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Solution of an Odds Inversion Problem

Robert K. Moniot

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Abstract. “From a bag containing red and blue balls, two are removed at random. The chances are 50-50 that they will differ in color. *What were the possible numbers of balls initially in the bag?*” This problem appeared in the National Museum of Mathematics’ *Varsity Math* puzzle, week 117. It is quite easy to solve, but what if we generalize to arbitrary odds? In this article, we characterize the solutions of the general case. We show that for most odds values that are at most 50-50, there is an infinite number of solutions, while for a certain well-defined class of odds below 50-50 and for any odds greater than 50-50, the number of solutions is zero or finite. We also explore some other interesting and surprising properties of this problem.

1. INTRODUCTION. Given a bag containing known numbers of red and blue balls, we can easily calculate the odds that two balls drawn at random will be different colors. But suppose we invert this problem, and ask what must be the numbers of balls of each color so that the odds are some chosen value between nil and certainty. This problem, for the special case of 50-50 odds, appeared in the *Varsity Math* puzzle feature of the National Museum of Mathematics in New York City [4]. The author of that puzzle is Dick Hess of the museum staff [1]. Here we solve the problem for arbitrary odds.

2. PRELIMINARIES. Let x and y be the number of red and blue balls, respectively, in the bag. The probability of drawing out two balls of different colors is a rational number, which we write as p/q in lowest terms. Then we have

$$\frac{p}{q} = \frac{2xy}{(x+y)(x+y-1)}. \tag{1}$$

Our goal is to invert this equation to find integer values of x and y that will yield the given value p/q. Provided there are at least 2 balls in the bag, we can rearrange (1) as the quadratic Diophantine equation

$$px^2 - 2(q-p)xy + py^2 - px - py = 0. \tag{2}$$

In order to be admissible as solutions to the original problem, solutions to this equation must satisfy $x \geq 0$, $y \geq 0$, and $x + y \geq 2$.

Symmetry. Equation (2) is symmetric in x and y, so given a solution (x, y), then (y, x) is also a solution. In what follows, we will often impose the condition that $x \leq y$ to ensure the solutions are distinct.

Odds of 50-50. We can deal with the *Varsity Math* puzzle quickly. Setting $p = 1$ and $q = 2$ and rearranging, (2) becomes

$$x^2 - 2xy + y^2 = y + x. \tag{3}$$

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The left-hand side is $(y-x)^2$. Let $y-x=v$; then $y+x=v^2$. Solving and assuming $x \le y$, we obtain

$$x = \frac{v(v-1)}{2}, \qquad y = \frac{v(v+1)}{2}.$$

This shows that x and y are successive triangular numbers. Their sum, the total number of balls, is a square. The number of solutions is infinite, with a unique solution (up to ordering of x, y) corresponding to each integer value of $v > 1$.

Trivial solutions. Transforming (1) to obtain (2) requires the denominator of the right-hand side of (1) to be nonzero. There are three solutions of (2) that violate this assumption, namely $(x, y) = (0, 0)$, $(0, 1)$, and $(1, 0)$. They satisfy (2) for any values of p and q. These trivial solutions yield an undefined value $0/0$ for the probability in (1). While these are not admissible as solutions to the problem (since there are not two balls to draw), they will prove useful in obtaining admissible solutions and in proving some results.

If $p = 0$, the odds of drawing different-color balls are nil, which can only be the case if all the balls are the same color, i.e., $x = 0$ or $y = 0$, and the other color any integer greater than 1. Hereafter we will assume $p > 0$.

Some examples. As a way of getting acquainted with the problem, we can find solutions of (2) by inserting various values of x and y into (1) and calculating the corresponding values of p/q. Here are some arbitrarily chosen examples from such a search using values of x and y less than 1000, and selecting ratios with p and q in single or double digits. In each case, all the distinct solutions found by the search are shown.

1/5 :	(1, 9)	(9, 72)	(72, 568)			
2/5 :	(1, 4)	(4, 12)	(12, 33)	(33, 88)	(88, 232)	(232, 609)
3/5 :	(2, 3)	(3, 3)				
7/16 :	(310, 651)					
7/18 :	(2, 7)	(7, 21)	(21, 60)	(95, 266)	(266, 742)	
9/22 :	(3, 9)	(9, 24)	(50, 126)	(126, 315)	(315, 785)	
8/15 :	(2, 4)	(4, 6)	(7, 8)	(8, 8)		
7/13 :	(6, 7)	(7, 7)				
4/7 :	(3, 4)	(4, 4)				

An interesting pattern in the solutions of (2) in the preceding table is that for a given p/q they often occur in sequences in which the larger number of balls in one solution reappears as the smaller number of balls in the next larger solution. The *Varsity Math* case exhibits this pattern continuing indefinitely, with solutions being successive triangular numbers. The ratio 2/5 is another example where the pattern continues indefinitely. The table above is too limited to show clearly that in many other cases these sequences form triplets of solutions. Here is an example: $p/q = 5/11$, for which the smallest nine admissible solutions of (2) are

(7, 15)	(15, 30)	(30, 58)
(184, 345)	(345, 645)	(645, 1204)
(3718, 6930)	(6930, 12915)	(12915, 24067)

For some ratios, like this one, all solutions are members of triplets that follow this pattern. Below we shall see a reason for their appearance. But not all solutions appear in such triplets; for instance, for the ratio 9/22 the first two solutions form a doublet.

It is also interesting to note that some ratios with small p and q do not appear in the results of this search, for instance, $7/17$, $4/9$, and $4/5$. We shall see that there is a different reason for each of these three ratios' nonappearance.

A "recycling" recurrence. As noted in the previous section, often solutions for a given p/q occur in a sequence where a value in one solution reappears in the next. We can obtain a recurrence to generate the solutions in such sequences. Assuming that the probability of drawing different colors for (x_i, y_i) is the same as for (x_{i+1}, y_{i+1}) with $x_{i+1} = y_i$ and $y_{i+1} \neq x_i$, inserting these into (1), and equating the two expressions for the odds (assuming all three numbers are nonzero), one obtains the recurrence

$$x_{i+1} = y_i, \quad y_{i+1} = \frac{y_i(y_i - 1)}{x_i}.$$

Because one of the values in one solution is re-used in the next, I call this the "recycling recurrence" to distinguish it from other recurrences that can be obtained for the solutions of (2). It is not guaranteed to yield integer values *ad infinitum,* and, in fact, except for certain special categories, the values it yields become fractional after a few iterations. For example, for the probability $7/18$, the table above shows three solutions $(2, 7)$, $(7, 21)$, and $(21, 60)$, but the next value given by the recycling recurrence is $(60, 1180/7)$.

If $y_i - 1 > x_i$ then $y_{i+1} > y_i$, meaning the recurrence advances to a larger solution. Swapping x_i and y_i runs the recurrence in reverse.

A special case. There is a class of probability values for which solutions are readily found, namely when $p = 1$ or $p = 2$. One can easily verify by substitution into (1) that if $p = 1$, then the smallest solution is $(1, 2q - 1)$, while if $p = 2$, the smallest solution is $(1, q - 1)$. We can also show that for $p = 1$ or $p = 2$ the recycling recurrence always gives integer solutions. This requires that each x divide $y(y - 1)$. But from (1), $p(x + y)(x + y - 1) = 2qxy$, and so whether $p = 1$ or $p = 2$, x must divide $(x + y)(x + y - 1)$, which implies that it divides $y(y - 1)$. And for $p/q \leq 1/2$, the recycling recurrence continues to yield admissible solutions indefinitely. The proof is deferred to the next section.

3. GENERAL SOLUTION.

Change of variables. For solving the general case, it is useful to make a change of variables. Let $t = y + x$ and $v = y - x$. Then (2) becomes

$$(q - 2p)t^2 + 2pt - qv^2 = 0. \tag{4}$$

The case $p/q = 1/2$ is the *Varsity Math* problem, (3), which we solved above. Otherwise, $q - 2p \neq 0$, and we can eliminate the linear term by completing the square. Multiply (4) by $q - 2p$ to make the coefficient of t square, and add p^2 to both sides. This yields

$$((q - 2p)t)^2 + 2p(q - 2p)t + p^2 - q(q - 2p)v^2 = p^2,$$

or

$$((q - 2p)t + p)^2 - q(q - 2p)v^2 = p^2. \tag{5}$$

Let

$$u = (q - 2p)t + p$$

and

$$D = q(q - 2p).$$

Then (5) becomes

$$u^2 - Dv^2 = p^2. \tag{6}$$

Both u and v may be of either sign. Different signs of u yield distinct solutions (x, y), while $v < 0$ yields the same solution as for $v > 0$ except with the order of (x, y) reversed.

We note that the trivial solution $(x, y) = (0, 0)$ corresponds to $(u, v) = (p, 0)$, while the other two trivial solutions $(x, y) = (0, 1)$ and $(1, 0)$ correspond to $(u, v) = (q - p, \pm 1)$. These satisfy (6) for all values of p and q. Changing the sign of u yields three more solutions of (6), but we will see below that, except for a special class of probabilities, they do not yield admissible solutions of (2).

Categorizing the solutions. The nature of (6) depends on the sign of D. If $D > 0$, i.e., $p/q < 1/2$, then the equation is a hyperbola, and may potentially have an infinite number of solutions. If $D < 0$, i.e., $p/q > 1/2$, it is an ellipse, and can have at most a finite number of solutions. The case $D = 0$ corresponds to $p/q = 1/2$, the *Varsity Math* case solved above. The change of variables to u, v breaks down in this case, and (4) is a parabola. (Since the transformation of variables from (2) to (4) and (6) is linear, all three equations are the same category of conic.)

Recycling recurrence revisited. Above we noted that if $y - 1 > x$, the recycling recurrence yields a larger solution, but this does not guarantee that it will continue to do so. We can now see that if (2) is elliptical, solutions generated by this recurrence cannot increase indefinitely, since they must all fall on the finite ellipse. Only if the equation is parabolic or hyperbolic can solutions increase indefinitely. We can show that in fact they do. From (1), for $1 \le x \le y$, the requirement that $p/q \le 1/2$ corresponds to

$$y \ge x + \frac{1}{2}\left(1 + \sqrt{1 + 8x}\right). \tag{7}$$

The radical is greater than 1, which ensures that $y - 1 > x$, as required for the next solution to be larger. And since the recycling recurrence preserves p/q, the next solution will also satisfy (7), guaranteeing that the increase will continue. Combining this result with the fact shown earlier that, for $p = 1$ or $p = 2$, the recycling recurrence always yields integers, we conclude that the recurrence generates admissible solutions, starting from the smallest nontrivial solution, *ad infinitum* if $p = 1$ or $p = 2$ and $p/q \le 1/2$. For $p > 2$ the proof that the iterates of the recycling recurrence are integer does not hold. I do not have a proof that in this case the recycling recurrence iterates always become fractional after a few steps, but I have not observed any examples having more than just a few successive integer iterates.

Elliptical case. When $p/q > 1/2$, $D < 0$ and the equation is an ellipse, so the solution space is bounded. We now look at ways to find solutions when they exist, and also at ways to rule out the existence of solutions.

Character of the ellipse. The ellipse (6) is a unit circle for $p/q = 1$, and elongates as p/q approaches $1/2$ from above. This is shown in Figure 1.

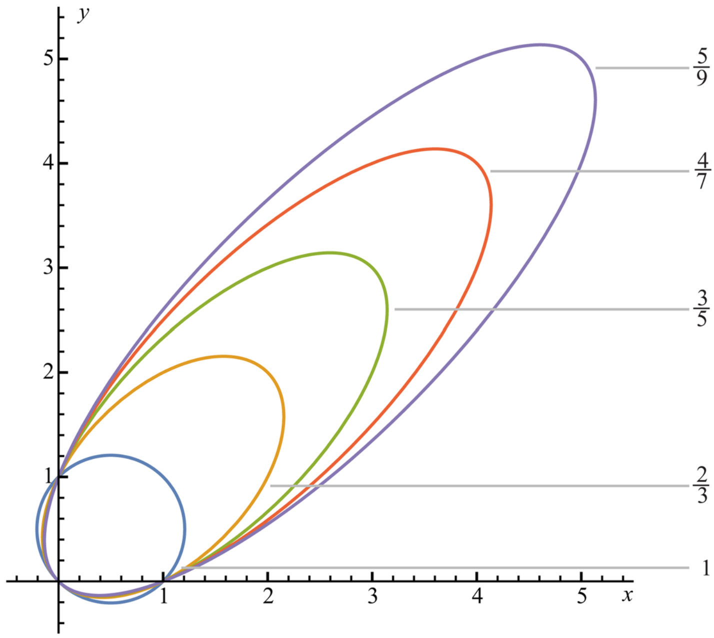

Figure 1. Family of ellipses defined by (2) for ratios of the form $p/(2p-1)$.

Direct search. Searching by testing all values of v turns out to be surprisingly efficient up to fairly large solution sizes. In (6), the maximum value of v occurs when $u = 0$, so

$$|v| \leq \frac{p}{\sqrt{q(2p-q)}} = \frac{p/q}{\sqrt{2p/q - 1}}.$$

Supposing the probability is quite close to 1/2, let $p/q = 1/2 + \epsilon$; then asymptotically for small ϵ,

$$v_{\max} \approx \frac{1}{2\sqrt{2\epsilon}},$$

so $v_{\max}$ grows only as $\epsilon^{-1/2}$. For instance, if $p = 10^6 + 1$ and $q = 2 \times 10^6$, then $\epsilon = 5 \times 10^{-7}$, and the maximum number of balls is approximately 10^6 but the number of values of v to search is only 500. Solutions involving even 10^{12} balls are in range for a modest personal computer to find in a few minutes of computing time.

Cases that always have solutions. There is a class of probability ratios in the elliptical regime, namely those of the form $p/(2p-1)$, for which solutions always exist. As mentioned earlier, the trivial solutions in (x, y)-space convert to $(u, v) = (p, 0)$ and $(q-p, \pm 1)$. But $-u$ also satisfies (6). Negating u in these three trivial solutions and mapping back to (x, y)-space yields three other points at the opposite end of the ellipse, furthest from the origin, symmetric counterparts to the three trivial solutions around the origin. The point furthest from the origin has $x = y = p/(2p-q)$, which is integer and positive if and only if $q = 2p - 1$, in which case the three solutions are (p, p), $(p-1, p)$, and $(p, p-1)$. Other solutions may also exist for ratios of this form, for example for the ratio 8/15 in the table above.

Bounds on p and q. We can rule out the existence of solutions for p/q ratios in the elliptical regime in which p or q individually exceed bounds calculated as follows.

In (1), the factor of 2 in the numerator always divides the denominator, so we have $p \le xy$ and $q \le \frac{1}{2}(x+y)(x+y-1)$. (Cancellation of common factors may reduce p and q further.) These expressions are maximized at the far end of the ellipse, where $x = y = p/(2p-q)$. Thus we have bounds

$$p \le \frac{z^2}{(2z-1)^2}, \quad q \le \frac{z}{(2z-1)^2}$$

where $z = p/q$. Ratios in lowest terms violating these inequalities can be excluded *a priori*. For example, there cannot be a solution for $p/q = 200/381$ because p exceeds the bound of 110 for this ratio.

Exhaustive enumeration. It was noted earlier that as p/q increases towards 1, the ellipse (2) shrinks. This means that for p/q above a chosen threshold, one can enumerate all solutions. For instance, the probability ratio for $(x, y) = (5, 5)$ is $p/q = 5/9$. For probabilities greater than or equal to this, all solutions must have x and y at most 5. Calculating the probabilities given by (1) for all pairs of integers $1 \le x \le y \le 5$ and excluding those below the threshold, we find the complete set of solutions:

$5/9:$	$(4, 5),$	$(5, 5)$
$4/7:$	$(3, 4),$	$(4, 4)$
$3/5:$	$(2, 3),$	$(3, 3)$
$2/3:$	$(1, 2),$	$(2, 2)$
$1/1:$	$(1, 1)$	

No other probabilities $p/q \ge 5/9$ give solutions. The paucity of solutions for $p/q > 1/2$ reflects the fact that it is harder to make it likely that the balls differ in color than that they be the same. This result provides the explanation for the absence of $4/5$ from the search results described in the "Some examples" section. It exceeds $5/9$ and is not in the above list.

Hyperbolic case. When $p/q < 1/2$, $D > 0$ and (6) is a hyperbola. In this case, $u < 0$ yields negative x, y. Since admissible x, y must be positive, u must be as well. A family of hyperbolic cases with $q = 2p + 1$ is plotted in Figure 2. As the ratios decrease, the asymptotes move closer to the coordinate axes and the two branches approach each other near the origin, the curves becoming identical with the coordinate axes when $p/q = 0$. As the ratios increase towards $1/2$, the hyperbola narrows and the two branches move apart, becoming a parabola at $p/q = 1/2$.

Solving the hyperbolic case completely is fairly complicated. We will limit ourselves to showing that there can be at most a finite number of solutions if D is square, and that for all cases where D is nonsquare, the number of admissible solutions is infinite. We will also find the explanation for the occurrence of triplets of solutions related by the recycling recurrence.

Case where D is square. If D is square, then the left side of (6) factors, and we have

$$(u - \sqrt{D}v)(u + \sqrt{D}v) = p^2, \tag{8}$$

where $\sqrt{D}$ is an integer by assumption. This equation represents a factorization of p^2. The method of solution is therefore to obtain the list of divisors d_i of p^2, $i = 1, \ldots, n$. For each d_i, equate one of the factors in (8) to d_i and the other to p^2/d_i. This gives two linear equations in the two unknowns u, v. It suffices to search only $1 \le d_i \le p$. Solutions that do not yield admissible values of x, y are discarded. The total number of solutions for any given probability value is finite, at most $\lceil n/2 \rceil$, and there may be no solutions other than the trivial ones.

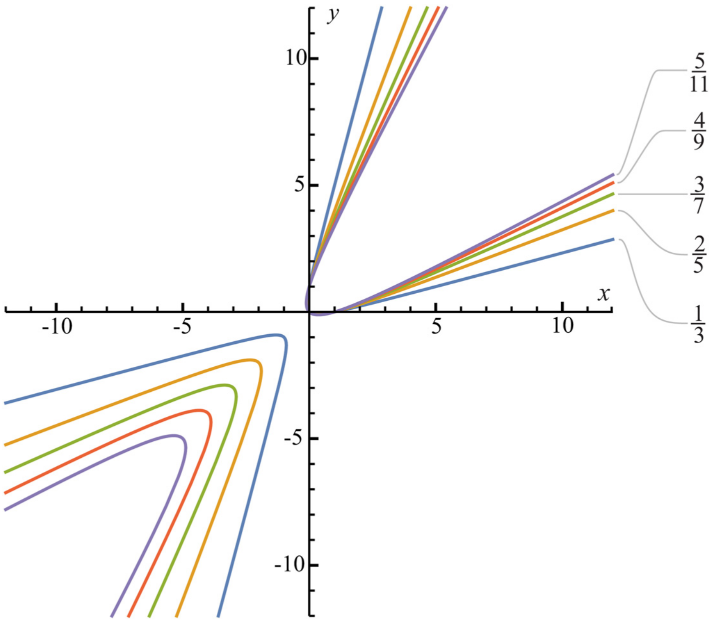

Figure 2. Family of hyperbolas defined by (2) for ratios of the form $p/(2p+1)$.

This is the explanation for the absence of 4/9 from the search results as mentioned in the section "Some examples." For this case $D = 9$ is a square, and none of the solutions found by solving (8) turn out to be admissible. For $p/q = 12/25$, $D = 25$, and p^2 has 15 divisors, but there is only one distinct admissible solution (9, 16).

Case where D is nonsquare. We can show that for $D > 0$ nonsquare, an infinite number of solutions of (2) always exists. Dividing (6) by p^2 and setting $r = u/p$ and $s = v/p$ transforms the equation into

$$r^2 - Ds^2 = 1. \tag{9}$$

This is the well-known Pell equation, and for $D > 0$ nonsquare, it always has an infinite number of integer solutions. These can be found using the method of continued fractions [**2**]. However, because the mapping from u to t involves division by $q - 2p$ and the mapping from (t, v) to (x, y) involves division by 2, it is not in general guaranteed that the solutions to (9) will yield admissible solutions to the original problem. In fact, in many cases, the smallest Pell solutions do not yield admissible (x, y). But we can show that the next larger solution will always be admissible.

As an example, for $p/q = 4/11$, $D = 33$ and solving the Pell equation (9) yields, as the first nontrivial solution, $(r, s) = (23, 4)$ which implies $(u, v) = (92, 16)$. This yields the inadmissible $t = 88/3$. The next larger solution of (9) is $(r, s) = (1057, 184)$, giving $(u, v) = (4228, 736)$, $(t, v) = (1408, 736)$, and the admissible $(x, y) = (336, 1072)$. To show that admissible (x, y) always occurs for the second solution, we need to develop a recurrence for the solutions of (6).

Let (r, s) be the fundamental solution to the Pell equation (9), defined as the solution for which r and s are positive and minimal. All positive solutions of (9) are then given (see [**2**]) by equating rational and irrational parts on each side of

$$(r_n + s_n\sqrt{D}) = (r + s\sqrt{D})^n, \quad n \in \mathbb{N}. \tag{10}$$

From this it follows that if (u, v) is a solution of (6), then (u', v') satisfying

$$u' + v'\sqrt{D} = (u + v\sqrt{D})(r + s\sqrt{D})$$

is also a solution. This leads to the recurrence

$$u_{n+1} = ru_n + Dsv_n, \quad v_{n+1} = su_n + rv_n. \tag{11}$$

I will call (11) the "Pell recurrence" since it is derived from the Pell equation, although it can also be used on solutions that were not obtained from the Pell equation.

Now suppose (x_n, y_n) is an integer solution of (2). Transforming it to (u_n, v_n) and applying two iterations of (11) yields, after simplication,

$$u_{n+2} = (r^2 + Ds^2)\,[p + (q - 2p)(x_n + y_n)] - 2Drs(x_n - y_n),$$
$$v_{n+2} = 2rs\,[p + (q - 2p)(x_n + y_n)] - (r^2 + Ds^2)(x_n - y_n).$$

Transforming this back to (x_{n+2}, y_{n+2}) yields expressions with a term $q - 2p$ in the denominator. Substituting $r^2 = 1 + Ds^2$ and using $D = q(q - 2p)$ provides the necessary factor to cancel that term, and we obtain

$$x_{n+2} = x_n + s^2q\,[p + 2(q - 2p)x_n] + rs\,[p(2(x_n + y_n) - 1) - 2qx_n]\,,$$
$$y_{n+2} = y_n + s^2q\,[p + 2(q - 2p)y_n] - rs\,[p(2(x_n + y_n) - 1) - 2qy_n]\,.$$

Both of these expressions are integer in form. It is clear from (11) that starting with all quantities positive gives u_{n+2} also positive, ensuring that x_{n+2} and y_{n+2} are positive. Therefore they are admissible solutions of (2).

Observe that inserting $(u_1, v_1) = (p, 0)$, corresponding to the trivial solution $(x_1, y_1) = (0, 0)$, into the Pell recurrence yields $(u_2, v_2) = (pr, ps)$, which is the first nontrivial solution given by the Pell equation. Since the trivial solution is integer, the second solution from the Pell iteration will be admissible. Continuing the recurrence therefore will yield infinitely many admissible solutions of (2), on at least every other iteration.

We can now see the reason for the nonappearance of a solution for the case $p/q = 7/17$ mentioned in the section "Some examples." For this ratio $D = 51$, which is nonsquare, so this example does indeed have admissible solutions, but the smallest solution is $(x, y) = (1380, 3381)$, which is simply outside of the region $x < 1000$, $y < 1000$ used for our search.

Trivial solutions generate solutions related by recycling recurrence. For all probabilities $p/q < 1/2$ and D nonsquare, there are always triplets of solutions related by the recycling recurrence. An example, $p/q = 5/11$, was shown earlier.

For ratios having $p = 1$ or $p = 2$, successive sets of triplets are contiguous, i.e., the recycling recurrence takes the last member of one triplet to the first member of the next, forming an unbroken recycling sequence. But for $p > 2$ the triplets are isolated from one another as in this example. These triplets turn out to be produced from the three trivial solutions by the Pell recurrence (11), and there are infinitely many of them as we shall now see.

As seen in the previous section, applying the Pell recurrence to the trivial solution $(x, y) = (0, 0)$ yields solutions corresponding to those obtained from the solutions of the Pell equation. The other two trivial solutions $(x, y) = (0, 1)$ and $(1, 0)$ correspond to $(u, v) = (q - p, \pm 1)$. These are not divisible by p if $p > 1$, and so do not map to solutions of the Pell equation. Although these trivial solutions are not themselves admissible, applying the Pell recurrence to them yields other solutions that are admissible. The three solutions generated this way from the trivial solutions form a recycling triplet. The proof of that fact is lengthy and is omitted. The solution arising from $(x_1, y_1) = (1, 0)$ is the smallest member of the triplet, that from $(0, 0)$ is the middle member, and the one from $(0, 1)$ is the largest.

The three solutions obtained this way from the trivial solutions are not necessarily integer. But above we showed that starting with integer (x_1, y_1) yields integer solutions at least on every other iteration of the Pell recurrence. The trivial solutions are integer. Hence as the Pell recurrence is applied repeatedly to obtain further triplets, integer solutions will be generated without limit. Furthermore, it can be shown that if any two solutions of (2) are related by the recycling recurrence, then the solutions obtained from them via the Pell recurrence are also related by the recycling recurrence.

As an example, we obtain the solutions for the case $p/q = 5/11$. Here $D = 11$, and the solution to the Pell equation (9) is $(r, s) = (10, 3)$. Inserting the solutions $(u, v) = (6, -1), (5, 0)$, and $(6, 1)$, corresponding to the trivial solutions $(x, y) = (1, 0), (0, 0)$, and $(0, 1)$, respectively, into the Pell recurrence (10) yields $(u, v) = (27, 8), (50, 15)$, and $(93, 28)$, respectively. These map to $(x, y) = (7, 15), (15, 30)$, and $(30, 58)$ as in the first row of the table for this example in the section "Some examples." Repeating the Pell recurrence generates the next rows.

Completeness of solutions. For hyperbolic cases with $p = 1$, (6) is the Pell equation, for which all positive solutions are generated by (10). For $p = 2$, [**2**] provides a similar recurrence that also generates all solutions. It can be shown that in both cases, the method described earlier of applying the recycling recurrence to the starting solutions $(1, 2q - 1)$ or $(1, q - 1)$, respectively, generates the same solutions.

For $p > 2$ we saw that solutions can be generated by applying the Pell recurrence to the three trivial solutions. However, this method is, in general, not complete. For instance, in the section "Some examples," the case $9/22$ has a pair of solutions $(3, 9)$ and $(9, 24)$ that are not part of a recycling triplet. These therefore cannot be generated from the trivial solutions via the Pell recurrence.

Hua [**2**] and Nagell [**3**], among others, provide methods that are capable of finding all solutions of (6) for the hyperbolic case. The interested reader is referred to those.

4. CONCLUSION.

The problem of finding x and y to yield a given probability in (1) has turned into the problem of solving the Diophantine equation (6), which is closely related to the well-studied Pell equation (9). We presented some methods for finding solutions, or for ruling them out.

Results. For probability ratios $p/q > 1/2$, there are at most finitely many solutions of (2) for each case. For many ratios in this range, there are no solutions. The density of ratios for which there are solutions decreases dramatically as the ratios increase toward 1: only five ratios in the range $5/9$ to 1 yield solutions. This reflects the fact that it is difficult to arrange for the two balls to be highly likely to differ in color unless the number of balls is very small.

For probability ratios $p/q < 1/2$ having $D = q(q - 2p)$ square, the number of solutions is finite, and many cases have no solution. This does not seem to have a physical explanation, but simply reflects the constraints of having a finite number of factors of p^2 to work with.

For probability ratios $p/q \leq 1/2$, and having D nonsquare, there are always an infinite number of solutions. This reflects the fact that the balls are likely to be the same color if the number of balls of one color is much smaller than the other, which can be achieved in many ways. Furthermore, there are always "recycling" triplets of solutions, i.e., triplets in which a number in one member of the triplet reappears in another member. These triplets arise from the three trivial solutions of (2) via a recurrence based on the solution of the Pell equation.

Open questions. The problem is essentially solved, but there remain some interesting questions that could be pursued.

Are there only singlets, doublets, and triplets, but not quadruplets? In all hyperbolic cases with nonsquare D and $p > 2$, admissible solutions occur in doublets and triplets of solutions that are related to each other via the recycling relationship, as well as singlets that have no such relationship to another solution. I have not encountered any examples with larger tuples, but I do not have a proof that three is the maximum. These relationships are preserved in successive generations of the Pell recurrence. That is, singlets give rise to singlets, doublets to doublets, and triplets to triplets. Therefore, it would suffice to prove the hypothesis for the smallest solutions.

Are there more special cases? While exploring this problem, I discovered some special cases that have predictable solutions. For instance, we saw above that probability ratios of the form $p/(2p-1)$ (elliptical regime) always have solutions at the opposite end of the ellipse from the trivial solutions, namely at (p, p) and $(p-1, p)$. If furthermore $p = 2k^2$, where k is any positive integer, then there are solutions near the middle part of the ellipse as well, of the form $(k^2, k^2 \pm k)$.

If $p = (k^2-1)(k^2-4)/4$ where k is an integer greater than 2, and $q = 2p+1$ (hyperbolic regime), then p and q are relatively prime integers, and there is always a solution $(k(k-1)/2-1, k(k+1)/2-1)$. Interestingly, the two values are successive triangular numbers less 1, an echo of the *Varsity Math* solutions. There is also a pair of solutions in each of which one value equals p, namely $(p-k^2+4, p)$ and $(p, p+k^2-1)$. For instance, for $k = 4$, $p/q = 45/91$, for which (2) has solutions $(5, 9)$, $(33, 45)$, and $(45, 60)$. No doubt there are other families of special cases waiting to be discovered.

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REFERENCES

[1] Hess, D. (2018). Private communication.
[2] Hua, L. (1982). *Introduction to Number Theory.* (Shiu, P., trans.) Berlin-Heidelberg-New York: Springer-Verlag.
[3] Nagell, T. (1964). *Introduction to Number Theory*, 2nd ed. New York: Chelsea Publishing Co.
[4] National Museum of Mathematics in New York City (2017). *Varsity Math*, Week 117. momath.org/home/varsity-math/varsity-math-week-117/

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