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NOTES

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The Midpoints Between Roots Reveal the Quartic Equation

Javier Sánchez-Reyes

Abstract. The midpoints between roots provide the key to understanding the geometry, in the complex plane, behind the roots of a quartic polynomial. In reduced form (i.e., with no cubic term), midpoints come in three pairs, with opposite signs, as solutions to a resolvent cubic. At any midpoint, a startlingly simple expression of the polynomial derivative indicates the vectors from the midpoint to the corresponding pair of roots. This approach simplifies Euler's method for solving the quartic, since there is no need to make a suitable choice of the plus or minus signs in the pairs of midpoints.

1. INTRODUCTION. The quartic equation arises in a number of relevant geometrical problems, in particular those involving conic sections, such as computing the intersection of two conics, their overlapping areas, or the orthogonal projection of a point onto a conic. The quartic is the highest degree polynomial equation admitting a closed-form solution in terms of radicals. Basically, analytical methods [10–12, 18] for solving the quartic involve inspired algebraic manipulation, finding the solutions of a certain resolvent cubic, and extracting square roots.

Unfortunately, in general such methods are not endowed with any clear geometric meaning. Thus, already in the 19th century, a number of mathematicians [7–9, 19] advocated transforming the analytical root-finding problem into its geometric counterpart of finding the intersection of two conics. In articles published in this MONTHLY, this technique was also employed by Glenn [5], Graustein [6], and Running [17], and more recently by Faucette [4] and Auckly [1], who apply methods from algebraic geometry. If all roots are real, they also admit an intuitive representation, put forward by Nickalls [14] and Northshield [15], as coordinates of the vertices of a regular tetrahedron in $\mathbb{R}^3$.

In a manner different from those used in these previous works, we resort to simple geometric concepts in the complex plane $\mathbb{C}$, tackling the general case of complex roots. First, we borrow a key idea from another inspiring paper by Nickalls [13], giving a modern exposition of Euler's method for solving the quartic: Instead of directly finding the roots, we try instead to find a midpoint between two roots. Second, we compute the vectors pointing from a midpoint to the corresponding roots, rather than finding a proper combination of signs as in Euler's method.

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Color versions of one or more of the figures in the article can be found online at www.tandfonline.com/uamm.

2. FINDING THE ROOTS OF A QUARTIC. We assume a quartic polynomial P is expressed in reduced form, i.e., with no cubic term:

$$P(z) = z^4 + pz^2 + qz + r. \tag{1}$$

If the quartic has a cubic term az^3, it can be rewritten in reduced form by the change of variable $z \to z - a/4$. By Vieta's formulas [20], this is tantamount to setting the centroid of the roots z_k at the origin:

$$z_1 + z_2 + z_3 + z_4 = 0. \tag{2}$$

We also assume that at least one of the coefficients $p, q, r \in \mathbb{C}$ does not vanish, to avoid the singular case $P(z) = z^4$, where all roots and midpoints coalesce to the origin.

Vectors pointing at a pair of roots from their midpoint. Consider a midpoint ζ between two roots, let us say $\zeta = \frac{1}{2}(z_1 + z_2)$ midway between z_1, z_2, and factor P as the product

$$P(z) = Q(z)R(z), \quad \text{where} \quad \begin{array}{l} Q(z) = (z - z_1)(z - z_2), \\ R(z) = (z - z_3)(z - z_4). \end{array}$$

As Q is a quadratic polynomial with roots z_1, z_2, its critical point, where the derivative Q' vanishes, coincides with the midpoint ζ, so $Q'(\zeta) = 0$. Since $R'(\zeta) = 4\zeta$ from equation (2), by differentiating the product $P(z)$, we see that

$$P'(\zeta) = Q(\zeta)R'(\zeta) = -4\zeta v^2(\zeta), \quad v(\zeta) = \tfrac{1}{2}(z_1 - z_2),$$

where $\pm v(\zeta)$ indicates the vectors emanating from ζ and pointing at the pair of roots z_1, z_2 (Figure 1a). Solving for $v(\zeta)$, we have

$$v(\zeta) = \tfrac{1}{2}\sqrt{-P'(\zeta)/\zeta}, \qquad \zeta \neq 0. \tag{3}$$

Note that expression (3), characterized exclusively by the derivative $P'(\zeta)$, holds for any pair of roots and their midpoint ζ. In fact, it makes sense for all $\zeta \neq 0$, which justifies the function notation $v(\zeta)$. The reader may also wonder what happens for midpoints $\zeta = 0$. We advance that a midpoint $\zeta = 0$ exists if and only if $q = 0$, putting us in the biquadratic case, which deserves a particular discussion later. Clearly, not all midpoints can be $\zeta = 0$, since this leads to the case $p = q = r = 0$, which was ruled out in the beginning.

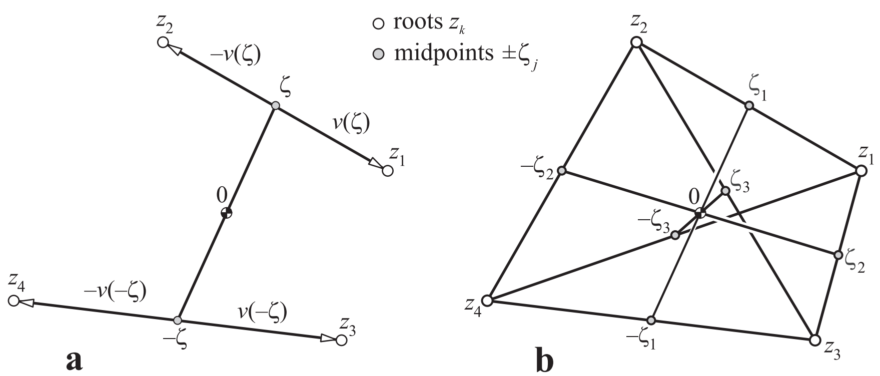

Figure 1. Geometry of the roots z_k of a quartic equation: a) Vectors $v(\zeta)$. b) Pairs $\pm\zeta$ of midpoints.

The resolvent cubic yielding the midpoints. Centering the roots at the origin, (2) implies that the midpoints come in pairs $\pm\zeta$, for instance, $\frac{1}{2}(z_1+z_2) = \frac{1}{2}(z_3+z_4)$, as Figure 1a shows. Consequently, if we find a pair $\pm\zeta$, the four roots z_k are given by formulas with a great deal of symmetry:

$$\{z_1, z_2\} = \zeta \pm v(\zeta), \quad \{z_3, z_4\} = -\zeta \pm v(-\zeta). \tag{4}$$

Four roots furnish $\binom{4}{2} = 6$ midpoints, grouped as three pairs $\pm\zeta_1, \pm\zeta_2, \pm\zeta_3$ (Figure 1b). But, how can we find them? By Vieta's formulas, the product of all roots z_k in (4) is equal to the constant term r in the reduced form (1) of $P(z)$:

$$(z_1 z_2)(z_3 z_4) = r \;\to\; [\zeta^2 - v^2(\zeta)][\zeta^2 - v^2(-\zeta)] = r. \tag{5}$$

Next, in this expression we compute $-v^2(\zeta)$ in terms of the remaining polynomial coefficients p, q:

$$-v^2(\zeta) = \frac{P'(\zeta)}{4\zeta} = \frac{4\zeta^3 + 2p\zeta + q}{4\zeta},$$

and analogously for $-v^2(-\zeta)$. Finally, after multiplying by ζ^2 and using straightforward algebraic manipulations, we see that equality (5) generates a resolvent sextic in ζ. With the substitution $\zeta^2 \to w/4$, the sextic transforms to a cubic in w:

$$w^3 + 2pw^2 + (p^2 - 4r)w - q^2 = 0, \tag{6}$$

whose three roots $\{w_1, w_2, w_3\}$ furnish the corresponding pairs $\pm\zeta = \pm\frac{1}{2}\sqrt{w_j}$ of midpoints.

Let us summarize these results in a formal way.

Theorem 1. *The four roots z_k of a reduced quartic $P(z)$ (1) are $\zeta_j \pm v(\zeta_j)$ and $-\zeta_j \pm v(-\zeta_j)$, where $v(\zeta) = \frac{1}{2}\sqrt{-P'(\zeta)/\zeta}$, $\zeta_j = \frac{1}{2}\sqrt{w_j}$, and w_j is any nonvanishing root of the resolvent cubic (6).*

Expression (3) has a singularity at the origin $\zeta = 0$, which determines the choice of w_j:

- The condition $w_j \neq 0$ is required to avoid the singularity. If all the w_j's vanish, the resolvent cubic simplifies to $w^3 = 0$, and then $p = q = r = 0$, but we ruled out this case from the beginning.
- In exact arithmetics, any $w_j \neq 0$ would do the job. However, the computation of $v(\zeta)$ is clearly ill-conditioned for ζ close to the origin, so roots w_j close to this singularity must be avoided in a floating-point environment.

In essence, Theorem 1 reinterprets Euler's method [**13**] for solving the quartic, where the four roots must be identified among the eight possible combinations $\pm\zeta_1 \pm \zeta_2 \pm \zeta_3$. While this sign allocation boils down to a straightforward analysis, we circumvent it by directly computing the roots from any midpoint $\zeta_j \neq 0$.

3. THE CASE OF THE BIQUADRATIC EQUATION. Finally, we remark on how the geometry of Figure 1b simplifies, becoming even more symmetrical, for a biquadratic equation. In this case, $q = 0$ in the quartic (1), which can hence be rewritten as

$$P(s) = s^2 + ps + r = 0, \quad s = z^2.$$

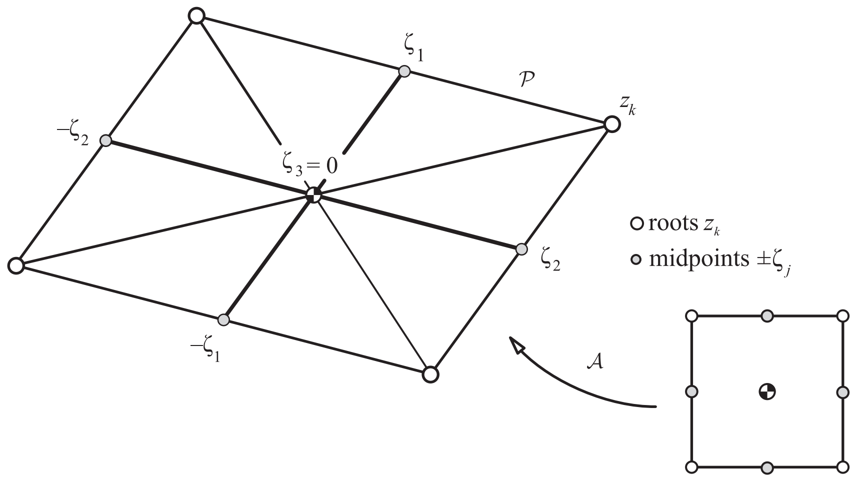

Figure 2. Biquadratic equation: Configuration of the roots z_k and midpoints ζ_j.

Therefore, the roots come in pairs $\pm\sqrt{s_j}$ from the roots s_j of a quadratic polynomial, thereby defining a parallelogram $\mathcal{P}$ (Figure 2). This is precisely the case already analyzed by Clifford and Lachance [**2**], and implicit in the papers by Parish [**16**] and Clifford and Lachance [**3**], where $\mathcal{P}$ can be generated by an affine transformation $\mathcal{A}$ of a square. Two pairs $\pm\zeta_1$, $\pm\zeta_2$ correspond to the midpoints of the edges of $\mathcal{P}$, whereas the remaining pair $\pm\zeta_3$, the midpoints of the two diagonals of $\mathcal{P}$, coalesce to the origin. This is tantamount to a root $w_3 = 0$ of the resolvent cubic (6), a property also clear since the biquadratic condition $q = 0$ means a constant term $q^2 = 0$ of this cubic.

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