2016

The Influence of Instruction in Base 8 on Prospective Teachers' Mathematical Knowledge for Teaching

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THE INFLUENCE OF INSTRUCTION IN BASE 8 ON PROSPECTIVE TEACHERS' MATHEMATICAL KNOWLEDGE FOR TEACHING

by

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A dissertation submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy
in the College of Education and Human Performance
at the University of Central Florida
Orlando, Florida

Summer Term
2016

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ABSTRACT

The focus of this research project was to extend existing research literature by providing insight to how prospective teachers with differing levels of mathematical knowledge developed their conceptual understanding of whole number concepts and operations. Prospective teachers of interest were enrolled in an Elementary School Mathematics Course for Teachers (in the College of Education) with a whole number concepts and operations unit set entirely in base 8. In this mixed-mode study, participants were given the MKT, a measure designed to measure teacher mathematical knowledge for teaching, before and after taking the base 8 unit. The researcher focused on two specific constructs: Common Content Knowledge (CCK) and Specialized Content Knowledge (SCK). The qualitative portion of the study involved carefully constructed student interviews which allowed the researcher to deeply explore how prospective teacher conceptual understanding changed as a result of taking the unit in base 8.

Four participants with varying levels of knowledge were selected to be interviewed based upon initial scores on the MKT: 1) low CCK and low SCK, 2) low CCK and high SCK, 3) high CCK and low SCK, 4) high CCK and high SCK. Results of the interviews were used to help explain results from the MKT.

Quantitative and qualitative results showed that participants did not significantly increase their CCK, but did experience an increase in their conceptual understanding (SCK) as a result of taking the unit in base 8. Most prospective teachers entering the Elementary School Mathematics Course already had high procedural knowledge of the algorithms associated with elementary mathematics, which could account for the non-significant increase in CCK. Prospective teachers

all showed deeper conceptual understanding of whole number concepts and operations at varying levels by the end of the base 8 unit. Prospective teachers showed their increased understanding by way of explanations, justifications, and alternate solution strategies.

I would like to dedicate this dissertation to my family. You have been my pillars of support and encouragement throughout my entire life. I struggle now to find words that could possibly express the immense amount of gratitude and love that I have for you. Mom, Dad, and Mary: Thank you for helping me through the countless different emotions I have experienced throughout the course of my doctoral studies. From anger to joy, anxiousness to relief, and defeat to pride, you have been there. Whenever I wanted to quit, knowing that you were there rooting for me is what got me through. I will never stop trying to make you proud. Fernando, this dissertation has brought out my ugly side more than I'd like to admit. Thank you for persevering, loving me, and helping me put things in perspective. Now let's go do something fun!

ACKNOWLEDGMENTS

First and foremost, I'd like to thank to my committee chair, Dr. Erhan Selcuk Haciomeroglu, for the immense amount of support and encouragement throughout this process. Thank you for putting up with my frantic calls, e-mails, and meeting requests for nearly a year, and for always responding quickly and calmly. I'd also like to thank my committee members, Dr. Janet Andreasen, Dr. Farshid Safi, Dr. Nancy Lewis, and Dr. Lea Witta for providing me with the utmost constructive feedback on my study, both before and after it was carried out. Also, thank you to my fellow doctoral students Laura Tapp, Rebecca Gault, and Heidi Eisenreich. Words cannot describe how lucky I feel to have gone through this with anyone else but you. All I had to do was call, and there would be someone on the other end who would empathize with my physical and emotional feelings, and someone to offer me tremendous support and words of encouragement. If it weren't for you girls, I may have never persevered through that first year. Thank you from the bottom of my heart, and although we are scattering around the country due to jobs and family, I know we will never lose contact. Remember I am still just a phone call away.

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ACRONYMS

CCK	Common Content Knowledge
CCSI	Common Core State Standards Initiative
CGI	Cognitively Guided Instruction
HLT	Hypothetical Learning Trajectory
IRB	Institutional Review Board
MKT	Mathematics Knowledge for Teaching
NCTM	National Council of Teachers of Mathematics
NRC	National Research Council
RME	Realistic Mathematics Education
SCK	Specialized Content Knowledge

CHAPTER 1:
THE PROBLEM AND ITS CLARIFYING COMPONENTS

Introduction

Mathematics Education faculty members at a college located in the southeast United States designed an Elementary School Mathematics Course for Teachers (in the College of Education) in which prospective teachers learn number concepts and operations in base 8 rather than the traditional base 10. Instruction in an alternate base intends to minimize prospective teachers' formal understanding of mathematics so that they experience cognitive dissonance and have to re-learn number concepts and operations conceptually. Additionally, the hope is that this experience will allow them to see things from an elementary student's perspective, allow them to better able to understand challenges and foresee error patterns that may arise for their future students, and have a deeper, more conceptual understanding of the mathematics that they will be teaching.

Teacher preparation is currently an area in which there should be continuous revamping. Research is constantly showing better ways to educate students, so in turn, teacher preparation programs should be changing as well. While there has been a call for a reform in the teaching of mathematics for decades (Common Core State Standards Initiative (CCSSI, 2010; NCTM, 1989, 1991, 2000, 2014), research suggests that teaching practices are not changing (NRC, 2001). The Common Core State Standards call for teachers to implement a more rigorous, coherent, and focused curriculum. Emphasis is also placed on students having a deep conceptual understanding of mathematics topics (CCSSI, 2010). Teachers should be educated on instructional methods that focus on teaching conceptually so that their students may have a deep understanding of mathematics. Instructors of the Elementary School Mathematics Course for Teachers (in the College of Education) model conceptual teaching while teaching in base 8.

Conceptual teaching can stem from constructivist learning theory. Traditionally mathematics has been taught in a lecture fashion, with the teacher's role as the primary source of knowledge (National Research Council, 2001). Constructivists believe that students learn best when they are given the opportunity to construct their own ideas and meaning of the mathematics (Cobb, 1988). The traditional approach assumed that learning facts and algorithms is a prerequisite for learning mathematics, however this is highly optimistic. For many students, conceptual learning never occurs. (Davis & Noddings, 1990). "Conceptual understanding (i.e., the comprehension and connection of concepts, operations, and relations) establishes the foundation, and is necessary, for developing procedural fluency (i.e., the meaningful and flexible use of procedures to solve problems) (NCTM, 2014, p.7). When students are given the opportunity to construct their own understanding of a mathematical concept, they are more likely to comprehend the material conceptually and achieve mathematical proficiency. When students are taught rules without meaning, conceptual understanding is unlikely to occur. Rules and procedures are easily forgotten, and students may have difficulty in the future with mathematics if they learn in this manner (Cobb, 1988; NRC, 2001). If scholars agree that students must be taught the conceptual component of mathematics, why do teachers continue to teach students only rules and procedures? It can be argued that teachers simply teach the way they were taught. In addition, pre-service teachers enter universities with certain beliefs and attitudes which may not align with conceptual instruction techniques that facilitate making sense of problems (Stohlmann, Cramer, Moore, & Maiorca, 2014). Maybe this could be a result of teachers not having a conceptual understanding of mathematics. You cannot teach what you don't know. It is often difficult to convince in-service and pre-service teachers to instruct mathematics conceptually rather than procedurally (Naylor, & Keogh, 1999). Therefore, the approach to teacher education must be changed.

Many pre-service elementary school teachers lack conceptual knowledge in the mathematics area of whole number and operations (Thanheiser, Whitacre, & Roy, 2014). When asked to solve a subtraction problem for example, pre-service teachers reason through their solutions poorly, and those solutions are tied to standard algorithms. Their conceptual knowledge is lacking. Current national and state adopted standards replace rote learning with exploratory arguments, focus on rigor and real world problems, and assert that students must explain and justify their thinking (CCSSI, 2010). If students are required to develop this level of knowledge, teachers must deliver instruction supporting a deeper understanding of the mathematics. Cobb (1988) stresses, "Ideally the teacher should have a deep, relational understanding of the subject matter and be knowledgeable about possible courses of conceptual development in specific areas of mathematics" (p. 99). Quality teachers not only know basic mathematics concepts, but also possess a conceptual understanding of the content (Ball, 1990). How should mathematics educators prepare pre-service teachers for such a demanding task?

Instructors of the Elementary School Mathematics Course for Teachers (in the College of Education) utilized in this study teach number concepts and operations in base 8 instead of base

10. This method of instruction supports prospective teachers' relearning of basic concepts as if they were elementary students. Research suggests that cognitive dissonance acts as a catalyst for understanding. "Students operating on the frontiers of their conceptual knowledge have no reason to build new conceptual structures unless their current knowledge results in obstacles, contradictions, or surprises" (von Glasersfeld, 1983). It is human nature to believe something is true until that belief is challenged. In this base 8 unit, prospective teachers start out believing they have an understanding of number concepts and operations because they remember the algorithms and procedures. Once they are subjected to a different number system, however, those beliefs are challenged and prospective teachers realize they did not understand as well as they initially thought. This opens the door for them to develop a deeper understanding of the mathematics.

Research has shown that quality mathematics teachers possess a knowledge that goes deeper than that of a typical adult in another profession. (Ball, Thames, & Phelps, 2008). Teachers require a *specialized content knowledge* (SCK), which is distinct to teaching, requiring educators to correctly demonstrate mathematics concepts, provide explanations and justifications for solutions, evaluate and explain atypical student solutions, and reason through why algorithms work. Conversely, *common content knowledge* (CCK) is the knowledge of mathematics that is used in other professions that use mathematics (Hill, Rowan, & Ball, 2005). In order to produce high quality teachers, teacher preparation programs must afford prospective teachers the ability to develop pertinent mathematical knowledge (Lee, Meadows, & Lee, 2003). Most teacher preparation programs require their prospective teachers to develop mathematical proficiency by taking general courses in the mathematics department, however these types of courses typically do not address SCK (Ball, 2003; Ball et al., 2008). Therefore, university teacher preparation programs should offer courses that strive to aid prospective teachers in their mathematical development of CCK and SCK. In addition, these courses should focus on a making sense of the mathematics rather than simply rule following (Battista, 1994), and afford prospective teachers the opportunity to explore mathematics in a way that develops their conceptual understanding of the content. Instructing in a different base system may be a way for teachers to develop their conceptual understanding of number concepts and operations.

The significance in this study lies in the small body of research available in regards to instructing pre-service elementary teachers in a different base. Three studies in particular focus on pre-service elementary classes taught in base 8. While the information gleaned in these studies shows teaching in base 8 is helpful in preparing teachers, they are both qualitative in nature, and focus only on one class. Results were obtained primarily from observations and student interviews. In this study, the researcher hopes to gather quantitative data that shows how prospective teachers' mathematical knowledge in regards to number operation is impacted with instruction in base 8 using the Mathematical Knowledge for Teaching Measures (MKT), a measure specifically designed to measure CCK and SCK.

Statement of the Problem

New standards require elementary students to have a deep conceptual understanding of mathematics. It is necessary for pre-service elementary teachers to be knowledgeable of such material upon entering the field. In order to be prepared to teach to the depth of new standards, pre-service elementary teachers knowledge must surpass common content knowledge of the material to include a specialized knowledge for teaching mathematics (Hill et al. 2004). Teacher preparation courses can afford prospective teachers opportunities to develop both their common content knowledge of mathematics as well as their specialized knowledge for teaching mathematics. The purpose of this study is to measure the impact of an elementary mathematics whole number operations unit taught in base 8 on pre-service teachers mathematical common content knowledge and specialized knowledge for teaching of elementary number concepts and operations.

Research Questions

This study is focused on several research questions regarding the measurement of two constructs: common content knowledge (CCK) of elementary number concepts and operations and specialized content knowledge (SCK) of elementary number concepts and operations. The measure was chosen from the Mathematical Knowledge for Teaching (MKT) Measures developed by the University of Michigan's Learning Mathematics for Teaching Project, and is designed to specifically measure CCK and SCK (Ball, Hill, & Bass, 2005; Ball et al., 2008; Hill & Ball, 2004; Hill et al., 2005; Hill et al., 2004). The researcher hopes to answer the following research questions:

- 1) What impact does an elementary mathematics unit taught in base 8 have on pre-service teachers' mathematical common content knowledge of elementary number concepts and operations?
- 2) What impact does an elementary mathematics unit taught in base 8 have on pre-service teachers' specialized content knowledge of elementary number concepts and operations?
- 3) How do prospective teachers' participation in the instructional unit help explain the change, if any, in their common content knowledge and specialized content knowledge?

Contributions

Pre-service teachers' comprehension of mathematics is rooted in rote memorization, rules, and procedures, which is insufficient for them to be successful as a teacher in the classroom. Quality mathematics teachers not only know the basics, but also have a conceptual understanding of the content (Ball, 1990). Current reforms in mathematics education require students to have a deep understanding of the material, and the ability to explain and justify their reasoning (CCSSI, 2010). Research has shown that pre-service teachers are not prepared to teach to the depth of current standards (Chapman, 2007; Thanheiser et al., 2014) and preparing teachers to teach conceptually versus procedurally could influence students to comprehend the mathematics at a deeper level (Cobb et al. 1991). Teachers must have a specialized knowledge unique to teaching. It includes a conceptualization of the mathematics (Ball, 1990; Fennema & Franke, 1992; Lee et al., 2003; Shulman, 1986). One possible option for increasing teacher candidates' conceptual knowledge is teaching them number concepts and operations in base 8.

There is a small body of research that shows teaching in base 8 can increase a teacher candidates' ability to reason mathematically and increase their conceptual fluency of mathematics (Andreasen, 2006; Fasteen et al., 2015; McClain, 2003; Roy, 2008; Safi, 2009; Yackel et al., 2007).

Safi (2009) specifically examined the development of two prospective teachers' understanding of whole number concepts and operations in the same base 8 unit utilized by this study. The participants selected were classified as either high performing or low performing. The current study aims to build on prior research by further breaking down the different types of incoming prospective teacher knowledge, and examining how it changes with a specific classification and focus on CCK and SCK. This study could influence the instruction individuals with differing incoming knowledge receive. Additionally, results could encourage other teacher preparation programs to consider redesigning their curriculum to include instruction in a different base, thereby increasing pre-service teachers' conceptual understanding of number concepts and operations.

CHAPTER 2:
REVIEW OF THE LITERATURE

Need for Reform in Teacher Training

For the past several decades, teacher mathematical knowledge has been a major focus of interest. Researchers have sought to determine key aspects of teacher mathematical knowledge that best aid in student comprehension and achievement, and there is a general consensus that teacher knowledge is one of the greatest influences (Fennema & Franke, 1992; Ma, 1999; Rech, Hartzell, & Stephens, 1993). Effects of these efforts can be seen in standards published by teaching organizations (CCSSI, 2010; NCTM, 1989, 1991, 2000, 2014), state-led efforts to develop standards (Common Core State Standards Initiative, 2010) and national policies regarding education (No Child Left Behind, 2001). Given the aforementioned standards and policies, one might believe that there is a general consensus regarding what knowledge a teacher must possess. A closer look at research reveals discrepancies. The U.S. Department of Education (2002) reported that content knowledge was an important aspect of a high quality teacher. Shulman (1986) believed that there were three main components to teacher knowledge: content knowledge, pedagogical content knowledge, and curriculum knowledge. He emphasized the importance of pedagogical content knowledge in teaching. Ball (1990) discussed the significance not only of pre-service teachers having mathematical knowledge, but a conceptual understanding of mathematics so that they may answer the *why* questions that students will inevitably ask. Shulman (1987) also stressed the importance of pedagogical content knowledge, explaining that "it represents the blending of content and pedagogy into an understanding of how particular topics, problems, or issues are organized, represented, and adapted to the diverse interests and abilities of learners, and presented for instruction" (p. 8). Researchers seem to agree that teachers must know the content, however, there have been disagreements and ever changing developments in what constitutes a teachers specialized mathematical knowledge.

Teacher Mathematical Knowledge

In the mid 1980's, as reforms began calling for a different approach to mathematics instruction, scholars became increasingly interested in how different forms of teacher knowledge related to instruction (Ball, 1988; 1990; Ma, 1999; Shulman, 1986). Shulman (1986; 1987) was among the first to identify and stress the importance of pedagogical content knowledge, a knowledge described as going beyond subject matter knowledge; a knowledge specifically for teaching. Researchers began to place increasing emphasis on teachers' knowledge for teaching mathematics (Ball & Bass, 2000; Simon, 1995) and these studies helped to break down what that knowledge entails.

Specialized Content Knowledge

Research has illuminated the importance of teachers to have a deep understanding of mathematical content (Ball 1988, 1990; Ma, 1999; Thanheiser et al., 2014). Pre-service teachers must possess strong mathematical content knowledge, and in addition, they must have a strong *conceptual* content knowledge (Ball, 1990). In a study examining subject matter knowledge of prospective elementary and secondary teachers, Ball challenged the assumptions that traditional mathematics content knowledge is not difficult, that k-12 mathematics provides pre-service teachers with the information they need to know to teach, and that majoring in mathematics is enough for pre-service teachers to be successful in their careers. Ball drew on data from the longitudinal Teacher Education and Learning to Teach Study from Michigan State and focused on teacher candidates' ($N = 252$) understandings of mathematics.

The study utilized both questionnaires and interviews designed to "explore participants' ideas, feelings, and understandings about mathematics and writing, about the teaching and learning of mathematics and writing, and about students as learners of these subjects" (Ball, 1990, p. 451). Elementary as well as secondary candidates had difficulty representing and conceptualizing fraction division, and most prospective teachers' understanding of mathematics was based upon rules and procedures. Ball stressed the importance of children being given the opportunity to explore mathematics content, make conjectures, and validate solutions. In turn, prospective teachers need to understand mathematics deeply in order to facilitate their students' understanding. Prospective teachers must also be able to participate in group and class discourse, justify their claims, and explain the claims of others in order for a true conceptual understanding to occur. Just because someone can do mathematics does not imply an understanding of the mathematics (Ball, 1988). Many adults, including teachers, perform mathematical operations without truly understanding the principles or connections behind the mathematics. Teacher candidates' understandings are rooted in rote memorization, rules, and procedures, which is insufficient for them to be successful in the classroom (Ball, 1990; Southwell & Penglase, 2005; Thanheiser, 2012; Thanheiser et al., 2014). Being able to reason through solutions and justify actions is essential to prospective teachers' conceptual understanding of mathematics. In turn, teachers must also have a deep conceptual understanding of the content, so that they may facilitate the type of instruction that is conducive to this type of understanding in their students.

With the emerging research in the area of teacher knowledge, the ability to measure these different facets of knowledge also came into focus. Beginning attempts to measure teacher knowledge tended to indirectly measure this knowledge or measured it in ways that loosely related to how pedagogical knowledge impacted teacher performance (Rowan, Schilling, Ball, & Miller, 2001). In response to these issues, the Study of Instructional Improvement began in 1999 at the University of Michigan to design measures of elementary teachers' knowledge for teaching mathematics (Hill, Rowan, & Ball, 2005). Realizing this discrepancy, Rowan and colleagues (2001) began creating questions from scratch for a survey designed to measure teacher pedagogical knowledge in two areas: elementary mathematics and reading/language arts.

Researchers chose initial topics of number concepts, place value and operations. Three domains of pedagogical content knowledge were identified: content knowledge, knowledge of students' thinking, and knowledge of pedagogical strategies. Items on the questionnaire were developed to represent scenarios that teachers might encounter in the classroom. Solutions were selected based upon research on teaching and learning in aforementioned domains. Researchers developed two alternate forms of the questionnaire. Each multiple choice question was intended to measure a certain aspect of a teacher's pedagogical content knowledge (Rowan et al., 2001).

Data was gathered as part of this pilot study over two semesters. One hundred twenty three teachers were mailed the questionnaire, of which, 104 responded and became the participants of the study. For mathematics, 26 teachers completed both forms of the questionnaire, 33 teachers completed form A, and 24 completed form B. Scales were built from the data fitting items into different categories of teacher knowledge using the computing program BILOG. Additionally, researchers utilized this program to assign questions a level of difficulty (Rowan et al., 2001).

While analyses were limited to the small sample selected for the study, researchers were successful in creating scales with adequate reliabilities for some of the domains of teacher knowledge. There was success in developing items that loaded under content knowledge, knowledge of students thinking, and pedagogical content knowledge, although reliabilities were mixed. Researchers concluded that it *is* possible to develop reliable scales that measure certain aspects of teacher pedagogical content knowledge, although the process to do so is difficult (Rowan et al., 2001). Suggestions for future research included further examination of the dimensionality of pedagogical content knowledge to better understand the structure of said knowledge. Given the difficulty researchers experienced in developing the scales, the question still remained as to whether it was even possible to develop a summative survey of pedagogical content knowledge.

Building on the aforementioned research regarding measuring teacher pedagogical content knowledge, Hill and colleagues sought to answer the question "Is there one construct that can be called 'mathematics knowledge for teaching' and that explains patterns of teachers' responses, or do these items represent multiple constructs and thus several distinct mathematical competencies of elementary mathematics teachers?" (Hill, Schilling, & Ball, 2004, p. 12). They also sought to construct scales that could measure such constructs reliably. Researchers wrote and later piloted a number of multiple choice items intended to represent the knowledge that was required to teach elementary school mathematics. Items were pulled and sometimes modified from other researchers' work to create different categories. One hundred thirty eight mathematics items were developed and organized into categories including number concepts, operations, and patterns, functions, and algebra. Researchers crossed these three content areas with two domains of teacher knowledge (1) knowledge of content and (2) knowledge of students and content, creating six cells. Items were used in a pilot forms, which were then used to conduct statistical analysis. Factor analysis was used to identify patterns. Results indicated that responses to the items written to represent SKC was likely to be explained by the content knowledge factor, and supported the hypothesis that some content knowledge is specific to the teaching profession. Items specific to the SKC factor included "(*a*) analyzing alternative algorithms or procedures, (*b*) showing or representing numbers (e.g., 10.05) or operations (e.g., 1/2 x 2/3) using manipulatives, and (*c*) providing explanations for common mathematical rules (e.g., why any number can be divided by 4 if the last two digits are divisible by 4)" (Hill et al., 2004, p. 24). These findings provide evidence that there is a specialized mathematical content knowledge that is particular to teaching. Additionally, these results show that other theorists are at least partially correct. As far as constructing scales to measure teacher mathematical knowledge, researchers established that their analysis was the first step in the development process, acknowledging that the data set was less than ideal, and that results should be replicated. Researchers concluded that if their results hold, findings contribute to teacher recruitment and preparation program policies. These findings provide evidence that there is a difference in CCK and SCK, and that the two constructs can be reliably differentiated and measured.

Researchers have successfully identified two separate constructs of mathematical knowledge for teaching. Teachers need not only knowledge of the mathematics content they teach (CCK) such as long division, decimals, fractions, etc. They also need specialized content knowledge (SCK) for teaching. This knowledge, as defined by Hill and colleagues (2005) is mathematical knowledge, not pedagogy. "It includes knowing how to represent quantities such as 1/4 or .65 using diagrams, how to provide a mathematically careful explanation of divisibility rules, or how to appraise the mathematical validity of alternative solution methods for a problem such as 35 × 25" (Hill et al., 2005, p. 377). Having identified those important constructs of teacher knowledge, researchers sought to determine their effect on student achievement.

Researchers gathered data from 115 elementary schools from 2000 – 2004. Schools varied in policy and social environments, and students varied in social backgrounds. The two cohorts of students consisted of kindergarteners who were followed through second grade, and third graders who were followed through fifth grade. Students were administered assessments in the fall and spring of each year. Data was also collected through parent interviews. Teacher data was collected through a log and also through the content knowledge for teaching mathematics measure as developed by Hill and colleagues (2004).

Results showed that teachers' mathematical knowledge for teaching positively predicted student gains in achievement. In the analysis, researchers did not make a distinction between knowledge types, CCK or SCK, and how each type influenced student achievement. Researchers did state that it was possible that affects may differ across knowledge types (Hill et al., 2005). The current study endeavors to add to the existing research and capture how an intervention affects the two constructs of teacher knowledge, CCK and SCK, separately.

Procedural Knowledge VS. Conceptual Knowledge

The types of teacher knowledge discussed in above relate closely to what some researchers call procedural knowledge and conceptual knowledge. Researchers and theorists have many explanations distinguishing the two types of understanding/knowledge. Skemp (1978) was among the first to identify two types of understanding: relational understanding and instrumental understanding. He defined relational understanding as "knowing both what to do and why" (p. 21) and *instrumental understanding* as "rules without reason" (p. 21).

Hiebert and Lefevre (1986) described two types of understanding/knowledge as well: conceptual and procedural. Conceptual knowledge was defined as:

> Knowledge that is rich in relationships. It can be thought of as a connected web of knowledge, a network in which the linking relationships are as prominent as the discrete pieces of information. Relationships pervade the individual facts and proposition so that all pieces of information are linked to some network. (pp. 3–4)

There were two aspects of procedural knowledge: (1) "familiarity with the individual symbols of the system and with the syntactic conventions for acceptable configurations of symbols," (p. 7) and (2) "rules or procedures for solving mathematical problems" (p. 7).

A third distinction involves orientations for teaching and comes from Thompson, Philipp, Thompson, and Boyd (1994) in the form of calculational orientation and conceptual orientation. Teachers who have a calculational orientation focus on “the application of calculations and procedures for deriving numerical results.” (p. 86). Conceptually oriented teachers “focus students’ attention away from the thoughtless application of procedures towards a rich conception of situations, ideas, and relationships among ideas” (p. 86).

The National Research Council describes conceptual understanding as the comprehension of mathematical concepts, operations, and relations, unlike procedural understanding which focuses on rote memorization and rule following. Researchers at the University of Michigan describe types of teacher knowledge that are similar to above definitions: specialized content knowledge requires teachers to correctly reason through mathematics concepts, provide explanations and justifications for solutions, evaluate and explain multiple student solution strategies, and make connections to algorithms (Hill et al., 2005). Common content knowledge is knowledge is the basic procedural knowledge that allows one to solve a math problem. “This common knowledge is not unique to teaching; bankers, candy sellers, nurses, and other non-teachers are likely to hold such knowledge” (Hill & Ball, 2004, p. 333.)

Upon synthesizing the above descriptions of types of knowledge/understanding, one can see two perceptions surface: One describes a knowledge/understanding rich in conceptual connections and sense making. The other describes a knowledge built on rules without meaning and memorization without deep understanding. Because the definitions for procedural and conceptual understanding/knowledge are so similar, for the purposes of this study, we will take conceptual understanding/knowledge to mean specialized content knowledge (SCK), and procedural understanding/knowledge to mean common content knowledge (CCK).

Conceptually Based Instruction

Typically, university teacher preparation programs do not prepare teachers to teach to the depth of current standards, nor do they address the knowledge identified by Hill et al. (2004) specific to the teaching of mathematics (Ball, 2003; Ball et al., 2008). A change in teacher preparation programs needs to occur. This notion is not new. There has been a call for reform in teaching practices, and therefore teacher preparation, for several decades (CCSSI; 2010; NCTM, 1989, 1991, 2000, 2014). New standards require elementary students to have a deep conceptual understanding of mathematics. NCTM defines conceptual understanding as "the comprehension and connection of concepts, operations, and relations" (2014, p. 7). This means that when students do mathematics, they not only need to understand what they are doing, but the reasoning behind what they are doing, and the connections to procedures and algorithms. Teachers can help students build their conceptual understanding. NCTM suggests, "Teachers must regularly select and implement tasks that promote reasoning and problem solving. These tasks encourage reasoning and access to the mathematics through multiple entry points, including the use of different representations and tools, and they foster the solving of problems through varied solution strategies" (2014, p. 17). The constructivist learning theory supports this type of instruction.

Constructivist Learning Theory

Constructivists believe that students should be given the opportunity to reason through mathematics and build their own conceptions (Cobb, 1988). With traditional instruction, students are taught rules and procedures under the assumption that knowing algorithms means knowing the mathematics. Research has shown quite the opposite (Davis & Noddings, 1990). Rules and procedures are easily forgotten or confused, which makes understanding future topics frustrating to students who cannot remember how to do foundational material. Learning mathematics does not have to be a linear process, and students must have a strong conceptual foundation of mathematics in order to be successful. If students are allowed to build their own understanding of the concepts, they are more likely to remember them and make the connections to algorithms later (Carpenter, Franke, Jacobs, Fennema, & Empson, 1998). The Elementary School Mathematics Course utilized in this study uses the ideology of constructivism in its design. Prospective teachers are placed in groups and given problem solving scenarios in which they must develop a solution. Research supports the benefits of allowing students to construct their own meaning of the mathematics.

A yearlong study involving 18 second-grade classrooms and teachers supported the ideas of constructivism. In 1989 the National Council of Teachers of Mathematics and the National Research Council supplied reports suggesting what mathematics education should encompass. Noticing a lack of empirical research that supported such reports, Cobb et al. (1991) conducted a research study designed to utilize the recommendations of the NCTM and NRC in order to understand how students learn mathematics, their beliefs and motivations, and their pedagogical beliefs. The study consisted of 10 project classrooms and eight control classrooms. Using constructivist views of learning as a guide, researchers developed a yearlong set of classroom instructional activities for all areas of second grade mathematics that were delivered in the treatment classrooms. Students were tested on arithmetical competence and completed a goals and beliefs questionnaire. Teachers were given a pedagogical beliefs questionnaire.

Arithmetical competence was measured through a multiple choice standardized test (ISTEP) that consisted of two parts for mathematics and the Project Arithmetic Test, which was developed by research staff. For the standardized test, the Computation portion assessed addition and subtraction in typical textbook fashion. The Concepts and Applications portion consisted of items that centered on specific illustrations highlighted in textbooks and items that had students associate number sentences with numerals. The Project Arithmetic Test consisted of two scales: Instrumental and Relational. The Instrumental portion was labeled as such because it contained items that were possible to complete using standard algorithms and did not require conceptual understanding. The Relational portion was designed to measure conceptual understanding and included contextual, non-textbook type problems.

Based upon performance on the two arithmetical tests, statistical results showed that both project and control students had a similar ability to perform computational tasks, however, project students developed a higher level of mathematical reasoning (per performance on the Relational portion of the Concepts and Applications test). Project students were also less familiar with textbook conventions typical to elementary school textbooks. The unfamiliarity was significant because those students also were more likely to supply answers that indicated a conceptual understanding of the material rather than rule memorization. In regards to beliefs, control students were more likely to believe that conformity leads to success. More specifically, they tended to follow solution patterns of the teacher and their peers, unlike the project students, who were more likely to understand the importance of collaboration and exploring all solution avenues. Treatment students were frequently given the opportunity to arrive at their own conclusions and teachers emphasized the importance of discussing solution strategies with their peers, a teaching technique that corresponds to constructivist learning theory. The results of this study indicated that it was feasible to introduce instruction compatible with the constructivist learning theory into the public school system.

A notable program called Cognitively Guided Instruction (CGI) complimented and strengthened the results of the aforementioned work of Cobb and colleagues (1991). The purpose of the CGI project was to investigate the effects of providing teachers with explicit knowledge regarding their students thinking in certain content domains (Carpenter, Fennema, Peterson, Chiang & Loef, 1989; Carpenter, Fennema, & Franke, 1996). Participants in the study ($N = 40$) consisted of first and second grade teachers. Researchers provided research-based training to CGI project teachers that endeavored to guide their development of instructional practice compatible with constructivist learning theory. Half of the teachers that participated in the study received treatment while half did not. Researchers observed treatment and non-treatment teachers and their students for the entire school year. Results showed that treatment teachers spent significantly more time talking with their students about problem solving than did control teachers. Treatment teachers also spent more time discussing alternate strategies for solving problems, while control teachers had students working on problems independently. Students were given achievement scales before and after the school year. Researchers found that students in treatment and control groups performed similarly on low-level problems, but treatment students performed better on more complex problems. This could be attributed to the greater amount of time spent talking about problem solving and discussing alternate solutions (Carpenter et al., 1989; Carpenter et al., 1996; Sahin, 2015).

In addition to the students' achievement, teachers increased their mathematical content knowledge as well as their knowledge of their students' thinking. Another notable result was that CGI teachers tended to let their students struggle with the mathematics before asking for different solutions to the problem. They also focused much more on the process of their students thinking and their steps for solving the problem. This method of teaching reflects constructivist learning theory in that students are allowed to construct their own solutions and make meaning of the mathematics. It contrasts with traditional instruction where teachers show students how to solve a problem and then have them practice. Results from this study indicated that if teachers are provided with research-based instruction regarding how students think, it could improve their mathematical knowledge, classroom practices, and most importantly, their students' success (Carpenter et al., 1989; Carpenter et al., 1996; Sahin, 2015).

Both the CGI project and the Second Grade Mathematics Project completed by Cobb and colleagues rejected traditional teaching techniques, which imply a "separation of acquisition from application" (Cobb et al., 1991, p. 25). Instead, these studies stressed instructional strategies that were guided by constructivist learning theory. In both studies the goal was for the teacher to extract what knowledge students brought to the table, and help them build on that knowledge by providing instruction that both challenged them, and encouraged them to develop their own mathematical understanding. The teacher did not act as a lecturer and did not provide ready-made solutions and explanations. Rather, the teacher acted as a facilitator of knowledge, supporting students as they worked together to explore all avenues of the solution. Students were encouraged to explain and justify their solution strategies and ask questions of other students' arguments. In both studies, treatment students had a deeper knowledge of the content and they were able to utilize many different solution strategies to arrive at conclusions. Additionally, the teachers' knowledge of mathematics content increased. The studies provided empirical research that supported NCTM and the NRC reports suggesting what mathematics education should encompass, which includes a conceptual understanding of content (Carpenter et al., 1989; Carpenter et al., 1996; Cobb et al., 1991; Sahin, 2015). If conceptually based instruction is more beneficial to students, teacher preparation programs should provide an environment where pre-service teachers can develop this type of instruction.

Teacher Preparation Program Reform

Although research suggests that programs and professional development for teachers can develop a deep understanding of mathematical content (Carpenter et al., 1996; Cobb et al., 1991; Hill & Ball, 2004), studies have not shown a great deal of change in teaching practices. Trends in International Mathematics and Science Study (TIMSS, 2007). Teachers believed in reform, and also believed that they were practicing reform strategies in their classrooms, yet observations from the study showed otherwise. Students were not learning mathematics conceptually. Teachers were still teaching mathematics using rote memorization, rules, and procedures. This could be a result of teachers not having a conceptual understanding of mathematics. One cannot teach what one does not know.

Prospective teachers lack the mathematical knowledge needed for teaching (Thanheiser et al., 2014). This knowledge is described by Hill and colleagues (2005) as *specialized content knowledge* (SCK), which is distinct to teaching. It requires educators to correctly demonstrate mathematics concepts, provide explanations and justifications for solutions, evaluate and explain multiple student solution strategies, and reason through why algorithms work. Research shows that prospective teachers lack the conceptual knowledge of whole numbers, operation, and place value in the base ten system. Place value concepts are the most problematic for pre-service teachers. (Southwell & Penglase, 2005; Thanheiser, 2012; Zazkis & Khoury, 1993). In another project, nearly half of pre-service teacher participants could not recognize that the 2 in 25 represented 20 (Ross, 2001). Most importantly, many pre-service teachers are not able explain the meaning behind the rules and procedures that are taught to students in their mathematics classes (Ball 1988, 1990; Ma, 1999; Thanheiser et al., 2014). It will be difficult, if not impossible, for these teachers to explain the concepts behind the mathematics to their students.

Researchers agree that teachers should hold a unique knowledge separate from content knowledge. (Ball, 1990; Fennema & Franke, 1992; Lee et al., 2003; Shulman, 1986). In teaching mathematics, how well a teacher knows the content may matter much more than how much content they know. Furthermore, “teaching quality might not relate so much to performance on standard tests of mathematics achievement as it does to whether teachers' knowledge is procedural or conceptual, whether it is connected to big ideas or isolated into small bits, or whether it is compressed or conceptually unpacked" (Hill & Ball, 2004, p. 332).

There is a widespread agreement that teachers have a specialized knowledge unique to teaching (Ball, 1990; Fennema & Franke, 1992; Lee et al., 2003; Shulman, 1986).

Current calls for reform, such as the Common Core State Standards Initiative (2010) and NCTMs Principles to Actions: Ensuring Mathematical Success for All (2014) emphasize that students must have a conceptual foundation to truly understand mathematics. Research suggests that preparing teachers to teach conceptually versus procedurally could influence students to understand mathematics in a conceptual manner (Carpenter et al., 1996; Cobb et al., 1991; Sahin, 2015). This means that pre-service teachers, who were mostly taught in the traditional fashion of memorization and procedures, must also conceptually understand the mathematics content. Unfortunately, research has shown that pre-service teachers are not prepared to teach in a conceptual fashion (Chapman, 2007; Thanheiser et al., 2014). Teacher preparation programs vary widely in their content. This leads one to ask: What is the best method for preparing teacher candidates to teach mathematics conceptually?

Elementary Mathematics Content Course Design

The National Council for Accreditation of Teacher Education (NCATE) indicates that pre-service teachers must demonstrate both content and pedagogical knowledge of their field (NCATE, 2002). In order to meet this standard, universities typically have pre-service teachers take mathematics content courses either through a college of education, or through the mathematics department. Unfortunately, many times these classes are a continuation of the same algorithm and procedure laden mathematics courses seen in the k-12 school system.

> "It seems, then, that we are caught in a vicious cycle: poor K-12 mathematics instruction produces ill-prepared college students, and undergraduate education often does little to correct the problem. Indeed, some universities mandate next to no mathematics coursework for the prospective elementary teacher. However, simply increasing the number of required credit hours is no solution—courses that allow students to get by using the same strategies that got them through K-12 just perpetuate the problem" (Conference Board of the Mathematical Sciences, 2001, p. 55-56)

It is important that prospective teachers entering the field have a conceptual understanding of the material that they will be teaching. Having them take multiple mathematics classes that stress rules and procedures is not beneficial to that understanding. What could be beneficial is a course designed to aide pre-service teachers in the unpacking of this knowledge so that they may enter classrooms with ability to transfer it to their students.

Faculty members at a college located in the southeast United States in the mathematics education program took on this challenge and designed an Elementary School Mathematics Course for Teachers (in the College of Education) using theories and strategies deemed effective in the instruction of mathematics. Research regarding how children learn place value and number operation was utilized since pre-service teachers were essentially re-learning the material. The Realistic Mathematics Education instructional theory was utilized to create the instructional sequence and hypothetical learning trajectories (HLTs) of the Elementary School Mathematics Course (Andreasen, 2006). Faculty members also considered research regarding teaching pre-service teachers in a different base when designing the course.

How Children Learn Place Value and Number Operation

The foundation for children's understanding of place value and number operation is their number sense. A key aspect of number sense is conservation of number, which is the comprehension that if objects are grouped differently, the number of objects does not change (Ross, 1986). For example, a child is shown 15 snap cubes grouped as ten and five, and asked how many. The child counts out all snap cubes and responds 15. Then the snap cubes are rearranged into groups of eight and seven and the child is asked how many. A child in the early stages of conservation of number would still count out all of the snap cubes even though the cubes were arranged right before his eyes and the number of objects clearly did not change. A child who has mastered conservation of number would not have to count the snap cubes again. Establishing conservation of number is an important building block for children to develop effective counting strategies and the comprehension of place value and number operations.

Another important building block is the concept of ten. Researchers have identified a general progression of a child's conceptual development in regards to the concept of ten (Ross, 1986; Steffe, 1983). Children in the beginning stage cannot identify a ten as a single entity, and instead, focus on the units that make up the ten. For example, if a child is shown 24 snap cubes grouped as two tens and four ones, they will still count out each snap cube individually to tell how many are in the set. They cannot identify each ten as a unit and do not recognize that each ten is comprised of ten ones. Naturally, the next level in the progression is being able to identify each ten as a single entity and understanding that there are ten ones in a ten. The final stage in conceptual development includes children being able to complete tasks more abstractly (Ross, 1986; Steffe, 1983). For example: Suppose a student is shown a collection of base ten blocks as follows: 5 tens and 3 ones. A child in one of the later phases of conceptual development would be able to count each ten as "ten, twenty, thirty, forty, fifty", and then the ones as "fifty one, fifty two, fifty three", as opposed to a child in the early stage of conceptual development who would have to count each individual ones block, or would accidentally count the tens as ones and count eight total. (Ross, 1986; Steffe, 1983).

Once children develop a conceptual understanding of number sense they may move on to number operations. Initially, children begin with direct modeling, adding and subtracting single digit whole numbers by counting all objects in the set, or by counting on from one of the addends (Fusion, 1992). Subsequently, children progress to counting strategies that may include counting up, taking away, and counting back. These counting strategies are more abstract and require a greater conceptual understanding by the child. (Carpenter et al. 1996). Once children begin multi-digit addition and subtraction they may make errors if they do not have the foundational knowledge of how whole numbers are composed. Cobb and Wheatly (1988) found that Children who were able to construct ten as an abstract composite unit saw horizontal number sentences as vertical number sentences. Their approach to solving the number sentences was procedural and had errors. Additionally, children may treat multi-digit numbers as two single digit numbers that are side by side (Fuson, 1992; Steffe, 1983). Research shows that students are less likely to make errors if they are afforded the opportunity to develop their own strategies for multi-digit addition and subtraction problems. Allowing children to invent their own strategies deepens their understanding of the concepts, and children are more likely to be able to make the connection to rules and procedures later (Carpenter et al., 1998). Once students can make a connection between concepts and algorithms, deep understanding has taken place (Carpenter, 1986).

When introduced to multiplication and division, children can build on what they already know about repeated addition. A common model for multiplication is one factor representing the number of groups, and the other factor representing number of pieces in each group (Steffe, 1994). For example, in the problem 3×5, 3 represents the number of groups and 5 represents the number of pieces in each group. There are 3 groups, with 5 pieces in each group. As a child's continues to invent strategies for multiplication, those strategies advance to more conceptually based partitioning and compensation strategies (Baek, 1998). Partitioning involves breaking down one for both factors into smaller numbers to make a problem easier. An upper elementary student solved the problem of finding how many total apples if there were 15 boxes with 177 apples in each box. She used the fact that 15 = 5 x 3 to make the problem more manageable. Then the problem could be viewed as (5 x 3) x 117, or 5 x (3 x 177). She first tripled 177 by breaking it down by place value as 100, 70, and 7. Then she added each column to arrive at 531. She then found five groups of 531 by adding 531 to itself twice, and then adding another 531 (Baek, 1998). While the path the student took to arrive at the solution was far from efficient, it was complex and demonstrated a conceptual understanding of multiplicative properties and additive properties.

Compensation also makes multiplication easier by manipulating the original problem. For example, when multiplying 5 x 350, a compensation strategy is to reason that 5 is half of 10, and if we make the 5 a ten, we have to divide 350 by 2. Now 5 x 350 is the same as 10 x 175, and the answer s 1570 (Baek, 1998). It is easier to multiply by ten than five. With compensation strategies, the student understands the connection between the multiplier and multiplicand and uses those connections to break down the problem so it is easier to solve.

Children use what they know about multiplication to develop division concepts. When presented in context, and students are afforded the opportunity to direct model, students may be able to identify for themselves the different types of division problems: partitive and quotitive, or sharing and measurement respectively. While children can make the distinction between the two problem types, adults typically only recognize problems as division because they are not thinking about the actions or relationships being depicted in the problem (Carpenter et al., 1996). Therefore, it is important that pre-service be exposed to both types of contextual division problems, so that they may see the difference for themselves, and be able to provide their students with the experiences they need to develop their understanding of division concepts.

Referring back to the multiplication model of groups and number of pieces in each group, in sharing division problems, the number of groups is known and the number in each group is unknown. For example: *Jamie has 63 candies and wants to give to each of her 3 friends. How many candies does each friend receive?* In this case, the number of friends would be known and the number of candies each friend received would be unknown. Another way to think about a sharing scenario is an object or group of objects being divided into a number of equal parts. With sharing division, the dividend has to be larger than the divisor, the divisor has to be a whole number, and the quotient has to be smaller than the dividend. Sharing division is the most commonly recognized problem type and is easily modeled, which could explain why it is the most commonly taught in schools (Simon, 1993).

Measurement division is more difficult to recognize, which could explain why it is less frequently seen in classrooms (Simon, 1993). In measurement division, the number of pieces in each group is known and the number of groups is unknown. For example: *Jamie has 63 candies. She wants to give each of her friends 21 candies. How many friends can she share with?* In this example, the number of candies is known, and the number of friends is unknown, which is more difficult than being able to partition out to each friend as in sharing division (Carpenter et al., 1996). Another way to think about measurement division is determining how many times a given quantity is contained in a larger quantity. With measurement division, the only constraint is that the dividend must be larger than the divisor. Also, if the quotient is a whole number, a problem can be viewed as a repeated subtraction.

With sharing division, when dividing whole numbers, the quotient is always smaller than the dividend. With measurement division, the only constraint is that the dividend must be larger than the divisor. When dealing with fraction division, however, the quotient is usually larger than the dividend (Simon, 1993). When elementary students learn whole number division, if they are only exposed to sharing division, it could lead to the misconception that multiplication always makes bigger and division always makes smaller (Greer, 1992), which is not true for fraction division. This misconception during whole number division will be detrimental to the conceptual development of future division concepts when learning fractions and integer division for example.

The distinction between sharing and measurement division is important in students' conceptual development of division concepts, and in turn, is also a critical development for pre-service teachers. If pre-service teachers do not properly develop the conceptual knowledge of these important concepts, it is unlikely they will provide learning experiences for their students that lead to this type of understanding (Ma, 1999).

Children progress through stages as they develop a conceptual understanding of number. Conservation of number serves as the foundation for number sense. Then children must be able to recognize ten as a single entity, comprised of ten ones. Once this understanding occurs, children can then progress to solving problems without manipulatives. Other developmental milestones include understanding the meaning of each digit in a two-digit number and the ability to compose and decompose those numbers (Ross, 1986). Once children develop a conceptual understanding of number sense they may move on to number operations. Research shows that children are less likely to make errors if they are afforded the opportunity to develop their own strategies for number operation. Additionally, allowing children to invent strategies deepens their understanding of the concepts, and they are more likely to be able to make the connection to algorithms later (Carpenter et al., 1998). The aforementioned research regarding how children learn place value and number operation was utilized in the creation of the Elementary School

Mathematics Course. Additionally, Realistic Mathematics Education (RME) theory and hypothetical learning trajectories (HLTs) were used to design the course.

Realistic Mathematics Education

Realistic Mathematics Education (RME) is an instructional theory that allows students to deepen their conceptual knowledge by exploring mathematics to build an understanding of the content. The theory rejects traditional instructional design, which follows a task analysis approach where students are given a series of steps to follow to solve a problem. RME is comprised of three design heuristics including guided reinvention, didactical phenomenology, and emergent modeling (Gravemeijer, 2004). Guided reinvention is a key element of RME and asserts that students should be afforded the opportunity to invent the mathematics for themselves. Rather than give them a pre-mapped out procedure for solving a problem, students should be guided to develop their own solution strategies. By providing contextual problems for students, they are able explore the mathematics on their own and draw their own conclusions, providing a strong conceptual base of knowledge (National Research Council, 2001; NCTM, 2014). When using guided reinvention as a means of developing a learning trajectory, instructors consider how children historically build their understanding of mathematics.

Similar to guided reinvention is didactical phenomenology, which considers the tasks and activities utilized in conjunction to the intended learning trajectory. More specifically, the tasks and activities should be experientially real, making the mathematics more meaningful to the students. For example, rather than giving students a number sentence like 7 + ? = 12, students would be presented with a real world context such as *Marcus has 7 pencils. Shana also has some pencils. Together they have 12 pencils total. How many pencils does Shana have?* This problem involves a realistic situation in which students can relate to the mathematics. Additionally, these situations allow for more emergent modeling, by use of drawings, diagrams, or manipulatives; the final component of RME (Cobb, 2000; Gravemeijer, 2004). Emergent modeling refers to the levels of modeling that play out as students become more and more sophisticated in their reasoning. Low level modeling could involve drawings, or use of manipulatives. Students eventually are able to progress to higher level, more general, abstract modeling, which could be the use of algorithms (Gravemeijer, 2004). Real world situations afford students the opportunity to build understanding for themselves, and provides a conceptual base of understanding, which is essential for procedural fluency (National Research Council, 2001; NCTM, 2014).

The abovementioned Elementary School Mathematics Course for education majors utilizes elements of RME by integrating the base 8 number system into instruction. Prospective teachers are told they are living in base 8 world and need to think and perform operations in base 8. Although living in base 8 world is not necessarily “real” to prospective teachers, an activity called “Candy Shop” is something they can envision (Bowers, Cobb & McClain, 1999). Candy Shop is a candy packaging activity that aids in the understanding of place value and operations. In Candy Shop, prospective teachers are presented with pictures of boxes, rolls, and pieces of candy that they can compose and decompose to solve problems in and out of context. Research has shown that children exposed to the candy shop scenario improved their understanding of place value, number operations, and conceptual understanding of standard algorithms (Bowers et al., 1999; McClain, 2003).

Initial research on the candy packaging activity was done with children in base 10. One box contained 100 rolls and one role contained 10 pieces of candy (Bowers et al., 1999). Because this study is with pre-service teachers in a base 8 environment, one box contains 64 rolls and one roll contains 8 pieces of candy. A special language is utilized so that prospective teachers will not confuse base eight with base 10. For example, "one-hundree", written as 100, denotes 64 pieces, and "one-ee-zero", written as 10, denotes eight pieces (Andreasen, 2006). Instruction in base eight was also chosen because prospective teachers typically enter the Elementary School Mathematics Course with knowledge of algorithms and procedures typical from exposure to the K-12 school system (Conference Board of the Mathematical Sciences, 2001). It can be difficult for them to gain a conceptual understanding of the mathematics if they already have those rules and procedures in their problem-solving arsenal. RME instructional design helps prospective teachers develop a conceptual understanding of the content as well as mathematical reasoning in the sophisticated manner called for by current reforms.

Hypothetical Learning Trajectory

In addition to RME, a hypothetical learning trajectory (HLT) was utilized in the design of the Elementary School Mathematics Course. A HLT is "the teacher's prediction as to the path by which learning might proceed" (Simon, 1995, p. 135). The learning trajectory is hypothetical because one cannot be certain of prospective teachers' progression of learning until they are observed completing tasks. Upon completion of the tasks, the hypothetical learning trajectory becomes an actualized learning trajectory (ALT). The ALT for this study is based upon research conducted in previous iterations of the course (Andreasen, 2006; Roy, 2008) and upon research on children's progression of learning whole number concepts and operations. Research on children was used due to the limited research regarding pre-service teachers in this context (Safi, 2009). The goal of the base 8 unit in the Elementary School Mathematics Course is to influence pre-service teachers' mathematical knowledge in regards to number concepts and operations. In order to achieve this goal, the ALT used for this study is comprised of three pertinent phases as illustrated in Table 1.

Table 1-Actualized Learning Trajectory, Roy (2008)

HLT phase	**Learning goal**	**Supporting tasks for instructional sequence**	**Supporting tools**
Phase One	Counting	Counting; Skip Counting; Open number line Problems; Counting Problem Set #1; Counting Problem Set # 2	Double 10-Frames, and Open number line
Phase Two	Unitizing, Flexible representations of numbers	Estimating with Snap Cubes; Candy Shop 1; **Candy Shop Exercise 1 (Exit Question); Candy Shop Exercise #2**; Torn Forms	Snap Cubes; Boxes, rolls and pieces; Inventory forms

HLT phase	Learning goal	Supporting tasks for instructional sequence	Supporting tools
Phase Three	Invented Computational Strategies	Candy Shop 2; Candy Shop Inventory; Candy Shop Addition and Subtraction; Inventory Forms for addition and subtraction (in context); Inventory Forms for addition and subtraction (out of context); 10 Dot Frames; Broken Machine; Multiplication Scenario, Multiplication Word Problems; Division Word Problems	Boxes, rolls, and pieces; Inventory forms; dot arrays; and Open number line

Prospective teachers begin with counting in base eight. They practiced using tasks such as skip counting and open number line problems with tools including double 10-frames and the open number line. The second phase includes unitizing and flexible representations, where prospective teachers are introduced to the Candy Factory scenario. Their tasks include estimation and problems involving packaging and un-packaging candy. Prospective teachers utilize snap cubes, the idea of boxes, rolls, and pieces, and inventory forms to help them complete the aforementioned tasks. In phase three prospective teachers develop invented computational strategies in base eight. They continue to use the Candy Factory scenario and inventory forms to perform transactions. They are introduced to multiplication scenarios and work with multiplication and division word problems. Prospective teachers are given access to previously mentioned supporting tools as well (Roy, 2008). All tasks were completed in base-8.

The HTL was developed to provide pre-service teachers with experiences similar to those children experience when developing number sense. The course intended to aide in the development of their conceptual understanding of "ten" by exposing them to an alternate base. Andreasen (2006) found that when exposed to base eight, pre-service teachers tended to progress through similar stages of conceptual development as children.

Base 8 Instruction

Pre-service teachers typically enter universities with a traditional mathematics background in which they learned rules and algorithms. It can be difficult to get them to see the value in learning mathematics in a conceptual way because they feel they already understand how the mathematics works. Research has shown that to break these cognitive patterns, prospective teachers must be placed in situations where their beliefs are challenged. This will lead them to build new conceptual structures (von Glasersfeld, 1983). One possible option to challenge pre-service teachers pre-existing beliefs is to teach them in a different base. This forces them to learn the mathematics as if they were learning it for the first time. Additionally, base 8 was selected rather than a different base, like base 4 for example, because the values in this particular base do not immediately reach their maximum place value, or ten. Another reason is that both base-8 and base-10 are even, so they share common base system characteristics and eventually lead to common strategies used to solve whole number operations. There is a small body of research on instructing pre-service teachers in a different base.

McClain (2003), whose research efforts helped shape the design of the course in this study, focused on a methods course ($N = 45$) that was designed to prepare pre-service elementary teachers to implement mathematics curriculum in a conceptual manner. Emphasis was placed on solution processes and participants' explanations and justifications of solutions. The vehicle for such practices was an instructional sequence called Candy Factory, which involved learning place value by thinking of a candy shop. In McClain's study, single pieces of candy were packaged in rolls of 8, and there were 8 rolls in a box, just as the pieces in this study are packaged. This was similar to the third grade activity utilized in the aforementioned Bowers et al. (1999) study, in which candies were packaged in tens to help develop place value and number concepts in children. McClain used base 8 for pre-service teachers to "problematize the mathematics" (McClain, 2003, p. 286). Data was collected via video recordings, student work samples, and a daily reflective journal. Results indicated that teaching in this fashion resulted in an increase in pre-service teachers' ability to reason mathematically. It forced them to assume a dual role, one of student, and one of prospective teacher. They came to realize the importance of teaching mathematics in a conceptual way.

In the second study ($N = 24$), researchers observed a mathematics content course in which number concepts and operations were taught using base 8 (Yackel, Underwood, & Elias, 2007). As in this study, Yackel and colleagues adopted special wording for base 8, i.e. eight was referred to as one-e. A university goal in this study was to develop a belief in pre-service teachers that "mathematical knowledge is not a pre-given, external body of knowledge to be acquired, but rather is built up by cognizing individuals as they engage in mathematical activity, including discussions of their own and others' mathematical actions" (Yackel et al., 2007, p. 363). In other words, mathematical knowledge is not something that one can be told and one can absorb. In order for one to glean mathematical knowledge, he must spend time developing concepts, making connections, and constructing an understanding for himself. Another goal of the university was for pre-service teachers to develop conceptual basic number facts, number relationships, grouping strategies, and thinking strategies. Researchers concluded that both goals were achieved. They arrived at this conclusion based upon the changes they witnessed in prospective teachers over the course of the class, and also how they performed and responded in subsequent courses. Researchers reported that prospective teachers were consistently telling their instructors how learning in base 8 gave them a new view of arithmetic and how it can be taught.

Three significant research projects regarding instruction in base 8 took place in a similar environment to this study. Andreasen (2006), Roy (2008), and Safi (2009) conducted research that examined prospective teachers understanding of place value and number operation albeit in different iterations of the elementary mathematics course. Similar to McClain (2003), Andreasen used the theoretical framework of how children learn place value and number operations to design a hypothetical learning trajectory and mathematical tasks. The focus of her research was on the instructional sequence and the social context of the classroom and ways in which both contributed to the development of pre-service teachers' understanding of place value and number operations. Unlike McClain (2003), the majority of the place value and number operations unit in Andreasen's study was in base 8. Pre-service teachers were expected to think and reason through problems entirely in base 8, and only switched over to base 10 at the end of the unit. The reasoning for the switch was to encourage a conceptual connection between reasoning in base 8 and base 10. Additionally, the switch provided an opportunity for prospective teachers to discuss alternate problem solving strategies and potential comprehension strategies of children (Andreasen, 2006). Traditional and alternative algorithms were also presented in accordance to prospective teachers' reasoning. Andreasen (2006) established the following social norms in the classroom: "1) explaining and justifying one's solution and solution processes and 2) making sense of other prospective teachers' solutions by asking questions of classmates or the instructor" (p. 160). Sociomathematical norms that were partially established during the teaching experiment were determining: "1) what constituted a different mathematical solution and 2) what made a good explanation" (p. 160). Also established in the classroom were mathematical practices including unitizing, flexibly representing whole numbers, and reasoning about operations. The hypothetical learning trajectory was realized through pre-service teachers' learning, which was supported by tasks incorporated as part of the instructional sequence (Andreasen, 2006).

Using the same environment and the same learners, Roy (2008) built on Andreasen's work, focusing on mathematical practices established by pre-service teachers in a learning environment positioned entirely in base 8. Roy's research was centered on individual pre-service teacher's conceptual development of whole number concepts and operation as it occurred within the established classroom environment. Roy refined the HLT described by Andreasen (2006) to an ALT (see Table 1) that is utilized in this study. Roy (2008) concluded that the revised instructional sequence supported the following normative mathematical practices "(a) developing small number relationships using Double 10-Frames, (b) developing two-digit thinking strategies using the open number line, (c) flexibly representing equivalent quantities using pictures or Inventory Forms, and (d) developing addition and subtraction strategies using pictures or an Inventory Form" (p. 136).

Like the current study, Roy (2008) utilized the research efforts of Hill and colleagues (2004) in developing a measure designed to measure mathematical content knowledge necessary to teach elementary mathematics. As in this study, Roy (2008) selected and administered items from the Mathematical Knowledge for Teaching (MKT) Measures (Hill et al., 2004) before and after the base 8 unit of the elementary mathematics course. There was a statistically significant difference in mean scores on items from the MKT prior to and subsequent to the instructional sequence (Roy, 2008), indicating the possibility that pre-service teachers' content knowledge for teaching increased as a result of classroom instruction.

Safi (2009) extended the research of Roy (2008) and Andreasen (2006) by examining the way individual pre-service teachers' progress in their learning of number concepts and operations. Using the emergent prospective as theoretical framework, Safi outlined and analyzed the mathematical conceptual understanding of two pre-service teachers. Like the current study, Safi selected the two participants based upon their scores on the MKT (Hill et al., 2004), one scoring lower and one higher.

Cordelia scored in the low mathematical content knowledge category on the MKT. She was interviewed before the base 8 unit and showed a very procedural, rule based understanding of how to complete problems. By the end of the unit, she had progressed in her understanding of how to solve problems. She was able to explain how to solve the problems, however she lacked the ability, and therefore seemingly the conceptual understanding, to justify her solution strategies.

Claudia scored in the upper quartile on the MKT. In the pre-interview, she demonstrated the ability to understand the procedure, and also the conceptual component to addition and subtraction problems. She was able to make a connection to place value in regard to multiplication. By the end of the base 8 unit, Claudia was able to make a solid conceptual connection for all problems, i.e. she was able to not only understand place value concepts in the multiplication problem, but also break down digits into partial products. Additionally, she illustrated the ability to make connections between different solution strategies, and explore and make sense of others' solutions. Safi's findings provided a psychological perspective as it relates to the social perspective's mathematical practices.

In addition to the literature in which base 8 was utilized as an instructional tool, there was a research project that utilized base 5 as a tool for developing prospective teachers' mathematical knowledge (Fasteen, Melhuish , & Thanheiser, 2015, p. 85). The setting for this study was a three-term mathematics content course sequence designed for pre-service K–8 teachers. Place value and number operation were taught in base 5 as well as base ten, with a focus on reasoning through solution methods. Researchers observed one small group ($N = 5$) during the multiplication portion of the course and collected data in the form of field notes and video observations. Data gleaned from a pre-test multiplication task revealed that the pre-service teachers' knowledge regarding what authors refer to as the "Times Base Rule" was limited. The Times Base Rule refers to the concept that "multiplying a number by 10 causes all of the digits in that number to be shifted to the left one place value and a zero to be appended at the end of the number (in the ones place)" (Fasteen et al., 2015, p. 85). When given a task and asked to justify their solution, pre-service teachers could only explain what they did and not why they did it. Most justifications were explanations of the multiplication algorithm and prospective teachers expressed discomfort with having to justify their solutions. Researchers reported that it was difficult for pre-service teachers to make the conceptual shift on their own. It was through instructor prompts that they were able to make the shift from explaining their solutions via the standard algorithm, to explaining via the multiplicative structure of the place value system (Fasteen et al., 2015). Researchers concluded that instruction in base five placed pre-service teachers in a situation where they could not rely on prior procedural knowledge and gave them an opportunity to focus on sense making. Instruction in an alternate base provided a context to explore the conceptual understanding of whole number multiplication, an experience that may not be typical in university mathematics content courses.

Both McClain (2003) and Yackel et al. (2007) conducted practitioner based studies on a pre-service teacher education course in which teachers learned number concepts and operations in base 8. Both studies utilized a candy shop type activity in which pre-service teachers work in the context of boxes, rolls, and pieces. Eight candies are in a roll, eight rolls are in a box. Candies can be packaged and unpackaged accordingly. McClain (2003) used the candy shop scenario "(a) to develop an understanding of the multiplicative relations within place value, and (b) to develop ways of symbolizing transactions in the Candy Factory dealing with buying or adding and selling or subtracting candies" (p. 286) the same goals that are set for elementary prospective teachers. McClain initially focused on addition and subtraction problems in base 8, which was unsuccessful because prospective teachers began discovering methods to convert back to base 10 and still use their old algorithms. In a second attempt McClain focused on grouping and regrouping the candy, which was much more successful in helping prospective teachers see how to perform calculations in base 8. McClain saw that when working with base 8 prospective teachers came up with invented algorithms, and shared them with their group members, which fostered a deeper, more conceptual understanding. Yackel and colleagues (2007) learned from McClain's (2003) struggles and began the base 8 unit with practicing counting in base 8, then moved on to the candy shop activity involving grouping and regrouping.

While McClain (2003) and Yackel and colleagues (2007) focused on addition and subtraction operations, Fasteen and colleagues (2015) built on that work to include multiplication operations. Additionally, data collection and analysis was more rigorous and structured in that researchers conducted interviews from a specific group of prospective teachers. This qualitative study demonstrated that making sense of multiplication in a different base was not trivial, and provided insight as to how one small group of pre-service teachers was able to make sense of the multiplicative structure of the place value system.

Andreasen (2006), Roy (2008), and Safi (2009) focused on all number operations, and how instruction in base 8 influenced prospective teachers' understanding of place value and operations in different iterations of an elementary mathematics course. Andreasen (2006) designed an instructional sequence based upon prior research regarding instruction in base 8 and children's understanding of place value and whole number operations. Her research focused on the social context of the classroom. Roy (2008) built on Andreasen's work and concentrated on individual pre-service teacher's conceptual development by taking into account specific learning goals established in the classroom. Safi (2009) extended their work by providing a deeper insight into the way *individual* prospective teachers conceptual understanding of whole number concepts and operations develops in a social context. All three researchers reported that pre-service teachers experienced an increase in their conceptual understanding as a result of taking the course.

One key aspect of elementary teacher preparation programs is mathematics content courses. These courses are traditionally taught in base 10. One could argue that teacher candidates already have a firm grasp on their elementary arithmetic and would find these content classes boring or not useful. Even if the classes were taught in a fashion that fostered a conceptual understanding, prospective teachers could still get through using rules and procedures. Little research has been conducted which involves pre-service teachers learning number concepts and operations in a different base. Further research is needed in the field of teacher preparation, specifically in regards to teaching prospective teachers in base 8 to increase teacher knowledge. If this topic is more thoroughly researched, it could mean a lot for the improvement of teacher preparation programs in accordance with current standards and calls for reform (CCSSI, 2010; NCTM, 2014). Research has shown that teachers' mathematical knowledge for teaching is significantly related to student gains in achievement (Hill et al., 2005). This study aims to add to the small pool of research regarding pre-service instruction in an alternate base by measuring the construct of teacher knowledge.

The conclusions in the aforementioned base 8 studies are interesting and offer solutions for pre-service teacher preparation program reform. The studies were mostly qualitative in nature and pose limitations and challenges. Small sample sizes make results difficult to generalize to the general population, it is difficult to collect and analyze data appropriately, and researchers' personal bias could influence results (Creswell, 2007). The researchers in these studies discuss how prospective teachers' abilities to reason mathematically and conceptual fluency improved. This study aims to add to the existing research by adding the analysis of prospective teacher CCK and SKC in hopes of continuing improving teacher education programs.

CHAPTER 3:
RESEARCH DESIGN AND METHODOLOGY

Research Design

This study utilized a mixed methods design. The quantitative intervention study was quasi experimental in design. Due to time and participant availability constraints, the researcher used a nonequivalent design in which there was no random assignment of participants (Gall, Gall, & Borg, 2007). There was not a control group. Design followed O_1 X O_2 format in that all participants received a pre-test, treatment, and post-test (Shadish, Cook, & Campbell, 2002). The researcher hoped to utilize this design to draw inferences in regards to the influence of the treatment on mathematical knowledge for teaching.

The researcher will utilize the grounded theory method to help explain quantitative results. Interviews will be conducted to help paint the picture of how prospective teachers' knowledge changes as a result of the course. Additionally a series of open coding will be utilized to build a theory.

Population and Sampling

Convenience sampling was employed, as participants were selected from readily available Elementary School Mathematics Content Course sections at a college located in the southeast United States (Gall et al., 2007). Courses were chosen based upon the method used to teach number concepts and operations. There were three Elementary School Mathematics Course sections available with the number of participants totaling 95. Of the 95 students enrolled, 91 students participated in the study. There was a different instructor for each section of the course. Two instructors were faculty members in the college of education and one was a doctoral student in the college of education. Only courses where the instructor taught number concepts and operations in base 8 were selected.

Power analysis for this study suggested a minimum of 90 participants, a significance level of .05, a power of .9, and a medium effect size of .25 (Faul, Erdfelder, Buchner, & Lang, 2009). A significance level of .05 was appropriate as it reduced the chances of making a type one error. The medium effect size of .25 demonstrated practical significance. The power of .9 implied that there is a 90% chance that the researcher did *not* reject the null hypothesis when she should have.

Course Materials

In the base 8 unit, students engaged in many different activities which were designed to develop their conceptual understanding of place value and whole number operations. Candy Shop was the main vehicle for aiding in this conceptual development. In Candy Shop, prospective teachers are presented with pictures of boxes, rolls, and pieces of candy that they can compose and decompose to solve problems in and out of context. Additional materials were utilized in each phase of the course. In phase 1, which consisted of re-learning how to count in base 8, students had tasks of counting, skip counting, and open number line problems. They solved these tasks using double 10-frames, and open number lines. In the second phase, students were introduced to unitizing and flexible representations. In this phase, prospective teachers were introduced to the Candy Factory scenario. Their tasks included estimation and problems involving packaging and un-packaging candy. Prospective teachers utilized snap cubes, the idea of boxes, rolls, and pieces, and inventory forms to help them solve problems and develop place value and number sense. In phase three prospective teachers advanced to develop invented computational strategies in base 8. They continued to use the Candy Factory scenario and inventory forms to perform transactions. They were introduced to multiplication scenarios and worked with multiplication and division word problems. Prospective teachers were given access to previously mentioned supporting tools as well (Roy, 2008). All tasks were completed in base-8.

Data Collection and Procedures

Research took place at a college located in the southeast United States in an Elementary School Mathematics Course for Teachers (in the College of Education) for undergraduate elementary education prospective teachers in the spring of 2016. Before collecting data the researcher obtained approval from the institutional review board (IRB), and requested permission from each of the elementary mathematics content course instructors to conduct research in his/her class. Once the researcher obtained permission from the course instructors, she visited individual classrooms to speak with prospective teachers in each class about aspects of the study and explain confidentiality. Prospective teachers were also given the opportunity to opt out of the study.

Items in the Mathematical Knowledge for Teaching (MKT) Measures

Hill and colleagues sought to determine what subject matter knowledge is required for teaching, and to develop a measure that could examine those constructs reliably. In 2001 they created 138 test items in two content areas (1) number concepts and operations, and (2) patterns, functions and algebra. Items also crossed with two domains (1) knowledge of content, and (2) knowledge of students and content. Three different forms were constructed and tested on in-service teachers. Focusing on number concepts and operations, researchers ran factor analysis on the data and were able to determine that there is a type of knowledge specific to teaching (SCK), and that it is possible to write items that represent each factor, CCK and SCK. Results indicated that CCK explained between 72% and 77% of the variation in responses on each of the three forms, and SCK explained between 12% and 23% of all test items. Percentages do not add up to 100 because there is some overlap in questions that could represent both CCK and SCK. Reliabilities were computed using BILOG, a program that enables item response theory analyses (Hambleton, Swaminathan, & Rogers, 1991), and results showed reliabilities were “good to excellent, ranging from 0.71 to 0.84”. Researchers also chose not to divulge which items specifically measured SCK and CCK because of the overlap in questions that could represent both constructs (Hill, et al. 2004).

Mathematical knowledge for teaching is the mathematical knowledge that separates teachers from other professions that use mathematics (Ball, Hill, & Bass, 2005). This is the type of knowledge that the MKT is designed to measure. Developers of the MKT sought to design items that measured the different types of knowledge required for teaching elementary mathematics (Hill et al., 2004). Multiple forms of the measure were developed, each having a reliability that ranged from .75 - .85 (Hill, 2007). For this study, the Elementary Number Concepts and Operations measure was chosen because items dealt with content presented in the base 8 unit of the course utilized in the research. The Elementary Number Concepts and Operations measure contains items that can be differentiated into common content knowledge or specialized content knowledge. Common content items were constructed to measure mathematical knowledge such as calculating, making correct mathematical statements and the ability to solve problems accurately (Ball, Hill, & Bass, 2005). See Figure 1 for an example of a common content knowledge item.

Ms. Dominguez was working with a new textbook and she noticed that it gave more attention to the number 0 than her old book. She came across a page that asked students to determine if a few statements about 0 were true or false. Intrigued, she showed them to her sister who is also a teacher, and asked her what she thought.

Which statement(s) should the sisters select as being true?
(Mark YES, NO, or I'M NOT SURE for each item below.)

	Yes	No	I'm not sure
a) 0 is an even number.	1	2	3
b) 0 is not really a number. It is a placeholder in writing big numbers.	1	2	3
c) The number 8 can be written as 008.	1	2	3

Figure 1 Common Content Knowledge Item, Ball, Hill, and Bass (2005)

Researchers also developed items designed to measure specialized content knowledge. For these items, test takers needed knowledge of representing mathematical ideas and operations, giving alternatives, suggesting explanations, and understanding student invented algorithms (Ball, Hill, & Bass, 2005). See Figure 2 for an example of a specialized content knowledge item.

Imagine that you are working with your class on multiplying large numbers. Among your students' papers, you notice that some have displayed their work in the following ways:

Student A	Student B	Student C
35	35	35
× 25	× 25	× 25
125	175	25
+ 75	+ 700	150
875	875	100
		+ 600
		875

Which of these students is using a method that could be used to multiply any two whole numbers?

	Method would work for all whole numbers	Method would NOT work for all whole numbers	I'm not sure
a) Method A	1	2	3
b) Method B	1	2	3
c) Method C	1	2	3

Figure 2 Specialized Content Knowledge Item, Ball, Hill, and Bass (2005)

Procedure

In this study, the MKT was administered to all prospective teachers in the study once at the beginning of the base 8 unit, and once at the conclusion of the base 8 unit. Prospective teachers were given as long as they needed to complete the measure, with most prospective teachers taking approximately 30 minutes to finish all questions. Items were selected for the measure in regard to their emphasis on either common content knowledge (CCK) or specialized content knowledge (SCK). Writers and researchers of the MKT did not provide classifications for each individual item on the measure. They only indicated that items could be classified as CCK and SCK. They also indicated that some items could classify as both CCK and SCK, creating overlap. As a result, the researcher synthesized prior researchers' definitions of CCK and SCK (Ball et al., 2005; Hiebert & Lefevre, 1986; Hill & Ball, 2004; Skemp, 1978; Thompson et al., 1994) in order to determine which questions fit into each category. There were 14 items on the MKT with sub questions totaling 25 questions. The researcher selected 11 of the 25 items to represent CCK and 6 to represent SCK. The remaining 8 items consisted of items that were either not considered whole number operations, or were determined to overlap CCK and SCK.

CCK items were selected if they measured mathematical knowledge such as calculating, making correct mathematical statements, and the ability to solve problems using standard algorithms, rules, or procedures. There were three items on the MKT that were selected to represent CCK. Those items had sub questions, which totaled 11 questions that were used to measure CCK. SCK items were selected if they required participants to represent mathematical ideas and operations, provide justifications, or to understand student invented algorithms (Ball et al., 2005). There were six items total selected to measure SCK. A demographics questionnaire was also administered along with the pre-test. Once each administration of the measure was completed, the researcher input the raw data into a spreadsheet. Raw data was then scored as correct or incorrect, and total scores were calculated.

In addition to the quantitative piece, the researcher used interview data to help explain results from the measure and to more deeply explore prospective teacher understanding. Since there were two constructs of interest, CCK and SCK, and the researcher examined how varying levels of each type of knowledge changed as a result of the unit, data from four prospective teachers was ultimately utilized: (1) a student with initial low CCK and low SCK, (2) a student with initial high CCK and low SCK, (3) a student with initial low CCK and high SCK, and (4) a student with initial high CCK and high SCK. In order to ensure that each type of student was clearly represented, the researcher initially selected 16 prospective teachers to interview: four for each aforementioned category. Pretest results for each type of knowledge (CCK and SCK) was utilized in the selection of the 16 prospective teachers. If a student scored low on CCK questions and high on SCK questions on the pre-test, that person was selected to be interviewed the category initial low CCK and high SCK.

Interviews

Prospective teachers were interviewed at the beginning of the base 8 unit and again at the end of the unit. The individual interviews were conducted by this researcher and lasted approximately 45 minutes. All interviews were audio recorded and later transcribed. Prospective teachers were given a sheet with the problems pre-printed and encouraged to show their work and their thought process on the sheet. As prospective teachers were solving each problem, the researcher asked pre-determined, open-ended questions to gain the deepest insight into their thinking. Prospective teacher responses guided the researcher's questions, and as a result, not all questions were asked of all interviewees. A copy of the questions used in the interviews is included in Appendix C. As prospective teachers worked and explained their answers, the researcher took detailed field notes. Student work, transcriptions, and field notes were all included as part of the data analysis.

The researcher used data collected from the interviews as additional support for measure results. The data provided a rich insight into prospective teachers' thinking and their understanding of place value and number operations before and after being exposed to the base 8 unit. Additionally, this data helped answer research question 3: How do prospective teachers' participation in the instructional unit help explain the change, if any, in their common content knowledge and specialized content knowledge?

During the interviews, prospective teachers were given story problems representing addition, subtraction, multiplication, and division situations, which were the four operations covered in the base 8 unit of the course. These questions were answered in base 10, as one of the purposes of the unit was to encourage a conceptual connection between reasoning in base 8 and base 10, and the researchers purpose was to analyze the effects of instruction in base 8 on prospective teachers' understanding of place value and number operations in base 10. Interviews were audio recorded and transcribed.

As prospective teachers were working through the problems, the researcher asked probing questions to glean rich data: How did you decide what to do to solve the problem? Can you explain a little more what you did here? Why did you cross out that number? Did you change the value of the larger number? Why did you put a 1 next to that number? Can you explain your grouping strategy in this problem? Can you explain why your method works? Will that strategy always work? How do you know your answer is correct? Can you write a number sentence to describe the problem? Can you solve the problem a different way? How might you teach a child to solve this problem? For division problem: What does this group represent? Additional questions were asked depending upon unique student responses (Thanheiser, 2012).

For the interviews, prospective teachers were given problems represented in Table 2.

Table 2 Interview Questions

Question 1	Tyler's bakeshop sells chocolate cupcakes. Tyler sells 255 chocolate cupcakes to a customer. Now there are 261 chocolate cupcakes in the bakeshop. How many chocolate cupcakes were in the bakeshop to start with?
Question 2	There are 253 people on a cruise ship. Some more people board the cruise ship. Now there are 520 people on the cruise ship. How many people boarded the cruise ship?
Question 3	There are 16 sports drinks in a package. How many sports drinks are in 12 packages?
Question 4	Alex has 134 stickers. He wants to give each of his friends 12 stickers. How many friends can he share with? Does he have any stickers left for himself? If so, how many?
Question 5	This was a question from the MKT which analyzes prospective teachers' ability to analyze student solution strategies. The researcher was not able to obtain permission from writers to provide a copy of this question, but was advised to provide a detailed description of the question. Question 5 showed three different student solution methods to a three digit subtraction problem where, if one were to perform the standard algorithm, regrouping was required from the tens place to the ones place, and from the hundreds place to the tens place. The first method showed an invented counting up strategy where the student began with the smaller number and added on until the larger number was reached. The student then added up all of the numbers he had added

	on to arrive at the answer. In the second method, the student broke up the smaller number into hundreds, tens, and ones. The student then subtracted each part from the larger number. In the third method, the student set up the problem as one would a standard algorithm. He then added the same number to the top and bottom numbers until the bottom number had zeros in the tens and ones place. This strategy allowed the student to subtract without regrouping because the tens and ones place on the bottom number were zero. Prospective teachers were required to tell whether each method was correct, and whether the method would work for any two whole numbers.

Questions 1-4 were developed to represent each of the four operations covered in the base 8 unit of the course. Additionally, they were designed to represent CCK because they require basic procedural knowledge that allows one to solve a math problem (Hill & Ball, 2004). Through questioning, the researcher determined if prospective teachers have conceptual understanding (SCK) of the mathematics presented i.e. can they reason through why their solutions/algorithms work, represent mathematical ideas and operations (i.e. write a number sentence), solve the problem in another way, and explain and justify their solutions (Hill, Ball, & Bass, 2005; NCTM, 2014)? Question five was chosen specifically to represent the construct SCK because it requires prospective teachers to provide explanations and justifications for solutions, evaluate and explain atypical student solutions, and reason through why algorithms work (Hill et. al, 2005).

Questions 1-4 were meticulously constructed in order to collect the richest data. Results from Cognitively Guided Instruction research were utilized in selecting questions as well as the questions ability to help answer the research questions. In regards to problem 1 and 2 (addition and subtraction respectively), action problems were chosen because in order to analyze interviewees responses from a conceptual/non-conceptual lens, interviewees need to be able to decontextualize the problem. Start unknown and change unknown problems were selected so that the researcher could analyze whether or not the interviewee could decontextualize the problem and also because start unknown and change unknown problems are more challenging than result unknown (Carpenter, Fennema, Franke, Levi, & Empson, 1999), allowing the researcher to collect rich data in regards to prospective teachers' understanding.

In question three of the interview protocol, the multiplication scenario that was chosen was a double digit by a double digit. In order to analyze prospective teachers' conceptual understanding of multiplication, the researcher used the notion of partitioning. Partitioning is a conceptually based strategy that involves breaking down one for both factors in a multiplication problem into smaller whole numbers to make a problem easier (Baek, 1998). In this study, the simpler multiplication problems resulting from breaking apart numbers in a more complicated multiplication problem are referred to as partial products. For example, in question three, the number sentence is 12x16. There are several strategies for breaking apart this problem to create a simpler problem. One could also break apart the 12 into 10 and 2, in which case one would be able to then multiply the 16 by both the 10 and 2, i.e. 12x16 = (10+2)x16 = (10x16) + (2x16). One could break apart the 16 into 10 and 6, in which case one would be able to then multiply the 12 by both 10 and 6, i.e. 12x16 = 12x(10+6) = 12x10 + 12x6. Finally, one could also break apart both numbers, i.e., 12x16 = (10+2) x (10x6), giving the following partial products: 10x10=100, 10x6=60, 2x10=20 and 6x2=12. Adding up all of the products yields the final solution.

Prospective teachers were also asked if it matters whether one multiplies 12x16 or 16x12.

A component of conceptual understanding involves the ability to make connections between contextual and symbolic representations of a mathematics problem (NCTM, 2014). The capability to decontextualize a problem into the proper symbolic number sentence, representing groups times number of pieces in each group, is an indicator of conceptual understanding (Steffe, 1994). The multiplication problem in this study describes a situation in which there are 16 sports drinks and one is asked to find how many sports drinks are in 12 packages. In this case the groups are represented by packages and sports drinks represent the number of groups. The correct number sentence for this problem is 12x16; 12 groups of 16. Additionally prospective teachers were asked if there was another way to solve the problem, as understanding multiple solution strategies is a component of SCK (Hill et al., 2004).

The division scenario in question four was written as measurement, or quotitive division. In measurement division, the number of pieces in each group is known and the number of groups is unknown. In question four, the number of stickers is known and the number of friends is unknown, which is more difficult than being able to partition out to each friend as in sharing division (Carpenter et al., 1996). This type of problem was selected because it is difficult for adults to distinguish between the two types of division. Most adults only recognize problems as division because they do not think about the actions or relationships happening in the problem (Carpenter et al., 1996). Additionally, sharing division is the most commonly recognized problem type and is easily modeled, so measurement division was chosen because since it is more difficult, it will provide for rich data. If interviewees can make the distinction, it will strengthen the argument that they conceptually understand the mathematics (Simon, 1993).

Question 5 was selected as it was one of the items on the MKT that was identified to measure SCK. It follows the description of SCK, as it requires prospective teachers to "appraise the mathematical validity of alternative solution methods for a problem" (Hill et al., 2005, p. 377). Prospective teachers were asked if the solution methods would work for any two whole numbers.

Theoretical Framework

Questions 1-4 were developed to represent each of the four operations covered in the base 8 unit of the course. Prospective teachers were considered to have high CCK if they could answer questions 1-4 correctly using the standard algorithm. The questions were designed to represent CCK because they require basic procedural knowledge that allows one to solve a math problem (Hill & Ball, 2004).

While many pre-service teachers can perform the standard U.S. algorithms for addition and subtraction, they fail to understand the meaning of the algorithms (Ball, 1988; Southwell & Penglase, 2005). For example, a pre-service teacher might add 5 and 8 but not understand why they write the 3 below and regroup the 1 to the next column. If pre-service teachers have conceptual understanding, they recognize "the essence of the numeration system—the numerals have different *values* depending on their *place*" (Ball, 1988, p. 49).

Conceptual understanding for addition and subtraction will be analyzed using Thanheiser's (2009) model for categorizing prospective teachers' understanding of multi-digit whole numbers. In this model, prospective teachers may be categorized into one of four groups.

1. *Reference units.* PSTs with this conception reliably1 conceive of the reference units for each digit and relate reference units to one another, seeing the 3 in 389 as 3 *hundreds* or 30 *tens* or 300 ones, the 8 as 8 *tens* or 80 *ones*, and the 9 as 9 *ones*. They can reconceive of 1 *hundred* as 10 *tens*, and so on.

2. *Groups of ones*. PSTs with this conception reliably conceive of all digits in terms of groups of ones (389 as 300 *ones,* 80 *ones*, and 9 *ones*) but not in terms of reference units; they do not relate reference units (e.g., 10 *tens* to 1 *hundred*).

3. *Concatenated-digits plus.* PSTs with this conception conceive of *at least one* digit as an incorrect unit type at least sometimes. They struggle when relating values of the digits to one another (e.g., in 389, 3 is 300 ones but the 8 is only 8 ones).

4. *Concatenated-digits only*. PSTs holding this conception conceive of *all* digits in terms of *ones* (e.g., 548 as 5 *ones,* 4 *ones*, and 8 *ones*). (Thanheiser, 2009, p. 263).

The categories are ordered hierarchically from most sophisticated to least sophisticated conception. Explanations that depict reference units or groups of ones conception are correct. Explanations that depict concatenated-digits plus or concatenated-digits only are considered incorrect (Thanheiser, 2009).

For example, consider the following subtraction problem in Figure 3:

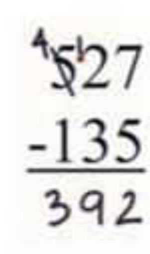


Figure 3 Student Work, Thanheiser (2009)

A prospective teacher with reference units understanding would explain regrouping by relating the value of 100 as 10 tens and 100 ones. Thanheiser (2009) describes an explanation that would be considered reference units understanding: "[the value of the regrouped digits in the hundred's place is] 4 hundreds ... the hundreds is the same as 10 tens, okay [thus the value of the 12 in the ten's place], 12 tens…is the same as 120 ... I changed it [the hundred] into 10 tens" (p. 9). The prospective teacher saw the one hundred she took from the hundreds place as 10 tens and also explained it as 100 ones when she said the value of the 12 was 120.

A prospective teacher with groups of ones understanding might be able to explain how the regrouping works from the hundred's place to the ten's place, but get confused about what the 12 means in the ten's place. A prospective teacher with a concatenated-digits-plus conception could explain that 100 is taken from 500, but not see where that 100 goes because she sees it regrouped in the tens place as 10. Finally, a prospective teacher with a concatenated-digits-only understanding may not understand that they are taking 100 from the hundred's place and refer to it as 1, and consequently also think of all other digits as singular, disregarding their place value (Thanheiser, 2012). For the pre-test of this study, prospective teachers with concatenated-digits only and concatenated-digits plus conception were considered to have low SCK, as they did not have a conceptual understanding of regrouping. Prospective teachers with groups of ones and reference units understanding were considered to have high SCK, as they did have a conceptual understanding of regrouping.

The standard multiplication algorithm is far removed from any conceptual foundation. Single digits are multiplied without any regard to their place value, and without reference as to why one can break apart multi-digit numbers and multiply the single digits to get a solution (Baek, 1998). This type of knowledge involves doing things with whole numbers and symbols according to a set of rules (Lampert, 1986). For example, when one multiplies $12x16$, using the standard algorithm, one would need to know it should be written as follows:

$$\begin{array}{r} 16 \\ \underline{x\ \ 12} \end{array}$$

Then, working right to left, one would multiply $2x6$ to get 12, because at this point one should have memorized his times tables. Then one would write a 2 under the right hand column and "carry" the 1 to the top of the left hand column. Next, one would multiply $2x1$ to get 2 and add the carried 1 to get 3, which one would place to the right of the 2 as follows:

$$\begin{array}{r} 16 \\ \underline{x\,12} \\ 32 \end{array}$$

Then, one would add a zero to the second solution line, move to the next column, and follow the same procedure for multiplying the 1 by both the 6 and the 1, placing the solutions to the left of the zero as follows:

$$\begin{array}{r} 16 \\ \underline{x\,12} \\ 32 \\ 160 \end{array}$$

To get the final answer, one would then add the two numbers below the problem to get the final solution of 192:

$$\begin{array}{r} 16 \\ x\,12 \\ \hline 32 \\ 160 \\ \hline 192 \end{array}$$

In order to get the correct answer, one does not need to know that the 1 in the 16 and in the 12 actually means 10, nor that one is actually adding 10 to 20 (the product of 10 and 2) to get the 30 within the 32 on the first solution line. The only thing one must know is when to add or multiply, and the location and order of the digits. Additionally, one can follow rules to check an answer. It is a rule that when a two-digit number is multiplied by a two-digit, the product is at least three-digits. So if one followed the procedure incorrectly, and forgot to place a zero in the second line,

$$\begin{array}{r} 16 \\ x\,12 \\ \hline 32 \\ 16 \\ \hline 48 \end{array}$$

he could visually see that he made a mistake and go back to fix the calculations. This type of understanding does not involve realizing that 48 is much too small to be a sensible product, rather following a rule that there are not enough digits (Lampert, 1986). Prospective teachers that describe the multiplication problem in this study as such will be considered to have low SCK.

Invented algorithms, on the other hand, require one to possess fundamental concepts of multiplication (Baek, 1998, Lampert, 1986). With this sort of conceptual knowledge, one would know that there are many ways to break apart the digits in 12x16 to get a correct solution. The following principles signify a conceptual knowledge of multi-digit multiplication:

- "The way digits are lined up in a number has meaning" (Lampert, 1986, p. 13). For example, the 1 in 16 and the 1 in 12 means 10 because it is in the tens place.
- "Numbers are composed by addition and can be decomposed in many different ways without changing the total quantity" (Lampert, 1986, p. 13). For example, 16 can be thought of as 10 + 6, or 2 + 8, or 4 + 4 + 4 + 4.
- "Numbers are also composed by multiplication in a sense that they can be decomposed into equal groups" (Lampert, 1986, p. 14). For example, 16 can be thought of as 8 + 8, which is 2 x 8.
- "The elements of those multiplicative compositions can be grouped and multiplied in different ways without affecting the total quantity" (Lampert, 1986, p. 14). For example, 16 = (2x2) x 4 = 4x4 = 2 x (2x4)
- "Numbers to be multiplied can be decomposed additively, each of the elements operated on separately, and the product obtained from recomposing the 'partial products'" (Lampert, 1986, p. 14). For example, 12x16 = (10+2)x16 = (10x16) + (2x16), so 160+32=192. Also, 12x16 = 12x(10+6) = 12x10 + 12x6, so 120+72=192. Finally,12x16 = (10+2) x (10x6), giving the following partial products: 10x10=100, 10x6=60, 2x10=20 and 6x2=12, and 100+60+20+12=192.

The principles are considered building blocks to a conceptual understanding of multiplying multi-digit whole numbers and do not involve any rule following (Lampert, 1986). When a prospective teacher shows how a multi-digit number can be broken apart, displaying the partial products, one can see that conceptual understanding is evident. Breaking down a multi-digit multiplication problem makes knowledge of place value concepts, the distributive property, and the associative property more apparent (Baek, 1998). Students in the base 8 unit of this study learned how to break apart multi-digit numbers to reveal partial products, and use the distributive property to multiply them together. The first four aforementioned bullet points are embedded in the fifth bullet point. Therefore, for the purposes of this study, prospective teachers who were able to break down either or both digits in question three to reveal partial products, and use the distributive property to find the product were considered to have SCK.

Another criteria used to determine SCK was a prospective teacher's ability to make connections between contextual and symbolic representations of a mathematics problem (NCTM, 2014). Current standards assert that students must be able to "decontextualize—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to contextualize, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved" (CCSSI, 2010, p. 9). In order for prospective teachers to deeply understand the mathematics, they need to create a representation of a problem at hand, and be able to explain the meaning of the quantities. This criteria was utilized to show additional evidence of SCK. If a prospective teacher could make the connection between the context and number sentence, they were considered to have SCK.

The most common division problem type taught in schools is sharing, perhaps because it is easily modeled (Simon, 1993). Prospective teachers were not expected to enter the unit in this study with the knowledge of sharing and measurement division because they were unlikely exposed to it in their own school experiences. However, in the unit, many division situations are presented and the importance of understanding the distinction between the two is discussed. Research has shown that when asked to interpret division satiations, prospective teachers had a definite tendency to interpret situations to be sharing (Silver, 1986). If prospective teachers cannot make the distinction between sharing and measurement division, they may not have a conceptual understanding of division, for example, they may be under the impression that multiplication always makes bigger and division always makes smaller (Greer, 1992). A criterion to determine whether one has attained conceptual understanding is to give examples and have participants make the distinction (Silver, 1986). If prospective teachers were able to make the distinction in the post-interview, they were considered to have SCK. It is imperative that teachers understand the difference between the two types so that they may present scenarios representing both situations to their future prospective teachers (Simon, 1993).

Specialized content knowledge requires prospective teachers to "appraise the mathematical validity of alternative solution methods for a problem" (Hill et al., 2005, p. 377). Question five was taken directly from the MKT and was categorized to measure SCK (Hill et al., 2005). It presented three different solution strategies for a three-digit subtraction problem, and required participants to identify if each method was correct and whether each method would work for any two whole numbers. Prospective teachers should be able to identify what the child for each method and whether that method would work for any two whole numbers if they have SCK (Hill et al., 2004). For this question, every method was correct and every method would work for any two whole numbers. Prospective teachers were considered to have SCK if they were able to identify what the student did and whether the solution methods would work in any subtraction scenario for whole numbers for the majority, 2 of the 3, methods.

Selection of Cases

The researcher used data from the pre-test to select interviewees. Prospective teachers were selected based upon how they performed on items measuring CCK and items measuring SCK. Prospective teachers were considered to have high CCK if they answered over 50% of the items identified as CCK correctly. Since there were 11 CCK items on the measure, prospective teachers were considered to have high CCK if they answered 6 of the 11 questions correctly. Prospective teachers were considered to have high SCK if they answered 50% or more of the items identified as SCK correctly. Since there were 6 SCK items on the measure, prospective teachers were considered to have high SCK if they answered 3 of the 6 questions correctly.

The researcher did not make an effort to make sure each of the three course sections were represented, but based upon pre-test data, participants just so happened to come from each section. Sixteen prospective teachers were initially interviewed: Four prospective teachers who scored as low CCK and low SCK, four prospective teachers who scored as low CCK and high SCK, four prospective teachers who scored as high CCK and low SCK, and four prospective teachers who scored as high CCK and high SCK. Once all 16 prospective teachers were interviewed, the researcher synthesized interview responses and selected 8 prospective teachers that fit strongly into one of the four aforementioned categories of knowledge; two in each category. Two prospective teachers were selected for each knowledge category in case a student was unable to be interviewed again, and to assure the richest data available was reported. Those eight prospective teachers were interviewed again at the completion of the base 8 unit.

For example, upon analyzing at pre-test data, if a participant was deemed to have high CCK and low SCK, that person was selected to be on of the 4 students interviewed for the category High CCK and Low SCK. That person, along with 3 other prospective teachers deemed to have high CCK and low SCK by pre-test results, was interviewed. The researcher then selected 2 of those prospective teachers to be interviewed again at the close of the base 8 unit based upon how closely their interview results matched their pre-test results. Some students were not interviewed again if, through the interview, they were deemed not to fit that category. Of the 2 students who were interviewed again, the researcher selected the interviewee who provided the richest data, which in this case, meant the most detailed explanations and justifications to their solutions. The researcher ultimately selected four prospective teachers, one from each category of knowledge, from which to report data.

Prospective teachers were not expected to enter the base 8 unit in this study with the knowledge of sharing and measurement division because they were unlikely exposed to it in their own school experiences. As a result, the researcher only used addition, subtraction, multiplication and question 5 from the MKT to classify high or low SCK in the pre-interview. The ability to make the distinction between measurement and sharing division this distinction was only used to classify high or low SCK in the post-interview.

The first participant, called Brenda, was considered to have low CCK and low SCK. Brenda could not solve three of the four problems using the standard algorithm in the pre interview, demonstrating low CCK. She was placed in the concatenated-digits-plus category for the addition problem and concatenated-digits-only for the subtraction problem, both of which were considered incorrect explanations (Thanheiser, 2009). Brenda was very procedural in her explanation of the multiplication problem, and did not have knowledge of partial products, indicating low SCK (Lampert, 1986). Her representation did not match the context of the problem as she showed 16 groups of 12 rather than the other way around, signifying low SCK (CCSI, 2010). She did not understand what the child did for any of the methods in question 5, and did not think the methods would work for any two whole numbers, also indicating low SCK (Hill et al., 2004; Hill et al., 2005). Because Brenda was considered to have low SCK for all four questions analyzed, she had overall low SCK.

The second participant, called Mary, was classified as high CCK and low SCK. She was considered to have high CCK because she correctly solved all problems using the standard algorithm for the pre-interview. She was considered to have low SCK because her explanation for the addition and subtraction problem fell into concatenated-digits only category and therefore were considered incorrect (Thanheiser, 2009). For the multiplication problem, she had no conceptual understanding of partial products. All whole numbers in the problem were considered single digits and place value did not mean anything, therefore she was considered to have low SCK (Lampert, 1986). Mary represented the problem in the correct context, but indicated that it did not matter if she wrote 12x16 or 16x16, showing she could not make a connection between the context of the problem and the number sentence she wrote. She was classified as having low SCK in terms of her ability to decontextualize and contextualize (CCSI, 2010). For question 5

Mary could only understand what the child did in method B, and was not sure it would work for any two whole numbers, demonstrating low SCK (Hill et al., 2004; Hill et al., 2005). Mary had low SCK for all four questions analyzed, and so overall, she was considered to have low SCK.

The third participant, called Terri, was classified as having low CCK and high SCK. Terri struggled with the multiplication and division algorithms in the pre-interview showing low CCK. She was considered to have high SCK because her explanation for the addition and subtraction problem fell into the groups-of-ones category and therefore were considered correct (Thanheiser, 2009). Terri did not explain partial products and was therefore considered to have low SCK for the multiplication problem (Lampert, 1986). In her representation of the multiplication problem, Terri added 12 sixteen times, which did not match the context of the problem, showing low SCK (CCSI, 2010). For question five, she identified what the child did for every method and said every method would work for any two whole numbers, demonstrating high SCK (Hill et al., 2004; Hill et al., 2005). Because Terri had high SCK for three of the four questions analyzed, she was considered to have high SCK overall.

The fourth participant, called Dottie, was considered to have high CCK and high SCK. Dottie correctly solved all problems except the addition problem using the standard algorithm. Although she could not solve all of the problems, Dottie was still considered to have high CCK. She seemed nervous and shy, which could have contributed to her inability. Also upon analyzing her high level responses to subsequent problems, it was determined that she really did have high CCK. Dottie had high SCK because she was placed in the groups-of-ones category of Thanheiser's (2009) model for the addition and subtraction problems, both of which were considered correct. She also showed high SCK because she was also able to break apart the partial products for the multiplication problem, explaining 10x10=100, 10x60=60, 10x2=20, and 2x6=12 (Lampert, 1986). Dottie also correctly matched the context of the problem to her representation, demonstrating high SCK (CCSI, 2010). Finally, for question 5, the Dottie identified what the child did for every method and said all methods would work for any two whole numbers, indicating high SCK (Hill et al., 2004; Hill et al., 2005). Because Dottie had SCK for all four questions analyzed, she was considered to have SCK overall.

Data Analysis

Quantitative data was analyzed using a dependent t-test to examine significance for each construct of knowledge: CCK and SCK. This test was selected because the researcher examined significance using a pre-post test design and data followed a normal distribution.

For the qualitative data, first each case was analyzed based upon prospective teacher responses to each task during the pre and post interviews. Then a cross case analyses was performed. The researcher looked for common themes and utilized an open coding method. The researcher employed the constant comparative method to saturate themes (Creswell, 2007). Repetitive instances that represented a theme was noted and the researcher continued to look for those instances until no further could be produced from the data. Sub-themes emerged from the main themes, which represented multiple perspectives in regards to the themes. The researcher then reviewed the themes that were detected and used the information to support the quantitative data.

Potential Limitations

The quasi-experimental nonequivalent research design of the quantitative piece presented a limitation to this study. There was threat to internal validity since there was a chance that any statistical significance found was due to pre-existing group differences rather than the treatment. These differences could have included prior experience working with children, a stronger mathematics background, or general classroom climate.

Other limitations included data collection and sampling method. In regards to data collection, prospective teachers could have left test items blank or test scores could have been entered incorrectly. In attempt to decrease error, the researcher had a colleague look over the statistics tables. In regards to the sampling method, the researcher only had access to certain classes, and even then, not every student volunteered to be part of the study. Also, prospective teachers who were willing to stay after class to be interviewed could have been more motivated and eager learners.

Other possible threats to validity included maturation, testing validity, and statistical conclusion validity. Maturation was a threat because there was no control group. Prospective teachers also could have been learning things in other classes that caused a change in scores. Testing validity refers to prospective teachers performing better on the post-test because it is similar to the pre-test, and they remember certain types of questions. Over a month of time passed in between interviews and prospective teachers were exposed to so many different problems in the unit, they did not seem to remember the problems from the pre-test. Finally, statistical conclusion validity refers to the chance that the required sample size is not met, affecting the statistical power. This was not an issue, as the sample size was met.

CHAPTER 4:
FINDINGS

The purpose of this study was to measure the impact of an elementary mathematics unit taught in base 8 on pre-service teachers mathematical common content knowledge (CCK) and specialized content knowledge (SCK) for teaching of elementary number concepts and operations. Literature was synthesized regarding descriptions of types of knowledge/understanding. Two perceptions surfaced: Conceptual knowledge is knowledge/understanding rich in conceptual connections and sense making which, for the purposes of this study, we take to mean specialized content knowledge (SCK). Procedural knowledge is a knowledge built on rules without meaning and memorization without deep understanding which, for the purposes of this study, we take to mean common content knowledge (CCK). Participants were given the Mathematical Knowledge for Teaching (MKT) Measure on the first day of the base 8 unit, and after they took the last exam of the unit (Hill et al., 2004). Items were selected for the measure in regard to their emphasis on either CCK or SCK. Data was analyzed using a dependent t-test to examine significance for each construct of knowledge: CCK and SCK. A dependent *t*-test was conducted to analyze student responses on the pre- and post-instructional administrations the measure.

Quantitative Findings:

Results from the Mathematical Knowledge for Teaching (MKT) Measures

The MKT was administered to help answer the following research questions:

- 1) What impact does an elementary mathematics unit taught in base 8 have on pre-service teachers' mathematical common content knowledge of elementary number concepts and operations?
- 2) What impact does an elementary mathematics unit taught in base 8 have on pre-service teachers specialized content knowledge of elementary number concepts and operations?

Upon completion of a pre-test and a post-test of the MKT, student responses on each measure were graded either correct or incorrect. A total number correct was obtained for items selected to measure each construct, CCK and SCK, on both the pre-test and post-test. This data was entered into the SPSS statistical software package and a dependent t-test was run on each construct. This test was selected because data was normally distributed and because of the pre-post test design.

For Common Content Knowledge Items

H_0 - There is no statistically significant difference in mean scores in pre- and post-administrations of items measuring Common Content Knowledge from the Mathematical Knowledge for Teaching Measures

Table 3 CCK Dependent T-Test and Descriptive Statistics

Results of Paired Samples Test and Descriptive Statistics for CCK

	Pre-Test		Post-Test		95% CI for Mean Difference			
	M	SD	M	SD	lower, upper	*t*	*df*	*p*
CCK Pre-Post	6.46	1.772	6.41	1.549	-.323, .433	.289	90	.774*

* p > .05.

We fail to reject the null hypothesis. There is no statistically significant mean difference in items measuring Common Content Knowledge from the Mathematical Knowledge for Teaching Measures (t= 289, df=90, p>0.05) between pre-test (mean=6.5, s=1.8) and post-test administrations (mean=6.4, s=1.5).

For Specialized Content Knowledge Items

H_0 - There is no statistically significant difference in mean scores in pre- and post-administrations of items measuring Specialized Content Knowledge from the Mathematical Knowledge for Teaching Measures

The null hypothesis was rejected as p = 0.036. Since p<0.05, there is a statistically significant mean difference in items measuring Specialized Content Knowledge from the Mathematical Knowledge for Teaching Measures (t= -2.140, df=90). The pre-test mean was 1.68 with a standard deviation of 1.2. The post-test mean was 1.99 with a standard deviation of 1.5.

Quantitative results reveal that there was no statistically significant difference in mean scores in pre- and post- administrations of items measuring Common Content Knowledge from the Mathematical Knowledge for Teaching Measures, however there was a statistically significant difference in mean scores in pre- and post- administrations of items measuring Specialized Content Knowledge from the Mathematical Knowledge for Teaching Measures. The results imply that while there was not a statistically significant change in prospective teachers' CCK, there was a statistically significant increase in their SCK (pre-test mean=1.68, post-test mean=1.99). It is interesting that mean CCK scores were lower on the post-test than on the pre-test. A possible explanation for this could be that students were so used to inventing their own algorithms since being submerged in the base 8 unit, that they may have forgotten the rules and procedures they entered the course with.

The CCK questions were rule based and required only procedural knowledge in order to get the correct answer. Of the 11 questions, the researcher determined that in order to be considered to have high CCK, one should answer the majority of questions correctly, which would be six questions. Sixty-seven participants answered six or more of the 11 CCK questions correctly as indicated by table 4.

Table 4 CCK Pre-Test Frequencies

Frequency Table for CCK Pre-Test

		Frequency	Percent
Valid	11	1	1.1
	10	1	1.1
	9	7	7.7
	8	18	19.8
	7	20	22.0
	6	20	22.0
	5	11	12.1
	4	9	9.9
	3	2	2.2
	2	1	1.1
	1	1	1.1
	Total	91	100.0

Adding rows 1-6, one can see that 74 percent of participants got six or more CCK questions correct on the pre-test. Results support the claim that most prospective teachers do not enter the unit with low CCK. A non-significant change in CCK could be explained by prospective teachers' educational background. Many prospective teachers enter college with the knowledge of standard algorithms (Thanheiser et al., 2014). In general, prospective teachers entering the unit had educational backgrounds consisting of rule and procedure based mathematics. Solving addition, subtraction, multiplication, and division problems with algorithms is something that prospective teachers learned in elementary school and used throughout their k-12 academics. They would presumably still have this knowledge by the time they arrived at a university. Because they already had this knowledge at the beginning of the unit, there is not much room for growth, which could explain why there was no change in CCK by the end of the unit.

Individual scores of interviewed prospective teachers were as follows: Dottie, the prospective teacher with high CCK and high SCK began with answering 8 questions correct for CCK and answered 8 questions correct on the post-test. She answered 4 SCK questions correctly initially and 3 correctly on the post test. Mary, the prospective teacher with high CCK and low SCK, began answering 7 CCK questions correctly and answered 9 correctly on the post-test. She did not initially answer any SCK questions correctly answered 1 correctly on the post-test. Terri, the prospective teacher who began with low CCK and high SCK initially answered 5 CCK questions correctly and answered 9 correctly on the post-test. She answered 3 SCK questions correctly on the pre-test and 4 correctly on the post-test. Brenda, the prospective teacher with beginning low CCK and low SCK answered 5 CCK questions correctly initially, and on the post-test answered 6 correctly. She initially answered two SCK questions correctly and answered 2 correctly on the post-test.

Qualitative Findings:

Analysis of Interview Data

In addition to the quantitative data, individual student interviews were conducted to help explain results from the MKT and to more deeply explore prospective teacher understanding. Prospective teachers' knowledge development was described using the conceptual framework for SCK and CCK (Ball et al., 2005; Ball et al., 2008; Hill & Ball, 2008; Hill et al., 2004; Hill et al., 2005), and conceptual framework for conceptual understanding of number concepts and operations (Baek, 1998; Carpenter et al., 1996; Cobb & Wheatly, 1988; Ross, 1986; Simon, 1993; Steffe, 1994; Thanheiser, 2009; Sahin, 2015).

Interview data was obtained to examine how varying levels of each type of knowledge changed as a result of the unit. Participants were selected based upon their responses to interview questions about whole number operations and a question representing SCK directly off of the MKT. Data from four different types of prospective teachers was utilized: (1) a student with beginning low CCK and low SCK, who will be referred to as Brenda (2) a student with beginning high CCK and low SCK, who will be referred to as Mary (3) a student with beginning low CCK and high SCK, who will be referred to as Terri, and (4) a student with beginning high CCK and high SCK who will be referred to as Dottie. Results were organized and analyzed using the constant comparative method (Creswell, 2007) in attempt to saturate themes. Findings will be presented from each case for the first and post-interview, and themes will be discussed.

Single Case Analysis

Case 1: Brenda, a student with beginning Low CCK and Low SCK

Question 1: Tyler's bakeshop sells chocolate cupcakes. Tyler sells 255 chocolate cupcakes to a customer. Now there are 261 chocolate cupcakes in the bakeshop. How many chocolate cupcakes were in the bakeshop to start with?

In the pre-interview, Brenda immediately set up the problem vertically as 261 – 255. She got confused with the subtraction, and questioned herself because she was getting 006. Eventually she decided to re-read the problem and decided to try addition. When she got the answer of 516, she said it made more sense with a larger number. Because Brenda struggled to follow the procedure of setting up and solving the problem, her CCK seemed to be lacking. When she realized that she needed to add, and after she solved the problem, the researcher asked her to explain what she did. There did not seem to be any conceptual understanding of the carried 1 in the algorithm, showing a low level of SCK, as seen in the following conversation:

> Brenda: I set it up like number-to-number addition. And then, it's simple. One plus 5 is 6. And then 5 plus 6, you have to carry the 1 over to the next column.
>
> Researcher: What do you mean you have to 'carry' the 1?
>
> Brenda: Because it's greater than 10— or 9. I don't really know how to explain how you carry numbers. But since you would have two digits, one has to go over here and you add that to the next number.
>
> Researcher: Okay. So when you added— what column were you adding? I'm sorry.
>
> Brenda: So 6 plus 5 is 11. So the 1 would go over here.
>
> Researcher: And what does the 1 represent?

Brenda: Like, a 10.

Brenda saw all of the numbers in terms of ones, not their actual place value. Additionally when explaining how to add in the hundreds column, she indicated that she needed to add the two 2's and the carried 1. When asked what those numbers represented, Brenda explained, "The numbers you need to add together", implying that she did not have the conception that the numbers represented hundreds. Brenda would be categorized as having concatenated-digits plus understanding, because she knew the 1 represented something other than one, but it was an incorrect understanding (Thanheiser, 2009).

In the post-interview, Brenda had no issue solving the problem using the standard algorithm and could explain why she needed to add: "Because I know that my answer 516 minus 255 equals 261. So, this plus this equals 516. So, it's the same as like flipping the operations in order to get an answer". Brenda easily set up and solved the problem correctly using the standard algorithm. In the pre-interview, she could not solve the problem, demonstrating that her CCK had increased.

As in the post-interview, Brenda still struggled with place value concepts. The following transcription represents her response when asked to explain the mathematics behind her algorithm.

> Brenda: Okay. So, you look at the ones place and you see 1 plus 5 is 6. Then, you go to the tens place and 6 plus 5 is 11 so you can carry that 1 and then 2 plus 2 is 4. Four plus 1 is 5.
>
> Researcher: Why did you carry the one?

> Brenda: Because you've got a number that was greater than like 1 through 9. So, if you have a number that uses two-digits which represents the 10's place. Then, you have to carry one of those digits over.
>
> Researcher Okay. What does the 1 represent in this instance?
>
> Brenda: Well, it actually represents--I think that's the 10. No, no, no. That's the 10 and this is the 1 that you are carrying. So, like the 1's place. I don't know, but it's just--there is a 1 right there.

Brenda was still unable to explain the concept of the carried one. Like in the pre-interview, said it was a ten. She did not advance according to Thanheiser's (2009) model. She was unable to explain the 10:1 relationship between adjacent digits. Perhaps this was because visually the digit in the tens place looked like a 10, but when regrouped to the hundreds column, it did not look like 100.

In the pre-interview, Brenda only showed the standard algorithm. For the post-interview, she mentioned the number line strategy, although got frustrated and abandoned trying to explain it. She also tried to discuss the Candy Shop scenario without a picture, but got confused with the regrouping. The following transcription represents Brenda's explanation of Candy Shop:

> Brenda: Well, basically we have like two numbers that you're adding together. So…261 and 255. You line up--that is not like totally in line, but you line up each place value and so…we just learned this in class. Like, this is boxes, rolls, and pieces. Boxes represent 100. Rolls represent tens and the ones place represents like 1 single-digit piece. So, those will give you your place value numbers. And, 1 piece plus 5 pieces equals 6 pieces. So, you basically take what's on top, add it to the bottom and to column by column.
>
> Researcher: Can you keep going for each column?
>
> Brenda: Mm-hmm. So, then the next column we have 6 adding 5. So, 6 plus 5. That's 11. But, since 11 is two-digits, you take the 1 and carry it over here because you can't put two-digits there, right? It's only the tenths place. So, this is like 10. Then, you carry that 1, that 1 piece over to here to the next column.

Brenda explained Candy Shop much like she would explain a rule based procedure. There did not seem to be a conceptual understanding of place value.

> Researcher: So, what does--what does the 11 represent in terms of you were just talking about: boxes, rolls, and pieces?
>
> Brenda: So, the 11 means 11 pieces, but it's really 1 roll and 1 piece. A roll is worth 10. So, basically that equals a 10 because it's in the 10's place.
>
> Researcher: And what does this one represent (the 1 that was regrouped)?
>
> Brenda: One piece.

Brenda identified that the 11 represented one role and one piece, which is incorrect. The pieces she was talking about were really rolls, each representing 10 pieces. Brenda did not understand that since she was working with the tens place, the 5 and the 6 represented 50 and 60, or 110 together, not 11. She was still conceiving most digits as ones rather than by their place value.

> Researcher: One piece. Okay. And, then what?
>
> Brenda: Oh, well no. That would be another box because that's like 2 plus 2 is 4. Four equals 5, but it's really in the 500's. So, that's adding out of the box. A box is worth 100 pieces. So, it's adding to the 100's place.
>
> Researcher: Okay. So, what does the 1 represent not in terms of boxes, rolls, and pieces then?
>
> Brenda: Now I'm confused and I don't really know--this is what I'm frustrated with in class. Like, I don't know what carrying something means. So, every time it's like my mind is changing. I just know I need to carry it, but it's like--she was trying to explain it to me the other day and it didn't make sense to me. I know it.

Like in the pre-interview, Brenda still could not explain the regrouped 1 correctly. She initially said it represented one, but then changed her answer and said she knew she was in the hundreds place. Even though she knew she was in the hundreds place, she could not conceptually explain why the 1 represented 100. Brenda remained in the concatenated-digits plus category (Thanheiser, 2009) and did not improve in her SCK.

Question 2: There are 253 people on a cruise ship. Some more people board the cruise ship. Now there are 520 people on the cruise ship. How many people boarded the cruise ship?

In the pre-interview, Brenda set up the problem horizontally and added the two quantities in the problem. She solved it using the standard algorithm. Since the problem was subtraction, she got an incorrect answer indicating low CCK, however, once she was asked to explain her solution, she was able to catch her mistake because her answer did not make sense. Brenda was classified to have low CCK for the pre-interview.

Brenda did not appear to have a conceptual understanding of regrouping from the tens place to the ones place. She explained the following algorithm in Figure 4.

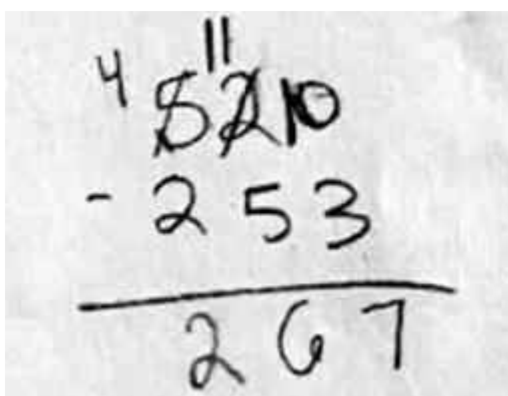


Figure 4 Brenda Work Question 2 Pre

Rather than taking a ten and adding it to the zero ones, Brenda explained that you put a 1 in front of the zero and it "becomes" ten. There also did not appear to be a conceptual understanding of regrouping from the hundreds place to the tens place.

> Brenda: So then 1 minus 5 is the same thing that we just did in this column. You go to the next column and you take one less than the number that's there…so 4…4 is one less than 5. And you're going to add a ones place to that number (referring to the ten, represented as a 1, in the tens place). So 11 minus 5 is 6. And then you can go to this column and it's 5 and there's nothing you need to take away, so 4 minus 2 is 2.

Brenda had a very procedural explanation of her solution. She perceived every number in the problem in terms of ones. The regrouped 1 that she was taking from the hundreds column did not represent anything other than a single digit which was placed in the "ones place" of ten in the tens column. She did not conceptualize the value of each number based upon place value. Brenda had a concatenated-digits only understanding of place value, demonstrating low SCK (Thanheiser, 2009).

In the post-interview, the Brenda had no issue setting up the problem as subtraction and solved it using the standard algorithm, signifying improved CCK from the pre-interview. When asked to explain her solution, Brenda was able to conceptually explain regrouping from the tens to the ones place, but struggled from the hundreds to the tens place.

> Brenda: So, like I was saying up here, you just line up top number to bottom number, 1's, 10's, 100's, and you have 0 minus 3, but you can't subtract 0 minus 3. So, you need to borrow a 1--or borrow a 10, which is--you put a 1 on there, but it is representative of 10…Then, 10 minus 3 is 7. Then, you have 2 minus 5 and you can't subtract that because you only have 2 numbers so how can you take away 5 numbers? Then, so you took away that 1 so it became--the 2 became 1. Now--

Researcher: What did the 1 represent?

Brenda: The 1 represents a 10. So, you can borrow from over here and then add it to that 1 or add it to that 10 to make it 11. So, 11 minus 5 is 6. Then, over here because you've borrowed now your 4 instead of 5 and you don't need to borrow those 5's.

Brenda was able to conceptually explain that she regrouped, taking ten from the tens place and moving it to the ones place, and also recognized that she had ten leftover in the tens place. She no longer saw all digits in terms of ones, but by their proper place value.

Researcher: Okay, when you crossed off the 5 and made it a 4, I noticed you put a 1 there. What did that 1 represent that you took from the 5?

Brenda: A 10.

Researcher: Okay. So, when you crossed off the 5, what did you have left?

Brenda: I have 4. So, it was like almost taking away 100. So, 4 minus 2 is 2 which gave me 267. So, even though there's not like 400 like that means 400.

Brenda understood that when she regrouped from the hundreds she was taking 100, but could not explain why the borrowed 1 represented 100. She could not justify that there were 10 tens in 100. Although she was unable to explain regrouping from the hundreds to the tens place, her increased understanding of regrouping from the tens to the ones place placed her in a higher category in Thanheiser's (2009) model: concatenated-digits plus. While Brenda was still considered to have low SCK, she did increase one category in the hierarchy of Thanheiser's model.

In the pre-interview, Brenda did not show an alternate strategy for solving the problem. She only solved it using the standard algorithm. In the post-interview she demonstrated an adding up strategy using an open number line, indicating that she was able to grasp and apply strategies she learned during the base 8 unit. Her work can be seen in figure 5.

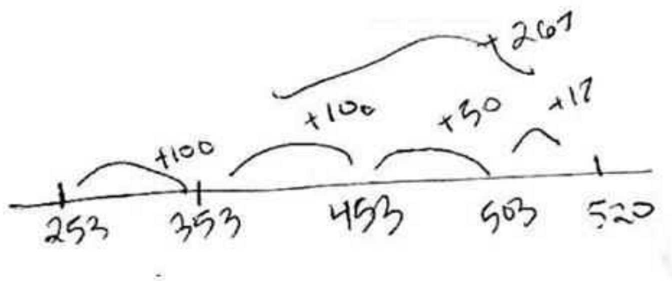


Figure 5 Brenda Work Question 2 Post

Question 3: There are 16 sports drinks in a package. How many sports drinks are in 12 packages?

Brenda had no issue recognizing the problem as multiplication in the pre-interview and solving it correctly using the standard algorithm, demonstrating high CCK as seen in figure 6.

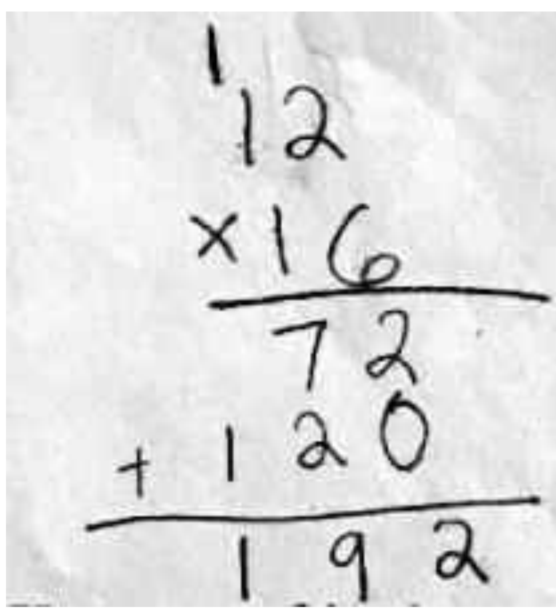


Figure 6 Brenda Work Question 3

Her explanation of the problem was very procedural. When asked about the carried 1, she explained that one cannot put two digits on the bottom. There was no understanding of partial products, and when asked why she placed a zero on line two of the problem, she indicated that it was a placeholder, but could not explain what that meant. Her limited knowledge of partial products and the conceptual meaning of the standard algorithm demonstrated low SCK in regards to this problem (Lampert, 1986).

In the post-interview, Brenda again solved the problem using the standard algorithm, stacking the 12 on top of the 16. She was able to explain that the carried 1 represented ten. Additionally she saw that when multiplying by the 1 in the 16, it was a ten, and attempted to make a connection between that and why a zero is needed on line 2.

> Researcher: Okay. So, when you covered the 6, what did your 1 represent in your problem?
>
> Brenda: Well, I know I'm just multiplying it 12 times 1. But, the equation is like 16, which means 10 plus 6. So, it's really a 10, but that's like kind of where that 0 is.
>
> Researcher: What do you mean? Can you elaborate a little bit?
>
> Brenda: I don't really know how to explain kind of like why we put the 0 there, but I don't know. I just know that you need to multiply 12 times 1.

Although Brenda did show an increased knowledge of the 1 in the 16 representing 10, she struggled to explain much farther. She was not able to show the break down of either two-digit number or what the partial products would be, therefore her SCK did not increase from the pre-interview (Lampert, 1986).

A key component to conceptual understanding is the ability to represent mathematical ideas and operations in a number of different ways (CCSSI, 2010; Hill et al., 2005; NCTM, 2014). This includes decontextualizing and contextualizing a mathematics problem. In order to evaluate a prospective teachers' ability in regards to this criteria, the researcher asked Brenda whether it mattered if one wrote the problem as 16x12 or 12x16. Brenda replied that "with adding and multiplying you can switch it out and it wouldn't affect the operation of the numbers" which is simply a rule, indicating CCK but not SCK. Also she gave an example of repeated addition as an alternative strategy in the pre-interview, adding 12 16 times. This strategy did not match the context of the multiplication problem, which described 16 sports drinks in 12 packages. Repeated addition for the problem should be adding 16 sports drinks 12 times. There was no mention of the number sentence and how one could write it to match the context of the problem, showing a lack of SCK ways (CCSSI, 2010; NCTM, 2014).

In the post-interview, Brenda first explained how to set up the problem. "So, for this one I multiplied it by what we were given. So, there's 12 packages. Each one has 16. So, you need to find out 12 times 16". She also explained that order of the number sentence didn't matter because "multiplication can be reversed and it wouldn't affect anything. But, if you wanted to be like specific in knowing how many groups you're solving for, or I mean that you have 12 groups of 16--but if I wrote it 16 times 12 it would just mean 16 sports drinks 12 times". Brenda clearly understood the context of the problem and how it should be written in regards to groups times number of pieces in each group, demonstrating SCK in regards to matching the context, which is a deeper understanding from the pre-interview (CCSSI, 2010; Steffe, 1994).

Question 4: Alex has 134 stickers. He wants to give each of his friends 12 stickers. How many friends can he share with? Does he have any stickers left over for himself? If so, how many?

The division scenario in question four was written as measurement, or quotitive division. In measurement division, the number of pieces in each group is known and the number of groups is unknown, which is more difficult than being able to partition out to each friend as in sharing division (Carpenter et al., 1996). Prospective teachers who can make the distinction, it will strengthen the argument that they conceptually understand the mathematics (Simon, 1993). Because prospective teachers most likely did not learn about measurement and sharing division until they took the base 8 unit, they were only asked about the distinction in the post-interview.

Brenda solved this problem using the standard algorithm, but got confused because her procedure doesn't seem to work. She performed the standard algorithm until she had 11 friends with a remainder of 2, but the remainder threw her off. She thought she had to keep going with the algorithm so she tried adding a zero behind the 134 to create a decimal answer. Then she brought down the zero after the 2 to make 20, but immediately aborted the problem because she said it did not make sense. Brenda's inability to solve the problem using the standard algorithm indicated low CCK. She aborted the algorithm and solved the problem with multiplication explaining that 12x10=120, so 12x11=132, therefore Alex can share with 11 friends and have 2 leftover for himself. When asked what her numbers represented, she explained that 12 represented how many stickers he could give to each friend, and 11 represented how many friends to which he could give stickers. By multiplying, Brenda first tried giving 12 stickers to 10 friends, and still had more stickers to give. Then she tried giving 12 stickers to 11 friends, and realized that she couldn't give 12 more, but would have 2 leftover.

In the post-interview, Brenda correctly solved the problem with the algorithm indicating CCK. Her explanation of her solution was very procedural. She described an alternate strategy of representing the problem pictorially, drawing 12 circles and saying to disperse the stickers amongst the friends. While her picture would yield the correct answer, it did not match the context of the problem. She said each circle would represent a friend, but the problem was asking how many friends would get stickers, or how many groups could be created. The problem was a measurement problem and her drawing represented a sharing problem. When asked whether the problem was measurement or sharing, Brenda guessed sharing, and then could not write a measurement problem. Brenda could not match her picture to the context of the problem, and could not distinguish between measurement and sharing division, suggesting low SCK (CCSSI, 2010; NCTM, 2014; Silver, 1986).

Question 5: SCK item from MKT

This question follows the description of SCK, as it requires prospective teachers to "appraise the mathematical validity of alternative solution methods for a problem" (Hill et al., 2005, p. 377). Prospective teachers were identified as having SCK if they could identify what the student did and whether the solution methods would work in any subtraction scenario for whole numbers.

For Method A, in the pre-interview, Brenda said that the child's mathematics was correct, but did not think the solution method would work in every instance showing low SCK. In the post-interview, she not only knew what the child did, but was able to explain why their solution method would work in every scenario, displaying high SCK.

> Student: I see how they are counting up, what they did with like the number line. But, you don't know that you're counting to 576. So, it's like just kind of weird, but it's correct. And the same with this one. So, they know they're counting up from 365 to 932, okay? To try and get the in between number. Then, that's right. Yeah. That's totally right.

For Method B, in the pre-interview, Brenda did not understand what the child did commenting "Why did they take away 300 all of the sudden and then 50 and then 6? It's almost like they were trying to get down to that smaller number and picked random numbers". She did not think the solution method would work in every instance, demonstrating low SCK. In the post-interview, Brenda not only knew what the child did, but was able to explain why their solution method would work in every scenario, displaying high SCK.

For Method C, in the pre-interview, Brenda did not understand what the child did nor did she think the solution method would work in every instance. In the post-interview, she was able to correctly identify what the child did and she said the method would work in every instance, showing high SCK (Hill et al., 2005).

Summary

Brenda could not solve three of the four problems using the standard algorithm in the pre-interview, demonstrating low CCK. In the post-interview, she could correctly set up and solve all problems using the standard algorithm, indicating high CCK.

Brenda did not advance in SCK in regards to Thanheiser's model (2009). In the preliminary interview, she was placed in the concatenated-digits-plus category for question 1 and concatenated-digits-only for question 2. She did not progress in the post-interview for question 1, but advanced a category for question 2, keeping her in the two incorrect categories in regards to the hierarchy of Thanheiser's model. Brenda was still considered to have low SCK in regards to place value concepts for addition and subtraction problems.

When asked for an alternate solution strategy, the participant did not show examples for two of the four problems during the pre-interview. She demonstrated pictorial representations for the other two problems. Drawings are considered low level modeling, which indicate less sophisticated reasoning (Gravemeijer, 2004). By the post-interview, Brenda demonstrated the number line strategy, and attempted to explain boxes, rolls, and pieces, indicating that she was attempting to use strategies introduced in the base 8 unit.

For the multiplication problem, Brenda initially could not identify any partial products. She thought of all digits as individual, absent of place value. During the post-interview, Brenda showed an increased knowledge of place value, indicating that the 1 in 16 represents 10, but she struggled to explain farther. She was not able to show the breakdown of either two-digit number or what the partial products would be, so her SCK did not increase from the pre-interview (Lampert, 1986).

In the pre-interview, Brenda could not relate a number sentence to the context of the problem. She gave an example of repeated addition as an alternative strategy in which she added 12 sixteen times. Her example did not match the context of the problem, indicating low SCK. During the post-interview, Brenda gave a conceptually based explanation of how a number sentence should represent the context of the problem, demonstrating high SCK in regards to matching the context, which is a progression from the pre-interview (CCSSI, 2010; Steffe, 1994).

Brenda could not identify the division problem as measurement. She confused it with sharing. Additionally she could not match her picture to the context of the problem, suggesting low SCK for division (Silver, 1986).

Question 5 was an item off of the MKT Measures and given specifically to measure SCK. Brenda initially did not understand what the child did for any of the methods. She also did not think the methods would work for any two whole numbers, demonstrating low SCK. By the post-interview she could identify what the child did for every method and also said that all methods would work in every instance, indicating high SCK (Hill et al., 2005).

Brenda's CCK increased by the end of the base 8 unit, but overall, her SCK did not increase. She went from not being able to solve most problems using the standard algorithm to solving all of them correctly. She did not improve in place value understanding according to Thanheiser's model (2009). Brenda did not advance in SCK regarding recognizing partial products nor her decontextualizing ability. Question 5 showed an increase in SCK as Brenda went from not understanding two of the three methods to understanding what the child did in every scenario, plus indicating that every method would work for any two whole numbers.

Overall, Brenda went from having low CCK to having high CCK and only progressed in SCK for question 5. Findings for Brenda were similar to Safi's (2009) findings in that a student with beginning low mathematical content knowledge only progressed in ability to explain how to solve problems (CCK) and not in the ability to justify solutions (SCK).

Case 2: Mary, a student with beginning High CCK and Low SCK

Question 1: Tyler's bakeshop sells chocolate cupcakes. Tyler sells 255 chocolate cupcakes to a customer. Now there are 261 chocolate cupcakes in the bakeshop. How many chocolate cupcakes were in the bakeshop to start with?

In the pre-interview, Mary attempted to solve this problem using the standard algorithm with several combinations of whole numbers. She indicated that she was looking for a solution that made sense. Mary struggled to follow the procedure of setting up the problem correctly, however, her mathematics was correct for all attempted combinations of numbers. She arrived at the correct solution eventually and was deemed to have high CCK. To evaluate SCK, Mary was asked to explain the mathematics she did. She did not understand the concept of the carried 1, demonstrating low SCK.

> Mary: I took 261 plus 255, so I started with 5 plus 1 and got 6. Then 6 plus 5 and I got 11. And then I carried over the 1 that was leftover from 6 plus 5.
>
> Researcher: What do you mean you "carried over"?

Mary: I took the leftover 1 and then put it over, in…the first column. Which gave me 2 plus 2 plus 1, which is 5.

Mary spoke of all digits as ones, and not their appropriate place value, indicating low conceptual understanding of place value.

Researcher: Okay. So when you carried the 1, what did that 1 represent?

Mary: What was leftover from 6 plus 5.

Researcher: Okay. In your answer, the 516, what does the number 5 represent?

Mary: Hundreds.

Researcher: What about the 1?

Mary: Tens. And then the 6 would be ones.

While Mary did indicate the correct place value of each column, she did not relate the numbers in the problem to their correct place value, i.e., the 5 in 516 was not conceived to be 500. She explained that she shifted a 1 over from the tens place to the hundreds place, and while she recognized that column to be hundreds, she still saw the regrouped 1 as a single digit, placing her in Thanheiser's (2009) concatenated-digits only level of understanding, a category representing an incorrect understanding and low SCK.

In the post interview, Mary had no issue setting up the problem as addition and solved it correctly using the standard algorithm, demonstrating high CCK from the pre-interview. She had a conceptual understanding of the regrouped 1, although she struggled to explain it.

Mary: I did 5 +1 which is six. And, then, 6 + 5 which is 11. And, then, I knew that there was 1 left over. So, I regrouped it over in the hundreds.

Researcher: What do you mean "one left over"?

> Mary: Because--okay, there--because the tens place--if you have like one group of ten, it can be made into one group of one hundred. So, that one left over is the one that you couldn't group with anything else to be able to make another hundred. That sounds really confusing. It's the one left over that you couldn't make a group with because you have to have ten to be able to add on to the hundreds. So, it's the left over because you can't make another group to be able to group it over into the hundreds.

Mary demonstrated an understanding of regrouping between place values, but her explanation was not exactly clear. She said one needed 10 tens to be able to add to the hundreds, but did not explain why. She did not have a conceptual understanding of referent units.

> Researcher: What group are you talking about?
>
> Mary: Like grouping them by tens and then the--the hundreds because you group one group of ten and then you move it over to the hundreds.
>
> Researcher: What do you mean, group of ten?
>
> Mary: Because like in--like 261, this number is really 60, this is really one and the two is really two hundred. When you're adding the six and the five it's really sixty and fifty. So, it's really sixty and fifty and if you were to add sixty and fifty together, it gives you 110. So, this one is like the ten that's there and then you take the one hundred over to the other hundreds so then instead of 200 plus 200 you have 200 plus 200 plus 100, which gives you five hundred.
>
> Researcher: So the one up here represents what?
>
> Mary: The one hundred that you regrouped.

Mary understood that the carried 1 represented 100. She was also able to explain the concept of 6+5 in the tens place conceptually meaning 60+50 as can be seen in figure 7.

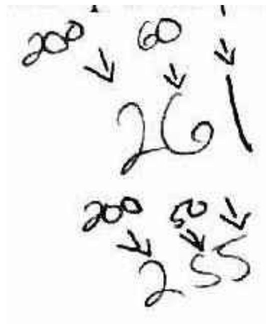


Figure 7 Mary Work Question 1

She correctly indicated that 110 could be broken into 100 and 10, and that the 100 could be regrouped to the hundreds column and 10 remained in the tens column. She talked about a group of ten moving to the hundreds column, but did not indicate what comprised that group. She did not say there were ten tens in one hundred, the referent units. Her understanding in the post-interview advanced her two categories in Thanheiser's (2009) model to groups-of-ones, which represented a correct understanding and high SCK.

When asked to show an alternate strategy to solve the problem, Mary could not during the pre-interview. In the post-interview, she explained a rounding strategy, saying "They could maybe like round the numbers off to make them whole numbers to work with. Then, once they solve it go back and add what they had rounded". Perhaps as a result of so many strategies discussed in the base 8 unit, Mary was able to give an alternate solution strategy in the post-interview.

Question 2: There are 253 people on a cruise ship. Some more people board the cruise ship. Now there are 520 people on the cruise ship. How many people boarded the cruise ship?

During the pre-interview, Mary correctly set up and solved the problem. She used the standard algorithm, but instead of regrouping first with the tens place, she regrouped first with the hundreds place, then the tens. The only reason her algorithm worked was because regrouping was required in the tens place. Had it not been required, she would have been confused because she would have gotten a two digit number in the tens place and not known what to do with it. Overall, Mary demonstrated high CCK because she correctly set up the problem and got a correct answer. Mary did not conceptually understand regrouping from the tens place to the ones place, nor from the hundreds place to the tens place.

> Mary: I took 1 from the 5 which left me with 4, and then I added it over here to the 2 which would make the 2 a 12. But then you still have 0 minus 3, so then you have to subtract another 1 from the 12 which would make it 11, and the 1 you subtracted would come over and make the 0 a 10.

When asked what certain whole numbers represented, Mary referred to them as single digits, not in regard to their proper place value. For example, when she crossed off the 5 and made it a 4, she was taking 1 from 5, not 100 from 500. When she crossed off the 12 and made it 11, she was moving 1 to the ones place, not 10. This level of understanding placed her in Thanheiser's (2009) concatenated-digits only, as all digits were perceived in terms of ones, indicating her SCK was low.

In the post-interview, Mary had no trouble setting up the problem. She followed the standard algorithm from left to right, as in the pre-interview, and got the correct answer designating no change in CCK. When asked to explain the mathematics of her solution, she struggled to explain the concept of regrouping.

> Mary: So, I took ten--I took ten from--no. I took one hundred--yeah. I took 100 from the 500 which gave me 400. I should have done this another way. It would be easier to explain it. Yeah, okay. So I took 100 from the 500 and that left me 400 and I took that 100 and added it to the tens. So, instead of twenty it—okay--I don't know.
>
> Researcher: So, when you crossed off the two here and made it a one--
>
> Mary: Mm-hmm. It was really…it became an eleven, not a one.
>
> Researcher: Oh, okay. And, what does the 11 represent?
>
> Mary: The number of--the number of tens? I don't know.

Mary demonstrated an understanding of what each digit in the setup of the problem represented in regards to place value i.e., the 5 in 520 represented 500 and the 2 represented 20. She did not conceptually understand what happened to the 100 she took from the hundreds place. When she wrote a 1 in the tens place to represent the 100, she did not see that it was still a 100. She did not have a concept of referent units.

> Researcher: So, when you crossed off the two and made it a one, what did that one represent?
>
> Mary: The one that I took from the five because I took one from the five and made it four. Then, I regrouped it over with the twelve and made it an eleven. Because, yeah, because originally this should have been a twelve because I should have taken a one and added it because it was ten more so I should have added it to make that--but then, I still had to do something about the one's place. So, then, the twelve became and eleven and the ten I took from here I moved over here. Yeah.

Mary understood the concept that she was taking 100 from the hundreds, but got confused as to where it went when she got to the tens, because it didn't look like 100 anymore. She initially said she was taking a ten, because when regrouping from the hundreds place to the tens place, one writes a 1 in front of the digit in the tens place, so the regrouped 100 looks like 10. Mary could not see that the regrouped 100 represented 10 tens. She was able to explain that the 2 in the tens place represented 20, but when she crossed off the 12 she created to get 11, she was unable to explain conceptually what the 11 meant. Because Mary recognized that she was taking 100 from the hundreds place, but unable to see where it went once regrouped, she did not have an understanding of referent units. Her understanding placed into Thanheiser's (2009) groups of ones category, which was a step up from where she was in the pre-interview. Her increased knowledge of place value concepts demonstrated correct understanding and high SCK.

When asked if there was an alternate strategy to solve the problem, Mary could not come up with one during the pre-interview, but described a rounding strategy during the post-interview similar to the addition problem in the same interview, which could be a result of all of the solution strategies discussed in the base 8 unit of the course.

Question 3: There are 16 sports drinks in a package. How many sports drinks are in 12 packages?

In the pre-interview Mary recognized the problem as multiplication and solved it correctly using the standard algorithm, exhibiting high CCK. She explained the problem in a rule-based manner. When performing the multiplication algorithm, she first multiplied the 6 and the 6 to get 12. She wrote a 2 on her solution line and carried the 1 to the tens column. When asked about the carried 1 in her procedure, Mary couldn't explain that it represented 10. She demonstrated no knowledge of partial products, that is, when multiplying the 2 in the 12 by the 1 in 16, she had no conception that she was really multiplying 2x10. To her it was 2x1. Mary demonstrated no SCK of the multiplication algorithm (Lampert, 1986). When asked about an alternate strategy, Mary said one could draw a picture with 12 different groups of 16, which did represent the context of the problem. When asked about the connection to the problem, however, Mary said that it didn't matter, explaining, "With adding and multiplying you can switch it out and it wouldn't affect the operation of the numbers. Like if it were 12 divided by 16…you can't just do 16 divided by 12. It would be a different answer." Mary's response demonstrated a rule based explanation, with no conceptual connection to the context of the problem, indicating low SCK (CCSI, 2010; Lampert, 1986).

When presented with this problem in the post-interview, Mary first demonstrated the solution pictorially, drawing 12 circles with 16 dots in each one. She identified the circles as packages and the dots as sports drinks, so her pictorial representation was again consistent with the context of the problem. Another strategy she demonstrated was the distributive property which can be seen in figure 8.

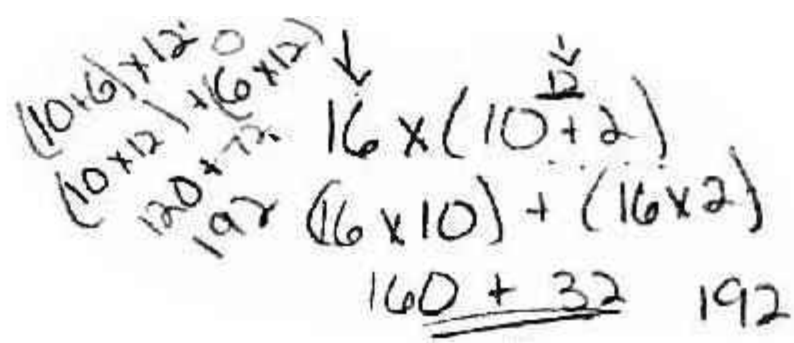


Figure 8 Mary Work Question 3

She broke 16 into 10+6 and multiplied it by 12. When she was asked to explain what she did, she stopped and made a statement about the context of the problem.

Researcher: So, what's this strategy?

Mary: Distributing. You're breaking up the twelve into--wait. There's twelve groups of sixteen. Yeah. This should be flopped.

Researcher: It has to be--?

Mary: Well, not necessarily. You can have--the way the question reads, this is the number of groups and then this twelve would be the number in each group, but there's twelve groups of sixteen. So, it should really be--according to the question, it should be twelve groups and then this should be ten plus six to make the sixteen.

Mary originally wrote 16x12, but when she began explaining the problem, she indicated that since the context of the problem was 12 groups of 16, the number sentence should be 12x16. She also explained that numerically, it wouldn't matter which way one wrote it, one would still get the correct answer, demonstrating a clear conception of how a numeric representation relates to contextual representation, which indicated high SCK (CCSS, 2010, NCTM, 2014).

In addition to explaining the connection between number sentence and context, Mary showed partial products using the distributive property as revealed in the following transcription.

Researcher: Okay. So, with your math here, you said you used distributive property. How do you know that method works?

Mary: Because you're still accounting for the twelve. You've just broken it up to make the number easier to work with and then--like you would do sixteen times ten and get your answer and sixteen times two and get your answer, and you're going to add these together which is going to account for the fact that you split up the twelve into ten and two.

Interviewer: Could you split up the sixteen if you wanted to?

Mary: Yeah. You could have like ten plus six times twelve.

Mary explained the distributive property saying that one could either break up the 12 into 10 and 2 to multiply by 16, or break up the 16 into 10 and 6 and multiply by 6. Because Mary showed how she would break up either digit to reveal partial products, she was considered to have SCK, a progression from the pre-interview (Lampert, 1986).

Question 4: Alex has 134 stickers. He wants to give each of his friends 12 stickers. How many friends can he share with? Does he have any stickers left over for himself? If so, how many?

In the pre-interview, Mary solved this problem using the standard division algorithm and got the correct answer indicating high CCK. She was able to identify that her solution represented the number of friends Alex could give stickers to, and that he would have 2 stickers left over for himself. She also demonstrated an alternative solution strategy of drawing out all 134 stickers and circling groups of 12, which would represent friends, until there were no more groups of 12 to circle.

In the post-interview, Mary provided similar solutions, correctly solving the algorithm and explaining similar solution strategy of drawing 134 stickers and circling groups of 12. When asked if she could explain why the algorithm worked, she said, “Because you’re just dividing--you are dividing it without using like a picture. You could do--I can go back and check, you could do twelve times eleven and that would give you 132 and then you would take the two left over and you would add it to the 132 and it would give you the 134”. Mary could not explain the conceptual connection to the algorithm, but was able to describe how one could use multiplication to check a solution, possibly showing she understood the connection between groups and number of pieces in each group for division since children use multiplication concepts to develop division concepts (Carpenter et al., 1996).

When asked whether the problem was a sharing or measurement division, she said sharing because, “The question itself is breaking it into groups. So, each group can have a certain amount which is sharing”. Mary could not differentiate between measurement and sharing division and was deemed to have low SCK (Silver, 1986; Simon, 1993).

Question 5: SCK item from the MKT

For Method A, in the pre-interview, Mary could not identify what the child did. She was completely confused and thought the child made up a new problem demonstrating low SCK. In the post-interview, she explained exactly what the child did, and even connected it to an adding up strategy she learned in class. Mary indicated that the method would work every time, and she was able to explain why their solution method would work in every scenario, displaying high SCK (Hill et al., 2005).

For Method B, in the pre-interview, Mary correctly explained that the child was, “splitting this bottom number (356) up into hundreds then tens and then ones and then subtracting each time” showing that she understood what the child did. She did say she thought it would work every time, but struggled to explain why. In the post-interview, Mary not only knew what the child did, but was able to explain why the solution method would work in every scenario, stating, “They’re taking--they're accounting for the 100’s the 10’s and the ones but they’re doing it in different steps. So, instead of accounting for it all at once, they’re just breaking it down into the groups and then--or to the values". Because Mary could explain what the student did and why it would work in every instance, she was deemed to have high SCK.

For Method C, in the pre-interview, Mary was completely at a loss as to what the child did, and thought that the child made up a completely new and different problem. In the post-interview, she understood what the child did, adding 4 to both numbers, then 40 to both numbers, but was not sure why a child would do that. She thought that the child's strategy changed the problem, i.e., because the numbers changed, the problem changed even though the child got the correct answer. Mary was not sure the method would work for every two whole numbers. Although Mary progressed in her understanding of what the child did, she was not sure it would work in every instance, and therefore was considered to have low SCK (Ball et al., 2005).

Summary

Mary correctly solved all problems using the standard algorithm for the pre-interview and the second. She demonstrated no change in CCK because she entered the class with high CCK from the beginning.

Her conceptual knowledge of place value operations increased from low SCK to high SCK as demonstrated by Thanheiser's model (2009). She began in the concatenated-digits-only category for both problems and advanced two categories by the post-interview, advancing from an incorrect understanding to a correct understanding.

For the multiplication problem, Mary began with a procedural explanation and no understanding of the connection to the number sentence to the context of the problem, and was considered to have low SCK. In the post-interview she correctly explained partial products and demonstrated the capability to decontextualize a problem into the proper symbolic number sentence, progressing to high SCK (CCSSI, 2010; Lampert, 1986; NCTM, 2014). Mary could not differentiate between sharing and measurement division, indicating low SCK (Silver, 1986). Additionally, she could not come up with alternate solution strategies for half of the problems in the pre-interview. She showed low level modeling, with pictorial examples (Gravemeijer, 2004) for the rest.

For question 5 in the pre-interview, Mary could only understand what the child did in method B, and was not sure it would work for any two whole numbers. By the post-interview she could recognize what the child did for all three methods. She was able to explain why the methods would work in any instance for methods A and B. She still struggled with method C. Mary did show an increase in her conceptual understanding of 2 of the three methods, so she was considered to have high SCK for the post-interview in regards to question 5 (Hill et al., 2005).

Mary demonstrated no change in CCK and demonstrated improved SCK upon completion of the base 8 unit. Her CCK did not change because she began with high CCK. She improved in her place value understanding for questions 1 and 2. By the post-interview was able to explain partial products and decontextualize the problem. She went from having low SCK to high SCK for addition, subtraction, and multiplication. She did not advance in regards to the division problem but she did advance her SCK in regards to question 5. Overall, Mary went from having low SCK to having high SCK.

Case 3: Terri, a student with beginning Low CCK and High SCK

Question 1: Tyler's bakeshop sells chocolate cupcakes. Tyler sells 255 chocolate cupcakes to a customer. Now there are 261 chocolate cupcakes in the bakeshop. How many chocolate cupcakes were in the bakeshop to start with?

In the pre-interview, Terri set up the problem correctly and correctly solved it using the standard algorithm, demonstrating that she had high CCK. When asked to explain the mathematics, Terri first explained it very procedurally, saying one must carry the 1 because one can't place two numbers underneath. When asked what the carried 1 represented however, Terri had a clear conception.

> Terri: The 1 represents the 100 that I took over to the hundreds place from the tens place. Cuz if you do, if it was 60 and 50 it would go into 110 and you just add it to the hundredths place.
>
> Researcher: Can you say that again? Sorry my brain is trying to process.
>
> Terri: So ignore these numbers (ones place)…so if you just had 50 and 60 it would be like 110, so instead of just putting 110 here, I would put a 1 in the tens place and then carry the 100 over here to add to the already 400 here.

Terri appeared to have a conceptual understanding of what digits represented in regards to their place value. She also conceptually understood regrouping, because she explained that the 1 she was carrying from the tens to the hundreds place represented 100. She also indicated that the 100 came from the 50 and 60 which equaled 110 in the tens place. Although she did not indicate that there were 10 tens in 100, she had an understanding of all digits in terms of ones, placing her in the groups-of-ones category of Thanheiser's model (2009), indicating a correct understanding and high CCK.

In the post-interview Terri still solved the problem correctly and used the standard algorithm. When she explained the mathematics, she demonstrated the same understanding that she had in the pre-interview.

> Terri: Tyler sells 255 chocolate cupcakes to a customer and now there's 261 chocolate cupcakes left in the bakeshop. How many chocolate cupcakes were in the bake shop to start with? So I took the two numbers knowing that he originally had…we don't know the original number, and he sold 266 so that original number was in the total number before. Now he has 255 chocolate cupcakes left. So I take that as an addition problem and I added them up. I put the 255 and the 261…you can write it interchangeably. It doesn't matter which one's on top… and I went to the one's place and I added together. So 1+5 is 6 and then I went to the tens place and… 6… so it's 60 +50 is 110… but you're in the tens place, so you bring down the 10 and you would add the 100 into the hundreds place. Then once you add the number one to the one hundredths place… that signifies the 100 that you got from the tenths place… so then you would add the two 200s plus the 100 you originally had in the tens place to get 516.

Terri understood that she was adding 60+50 rather than 6+5 because it was in the tens place. She understood that she had 110 in the tens place and that she could regroup 100 to the hundreds place and that 10 would be left over. She still did not conceive of the reference units between digits, and therefore was still placed in the groups-of-ones category of Thanheiser's model (2009), demonstrating high SCK and a correct understanding.

Terri was also asked to provide an alternate strategy to solve the problem in both interviews. In the pre-interview Terri explained a combining tens and ones strategy (Carpenter et al., 1996) where she broke each number down by place value and added each separately. Terri presented her understanding of place value as well as her ability to represent the problem in a different fashion. She demonstrated the same strategy in the post-interview as seen in figure 9.

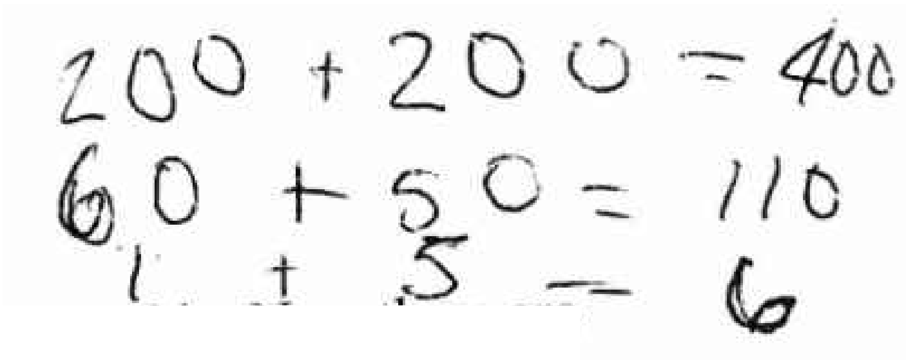


Figure 9 Terri Work Question 1

Question 2: There are 253 people on a cruise ship. Some more people board the cruise ship. Now there are 520 people on the cruise ship. How many people boarded the cruise ship?

In the pre-interview, Terri correctly set up the problem and solved it using the standard algorithm showing she has high CCK. She had a conceptual understanding of regrouping from the tens place to the ones place, and also from the hundreds place to the tens place.

> Terri: In the beginning there were only 253 people so after more people boarded, a total of 520 people were on. You would need to subtract the 253 from the 520 to see how many people were added on to the cruise ship. So I kinda like, I did the process as the first one but subtraction. So I wrote the 520 over the 253 kinda the way I learned. If you have a 0 or less than the one under it (gestures to ones place) you cross it out, go one under (makes the 2 a 1) and that makes it a ten. So then 10 – 3 is 7.
>
> Researcher: Why did you cross out the 2 again?
>
> Terri: So the 2…I crossed out the 2 because I wanted to add more in the ones place. So if I put a ten here, um, if there was a 20 here and I made it a 10 and moved it into the ones place, a ten, it would cover, um, it would cover the ones place so I could subtract the 3.
>
> Researcher: So what did you take from the 2?
>
> Terri: A ten. So like you would change it. So instead of it being 20 it becomes a ten. I mean a 1 in that place and the 1 that you took away would go into the ones place since

> you can't really have a ten in the ones place. It kinda crosses over so you take that away. And I kinda did the same thing in the tens place in that you cross out the 1 and you took away from the 5 to make it a 4. So like instead of 500 it would become 400 and you would add it to make it 11 here and you subtract that from 5 to get the 6.

Terri demonstrated a conceptual understanding of regrouping from the tens place to the ones place. She understood digits in terms of their place value, i.e., the 2 in 520 represented 20, and she took 10 from the 20 and moved it to the ones place, leaving 10 left in the tens place. Additionally the 5 in 520 represented 500 rather than 5, indicating that Terri could represent each digit in terms of ones.

> Researcher: So you said you crossed out the 5 here to make it a 4 and you put the 1 here?
>
> Terri: Yes to make it an 11.
>
> Researcher: What does the 11 represent?
>
> Terri: I basically did the same as I did in the first place, so when I took the 4 away, I added a ten back into the tens place but since I already had a 1 there you add the ten and the one so in essence the 11 here is 110. Minus 50.
>
> Researcher: How do you know it's 110?
>
> Terri: Um, This is a lot harder to explain than I thought. So I know because when I took the ten away in the ones place I left one in the tens place so that would equal ten. So when you take 100 away from the 500 to make it 400, you're adding essencely the 100 and the 10 together in the tenths place. Which makes it 11.

Terri conceptually described the process in great detail when regrouping from the tens place to the ones place. She understood that the 2 in the tens place represents 20, and that she was taking ten from the tens place and adding it to the ones place. She explained that there were ten leftover in the tens place. Terri initially incorrectly explained that she added a 10 to the tens place when she regrouped from the hundreds to the tens, but still said she would then have 110 in the tens place. When she was asked how she knew she had 110, she corrected herself and said she took 100 from the hundreds place. Terri was able to conceptually explain that she took 100 from the hundreds place and regrouped to the tens place. She clarified that she had 10 plus 110, minus 50 in the tens column. She was unable to explain why the 110 was represented as 11, and did not indicate that there were 10 tens in 100, demonstrating that she did not conceive the reference units between digits. Terri's understanding placed her in the groups-of-ones category in Thanheiser's model (2009), representing an initial high level of SCK.

In the post-interview, Terri again correctly set up and solved the problem using the standard algorithm, indicating no shift in CCK. She also conceptually explained the problem much of the same way as she did in the pre-interview.

> Terri: when I have the 520 minus the 253 original people on board, you have a 0-3 so you can't take three from nothing, So you go back into the tens place… This is like 20 here, So then if you borrow… you borrow from the tens place and you take one to make it a 10… like it's opening a tens place and you're adding 10 more into the one's place. So you take a 10 from here and you add it into the one's place so you can subtract the three. So once you take it you have 10-3 which gives you seven. So when you go to the next place you have that one because you borrowed a 10. So you basically have 10-50, but you can't do that either because you can't have negative numbers… like you can't take 10 people away from 50 people… I mean 50 people away from 10 people so you would go to the hundreds place and you will also borrow, so instead of having the 500 you would borrow 100, which would leave you with 400. So you would borrow it, and since you're taking the 100 and you're adding it into the tens place you would have 11. Well, 110. See you're taking 110-50 people. That would give you 60 people. So then once you go to the next place you have 400 left minus the 200 people so then 400-200 would be 200. So you have 267 people that boarded the cruise ship.

Like in the pre-interview, Terri understood each digit in terms of its place value, and conceptually understood regrouping from the hundreds place to the tens place, and the tens place to the hundreds place. She understood the amount she was taking from each column in regard to place value.

> Researcher: So you crossed off the five and moved 100 people over to the tens place but you only wrote a one. How does the one relate to 100?
>
> Terri: The one that I placed would represent the 100. But since you already have another 10 here that you originally had…plus the 100…you would have 110 here. So from the five you're taking the 100 people to the tens place so that you can subtract 50 people.
>
> Researcher: I'm still not quite sure how the 100 that you took is a one… So it doesn't look like 100
>
> Terri: I guess I could add a zero to make it look like 110 or like put arrows.

Terri was able to conceptually explain regrouping from the tens to the ones place, and also able to explain that she was taking 100 from the hundreds place, but even with further prompting she was unable to explain why the 110 became 11. She was unable to explain that there were 10 tens in 100. Her knowledge did not shift for this problem, placing her in the same groups-of-ones category in Thanheiser's model, and also not advancing her SCK of place value.

Terri did not show an alternate strategy during the pre-interview but explained the boxes rolls and pieces strategy using pictures during the post-interview. Her work can be seen in figure 10.

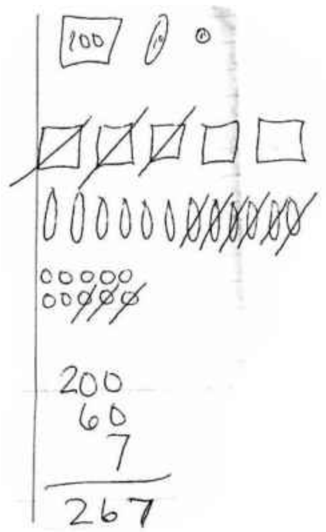


Figure 10 Terri Work Question 2

> Terri: So if we were doing it in boxes and rolls… we can teach them that a box represents 100 people, and a roll represents 10 people, and a piece would represent one person. So that would be the basis. So if a student wanted to visualize it they would write five boxes to represent the 500 people are originally. Then the 20 people you would represent it with the rolls. For the 20 represented two rolls… Which has 10 and each roll. So if you wanted to figure out how many people boarded from here, you would subtract the 253. So for the 200 you can just cross out two boxes but then you have 53 left. You don't have enough rolls to take the 53, so you would cross off one more box to open up 10 rolls plus the two, so that's 112. So you would cross off five of the rolls and we have no pieces, so you would open up one more roll, which gives you 10 pieces from that opened up roll. Then you still have the three, so you would cross out three pieces, which is three people out of the 253 people. So then the ones you have leftover you would count. So two boxes is 200. You have six rolls which is 60. And then you have seven pieces. So it's 267 pieces left…people left.

Terri's explanation shows that she can demonstrate an alternate representation of the problem. She explains that there are 10 rolls in a box, and that she needs to cross off a box to make 10 more rolls so she can subtract her rolls. She represented the solution to the problem using drawings; which shows conceptual understanding, and is considered a low level reasoning strategy (Gravemeijer, 2004). Because Terri was not able to conceive reference units when moving from hundreds to tens when she explained the algorithm, one cannot assume that she was making a connection between the candy shop scenario and the standard algorithm.

Question 3: There are 16 sports drinks in a package. How many sports drinks are in 12 packages?

In the pre-interview, Terri was unable to perform the standard algorithm for multiplication, indicating low CCK for this type of problem.

> Terri: I'm really not good with multiplying. When I do the 16 times 12 I know in essence you're just taking 12 and adding it together 16 times. So in my brain, the highest multiplication problem I know for the number 12, is 12 times 12 which is 144, so if it's 16 you still have to add 12 more 4's. Four times 12 is 38, so I just added the 144 plus 48 to make the 192.

Terri demonstrated she understood multiplication to be repeated addition, but did not mention partial products demonstrating low SCK (Lampert, 1986). She did not make a connection between the number sentence and context of the problem, writing the number sentence as 12x16, which does not represent groups times number of pieces in each group. Terri explained that she wrote the number sentence as 12x16 because it was the way the problem was written, and her solution strategy of adding twelve sixteen times did not match the context of the problem. She also said that order of the number sentence did not matter, further indicating no connection between the number sentence and problem context. Because of Terri's inability to recognize partial products and inability to decontextualize, she was classified to have low SCK (CCSI, 2010; Lampert, 1986).

In the post-interview, Terri again explained how she would solve the problem using multiplication.

> Terri: So you could also do this in representation. So how I did this was I broke it up, because I couldn't figure out what 12×16 was. So I broke it up into 12×12 is 144 and if you subtract 16 from 12 you still have four packages… Wait let me rethink how I'm explaining this. So yeah, I had 12×12 which represented the 12 Sports drinks … I mean 12 packages and 12 sports drinks, so you still have 4 sports drinks in each package. So you would take the 4 sports drinks that were still left in the packages and a multiply it by 12 because we still have 12 packages in total. I broke it up because I couldn't think of what 12×16 was off the top of my head. So then you multiply 12×12 which is 144 and then 4×12 is 48. Then I just took the two numbers and I added it up to get 192.
>
> Researcher: What does each number represent when you broke it down?
>
> Terri: 12 would be the packages…the first 12. And the second 12 would represent some of the sports drinks that were in each package. In the second… wait… the first 12… I should probably switch that. It's going to be 12×4. Yeah, the first one was the number of drinks and then the packages. And then with this…12x4, you still had four drinks leftover from each package that we did not count. There's 12 packages, so you would multiply the 12×4 to get in total what was left over in each package. So the 48 represents the drinks that we didn't count when we did 12×12.

Terri initially solved the problem similarly to the way she did in the pre-interview, with multiplication, however she more conceptually explained the whole numbers in terms of the context of the problem, describing which numbers would represent drinks and which would represent packages. Additionally, she explained how the context of the problem related to the number sentence representing the problem.

> Terri: When you're just using numbers and multiplying it I don't feel like it matters. But when you're explaining it in context… in the wording it would matter. It matters in this case because it wouldn't make sense if you said there was 16 packages and 12 sports drinks… it's grouping in a certain way. You don't want to switch the numbers and say oh well there was 192 packages instead of 192 drinks.

Terri conceptually explained her solution strategy to include representation of the context to the problem. She has a clear understanding of multiplication being groups, times number of pieces in each group (Steffe, 1994). Additionally, she gave another example of a solution strategy where a student might draw 12 circles representing the 12 packages, and 16 marks in each one, representing sports drinks, signifying her ability to give an alternate representation of the problem.

When Terri was asked to solve the problem using the standard algorithm, she was unable to do so, but she did explain how to break up 12 and 16 to reveal partial products.

> Terri: I honestly don't even remember the standard one. I guess you could do 10×10 would be 100, 10×2 would be 20, 10×6 would be 60, and 2×6 would be 12. I mean you could foil… And that would be the 16×12. So 10×10 would be 100, then 10×2 would be 20, the 6×10 would be 60, And then the 6×2 would be 12. So then you add them up together and you would get 192.

Because Terri did not know how to do the standard multiplication algorithm, her CCK did not increase. She was still considered to have low CCK. Terri conceptually explained how to break up both 12 and 16 to multiply by partial products. She also matched the number sentence to the context of the problem, therefore she was considered to have high SCK by the post-interview (Lampert, 1986; CCSSI, 2010; NCTM, 2014).

Question 4: Alex has 134 stickers. He wants to give each of his friends 12 stickers. How many friends can he share with? Does he have any stickers left over for himself? If so, how many?

Terri attempted to solve the problem using the standard division algorithm in the pre-interview. She got confused however and decided to solve it another way. Because she was unable to solve using the procedure of the standard algorithm, she had beginning low CCK. Terri decided to solve the problem using multiplication instead. She multiplied 12x10, with 10 representing friends, and 12 representing how many stickers each friend got. Then she explained that she still had 14 stickers, so she could give 12 stickers to one more friend, with 2 stickers leftover for Alex. She clearly had a conceptual understanding (SCK) of what it meant to divide, and she was able to match her solution strategy and answer to the context of the problem.

In the post-interview, Terri again set up the standard division algorithm and was unable to solve it, showing no change in CCK. Her alternate strategy was the same as in the pre-interview as she determined how many times 12 could go into 134 without going over. She again correctly identified her numbers in terms of friends and stickers.

When asked if the problem is measurement or sharing, Terri incorrectly identified it as sharing explaining, "I know it's sharing because he already knows the amount that he can give to each person. So it's not a fact of grouping it… It's a fact of how many you can give out". Terri described a measurement scenario, but identified the problem as sharing. It is difficult to tell from this information whether Terri can make the distinction, however, Terri also wrote a sharing problem when asked to write a measurement problem. Because Terri identified the incorrect type of division scenario and also wrote the wrong type of problem, the researcher concluded that Terri could not distinguish between sharing and measurement division, indicating low SCK (Silver, 1986).

Question 5: SCK Item from the MKT

For Method A, in the pre-interview, Terri could identify what the child did and explained why it would work every time indicating high SCK. She explained that the child was adding up to solve a subtraction problem. In the post-interview, Terri could still explain why the method would work and that it would work in every instance indicating no change in SCK (Ball et al., 2005). Also, she connected the method the child used to the counting up method she learned in class, demonstrating her ability to make connections between different representations of the problem. She also explained why the child might have chosen Method A instead of the standard algorithm.

> Terri: Because it's confusing for the child… Like crossing out numbers and stuff. Like how I was taught. It's more confusing to than understanding that the number you're getting is the number that's in between the two numbers. Maybe this is easier for them to understand…to get to an easier number for them to add up to. Like here they're gaining their understanding of what's in the middle. In essence all you're doing is going up to find the difference in between the two numbers. I think it'll work every time because you're not really changing the numbers, you're just finding the difference in between. So instead of subtracting you're seeing the specific numbers in between.

Terri conceptually explained why the method would work every time and also identified a reason a student may choose this particular strategy. Although by the criteria set forth for SCK, Terri's

SCK did not change, she did show that she could make connections between different representations of the problem and explained why the child might have chosen Method A instead of the standard algorithm.

For Method B, in the pre-interview, Terri conceptually explained the mathematics and reasoned through why it would work every time. In the post-interview, Terri went further into detail as to why the child's method would work. She also said why the method might be more beneficial to a child's understanding.

> Terri: They're taking the 932 and they're subtracting by the different places. So they're subtracting the 300 first, and then they're taking the tens place… The 50… and subtracting it from the 632. Then they're taking the six… The last place… and subtracting it from 582. So in essence they're just breaking down, the 356, by place value. It always works because you're still subtracting the same number. It's just that you're breaking it down into the places so maybe it's easier for the child to understand that. Instead of 356 you're subtracting 300 first, then 50, and then six. I feel like it's a longer way to do it, but it could be easier for them instead of doing the algorithm like how I do.

For Method C, in the pre-interview, Terri was able to explain what the child did, and said that as long as they are consistent with their mathematics, the method would work in every instance. During the post-interview, she also explained why the method would work for any two whole numbers.

> Researcher: How come whatever they do to the bottom they have to do to the top?
>
> Terri: If you don't do it then you'll get the wrong number. Whatever you do to the bottom you have to also do to the top so that you can keep the number range… the number in the middle that you're trying to get it to…the difference… the same. If you just add 4 to one number and not the other, you get the wrong answer. You're messing up the equation and you're not going to get the same answer.
>
> Researcher: Are they changing the problem?

> Terri: I mean they are kind of changing the problem… Just the numbers of the problem. Like if it's just numbers… as long as whatever they do to the top number they also do to the bottom number, it's fine. But if you put it into a context like the cookies or something, I feel like that's more confusing because you're like, you're diluting the number of cookies or the number of candies or something like that.
>
> Researcher: Will this work every time?
>
> Terri: I think it will work every time because… it's hard for me because I don't think like this… is that they're trying to make the problem easier for themselves. Instead of 356… I know that they're adding onto each one to get to 400. Like they're adding four and then the 40. On the top they're doing the same thing. The gap… The difference isn't getting bigger or smaller, it's keeping it the same.

Terri conceptually explained why the method would work, and the reason one had to be consistent with adding the same amount to the top as well as the bottom numbers. She also discussed how the child was not changing the problem numerically, but that changing the whole numbers did change the amounts if one was considering a context, further showing her understanding of contextually representing a story problem. Although by the criteria set forth in the theoretical framework Terri's SCK did not change, she went further in her explanation of why the methods would work for the post-interview.

Summary

Although Terri did use the standard algorithm for addition and subtraction correctly, she struggled with the multiplication and division algorithms in the pre-interview, which is why she was classified as having low CCK. By the post-interview, she still struggled to solve the multiplication and division problems using the standard algorithm, indicating that her CCK had not progressed.

Her understanding of place value began high, and did not change throughout the base 8 unit. The only reason she did not move up in a category in Thanheiser's (2009) model is because she could not explain that there were ten tens in one hundred. Perhaps this could be due to the fact that she was not able to make a connection between the algorithm in base 10 and strategies she learned in base 8. She did explain the candy shop scenario, but she did so pictorially rather than abstractly, demonstrating less sophisticated reasoning (Gravemeijer, 2004). There was no connection between the candy shop scenario and the standard algorithm.

Terri began by not understanding that there was a connection between the context of the problem and writing a number sentence for the multiplication problem. She also did not recognize partial products, showing low SCK. She improved to high SCK in the post-interview because she correctly decontextualized the problem. She also demonstrated how to break apart both digits to get the partial products, also demonstrating high SCK (Lampert, 1986). Terri demonstrated low SCK in that she could not differentiate between sharing and measurement division (Silver, 1986).

When asked for an alternate solution strategy, Terri originally demonstrated low-level modeling with pictures as alternate solution strategies for most problems. In the post-interview, she showed some examples with pictures. For example, she conceptually explained regrouping with boxes, rolls, and pieces, but did not connect the representation with the standard algorithm indicating low-level modeling. She also explained some more abstract concepts, such as the distributive property indicating higher level and more sophisticated reasoning (Gravemeijer, 2004).

In the pre-interview, Terri could identify what the child did for every method in question 5. She said every method would work for any two whole numbers but did not explain why. By the post-interview, she was able to explain why for all methods. She also correctly described why method C was not changing the problem, indicating high SCK (Ball et al., 2005).

Terri progressed in CCK but not SCK upon completion of the base 8 unit. She started off struggling with some algorithms and ended up solving all correctly by the post-interview. While there was no progress with her place value understanding, but she improved in her ability to decontextualize. She demonstrated low SCK in her ability to differentiate between sharing and measurement division and she also did not advance in SCK for question 5. Overall, based upon the criterion established in the theoretical framework, Terri did not progress in SCK. Although she did not progress if one strictly followed the criterion, perhaps because she began with high SCK, Terri did show more solution strategies, and she went deeper in her explanations and justifications for methods in question 5.

Case 4: Dottie, a student with beginning High CCK and High SCK

Question 1: Tyler's bakeshop sells chocolate cupcakes. Tyler sells 255 chocolate cupcakes to a customer. Now there are 261 chocolate cupcakes in the bakeshop. How many chocolate cupcakes were in the bakeshop to start with?

Dottie was not able to solve the problem during the pre-interview, questioning, "Is there an issue with the problem? If he sells 255 and now there are 261… so he had to bake them. The problem doesn't make sense. Tyler's bake shop sells chocolate cupcakes. Tyler sells 255. Now there's more than there were originally. Now there's 261. There's not enough information to tell that. Because there's a negative number. From 261 to 255 if he's selling them now. Tyler sells 255 cupcakes. Now there's 261 cupcakes". Because Dottie could not set up the problem correctly due to the context, it was initially assumed that she had low CCK, however, as she answered subsequent questions it was determined that she had high CCK as the reader will see in question 2. During the pre-interview, Dottie seemed nervous and shy, which could be the reason she was confused and could not answer the first question.

In the post-interview, Dottie immediately knew how to set up the problem. She knew she was adding, explaining, "Because it asked for how many they had to start with. I knew that if I'm starting with something, then I know I need to put all the numbers together to make a total". She also began with an alternate strategy to the algorithm, drawing a number line and counting up. She explained that numerically it didn't matter if one started with 261 or 255, and she chose to start at 261 and counted up 255, getting the correct answer of 516. Additionally, Dottie indicated that she wanted to show the number line strategy because she knew it was something her future students might do to solve a problem. She explained that it would be easier for some students to solve a problem using a number line because they could visualize the jumps.

Dottie's second representation of the problem included a procedural explanation of the algorithm. Upon further questioning, it was revealed that Dottie had a high level of SCK because she was able to conceptually explain regrouping. When she added the 5+6 in the tens column, she explained that she had 11 tens, and that since there were 10 tens in one hundred, she could regroup by taking 10 of the 11 tens to the hundreds, as 100, and leaving ten. Her understanding of reference units for each digit placed her in Thanheiser's highest category reference-units (2009), indicating a correct understanding and high SCK. Dottie's explanation of regrouping led her to show another strategy a student might use. She described how a student might draw lines to signify each column, and add straight down each column. This strategy mimicked the out of context inventory forms examples she completed in the base 8 unit.

> Dottie: In the one's column they would have six. In the tens column they would have 11. And in the hundreds column they would have four. But because I can't have this number (the 11 in the tens column) I have to regroup. Like if it were boxes, rolls, and pieces I would have to move 10 of these rolls over to the hundreds, and I would have to mark that off and make it a one, and then I would have my five by regrouping.

Dottie explained regrouping in terms of boxes rolls and pieces. Rather than drawing out the boxes, rolls, and pieces, she described it more abstractly, indicating higher-level modeling (Gravemeijer, 2004). She was able to connect the standard algorithm to a strategy she learned in class, which is an understanding not achieved by any other participants interviewed. Once prospective teachers can make a connection between concepts and algorithms, deep understanding has taken place (Carpenter, 1986).

Question 2: There are 253 people on a cruise ship. Some more people board the cruise ship. Now there are 520 people on the cruise ship. How many people boarded the cruise ship?

In the pre-interview, Dottie initially solved the problem using the standard algorithm and explained it procedurally demonstrating high CCK. When asked to explain the mathematics she performed, Dottie displayed high SCK as she explained regrouping.

> Dottie: I know that zero is less than three so I have to borrow from… I know the term isn't borrow anymore… But I need to take 10 from the tens place and move it over here (gestures to ones place) and that reduces the 2 down to 1 and then I subtracted 10-3 and then also 1 is less than 5 so I need to go ahead and do that again and take 10 away from… I mean take 10 over from the hundreds to the tens place, make that 11, do my subtraction of 11-5 to get six. That reduced my five in the original number to four hundred and then subtract the 200.
>
> Researcher: When you crossed off the two and made it a one, why did you do that?
>
> Dottie: Because it's in the 20s. That's in the tens place, so the place value is 20 so if I remove 10 and I move it over to the ones place to increase the 0 to 10, then I'm reducing my 20 down to 10. So the one is really a place value, so instead of 20 now it's worth 10. And 10 is still less than 50, so that's what I would have to borrow again from the five. The 500.

Dottie conceived each digit in terms of groups of ones i.e., the 5 in 520 represented 500, the 2 represented 20, and the 0 represented zero ones.

> Researcher: So what did you take?
>
> Dottie: I took another… I don't know how to explain that… but I moved 10 over to the two. I mean to the one to make it 11.

Dottie conceptually was able to describe regrouping from the tens place to the ones place, but struggled when explaining regrouping from the hundreds to the tens place. She knew that she was reducing the 500 by 100, but she struggled to relate the 11 to it's place value of 110. Because she understood each digit in terms of groups of ones, but did not understand reference units, she was placed in the groups of ones category in Thanheiser's model (2009) indicating correct understanding and high SCK.

In the post-interview, Dottie first solved the problem using alternate strategies to the standard algorithm. She explained several number line strategies. Her work can be seen in figure 11.

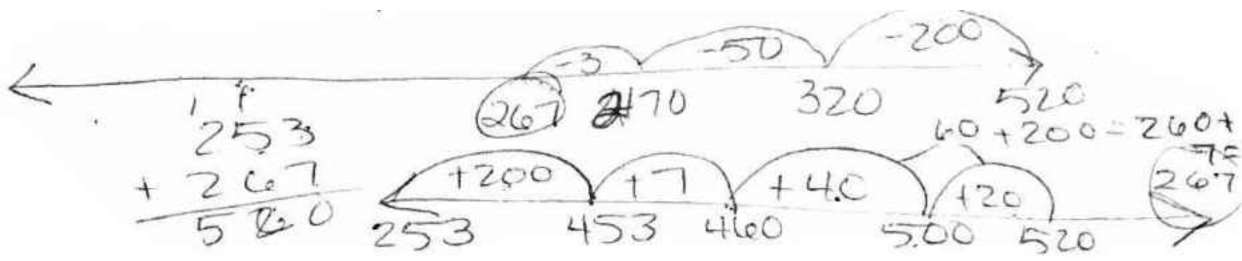


Figure 11 Dottie Work Question 2

> Dottie: If I want to find out the missing part I can also do another number line. I don't know if I want to takeaway or… yes I can do take away and take away the 253. So if I start with 520 and go backwards 200, I get 320. Then I'll take away 50 and that's going to leave me at 270. And then minus 3 more leaves me at 267 so 267 people boarded the cruise ship. Let me just check my answer.
>
> Researcher: How did you know to take away?
>
> Dottie: I could have done take away or I could've done count back. I just chose one. So I just took away the amount of people that were already on the cruise ship. And I knew I would land on the number of people that came on the cruise ship later. If I wanted to I could have just jumped back until I landed on 253 and that would have been counting back. Then, however many jumps I had, I just would've had to add up those numbers in the jumps. I chose take away because it's one less method for me to go through…one less step in order to find my answer.

Dottie showed a conceptual understanding of how to solve the problem and also demonstrated several solution strategies that a child might use instead of the standard algorithm. She also made connections between different representations of the problem since she compared the two strategies (NCTM, 2014).

> Researcher: You said take away and count back, which are both subtraction. Would you have to subtract to solve this problem?

> Dottie: No. I think some kids would not do that because they would not make that connection that addition and subtraction are opposite. They would have possibly counted up from 253. So they would have started at 253 and then they would have wanted to end on 520. So then they could have just gone up… so say they went up 200 to land on 453. Their goal is to get to 520, so plus 7 would give them 460. Plus 40 more would be 500, plus 20 would be 520. And then they can add up these numbers so 40 and 20 is 60, plus 200 and 260+7 equals their answer of 267 people boarded the ship.

Dottie accurately described all three number line methods for subtraction. She also explained how and why a student might want to use addition to solve a subtraction problem. When asked to solve the problem using an algorithm, Dottie deeply explained the concept of regrouping.

> Dottie: I could use an algorithm and do 520-253. Then I could subtract that way. There's a lot of regrouping. Let's see, it's not called borrowing anymore. I think it's just called regrouping. In order to make the numbers on the top more than the numbers on the bottom… So you're not getting negative numbers… you would increase the value of the numbers by borrowing from the tens and the hundreds place. So to make zero more, you want to borrow 10 from the tens. This causes you to decrease your 2 by 1. So your tens place is 2 which is equal to 20. You're decreasing that down to a 1, which is 10. So you are using that 10 from the 20 and putting it in the one's place, making it 10 instead of zero. So 10-3 is seven. The 2 you had decreased to 1, which is still less than five. So you need to increase that by a 10 which you would get from the hundreds place… decreasing your 5 to 4… moving that 10 over making the one an 11. Then 11-5 is six. 4-2 is two, which gives you your answer of 267.
>
> Researcher: You said you took one from the five. What does the one represent?
>
> Dottie: One hundred, which if you breakdown 100 by 10… Because you're putting it in the tens place… there's 10 tens in 100. So then you're putting 10 tens into your rolls… or your tens place. Like you're breaking your box of 100 into 10 rolls and making that (the one ten) 11 rolls… Instead of five boxes now you have four boxes. Boxes represent 100 rolls and rolls represent 10 pieces.

Dottie had a deeply conceptual understanding of regrouping. She was able to explain that there were 10 ones in 10, and 10 tens in 100. She understood what each place value meant, describing that the 2 in the tens place represented 20, and the digits in the hundreds place represented hundreds. Her justification and conception of reference units placed her in the highest category of place value comprehension in Thanheiser's model (2009): Reference-units conception, which indicated correct understanding and high SCK. She correctly made the connection between regrouping and an alternative solution strategy, also indicating high SCK (NCTM, 2014).

Question 3: There are 16 sports drinks in a package. How many sports drinks are in 12 packages?

Dottie displayed both high CCK and high SCK in the pre-interview. She initially solved the problem using the standard algorithm, referring to place value when she multiplied. She set up the standard algorithm by horizontally stacking 16 over 12. She first multiplied 6x2 and got 12. She placed the 2 underneath as part of her solution and carried 1, correctly explaining it as meaning 10. She explained that she needed to place a zero in the second line of the solution because she was no longer multiplying ones; she was multiplying tens.

> Researcher: When you wrote down the 160…how did you get that answer?
>
> Dottie: Because I put my zero in the place value and then I multiplying from my tens place. So I'm multiplying by 10 instead of by 1, so 10×6 is 60 and then 10×10 is 100. So that's where that would come from. That's technically a 10 and that's technically a 10. You're multiplying 1×1 which will give you one but because of where the place value is the one is actually at 10 and this one is actually a 10. 10×10 is 100 so if you don't put the place value here then you're going to put the one in the tens place and that's going to give you 10 instead of 100.

Dottie correctly broke up both numbers in order to talk about partial products. She understood the concept of the standard algorithm and why it worked, displaying high SCK (Ball et al., 2005).

This problem describes a situation where there are 16 sports drinks in 12 packages, or 12 packages with 16 drinks in each group. Since multiplication number sentences are represented as groups times number of pieces in each group, students should decontextualize the problem as 12x16. Writing 16x12 represents 16 groups of 12, which is not a description of the context of the problem. Dottie also described an alternate solution strategy in which a child might draw an array with 12 tallies across and 16 down. She also said a student might draw a picture with 12 groups of 16 drinks. Her multiple representations of the problem showed high SCK (NCTM, 2014). Dottie also said she didn't think it mattered if one multiplied 12x16 or 16x12, and checked to make sure by quickly doing the standard algorithm. She explained that she had always been programmed to perform the multiplication with the larger number on top, indicating low SCK because she did not make a connection to the context of the problem (CCSSI, 2010; NCTM, 2014).

For the post-interview, Dottie first solved the problem using the distributive property as can be seen in figure 11. She explained how she broke each 2-digit number down to make it easier to multiply as seen in figure 12.

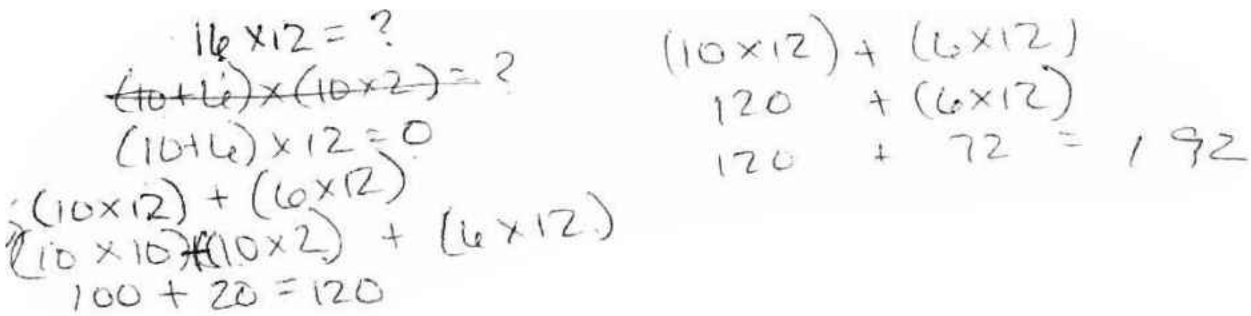


Figure 12 Dottie Work Question 3

> Dottie: I'm going to do 10 and 6, times 12. Then do 10×12 plus 6×12. Now I know that I can break my 12 down… say if I wanted to do 10… So I know that 10×10 is 100 and I know that 10×2 is 20 so that would mean that that's 120, so I know that 10x12 is 120 plus 6×12…and to find that I can do a number line and do six jumps of 12. Starting at zero… plus 12 is going to be 12, plus another 12 is going to be 24, plus another 12 is going to be 36… I get 72 in the end. So now I can add 120+72 and that would be 192.
>
> Researcher: How do you know that this method works? What are you actually doing?
>
> Dottie: Well these are larger numbers and even though I know my tens, I want to put myself in my students' shoes, and I know that like after 10 they may get confused. So I want to break it down so that it is as easy as possible. So if I'm multiplying by 12 I want to get it 10 or less. So I broke it down… so 16 is 10+6. When I multiply the 12 I distributed it to the 10 and the 6. So 12×10 and 12×6. So with the 12×10… You really don't want to go over 10… so I'm broke my 12 down again and did another distributive property. So I did 10 plus 10 times 2. Which would be 10×10 plus 10×2. So I know 10×10 is 100 and 10×2 is 20. So that just made that easier because I broke down one of the distributions. So now I only have to worry about my easier one of 6×12.

Dottie correctly broke up both digits to show partial products demonstrating conceptual understanding of multiplication and high SCK (Lampert, 1986). She was also able to see why this method may be easier for a child to understand.

When asked to solve the problem using the standard algorithm, Dottie again explained her process using partial products. She exhibited a deep understanding of why the multiplication algorithm worked. She also was able to make a connection between her distributive strategy, partial products, and the standard algorithm.

> Dottie: You're still breaking down the problems …so like here in the algorithm I'm breaking my 16 and my 12 apart, So really my top number is 10+6 and then my bottom number is 10+2 and then I am multiplying by those numbers. So I'm multiplying the two times the six and I'm multiplying the two times the 10, And then I'm going the other way and multiplying the 10 by the six and the 10 by the 10.

Because Dottie made a connection between concepts and algorithms, deep understanding had taken place (Carpenter, 1986). Additionally, she explained how numerically, it didn't matter how one wrote the number sentence for this problem. She said that if one were trying to match the context of the problem, order would matter, as one would be swapping drinks and packages if written incorrectly. Her SCK improved from the pre-interview in regards to decontextualizing (CCSI, 2010). Because she explained partial products for both interviews, her SCK did not improve (Lampert, 1986). Dottie's SCK did not improve in regard to the theoretical framework regarding partial products, however, one can not discount her improved ability to make a connection between all of her solution strategies, which is something no other participant was able to do.

Question 4: Alex has 134 stickers. He wants to give each of his friends 12 stickers. How many friends can he share with? Does he have any stickers left over for himself? If so, how many?

Dottie performed the standard division algorithm for the pre-interview and got the correct answer, showing her high CCK. Her explanation of the standard algorithm was procedurally based indicating low SCK, however she could correctly place her solution into context. She explained that 11 represented the number of friends Alex could share with and the 2 represented how many stickers he would have left over for himself. Dottie also made a comment that may offer insight as to why she began with high CCK and SCK. She explained how she had been observing in her son's classroom and was learning new methods of instruction.

> Dottie: I haven't observed a lot of division with my son. I've only been there multiplication with him right now. And in the classrooms… I've only seen multiplication so I haven't seen how kids are learning it now so that's the only way I know how to do it.

Dottie's initial high CCK and SCK could be explained by her time spent in her son's class. It's possible that her son's teacher taught conceptually rather than procedurally, or it could just be the refresher that caused Dottie's conceptual understanding.

In the post-interview, Dottie initially drew 12 circles and began making tally marks in each circle. She said she would keep distributing the tally marks until she could not give any more evenly. She arrived at the correct answer of 11 friends and 2 leftover for Alex. To check the answer she got from her drawing, she solved the problem using the standard algorithm for division and showed no change in CCK. She was again unable to conceptually explain why the algorithm worked.

Her picture did not seem to match the context of the problem until she provided further explanation.

> Dottie: So the answer the problem gives me is 134, and 12, so even though I'm trying to find… I'm trying to group of them… So if anyone looked at this they would think that the groups were the friends, but I had 12 so that that I know that I needed to distribute it by 12 so if I did it this way then I wouldn't have to go through and have to… so the other way that I could've done it was to make 134 lines and then draw circles around 12 each so that each line would represent a sticker and then by circling them that as a friend. So I

> did not have to draw all of my figures, I just wanted to already group them into 12. And then I counted inside and that told me how many friends would get 12 stickers. Mine works, but it is hard to explain to anyone because really you're trying to give them to friends and in this picture each line… each line represents a friend but you don't have 134 for friends, you have 134 stickers, so my circles are my groups.

Dottie recognized that her solution strategy did not represent the context of the problem. She was able to describe why her method made sense in her mind, as well as how she could have changed her picture to fit the context. Her exceptional explanation of how she connected her drawing to the context of the problem indicated high SCK (CCSI, 2010).

Dottie was the only interviewed participant to correctly identify the problem as measurement.

> Student: It's measurement because I need to find how many groups there are, not how many pieces are in each group.
>
> Researcher: Can you write me a sharing problem?
>
> Student: Alex has 134 stickers. He wants to give his 11 friends the same amount of stickers. How many stickers does each friend get? So if I were to solve this then I would have drawn 11 circles which would have represented each of the friends, And that I would've counted out the hundred and 34 stickers which would've gone to each friend.
>
> Researcher: What type of problem do you think is easier to solve?
>
> Student: The sharing problem. And my method would have been easier because the groups are the same.

Not only was Dottie able to differentiate between sharing and measurement division, she identified which would be easier for a child to solve and why indicating high SCK (Greer, 1992; Silver, 1986). She also recognized that measurement problems were more difficult than being able to partition out to each friend as in sharing division (Carpenter et al., 1996). Her high SCK in regards to this type of problem surpassed that of all other participants interviewed.

Question 5: SCK Item from the MKT

For Method A, in the pre-interview, Dottie identified what the child did and said she thought it would work every time. She provided a conceptual reasoning as to why. She also discussed that a student may choose this method because it was easier to add numbers that would make each place value zero. In the post-interview, Dottie also connected the solution strategy to the number line strategy, explaining that it was like adding up. She also discussed how a student might choose to solve a subtraction problem using addition.

For Method B, in the pre-interview, Dottie described what the student did and said she thought it would work for any two whole numbers. She did not discuss why it would work in every instance.

> Dottie: With Method B, I think that would also work every time with whole numbers, because they're just breaking their numbers down. They're just doing three different subtraction problems for one by going down to the nearest hundreds or tens place and then subtracting what's remaining.

By the post-interview, she did have a conceptual explanation as to why. Dottie began and ended with high SCK.

For Method C, in the pre-interview, Dottie explained what the child did and thought that one might choose this method because it would give them an easy equation to work with. She also said it would work with every two whole numbers but did not explain why. During the post-interview, Dottie explained why the method would work in every instance.

> Dottie: They have to increase their value in the first number because if you don't do that then your difference is going to have a completely different value. It changes the problem. Here, they're not changing the difference. They're just changing the numbers.

> So just by increasing…kind of like fractions. Like 3/12 is the same is 1/4. You have different numbers but it's still the same amount.

Dottie conceptually explained that in order to avoid changing the problem, one had to be consistent with adding to the top and bottom numbers. Dottie explained that if one added to only the top or only to the bottom, the difference between the numbers would change, indicating that she understood that while the numbers in the problem would change, the difference between the numbers, and therefore the solution, would not change. Based upon the theoretical framework, there was no change in Dottie's SCK in regard to question 5 (Hill et al., 2004). It should be noted that by the post-interview, Dottie was able to more deeply explain why the methods would work in every instance.

Summary

Dottie did not demonstrate a change in CCK because she began being able to correctly set up and solve the problems correctly. There was no further improvementfor her. One might conclude that her CCK increased because she was not able to solve problem 1 in the pre-interview, but it was determined that she could not solve it because of nerves, rather than a lack of understanding. One major difference was that in the post-interview, she first solved the problems showing an alternate solution strategy, not the standard algorithm. When asked to solve the standard algorithm, she was able to solve it and also make a connection to her alternate representation, which demonstrated deep conceptual understanding (Carpenter, 1986; Sahin, 2015).

In regard to place value understanding, Dottie began at a high level for questions 1 and 2 in Thanheiser's (2009) model: groups-of ones. By the post-interview, she improved to the highest level in the model for both questions: referent units. While Dottie only advanced one category, which is a similar improvementto the rest of the participants, she had the deepest understanding of anyone by the end of the unit. She was the only participant to fully understand regrouping regarding referent units from the hundreds column to the tens column, and she was able to connect the algorithm to an abstract explanation of candy shop.

Dottie was able to break apart the digits in the multiplication problem and identify the partial products for both the pre and post interviews, so there was no change in her already high SCK (Lampert, 1986). She began having low SCK in regard to decontextualizing, because she was following a rule for always placing the larger number first in multiplication. She progressed to explaining that if one were trying to match the context of the problem, order would matter, as one would be swapping drinks and packages if written incorrectly. Her SCK improved from the pre-interview in regards to decontextualizing (CCSI, 2010).

Dottie was the only participant who could differentiate between sharing and measurement division. She also identified which would be easier for a child to solve and why indicating high SCK (Silver, 1986). She also recognized that measurement problems were more difficult than being able to partition out to each friend as in sharing division (Carpenter et al., 1996). Her high SCK in regards to this type of problem was higher than any other participant.

In regard to alternate solution strategies, Dottie showed pictorial examples for two of the three problems during the pre-interview suggesting low-level reasoning. In the post-interview, she showed multiple methods and exhibited high-level modeling (Gravemeijer, 2004). For example, questions 1 and 2 were described via a number line, the candy shop scenario, and the taking inventory scenario, which she described more abstractly.

For question 5, Dottie identified what the child did for every method in the pre-interview. She said all methods would work for any two whole numbers but did not explain why. By the post-interview, she was able to explain why for all methods. She also correctly identified why method C was not changing the problem. Following the theoretical framework, Dottie began with high SCK and also ended with SCK, indicating no change (Hill et al., 2005). It should be noted that she did progress in her ability to explain deeply why the child's methods would work in every instance.

Overall, Dottie did not progress in CCK and she did progress in SCK upon completion of the base 8 unit. Her CCK did not change because she began with high CCK. Her SCK progressed in regard to place value of addition and subtraction problems. Her SCK improved in regards to decontextualizing the multiplication problem. Her SCK did not progress in regard to her understanding of partial products, as she already had high SCK in the beginning of the base 8 unit. Dottie's SCK did not progress in SCK for question 5 because she already could explain what every child did and whether each method would always work in the pre-interview. Additionally, Dottie was the only participant who was able to make conceptual connections between her representations and the standard algorithm, demonstrating high SCK (Carpenter, 1986). Findings are similar to Safi's (2009) in that a student with high mathematical content knowledge demonstrated full conceptual understanding by the end of the base 8 unit, and also possessed the ability to make connections between different solution strategies to problems.

Cross Case Analysis

In addition to analyzing each participant as a single case before and after instruction in base 8, themes were identified in a cross case analysis. Analysis was performed utilizing the following table:

Table 5 Cross Case Analysis

		SCK	
		Low	High
CCK	Low	Low CCK; Low SCK	Low CCK; High SCK
	High	High CCK; Low SCK	High CCK; High SCK

Although there are 6 possible scenarios, only several will be analyzed, as the others results are as expected. The following 4 scenarios will be considered:

1) How change in knowledge differs between a participant with beginning low CCK and low SCK (Brenda) and a participant with beginning low CCK and high SCK (Terri).
2) How change in knowledge differs between a participant with beginning high CCK and low SCK (Mary) and a participant with beginning high CCK and high SCK (Dottie).
3) How change in knowledge differs between a participant with beginning low CCK and low SCK (Brenda) and a participant with beginning high CCK and low SCK (Mary).
4) How change in knowledge differs between a participant with beginning low CCK and high SCK (Terri) and a participant with beginning high CCK and high SCK (Dottie).

Scenario 1: How change in knowledge differs between a participant with beginning low CCK and low SCK (Brenda) and a participant with beginning low CCK and high SCK (Terri).

Both participants in this scenario began with low CCK. In the pre-interview, they both struggled with standard algorithms for most or all operations problems. Sometimes it was setting up the problem that was the issue, for example a participant may have added in problem 2 rather than subtracted. Other times they missed a step when following a procedure, such as trying to add a zero where it didn't belong in the standard division algorithm. In the post-interview, Brenda was able to correctly set up and solve all problems using the standard algorithm indicating an increase from low CCK to high CCK. Terri however, still could not solve the multiplication and division problems using the algorithm, indicating that her CCK did not increase.

There was a difference in progress for SCK in reference to Thanheiser's model (2009). Brenda was placed in the concatenated-digits-plus category for question 1 and concatenated-digits-only for question 2. She had only advanced a category for question 2 by the post-interview. Terri began at a higher level in Thanheiser's model than Brenda, but did not progress in the post-interview. It could be due to the fact that she struggled to make connections between alternative solution strategies to algorithms. Also, Terri already had a high conceptual understanding of place value. She struggled to explain regrouping from the hundreds place to the tens place, but did perceive all digits in terms of groups of ones, a correct and high SCK category. Perhaps her SCK did not progress because she was not able to make the connection between the standard algorithm and solution strategies she learned in class. Brenda had more room to grow, which could explain why she progressed more.

For the multiplication problem, Brenda initially could not identify any partial products but progressed in the post-interview by showing an increased knowledge of place value. She could explain that the 1 in 16 represented 10, but she struggled to explain farther. She was not able to show the break down of either two-digit number or what the partial products would be, so her SCK did not increase from the pre-interview (Lampert, 1986). Terri did not recognize partial products in the pre-interview, showing low SCK, but was able to break apart the digits by the post-interview, showing an increase to high SCK.

In the pre-interview, Brenda could not relate a number sentence to the context of the problem, but during the post-interview, she gave a conceptually based explanation of how a number sentence should represent the context of the problem, demonstrating high SCK in regards to matching the context, which is a progression from the pre-interview (CCSSI, 2010; Steffe, 1994). Terri also could not make a connection between the context of the problem and her number sentence initially, but by the end of the unit she could correctly decontextualize the problem and explain the connection indicating a shift to high SCK. Both participants demonstrated an increase from low SCK to high SCK in regard to decontextualizing. Neither participant was able to differentiate between sharing and measurement division at the post-interview, indicating low SCK.

When asked for an alternate solution strategy, Brenda did not show examples for two of the four problems and showed pictorial representations for the other two problems. Drawings are considered low level modeling, which indicate less sophisticated reasoning (Gravemeijer, 2004). In the post-interview Brenda drew pictures to show the solution for all problems. Terri demonstrated low-level modeling with pictures as alternate solution strategies for most problems in the pre-interview, and in the post-interview, she showed some pictures and some more abstract strategies. She did conceptually explain regrouping with boxes, rolls, and pieces, but in terms of a drawing, indicating less sophisticated reasoning (Gravemeijer, 2004). Both participants showed an increase in number of solution strategies in the post-interview. Brenda went from showing none, to solving with drawings. Terri began with low-level modeling, and went to showing a few more abstract representations signifying more sophisticated reasoning (Gravemeijer, 2004). No connection was made between algorithms and alternate representations.

Question 5 was given specifically to measure SCK, as it required prospective teachers to "appraise the mathematical validity of alternative solution methods for a problem" (Hill et al., 2005, p. 377). Brenda began by not understanding what the child did for two of the three methods. She also did not think the methods would work for any two whole numbers. She was initially classified to have low SCK. By the post-interview she could identify what the child did for all three methods. She could also conceptually explain why the strategies would work for methods A and B. She still struggled with method C. Brenda progressed from low SCK to high SCK. In the pre-interview, Terri could identify what the child did for every method. She said every method would work for any two whole numbers but did not explain why. She was considered to have high SCK. By the post-interview, she was able to explain why each method would work for any two whole numbers.

Brenda and Terri both began with CCK but only Brenda progressed. The base 8 unit was designed to encourage invented algorithms and strategies, not standard algorithms, which could explain why Terri did not progress. Also, 3 out of the 4 participants interviewed struggled to connect their invented algorithms to standard algorithms, suggesting that more emphasis should be placed upon making that important connection in the future. According to the theoretical framework criteria, Terri did not show progress in SCK, her advanced reasoning of why each method worked should be noted. Brenda appeared to advance more, although this is most likely due to the fact that there was more room for growth than Terri.

Scenario 2: How change in knowledge differs between a participant with beginning high CCK and low SCK (Mary) and a participant with beginning high CCK and high SCK (Dottie).

Both participants in this scenario began with high CCK. For the pre-interview, they both easily solved the operation problems correctly with the standard algorithm demonstrating high

CCK. There was no change in their CCK for the post-interview, as they were still able to correctly solve all of the problems using the procedure. One major difference was that in the post-interview, Dottie first solved the problems showing an alternate solution strategy, not the standard algorithm. When asked to solve the standard algorithm, she was able to solve it and also make a connection to her alternate representation, demonstrating a deep conceptual connection (Carpenter, 1986).

Both participants demonstrated that their SCK deepened, but in different ways. Mary began in the concatenated-digits-only for the pre-interview and advanced two categories by the post interview (Thanheiser, 2009). She went from having low SCK to high SCK. Dottie began at a higher level in Thanheiser's model than Mary, and only advanced one category by the post-interview. Dottie began with high SCK and ended with high SCK. It should be noted that she was the only participant to be able to explain reference units by the end of the base 8 unit.

For the multiplication problem, Mary began with a procedural explanation and no understanding of the connection to the number sentence to the context of the problem, and was considered to have low SCK. In the post-interview she correctly explained partial products and demonstrated the capability to decontextualize a problem into the proper symbolic number sentence, progressing to high SCK in regards to both partial products and decontextualizing. Dottie was able to break apart the digits in the multiplication problem and identify the partial products for both the pre and post interviews, so there was no change in her already high SCK. She began having low SCK in regard to decontextualizing, because she was following a rule for always placing the larger number first in multiplication, but progressed to high SCK in the post-interview because she could conceptually explain why her number sentence had to match the context of the problem. Mary could not differentiate between sharing and measurement division, indicating low SCK. Dottie was the only participant interviewed who could differentiate between sharing and measurement division. She also explained why sharing would be an easier problem type for a child to solve indicating high SCK (Greer, 1992; Silver, 1986). Her high SCK in regards to this type of problem surpassed that of all other participants interviewed.

Mary could not come up with alternate solution strategies for half of the problems in the pre-interview. For the other half, she showed low level modeling, with pictorial examples (Gravemeijer, 2004). By the post-interview, her representations changed very little. She only mentioned an estimation strategy but did not go into detail. Dottie showed pictorial representations for 2 problems during the pre-interview. In the post-interview, she showed multiple methods and exhibited high-level modeling (Gravemeijer, 2004). For example, questions 1 and 2 were described via a number line, the candy shop scenario, and the taking inventory scenario, which she described more abstractly.

Participants also showed how their SCK changed via question 5. Mary initially could only understand what was happing in method B, and was not sure it would work for any two whole numbers. By the post-interview she could identify what the child did for all three methods. She was able to explain why the methods would work in any instance for methods A and B. She still struggled with method C. Her SCK progressed from low to high. Dottie identified what the child did for every method in the pre-interview. She said every method would work for any two whole numbers but did not explain why. By the post-interview, she was able to explain why for all methods. She correctly described why method C was not changing the problem.

Neither prospective teacher progressed in CCK, although they began with high CCK, so there was little room to improve. According to the evaluation criteria, Mary progressed in SCK and Dottie did not, and that was because Dottie already had high SCK to begin with. It should be noted that Dottie provided deeply conceptual explanations of why each method would work in every instance by the post-interview. Results showed that a prospective teacher with high CCK and low CCK can benefit greatly from a unit taught in base 8. Since Mary began with high CCK, that could have contributed to her success in developing her SCK. Because she was already proficient in the procedural mathematics, she could focus all of her attention on her conceptual development of the mathematics. Like Safi (2009), results also indicated that a student with high CCK and SCK, Dottie, could progress even further and fill any conceptual gaps upon completing a unit taught entirely in base 8.

Scenario 3: How change in knowledge differs between a participant with beginning low CCK and low SCK (Brenda) and a participant with beginning high CCK and low SCK (Mary).

In this scenario, Mary began with high CCK and Brenda began with low CCK. In the pre-interview, Brenda struggled with the standard algorithm for all operations problems except multiplication. In the post-interview, the she was able to correctly set up and solve all problems using the standard algorithm, demonstrating a progression to high CCK. Mary easily solved the operation problems correctly with the standard algorithm for the pre-interview, demonstrating high CCK. There was no change in her CCK for the post-interview, as she was still able to correctly solve all of the problems.

In regards to place value concepts, Brenda began in the concatenated-digits-plus category for both questions 1, and the concatenated-digits-only category for question 2. She only advanced one category for question 2 by the post-interview (Thanheiser, 2009). Even though Brenda advanced one category, she was still in the bottom 2 categories, which indicate low SCK. Mary began in the concatenated-digits-only for the pre-interview and advanced two categories (Thanheiser, 2009) indicating a greater progression than Brenda. Perhaps because her CCK was initially high, she had less to progress in that area, allowing her more time to learn and develop her SCK.

For the multiplication problem, Brenda initially could not identify any partial products but progressed in the post-interview by showing an increased knowledge of place value. She could explain that the 1 in 16 represented 10, but she struggled to explain farther. She was not able to show the break down of either two-digit number or what the partial products would be, so her SCK did not increase from the pre-interview (Lampert, 1986). Mary began with a procedural explanation and no understanding of the connection to the number sentence to the context of the problem, and was considered to have low SCK. In the post-interview she correctly explained partial products and demonstrated the capability to decontextualize a problem into the proper symbolic number sentence, progressing to high SCK. Both participants progressed in their understanding of partial products and demonstrated an increase in SCK.

Brenda initially could not relate a number sentence to the context of the problem (low SCK), but during the post-interview, she gave a conceptually based explanation of how a number sentence should represent the context of the problem, demonstrating high SCK in regards to matching the context (CCSSI, 2010; Steffe, 1994). Mary also struggled with relating her number sentence to the context of the problem in the pre-interview, but demonstrated the capability to decontextualize a problem into the proper symbolic number sentence by the post-interview, progressing to high SCK.

When asked for an alternate solution strategy, Brenda did not show examples for two of the four problems and showed lower level modeling examples for the other two problems. In the post-interview, she modeled with pictures for all problems. Mary could not come up with examples for half of the problems in the pre-interview. For the other half, she showed low level modeling strategies (Gravemeijer, 2004). By the post-interview, her representations did not change significantly. Both participants showed more solution strategies by the end of the base 8 unit.

Participants showed similar progress in question 5, which was administered specifically to measure change in SCK. Brenda initially did not understand what the child did for methods A and B. She also did not see how any of the methods would work for any two whole numbers. When interviewed after the base 8 unit, she could identify what the child did for all three methods and also conceptually explain why the methods would work for methods A and B. She still struggled to explain method C. Mary explained only what the child did in method B for the pre-interview. She was not sure it would work for any two whole numbers. Mary could identify what the child did for all three methods by the post-interview. She was able to explain why the methods would always work for methods A and B. She still struggled to explain method C.

Brenda began the unit with low CCK and made the progression to high CCK. Mary began with already high CCK, so she did not progress. Both participants began with low SCK in the pre-interview and progressed to high SCK in the post-interview. Although overall, both participants progressed to high SCK due to evaluation criterion, it should be noted that for several problems Mary showed a deeper conceptual understanding than Brenda. For example, she advanced 2 entire categories according to Thanheiser's model (2009), and she was able to explain partial products for the multiplication problem, where Mary could not. Perhaps entering the course with already high CCK is more beneficial in developing SCK. Perhaps if students enter the unit already proficient in the procedural mathematics, they could focus all of their attention on the conceptual development of the mathematics, as was also the case in scenario 2.

Scenario 4: How change in knowledge differs between a participant with beginning low CCK and high SCK (Terri) and a participant with beginning high CCK and high SCK (Dottie).

In this scenario, Terri began with low CCK and Dottie began with high CCK. In the pre-interview, Terri could not perform the standard algorithm for the multiplication and division problems. She could not remember the procedure and solved using alternate solution strategies, demonstrating low CCK. By the post-interview, she still struggled to solve the multiplication and division problems using the standard algorithm, showing no progression in CCK. Dottie could not solve the first problem in the initial interview, however it was assumed that she was nervous, and truly had high CCK due to her responses in subsequent problems. For the post-interview, she solved all problems correctly, indicating little if any change in CCK. She did, however, first solve the problems showing an alternate solution strategy, not the standard algorithm. When asked to solve using the standard algorithm, she was able to solve it and also make a connection to her alternate representation.

Both participants showed that they enhanced their SCK regarding place value, but at different rates. Terri began in the groups-of-ones category in Thanheiser's (2009) model for both questions 1 and 2, but did not advance categories in the post-interview. Dottie began at the same level as Terri, but advanced one category to the highest placement: reference-units. Based upon the criteria for analyzing interview questions, both participants began with high SCK and ended with SCK, indicating no change. It should be noted however, that Dottie did advance to the highest category of understanding. This could be due to the fact that she had a son in second grade and had been spending time in his classroom observing his class. Some of the strategies she used indicated that her son's teacher instructed conceptually rather than procedurally.

Dottie was able to break apart the digits in the multiplication problem and identify the partial products for both the pre and post interviews, so there was no change in her already high SCK. Terri did not identify partial products in the pre-interview, demonstrating low SCK, but was able to break apart the digits by the post-interview to reveal partial products, showing an increase to high SCK.

Dottie could not make a connection between the number sentence and the context of the problem and was classified to have low SCK in the pre-interview. She progressed to high SCK in the post-interview because she could conceptually explain why her number sentence had to match the context of the problem. Terri also could not make a connection between the context of the problem and her number sentence in the pre-interview, but by the end of the unit she could correctly decontextualize the problem and explain the connection and was classified to have high SCK (CCSSI, 2010; Steffe, 1994). Both participants demonstrated an increase from low SCK to high SCK in regard to decontextualizing.

Terri could not differentiate between measurement and sharing division, unlike Dottie who was the only participant interviewed who could make the distinction. She also explained why sharing would be an easier problem type for a child to solve indicating high SCK (Greer, 1992; Silver, 1986). Her SCK in regards to this type of problem was higher than all other participants.

When asked for an alternate solution strategy, Terri demonstrated low-level modeling with pictures as alternate solution strategies for most problems in the pre-interview, and in the post-interview, she showed some pictures and some more abstract strategies. She did conceptually explain regrouping with boxes, rolls, and pieces, but in terms of a drawing, indicating low-level modeling. Dottie showed solution strategies for two problems during the pre-interview, both represented pictorially. In the post-interview, she showed multiple methods on a spectrum of reasoning levels (Gravemeijer, 2004); more methods than Terri. Dottie was also able to make conceptual connections between her representations and the standard algorithm, demonstrating high SCK (Carpenter, 1986).

For question 5, designed to specifically measure SCK, Terri could identify what the child did for every method in the pre-interview. She said every method would work in any instance of whole numbers indicating high SCK, but did not explain why. She was able to explain why for all methods in the interview after the base 8 unit. She conceptually explained why method C was not changing the problem. Dottie identified what the child did for every method in the pre-interview. She believed every method would work for any two whole numbers also indicating high SCK, but did not explain why. She was able to explain why for all methods during the post-interview. She correctly described why method C was not changing the problem as well.

Terri began the course with low CCK and did not progress by the end of the base 8 unit. As noted before, this could be attributed to the nature of the unit, which was to focus on invented algorithms and strategies, not the standard algorithm. Dottie began with high CCK and ended with high CCK as there was no room to progress. She did answer one more question correctly in the post-interview. Both participants began with high SCK and ended with high SCK in regards to the criterion set forth in the methods section. It should be noted that in the post-interview, both participants provided a detailed explanation of why all methods would work in every instance. It should also be noted that, although both ended with high SCK, Dottie showed a deeper conceptual understanding at the end of the unit. Dottie was the only prospective teacher who could make the important connection between the standard algorithm and her invented strategies, demonstrating a conceptual understanding than any of the other participants. Similar to Safi (2009), results from this study showed that a prospective teacher who entered the unit with already high conceptual knowledge, could benefit from instruction in base 8.

One of the scenarios that was not analyzed in depth was how the change in knowledge differed between a participant with beginning low CCK and low SCK (Brenda) and a participant with beginning high CCK and high SCK (Dottie). Brenda progressed more in her SCK since she was so low to begin with, although her conceptual understanding was still considerably low by the end of the base 8 unit. Dottie began with high SCK, and she extended her knowledge to provide more detailed and complete justifications for her solutions than anyone interviewed. Dottie began with several gaps, and she closed them by the end of the base 8 unit. Findings are similar to Safi (2009) in that the participant with the lowest understanding at the beginning of the unit did not progress much by the end of the unit and still did not have a complete conceptual understanding of the mathematics. The student who began the highest, Dottie, had the most complete conceptual understanding of anyone by the end of the base 8 unit.

Another scenario not analyzed in depth was how change in knowledge differed between a participant with beginning high CCK and low SCK (Mary) and a participant with beginning low CCK and high SCK (Terri). As one might expect, the changes in knowledge between these two participants was the most noticeable, since these participants were on opposite ends of the spectrum in terms of knowledge. Mary began with high CCK and Terri began with low CCK. Mary did not progress in CCK because she was already high in that area of knowledge. Surprisingly, Terri did not progress either, which could be explained by the course focusing on invented algorithms and strategies, not standard algorithms. Mary made progress from low SCK to high SCK, but since Terri had high SCK to begin with, she was classified as showing no change. It should be noted however, that she was able to more deeply explain why each method would work by the post-interview.

Summary: Common Content Knowledge

Participants who could solve the addition, subtraction, multiplication, and division questions correctly using the standard algorithm were deemed to have high CCK. Qualitative results showed that Brenda and Terri both began with low CCK and Terri and Dottie both began with high CCK. Brenda went from struggling with three of the four problems incorrectly, to solving them all correctly, demonstrating a shift from low CCK to high CCK. Terri struggled with the multiplication and division problem for the pre and post interview, indicating no change in CCK. Mary and Dottie entered the base 8 unit with high CCK, so there was no room for improvement, and no change in their CCK. Terri's progress could be accounted for by the base 8 unit positively affecting her CCK. Quantitative results indicated that overall, prospective teachers did not show a significant change in CCK by the end of the base 8 unit. Quantitative results also showed that the majority of prospective teachers enter the base 8 unit with high CCK already. There were eleven questions on the MKT that specifically measured CCK. The researcher determined that in order to be considered to have high CCK, participants must be able to answer at least six of the 11 questions correctly. Seventy-four percent of participants were deemed to have high CCK, indicating that most prospective teachers do not enter the base 8 unit with low CCK, so there is little room for improvement, which could explain why both quantitative and qualitative results indicate little progress in CCK.

Many pre-service teachers enter the university system with knowledge tied to standard algorithms (Thanheiser et al., 2014). Prospective teachers entering the base 8 unit have had previous experiences in their schooling, which was mostly procedural. Solving addition, subtraction, multiplication, and division problems with algorithms is something that prospective teachers learn in elementary school and used throughout their academic history in mathematics. Therefore, because solving these problems could be considered elementary, the majority of prospective teachers entering the base 8 unit would presumably still have this knowledge by the time they arrive at a university. Because they already had this knowledge at the beginning of the base 8 unit, there was not much room for progression, which could account for the fact that quantitative results showed no change in CCK by the end of the unit. Additionally the focus of the base 8 unit was not on standard algorithms, rather the focus was on fostering a conceptual understanding of whole number operations.

One would think that a unit designed to develop prospective teachers' concepts of whole numbers and operations would increase the CCK of participants who entered with low CCK. Terri's experiences may help explain why prospective teachers who entered with low CCK did not progress. She did not show signs of progression in CCK by the end of the unit. When she was interviewed the second time, she used the standard algorithm for addition and subtraction, but not for multiplication and division. She demonstrated a pictorial representation for the addition and subtraction problems, and was unable to make the connection between the standard algorithm and her representation. For multiplication and division, she only showed alternate solution strategies, not the standard algorithm. When asked to perform the standard algorithm for multiplication she responded, “The standard algorithm? I don’t know that means.” The researcher explained that it was the traditional way of multiplying when the two numbers are stacked. Terri replied, “I honestly don’t even remember the standard one…as a kid I never got it that way.” Her responses could indicate that the actual procedure of the standard algorithm was either not covered during the base 8 unit, or was not covered in depth. The base 8 unit was designed to develop prospective teachers’ conceptual understanding of whole number concepts and operations, and not necessarily procedural fluency. Her responses could also indicate that sufficient time was not taken to make the important connection between the standard algorithm and alternate representations. To further back up this theory, Mary and Brenda were also not able to make a connection between alternate solution strategies and the standard algorithm.

Summary: Specialized Content Knowledge

Qualitative results were also consistent with quantitative findings in that most prospective teachers did show improved SCK. Regardless of the level of understanding with which a student entered the base 8 unit, there was some progress in SCK. Some prospective teachers progressed more than others. In general, participants who began with low SCK seemed to progress more than prospective teachers with high SCK. This could be explained by the fact that prospective teachers with low SCK had more room to progress than prospective teachers with high SCK, since prospective teachers with high SCK already entered with high conceptual understanding. Interviews helped explain why some prospective teachers may not have progressed as much as others in their SCK.

One area of interest was participants' change in understanding of place value concepts, which was analyzed using Thanheiser's (2009) model. In this model, prospective teachers were categorized into one of four groups.

1. *Reference units.* PSTs with this conception reliably1 conceive of the reference units for each digit and relate reference units to one another, seeing the 3 in 389 as 3 *hundreds* or 30 *tens* or 300 ones, the 8 as 8 *tens* or 80 *ones*, and the 9 as 9 *ones*. They can reconceive of 1 *hundred* as 10 *tens*, and so on.

2. *Groups of ones*. PSTs with this conception reliably conceive of all digits in terms of groups of ones (389 as 300 *ones,* 80 *ones*, and 9 *ones*) but not in terms of reference units; they do not relate reference units (e.g., 10 *tens* to 1 *hundred*).

3. *Concatenated-digits plus.* PSTs with this conception conceive of *at least one* digit as an incorrect unit type at least sometimes. They struggle when relating values of the digits to one another (e.g., in 389, 3 is 300 ones but the 8 is only 8 ones).

4. *Concatenated-digits only*. PSTs holding this conception conceive of *all* digits in terms of *ones* (e.g., 548 as 5 *ones,* 4 *ones*, and 8 *ones*). (Thanheiser, 2009, p. 263).

The categories were ordered hierarchically from most sophisticated to least sophisticated conception. Explanations that represented reference units or groups of ones conception were considered high conceptual (SCK) understanding and correct. Explanations that represented concatenated-digits plus or concatenated-digits only were considered low conceptual understanding and incorrect (Thanheiser, 2009).

Brenda, who began with low SCK, was placed in the concatenated-digits-plus category for question 1 and concatenated-digits-only for question 2. She had only advanced a category for question 2 by the post-interview, placing her in the low SCK, incorrect group indicating no shift in SCK. Mary also began with low SCK in the concatenated-digits-only for the pre-interview. She advanced two categories by the post interview (Thanheiser, 2009). She went from having low SCK to high SCK. It is possible that the reason Mary was able to make more conceptual connections is because she began with high CCK, and already knew the procedures for the standard algorithm which allowed her to focus more on making connections.

Terri began with high SCK in the groups-of-ones category in Thanheiser's (2009) model for both questions 1 and 2, but did not advance categories in the post-interview, indicating no shift in SCK. It could be due to the fact that she struggled to make connections between alternative solution strategies to algorithms as she began with low CCK. Also, Terri already had a high conceptual understanding of place value, so there was little room for growth. Dottie, who also began with high SCK, began with a correct conceptual understanding of place value in the groups of ones category. She advanced one category by the post-interview to the highest level of understanding, reference units. Dottie began with high SCK and ended with high SCK.

As a trend, participants tended to advance in their place value understanding by one to two categories in Thanheiser's (2009) model by the end of the unit. For example, Brenda advanced from concatenated digits only to concatenated digits plus for subtraction. Mary began in the concatenated digits only category and progressed to groups of ones understanding. Dottie began in the groups of ones category and ended in referent units. Terri was the only one who did not make progress. Another trend was that by the post-interview, prospective teachers were able to identify each digit in terms of its place value. For example in the number 520, prospective teachers could identify the 5 as representing 500, the 2 as 20 and the 0 as representing zero ones. Also, most prospective teachers could conceptually explain regrouping from the tens to the ones place. Issues arose when regrouping from the hundreds to the tens column. Participants understood that they were moving 100 over from the hundreds place, but could not explain why the 100 looked like a ten in the tens place. They were unable to explain that there were 10 tens in 100 which is consistent with Thanheiser's (2012) findings of prospective teachers having a conceptual gap between regrouping from hundreds to tens versus tens to ones. Dottie was the only participant who was able to conceive 100 as 10 tens.

Participants were also evaluated in their ability to conceptually explain a multiplication problem. "Numbers to be multiplied can be decomposed additively, each of the elements operated on separately, and the product obtained from recomposing the 'partial products'" (Lampert, 1986, p. 14). When a prospective teacher shows how a multi-digit number can be broken apart, displaying the partial products, one can see that conceptual understanding is evident. Therefore prospective teachers who were able to break down either or both digits in question three to reveal partial products were considered to have SCK. A trend for the multiplication component of the pre-interview was that prospective teachers were unaware that they could break apart the digits and multiply by numbers in regard to their place value. They only had procedural knowledge of the standard algorithm and could not explain any conceptual connection to the algorithm. By the post-interview, prospective teachers showed how digits could be broken up into partial products and conceptually explained the multiplication. Dottie was the only participant who broke apart the numbers in the pre-interview. Consistent with Safi's (2009) findings, Dottie, as a participant with high mathematical content knowledge, went above and beyond other prospective teachers in the post-interview and was able to make a connection between partial products and the standard algorithm. She also discussed how the method would make multiplication easier for her future students to understand, indicating that the instructional unit appeared to have improved her ability to address the needs of her future students.

Another criteria used to determine SCK was a prospective teacher's ability to make connections between contextual and symbolic representations of a mathematics problem (NCTM, 2014). Current standards assert that students must be able to "decontextualize—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to contextualize, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved" (CCSS, 2010, p. 9). If a prospective teacher could make the connection between the context and number sentence, they were considered to have SCK.

All prospective teachers interviewed initially could not make a connection between the context of the multiplication problem and writing a number sentence to represent the context. Even Dottie, who had higher SCK than anyone going into the base 8 unit, lamented that she was following a rule and multiplying in the order the numbers appeared in the problem because that is what her teachers in the past had taught her to do. By the post-interview, all prospective teachers could explain how the order of the number sentence would matter in regards to the context of the problem. They could identify packages as groups, and the sports drinks as the number of pieces in each group. The capability to decontextualize a problem into the proper symbolic number sentence, representing groups times number of pieces in each group, is an indicator of conceptual understanding (Steffe, 1994).

Research has shown that when asked to interpret division satiations, prospective teachers have a definite tendency to interpret situations to be sharing (Silver, 1986). If prospective teachers cannot make the distinction between sharing and measurement division, they may not have a conceptual understanding of division (Greer, 1992). If prospective teachers were able to make the distinction in the post-interview, they were considered to have SCK. Qualitative results revealed an overwhelming trend of prospective teachers not being able to make the distinction. Participants who guessed always guessed sharing and said that it was because the student in the scenario was sharing stickers. This could be a result of the question actually saying the word share. It could also be because conceptually understanding the division algorithm is difficult for prospective teachers (Simon, 1993). Another reason for the lack of understanding could be because there was not enough time spent on this concept in the base 8 unit. One student said she did not recall the instructor talking about measurement and sharing division nor the alternate labels; partitive and quotative division. It is possible that instructors run out of time and do not thoroughly cover whole number division since it is the last topic in the unit. Difficulty differentiating units may contribute to difficulty conceptualizing division (Simon, 1993), so there should be ample time devoted to this concept.

Prospective teachers demonstrated no conceptual connection between groups, number of pieces in each group, and which is known or unknown. Results are consistent with research on division types in that while children can make the distinction between the two problem types, adults can not because they are not thinking about the actions or relationships being depicted in the problem (Carpenter et al., 1996). Dottie was the only prospective teacher who could make the distinction and provide a conceptual description regarding groups and number of pieces in each group in regard to measurement and sharing division. Another similarity was that while prospective teachers could perform the standard division algorithm, they could not explain a conceptual connection of why it worked.

Question five was taken directly from the MKT and was categorized to measure SCK (Hill et al., 2005). It presented three different solution strategies for a three-digit subtraction problem, and required participants to identify if each method was correct and whether each method would work for any two whole numbers. Prospective teachers should be able to identify what the child for each method and whether that method would work for any two whole numbers if they have SCK (Hill et al., 2004). Prospective teachers were considered to have SCK if they were able to identify what the student did and whether the solution methods would work in any subtraction scenario for whole numbers for the majority, 2 of the 3, methods.

Qualitative data revealed several trends in regards to question 5. Both of the prospective teachers who began with low SCK, Brenda and Mary, initially did not understand what the child did for most of the methods. They both did not think the methods would work for any two whole numbers. By the post-interview both prospective teachers could identify what the child did for every method and also said that all methods would work in every instance, indicating a shift to high SCK.

Dottie and Terri, who began with high SCK, could identify what the child did for every method in the pre-interview. They also indicated that all methods would work every time in the case of any two whole numbers. Although going by the theoretical framework criteria, there was no room for improvement, it should be noted that both prospective teachers went above identifying whether the methods would always work. They also went into detail as to why each method would work, in particular method C, which was the most difficult to understand.

Another area of interest, that was not evaluated directly, but was apparent in interviews, was how prospective teachers' use of alternate solution strategies changed. In general, participants with initial low SCK were either not able to come up with alternate strategies, or explained a few low level modeling strategies such as pictorial representations. Participants with higher SCK began with mostly pictorial representations. The participants with low SCK were able to demonstrate pictorial representations as well as a few abstract representations by the end of the unit. Participants with high SCK progressed to more sophisticated reasoning strategies (Gravemeijer, 2004). Participants with higher initial SCK demonstrated more abstract and sophisticated reasoning strategies because they were further along initially in their knowledge. For example, Dottie discussed the Candy Shop scenario for the subtraction problem without drawing pictures of boxes, rolls, and pieces. She referred to the boxes, rolls, and pieces as she was referencing each place value column in solving using the standard algorithm. Another notable observation was that overall, prospective teachers showed more strategies for solving the problems in the post-interview.

The prospective teacher who had the most conceptual understanding of number concepts and whole number operations by the end of the unit was Dottie, which is consistent with the findings of Safi (2009). One would expect this result, as she was already so knowledgeable when she entered the base 8 unit. She entered the base 8 unit with high SCK, although there were some gaps, and she filled them. She was the only student interviewed who cold fully explain regrouping from the hundreds to the tens, and she was the only student interviewed who made a conceptual connection between the standard addition and subtraction algorithms and the candy shop scenario. The connection demonstrated true conceptual understanding had occurred (Carpenter, 1986). Dottie was the only participant who could differentiate between sharing and measurement division. She also showed the most variety of solution strategies and the most sophisticated reasoning for her alternate representations of the problems (Gravemeijer, 2004). Dottie ended up with the deepest understanding of whole number concepts because she entered with high SCK. Another explanation for her initial deep understanding is that she was spending time in her son's elementary classroom, and was learning concepts conceptually in that setting.

Also consistent with Safi's (2009) findings is that the prospective teacher who had the least conceptual understanding of number concepts and whole number operations by the end of the unit was Brenda. Brenda entered the base 8 unit with low SCK in that she had strictly procedural knowledge of number operations. She was placed in the lowest categories in Thanheiser's (2009) model, representing low conceptual understanding. She remained in those categories in the post-interview, indicating no improvement in SCK. While Brenda did improve in her ability to decontextualize, she was unable to deeply explain partial products. The most improvement could be seen in question 5, where Brenda progressed from not understanding what the child did for any method in the problem, to understanding not only what the child did, but indicating that the method would work for any two whole numbers. While overall Brenda's understanding was the lowest, she did progress from having low SCK to having high SCK in question 5 and also was able to demonstrate more solution strategies by the close of the base 8 unit.

The participant who progressed the most, in terms of who made the largest gains in understanding was Mary. She was the only participant to jump two categories in Thanheiser's (2009) model for place value understanding. The only thing keeping her from achieving the highest category was that she struggled to explain why the 100 looked like a 10 when regrouping from the hundreds to the tens place. She progressed from initially having a strictly procedural explanation of double digit multiplication to breaking down the two digits to show the conceptual piece of partial products. She also went from being unable to decontextualize a problem in context to understanding how to match the number sentence to the context of the problem. Mary also went from struggling to show alternate solution strategies, to modeling all four. Finally, she showed a more complete understanding in regards to question 5. She went from not understanding most methods to understanding all methods and even explaining why they would work for any two whole numbers. Perhaps the reason she was able to progress so significantly is because she already had high CCK, and therefore was able to focus all of her attention to gaining the conceptual piece, which would explain why she showed more progress than other prospective teachers. Mary's development is an interesting finding of this study because she represents the typical student entering the base 8 unit, a student with high CCK and low SCK. Most prospective teachers enter the base 8 unit with high CCK because solving addition, subtraction, multiplication, and division problems could be considered elementary, and the majority of prospective teachers entering the base 8 unit would presumably still have this knowledge by the time they arrive at a university. Many pre-service elementary school teachers lack conceptual knowledge in the mathematics area of whole number and operations and their understandings are rooted in rote memorization or rules and procedures (Thanheiser et al., 2014). The finding is positive in that because this student fits the typical profile of a teacher candidate entering the base 8 unit, there is a chance that the majority of prospective teachers taking the base 8 unit will experience a more complete, conceptual understanding of mathematics.

CHAPTER 5: CONCLUSION

The principal purpose of this study was to measure the impact of an elementary mathematics course unit taught in base 8 on pre-service teachers' mathematical common content knowledge and specialized knowledge for teaching of elementary number concepts and operations. The researcher also sought to explore how prospective teachers' participation in the instructional unit explain the change, if any, in their common content knowledge and specialized content knowledge via individual student interviews. High quality mathematics teachers not only know algorithms, but also have a conceptual understanding of the content (Ball, 1990). Recent reforms in mathematics education require students to have a deep understanding of the material with the ability to explain and justify their reasoning (CCSSI, 2010; NCTM, 2014).

Research has shown that prospective teachers do not enter the classroom prepared to teach to the depth and rigor of current standards (Chapman, 2007; Thanheiser et al., 2014). Preparing teachers to teach conceptually versus procedurally could influence students to understand mathematics at a deeper level (Cobb et al. 1991). Teachers must possess a specialized knowledge unique to teaching. It includes a conceptualization of the mathematics content (Ball, 1990; Fennema & Franke, 1992; Lee et al., 2003; Shulman, 1986). One possible option for increasing teacher candidates' conceptual knowledge is teaching them number concepts and operations in base 8.

Prior to this study, there was a small body of research on instructing pre-service teachers in a different base. McClain (2003) and Yackel and colleagues (2007) focused on addition and subtraction operations. Fasteen and colleagues (2015) built on that work to include multiplication operations. Andreasen (2006), Roy (2008), and Safi (2009) focused on addition, subtraction, multiplication, and division operations, and how instruction in base 8 influenced prospective teachers' understanding of place value and operations in different iterations of an elementary mathematics base 8 unit. Andreasen (2006) developed an instructional sequence based upon prior research on children's understanding of place value and whole number operations. She also utilized prior research regarding instruction in base 8. Her research concentrated on the social context of the classroom. Roy (2008) built on Andreasen's work. He focused on individual pre-service teacher's conceptual development in regards to specific learning goals established in the classroom. Safi (2009) extended the work of Andreasen and Roy by providing a deeper insight into the way *individual* prospective teachers' conceptual understanding of whole number concepts and operations developed in a social context. Safi (2009) specifically examined the development of two prospective teachers' understanding of whole number concepts and operations in the same base 8 unit utilized by this study. The participants selected were classified as either high performing or low performing. The current study aims to build on prior research by further breaking down the different types of incoming prospective teacher knowledge, and examining how it changes with a specific classification and focus on CCK and SCK.

In this mixed methods study the researcher examined the impact of an elementary mathematics unit taught in base 8 on prospective teachers' knowledge of elementary number concepts and operations. Teacher knowledge was measured quantitatively through administration of the Mathematical Knowledge for Teaching (MKT) measures at the beginning and end of the base 8 unit, and also through in depth individual student interviews. The researcher used interview data to help explain results from the measure and to more deeply explore prospective teacher understanding. There were two constructs of interest, CCK and SCK, and the researcher examined how varying levels of each type of knowledge changed as a result of the base 8 unit. The researcher interviewed four students: (1) a student with initial high CCK and low SCK, (2) a student with initial low CCK and high SCK, (3) a student with initial low CCK and low SCK, and (4) a student with initial high CCK and high SCK. Interview data was analyzed case by case, and also across cases.

In previous research efforts (Roy, 2009; Safi, 2008), the MKT was administered to measure overall mathematical knowledge needed for teaching, which is comprised of different types of knowledge, including CCK and SCK. This study aimed to break down the types of knowledge needed for teaching and focused specifically on CCK and SCK. Writers and researchers of the MKT did not provide classifications regarding CCK or SCK for each individual item on the measure. They only indicated that items could be classified as CCK and SCK. Additionally they explained that some items would classify as both CCK and SCK. As a result, the researcher synthesized prior researchers' definitions of CCK and SCK (Ball et al., 2005; Hiebert & Lefevre, 1986; Hill & Ball, 2004; Skemp, 1978; Thompson et al., 1994) in order to determine which questions fit into each category of knowledge. CCK items were selected if they measured mathematical knowledge such as calculating, making correct mathematical statements, and the ability to solve problems using standard algorithms, rules, or procedures. Three items on the MKT were classified to represent CCK. The three questions had sub questions, totaling 11 questions selected to measure CCK. SCK items were selected if they required participants to represent mathematical ideas and operations, provide justifications, or to understand student invented algorithms (Ball et al., 2005). There were six items total selected to measure SCK.

In regard to CCK, quantitative results indicated that overall, prospective teachers did not show a significant change in CCK by the end of the base 8 unit ($p = .774$). Descriptive statistics indicated that the majority of participants entered the base 8 unit with high CCK. Eleven questions on the MKT measured CCK. The researcher selected these questions because they most closely matched other researchers' definitions and descriptions of CCK. Questions selected were rule based and required only procedural knowledge in order to get the correct answer. Of the 11 questions, the researcher determined that in order to be considered to have high CCK, one should answer the majority of questions correctly, which would be six questions. Seventy-four percent of participants got six or more CCK questions correct on the pre-test. Results supported the claim that most prospective teachers do not enter the base 8 unit with low CCK. Quantitative results showed no significant change in CCK by the end of the base 8 unit. Perhaps the reason there was not a significant change is because the majority of participants already enter with high CCK, so there is little room for improvement, which could explain why both quantitative results indicate little progress in CCK.

Overall, interview also did not show change in participants' CCK. Qualitative interviews showed that both Brenda and Terri began with low CCK and Terri and Dottie both began with high CCK. Brenda went from struggling with three of the four problems, to solving them all correctly, demonstrating a shift from low CCK to high CCK. Terri struggled with the multiplication and division problem for the pre and post interview, indicating no change in CCK. Mary and Dottie began and ended the unit with high CCK. These results are inconsistent with measure results in that measure results showed everyone but Dottie got one more CCK question correct in the post-test than in the pre-test. One possible explanation the results are different is because CCK items on the measure were more rule based questions and the interview questions resulted in prospective teachers having to follow an algorithm.

Mary and Dottie's lack of progress could be explained by their already high CCK. Because they already had high CCK, like the majority of students entering the base 8 unit, there was little room for improvement, and therefore no change in their CCK. Brenda made a shift from low CCK to high CCK, indicating that the base 8 unit may help develop participants' CCK. The base 8 unit instruction focused on conceptual understanding (SCK) rather than procedural knowledge (CCK), which might have enabled Brenda to connect procedures and algorithms and understand reasons underlying the procedure, therefore helping her develop her CCK.

Terri had beginning low CCK and did not make a shift to high CCK. Terri was unable to make the connection between the standard algorithm and her representation for the addition and subtraction problems, which could have been the reason her CCK did not advance. For multiplication and division, she only showed alternate solution strategies, not the standard algorithm. When asked to perform the standard algorithm for multiplication she responded, "The standard algorithm? I don't know that means." The researcher explained that it was the traditional way of multiplying when the two whole numbers are stacked. Terri replied, "I honestly don't even remember the standard one…as a kid I never got it that way." The base 8 unit was designed to develop prospective teachers' conceptual understanding of whole number concepts and operations, and not necessarily procedural fluency. Her responses could mean that either she could not grasp the connection between the standard algorithm and conceptual connections, or perhaps the connection was not covered in depth during the base 8 unit. Her responses could also indicate that sufficient time was not taken to make the important connection between the standard algorithm and alternate representations. Mary and Brenda were also not able to make a connection between alternate solution strategies and the standard algorithm, which backs up the claim that the reason some students do not progress in CCK is the lack of ability to connect standard algorithms to conceptual concepts.

Similar to results of Andreasen (2006), Roy (2008), and Safi (2009), prospective teachers in this study experienced an increase in their conceptual understanding (SCK) as a result of taking the base 8 unit. Overall, quantitative results for all 91 participants who took the measure showed a significant difference in pre-test and post-test scores ($p = .035$) on the MKT in regard to the six SCK questions selected. Quantitative results for only the participants interviewed showed that Dottie actually answered one less SCK question correctly. Dottie's results could be explained by her having an "off" day, as the qualitative results indicated otherwise. Brenda did not make a shift on the measure in regard to SCK. Mary and Terri both answered 1 more SCK question correctly on the post-test. Qualitative interview results showed that all participants interviewed in this research effort enhanced their conceptual understanding of whole number concepts.

Most participants progressed in their place value understanding by one to two categories in Thanheiser's (2009) model by the end of the unit. Also, most participants could conceptually explain regrouping from the tens to the ones place. Participants experienced difficulty when regrouping from the hundreds to the tens column. As a trend, most understood that they were moving 100 over from the hundreds place, but could not explain why the 100 looked like a ten in the tens place. They were unable to explain that there were 10 tens in 100 which is consistent with Thanheiser's (2012) findings of students having a conceptual gap between regrouping from hundreds to tens versus tens to ones. Dottie was the only participant who was able to conceive 100 as 10 tens.

A trend for the pre-interview multiplication problem was that prospective teachers explained the multiplication procedurally, and did not show the breakdown of digits to reveal partial products. As expected prior to conducting the research, prospective teachers only had rule-based knowledge of the standard algorithm and could not explain any conceptual connection to the algorithm. At the post-interview, prospective teachers showed how digits could be broken up by place value, and multiplied to show partial products. They were also able to conceptually explain the multiplication. Consistent with Safi's (2009) findings, Dottie, the participant with high mathematical content knowledge, went above and beyond other students in the post-interview and was able to make a connection between partial products and the standard algorithm. This could be explained by the fact that she observed in her son's classroom, where the teacher made conceptual connections in the mathematics.

Another criteria used to determine SCK was a prospective teacher's ability to make connections between contextual and symbolic representations of a mathematics problem (NCTM, 2014). All prospective teachers in the pre-interview could not make a connection between the context of the multiplication problem and writing a number sentence to represent the context. By the post-interview, all students could explain how the order of the number sentence would matter in regards to the context of the problem. They could identify packages as groups, and the sports drinks as the number of pieces in each group. The capability to decontextualize a problem into the proper symbolic number sentence, representing groups times number of pieces in each group, is an indicator of conceptual understanding (Steffe, 1994).

Overall, for the division scenario, prospective teachers showed no conceptual connection between groups, number of pieces in each group, and which is known or unknown during the pre-interview. Results are consistent with research on division types in that while children can make the distinction between the two problem types, adults typically only recognize problems as division because they are not thinking about the actions or relationships described in the problem (Carpenter et al., 1996). Dottie was the only participant who could make the distinction and provide a conceptual description regarding groups and number of pieces in each group in regard to measurement and sharing division in the post-interview. Another trend was that all participants could perform the standard division algorithm, but none could explain it conceptually.

Prospective teachers all progressed in their conceptual understanding of question 5, which was selected from the MKT to specifically measure SCK. The prospective teachers who began with low SCK, Brenda and Mary, made a shift from low SCK to high SCK. Dottie and

Terri, who began with high SCK, went above identifying whether the methods would always work for the post-interview. They also went into detail as to *why* each method would work, in particular method C, which was the most difficult to understand.

Dottie had the strongest conceptual understanding of number concepts and whole number operations by the end of the unit, which is consistent with the findings of Safi (2009). This result is expected, as she was already very knowledgeable when she entered the base 8 unit. She entered the base 8 unit with high SCK, although there were some gaps, and she filled them by the end of the base 8 unit. She was the only participant who was able to make the important connection between standard algorithms and the deeper mathematical concepts. Dottie's progress suggests that students who enter the base 8 unit with high CCK and high SCK are able to take the strategies and concepts they learn during the base 8 unit, and connect them to standard US algorithms. Once students can make a connection between concepts and algorithms, deep understanding has taken place (Carpenter, 1986; NRC, 2001). Furthermore, these students who have the strongest conceptual understanding are more likely to provide learning experiences for their students that lead to this type of understanding (Ma, 1999).

Also consistent with Safi's (2009) findings is that Brenda, the prospective teacher with initially the weakest conceptual understanding, had the least conceptual understanding of number concepts and whole number operations by the end of the unit. Brenda entered the base 8 unit with low CCK and low SCK. While by the end of the unit she had the ability to perform the procedures using the standard algorithm, she lacked conceptual understanding required to justify why the mathematics worked. It should be noted that although Brenda did not advance overall based upon criterion laid out in the methodology, Brenda did progress in her understanding in some areas. She initially could not decontextualize a problem, but after the base 8 unit, she gave a conceptually based explanation of how a number sentence should represent the context of the problem, demonstrating high SCK (CCSSI, 2010; Steffe, 1994). Also, for Question 5, Brenda initially could not understand what the child did for any of the methods, nor did she think the methods would work for any two whole numbers. By the post-interview she could identify what the child did for every method and also said that all methods would work in every instance, indicating a shift to high SCK. The two aforementioned areas do not require students to make a connection between a concept and a standard algorithm. Perhaps making that important connection is most difficult for students who enter the base 8 unit with low CCK and low SCK, which is why Brenda was able to advance in those two areas and not for the addition, subtraction, multiplication, and division operations problems. Qualitative results from Brenda's interview show that the base 8 unit was beneficial in developing her ability to decontextualize a problem and evaluating student work, but she still struggled to make important connections between standard algorithms and mathematical concepts.

Terri began with overall high SCK. Based upon the criterion in the methodology she did not advance in SCK by the end of the unit, but it should be noted that Terri did show more solution strategies, and she went deeper in her explanations and justifications of solutions. For example, in the post-interview, Terri provided a detailed explanation of the Candy Shop scenario for place value problems. Terri also demonstrated the ability to break apart digits in the multiplication problem and explained partial products. Although she began with high SCK for question 5, which examined student solution strategies, she went into greater detail of explaining why the students' strategies would work for any two whole numbers. She also gained the ability to decontextualize a problem. Results from Terri's interviews showed that the base 8 unit could be beneficial for students who enter the Elementary School Mathematics Course with low CCK and high SCK in regards to their conceptual understanding. Terri entered the base 8 unit with an overall high SCK, but had some gaps in her conceptual understanding. By the end of the unit she had filled those gaps. Terri did still struggle with performing the standard algorithm at the end of the unit, but the base 8 unit was designed to develop prospective teachers' conceptual understanding of whole number concepts and operations, and not necessarily procedural fluency. Her responses indicate that she was not able to make the important connection between the standard algorithm and alternate representations. Mary and Brenda were also not able to make the connection, which could indicate that making connections is the most difficult concept for students to grasp.

The participant who made the largest gains in conceptual understanding (SCK) was Mary. Mary could have progressed significantly because she already had high CCK, and therefore was able to focus all of her attention on developing a conceptual understanding of the mathematics. The focus might have enabled her to better connect procedures and algorithms to concepts, and understand the reasons underlying the procedures. Mary's development is an interesting finding of this study because she represents the typical student entering the base 8 unit, a student with high CCK and low SCK. The finding is encouraging because Mary fits the typical profile of a teacher candidate entering the base 8 unit. Therefore, it is possible that most students taking the base 8 unit will experience a more complete, conceptual understanding of mathematics.

The results of this study suggest that learning place value and whole number operations in base 8 can be beneficial for prospective teachers' conceptual understanding of the mathematics. Learning in a different base creates cognitive dissonance and prospective teachers cannot rely on engrained algorithms, rules, and procedures to help them through the mathematics. They must start from square one and re-learn mathematical concepts. The activities, tasks, and exploration of mathematics in the Elementary School Mathematics Course aide in the conceptual development of place value and whole number operations in base 8. It is suggested that university elementary education programs incorporate instruction in base 8 into their curriculum. Additionally, it is suggested that school districts incorporate base 8 into their elementary mathematics teacher professional development, as in-service teachers may benefit from re-developing place value and whole number operations. When teachers and prospective teachers experience a conceptual understanding of mathematics concepts, they have the opportunity to provide those types of experiences to their future students.

Limitations

This study incurred several potential limitations. The researcher employed convenience sampling, as participants were selected from readily available Elementary School Mathematics Content Course sections at a college located in the southeast United States (Gall et al., 2007). The sampling procedure resulted in no random sampling, and a small sample size, which is a threat to generalizability. The researcher only had access to certain classes, and even then, not every student volunteered to be part of the study. Additionally, with interviews, the researcher could only interview students who were willing to stay after class, and those students could have been more motivated and eager learners.

The design of the study also posed a threat to internal validity since there was a chance that any statistical significance found was due to pre-existing group differences rather than the treatment. These differences could have included prior experience working with children, a stronger mathematics background, or general classroom climate.

Other possible threats to validity included data collection and testing validity. Students could have left test items blank or test scores could have been entered incorrectly. In attempt to decrease error, the researcher had a colleague look over the statistics tables. Prospective teachers also could have been learning things in other classes that caused a change in scores. Testing validity refers to students performing better on the post-test because it is similar to the pre-test, and they remember certain types of questions. Over a month of time passed in between interviews and students were exposed to so many different problems in the base 8 unit. They did not seem to remember the problems from the pre-test.

The researcher used data from the pre-test to select interviewees. Prospective teachers were selected based upon how they performed on items measuring CCK and items measuring CCK. The researcher did not make an effort to make sure each of the three course sections were represented, but based upon pre-test data, participants just so happened to come from each section, therefore, interviewees had different instructors, which could have had an impact on results.

The researcher utilized the MKT Elementary Number Concepts and Operations measure to select cases for the qualitative data collection and analysis. The measure contained items that could be differentiated into common content knowledge or specialized content knowledge. Writers and researchers of this measure did not provide classifications for each individual item on the measure; they only indicated that items could be classified as CCK and SCK. As a result, the researcher synthesized prior researchers' definitions of CCK and SCK (Ball et al., 2005; Hiebert & Lefevre, 1986; Hill & Ball, 2004; Skemp, 1978; Thompson et al., 1994) in order to determine which questions fit into each category. CCK were selected if they measured mathematical knowledge such as calculating, making correct mathematical statements, and the ability to solve problems using standard algorithms, rules, or procedures. SCK items were selected if they required participants to represent mathematical ideas and operations, provide justifications, or to understand student invented algorithms (Ball et al., 2005). The researcher understands that in numerous research projects, a single investigator handles issues such as this, however a team of researchers could have analyzed the measure items to provide additional reliability.

Data from qualitative interviews was analyzed by looking for common themes. An open coding method was utilized. The researcher employed the constant comparative method to saturate themes (Creswell, 2007). Repetitive instances that represented a theme were noted and the researcher continued to look for those instances until no further could be produced from the data. A fellow doctoral student also reviewed the data and categorized it into themes. The researcher then compared the two analyses to reach a final conclusion. While the individual who analyzed the data is qualified and worked independently from the researcher, it would be preferable to have an entire team of researchers analyzing data, which would provide an additional measure of reliability.

Finally, due to time constraints, there was no opportunity to follow up with prospective teachers to member check. Member checking would allow the researcher to restate or summarize the information gleaned during data collection and discuss accuracy with the participant. Participants are given the opportunity to comment on findings. Member checking would have given more credibility to the study if participants had agreed with the conclusions found in the study.

Implications for Teaching and Learning

This study focused on the constructs CCK and SCK in regard to prospective teacher mathematical knowledge. Those constructs were utilized to determine research participants and analyzing data. The researcher attempted to add to the existing research regarding instruction in an alternate base system, and how it would impact prospective teachers' conceptual understanding of place value and whole number operations. Results of this study do not conclude the research on this topic, but serve to further explain how conceptual understanding is developed and to raise questions, which lead to future research. The goal of this study was to provide insight into prospective teachers' thinking and understanding of the mathematics needed for teaching. Results could influence the ever-changing world of teacher preparation.

Results of this study indicated that the majority of students did show enhanced SCK. Regardless of the level of understanding with which a student entered the base 8 unit, there was some progress in SCK. An interesting finding of the study was that the participant who progressed the most was Mary, the student who entered the unit with high CCK and low SCK. Mary fit the profile of a typical student entering the base 8 unit. Most students already have the procedural knowledge needed to perform a standard algorithm to get the correct answer. They lack the conceptual piece to help explain why the math works (Thanheiser et al., 2014). Mary's success is an interesting and optimistic finding of this study because if the majority of students entering the base 8 unit have Mary's beginning understanding, there is an implication that most students who take the base 8 unit will experience the same deep increase in conceptual knowledge.

Brenda entered the base 8 unit with low CCK and low SCK. She had the least conceptual understanding of number concepts and whole number operations by the end of the unit, although she did progress in her understanding in some areas. She improved her ability to decontextualize a problem, and she improved her ability to analyze student solution strategies: two areas which do not require participants to make a connection between a concept and a standard algorithm. It is possible that making the connection between concepts and algorithms is most difficult for most students taking the base 8 unit. The theory is further backed up by the fact that Mary and Terri also could not make the connection. This finding implies that instructors of the base 8 unit, as well as instructors of any elementary school mathematics education course should focus more attention on helping students discover the connection between mathematical concepts and their connection to procedures.

Terri began with overall low CCK and high SCK. Results from Terri's interviews showed that the base 8 unit could be beneficial for students who enter the Elementary School Mathematics Course with low CCK and high SCK in regards to their conceptual understanding. Although Terri entered the base 8 unit with high SCK, she had some gaps in her conceptual understanding. By the end of the unit she had filled those gaps, suggesting that the base 8 unit helped her deepen her conceptual knowledge. Terri still struggled with performing the standard algorithm at the end of the unit, but the unit was designed to develop prospective teachers' conceptual understanding of whole number concepts and operations, and not necessarily procedural fluency. She was not able to make the important connection between the standard algorithm and alternate representations, further supporting the need for mathematics educators to create environments for students to make this important connection. Perhaps if Terri had been able to make the connection, she would have progressed in her CCK as well.

Dottie had the deepest conceptual understanding of number concepts and whole number operations by the end of the unit. Dottie's progress proposes that students who enter the base 8 unit with high CCK and high SCK are the most likely to take the strategies and concepts they learn during the base 8 unit, and connect them to standard US algorithms. Once students can make a connection between concepts and algorithms, deep understanding has taken place (Carpenter, 1986; NRC, 2001). Furthermore, these students who have the strongest conceptual understanding are more likely to provide learning experiences for their students that lead to this type of understanding (Ma, 1999).

Prospective teachers' overall demonstrated a lack of understanding in regards to division concepts. Only Dottie was able to make the distinction between sharing and measurement division, and no participant could explain why the standard division algorithm worked. This is a particular area of concern because prospective teachers need to be able to provide both sharing and measurement scenarios to their future students as well as explain to them the underlying concepts of the division algorithm. Perhaps the reason prospective teachers did not demonstrate understanding in this area is because division is the last topic covered in the base 8 unit, and if instructors are running short on time, adequate time is not spent on this important concept. Results from the interviews may be beneficial to mathematics teacher educators in that they can plan adequate time and activities which foster making the connection between the standard algorithm and underlying concepts.

Another discovery worth noting is how prospective teachers' use of alternate solution strategies changed. In the pre-interview, participants struggled to come up with alternate strategies, or explained a few low level modeling strategies such as pictorial representations (Gravemeijer, 2004). By the post-interview, all participants demonstrated more strategies than the pre-interview. Participants with initial higher SCK progressed to more sophisticated and abstract reasoning strategies. This information may be valuable to mathematics teacher educators, because Dottie, who was the only participant who could make a connection between algorithms and mathematical concepts, made conceptual connections through her learned strategies. She used the Candy Shop scenario to explain place value in the addition and subtraction problems, she used the distributive property to conceptually explain the multiplication algorithm, and she was able to explain division concepts through a pictorial grouping strategy. Terri and Mary also used strategies to explain concepts, although still struggled to connect concepts to algorithms. Findings are encouraging because Dottie's experiences show that strategies could be catalyst for making a connection between concepts and algorithms, a connection that many students struggle to make.

Recommendations for Future Research

It would be interesting to see how participants interviewed developed in their conceptual understanding of place value and whole number operations as they advanced through the rest of their course work and even as they became teachers and had their own classrooms. This type of research would examine how well prospective teachers retained the knowledge that they gleaned during the base 8 class, and could be a positive indicator of the success of instruction in an alternate base system.

Another suggestion for future research involves examining how making a connection between solution strategies and the standard algorithm impacts conceptual understanding. The only student who could make a solid conceptual connection between alternate strategies and the standard algorithm was Dottie, who demonstrated the highest level of understanding at the end of the base 8 unit. Current reform movements in mathematics education stress the importance of not only conceptual understanding, but also procedural fluency (CCSSI, 2010; NCTM, 2014). Additionally, researchers stress that once students can make a connection between concepts and algorithms, deep understanding has taken place (Carpenter, 1986; NRC, 2001).

Further research regarding prospective teachers' development of division concepts is suggested. The majority of research regarding division concepts involves school age students (Carpenter et al., 1996; Greer, 1992; Silver, 1986; Sahin, 2015) and more research on pre-service teachers is needed. Only Dottie, the participant in the highest category of knowledge, was able to differentiate between sharing and measurement division. Also, students could not make connections between the standard division algorithm and why it worked. What strategies utilized in the base 8 unit were successful in helping students to conceptually understand division concepts? What connections needed to be made for that understanding to occur?

Finally, more research is needed in regard to the performance of students categorized as having beginning low mathematical content knowledge. Like Safi (2009) this researcher concluded that a student with beginning low mathematical content knowledge only progressed in ability to explain how to solve problems (CCK) and not in the ability to justify solutions (SCK). What strategies and instructional techniques could help a student of this nature be more successful? What experiences need to be provided in order to best prepare this type of student to be successful in the classroom?

Qualitative interviews were conducted in order to help explain results of the MKT, which was administered to measure change in CCK and SCK over the course of an elementary number concepts and operations unit in base 8. In addition to examining how qualitative results helped explain results from the MKT, qualitative interviews provided insight into how students with differing beginning categories of knowledge progressed as a result of the base 8 unit. Findings supported previous research endeavors in that students experienced cognitive dissonance due to taking a unit taught entirely in base 8, and as a result, gained a deeper conceptual understanding of whole number concepts and operations by the end of the unit. The biggest take away from the findings of this study is that students who enter with all types of knowledge benefitted in some fashion in their conceptual understanding of mathematics as a result of taking the base 8 unit. Because the majority of students interviewed struggled to make connections between procedures and their underlying concepts, mathematics educators should concentrate their efforts in helping prospective teachers make that important connection. Future research endeavors should consider the amount of progress each type of learner in this study made, and what could be done for the lower progressing students to ensure that they experience the same gains as students who enter with stronger conceptual understandings.

APPENDIX A:
INSTITUTIONAL REVIEW BOARD FORMS

IRB Approval Letter

University of Central Florida

University of Central Florida Institutional Review Board
Office of Research & Commercialization
12201 Research Parkway, Suite 501
Orlando, Florida 32826-3246
Telephone: 407-823-2901 or 407-882-2276
www.research.ucf.edu/compliance/irb.html

Approval of Exempt Human Research

From: **UCF Institutional Review Board #1**
FWA00000351, IRB00001138

To: **Katie Elizabeth Harshman**

Date: **November 19, 2015**

Dear Researcher:

On 11/19/2015, the IRB approved the following activity as human participant research that is exempt from regulation:

Type of Review:	Exempt Determination
Project Title:	Pedagogical Content Knowledge in Preservice Teachers
Investigator:	Katie Elizabeth Harshman
IRB Number:	SBE-15-11729
Funding Agency:	
Grant Title:	
Research ID:	N/A

This determination applies only to the activities described in the IRB submission and does not apply should any changes be made. If changes are made and there are questions about whether these changes affect the exempt status of the human research, please contact the IRB. When you have completed your research, please submit a Study Closure request in iRIS so that IRB records will be accurate.

In the conduct of this research, you are responsible to follow the requirements of the Investigator Manual.

On behalf of Sophia Dziegielewski, Ph.D., L.C.S.W., UCF IRB Chair, this letter is signed by:

Joanne Muratori

Signature applied by Joanne Muratori on 11/19/2015 10:50:21 AM EST

IRB Manager

Informed Consent Form

Pedagogical Content Knowledge in Preservice Teachers

Informed Consent

Principal Investigator:	Katie Harshman, MS
Faculty Advisor:	Erhan Haciomeroglu, PhD
Investigational Site(s):	University of Central Florida 4000 Central Florida Blvd. Orlando, FL 32816

Introduction: Researchers at the University of Central Florida (UCF) study many topics. To do this we need the help of people who agree to take part in a research study. You are being invited to take part in a research study that will include about 120 people at [redacted]. You have been asked to take part in this research study because you are a student at the University [redacted]. You must be 18 years of age or older to be included in the research study.

The person doing this research is Katie Harshman of the Mathematics Education Department of the UCF College of Education and Human Performance. Because the researcher is a doctoral student, she is being guided by Dr. Erhan Haciomeroglu, a UCF faculty advisor in the UCF College of Education and Human Performance.

What you should know about a research study:

- Someone will explain this research study to you.
- A research study is something you volunteer for.
- Whether or not you take part is up to you.
- You should take part in this study only because you want to.
- You can choose not to take part in the research study.
- You can agree to take part now and later change your mind.
- Whatever you decide it will not be held against you.
- Feel free to ask all the questions you want before you decide.

Purpose of the research study: The purpose of this study is to measure the influence of instruction in an alternate base system on the pedagogical content knowledge of pre-service teachers.

What you will be asked to do in the study: Participants will be asked to complete a questionnaire about knowledge in regard to mathematics content and an elementary student's learning of mathematics. Certain individuals may be asked to complete a brief interview.

Location: [redacted]

Time required: A questionnaire will administered twice. It is expected that it will take you 30 minutes to complete each questionnaire, which would be approximately 60 minutes total out of the semester.

Risks: There are no reasonably foreseeable risks or discomforts involved in taking part in this study.

Compensation or payment: There is no compensation or other payment to you for taking part in this study.

Confidentiality: We will limit your personal data collected in this study to people who have a need to review this information. We cannot promise complete secrecy.

Study contact for questions about the study or to report a problem: If you have questions, complaints contact Katie Harshman, Doctoral Student, University of Central Florida at (443) 928-9623 or katie.harshman@knights.ucf.edu, or Erhan Haciomeroglu, PhD, Faculty Supervisor, University of Central Florida at erhan.haciomeroglu@ucf.edu.

IRB contact about your rights in the study or to report a complaint: Research at the University of Central Florida involving human participants is carried out under the oversight of the Institutional Review Board (UCF IRB). This research has been reviewed and approved by the IRB. For information about the rights of people who take part in research, please contact: Institutional Review Board, University of Central Florida, Office of Research & Commercialization, 12201 Research Parkway, Suite 501, Orlando, FL 32826-3246 or by telephone at (407) 823-2901. You may also talk to them for any of the following:

- The research team is not answering your questions, concerns, or complaints.
- You cannot reach the research team.
- You want to talk to someone besides the research team.
- You want to get information or provide input about this research.

Withdrawing from the study: If you decide to leave the study, contact the investigator so that the investigator can remove your information from the study. You can email or call using the information above, or speak to your instructor who will relay the information.

APPENDIX B:
MEASURE ITEMS

Sample CCK Item from the MKT

Ms. Dominguez was working with a new textbook and she noticed that it gave more attention to the number 0 than her old book. She came across a page that asked students to determine if a few statements about 0 were true or false. Intrigued, she showed them to her sister who is also a teacher, and asked her what she thought.

Which statement(s) should the sisters select as being true?
(Mark YES, NO, or I'M NOT SURE for each item below.)

	Yes	No	I'm not sure
a) 0 is an even number.	1	2	3
b) 0 is not really a number. It is a placeholder in writing big numbers.	1	2	3
c) The number 8 can be written as 008.	1	2	3

Sample SCK Item from the MKT

Imagine that you are working with your class on multiplying large numbers. Among your students' papers, you notice that some have displayed their work in the following ways:

Student A	Student B	Student C
35	35	35
× 25	× 25	× 25
125	175	25
+ 75	+ 700	150
875	875	100
		+ 600
		875

Which of these students is using a method that could be used to multiply any two whole numbers?

	Method would work for all whole numbers	Method would NOT work for all whole numbers	I'm not sure
a) Method A	1	2	3
b) Method B	1	2	3
c) Method C	1	2	3

APPENDIX C:
INTERVIEW QUESTIONS

Interview Questions: Researcher Copy

Interview Protocol

Pre-interview question:

Do you like Math?

What kind of learner are you? Do you like it when teachers tell you how to solve a problem or do you like to try to figure it out yourself?

How do you feel about learning in base 8?

Directions. Solve and write a number sentence corresponding to the problem.

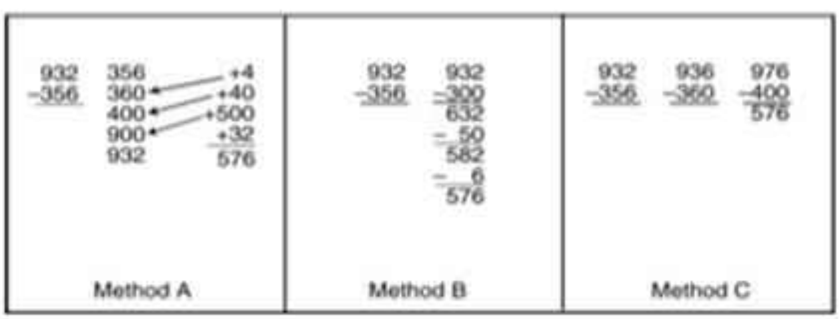
1) Tyler's bakeshop sells chocolate cupcakes. Tyler sells 255 chocolate cupcakes to a customer. Now there are 261 chocolate cupcakes in the bakeshop. How many chocolate cupcakes were in the bakeshop to start with?

2) There are 253 people on a cruise ship. Some more people board the cruise ship. Now there are 520 people on the cruise ship. How many people boarded the cruise ship?

3) There are 16 sports drinks in a package. How many sports drinks are in 12 packages?

4) Alex has 134 stickers. He wants to give each of his friends 12 stickers. How many friends can he share with? Does he have any stickers left for himself? If so, how many?

5) Imagine that you are working with your class on subtracting large numbers. Among your students' papers, you notice that some have displayed their work in the following ways:



Which of these students is using a method that could be used to subtract any two whole numbers? (Mark ONE answer.)

a) A only

b) B only

c) A and B

d) B and C

e) A, B, and C

f) I'm not sure.

Post-interview questions:

Why do you think your teacher has you learning in base 8?

Is there anything else you'd like to share with me?

As students are working through the problem, the researcher will ask questions that check for understanding such as: How did you decide what to do to solve the problem? Can you explain a little more what you did here? Why did you cross out that number? Did you change the value of the larger number? Why did you put a 1 next to that number? Can you explain your grouping strategy in this problem? Can you explain why your method works? Will that strategy always work? How do you know your answer is correct? Can you write a number sentence to describe the problem? Can you solve the problem a different way? How might you teach a child to solve this problem? For division problem: What does this group represent? Additional probing questions will be asked depending upon unique student responses.

Interview Questions: Prospective Teacher Copy

Directions. Solve and write a number sentence corresponding to the problem.

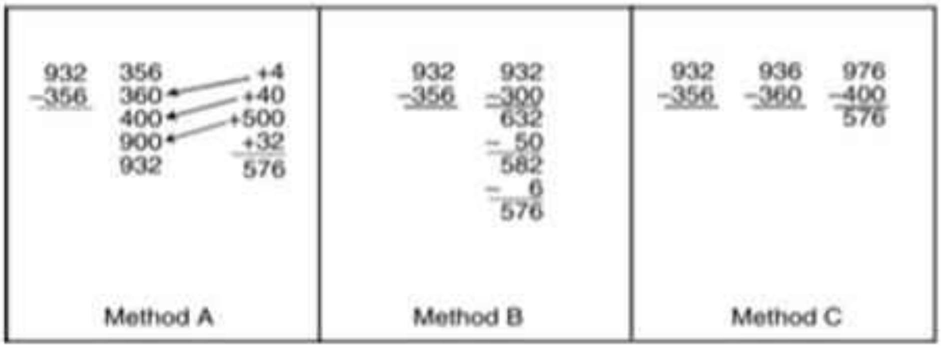
1) Tyler's bakeshop sells chocolate cupcakes. Tyler sells 255 chocolate cupcakes to a customer. Now there are 261 chocolate cupcakes in the bakeshop. How many chocolate cupcakes were in the bakeshop to start with?

2) There are 253 people on a cruise ship. Some more people board the cruise ship. Now there are 520 people on the cruise ship. How many people boarded the cruise ship?

3) There are 16 sports drinks in a package. How many sports drinks are in 12 packages?

4) Alex has 134 stickers. He wants to give each of his friends 12 stickers. How many friends can he share with? Does he have any stickers left for himself? If so, how many?

5) Imagine that you are working with your class on subtracting large numbers. Among your students' papers, you notice that some have displayed their work in the following ways:



Which of these students is using a method that could be used to subtract any two whole numbers?

REFERENCES

Andreasen, J. B. (2006). Classroom mathematical practices in a preservice elementary mathematics education course using an instructional sequence related to place value and operations. Unpublished Doctoral Dissertation, University of Central Florida.

Baek, J.M. (1998). Children's invented algorithms for multi-digit multiplication problems. In L. J. Morrow & M. J. Kenney (Eds.), *The Teaching and Learning of Algorithms in School Mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Ball, D. L. (1988). Knowledge and reasoning in mathematical what pedagogy: Examining prospective teachers bring to teacher education. Unpublished doctoral dissertation.

Ball, D. L. (1990). The mathematical understandings that prospective teachers bring to teacher education. *The Elementary School Journal*, *90*(4), 449 – 466.

Ball, D. L. (2003). *What mathematical knowledge is needed for teaching mathematics*. Paper presented at the Secretary's Summit on Mathematics.

Ball, D. L., & Bass, H. (2000). Interweaving content and pedagogy in teaching and learning to teach: Knowing and using mathematics. In J. Boaler (Ed.), *Multiple perspectives on the teaching and learning of mathematics.* (pp. 83-104). Westport, CT: Ablex.

Ball, D. L., Hill, H. C., & Bass, H. (2005). Knowing mathematics for teaching: Who knows mathematics well enough to teach third grade and how can we decide? *American Educator, 29*(1), 43 – 46.

Ball, D. L., Thames, M. H., & Phelps, G. C. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*(5), 389 – 407.

Bowers, J., Cobb, P., & McClain, K. (1999). The evolution of mathematical practices: A case study. *Cognition & Instruction, 17*(1), 25 – 64.

Chapman, O. (2007). Facilitating preservice teachers' development of mathematics knowledge for teaching arithmetic operations. *Journal of Mathematics Teacher Education, 10*(4), 341–349.

Carpenter, T. P., Fennema, E., & Franke, M. L. (1996). Cognitively guided instruction: A knowledge base for reform in primary mathematics instruction. *The Elementary School Journal*, *97*(1), 3 – 20.

Carpenter, T. P., Fennema, E., Franke, M. L., Levi, L., & Empson, L.B. (1999). *Children's mathematics: Cognitively Guided Instruction.* Portsmouth, NH: Heinemann.

Carpenter, T. P., Fennema, E., Peterson, P. L., Chiang, C., & Loef, M. (1989). Using knowledge of children's mathematics thinking in classroom teaching: An experimental study. *American Educational Research Journal*, *26*(4), 499 – 531.

Carpenter, T. P., Franke, M. L., Jacobs, V. R., Fennema, E., & Empson, S. B. (1998). A longitudinal study of invention and understanding in children's multidigit addition and subtraction. *Journal for Research in Mathematics Education, 29*(1), 3 – 20.

Cobb, P. (1988). The tension between theories of learning and instruction in mathematics education. *Educational Psychologist*, *23*(2), 87 – 103.

Cobb, P. (2000). Conducting teaching experiments in collaboration with teachers. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of Research Design in Mathematics and Science Education* (pp. 307 – 333). Mahwah, NJ: Lawrence Erlbaum Associates, Inc.

Cobb, P., Wood, T., Yackel, E., Nicholls, J., Wheatley, G., Trigatti, B., & Perlwitz, M. (1991). Assessment of a problem-centered second-grade mathematics project. *Journal for Research in Mathematics Education, 22*(1), 3 – 29.

Common Core State Standards Initiative (CCSSI). 2010. Common Core State Standards for Mathematics. Washington, D.C.: National Governors Association Center for Best Practices and the Council of Chief State School Officers. http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf

Conference Board of the Mathematical Sciences. (2001). *The Mathematical Education of Teachers*. Providence RI and Washington DC: American Mathematical Society and Mathematical Association of America.

Creswell, J.W. (2007). *Qualitative inquiry & research design: Choosing among five approaches* (3rd Ed.). Sage.

Davis, R. B., Maher, C. A., & Noddings, N. (1990). Chapter 12: Suggestions for the improvement of mathematics education. *Journal for Research in Mathematics Education. Monographs*, 4.

Faul, F., Erdfelder, E., Buchner, A., & Lang, A.G. (2009). Statistical power analyses using G*Power 3.1: Tests for correlation and regression analyses. *Behavior Research Methods, 41*(4), 1149 – 1160.

Fasteen, J., Melhuish, K., & Thanheiser, E. (2015). Multiplication by 10_{five}: Making sense of place value structure through an alternate base. *Mathematics Teacher Educator*, *3*(2), 83 – 98.

Fennema, E., & Franke, M. L. (1992). Teacher knowledge and its impact. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 147 – 164). New York: MacMillan.

Fuson, K. C. (1992). Research on whole number addition and subtraction. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning*. New York: MacMillan.

Gall, M. D., Gall, J. P., & Borg, W. R. (2006). Educational research: An introduction (8th ed.). Boston: MA: Allyn & Bacon.

Gravemeijer, K. (2004). Local instructional theories as a means of support for teachers in reform mathematics education. *Mathematical Thinking and Learning, 6*(2), 105 – 128.

Greeno, J., Riley, M., & Gelman, R. (1984). Conceptual competence and children's counting. *Cognitive Psychology, 16*, 94 – 143.

Greer, B. (1992). Multiplication and division as models of situations. In D. A. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning*. New York: MacMillan.

Hambleton, R. K., Swaminathan, H., & Rogers, H. J. (1991). *Fundamentals of item response theory.* Newbury Park, CA: Sage.

Hiebert, J., & Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics: An Introductory Analysis. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp. 1 - 27). Hillsdale, NJ: Erlbaum.

Hill, H. C., (2007). *Technical report on number and operations content knowledge items – 2001-2008* (Mathematical Knowledge for Teaching (MKT) measures). Retrieved from https://lmt.isr.umich.edu/administration/AssessmentInfoAdditionalDoc.aspx?openasnewpage=true

Hill, H. C., & Ball, D. L. (2004). Learning mathematics for teaching: Results from California's mathematics professional development institutes. *Journal for Research in Mathematics Education, 35*(5), 330 – 351.

Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal, 42*(2), 371 – 406.

Hill, H. C., Schilling, S. G., & Ball, D. L. (2004). Developing measures of teachers' mathematics knowledge for teaching. *Elementary School Journal*, *105*(1), 11 – 30.

Huck, S. W. (2012). Reading statistics and research. New York: Longman, 2012.

Lampert, M. (1986). Knowing, doing, and teaching multiplication. *Occasional Paper No. 97*.

Lee, J., Meadows, M., & Lee, J. O. (2003). *What causes teachers to implement high-quality mathematics education more frequently: Focusing on teachers' pedagogical content knowledge.* Paper presented at the Association for Childhood Education International (ACEI) International Annual Conference, Phoenix, AZ.

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Hillsdale, NJ: Erlbaum.

McClain, K. (2003). Supporting pre-service teachers' understanding of place value and multi-digit arithmetic. *Mathematical Thinking and Learning*, *5*(4), 281 – 306.

National Council of Teachers of Mathematics (NCTM). (1989). *Curriculum and evaluation* *standards for school mathematics.* Reston, VA: Author

National Council of Teachers of Mathematics (NCTM). (1991). *Professional standards for teaching mathematics.* Reston, VA: Author.

National Council of Teachers of Mathematics (NCTM). (2000). *Principles and standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

National Council of Teachers of Mathematics (NCTM). 2014. *Principles to actions: Ensuring mathematical success for all*. Reston, VA: NCTM

National Research Council. (2001). *Adding it up: Helping children learn mathematics.* J. Kilpatrick, J. Swafford, and B. Findell (Eds.). Mathematics Learning Study Committee, Center for Education, Division of Behavioral Sciences and Education. Washington, DC: National Academy Press.

Naylor, S., & Keogh, B. (1999). Constructivism in classroom: Theory into practice. *Journal of Science Teacher Education*, *10*(2), 93 – 106.

No Child Left Behind (NCLB) Act of 2001, Pub. L. No. 107-110, § 115, Stat. 1425 (2002).

Rech, J., Hartzell, J., & Stephens, L. (1993). Comparisons of mathematical competencies and attitudes of elementary education majors with established norms of a general college population. *School Science and Mathematics, 93*(3), 141 – 144.

Ross, S. H. (2001). Pre-service elementary teachers and place value: Written assessment using a digit-correspondence task. In C. N. Walter (Ed.), Proceedings of the twenty-third annual meeting of the north American chapter of the international group for the psychology of mathematics education (pp. 897 – 906) Snowbird, UT: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Rowan, B., Schilling, S.G., Ball, D. L., & Miller, R. (2001). *Measuring teachers' pedagogical content knowledge in surveys: An exploratory study.* (Research Note S-2). Ann Arbor M,I: Consortium for Policy Research in Education, Study of Instructional Improvement, University of Michigan.

Roy, G. J. (2008). Prospective teachers' development of whole number concepts and operations during a classroom teaching experiment. Unpublished Doctoral Dissertation, University of Central Florida.

Safi, F. (2009). Classroom mathematical practices in a preservice elementary mathematics education course using an instructional sequence related to place value and operations. Unpublished Doctoral Dissertation, University of Central Florida.

Sahin, N. (2015). The effects of cognitively guided instruction on students' problem solving strategies and the effect of students' use of strategies on their mathematics achievement. Unpublished Doctoral Dissertation, University of Central Florida.

Shadish, W. R., Cook, T. D., & Campbell, D. T. (2002). *Experimental and quasi-experimental designs for generalized causal inference*. Wadsworth Cengage Learning.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4 – 14.

Shulman, L. S. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review,* 57(1), 1 – 22.

Silver, E. (1986). Using conceptual and procedural knowledge: A focus on relationships. In J. Hiebert, (Ed.), Conceptual and procedural knowledge: The case of mathematics (pp. 181-198). Hillsdale, NJ: Lawrence Erlbaum Associates.

Simon, M. A. (1993). Prospective Elementary Teachers' Knowledge of Division. *Journal for Research in Mathematics Education, 24*(3), 233-254.

Simon, M. A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education, 26*(2), 114-145.

Skemp, R. R. (1978). Relational understanding and instrumental understanding. *Arithmetic Teacher, 26*, 9 – 15.

Southwell, B., & Penglase, M. (2005). Mathematical knowledge of pre-service primary teachers Vol. 4. H. L. Chick & J. L. Vincent (Eds.), International Group for the Psychology of Mathematics Education (pp. 209 – 216).

Steffe, L. P. (1983). Children's Algorithms as Schemes. *Educational Studies in Mathematics, 14*, 233 – 249.

Steffe, L. P. (1994). Children's multiplying schemes. In G. Harel & J. Confrey (Eds.), *The Development of Multiplicative Reasoning in the Learning of Mathematics* (pp. 3 – 39). Albany, NY: SUNY Press.

Stohlmann, M., Cramer, K., Moore, T., & Maiorca, C. (2014). Changing pre-service elementary teachers' beliefs about mathematical knowledge. *Mathematics Teacher Education & Development, 16*(2), 4 – 24.

Thanheiser, E. (2009). Preservice elementary school teachers' conceptions of multidigit whole numbers. *Journal for Research in Mathematics Education, 40*(3), 251 – 281.

Thanheiser, E. (2012). Understanding multi-digit whole numbers: The role of knowledge components, connections, and context in understanding regrouping 3+-digit numbers. *Journal of Mathematical Behavior*, *31*(2), 220 – 234.

Thanheiser, E., Whitacre, I., & Roy, G. J. (2014). Mathematical content knowledge for teaching elementary mathematics: A focus on whole-number concepts and operations. *Mathematics Enthusiast*, *11*(2), 217 – 266.

Thompson, A. G., Philipp, R. A., Thompson, P. W., & Boyd, B. A. (1994). Calculational and conceptual orientations in teaching mathematics. In D. B. Aichele & A. F. Coxford (Eds.), *Professional development for teachers of mathematics* (pp. 79-92). Reston, VA: National Council of Teachers of Mathematics.

Trends in International Mathematics and Science Study (TIMSS). U.S. Department of Education Institute of Education Science, 2007. http://nces.ed.gov/timss/.

U.S. Department of Education. (2002). *Meeting the highly qualified teachers challenge: The secretary's annual report on teacher quality.* Washington, DC: U.S. Department of Education, Office of Postsecondary Education, Office of Policy, Planning, and Innovation.

von Glassersfeld, E. (1983). Learning as a constructive activity. In J. C. Bergeron & N. Herscovics (Eds.), Proceedings of the fifth annual meeting of the North American

Chapter of the International Group for the Psychology of Mathematics Education (pp. 41 – 69). Montreal: Universite de Montreal, Faculte de Science de l'Education.

Yackel, E., Underwood, D., & Elias, N. (2007). Mathematical tasks designed to foster a reconceptualized view of early arithmetic. *Journal of Mathematics Teacher Education*, *10*(4), 351 – 367.

Zazkis, R., & Khoury, H. A. (1993). Place value and rational number representations: Problem solving in the unfamiliar domain of non-decimals. *Focus on Learning Problems in Mathematics, 15*(1), 38 – 51.