  -1  -1   1  -1   1   1  -1  -1  -1  -1   1

Columns 19 through 36

   1  -1  -1  -1  -1   1   1  -1   1  -1  -1  -1   1  -1  -1   1  -1  -1
  -1   1  -1  -1   1  -1  -1   1  -1   1  -1  -1  -1   1   1  -1  -1  -1
  -1   1   1  -1  -1  -1  -1  -1   1  -1   1  -1  -1  -1   1  -1  -1   1
   1  -1  -1   1  -1  -1  -1  -1  -1   1  -1   1  -1  -1  -1   1   1  -1
  -1  -1  -1   1   1  -1   1  -1  -1  -1   1  -1  -1   1  -1  -1   1  -1
  -1  -1   1  -1  -1   1  -1   1  -1  -1  -1   1   1  -1  -1  -1  -1   1
   1  -1  -1  -1   1  -1   1  -1  -1   1  -1  -1   1  -1   1  -1  -1  -1
  -1   1  -1  -1  -1   1  -1   1   1  -1  -1  -1  -1   1  -1   1  -1  -1
   1  -1   1  -1  -1  -1  -1  -1   1  -1  -1   1  -1  -1   1  -1   1  -1
  -1   1  -1   1  -1  -1  -1  -1  -1   1   1  -1  -1  -1  -1   1  -1   1
  -1  -1   1  -1   1  -1  -1   1  -1  -1   1  -1   1  -1  -1  -1   1  -1
  -1  -1  -1   1  -1   1   1  -1  -1  -1  -1   1  -1   1  -1  -1  -1   1
   1   1  -1  -1  -1  -1   1  -1  -1  -1   1  -1   1  -1  -1  -1  -1   1
   1   1  -1  -1  -1  -1  -1   1  -1  -1  -1   1  -1   1  -1  -1   1  -1
  -1  -1   1   1  -1  -1   1  -1   1  -1  -1  -1  -1   1   1  -1  -1  -1
  -1  -1   1   1  -1  -1  -1   1  -1   1  -1  -1   1  -1  -1   1  -1  -1
  -1  -1  -1  -1   1   1  -1  -1   1  -1   1  -1  -1  -1  -1   1   1  -1
  -1  -1  -1  -1   1   1  -1  -1  -1   1  -1   1  -1  -1   1  -1  -1   1
  -1   1   1   1   1   1   1  -1  -1  -1  -1   1   1  -1  -1  -1   1  -1
   1  -1   1   1   1   1  -1   1  -1  -1   1  -1  -1   1  -1  -1  -1   1
   1   1  -1   1   1   1  -1   1   1  -1  -1  -1   1  -1   1  -1  -1  -1
   1   1   1  -1   1   1   1  -1  -1   1  -1  -1  -1   1  -1   1  -1  -1
   1   1   1   1  -1   1  -1  -1  -1   1   1  -1  -1  -1   1  -1   1  -1
   1   1   1   1   1  -1  -1  -1   1  -1  -1   1  -1  -1  -1   1  -1   1
   1  -1  -1   1  -1  -1  -1   1   1   1   1   1   1   1  -1  -1  -1  -1
  -1   1   1  -1  -1  -1   1  -1   1   1   1   1   1   1  -1  -1  -1  -1
  -1  -1   1  -1  -1   1   1   1  -1   1   1   1  -1  -1   1   1  -1  -1
  -1  -1  -1   1   1  -1   1   1   1  -1   1   1  -1  -1   1   1  -1  -1
  -1   1  -1  -1   1  -1   1   1   1   1  -1   1  -1  -1  -1  -1   1   1
   1  -1  -1  -1  -1   1   1   1   1   1   1  -1  -1  -1  -1  -1   1   1
   1  -1   1  -1  -1  -1   1   1  -1  -1  -1  -1  -1   1   1   1   1   1
  -1   1  -1   1  -1  -1   1   1  -1  -1  -1  -1   1  -1   1   1   1   1
  -1  -1   1  -1   1  -1  -1  -1   1   1  -1  -1   1   1  -1   1   1   1
  -1  -1  -1   1  -1   1  -1  -1   1   1  -1  -1   1   1   1  -1   1   1
   1  -1  -1  -1   1  -1  -1  -1  -1  -1   1   1   1   1   1   1  -1   1
  -1   1  -1  -1  -1   1  -1  -1  -1  -1   1   1   1   1   1   1   1  -1
```

Appendix B: A program for finding polarities of a design

```
Let LA=[0 1 2 3 4 5 6 7;
        3 0 1 2 5 4 7 6;
        2 3 0 1 6 7 4 5;
        1 2 3 0 7 6 5 4;
        4 5 6 7 0 1 2 3;
        5 4 7 6 1 0 3 2;
        6 7 4 5 2 3 0 1;
        7 6 5 4 3 2 1 0];
```

be the Latin square used in the ADLS method, described in Appendix D, to obtain a 2-(64, 28, 12) Menon design. Using the following code in DreadnautB, we obtain all the automorphisms of the group. (Note: each design is associated with a bipartite graph whose vertices consist of the points and blocks of the design and two vertices are adjacent if and only if one is a block and the other is a point on that block)

```
> $=1
> n=128
> g
 1 : 65:

 65: 9     11     13     15     17     18     21     22     25     28     29     32
     33     34     35     36     41     43     46     48     49     50     55     56
     57     60     62     63;

66 : 10     12     14     16     17     18     21     22     26     27     30
     31     33     34     35     36     42     44     45     47     49     50     55
     56     58     59     61     64;

 67: 9     11     13     15     19     20     23     24     26     27     30
     31     33     34     35     36     41     43     46     48     51     52     53
     54     58     59     61     64;

68: 10     12     14     16     19     20     23     24     25     28     29
    32     33     34     35     36     42     44     45     47     51     52     53
    54     57     60     62     63;
```

69: 9 11 13 15 17 18 21 22 25 28 29
32 37 38 39 40 42 44 45 47 51 52
53 54 58 59 61 64;

70: 10 12 14 16 17 18 21 22 26 27 30
31 37 38 39 40 41 43 46 48 51 52
53 54 57 60 62 63;

71: 9 11 13 15 19 20 23 24 26 27 30
31 37 38 39 40 42 44 45 47 49 50
55 56 57 60 62 63;

72: 10 12 14 16 19 20 23 24 25 28 29
32 37 38 39 40 41 43 46 48 49 50 55
56 58 59 61 64;

73: 1 4 5 8 17 19 21 23 25 26 29
30 33 35 38 40 41 42 43 44 49 52 54
55 57 58 63 64;

74: 2 3 6 7 18 20 22 24 25 26 29
30 34 36 37 39 41 42 43 44 50 51
53 56 57 58 63 64;

75: 2 3 6 7 17 19 21 23 27 28 31
32 33 35 38 40 41 42 43 44 50 51
53 56 59 60 61 62;

76: 1 4 5 8 18 20 22 24 27 28 31
32 34 36 37 39 41 42 43 44 49 52
54 55 59 60 61 62;

77: 1 4 5 8 17 19 21 23 25 26 29
30 34 36 37 39 45 46 47 48 50 51
53 56 59 60 61 62;

78: 2 3 6 7 18 20 22 24 25 26 29
30 33 35 38 40 45 46 47 48 49 52
54 55 59 60 61 62;

79: 2 3 6 7 17 19 21 23 27 28 31
32 34 36 37 39 45 46 47 48 49 52
54 55 57 58 63 64;

80: 1 4 5 8 18 20 22 24 27 28 31 32
33 35 38 40 45 46 47 48 50 51 53 56 57
58 63 64;

```
81: 1     2     5     6     9     12    13    16    25    27    29 31
   33    34    39    40    41    44    46    47    49    50    51 52 57
   59    62    64;

82: 1     2     5     6     10    11    14    15    26    28    30 32
   33    34    39    40    42    43    45    48    49    50    51 52 58
   60    61    63;

83: 3     4     7     8     10    11    14    15    25    27    29 31
   35    36    37    38    42    43    45    48    49    50    51 52 57
   59    62    64;

 84: 3     4     7     8     9     12    13    16    26    28    30
    32    35    36    37    38    41    44    46    47    49    50
    51    52    58    60    61    63;

 85: 1     2     5     6     9     12    13    16    25    27    29
    31    35    36    37    38    42    43    45    48    53    54
    55    56    58    60    61    63;

86: 1     2     5     6     10    11    14    15    26    28    30 32
   35    36    37    38    41    44    46    47    53    54    55 56 57
   59    62    64;

87: 3     4     7     8     10    11    14    15    25    27    29 31
   33    34    39    40    41    44    46    47    53    54    55 56 58
   60    61    63;

88: 3     4     7     8     9     12    13    16    26    28    30 32
   33    34    39    40    42    43    45    48    53    54    55 56 57
   59    62    64;

89: 1     3     5     7     9     10    13    14    17    20    21 24
   33    36    38    39    41    42    47    48    49    51    54 56 57
   58    59    60;

90: 2     4     6     8     9     10    13    14    18    19    22 23
   34    35    37    40    41    42    47    48    50    52    53 55 57
   58    59    60;

91: 1     3     5     7     11    12    15    16    18    19    22 23
   34    35    37    40    43    44    45    46    49    51    54 56 57
   58    59    60;

92: 2     4     6     8     11    12    15    16    17    20    21 24
   33    36    38    39    43    44    45    46    50    52    53 55 57
   58    59    60;
```

93: 1 3 5 7 9 10 13 14 17 20 21 24
34 35 37 40 43 44 45 46 50 52 53 55 61
62 63 64;

94: 2 4 6 8 9 10 13 14 18 19 22 23
33 36 38 39 43 44 45 46 49 51 54 56 61
62 63 64;

95: 1 3 5 7 11 12 15 16 18 19 22 23
33 36 38 39 41 42 47 48 50 52 53 55 61
62 63 64;

96: 2 4 6 8 11 12 15 16 17 20 21 24
34 35 37 40 41 42 47 48 49 51 54 56 61
62 63 64;

97: 1 2 3 4 9 11 14 16 17 18 23 24
25 28 30 31 41 43 45 47 49 50 53 54 57
60 61 64;

98: 1 2 3 4 10 12 13 15 17 18 23 24
26 27 29 32 42 44 46 48 49 50 53 54 58
59 62 63;

99: 1 2 3 4 9 11 14 16 19 20 21 22
26 27 29 32 41 43 45 47 51 52 55 56 58
59 62 63;

100: 1 2 3 4 10 12 13 15 19 20 21
22 25 28 30 31 42 44 46 48 51 52 55
56 57 60 61 64;

101: 5 6 7 8 10 12 13 15 19 20 21
22 26 27 29 32 41 43 45 47 49 50 53
54 57 60 61 64;

102: 5 6 7 8 9 11 14 16 19 20 21
22 25 28 30 31 42 44 46 48 49 50 53
54 58 59 62 63;

103: 5 6 7 8 10 12 13 15 17 18 23
24 25 28 30 31 41 43 45 47 51 52 55
56 58 59 62 63;

104: 5 6 7 8 9 11 14 16 17 18 23
24 26 27 29 32 42 44 46 48 51 52
55 56 57 60 61 64;

```
105: 1    3    6    8    9    10   11   12   17   20   22
     23   25   26   31   32   33   35   37   39   49   52
     53   56   57   58   61   62;

106: 2    4    5    7    9    10   11   12   18   19   21
     24   25   26   31   32   34   36   38   40   50   51
     54   55   57   58   61   62;

107: 1    3    6    8    9    10   11   12   18   19   21
     24   27   28   29   30   33   35   37   39   50   51
     54   55   59   60   63   64;

108: 2    4    5    7    9    10   11   12   17   20   22
     23   27   28   29   30   34   36   38   40   49   52
     53   56   59   60   63   64;

109: 2    4    5    7    13   14   15   16   18   19   21
     24   27   28   29   30   33   35   37   39   49   52
     53   56   57   58   61   62;

110:    1    3    6    8    13   14   15   16   17   20   22
     23   27   28   29   30   34   36   38   40   50   51   54
     55   57   58   61   62;

111:  2    4    5    7    13   14   15   16   17   20   22
     23   25   26   31   32   33   35   37   39   50   51   54
     55   59   60   63   64;

112: 1    3    6    8    13   14   15   16   18   19   21
     24   25   26   31   32   34   36   38   40   49   52   53
     56   59   60   63   64;

113: 1    2    7    8    9    12   14   15   17   18   19
     20   25   27   30   32   33   34   37   38   41   44   45
     48   57   59   61   63;

114:    1    2    7    8    10   11   13   16   17   18 19
     20   26   28   29   31   33   34   37   38   42   43 46 47
     58   60   62   64;

115: 3    4    5    6    10   11   13   16   17   18   19
     20   25   27   30   32   35   36   39   40   42   43   46
     47   57   59   61   63;

116: 3    4    5    6    9    12   14   15   17   18   19
     20   26   28   29   31   35   36   39   40   41   44   45
     48   58   60   62   64;
```

117: 3 4 5 6 10 11 13 16 21 22 23
24 26 28 29 31 33 34 37 38 41 44 45
48 57 59 61 63;

118: 3 4 5 6 9 12 14 15 21 22 23
24 25 27 30 32 33 34 37 38 42 43 46
47 58 60 62 64;

119: 1 2 7 8 9 12 14 15 21 22 23
24 26 28 29 31 35 36 39 40 42 43 46
47 57 59 61 63;

120: 1 2 7 8 10 11 13 16 21 22 23
24 25 27 30 32 35 36 39 40 41 44 45
48 58 60 62 64;

121: 1 4 6 7 9 10 15 16 17 19 22
24 25 26 27 28 33 36 37 40 41 42 45
46 49 51 53 55;

122: 2 3 5 8 9 10 15 16 18 20 21
23 25 26 27 28 34 35 38 39 41 42 45
46 50 52 54 56;

123: 2 3 5 8 11 12 13 14 17 19 22
24 25 26 27 28 34 35 38 39 43 44 47
48 49 51 53 55;

124: 1 4 6 7 11 12 13 14 18 20 21
23 25 26 27 28 33 36 37 40 43 44 47
48 50 52 54 56;

125: 2 3 5 8 11 12 13 14 18 20 21
23 29 30 31 32 33 36 37 40 41 42 45
46 49 51 53 55;

126: 1 4 6 7 11 12 13 14 17 19 22
24 29 30 31 32 34 35 38 39 41 42 45
46 50 52 54 56;

127: 1 4 6 7 9 10 15 16 18 20 21
23 29 30 31 32 34 35 38 39 43 44 47
48 49 51 53 55;

128: 2 3 5 8 9 10 15 16 17 19 22
24 29 30 31 32 33 36 37 40 43 44 47
48 50 52 54 56;

```
>  c x @
```

The automorphisms of the graph that we obtain after we run this program, are then used in the following piece of code, using GAP, to obtain the polarities of the design.

```
gap> SizeScreen([120,]); gap> collineation_group:=Group([
                     (33, 37)(34, 38)(35, 39)(36, 40)(41, 45)(42, 46)
                     (43, 47)(44, 48)(49, 53)(50, 54)(51, 55)
   (52, 56)(57, 61)(58, 62)(59, 63)(60, 64)(65, 69)(66, 70)(67, 71)(68, 72)(73, 77)
   (74, 78)(75, 79)(76, 80)(81, 85)(82, 86)(83, 87)(84, 88)(89, 93)(90, 94)(91, 95)
   (92, 96),
 (9, 10)(11, 12)(13, 14)(15, 16)(25, 26)(27, 28)(29, 30)(31, 32)(41, 42)(43, 44)(45, 46)
   (47, 48)(57, 58)(59, 60)(61, 62)(63, 64)(65, 66)(67, 68)(69, 70)(71, 72)(81, 82)
   (83, 84)(85, 86)(87, 88)(97, 98)(99, 100)(101, 102)(103, 104)(113, 114)(115, 116)
   (117, 118)(119, 120),
(9,13)(10,14)(11,15)(12,16)(25,29)(26,30)(27,31)(28,32)(33,34)(35,36)(37,38)
   (39, 40)(41, 46)(42, 45)(43, 48)(44, 47)(49, 50)(51, 52)(53, 54)(55, 56)(57, 62)
   (58, 61)(59, 64)(60, 63)(73, 77)(74, 78)(75, 79)(76, 80)(89, 93)(90, 94)(91, 95)
   (92, 96)(97, 98)(99, 100)(101, 102)(103, 104)(105, 110)(106, 109)(107, 112)
   (108, 111)(113, 114)(115, 116)(117, 118)(119, 120)(121, 126)(122, 125)(123, 128)
   (124, 127),
   (17, 21)(18, 22)(19, 23)(20, 24)(25, 29)(26, 30)(27, 31)(28, 32)
   (33, 35)(34 ,36)(37, 39)(38, 40)(41, 43)(42, 44)(45, 47)(46, 48)(49, 55)(50, 56)
   (51, 53)(52,54)(57, 63)(58, 64)(59, 61)(60, 62)(81, 85)(82, 86)(83, 87)(84, 88)
   (89, 93)(90,94)(91, 95)(92, 96)(97, 99)(98, 100)(101, 103)(102, 104)(105, 107)
   (106,108)(109, 111)(110, 112)(113, 119)(114, 120)(115, 117)(116, 118)(121, 127)
   (122,128)(123, 125)(124 ,126),
   (3, 4)(7, 8)(9, 25)(10, 26)(11, 28)(12, 27)(13, 29)(14, 30)(15, 32)(16, 31)(19, 20)
   (23, 24)(35, 36)(39, 40)(41, 57)(42, 58)(43, 60)(44, 59)(45, 61)(46, 62)(47, 64)
   (48, 63)(51, 52)(55, 56)(67, 68)(71, 72)(73, 89)(74, 90)(75, 92)(76, 91)(77, 93)
   (78, 94)(79, 96)(80, 95)(83, 84)(87, 88)(99, 100)(103, 104)(105, 121)(106, 122)
   (107, 124)(108, 123)(109, 125)(110, 126)(111, 128)(112, 127)(115, 116)(119, 120),
    (3, 7)(4, 8)(11, 15)(12, 16)(19, 23)(20, 24)(27, 31)(28, 32)(33, 53)(34, 54)(35, 51)
   (36, 52)(37, 49)(38, 50)(39, 55)(40, 56)(41, 61)(42, 62)(43, 59)(44, 60)(45, 57)
   (46, 58)(47, 63)(48, 64)(65, 69)(66, 70)(73 ,77)(74, 78)(81, 85)(82, 86)(89, 93)
   (90, 94)(97, 113)(98, 114)(99, 119)(100, 120)(101, 117)(102, 118)(103, 115)
   (104, 116)(105, 121)(106, 122)(107 ,127)(108, 128)(109, 125)(110, 126)(111, 123)
   (112, 124),
   (3, 19)(4, 20)(7, 23)(8, 24)(9, 25)(10 ,26)(11, 15)(12, 16)(13, 29)(14, 30)(27, 31)
   (28, 32)(33, 49)(34 ,50)(35, 36)(37, 53)(38, 54)(39 ,40)(41, 57)(42, 58)(43, 48)
   (44, 47)(45, 61)(46, 62)(51, 52)(55 ,56)(59, 64)(60, 63)(67, 83)(68, 84)(71, 87)
   (72, 88)(73, 89)(74, 90)(75, 79)(76 ,80)(77, 93)(78 ,94)(91, 95)(92, 96)(97, 113)
   (98, 114)(99, 100)(101, 117)(102, 118)(103, 104)(105, 121)(106, 122)(107, 112)
   (108, 111)(109, 125)(110, 126)(115, 116)(119, 120)(123, 128)(124, 127),
   (2, 6)(4, 8)(10, 14)(12, 16)(18, 22)(20, 24)(26, 30)(28, 32)(33, 41)(34 ,46)(35, 43)
   (36, 48)(37, 45)(38, 42)(39, 47)(40, 44)(49, 57)(50, 62)(51, 59)(52, 64)(53, 61)
   (54, 58)(55, 63)(56, 60)(66, 70)(68, 72)(74, 78)(76 ,80)(82, 86)(84, 88)(90, 94)
```

```
  (92, 96)(97, 105)(98, 110)(99, 107)(100 ,112)(101, 109)(102, 106)(103, 111)
  (104, 108)(113, 121)(114, 126)(115, 123)(116, 128)(117, 125)(118, 122)(119, 127)
  (120, 124),
  (1, 2)(5, 6)(9, 25)(10, 26)(11, 28)(12 ,27)(13, 29)(14 ,30)(15 ,32)(16 ,31)(17 ,18)
  (21 ,22)(33 ,34)(37 ,38)(41, 57)(42, 58)(43, 60)(44, 59)(45, 61)(46, 62)(47, 64)
  (48 ,63)(49 ,50)(53, 54)(67, 68)(71 ,72)(73 ,90)(74 ,89)(75, 91)(76 ,92)(77 ,94)
  (78, 93)(79, 95)(80 ,96)(83, 84)(87, 88)(99 ,100)(103 ,104)(105 ,122)(106, 121)
  (107, 123)(108, 124)(109, 126)(110 ,125)(111 ,127)(112, 128)(115, 116)(119, 120),
  (1, 3)(2, 4)(5 ,7)(6, 8)(9 ,11)(10 ,12)(13 ,15)(14 ,16)(17,19)(18,20)(21,23)(22,24)
  (25 ,27)(26, 28)(29, 31)(30, 32)(33, 35)(34 ,36)(37, 39)(38 ,40)(41 ,43)(42 ,44)
  (45 ,47)(46, 48)(49, 51)(50, 52)(53 ,55)(54 ,56)(57, 59)(58 ,60)(61 ,63)(62, 64)
  (65, 67)(66 ,68)(69 ,71)(70 ,72)(73 ,75)(74 ,76)(77 ,79)(78 ,80)(81 ,83)(82 ,84)
  (85 ,87)(86 ,88)(89 ,91)(90 ,92)(93 ,95)(94, 96)(97 ,99)(98 ,100)(101 ,103)
  (102, 104)(105 ,107)(106 ,108)(109 ,111)(110 ,112)(113 ,115)(114 ,116)(117, 119)
  (118 ,120)(121 ,123)(122 ,124)(125, 127)(126 ,128) ]);

6 orbits; grpsize=4096; 10 gens; 14195 nodes (11899 bad leaves);
maxlev=7 tctotal=22817; canupdates=7; cpu time = 1.66 seconds
>

gap> full_group:=Group([
 (33, 37)(34, 38)(35, 39)(36 ,40)(41, 45)(42, 46)(43 ,47)(44 ,48)(49, 53)(50,54)(51,55)
 (52, 56)(57, 61)(58, 62)(59, 63)(60, 64)(65 ,69)(66, 70)(67, 71)(68 ,72)(73 ,77)
 (74 ,78)(75 ,79)(76 ,80)(81, 85)(82 ,86)(83 ,87)(84 ,88)(89, 93)(90, 94)(91, 95)
 (92 ,96),
 (9 ,10)(11, 12)(13, 14)(15, 16)(25, 26)(27, 28)(29 ,30)(31 ,32)(41, 42)(43,44)(45,46)
 (47, 48)(57 ,58)(59 ,60)(61, 62)(63, 64)(65, 66)(67, 68)(69 ,70)(71 ,72)(81 ,82)
 (83 ,84)(85 ,86)(87, 88)(97, 98)(99 ,100)(101 ,102)(103, 104)(113, 114)(115 ,116)
 (117 ,118)(119, 120),
 (9 ,13)(10, 14)(11, 15)(12, 16)(25, 29)(26, 30)(27 ,31)(28, 32)(33,34)(35,36)(37,38)
 (39 ,40)(41 ,46)(42, 45)(43, 48)(44, 47)(49, 50)(51, 52)(53 ,54)(55 ,56)(57 ,62)
 (58 ,61)(59 ,64)(60 ,63)(73, 77)(74 ,78)(75, 79)(76, 80)(89, 93)(90 ,94)(91, 95)
 (92, 96)(97, 98)(99 ,100)(101 ,102)(103 ,104)(105 ,110)(106 ,109)(107 ,112)
 (108, 111)(113, 114)(115, 116)(117, 118)(119, 120)(121, 126)(122, 125)(123, 128)
 (124, 127),
 (17, 21)(18, 22)(19, 23)(20, 24)(25 ,29)(26, 30)(27 ,31)(28 ,32)(33,39)(34,40)(35,37)
 (36, 38)(41, 47)(42, 48)(43 ,45)(44, 46)(49, 51)(50, 52)(53, 55)(54, 56)(57 ,59)
 (58 ,60)(61 ,63)(62 ,64)(65 ,69)(66, 70)(67 ,71)(68 ,72)(73, 77)(74 ,78)(75, 79)
 (76, 80)(97, 99)(98, 100)(101, 103)(102 ,104)(105 ,107)(106 ,108)(109, 111)
 (110, 112)(113, 119)(114, 120)(115, 117)(116, 118)(121 ,127)(122, 128)(123, 125)
 (124 ,126),
 (3 ,4)(7, 8)(9, 25)(10 ,26)(11, 28)(12 ,27)(13 ,29)(14 ,30)(15 ,32)(16 ,31)(19 ,20)
 (23, 24)(35 ,36)(39, 40)(41, 57)(42 ,58)(43 ,60)(44 ,59)(45 ,61)(46 ,62)(47 ,64)
 (48 ,63)(51, 52)(55 ,56)(67 ,68)(71, 72)(73 ,89)(74 ,90)(75 ,92)(76, 91)(77, 93)
 (78, 94)(79, 96)(80, 95)(83, 84)(87, 88)(99 ,100)(103 ,104)(105 ,121)(106 ,122)
 (107, 124)(108, 123)(109, 125)(110, 126)(111 ,128)(112, 127)(115, 116)(119, 120),
 (3, 7)(4, 8)(11, 15)(12, 16)(19, 23)(20 ,24)(27, 31)(28 ,32)(33, 49)(34, 50)(35 ,55)
 (36, 56)(37, 53)(38, 54)(39, 51)(40, 52)(41, 57)(42, 58)(43 ,63)(44 ,64)(45 ,61)
```

```
 (46 ,62)(47 ,59)(48, 60)(67 ,71)(68 ,72)(75 ,79)(76 ,80)(83 ,87)(84 ,88)(91, 95)
 (92, 96)(97, 113)(98, 114)(99 ,119)(100 ,120)(101 ,117)(102,118)(103 ,115)
 (104, 116)(105, 121)(106, 122)(107, 127)(108, 128)(109, 125)(110, 126)(111 ,123)
 (112, 124),
 (3 ,19)(4, 20)(7 ,23)(8, 24)(9 ,25)(10 ,26)(11 ,15)(12, 16)(13, 29)(14, 30)(27 ,31)
 (28 ,32)(33, 49)(34 ,50)(35, 36)(37, 53)(38, 54)(39, 40)(41, 57)(42, 58)(43 ,48)
 (44 ,47)(45, 61)(46, 62)(51 ,52)(55, 56)(59, 64)(60 ,63)(67, 83)(68 ,84)(71, 87)
 (72 ,88)(73, 89)(74, 90)(75, 79)(76, 80)(77 ,93)(78, 94)(91 ,95)(92 ,96)(97, 113)
 (98 ,114)(99, 100)(101, 117)(102, 118)(103, 104)(105, 121)(106, 122)(107, 112)
 (108 ,111)(109 ,125)(110, 126)(115, 116)(119 ,120)(123, 128)(124, 127),
 (2, 6)(4, 8)(10, 14)(12 ,16)(18 ,22)(20 ,24)(26, 30)(28 ,32)(33 ,41)(34 ,46)(35 ,43)
 (36, 48)(37, 45)(38, 42)(39, 47)(40, 44)(49 ,57)(50, 62)(51, 59)(52, 64)(53, 61)
   (54 ,58)(55 ,63)(56 ,60)(66, 70)(68, 72)(74, 78)(76 ,80)(82 ,86)(84, 88)(90 ,94)
   (92, 96)(97, 105)(98, 110)(99, 107)(100 ,112)(101 ,109)(102, 106)(103, 111)
   (104, 108)(113, 121)(114, 126)(115, 123)(116, 128)(117, 125)(118, 122)(119, 127)
   (120, 124),
   (1, 2)(5, 6)(9 ,25)(10, 26)(11, 28)(12 ,27)(13, 29)(14 ,30)(15 ,32)(16 ,31)(17, 18)
   (21, 22)(33 ,34)(37 ,38)(41 ,57)(42 ,58)(43, 60)(44, 59)(45 ,61)(46 ,62)(47 ,64)
   (48 ,63)(49 ,50)(53 ,54)(67, 68)(71, 72)(73 ,90)(74 ,89)(75 ,91)(76 ,92)(77, 94)
   (78, 93)(79, 95)(80, 96)(83 ,84)(87, 88)(99 ,100)(103 ,104)(105 ,122)(106 ,121)
   (107, 123)(108, 124)(109, 126)(110, 125)(111, 127)(112 ,128)(115, 116)(119, 120),
   (1, 3)(2, 4)(5, 7)(6 ,8)(9, 15)(10, 16)(11 ,13)(12, 14)(17 ,23)(18,24)(19,21)(20,22)
   (25 ,27)(26 ,28)(29 ,31)(30, 32)(33, 34)(35 ,36)(37 ,38)(39 ,40)(41 ,46)(42 ,45)
   (43 ,48)(44 ,47)(49 ,54)(50 ,53)(51, 56)(52 ,55)(57 ,58)(59, 60)(61, 62)(63, 64)
   (65, 67)(66, 68)(69, 71)(70, 72)(73, 79)(74, 80)(75, 77)(76 ,78)(81 ,87)(82 ,88)
   (83 ,85)(84 ,86)(89, 91)(90, 92)(93, 95)(94 ,96)(97 ,98)(99 ,100)(101, 102)
   (103 ,104)(105 ,110)(106, 109)(107 ,112)(108, 111)(113, 118)(114 ,117)(115, 120)
   (116 ,119)(121, 122)(123, 124)(125, 126)(127, 128),
   (1, 73)(2, 74)(3 ,75)(4, 76)(5 ,77)(6, 78)(7, 79)(8 ,80)(9, 65)(10,66)(11,67)(12,68)
   (13, 69)(14, 70)(15, 71)(16, 72)(17, 89)(18 ,90)(19 ,91)(20, 92)(21, 93)(22 ,94)
   (23, 95)(24 ,96)(25, 81)(26, 82)(27 ,83)(28 ,84)(29, 85)(30 ,86)(31 ,87)(32, 88)
   (33, 105)(34, 106)(35, 107)(36 ,108)(37 ,109)(38 ,110)(39 ,111)(40, 112)(41 ,97)
   (42 ,98)(43 ,99)(44 ,100)(45, 101)(46, 102)(47, 103)(48, 104)(49 ,121)(50 ,122)
   (51, 123)(52, 124)(53, 125)(54, 126)(55, 127)(56, 128)(57, 113)(58, 114)(59, 115)
   (60 ,116)(61 ,117)(62, 118)(63, 119)(64 ,120) ]);

level 1:  1 cell; 3 orbits; 1 fixed; index 32/128 3 orbits;
grpsize=8192; 11 gens; 23073 nodes (19735 bad leaves); maxlev=7
tctotal=32987; canupdates=12; cpu time = 2.47 seconds
                 gap> Size(collineation_group);
4096 gap>  Size(full_group); 8192 gap>
IsSubset(full_group,collineation_group); true
p:=(1,73)(2,74)(3,75)(4, 76)(5 ,77)(6, 78)(7, 79)(8 ,80)(9, 65)(10,
66)(11, 67)(12, 68)
   (13, 69)(14, 70)(15, 71)(16, 72)(17, 89)(18 ,90)(19 ,91)(20, 92)(21, 93)(22 ,94)
   (23, 95)(24 ,96)(25, 81)(26, 82)(27 ,83)(28 ,84)(29, 85)(30 ,86)(31 ,87)(32, 88)
   (33, 105)(34, 106)(35, 107)(36 ,108)(37 ,109)(38 ,110)(39 ,111)(40, 112)(41 ,97)
   (42 ,98)(43 ,99)(44 ,100)(45, 101)(46, 102)(47, 103)(48, 104)(49 ,121)(50 ,122)
```

```
   (51, 123)(52, 124)(53, 125)(54, 126)(55, 127)(56, 128)(57, 113)(58, 114)(59, 115)
   (60 ,116)(61 ,117)(62, 118)(63, 119)(64 ,120);

n:=0; for g in collineation_group do x:=g*p; if  Order(x) = 2 then
Print(  \n ,x); n:=n+1; fi; od; Print( \nNumber of polarities =
,n,\n);

gap> Print("\nNumber of polarities = ",n,"\n"); Number of polarities
= 576 gap>
```

Some comments regarding the above algorithms:

Nauty does not tell us whether there are polarities or, if they exist, how many there are. It only tell as that there are no correlations (and hence no polarities, i.e correlations of order 2) if the collineation group(i.e the automorphism group of a symmetric design with lamda=1) is the full group of automorphisms of the graph. If it is smaller, there is some hope that there are correlations, but we have to search for one. If we find a correlation x, then every product g*x, where g is a collineation, is a correlation and every correlation has this form. Among these elements, the polarities are the elements of order 2.

Graph automorphisms obtained by Nauty which interchange the sets {1,...,64} and {65,...,128} correspond to correlations of the design, since by definition, an isomorphism of a design onto its dual is called a correlation.

Appendix C: A subdesign of the resolvable design $\mathcal{D}'$ that is the affine plane of order 4

Firstly we give the algorithm used in MATLAB2008, in order to produce the design arising from $L_2(4)$ association scheme and its 60 4-arcs which we then use as the block set of the design $\mathcal{D}'$.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%A PROGRAM TO PRODUCE THE L2(4) ASSOCIATION SCHEME DESIGN AND FIND ITS ALPHA ARCS
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

adjacency=zeros(16,16);

counter_row=1; for i=1:4,
    for j=1:4,
        counter_column=1;
        for k=1:4,
            for l=1:4,

                if ((i==k)&&(j~=l))||((i~=k)&&(j==l)), %xor
                    adjacency(counter_row,counter_column)=1;
                    counter_column=counter_column+1;
                else
                    counter_column=counter_column+1;

                end %end ifelse
            end
        end
        counter_row=counter_row+1;
    end

end disp('this is the adjacecny matrix of the L2(4) design')
disp(adjacency)

% Find blocks of adjacency and put them in matrix BlockAdj
%BlockAdj=zeros(16,6);
```

```
for i=1:16,
    BB=zeros(16,16);
    %disp('B');
    m=1;
    for j=1:16,   %check every row of A for 1's
        if adjacency(i,j)==1,
            BB(i,j)=j;  %if you find a 1, store the corresponding element in B
        end
    end


      % disp('These are the blocks of RHM')

   for k=1:16,
       if BB(i,k)~=0,
           BlockAdj(i,m)=BB(i,k); %in Block, put all the nonzero elements of B
           m=m+1;
       end
   end



end disp('These are the blocks of L2(4)') disp(BlockAdj)



%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%FIND ALL POSSIBLE COMBINATIONS TO USE THEM FOR FINDING 4-ARCS OF THE L2(4)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

  counter_column=1;
counter_row=1; for i=1:13,


    for j=(i+1):14,

        for k=(j+1):15,
            counter_column=1;
            for l=(k+1):16,

                comb(counter_row,counter_column)=i;
                comb(counter_row,counter_column+1)=j;
                comb(counter_row,counter_column+2)=k;
                comb(counter_row,counter_column+3)=l;
                counter_row=counter_row+1;

            end
```

```
        end
    end
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%FIND THE ARCS
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

   counter=0;
       disp('these are the alpha arcs:');
       for i=1:1820,
           counter_alpha=0;
           for j=1:16,
             d=length(intersect(comb(i,:),BlockAdj(j,:)));
                if (d==0 || d==2)
                counter_alpha=counter_alpha+1;
                end
           end
            if(counter_alpha==16)
                counter=counter+1;
                disp(comb(i,:));
            end
       end
       disp('found so many alpha arcs:');disp(counter);
```

The previous program gave that the arcs of the L2(4) association scheme design are:

1	2	5	6
1	2	9	10
1	2	13	14
1	3	5	7
1	3	9	11
1	3	13	15
1	4	5	8
1	4	9	12
1	4	13	16
2	3	6	7
2	3	10	11
2	3	14	15
2	4	6	8
2	4	10	12
2	4	14	16
3	4	7	8
3	4	11	12
3	4	15	16
5	6	9	10

5	6	13	14
5	7	9	11
5	7	13	15
5	8	9	12
5	8	13	16
6	7	10	11
6	7	14	15
6	8	10	12
6	8	14	16
7	8	11	12
7	8	15	16
9	10	13	14
9	11	13	15
9	12	13	16
10	11	14	15
10	12	14	16
11	12	15	16
1	6	11	16
1	6	12	15
1	7	10	16
1	7	12	14
1	8	10	15
1	8	11	14
2	5	11	16
2	5	12	15
2	7	9	16
2	7	12	13
2	8	9	15
2	8	11	13
3	5	10	16
3	5	12	14
3	6	9	16
3	6	12	13
3	8	9	14
3	8	10	13
4	5	10	15
4	5	11	14
4	6	9	15
4	6	11	13
4	7	9	14
4	7	10	13

Using the above 60 arcs of the L2(4) association scheme design as blocks of the design D, Choosing the following 20 blocks of D, give a subdesign of D which is the affine design of order 4, i.e having parameters 2-(16,4,1).

B1= {1,2,5,6} B2= {1,3,9,11} B3= {1,4,13,16} B4={2,3,14,15}

```
B5={3,4,7,8} B6= {5,7,13,15} B7= {5,8,9,12} B8= {6,7,10,11}
B9={6,8,14,16} B10= {9,10,13,14} B11= {11,12,15,16} B12= {2,4,10,12}
B13= {2,9,7,16} B14= {2,8,11,13} B15= {1,7,12,14} B16= {1,8,10,15}
B17= {3,5,10,16} B18= {3,6,12,13} B19= {4,5,11,14} B20= {4,6,9,15}.
```

Appendix D: Implementation of the ADLS method using MATLAB

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%THIS PROGRAM PRODUCES MENON DESIGNS OF ORDER 16 USING THE ADLS METHOD
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%


P=[0 1 2 3 4 5 6 7];   %labelling of the point set

DA=zeros(64,64);

LA=[0 1 2 3 4 5 6 7;   %latin square with zero diagonal
    1 0 6 4 7 3 5 2;
    2 7 0 6 3 4 1 5;
    3 2 7 0 5 6 4 1;
    4 6 1 5 0 7 2 3;
    5 4 3 1 2 0 7 6;
    6 3 5 7 1 2 0 4;
    7 5 4 2 6 1 3 0];


C=[ 0 1 4 5;                 %parallel classes of the affine design
    2 3 6 7;
    0 2 4 6;
    1 3 5 7;
    0 3 4 7;
    1 2 5 6;
    0 1 2 3;
    4 5 6 7;
    0 1 6 7;
    2 3 4 5;
    0 2 5 7;
    1 3 4 6;
    0 3 5 6;
    1 2 4 7];


counter_row=1;
```

```
%disp('counter_row')
%disp(counter_row)

for i=1:8,
    for j=1:8,
        counter_column=1;
        %disp('counter_column')
        %disp(counter_column)
        for x=1:8,
            for y=1:8,

                if y~=j


                    index=LA(j,y);

                    a=length(intersect(P(1,i), C((index*2)-1,:)));
                    b=length(intersect(P(1,x), C((index*2)-1,:)));
                    c=length(intersect(P(1,i), C(index*2,:)));
                    d=length(intersect(P(1,x), C(index*2,:)));

                    if xor((a~=0 && b~=0),(c~=0 && d~=0)),
                       DA(counter_row, counter_column)=1;

                       counter_column=counter_column+1;

                    else
                 counter_column=counter_column+1;
                    end %end second if else
                else
                   counter_column=counter_column+1;
                end % end first if else



            end %end y

        end %end x
         counter_row=counter_row +1;

    end % end j


end % end i
```

```
% Find blocks of D and put them in matrix Block
BlockA=zeros(64,28);

for i=1:64,
    BB=zeros(64,64);
    %disp('B');
    m=1;
    for j=1:64,   %check every row of A for 1's
        if DA(i,j)==1,
            BB(i,j)=j;%if you find a 1,store the corresponding element in B
        end
    end


      % disp('These are the blocks of RHM')

   for k=1:64,
       if BB(i,k)~=0,
           BlockA(i,m)=BB(i,k);%in Block, put all the nonzero elements of B
           m=m+1;
       end
   end
   %disp(BlockA(i,:)) %these are the blocks


end disp('These are the blocks of D') disp(BlockA)
```

Appendix E: Kronecker Product of Hadamard Matrices

Theorem 5.26. *For $i = 1, 2$, let $M^{(i)}$ be a HM of order $n_i \geq 2$ and let $M = M^{(1)} \otimes M^{(2)}$. If M is a RHM, then it does not have a special triple.*

Proof. Let $\rho_1, \ldots, \rho_{n_1}$ denote the rows of $M^{(1)}$ and let $\sigma_1, \ldots, \sigma_{n_2}$ denote the rows of $M^{(2)}$. Suppose that rows $(\rho_{r_1}, \sigma_{s_1})$, $(\rho_{r_2}, \sigma_{s_2})$ and $(\rho_{r_3}, \sigma_{s_3})$ of M form a special triple, where row $(\rho_{r_j}, \sigma_{s_j})$ is obtained from row ρ_{r_j} of $M^{(1)}$ and row σ_{s_j} of $M^{(2)}$. Let $S = \{1, \ldots, n_1\}$ and $T = \{1, \ldots, n_2\}$. For $j = 1, 2, 3$, let $A_j^{(0)}$ be the set of positions in which row ρ_{r_j} of $M^{(1)}$ has -1, let $B_j^{(0)}$ be the set of positions in which row σ_{s_j} of $M^{(2)}$ has -1, let $A_j^{(1)} = S - A_j$ and let $B_j^{(1)} = S - B_j$.

Let $X_{\alpha\beta\gamma} = A_1^{(\alpha)} \cap A_2^{(\beta)} \cap A_3^{(\gamma)}$ and let $Y_{\alpha\beta\gamma} = B_1^{(\alpha)} \cap B_2^{(\beta)} \cap B_3^{(\gamma)}$, for all $\alpha, \beta, \gamma \in \{0, 1\}$.

The set C_j of the positions of row $(\rho_{r_j}, \sigma_{s_j})$ of M in which -1 occurs is given by $C_j = A_j^{(0)} \times B_j^{(1)} \cup A_j^{(1)} \times B_j^{(0)}$. The set of positions in which both rows $(\rho_{r_i}, \sigma_{s_i})$ and $(\rho_{r_j}, \sigma_{s_j})$ of M have -1 is

$$\begin{aligned} C_i \cap C_j &= (A_i^{(0)} \cap A_j^{(0)}) \times (B_i^{(1)} \cap B_j^{(1)}) \cup (A_i^{(0)} \cap A_j^{(1)}) \times (B_i^{(1)} \cap B_j^{(0)}) \\ &\quad \cup (A_i^{(1)} \cap A_j^{(0)}) \times (B_i^{(0)} \cap B_j^{(1)}) \cup (A_i^{(1)} \cap A_j^{(1)}) \times (B_i^{(0)} \cap B_j^{(0)}). \end{aligned}$$

Thus,

$$\begin{aligned} C_1 \cap C_2 &= (X_{000} \cup X_{001}) \times (Y_{110} \cup Y_{111}) \cup (X_{010} \cup X_{011}) \times (Y_{100} \cup Y_{101}) \\ &\quad \cup (X_{100} \cup X_{101}) \times (Y_{010} \cup Y_{011}) \cup (X_{110} \cup X_{111}) \times (Y_{000} \cup Y_{001}) \\ &= X_{000} \times Y_{110} \cup X_{001} \times Y_{111} \cup X_{010} \times Y_{100} \cup X_{011} \times Y_{101} \\ &\quad \cup X_{100} \times Y_{010} \cup X_{101} \times Y_{011} \cup X_{110} \times Y_{000} \cup X_{111} \times Y_{001} \cup Z, \end{aligned}$$

where

$$\begin{aligned} Z &= X_{000} \times Y_{111} \cup X_{001} \times Y_{110} \cup X_{010} \times Y_{101} \cup X_{011} \times Y_{100} \\ &\quad \cup X_{100} \times Y_{011} \cup X_{101} \times Y_{010} \cup X_{110} \times Y_{001} \cup X_{111} \times Y_{000}. \end{aligned}$$

Similarly,

$$\begin{aligned} C_1 \cap C_3 &= X_{000} \times Y_{101} \cup X_{001} \times Y_{100} \cup X_{010} \times Y_{111} \cup X_{011} \times Y_{110} \\ &\quad \cup X_{100} \times Y_{001} \cup X_{101} \times Y_{000} \cup X_{110} \times Y_{011} \cup X_{111} \times Y_{010} \cup Z \end{aligned}$$

and

$$\begin{aligned} C_2 \cap C_3 &= X_{000} \times Y_{011} \cup X_{001} \times Y_{010} \cup X_{010} \times Y_{001} \cup X_{011} \times Y_{000} \\ &\quad \cup X_{100} \times Y_{111} \cup X_{101} \times Y_{110} \cup X_{110} \times Y_{101} \cup X_{111} \times Y_{100} \cup Z. \end{aligned}$$

Since we are assuming that rows $(\rho_{r_1}, \sigma_{s_1})$, $(\rho_{r_2}, \sigma_{s_2})$ and $(\rho_{r_3}, \sigma_{s_3})$ of M form a special triple,
$C_1 \cap C_2 = C_1 \cap C_3 = C_2 \cap C_3$. Hence,

$$\begin{aligned} \emptyset &= X_{000} \times Y_{110} = X_{001} \times Y_{111} = X_{010} \times Y_{100} = X_{011} \times Y_{101} \\ &= X_{000} \times Y_{101} = X_{001} \times Y_{100} = X_{010} \times Y_{111} = X_{011} \times Y_{110} \\ &= X_{000} \times Y_{011} = X_{001} \times Y_{010} = X_{010} \times Y_{001} = X_{011} \times Y_{000} \\ &= X_{100} \times Y_{010} = X_{101} \times Y_{011} = X_{110} \times Y_{000} = X_{111} \times Y_{001} \\ &= X_{100} \times Y_{001} = X_{101} \times Y_{000} = X_{110} \times Y_{011} = X_{111} \times Y_{010} \\ &= X_{100} \times Y_{111} = X_{101} \times Y_{110} = X_{110} \times Y_{101} = X_{111} \times Y_{100}. \end{aligned} \tag{5.1}$$

Write $x_{\alpha\beta\gamma} = |X_{\alpha\beta\gamma}|$ and $y_{\alpha\beta\gamma} = |Y_{\alpha\beta\gamma}|$, for all $\alpha, \beta, \gamma \in \{0, 1\}$. Then, as $|A_1^{(0)}| + |A_1^{(1)}| = n_1$ and $|B_1^{(0)}| + |B_1^{(1)}| = n_2$,

$$\begin{aligned} n_1 &= x_{000} + x_{001} + x_{010} + x_{011} + x_{100} + x_{101} + x_{110} + x_{111} && (5.2) \\ n_2 &= y_{000} + y_{001} + y_{010} + y_{011} + y_{100} + y_{101} + y_{110} + y_{111} && (5.3) \end{aligned}$$

Since M is a RHM, there is a positive integer u, such that

$$n_1 n_2 = 4u^2 \tag{5.4}$$

From the equations in (5.1) we get

$$\begin{aligned} 0 = x_{000}y_{110} &= x_{001}y_{111} = x_{010}y_{100} = x_{011}y_{101} && (5.5) \\ = x_{000}y_{101} &= x_{001}y_{100} = x_{010}y_{111} = x_{011}y_{110} \\ = x_{000}y_{011} &= x_{001}y_{010} = x_{010}y_{001} = x_{011}y_{000} \\ = x_{100}y_{010} &= x_{101}y_{011} = x_{110}y_{000} = x_{111}y_{001} \\ = x_{100}y_{001} &= x_{101}y_{000} = x_{110}y_{011} = x_{111}y_{010} \\ = x_{100}y_{111} &= x_{101}y_{110} = x_{110}y_{101} = x_{111}y_{100}. \end{aligned}$$

For $i, j = 1, 2, 3$ with $i \neq j$, we have $|C_j| = 2u^2 - u$ and $|C_i \cap C_j| = u^2 - u$. Hence,

$$\begin{aligned} 2u^2 - u &= (x_{000} + x_{001} + x_{010} + x_{011})(y_{100} + y_{101} + y_{110} + y_{111}) && (5.6) \\ &\quad + (x_{100} + x_{101} + x_{110} + x_{111})(y_{000} + y_{001} + y_{010} + y_{011}) \\ &= (x_{000} + x_{001} + x_{100} + x_{101})(y_{010} + y_{011} + y_{110} + y_{111}) && (5.7) \\ &\quad + (x_{010} + x_{011} + x_{110} + x_{111})(y_{000} + y_{001} + y_{100} + y_{101}) \\ &= (x_{000} + x_{010} + x_{100} + x_{110})(y_{001} + y_{011} + y_{101} + y_{111}) && (5.8) \\ &\quad + (x_{001} + x_{011} + x_{101} + x_{111})(y_{000} + y_{010} + y_{100} + y_{110}) \\ u^2 - u &= x_{000}y_{111} + x_{001}y_{110} + x_{010}y_{101} + x_{011}y_{100} && (5.9) \\ &\quad + x_{100}y_{011} + x_{101}y_{010} + x_{110}y_{001} + x_{111}y_{000} \end{aligned}$$

Suppose first that all but one of the $x_{\alpha\beta\gamma}$ are 0. If $x_{\alpha'\beta'\gamma'}$ is the single one that is non-zero, then $A_1^{(\alpha')} \cap A_2^{(\beta')} \cap A_3^{(\gamma')} = S$. Since $A_j^{\alpha} \subseteq S$ for all $j \in \{1, 2, 3\}$ and $\alpha \in \{0, 1\}$, we get $A_1^{(\alpha')} = A_2^{(\beta')} = A_3^{(\gamma')} = S$. If any two of α', β' and γ' differ, then $M^{(1)}$ has a row with all entries equal to 1 and a row with all entries equal to -1. Since $M^{(1)}$ is a HM, this is impossible.

Hence, $\alpha' = \beta' = \gamma'$. If $\alpha' = \beta' = \gamma' = 0$, that is $n_1 = x_{000} > 0$ and all other x's are zero, then $y_{011} = y_{101} = y_{110} = 0$ from equations (5.5) and $y_{000} + y_{001} + y_{010} + y_{100} + y_{111} = 4u^2/n_1$ from (5.3) and (5.4).
Since row r_1 of $M^{(1)}$ has all entries equal to -1, any other row $r \neq r_1$ has exactly $n_1/2$ entries equal to -1 and $n_1/2$ entries equal to 1. Hence, the number of entries equal to -1 on row (r, s_1) of M is $(n_1/2)(y_{100} + y_{111}) + (n_1/2)(y_{000} + y_{001} + y_{010}) = 2u^2$, since row s_1 of $M^{(2)}$ has $y_{100} + y_{111}$ entries equal to 1 and $y_{000} + y_{001} + y_{010}$ entries equal to -1. This contradicts the fact that M is Regular.

A similar argument applies if x_{111} is the only nonzero $x_{\alpha\beta\gamma}$, or if y_{000} or y_{111} is the only nonzero $y_{\alpha\beta\gamma}$.

Now suppose that at least two of the $x_{\alpha\beta\gamma}$ are nonzero and at least two of the $y_{\alpha\beta\gamma}$ are nonzero. Then some pair of r_1, r_2 and r_3 differ. We may suppose that $r_1 \neq r_2$. Since $M^{(1)}$ is a HM, the inner product of ρ_{r_1} and ρ_{r_2} is 0. Thus, $|A_1^{(0)} \cap A_2^{(0)}| + |A_1^{(1)} \cap A_2^{(1)}| =$

$|A_1^{(0)} \cap A_2^{(1)}| + |A_1^{(1)} \cap A_2^{(0)}|$. So,

$$x_{000} + x_{001} + x_{110} + x_{111} = x_{010} + x_{011} + x_{100} + x_{101} > 0. \tag{5.10}$$

Similarly, some pair of s_1, s_2 and s_3 differ. We may suppose that either $s_1 \neq s_2$ or $s_1 \neq s_3$ and $y_{000} + y_{001} + y_{110} + y_{111} = y_{010} + y_{011} + y_{100} + y_{101} > 0$ or $y_{000} + y_{010} + y_{101} + y_{111} = y_{001} + y_{011} + y_{100} + y_{110} > 0$, respectively.

If $x_{000} > 0$ and $x_{001} > 0$, then $y_{011} = y_{101} = y_{110} = y_{111} = y_{100} = y_{010} = 0$ from (5.5). So $y_{000} > 0$ and $y_{001} > 0$, and hence $x_{011} = x_{101} = x_{110} = x_{111} = x_{100} = x_{010} = 0$ from (5.5). From (5.6), we get $2u^2 - u = 0$. This is impossible.

If $x_{000} > 0$ and $x_{010} > 0$, then $y_{011} = y_{101} = y_{110} = y_{001} = y_{111} = y_{001} = 0$. So $y_{000} > 0$ and $y_{010} > 0$, and hence $x_{011} = x_{101} = x_{110} = x_{001} = x_{111} = x_{001} = 0$. From (5.6), we get $2u^2 - u = 0$. This is impossible.

If $x_{000} > 0$ and $x_{100} > 0$, then $y_{011} = y_{101} = y_{110} = y_{001} = y_{010} = y_{111} = 0$. So $y_{000} > 0$ and $y_{100} > 0$, and hence $x_{011} = x_{101} = x_{110} = x_{001} = x_{010} = x_{111} = 0$. From (5.7), we get $2u^2 - u = 0$. This is impossible.

If $x_{001} > 0$ and $x_{011} > 0$, then $y_{111} = y_{100} = y_{010} = y_{101} = y_{110} = y_{000} = 0$. So $y_{001} > 0$ and $y_{011} > 0$, and hence $x_{111} = x_{100} = x_{010} = x_{101} = x_{110} = x_{000} = 0$. From (5.6), we get $2u^2 - u = 0$. This is impossible.

If $x_{001} > 0$ and $x_{101} > 0$, then $y_{111} = y_{100} = y_{010} = y_{011} = y_{000} = y_{110} = 0$. So $y_{001} > 0$ and $y_{101} > 0$, and hence $x_{111} = x_{100} = x_{010} = x_{011} = x_{000} = x_{110} = 0$. From (5.7), we get $2u^2 - u = 0$. This is impossible.

If $x_{010} > 0$ and $x_{011} > 0$, then $y_{100} = y_{111} = y_{001} = y_{101} = y_{110} = y_{000} = 0$. So $y_{010} > 0$ and $y_{011} > 0$, and hence $x_{100} = x_{111} = x_{001} = x_{101} = x_{110} = x_{000} = 0$. From (5.6), we get $2u^2 - u = 0$. This is impossible.

If $x_{110} > 0$ and $x_{010} > 0$, then $y_{100} = y_{111} = y_{001} = y_{000} = y_{011} = y_{101} = 0$. So $y_{110} > 0$ and $y_{010} > 0$, and hence $x_{100} = x_{111} = x_{001} = x_{000} = x_{011} = x_{101} = 0$. From (5.7), we get $2u^2 - u = 0$. This is impossible.

If $x_{101} > 0$ and $x_{100} > 0$, then $y_{011} = y_{000} = y_{110} = y_{010} = y_{001} = y_{111} = 0$. So $y_{101} > 0$ and $y_{100} > 0$, and hence $x_{011} = x_{000} = x_{110} = x_{010} = x_{001} = x_{111} = 0$. From (5.6), we get $2u^2 - u = 0$. This is impossible.

If $x_{110} > 0$ and $x_{100} > 0$, then $y_{000} = y_{011} = y_{101} = y_{010} = y_{001} = y_{111} = 0$. So $y_{110} > 0$ and $y_{100} > 0$, and hence $x_{000} = x_{011} = x_{101} = x_{010} = x_{001} = x_{111} = 0$. From (5.6), we get $2u^2 - u = 0$. This is impossible.

If $x_{111} > 0$ and $x_{011} > 0$, then $y_{101} = y_{110} = y_{000} = y_{001} = y_{010} = y_{100} = 0$. So $y_{111} > 0$ and $y_{011} > 0$, and hence $x_{101} = x_{110} = x_{000} = x_{001} = x_{010} = x_{100} = 0$. From (5.7), we get $2u^2 - u = 0$. This is impossible.

If $x_{111} > 0$ and $x_{101} > 0$, then $y_{011} = y_{000} = y_{110} = y_{001} = y_{010} = y_{100} = 0$. So $y_{111} > 0$ and $y_{101} > 0$, and hence $x_{011} = x_{000} = x_{110} = x_{001} = x_{010} = x_{100} = 0$. From (5.6), we get $2u^2 - u = 0$. This is impossible.

If $x_{110} > 0$ and $x_{111} > 0$, then $y_{000} = y_{011} = y_{101} = y_{001} = y_{010} = y_{100} = 0$. So $y_{110} > 0$ and $y_{111} > 0$, and hence $x_{000} = x_{011} = x_{101} = x_{001} = x_{010} = x_{100} = 0$. From (5.6), we get $2u^2 - u = 0$. This is impossible.

A similar argument can be applied to the pairs

(x_{000}, x_{001}),

(x_{000}, x_{010}), (x_{000}, x_{100}), (x_{001}, x_{011}), (x_{001}, x_{101}), (x_{010}, x_{011}), (x_{010}, x_{110}), (x_{100}, x_{101}), (x_{100}, x_{110}), (x_{011}, x_{111}), (x_{101}, x_{111}), and (x_{110}, x_{111}).

If $x_{000} > 0$ and $x_{011} > 0$, then $y_{110} = y_{101} = y_{011} = y_{000} = 0$ and at least two of y_{001}, y_{010},

y_{100} and y_{111} are non-zero. Hence, $x_{001} = x_{010} = x_{100} = x_{111} = 0$. So, $x_{000}+x_{110} = x_{011}+x_{101} = n_1/2$ from (5.10). From (5.8), $2u^2 - u = (n_1/2)(y_{001} + y_{111}) + (n_1/2)(y_{010} + y_{100}) = n_1n_2/2$. As $n_1n_2 = 4u^2$, this is impossible.

If $x_{000} > 0$ and $x_{101} > 0$, then $y_{110} = y_{101} = y_{011} = y_{000} = 0$ and at least two of y_{001}, y_{010}, y_{100} and y_{111} are non-zero. Hence, $x_{001} = x_{010} = x_{100} = x_{111} = 0$. So, $x_{000} + x_{110} = x_{011} + x_{101} = n_1/2$. From (5.8), $2u^2 - u = (n_1/2)(y_{001} + y_{111}) + (n_1/2)(y_{010} + y_{100}) = n_1n_2/2$. As $n_1n_2 = 4u^2$, this is impossible.

If $x_{000} > 0$ and $x_{110} > 0$, then $y_{110} = y_{101} = y_{011} = y_{000} = 0$ and at least two of y_{001}, y_{010}, y_{100} and y_{111} are non-zero. Hence, $x_{001} = x_{010} = x_{100} = x_{111} = 0$. So, $x_{000} + x_{110} = x_{011} + x_{101} = n_1/2$. From (5.8), $2u^2 - u = (n_1/2)(y_{001} + y_{111}) + (n_1/2)(y_{010} + y_{100}) = n_1n_2/2$. As $n_1n_2 = 4u^2$, this is impossible.

If $x_{001} > 0$ and $x_{010} > 0$, then $y_{001} = y_{010} = y_{100} = y_{111} = 0$ and at least two of y_{000}, y_{011}, y_{101} and y_{110} are non-zero. Hence, $x_{000} = x_{011} = x_{101} = x_{110} = 0$. So, $x_{001} + x_{111} = x_{010} + x_{100} = n_1/2$. From (5.8), $2u^2 - u = (n_1/2)(y_{011} + y_{101}) + (n_1/2)(y_{000} + y_{110}) = n_1n_2/2$. As $n_1n_2 = 4u^2$, this is impossible.

If $x_{001} > 0$ and $x_{100} > 0$, then $y_{001} = y_{010} = y_{100} = y_{111} = 0$ and at least two of y_{000}, y_{011}, y_{101} and y_{110} are non-zero. Hence, $x_{000} = x_{011} = x_{101} = x_{110} = 0$. So, $x_{001} + x_{111} = x_{010} + x_{100} = n_1/2$. From (5.8), $2u^2 - u = (n_1/2)(y_{011} + y_{101}) + (n_1/2)(y_{000} + y_{110}) = n_1n_2/2$. As $n_1n_2 = 4u^2$, this is impossible.

If $x_{001} > 0$ and $x_{111} > 0$, then $y_{001} = y_{010} = y_{100} = y_{111} = 0$ and at least two of y_{000}, y_{011}, y_{101} and y_{110} are non-zero. Hence, $x_{000} = x_{011} = x_{101} = x_{110} = 0$. So, $x_{001} + x_{111} = x_{010} + x_{100} = n_1/2$. From (5.8), $2u^2 - u = (n_1/2)(y_{011} + y_{101}) + (n_1/2)(y_{000} + y_{110}) = n_1n_2/2$. As $n_1n_2 = 4u^2$, this is impossible.

If $x_{010} > 0$ and $x_{100} > 0$, then $y_{001} = y_{010} = y_{100} = y_{111} = 0$ and at least two of y_{000}, y_{011}, y_{101} and y_{110} are non-zero. Hence, $x_{000} = x_{011} = x_{101} = x_{110} = 0$. So, $x_{001} + x_{111} = x_{010} + x_{100} = n_1/2$. From (5.8), $2u^2 - u = (n_1/2)(y_{011} + y_{101}) + (n_1/2)(y_{000} + y_{110}) = n_1n_2/2$. As $n_1n_2 = 4u^2$, this is impossible.

If $x_{010} > 0$ and $x_{111} > 0$, then $y_{001} = y_{010} = y_{100} = y_{111} = 0$ and at least two of y_{000}, y_{011}, y_{101} and y_{110} are non-zero. Hence, $x_{000} = x_{011} = x_{101} = x_{110} = 0$. So, $x_{001} + x_{111} = x_{010} + x_{100} = n_1/2$. From (5.8), $2u^2 - u = (n_1/2)(y_{011} + y_{101}) + (n_1/2)(y_{000} + y_{110}) = n_1n_2/2$. As $n_1n_2 = 4u^2$, this is impossible.

If $x_{011} > 0$ and $x_{101} > 0$, then $y_{110} = y_{101} = y_{011} = y_{000} = 0$ and at least two of y_{001}, y_{010}, y_{100} and y_{111} are non-zero. Hence, $x_{001} = x_{010} = x_{100} = x_{111} = 0$. So, $x_{000} + x_{110} = x_{011} + x_{101} = n_1/2$. From (5.8), $2u^2 - u = (n_1/2)(y_{001} + y_{111}) + (n_1/2)(y_{010} + y_{100}) = n_1n_2/2$. As $n_1n_2 = 4u^2$, this is impossible.

If $x_{011} > 0$ and $x_{110} > 0$, then $y_{110} = y_{101} = y_{011} = y_{000} = 0$ and at least two of y_{001}, y_{010}, y_{100} and y_{111} are non-zero. Hence, $x_{001} = x_{010} = x_{100} = x_{111} = 0$. So, $x_{000} + x_{110} = x_{011} + x_{101} = n_1/2$. From (5.8), $2u^2 - u = (n_1/2)(y_{001} + y_{111}) + (n_1/2)(y_{010} + y_{100}) = n_1n_2/2$. As $n_1n_2 = 4u^2$, this is impossible.

If $x_{100} > 0$ and $x_{111} > 0$, then $y_{001} = y_{010} = y_{100} = y_{111} = 0$ and at least two of y_{000}, y_{011}, y_{101} and y_{110} are non-zero. Hence, $x_{000} = x_{011} = x_{101} = x_{110} = 0$. So, $x_{001} + x_{111} = x_{010} + x_{100} = n_1/2$. From (5.8), $2u^2 - u = (n_1/2)(y_{011} + y_{101}) + (n_1/2)(y_{000} + y_{110}) = n_1n_2/2$. As $n_1n_2 = 4u^2$, this is impossible.

If $x_{101} > 0$ and $x_{110} > 0$, then $y_{110} = y_{101} = y_{011} = y_{000} = 0$ and at least two of y_{001}, y_{010}, y_{100} and y_{111} are non-zero. Hence, $x_{001} = x_{010} = x_{100} = x_{111} = 0$. So, $x_{000} + x_{110} = x_{011} + x_{101} = n_1/2$. From (5.8), $2u^2 - u = (n_1/2)(y_{001} + y_{111}) + (n_1/2)(y_{010} + y_{100}) = n_1n_2/2$. As $n_1n_2 = 4u^2$, this is impossible.

A similar argument can be applied to the pairs

(x_{000}, x_{011}),

(x_{000}, x_{101}), (x_{000}, x_{110}), (x_{001}, x_{010}), (x_{001}, x_{100}), (x_{001}, x_{111}), (x_{010}, x_{100}), (x_{010}, x_{111}), (x_{100}, x_{111}), (x_{011}, x_{101}), (x_{011}, x_{110}), and (x_{101}, x_{110}).

If $x_{000} > 0$ and $x_{111} > 0$, then $y_{011} = y_{101} = y_{110} = y_{001} = y_{010} = y_{100} = 0$ from (5.5). So $y_{000} > 0$ and $y_{111} > 0$, and hence $x_{011} = x_{101} = x_{110} = x_{001} = x_{010} = x_{100} = 0$ from (5.5). As this contradicts (5.10), this case is impossible.

If $x_{001} > 0$ and $x_{110} > 0$, then $y_{111} = y_{100} = y_{010} = y_{000} = y_{011} = y_{101} = 0$. So $y_{001} > 0$ and $y_{110} > 0$, and hence $y_{111} = y_{100} = y_{010} = y_{000} = y_{011} = y_{101} = 0$. As this contradicts (5.10), this case is impossible.

If $x_{010} > 0$ and $x_{101} > 0$, then $y_{100} = y_{111} = y_{001} = y_{011} = y_{000} = y_{110} = 0$. So $y_{010} > 0$ and $y_{101} > 0$, and hence $x_{100} = x_{111} = x_{001} = x_{011} = x_{000} = x_{110} = 0$. As this contradicts (5.10), this case is impossible.

If $x_{100} > 0$ and $x_{011} > 0$, then $y_{010} = y_{001} = y_{111} = y_{101} = y_{110} = y_{000} = 0$. So $y_{100} > 0$ and $y_{011} > 0$, and hence $x_{010} = x_{001} = x_{111} = x_{101} = x_{110} = x_{000} = 0$. As this contradicts (5.10), this case is impossible.

A similar argument can be applied to the pairs

(x_{000}, x_{111}),

(x_{001}, x_{110}), (x_{010}, x_{101}), and (x_{100}, x_{011}). This completes the proof of the theorem. $\square$

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