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ABSTRACT
This study is the first attempt to measure growth depending on the equation of the concave production possibility frontier, under Solow-neutrality. Although there is evidence that the nature of technological progress is Solow-neutral, Solow-neutrality is not compatible with the steady-state. This study solves that contradiction. To do this, we make three simple contributions. First, the natural rate of growth has been explained in harmony with the economic concepts such as constant returns to scale, full capacity and steady-state, under Solow-neutrality. Secondly, the equation of the concave production possibility frontier has been obtained when the nature of technological progress is Solow-neutral. Thirdly, economic growth has been explained based on the equation of the concave production possibility frontier, under Solow-neutrality. According to the first result of the study, Solow-neutrality becomes compatible with the long-run equilibrium growth and the steady-state under specific conditions. According to the second result of the study, positive economic growth occurs under specific conditions.

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1. Introduction

Assume that machines or artificial intelligence can work instead of labour, extensively. This issue is discussed from different perspectives in the literature. For example, Acemoglu and Restrepo (2020a) investigate the impact of the rise in robot usage in industrial production of the United States for the period 1990–2007. Their model consists of a competition between robots and workers. They find evidence that improvements in the technology of robots may result in a decline in employment. There is a rapidly expanding literature on that subject (see also for example Aghion et al. 2017; Autor et al. 2020; Barkai 2020). Following Acemoglu and Restrepo (2019) and Acemoglu and Restrepo (2020b) we can claim that this can happen when the displacement effect of the automation is greater than its reinstatement effect. In that case, machines would determine the long-term growth of a country. If so, the rate of growth of capital would determine the long-term growth or the natural rate of

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growth. Moreover, if we incorporate technology, the rate of growth of effective capital would determine the natural rate of growth. Thus, if machines or artificial intelligence can work instead of labour, extensively, then the long-run equilibrium growth or the natural rate of growth becomes the rate of growth of effective capital. Then, the nature of technological progress is assumed to be capital-augmenting or Solow-neutral.

What matters if one admits the long-run equilibrium growth or the natural rate of growth as the rate of growth of effective capital and the nature of technological progress is assumed to be capital-augmenting or Solow-neutral? According to Boskin and Lau (2000), the impact of technological progress on economic growth depends on the physical capital stock and human capital stock of that economy. In other words, the nature of technological progress is capital-augmenting rather than labour-augmenting or Harrod-neutral. On the other hand, Uzawa (1961) shows explicitly that the nature of technological progress, which is compatible with steady-state conditions, is Harrod-neutral. Thus, the evidence of Boskin and Lau (2000) and prove of Uzawa (1961) result in a contradiction: If the nature of technological progress is Solow-neutral, then how can we explain the long-run equilibrium growth and the steady-state conditions of an economy? In other words, as Harrod-neutrality is compatible with the long-run equilibrium growth and the steady-state but Solow-neutrality is not, then how can we adapt this evidence to the steady-state analysis? Our study solves that contradiction.

Economic growth occurs when production capacity of an economy increases. By assuming Solow-neutrality, it is admitted that economic growth occurs thanks to an increase in the rate of growth of effective capital. In other words, production capacity of an economy increases thanks to an increase in the rate of growth of effective capital. This causes an outward shift of a concave production possibility frontier, as it is frequently supposed. How can production possible frontier shift outward? What are the conditions for that shift? Different from other studies on the possible shapes of production possible frontier, this study is an attempt which aims to present the conditions of measurement of economic growth formally by using concave production possibility frontier under Solow-neutrality.

Thus, this article has three aims in order to make the following three contributions: (1) to define the natural rate of growth compatible with the nature of technological progress (especially with Solow-neutrality), steady-state and full capacity or full employment; (2) to obtain the equation of production possibility frontier under Solow-neutrality; (3) to explain the conditions of economic growth under Solow-neutrality. Especially, the first contribution may be very important for the near future, which will be characterised by extensive automation.

The article is organised as follows: Next section explains the main problem. After that the natural rate of growth is explained. Moreover, at the end of that section compatibility of the Solow-neutrality with the long-run equilibrium growth and the steady-state is shown. The following sections obtain the equation of production possibility frontier under Solow-neutrality and explain the conditions of economic growth under Solow-neutrality, respectively, which are followed by concluding remarks.

2. The main problem

The debates on what is the function of modern economic growth can be explained on the basis of the production function. In this case, growth becomes a function of production factors and productivity. The growth rate of productivity or technological progress is the factor that determines the growth rate of per capita output in the long-run. Studies that examine how to increase the rate of growth of technology are known as endogenous growth models. For example, Lucas (1988) proves that the steady-state growth rate will increase if the external effect of human capital is positive. Romer (1990) shows that when the workers in the research and development sector and productivity of workers in the research and development sector increase, the rate of growth of technology will increase. Barro (1990) indicates that the steady-state growth rate will increase if the rate of growth of government expenditures increases under certain conditions. With the help of Rebelo (1991), one can show that a rise in long-run economic growth is the result of an increase in the saving rate. Nordhaus (1991) investigates the connection between economic growth and environmental conditions within the context of greenhouse effect. These findings are mostly relying on the endogenous growth literature. One of the aims of the current study is to show that output growth is determined by the rate of growth of technology and capital, by using the equation of the production possibilities curve. Before doing the explanation with regard to production possibilities curve, in order to express the main idea, the connection between the sources of growth literature and Solow-neutrality is explained. Let us explain this issue.

In order to determine sources of economic growth one will ask the following question: Where does economic growth stem from? It is known that there are approximate and fundamental sources of economic growth. According to Szirmai (1993) contributions of growth of input and productivity growth are the approximate sources. By using growth accounting one can disentangle the contributions of per-labour capital growth and rate of growth of productivity to the output growth.

To implement growth accounting there is an identification problem: What is the nature of technological progress? Is it Hicks-neutral, Harrod-neutral or Solow-neutral? If the capital–labour ratio does not change while the ratio of factor prices is constant, then the nature of technological progress will be Hicks-neutral (Hicks, 1963). If the capital–output ratio does not change while the marginal productivity of per-labour capital stock is constant, then the nature of technological progress will be Harrod-neutral (Harrod, 1948). If output per-labour and the wage rate do not change under technological progress, then the nature of technological progress will be Solow-neutral (Solow, 1962). Solow (1957) broadens his growth model in his own work by supposing the nature of technological progress as Hicks-neutral. In that case, Hicks-neutral technological progress is called Solow residual. This concept is the part of growth of per capita output which cannot be accounted for per capita capital growth. For this reason Abramovitz (1956) defines the growth rate of technology as the measure of ignorance. Note that, if the nature of technological progress is Hicks-neutral, this means that technology has an impact on output via capital and labour. In that case, technology means total factor productivity. If there is Harrod-neutral technological progress, technology has an impact on output via only labour. In that case, technology means labour productivity. Finally, if there is Solow-neutrality, then technology will influence output via only capital. This means that technology is capital productivity.

This identification problem is discussed in some studies. For example, Gundlach (2005) points out this problem and explains that conventional studies on sources of growth presume the nature of technological progress as Hicks-neutral instead of Harrod-neutral. Acikgoz and Mert (2014) address the same problem. By using Harrod-neutral technological progress identification, they show that technological progress is the only source of growth for Hong Kong, the Republic of Korea, Singapore and Taiwan in the short-run. Klenow and Rodríguez-Clare (1997) find evidence that the most growth in per-worker output in Hong Kong, the Republic of Korea, and Taiwan stem from technological progress by assuming Harrod-neutrality. Finally, Mankiw et al. (1992) also assume a production function which admits the nature of technological progress as Harrod-neutral. Their empirical findings are based on Harrod-neutrality. There are also many studies on the sources of economic growth which assumes Hicks-neutrality (see, for example, Barger 1969; Barro 1999; Hulten 2000; Nishimizu and Hulten 1978; Senhadji 2000; Solow 1957; van der Eng 2010).

Which identification assumption should be made? Beckmann and Sato (1969) report evidence on the forms of production functions and on the types of technical progress for the United States, Japan, and Germany. They find that assuming Hicks, Harrod, and Solow-neutrality is at least as good as the other types of neutrality for the United States, Japan, and Germany. They also find evidence that Solow-neutral technological progress exerts particularly well. Kumbhakar (2003) analyzes manufacturing industries of the United States for the period 1959–1992 by using National Bureau of Economic Research panel data. Kumbhakar (2003) finds evidence that the nature of technological progress is not Hicks, Harrod, or Solow-neutral. In other words, Kumbhakar (2003) rejects the Hicks-neutral and Harrod-neutral specifications and Solow-neutral specification either. Caselli (2006) investigates development accounting assuming the nature of technological progress as simultaneously in a Harrod and Solow-neutral character. Extending Caselli (2006), Aiyar and Dalgaard (2009) investigate the validity of the assumption of Cobb-Douglas production function in the development accounting studies. They find that Cobb-Douglas production function fits to the development accounting analysis. According to them, if we assume the production technology as Harrod and Solow-neutral, the same conclusion is reached.

As it is indicated in the first section, Boskin and Lau (2000) investigate the validity of Hicks-neutrality, Harrod-neutrality and Solow-neutrality for G-7 countries post-World War II. They find evidence that supports Solow-neutrality. According to their findings, the nature of technological progress is capital-augmenting rather than labour-augmenting or Harrod-neutral. However, Uzawa (1961) proves that the nature of technological progress is Harrod-neutral under steady-state conditions.

As a consequence, the results of Boskin and Lau (2000) and Uzawa (1961) reveals a contradiction: How can we explain the long-run equilibrium growth and the steady-state conditions of an economy compatible with the Solow-neutral technological progress? As Solow-neutrality is not compatible with the long-run equilibrium growth and the steady-state, then, how can we adapt the finding of Boskin and Lau (2000) to the steady-state analysis?

In order to bring an explanation to that contradiction let us suppose that machines or artificial intelligence can work instead of labour, extensively. In such a world, the long-term growth of an economy would be determined by machines. This means that the long-term growth or the natural rate of growth would be determined by capital. Meanwhile, if technology is incorporated to the analysis, the long-term growth or the natural rate of growth would be determined by effective capital. As a result, one can solve the contradiction above by assuming the long-run equilibrium growth or the natural rate of growth as the rate of growth of effective capital.

What is the natural rate of growth? According to Harrod (1939, p. 30) the natural rate of growth 'is the maximum rate of growth allowed by the increase of population, accumulation of capital, technological improvement and the work/leisure preference schedule, supposing that there is always full employment in some sense'. In another study Harrod (1948, p. 87) defines the natural rate of growth as 'the rate of advance which the increase of population and technological improvement allow'. According to Harrod (1953, p. 554) the natural rate of growth is 'for something that may be regarded as corresponding to an optimal static pattern'; 'if the economy proceeded along the line of natural growth, people would be comfortably fully employed; at each point on the line they would feel that the balance between work and its reward, on the one hand, and leisure, on the other, corresponded to their preferences'. Harrod (1953, p. 554) admits 'the natural rate of growth as being adapted to absorb any increase of population and any adjustments required by technological progress'. According to Solow (1956, p. 67) because of 'exogenous population growth, the labour force increases at a constant relative rate n. In the absence of technological change population growth is Harrod's natural rate of growth' so, the neoclassical model is based on the term natural rate of growth which is defined by Harrod. There are also critical on that concept. As an example, according to Yeager (1954, p. 62), the natural rate of growth is not an actual entity in the real world but rather just freely chosen construct of an economist's mind.

There is a wide acceptance about the definition that the natural rate of growth is the sum of the rate of growth of labour and the rate of growth of labour productivity. Note that, if so, the production function should be compatible with Harrod-neutral or labour-augmenting technological progress. For example, according to Tobin (1965, p. 674) the natural rate of growth is based on labour force growth and labour-augmenting technological progress and this is a 'conventional growth model assumption'. Tobin (1968, p. 844) defines the natural rate of growth as 'the sum of the rates of population increase and technical progress'. According to Steedman (1972, p. 1390) the natural rate of growth is 'equal to the growth rate of the working population plus the rate of Harrod neutral technical progress'. Many other studies define or refer to the natural rate of growth similarly (see, for example Boianovsky and Hoover 2009; De-Juan 2007; Eltis 1963; Franke 2019; Grabowski and Shields 2000; Meade and Hahn 1965; Nelson 1966; Otani and Villanueva 1990; Palley 1996; Sasaki 2013).

The natural rate of growth can also be defined by the help of empirical analysis. Based on reverse relationship of Okun (1962), Thirlwall (1969) measures the natural rate of growth where there is no change in the percentage level of unemployment. Thus, according to his methodology, the natural rate of growth is the rate of output growth where there is no change in the percentage level of unemployment. León-Ledesma and Thirlwall, (2002), Vogel (2009), Libânio (2009), Dray and Thirlwall (2011) make empirical analysis based on that definition.

According to another natural rate of growth definition, the natural rate of growth means that the rate of growth of output and capital where capital–output ratio is constant over time. This definition implies that capital–output ratio is an attractor. For example, Matthews (1960) and Phelps (1961) show the natural rate of growth similar to each other, where capital–output ratio is constant.

Apart from these definitions, if it is admitted that machines or artificial intelligence can be utilised instead of labour, extensively, thus if the rate of growth of effective capital would determine the natural rate of growth, the nature of technological progress should be assumed as Solow-neutral rather than Harrod-neutral. The present study claims that by using the natural rate of growth as the rate of growth of effective capital one can explain the contradiction between findings of Boskin and Lau (2000) and steady-state analysis.

By assuming Solow-neutrality it is admitted that economic growth occurs thanks to an increase in the rate of growth of effective capital. In other words, production capacity of an economy increases thanks to an increase in the rate of growth of effective capital. This causes an outward shift of a concave production possibility frontier, as it is frequently supposed. How can production possible frontier shift outward? What are the conditions for that shift? Apart from other studies on the possible shapes of production possible frontier, (see for example Dalal 2006; Herberg and Kemp 1969; Panagariya 1980, 1981; Tawada 1989; Wong 1996) the present study is an attempt which aims to show the conditions of measurement of economic growth formally by using concave production possibility frontier under Solow-neutrality.

3. The natural rate of growth

The aim of this section is to show compatibility of the Solow-neutrality with the long-run equilibrium growth and the steady-state. To do this, we first explain the connection among Harrod-neutrality, steady-state and full capacity or full employment. Then, we explain the connection among Solow-neutrality, steady-state and full capacity or full employment.

Let us begin with the first one. Assume that production function for the commodity x is the following:

$$Q_x = f(K_x, T_x, L_x) \tag{1}$$

Q is the quantity of production, T is the technology level, K is the capital and L is the labour. All variables are function of time. By definition there should be a common rate of growth at steady-state conditions. However, three questions arise: (1) Which variables grow at a common rate?; (2) Do this make sense with regards to economic theory?; and (3) At which common rate should they grow?

According to the production function (1), Q, K and L should grow at a common rate. Technology (T) will be explained below. Growing of Q, K and L at a common rate make sense because at constant returns to scale conditions there are full capacity conditions. Thus, it is needed to emphasise that, if there is a steady-state solution based on a production function, there will be full capacity conditions.

Finally, at which common rate should they grow? The answer is related to the attractor. In a stable model there should be an attractor, so the system will move to that attractor. What is the attractor in a model of economic growth?

If we based on economic theory, the attractor is: (1) the natural rate of growth, where the nature of technological progress is Harrod-neutral; and (2) capital–output ratio, where the nature of technological progress is Harrod-neutral. The second one will be explained later. Before that explanation, what is the natural rate of growth?

As it is stated in the previous section, the natural rate of growth has four definitions. One is the Harrod's definition. The other one defines the natural rate of growth as the sum of rate of growth of labour and the rate of growth of technology. The third one defines the natural rate of growth as the rate of output growth where there is no change in the percentage level of unemployment. According to the fourth definition, the natural rate of growth means the rate of growth of output and capital where capital–output ratio is constant over time. All of these definitions point out the full capacity or full employment conditions. In other words, the natural rate of growth is a meaningful economic concept which strictly points out full capacity or full employment conditions of an economy.

Thus, if we based on economic theory, the attractor should be the natural rate of growth. Let us use the second definition above which states that the natural rate of growth is the sum of the rate of growth of labour and the rate of growth of technology. Then, capital and output grow at a common rate which is equal to the sum of the rate of growth of labour and the rate of growth of technology; i.e., the rate of growth of effective labour. Taking into account that fact, according to the production function (1), Q, K and TL should grow at a common rate at steady-state conditions, where TL is effective labour. That common rate should be the attractor, i.e., the rate of growth of effective labour. As a general result, at steady-state conditions, it should be:

$$\frac{dQ_x}{Q_x} = \frac{dK_x}{K_x} = \frac{dT_x}{T_x} + \frac{dL_x}{L_x}$$

Then the Cobb-Douglas form of the production function will be:

$$Q_x = K_x^{\mu_x} T_x^{\lambda_x} L_x^{\lambda_x}$$

Q is quantity of production, T is technology level, K is capital and L is labour. μ and λ are elasticity coefficients which are all positive. All variables are function of time.

We know that the nature of the technology is assumed to be Harrod-neutral and there are constant returns to scale conditions in order to guarantee steady-state conditions. In order to explain that briefly, we rewrite (1) for constant returns to scale conditions:

$$Q_x = K_x^{\mu_x} T_x^{1-\mu_x} L_x^{1-\mu_x} \tag{2}$$

Then, we rewrite (2) in terms of per-effective labour:

$$\frac{Q_x}{T_x L_x} = \frac{K_x^{\mu_x}}{T_x^{\mu_x} L_x^{\mu_x}} \tag{3}$$

Rearranging (3) in terms of rate of growth:

$$\left(\frac{dQ_x}{Q_x} - \left(\frac{dT_x}{T_x} + \frac{dL_x}{L_x}\right)\right) = \mu_x\left(\frac{dK_x}{K_x} - \left(\frac{dT_x}{T_x} + \frac{dL_x}{L_x}\right)\right) \tag{4}$$

By definition there should be a common rate of growth at steady-state conditions. Indeed, (5) confirms that definition: When $\frac{dK_x}{K_x} = \frac{dT_x}{T_x} + \frac{dL_x}{L_x}$ it should be $\frac{dQ_x}{Q_x} = \frac{dT_x}{T_x} + \frac{dL_x}{L_x}$. Thus, at steady-state there will be:

$$\frac{dQ_x}{Q_x} = \frac{dK_x}{K_x} = \frac{dT_x}{T_x} + \frac{dL_x}{L_x} \tag{5}$$

or

$$\frac{dQ_x}{Q_x} - \frac{dL_x}{L_x} = \frac{dK_x}{K_x} - \frac{dL_x}{L_x} = \frac{dT_x}{T_x} \tag{6}$$

Recognise that (6) means long-term per capita output growth equals to the rate of growth of technology. Since, the nature of technological progress is Harrod-neutral, one can also say that long-term per capita output growth equals to the rate of growth of labour productivity.

It is shown that there should be a common rate of growth at steady-state conditions. In other words, capital and output grow at a common rate. Can it be *any* common rate? It is depicted that capital and output grow at a common rate which is equal to the sum of rate of growth of labour and the rate of growth of technology; i.e., the rate of growth of effective labour.

However, Phelps (1961) reaches a different common rate. Let us explain the common rate of growth in Phelps (1961). Remember that according to Phelps (1961) the natural rate of growth is defined as the rate of growth of output and capital where capital–output ratio is constant over time. Note that, this definition points out the full capacity or full employment conditions. In other words, the natural rate of growth is a meaningful economic concept which strictly points out full capacity or full employment conditions of an economy. Now let us use the production function of Phelps (1961). Phelps (1961) uses a production function such as:

$$Q_t = A\left(e^{\lambda t} K_t\right)^{\alpha} \left(e^{\mu t} L_t\right)^{1-\alpha} \tag{7}$$

$$L_t = L_0 e^{\gamma t} \tag{8}$$

$$\rho = \alpha\lambda + (1-\alpha)\mu \tag{9}$$

$$Q_t = Ae^{\rho t}K_t^{\alpha}L_t^{1-\alpha} \tag{10}$$

If one divides both sides of (10) by K_t and takes Q_t/K_t as the attractor, the natural rate of growth (g) will be:

$$g = \frac{\rho + (1-\alpha)\gamma}{1-\alpha} \tag{11}$$

or

$$\frac{dK}{K} = \frac{\alpha\lambda}{1-\alpha} + \mu + \frac{dL}{L} = \frac{dQ}{Q} \tag{12}$$

In other words, Phelps (1961) shows that both capital and output grow at a rate of (11). Thus, the natural rate of growth is the rate of growth where both capital and output grow at the same rate, which is shown in (11) and (12). Matthews (1960) also reaches a similar natural rate of growth equation.

We need to strongly emphasise that, in that case, both capital and output grow at the same rate but not equal to sum of the rate of growth of labour and labour productivity. Thus, according to Phelps (1961) the rate of growth of effective labour is not the attractor. This reveals a contradiction about full capacity conditions. Let us explain that contradiction.

In order to remind the linkage between constant returns to scale and full capacity or full employment, let us write total cost (TC) and average cost (AC) equations as follows:

$$TC = we^{\mu t}L + re^{\lambda t}K \tag{13}$$

$$\frac{TC}{Q} = AC = w\frac{e^{\mu t}L}{Q} + r\frac{e^{\lambda t}K}{Q} \tag{14}$$

where w, L, r, K and Q are wage, labour, rate of return, capital and output, respectively. All variables are function of time. Then, (15) is written:

$$dAC = wd\left(\frac{e^{\mu t}L}{Q}\right) + rd\left(\frac{e^{\lambda t}K}{Q}\right) + \frac{e^{\mu t}L}{Q}dw + \frac{e^{\lambda t}K}{Q}dr \tag{15}$$

Rearranging (15):

$$dAC = w\frac{e^{\mu t}L}{Q}\frac{d\left(\frac{e^{\mu t}L}{Q}\right)}{\frac{e^{\mu t}L}{Q}} + r\frac{e^{\lambda t}K}{Q}\frac{d\left(\frac{e^{\lambda t}K}{Q}\right)}{\frac{e^{\lambda t}K}{Q}} + \frac{e^{\mu t}L}{Q}dw + \frac{e^{\lambda t}K}{Q}dr \tag{16}$$

Thus (16) will be:

$$dAC = w\frac{e^{\mu t}L}{Q}\left(\frac{d(e^{\mu t}L)}{e^{\mu t}L} - \frac{dQ}{Q}\right) + r\frac{e^{\lambda t}K}{Q}\left(\frac{d(e^{\lambda t}K)}{e^{\lambda t}K} - \frac{dQ}{Q}\right) + \frac{e^{\mu t}L}{Q}dw + \frac{e^{\lambda t}K}{Q}dr \tag{17}$$

Since $Q = f(e^{\lambda t}K, e^{\mu t}L)$, if there are constant returns to scale, then by definition:

$$\lambda + \frac{dK}{K} = \mu + \frac{dL}{L} = \frac{dQ}{Q} \tag{18}$$

Therefore (17) becomes:

$$dAC = \frac{e^{\mu t}L}{Q}dw + \frac{e^{\lambda t}K}{Q}dr \tag{19}$$

Since there are perfect competition conditions as in Phelps (1961), $dw = 0$ and $dr = 0$. Then, (17) becomes:

$$dAC = 0 \tag{20}$$

Equation (20) means that average cost reaches its minimum. So, by definition, full capacity or full employment is achieved. Note that if there were increasing to returns to scale, (17) would be negative, so the scale would be below the full capacity or full employment level.

However, a production function such as in Phelps (1961) imposes a state which is different from full capacity conditions, because $\frac{dK}{K} = \frac{\alpha\lambda}{1-\alpha} + \mu + \frac{dL}{L} = \frac{dQ}{Q}$. Thus, a contradiction arises if we use a production function such as in Phelps (1961); it should be $\lambda + \frac{dK}{K} = \mu + \frac{dL}{L} = \frac{dQ}{Q}$ for the full capacity conditions as it is shown above, however it is $\frac{dK}{K} = \frac{\alpha\lambda}{1-\alpha} + \mu + \frac{dL}{L} = \frac{dQ}{Q}$ according to Phelps (1961).

As a general result, we showed that if one assumes the attractor as the rate of growth of effective labour or capital–output ratio, Harrod-neutrality, constant returns to scale and full capacity or full employment conditions are closely connected with each other for an economically true steady-state analysis.

Now let us explain the connection among Solow-neutrality, steady-state and full capacity or full employment. Assume that machines or artificial intelligence can work instead of labour, extensively. In that case, machines would determine the long-term growth of a country. If so, the rate of growth of capital would determine the long-term growth or the natural rate of growth. Moreover, if we incorporate technology, the rate of growth of effective capital would determine the natural rate of growth. This statement is valid when the nature of technology is Solow-neutral. Then the production function for the commodity x will be:

$$Q_x = T_x^{\mu_x} K_x^{\mu_x} L_x^{1-\mu_x} \tag{21}$$

According to the production function (21), Q, TK and L should grow at a common rate at steady-state conditions which means full capacity conditions. Now let us assume that the attractor is the rate of growth of effective capital(TK). As a result, at steady-state conditions, it should be:

$$\frac{dQ_x}{Q_x} = \frac{dT_x}{T_x} + \frac{dK_x}{K_x} = \frac{dL_x}{L_x} \tag{22}$$

If we rewrite (21) in terms of per-effective capital:

$$\frac{Q_x}{T_x K_x} = T_x^{\mu_x - 1} K_x^{\mu_x - 1} L_x^{1-\mu_x} \tag{23}$$

Rearranging (23) in terms of rate of growth:

$$\left(\frac{dQ_x}{Q_x} - \left(\frac{dT_x}{T_x} + \frac{dK_x}{K_x}\right)\right) = (1 - \mu_x)\left(\frac{dL_x}{L_x} - \left(\frac{dK_x}{K_x} + \frac{dT_x}{T_x}\right)\right) \tag{24}$$

By definition there should be a common rate of growth at steady-state conditions. Indeed, (24) confirms that definition: When $\frac{dL_x}{L_x} = \frac{dK_x}{K_x} + \frac{dT_x}{T_x}$ it should be $\frac{dQ_x}{Q_x} = \frac{dT_x}{T_x} + \frac{dK_x}{K_x}$. Thus, at steady-state there will be:

$$\frac{dQ_x}{Q_x} = \frac{dT_x}{T_x} + \frac{dK_x}{K_x} = \frac{dL_x}{L_x} \tag{25}$$

or

$$\frac{dQ_x}{Q_x} - \frac{dK_x}{K_x} = \frac{dL_x}{L_x} - \frac{dK_x}{K_x} = \frac{dT_x}{T_x} \tag{26}$$

Recognise that (26) means long-term output growth per-capital stock equals to the rate of growth of technology. Since, the nature of technological progress is Solow-neutral, one can also say that long-term output growth per-capital stock equals to the rate of growth of capital productivity.

It is shown that capital and output grow at a common rate which is equal to the sum of the rate of growth of capital and the rate of growth of technology; i.e., the rate of growth of effective capital, at steady-state conditions.

In order to show the linkage between constant returns to scale and full capacity or full employment, let us write total cost (TC) and average cost (AC) equations as follows:

$$TC = wL + rTK \tag{27}$$

$$\frac{TC}{Q} = AC = w\frac{L}{Q} + r\frac{TK}{Q} \tag{28}$$

where w, L, r, TK and Q are wage, labour, rate of return, effective capital and output, respectively. All variables are function of time. Then, (29) is written:

$$dAC = wd\left(\frac{L}{Q}\right) + rd\left(\frac{TK}{Q}\right) + \frac{L}{Q}dw + \frac{TK}{Q}dr \tag{29}$$

Rearranging (29):

$$dAC = w\frac{L}{Q}\frac{d\left(\frac{L}{Q}\right)}{\frac{L}{Q}} + r\frac{TK}{Q}\frac{d\left(\frac{TK}{Q}\right)}{\frac{TK}{Q}} + \frac{L}{Q}dw + \frac{TK}{Q}dr \tag{30}$$

Thus (30) will be:

$$dAC = w\frac{L}{Q}\left(\frac{dL}{L} - \frac{dQ}{Q}\right) + r\frac{TK}{Q}\left(\frac{dT}{T} + \frac{dK}{K} - \frac{dQ}{Q}\right) + \frac{L}{Q}dw + \frac{TK}{Q}dr \tag{31}$$

Since $Q = f(TK, L)$, if there are constant returns to scale, then by definition:

$$\frac{dT}{T} + \frac{dK}{K} = \frac{dL}{L} = \frac{dQ}{Q} \tag{32}$$

Therefore (31) becomes:

$$dAC = \frac{L}{Q}dw + \frac{TK}{Q}dr \tag{33}$$

Since there are perfect competition conditions, $dw = 0$ and $dr = 0$. Then, (31) becomes:

$$dAC = 0 \tag{34}$$

Equation (34) means that average cost reaches its minimum. So, by definition, full capacity or full employment is achieved. Note that if there were increasing to returns to scale, (31) would be negative, so the scale will be below the full capacity or full employment level.

Thus, our general result which is explained above changes as follows: If one assumes the attractor as the rate of growth of effective capital or capital–output ratio, Solow-neutrality, constant returns to scale and full capacity or full employment conditions are closely connected with each other for an economically true steady-state analysis. As a consequence, Solow-neutrality becomes compatible with the long-run equilibrium growth and the steady-state.

4. Equation of production possibility frontier under Solow-neutrality

Now in the present section let us explain economic growth based on a concave production possibility frontier under Solow-neutrality.

Two sectors produce commodities x and y. Cobb-Douglas production functions are assumed:

$$Q_x = T_x^{\mu_x} K_x^{\mu_x} L_x^{\lambda_x} \tag{35}$$

$$Q_y = T_y^{\mu_y} K_y^{\mu_y} L_y^{\lambda_y} \tag{36}$$

Q is quantity of production, T is level of technology, K is capital and L is labour. μ and λ are elasticity coefficients. Suppose that the nature of the technology is Solow-neutral, and $\mu_x, \mu_y, \lambda_x, \lambda_y$ have a positive value. Sub-indices,x and$y$, are the sectors.

For the sector x, (37) is the isoquant curve:

$$L_x = m_x K_x^{-b_x} \tag{37}$$

where m and b are parameters ($m_x > 0$ and $b_x > 0$).

Let us write a tangent at any point on (37):

$$\frac{dL_x}{dK_x} = m_x(-b_x)K_x^{-b_x-1} \tag{38}$$

Likewise, for y:

$$Q_y = T_y^{\mu_y} K_y^{\mu_y} L_y^{\lambda_y} \tag{39}$$

$$L_y = m_y K_y^{-b_y} m_y > 0; b_y > 0 \tag{40}$$

$$\frac{dL_y}{dK_y} = m_y\left(-b_y\right)K_y^{-b_y-1} \tag{41}$$

Assuming $m_x > 0$, $m_y > 0, b_x > 0$ and $b_y > 0$, it is obvious that $\frac{dL_x}{dK_x} = m_x(-b_x)K_x^{-b_x-1}$ and $\frac{dL_y}{dK_y} = m_y(-b_y)K_y^{-b_y-1}$ have a negative value. Thus, the conditions for technical efficiency are valid.

Let us write the conditions for production efficiency; i.e., the point where isoquant curves are tangent to each other. Then, (41) equals to (38):

$$\frac{dL_x}{dK_x} = \frac{dL_y}{dK_y} \tag{42}$$

$$m_x(-b_x)K_x^{-b_x-1} = m_y\left(-b_y\right)K_y^{-b_y-1} \tag{43}$$

Rewriting (43):

$$K_x^{-b_x-1}/K_y^{-b_y-1} = m_y b_y/(m_x b_x) \tag{44}$$

Dividing both side of the $Q_x = T_x^{\mu_x} K_x^{\mu_x} L_x^{\lambda_x}$ by $Q_y = T_y^{\mu_y} K_y^{\mu_y} L_y^{\lambda_y}$:

$$Q_x/Q_y = T_x^{\mu_x} K_x^{\mu_x} L_x^{\lambda_x} / \left(T_y^{\mu_y} K_y^{\mu_y} L_y^{\lambda_y} \right) \tag{45}$$

$$Q_x \frac{T_y^{\mu_y}}{T_x^{\mu_x}} \frac{K_y^{\mu_y}}{K_x^{\mu_x}} \frac{L_y^{\lambda_y}}{L_x^{\lambda_x}} = Q_y \tag{46}$$

Note that $L_x = m_x K_x^{-b_x}$ and $L_y = m_y K_y^{-b_y}$:

$$Q_x \frac{T_y^{\mu_y}}{T_x^{\mu_x}} \frac{K_y^{\mu_y}}{K_x^{\mu_x}} \frac{\left(m_y K_y^{-b_y} \right)^{\lambda_y}}{\left(m_x K_x^{-b_x} \right)^{\lambda_x}} = Q_y \tag{47}$$

$$Q_x \frac{T_y^{\mu_y}}{T_x^{\mu_x}} \frac{K_y^{\mu_y - b_y \lambda_y}}{K_x^{\mu_x - b_x \lambda_x}} \frac{{m_y}^{\lambda_y}}{{m_x}^{\lambda_x}} = Q_y \tag{48}$$

Note that $\frac{K_x^{-b_x-1}}{K_y^{-b_y-1}} = \frac{m_y b_y}{m_x b_x}$. Then$K_y = (K_x^{-b_x-1} \frac{m_x b_x}{m_y b_y})^{\frac{1}{-b_y-1}}$.
(48) can be rearranged:

$$Q_x \frac{T_y^{\mu_y}}{T_x^{\mu_x}} K_x^{\frac{(-b_x-1)}{-b_y-1}\left(\mu_y - b_y \lambda_y\right) - \mu_x + \lambda_x b_x} \left(\frac{m_x b_x}{m_y b_y} \right)^{\frac{(\mu_y - b_y \lambda_y)}{-b_y-1}} {m_y}^{\lambda_y} {m_x}^{-\lambda_x} = Q_y \tag{49}$$

(49) is the equation of the production possibility frontier.

Recognise that $L_x = m_x K_x^{-b_x}$ and $L_y = m_y K_y^{-b_y}$. As a result, $\frac{dL_x}{dK_x} \frac{K_x}{L_x} = -b_x$ and $\frac{dL_y}{dK_y} \frac{K_y}{L_y} = -b_y$.

Because of elasticity of output with respect to capital is equal to $\frac{dQ_x}{dK_x} \frac{K_x}{Q_x} = \mu_x$, $\frac{dQ_y}{dK_y} \frac{K_y}{Q_y} = \mu_y$ and labour elasticity of output is equal to $\frac{dQ_x}{dL_x} \frac{L_x}{Q_x} = \lambda_x$, $\frac{dQ_y}{dL_y} \frac{L_y}{Q_y} = \lambda_y$, (50) and (51) are valid.

$$\frac{dQ_x}{dK_x} \frac{K_x}{Q_x} / \left(\frac{dQ_x}{dL_x} \frac{L_x}{Q_x} \right) = \frac{dL_x}{dK_x} \frac{K_x}{L_x} = \frac{\mu_x}{\lambda_x} \tag{50}$$

$$\frac{dQ_y}{dK_y} \frac{K_y}{Q_y} / \left(\frac{dQ_y}{dL_y} \frac{L_y}{Q_y} \right) = \frac{dL_y}{dK_y} \frac{K_y}{L_y} = \frac{\mu_y}{\lambda_y} \tag{51}$$

Due to $\frac{dL_x}{dK_x} \frac{K_x}{L_x} = -b_x$ and $\frac{dL_y}{dK_y} \frac{K_y}{L_y} = -b_y$, (52) and (53) are valid.

$$\mu_x/\lambda_x = -b_x \tag{52}$$

$$\mu_y/\lambda_y = -b_y \tag{53}$$

Since $\mu_x = -\lambda_x b_x$ and $\mu_y = -\lambda_y b_y$, (49) can be rewritten:

$$Q_x \frac{T_y^{\mu_y}}{T_x^{\mu_x}} K_x^{\frac{(-b_x-1)2\mu_y}{-b_y-1} - 2\mu_x} \left(\frac{m_x b_x}{m_y b_y}\right)^{\frac{2\mu_y}{-b_y-1}} m_y^{\lambda_y} m_x^{-\lambda_x} = Q_y. \tag{54}$$

One can leave alone m_x and m_y in (38) and (41), then (55) and (56) can be written.

$$\frac{dL_x}{dK_x} K_x^{b_x+1} \frac{1}{-b_x} = m_x \tag{55}$$

$$\frac{dL_y}{dK_y} K_y^{b_y+1} \frac{1}{-b_y} = m_y \tag{56}$$

The last two equations make (54) as (57):

$$Q_x \frac{T_y^{\mu_y}}{T_x^{\mu_x}} K_x^{\frac{(-b_x-1)2\mu_y}{-b_y-1} - 2\mu_x} \left(\frac{m_x b_x}{m_y b_y}\right)^{\frac{2\mu_y}{-b_y-1}} \left(\frac{dL_y}{dK_y} K_y^{b_y+1} \frac{1}{-b_y}\right)^{\lambda_y} \left(\frac{dL_x}{dK_x} K_x^{b_x+1} \frac{1}{-b_x}\right)^{-\lambda_x} = Q_y \tag{57}$$

Lastly, since $\lambda_x b_x = -\mu_x$ and $\lambda_y b_y = -\mu_y$ (57) can be rewritten:

$$Q_x \frac{T_y^{\mu_y}}{T_x^{\mu_x}} K_x^{\frac{(-b_x-1)2\mu_y}{-b_y-1} - 2\mu_x} \left(\frac{m_x b_x}{m_y b_y}\right)^{\frac{2\mu_y}{-b_y-1}} \left(\frac{dL_y}{dK_y}\right)^{\lambda_y} \left(\frac{dL_x}{dK_x}\right)^{-\lambda_x} \frac{(-b_x)^{\lambda_x} K_y^{-\mu_y+\lambda_y}}{(-b_y)^{\lambda_y} K_x^{-\mu_x+\lambda_x}} = Q_y \tag{58}$$

Since $K_y = (K_x^{-b_x-1} \frac{m_x b_x}{m_y b_y})^{\frac{1}{-b_y-1}}$, then,

$$Q_x \frac{T_y^{\mu_y}}{T_x^{\mu_x}} K_x^{\frac{(-b_x-1)(\lambda_y+\mu_y)}{-b_y-1} - \lambda_x - \mu_x} \left(\frac{m_x b_x}{m_y b_y}\right)^{\frac{\lambda_y+\mu_y}{-b_y-1}} \left(\frac{dL_y}{dK_y}\right)^{\lambda_y} \left(\frac{dL_x}{dK_x}\right)^{-\lambda_x} \frac{(-b_x)^{\lambda_x}}{(-b_y)^{\lambda_y}} = Q_y \tag{59}$$

Consequently, (59) is the equation of production possibility frontier under Solow-neutrality. By definition, (59) should have a negative slope. The possible shape of production possibility frontier is determined by the second derivative of (59).

5. Second derivative of production possibility frontier

5.1. Initials

i) Level of technology has a positive value:$\frac{T_y^{\mu_y}}{T_x^{\mu_x}}>0$. It does not change with a change in the quantity of production, so, $\frac{\delta\left(\frac{T_y^{\mu_y}}{T_x^{\mu_x}}\right)}{\delta Q_x} = 0$.ii) Technical efficiency occurs, then $\frac{dL_y}{dK_y}<0$ and $\frac{dL_x}{dK_x}<0$. As a result, for the rational values of $(\frac{dL_y}{dK_y})^{\lambda_y}$ and $(\frac{dL_x}{dK_x})^{-\lambda_x}$, $(\frac{dL_y}{dK_y})^{\lambda_y}<0$ or

$(\frac{dL_y}{dK_y})^{\lambda_y}>0$ and $(\frac{dL_x}{dK_x})^{-\lambda_x}<0$ or $(\frac{dL_x}{dK_x})^{-\lambda_x}>0$. As it will be assumed that labour elatsticies are equal, whether $(\frac{dL_y}{dK_y})^{\lambda_y}<0$ or $(\frac{dL_y}{dK_y})^{\lambda_y}>0$, it is $(\frac{dL_y}{dK_y})^{\lambda_y}(\frac{dL_x}{dK_x})^{-\lambda_x}>0$.iii) $\frac{(-b_x)^{\lambda_x}}{(-b_y)^{\lambda_y}} = \frac{(\mu_x/\lambda_x)^{\lambda_x}}{(\mu_y/\lambda_y)^{\lambda_y}}$ and all of the parameters are positive, then $\frac{(-b_x)^{\lambda_x}}{(-b_y)^{\lambda_y}}>0$. Because μ_x, μ_y, λ_x and λ_y have a constant value, they do not depend on changes in output: $\frac{\delta((-b_x)^{\lambda_x}/(-b_y)^{\lambda_y})}{\delta Q_x} = 0$ .iv) As a result, within the framework of these conditions, the followings are valid.

Since $\frac{\delta\left(\frac{T_y^{\mu_y}}{T_x^{\mu_x}}\right)}{\delta Q_x} = 0$, $\frac{\delta\left((m_x b_x/(m_y b_y))^{\frac{\lambda_y+\mu_y}{-b_y-1}}\right)}{\delta Q_x} = 0$, $\frac{\delta((-b_x)^{\lambda_x}/(-b_y)^{\lambda_y})}{\delta Q_x} = 0$ then

$$\frac{\delta Q_y}{\delta Q_x} = \frac{T_y^{\mu_y}}{T_x^{\mu_x}} K_x^{\frac{(-b_x-1)(\lambda_y+\mu_y)}{-b_y-1}-\lambda_x-\mu_x}\left(\frac{m_x b_x}{m_y b_y}\right)^{\frac{\lambda_y+\mu_y}{-b_y-1}}\left(\frac{dL_y}{dK_y}\right)^{\lambda_y}\left(\frac{dL_x}{dK_x}\right)^{-\lambda_x}\frac{(-b_x)^{\lambda_x}}{(-b_y)^{\lambda_y}}$$
$$+\frac{\delta\left(\left(\frac{dL_y}{dK_y}\right)^{\lambda_y}\left(\frac{dL_x}{dK_x}\right)^{-\lambda_x}\right)}{\delta Q_x}\frac{T_y^{\mu_y}}{T_x^{\mu_x}}K_x^{\frac{(-b_x-1)(\lambda_y+\mu_y)}{-b_y-1}-\lambda_x-\mu_x}\left(\frac{m_x b_x}{m_y b_y}\right)^{\frac{\lambda_y+\mu_y}{-b_y-1}}\frac{(-b_x)^{\lambda_x}}{(-b_y)^{\lambda_y}}$$
$$+\frac{\delta\left(K_x^{\frac{(-b_x-1)(\lambda_y+\mu_y)}{-b_y-1}-\lambda_x-\mu_x}\right)}{\delta Q_x}\frac{T_y^{\mu_y}}{T_x^{\mu_x}}\left(\frac{m_x b_x}{m_y b_y}\right)^{\frac{\lambda_y+\mu_y}{-b_y-1}}\left(\frac{dL_y}{dK_y}\right)^{\lambda_y}\left(\frac{dL_x}{dK_x}\right)^{-\lambda_x}\frac{(-b_x)^{\lambda_x}}{(-b_y)^{\lambda_y}} \quad (60)$$

is written.

Then if (60) is negative, production possibility frontier will be negatively sloped.

The possible shape of production possibility frontier will depend on the second derivative of (59).

5.2. The second derivative of (59)

$$\frac{\delta^2 Q_y}{\delta Q_x{}^2} = \left(\frac{m_x b_x}{m_y b_y}\right)^{\frac{\lambda_y+\mu_y}{-b_y-1}}\frac{(-b_x)^{\lambda_x}}{(-b_y)^{\lambda_y}}\frac{T_y^{\mu_y}}{T_x^{\mu_x}}$$

$$\left[\begin{array}{l} \frac{\delta\left(K_x^{\frac{(-b_x-1)(\lambda_y+\mu_y)}{-b_y-1}-\lambda_x-\mu_x}\right)}{\delta Q_x}\left(\frac{dL_y}{dK_y}\right)^{\lambda_y}\left(\frac{dL_x}{dK_x}\right)^{-\lambda_x}+\frac{\delta\left(\left(\frac{dL_y}{dK_y}\right)^{\lambda_y}\left(\frac{dL_x}{dK_x}\right)^{-\lambda_x}\right)}{\delta Q_x}K_x^{\frac{(-b_x-1)(\lambda_y+\mu_y)}{-b_y-1}-\lambda_x-\mu_x} \\ +\frac{\delta^2\left(\left(\frac{dL_y}{dK_y}\right)^{\lambda_y}\left(\frac{dL_x}{dK_x}\right)^{-\lambda_x}\right)}{\delta Q_x{}^2}K_x^{\frac{(-b_x-1)(\lambda_y+\mu_y)}{-b_y-1}-\lambda_x-\mu_x}+\frac{\delta\left(K_x^{\frac{(-b_x-1)(\lambda_y+\mu_y)}{-b_y-1}-\lambda_x-\mu_x}\right)}{\delta Q_x}\frac{\delta\left(\left(\frac{dL_y}{dK_y}\right)^{\lambda_y}\left(\frac{dL_x}{dK_x}\right)^{-\lambda_x}\right)}{\delta Q_x} \\ +\frac{\delta^2\left(K_x^{\frac{(-b_x-1)(\lambda_y+\mu_y)}{-b_y-1}-\lambda_x-\mu_x}\right)}{\delta Q_x{}^2}\left(\frac{dL_y}{dK_y}\right)^{\lambda_y}\left(\frac{dL_x}{dK_x}\right)^{-\lambda_x}+\frac{\delta\left(\left(\frac{dL_y}{dK_y}\right)^{\lambda_y}\left(\frac{dL_x}{dK_x}\right)^{-\lambda_x}\right)}{\delta Q_x}\frac{\delta\left(K_x^{\frac{(-b_x-1)(\lambda_y+\mu_y)}{-b_y-1}-\lambda_x-\mu_x}\right)}{\delta Q_x} \end{array}\right] \quad (61)$$

As $\frac{\delta^2\left(\left(\frac{dL_y}{dK_y}\right)^{\lambda_y}\left(\frac{dL_x}{dK_x}\right)^{-\lambda_x}\right)}{\delta Q_x{}^2}$ and $\frac{\delta^2\left(K_x^{\frac{(-b_x-1)(\lambda_y+\mu_y)}{-b_y-1}-\lambda_x-\mu_x}\right)}{\delta Q_x{}^2}$ are zero and as $\frac{dL_y}{dK_y} = -\frac{MP_{K_y}}{MP_{L_y}}$ and $\frac{dL_x}{dK_x} = -\frac{MP_{K_x}}{MP_{L_x}}$, (61) is rewritten as (62):

$$\frac{\delta^2 Q_y}{\delta Q_x{}^2} = \left(\frac{m_x b_x}{m_y b_y}\right)^{\frac{\lambda_y+\mu_y}{-b_y-1}} \frac{(-b_x)^{\lambda_x}}{(-b_y)^{\lambda_y}} \frac{T_y^{\mu_y}}{T_x^{\mu_x}}$$

$$\left[\begin{array}{l} \frac{\delta\left(K_x^{\frac{(-b_x-1)(\lambda_y+\mu_y)}{-b_y-1}-\lambda_x-\mu_x}\right)}{\delta Q_x}\left(-\frac{MP_{K_y}}{MP_{L_y}}\right)^{\lambda_y}\left(-\frac{MP_{K_x}}{MP_{L_x}}\right)^{-\lambda_x} \\ +\frac{\delta\left(\left(-\frac{MP_{K_y}}{MP_{L_y}}\right)^{\lambda_y}\left(-\frac{MP_{K_x}}{MP_{L_x}}\right)^{-\lambda_x}\right)}{\delta Q_x} K_x^{\frac{(-b_x-1)(\lambda_y+\mu_y)}{-b_y-1}-\lambda_x-\mu_x} \\ +\frac{\delta\left(K_x^{\frac{(-b_x-1)(\lambda_y+\mu_y)}{-b_y-1}-\lambda_x-\mu_x}\right)}{\delta Q_x}\frac{\delta\left(\left(-\frac{MP_{K_y}}{MP_{L_y}}\right)^{\lambda_y}\left(-\frac{MP_{K_x}}{MP_{L_x}}\right)^{-\lambda_x}\right)}{\delta Q_x} \\ +\frac{\delta\left(\left(-\frac{MP_{K_y}}{MP_{L_y}}\right)^{\lambda_y}\left(-\frac{MP_{K_x}}{MP_{L_x}}\right)^{-\lambda_x}\right)}{\delta Q_x}\frac{\delta\left(K_x^{\frac{(-b_x-1)(\lambda_y+\mu_y)}{-b_y-1}-\lambda_x-\mu_x}\right)}{\delta Q_x} \end{array}\right] \tag{62}$$

As $-\frac{MP_{K_y}}{MP_{L_y}}<0$, $(-\frac{MP_{K_y}}{MP_{L_y}})^{\lambda_y}$ and $(-\frac{MP_{K_x}}{MP_{L_x}})^{-\lambda_x}$ may have an irrational value. For simplicity, it is supposed that $\lambda_y = \lambda_x$, then $(-\frac{MP_{K_y}}{MP_{L_y}})^{\lambda_y}(-\frac{MP_{K_x}}{MP_{L_x}})^{-\lambda_x}$ is rewritten as $(\frac{MP_{K_y}}{MP_{L_y}}\frac{MP_{K_x}}{MP_{L_x}})^{\lambda}$ and it does not have an irrational value. So, (63):

$$\frac{\delta^2 Q_y}{\delta Q_x{}^2} = \left(\frac{m_x b_x}{m_y b_y}\right)^{\frac{\lambda+\mu_y}{-b_y-1}} \frac{(-b_x)^{\lambda}}{(-b_y)^{\lambda}} \frac{T_y^{\mu_y}}{T_x^{\mu_x}}$$

$$\left[\begin{array}{l} \frac{\delta\left(K_x^{\frac{(\lambda-\mu_x)(\lambda+\mu_y)+(-\lambda-\mu_x)(\lambda-\mu_y)}{\lambda-\mu_y}}\right)}{\delta Q_x}\left(\frac{MP_{K_y}}{MP_{L_y}}\frac{MP_{L_x}}{MP_{K_x}}\right)^{\lambda}+\frac{\delta\left(\left(\frac{MP_{K_y}}{MP_{L_y}}\frac{MP_{L_x}}{MP_{K_x}}\right)^{\lambda}\right)}{\delta Q_x}K_x^{\frac{(\lambda-\mu_x)(\lambda+\mu_y)+(-\lambda-\mu_x)(\lambda-\mu_y)}{\lambda-\mu_y}} \\ +\frac{\delta\left(K_x^{\frac{(\lambda-\mu_x)(\lambda+\mu_y)+(-\lambda-\mu_x)(\lambda-\mu_y)}{\lambda-\mu_y}}\right)}{\delta Q_x}\frac{\delta\left(\left(\frac{MP_{K_y}}{MP_{L_y}}\frac{MP_{L_x}}{MP_{K_x}}\right)^{\lambda}\right)}{\delta Q_x}+\frac{\delta\left(\left(\frac{MP_{K_y}}{MP_{L_y}}\frac{MP_{L_x}}{MP_{K_x}}\right)^{\lambda}\right)}{\delta Q_x}\frac{\delta\left(K_x^{\frac{(\lambda-\mu_x)(\lambda+\mu_y)+(-\lambda-\mu_x)(\lambda-\mu_y)}{\lambda-\mu_y}}\right)}{\delta Q_x} \end{array}\right] \tag{63}$$

Note that $\frac{MP_{K_y}}{MP_{L_y}}\frac{MP_{L_x}}{MP_{K_x}} = 1$. Then, (63) is rewritten as:

$$\frac{\delta^2 Q_y}{\delta Q_x{}^2} = \left(\frac{m_x b_x}{m_y b_y}\right)^{\frac{\lambda+\mu_y}{-b_y-1}} \left(\frac{b_x}{b_y}\right)^{\lambda} \frac{T_y^{\mu_y}}{T_x^{\mu_x}} \frac{\delta\left(K_x^{\frac{2\lambda(\mu_y-\mu_x)}{\lambda-\mu_y}}\right)}{\delta Q_x} \tag{64}$$

Table 1. Shape of the production possibility frontier for alternative cases.

Returns to Scale	$\frac{\delta\left(K_x^{\frac{2\lambda(\mu_y-\mu_x)}{\lambda-\mu_y}}\right)}{\delta Q_x}$	Shape
If $\mu_y = \lambda$ and $\mu_x \neq \mu_y$ If there are (1) constant returns only for producer of *y* for the case $\mu_y = \lambda = 0.5 \neq \mu_x$ or (2) increasing returns for both producers or (3) decreasing returns for both producers	$= 0$	Linear since (64) is equal to zero.
If $\mu_x = \mu_y$ and $\lambda \neq \mu_y$ If there are (1) constant returns for both producers except for the case $\mu_x = \mu_y = 0.5$ or (2) increasing returns for both producers or (3) decreasing returns for both producers	$= 0$	Linear since (64) is equal to zero.
If $\mu_x > \mu_y$ and $\lambda \neq \mu_y$ If there are (1) constant returns only for producer of *x* $(\mu_x + \lambda = 1)$ or only for producer of *y* $(\mu_y + \lambda = 1)$ or (2) increasing returns for both producers or (3) decreasing returns for both producers	(if $\mu_y > \lambda$) > 0 (if $\mu_y < \lambda$) < 0	Convex to the origin since (64) is positive. Concave to the origin since (64) is negative.
If $\mu_x < \mu_y$ and $\lambda \neq \mu_y$ If there are (1) constant returns only for producer of *x* $(\mu_x + \lambda = 1)$ or only for producer of *y* $(\mu_y + \lambda = 1)$ or (2) increasing returns for both producers or (3) decreasing returns for both producers	(if $\mu_y > \lambda$) > 0 (if $\mu_y < \lambda$) < 0	Convex to the origin since (64) is positive. Concave to the origin since (64) is negative.

Source: Author's own.

6. Possible shapes of production possibility frontier under Solow-neutrality

Shape of the production possibility frontier can be linear, concave to the origin and convex to the origin depending on the conditions which are shown in Table 1. Since our study depends on the concavity conditions, we use only them.

7. Economic growth depending on concave production possibility frontier under Solow-neutrality

Rearranging (59):

$$d\left[Q_x \frac{T_y^{\mu_y}}{T_x^{\mu_x}} K_x^{\frac{(-b_x-1)(\lambda_y+\mu_y)}{-b_y-1}-\lambda_x-\mu_x} \left(\frac{m_x b_x}{m_y b_y}\right)^{\frac{\lambda_y+\mu_y}{-b_y-1}} \left(\frac{dL_y}{dK_y}\right)^{\lambda_y} \left(\frac{dL_x}{dK_x}\right)^{-\lambda_x} \frac{(-b_x)^{\lambda_x}}{(-b_y)^{\lambda_y}}\right] = dQ_y \quad (65)$$

Assume that growing sector is x. Thus, assume that$dQ_y = 0$. So, we write (66):

$$d\left[Q_x \frac{T_y^{\mu_y}}{T_x^{\mu_x}} K_x^{\frac{(-b_x-1)(\lambda_y+\mu_y)}{-b_y-1}-\lambda_x-\mu_x} \left(\frac{m_x b_x}{m_y b_y}\right)^{\frac{\lambda_y+\mu_y}{-b_y-1}} \left(\frac{dL_y}{dK_y}\right)^{\lambda_y} \left(\frac{dL_x}{dK_x}\right)^{-\lambda_x} \frac{(-b_x)^{\lambda_x}}{(-b_y)^{\lambda_y}}\right] = 0 \quad (66)$$

Thus, followings:

$$\begin{aligned}
&\delta Q_x\left[\frac{T_y^{\mu_y}}{T_x^{\mu_x}}K_x^{\frac{(-b_x-1)(\lambda_y+\mu_y)}{-b_y-1}-\lambda_x-\mu_x}\left(\frac{m_xb_x}{m_yb_y}\right)^{\frac{\lambda_y+\mu_y}{-b_y-1}}\left(\frac{dL_y}{dK_y}\right)^{\lambda_y}\left(\frac{dL_x}{dK_x}\right)^{-\lambda_x}\frac{(-b_x)^{\lambda_x}}{(-b_y)^{\lambda_y}}\right]+\\
&\delta\frac{T_y^{\mu_y}}{T_x^{\mu_x}}\left[Q_xK_x^{\frac{(-b_x-1)(\lambda_y+\mu_y)}{-b_y-1}-\lambda_x-\mu_x}\left(\frac{m_xb_x}{m_yb_y}\right)^{\frac{\lambda_y+\mu_y}{-b_y-1}}\left(\frac{dL_y}{dK_y}\right)^{\lambda_y}\left(\frac{dL_x}{dK_x}\right)^{-\lambda_x}\frac{(-b_x)^{\lambda_x}}{(-b_y)^{\lambda_y}}\right]+\\
&\delta K_x^{\frac{(-b_x-1)(\lambda_y+\mu_y)}{-b_y-1}-\lambda_x-\mu_x}\left[Q\frac{T_y^{\mu_y}}{T_x^{\mu_x}}\left(\frac{m_xb_x}{m_yb_y}\right)^{\frac{\lambda_y+\mu_y}{-b_y-1}}\left(\frac{dL_y}{dK_y}\right)^{\lambda_y}\left(\frac{dL_x}{dK_x}\right)^{-\lambda_x}\frac{(-b_x)^{\lambda_x}}{(-b_y)^{\lambda_y}}\right]+\\
&\delta\left[\left(\frac{dL_y}{dK_y}\right)^{\lambda_y}\left(\frac{dL_x}{dK_x}\right)^{-\lambda_x}\right]\left[Q_x\frac{T_y^{\mu_y}}{T_x^{\mu_x}}K_x^{\frac{(-b_x-1)(\lambda_y+\mu_y)}{-b_y-1}-\lambda_x-\mu_x}\left(\frac{m_xb_x}{m_yb_y}\right)^{\frac{\lambda_y+\mu_y}{-b_y-1}}\frac{(-b_x)^{\lambda_x}}{(-b_y)^{\lambda_y}}\right]+\\
&\delta\left(\frac{m_xb_x}{m_yb_y}\right)^{\frac{\lambda_y+\mu_y}{-b_y-1}}\left[Q_x\frac{T_y^{\mu_y}}{T_x^{\mu_x}}K_x^{\frac{(-b_x-1)(\lambda_y+\mu_y)}{-b_y-1}-\lambda_x-\mu_x}\left(\frac{dL_y}{dK_y}\right)^{\lambda_y}\left(\frac{dL_x}{dK_x}\right)^{-\lambda_x}\frac{(-b_x)^{\lambda_x}}{(-b_y)^{\lambda_y}}\right]+\\
&\delta\frac{(-b_x)^{\lambda_x}}{(-b_y)^{\lambda_y}}\left[Q_x\frac{T_y^{\mu_y}}{T_x^{\mu_x}}K_x^{\frac{(-b_x-1)(\lambda_y+\mu_y)}{-b_y-1}-\lambda_x-\mu_x}\left(\frac{m_xb_x}{m_yb_y}\right)^{\frac{\lambda_y+\mu_y}{-b_y-1}}\left(\frac{dL_y}{dK_y}\right)^{\lambda_y}\left(\frac{dL_x}{dK_x}\right)^{-\lambda_x}\right]=0
\end{aligned}\tag{67}$$

As it is supposed $\delta(\frac{m_xb_x}{m_yb_y})^{\frac{\lambda_y+\mu_y}{-b_y-1}}=0$, $\delta\left[(\frac{dL_y}{dK_y})^{\lambda_y}(\frac{dL_x}{dK_x})^{-\lambda_x}\right]=0$ and $\delta\frac{(-b_x)^{\lambda_x}}{(-b_y)^{\lambda_y}}=0$ followings:

$$\begin{aligned}
&\delta Q_x\left[\frac{T_y^{\mu_y}}{T_x^{\mu_x}}K_x^{\frac{(-b_x-1)(\lambda_y+\mu_y)}{-b_y-1}-\lambda_x-\mu_x}\left(\frac{m_xb_x}{m_yb_y}\right)^{\frac{\lambda_y+\mu_y}{-b_y-1}}\left(\frac{dL_y}{dK_y}\right)^{\lambda_y}\left(\frac{dL_x}{dK_x}\right)^{-\lambda_x}\frac{(-b_x)^{\lambda_x}}{(-b_y)^{\lambda_y}}\right]+\\
&\delta\frac{T_y^{\mu_y}}{T_x^{\mu_x}}\left[Q_xK_x^{\frac{(-b_x-1)(\lambda_y+\mu_y)}{-b_y-1}-\lambda_x-\mu_x}\left(\frac{m_xb_x}{m_yb_y}\right)^{\frac{\lambda_y+\mu_y}{-b_y-1}}\left(\frac{dL_y}{dK_y}\right)^{\lambda_y}\left(\frac{dL_x}{dK_x}\right)^{-\lambda_x}\frac{(-b_x)^{\lambda_x}}{(-b_y)^{\lambda_y}}\right]+\\
&\delta K_x^{\frac{(-b_x-1)(\lambda_y+\mu_y)}{-b_y-1}-\lambda_x-\mu_x}\left[Q\frac{T_y^{\mu_y}}{T_x^{\mu_x}}\left(\frac{m_xb_x}{m_yb_y}\right)^{\frac{\lambda_y+\mu_y}{-b_y-1}}\left(\frac{dL_y}{dK_y}\right)^{\lambda_y}\left(\frac{dL_x}{dK_x}\right)^{-\lambda_x}\frac{(-b_x)^{\lambda_x}}{(-b_y)^{\lambda_y}}\right]+\\
&0\left[Q_x\frac{T_y^{\mu_y}}{T_x^{\mu_x}}K_x^{\frac{(-b_x-1)(\lambda_y+\mu_y)}{-b_y-1}-\lambda_x-\mu_x}\left(\frac{m_xb_x}{m_yb_y}\right)^{\frac{\lambda_y+\mu_y}{-b_y-1}}\frac{(-b_x)^{\lambda_x}}{(-b_y)^{\lambda_y}}\right]+\\
&0\left[Q_x\frac{T_y^{\mu_y}}{T_x^{\mu_x}}K_x^{\frac{(-b_x-1)(\lambda_y+\mu_y)}{-b_y-1}-\lambda_x-\mu_x}\left(\frac{dL_y}{dK_y}\right)^{\lambda_y}\left(\frac{dL_x}{dK_x}\right)^{-\lambda_x}\frac{(-b_x)^{\lambda_x}}{(-b_y)^{\lambda_y}}\right]+\\
&0\left[Q_x\frac{T_y^{\mu_y}}{T_x^{\mu_x}}K_x^{\frac{(-b_x-1)(\lambda_y+\mu_y)}{-b_y-1}-\lambda_x-\mu_x}\left(\frac{m_xb_x}{m_yb_y}\right)^{\frac{\lambda_y+\mu_y}{-b_y-1}}\left(\frac{dL_y}{dK_y}\right)^{\lambda_y}\left(\frac{dL_x}{dK_x}\right)^{-\lambda_x}\right]=0
\end{aligned}\tag{68}$$

$$\begin{aligned}&\delta Q_x\left[\frac{T_y^{\mu_y}}{T_x^{\mu_x}}K_x^{\frac{(-b_x-1)(\lambda_y+\mu_y)}{-b_y-1}-\lambda_x-\mu_x}\left(\frac{m_xb_x}{m_yb_y}\right)^{\frac{\lambda_y+\mu_y}{-b_y-1}}\left(\frac{dL_y}{dK_y}\right)^{\lambda_y}\left(\frac{dL_x}{dK_x}\right)^{-\lambda_x}\frac{(-b_x)^{\lambda_x}}{(-b_y)^{\lambda_y}}\right]+\\&\delta\frac{T_y^{\mu_y}}{T_x^{\mu_x}}\left[Q_xK_x^{\frac{(-b_x-1)(\lambda_y+\mu_y)}{-b_y-1}-\lambda_x-\mu_x}\left(\frac{m_xb_x}{m_yb_y}\right)^{\frac{\lambda_y+\mu_y}{-b_y-1}}\left(\frac{dL_y}{dK_y}\right)^{\lambda_y}\left(\frac{dL_x}{dK_x}\right)^{-\lambda_x}\frac{(-b_x)^{\lambda_x}}{(-b_y)^{\lambda_y}}\right]+\\&\delta K_x^{\frac{(-b_x-1)(\lambda_y+\mu_y)}{-b_y-1}-\lambda_x-\mu_x}\left[Q_x\frac{T_y^{\mu_y}}{T_x^{\mu_x}}\left(\frac{m_xb_x}{m_yb_y}\right)^{\frac{\lambda_y+\mu_y}{-b_y-1}}\left(\frac{dL_y}{dK_y}\right)^{\lambda_y}\left(\frac{dL_x}{dK_x}\right)^{-\lambda_x}\frac{(-b_x)^{\lambda_x}}{(-b_y)^{\lambda_y}}\right]=0\end{aligned}\tag{69}$$

Rearranging (69):

$$\begin{aligned}&\frac{\delta Q_x}{Q_x}\left[\frac{T_y^{\mu_y}}{T_x^{\mu_x}}K_x^{\frac{(-b_x-1)(\lambda_y+\mu_y)}{-b_y-1}-\lambda_x-\mu_x}\left(\frac{m_xb_x}{m_yb_y}\right)^{\frac{\lambda_y+\mu_y}{-b_y-1}}\left(\frac{dL_y}{dK_y}\right)^{\lambda_y}\left(\frac{dL_x}{dK_x}\right)^{-\lambda_x}\frac{(-b_x)^{\lambda_x}}{(-b_y)^{\lambda_y}}\right]+\\&\delta\frac{T_y^{\mu_y}}{T_x^{\mu_x}}\left[K_x^{\frac{(-b_x-1)(\lambda_y+\mu_y)}{-b_y-1}-\lambda_x-\mu_x}\left(\frac{m_xb_x}{m_yb_y}\right)^{\frac{\lambda_y+\mu_y}{-b_y-1}}\left(\frac{dL_y}{dK_y}\right)^{\lambda_y}\left(\frac{dL_x}{dK_x}\right)^{-\lambda_x}\frac{(-b_x)^{\lambda_x}}{(-b_y)^{\lambda_y}}\right]+\\&\delta K_x^{\frac{(-b_x-1)(\lambda_y+\mu_y)}{-b_y-1}-\lambda_x-\mu_x}\left[\frac{T_y^{\mu_y}}{T_x^{\mu_x}}\left(\frac{m_xb_x}{m_yb_y}\right)^{\frac{\lambda_y+\mu_y}{-b_y-1}}\left(\frac{dL_y}{dK_y}\right)^{\lambda_y}\left(\frac{dL_x}{dK_x}\right)^{-\lambda_x}\frac{(-b_x)^{\lambda_x}}{(-b_y)^{\lambda_y}}\right]=0\end{aligned}\tag{70}$$

$$\begin{aligned}&\frac{\delta Q_x}{Q_x}\left[\frac{T_y^{\mu_y}}{T_x^{\mu_x}}K_x^{\frac{(-b_x-1)(\lambda_y+\mu_y)}{-b_y-1}-\lambda_x-\mu_x}\left(\frac{m_xb_x}{m_yb_y}\right)^{\frac{\lambda_y+\mu_y}{-b_y-1}}\left(\frac{dL_y}{dK_y}\right)^{\lambda_y}\left(\frac{dL_x}{dK_x}\right)^{-\lambda_x}\frac{(-b_x)^{\lambda_x}}{(-b_y)^{\lambda_y}}\right]=\\&-\delta\frac{T_y^{\mu_y}}{T_x^{\mu_x}}\left[K_x^{\frac{(-b_x-1)(\lambda_y+\mu_y)}{-b_y-1}-\lambda_x-\mu_x}\left(\frac{m_xb_x}{m_yb_y}\right)^{\frac{\lambda_y+\mu_y}{-b_y-1}}\left(\frac{dL_y}{dK_y}\right)^{\lambda_y}\left(\frac{dL_x}{dK_x}\right)^{-\lambda_x}\frac{(-b_x)^{\lambda_x}}{(-b_y)^{\lambda_y}}\right]\\&-\delta K_x^{\frac{(-b_x-1)(\lambda_y+\mu_y)}{-b_y-1}-\lambda_x-\mu_x}\left[\frac{T_y^{\mu_y}}{T_x^{\mu_x}}\left(\frac{m_xb_x}{m_yb_y}\right)^{\frac{\lambda_y+\mu_y}{-b_y-1}}\left(\frac{dL_y}{dK_y}\right)^{\lambda_y}\left(\frac{dL_x}{dK_x}\right)^{-\lambda_x}\frac{(-b_x)^{\lambda_x}}{(-b_y)^{\lambda_y}}\right]\end{aligned}\tag{71}$$

Suppose that $R=\left(\frac{m_xb_x}{m_yb_y}\right)^{\frac{\lambda_y+\mu_y}{-b_y-1}}\left(\frac{dL_y}{dK_y}\right)^{\lambda_y}\left(\frac{dL_x}{dK_x}\right)^{-\lambda_x}\frac{(-b_x)^{\lambda_x}}{(-b_y)^{\lambda_y}}$:

$$\begin{aligned}&\frac{\delta Q_x}{Q_x}\left[\frac{T_y^{\mu_y}}{T_x^{\mu_x}}K_x^{\frac{(-b_x-1)(\lambda_y+\mu_y)}{-b_y-1}-\lambda_x-\mu_x}R\right]=-\delta\left(\frac{T_y^{\mu_y}}{T_x^{\mu_x}}\right)\\&\left[K_x^{\frac{(-b_x-1)(\lambda_y+\mu_y)}{-b_y-1}-\lambda_x-\mu_x}R\right]-\delta K_x^{\frac{(-b_x-1)(\lambda_y+\mu_y)}{-b_y-1}-\lambda_x-\mu_x}\left[\frac{T_y^{\mu_y}}{T_x^{\mu_x}}R\right]\end{aligned}\tag{72}$$

We leave alone $\frac{\delta Q_x}{Q_x}$:

$$\frac{\delta Q_x}{Q_x} = -\delta\left(\frac{T_y^{\mu_y}}{T_x^{\mu_x}}\right)\frac{\left[K_x^{\frac{(-b_x-1)(\lambda_y+\mu_y)}{-b_y-1}-\lambda_x-\mu_x}R\right]}{\left[\frac{T_y^{\mu_y}}{T_x^{\mu_x}}K_x^{\frac{(-b_x-1)(\lambda_y+\mu_y)}{-b_y-1}-\lambda_x-\mu_x}R\right]} - \delta K_x^{\frac{(-b_x-1)(\lambda_y+\mu_y)}{-b_y-1}-\lambda_x-\mu_x}\frac{\left[\frac{T_y^{\mu_y}}{T_x^{\mu_x}}R\right]}{\left[\frac{T_y^{\mu_y}}{T_x^{\mu_x}}K_x^{\frac{(-b_x-1)(\lambda_y+\mu_y)}{-b_y-1}-\lambda_x-\mu_x}R\right]} \tag{73}$$

$$\frac{\delta Q_x}{Q_x} = -\delta\left(\frac{T_y^{\mu_y}}{T_x^{\mu_x}}\right)\frac{T_x^{\mu_x}}{T_y^{\mu_y}} - \delta K_x^{\frac{(-b_x-1)(\lambda_y+\mu_y)}{-b_y-1}-\lambda_x-\mu_x}\frac{1}{K_x^{\frac{(-b_x-1)(\lambda_y+\mu_y)}{-b_y-1}-\lambda_x-\mu_x}} \tag{74}$$

Rewriting (74) as:

$$\frac{\delta Q_x}{Q_x} = -\left(\frac{\delta T_y^{\mu_y}}{T_y^{\mu_y}} - \frac{\delta T_x^{\mu_x}}{T_x^{\mu_x}}\right) - \frac{\delta K_x^{\frac{(-b_x-1)(\lambda_y+\mu_y)}{-b_y-1}-\lambda_x-\mu_x}}{K_x^{\frac{(-b_x-1)(\lambda_y+\mu_y)}{-b_y-1}-\lambda_x-\mu_x}} \tag{75}$$

Note that steady-state occurs at full capacity conditions. So, returns to scale conditions are constant for the growing sector. Thus (75) will be:

$$\frac{\delta Q_x}{Q_x} = -\left(\frac{\delta T_y^{\mu_y}}{T_y^{\mu_y}} - \frac{\delta T_x^{\mu_x}}{T_x^{\mu_x}}\right) - \frac{\delta K_x^{\frac{(-b_x-1)(\lambda_y+\mu_y)}{-b_y-1}-1}}{K_x^{\frac{(-b_x-1)(\lambda_y+\mu_y)}{-b_y-1}-1}} \tag{76}$$

Rewriting (76) as:

$$\frac{\delta Q_x}{Q_x} = -\left(\frac{\delta T_y^{\mu_y}}{T_y^{\mu_y}} - \frac{\delta T_x^{\mu_x}}{T_x^{\mu_x}}\right) - \frac{\delta K_x^{\frac{(-b_x-1)(\lambda_y+\mu_y)+b_y+1}{-b_y-1}}}{K_x^{\frac{(-b_x-1)(\lambda_y+\mu_y)+b_y+1}{-b_y-1}}} \tag{77}$$

As $\mu_x/\lambda_x = -b_x$ and $\mu_y/\lambda_y = -b_y$ (77) is rearranged:

$$\frac{\delta Q_x}{Q_x} = -\left(\frac{\delta T_y^{\mu_y}}{T_y^{\mu_y}} - \frac{\delta T_x^{\mu_x}}{T_x^{\mu_x}}\right) - \frac{\delta K_x^{\frac{\left(\frac{\mu_x}{\lambda_x}-1\right)(\lambda_y+\mu_y)-\frac{\mu_y}{\lambda_y}+1}{\frac{\mu_y}{\lambda_y}-1}}}{K_x^{\frac{\left(\frac{\mu_x}{\lambda_x}-1\right)(\lambda_y+\mu_y)-\frac{\mu_y}{\lambda_y}+1}{\frac{\mu_y}{\lambda_y}-1}}} \tag{78}$$

$$\frac{\delta Q_x}{Q_x} = -\left(\frac{\delta T_y^{\mu_y}}{T_y^{\mu_y}} - \frac{\delta T_x^{\mu_x}}{T_x^{\mu_x}}\right) - \frac{\delta K_x^{\frac{\left(\frac{\mu_x-\lambda_x}{\lambda_x}\right)(\lambda_y+\mu_y)-\left(\frac{\mu_y-\lambda_y}{\lambda_y}\right)}{\frac{\mu_y-\lambda_y}{\lambda_y}}}}{K_x^{\frac{\left(\frac{\mu_x-\lambda_x}{\lambda_x}\right)(\lambda_y+\mu_y)-\left(\frac{\mu_y-\lambda_y}{\lambda_y}\right)}{\frac{\mu_y-\lambda_y}{\lambda_y}}}} \tag{79}$$

Assuming $\lambda_x = \lambda_y$:

$$\frac{\delta Q_x}{Q_x} = -\left(\frac{\delta T_y^{\mu_y}}{T_y^{\mu_y}} - \frac{\delta T_x^{\mu_x}}{T_x^{\mu_x}}\right) - \frac{\delta K_x^{\frac{(\lambda_x-\mu_x)(\lambda_y+\mu_y)-(\lambda_y-\mu_y)}{\lambda_y-\mu_y}}}{K_x^{\frac{(\lambda_x-\mu_x)(\lambda_y+\mu_y)-(\lambda_y-\mu_y)}{\lambda_y-\mu_y}}} \tag{80}$$

As it is supposed $dQ_y = 0$, so, $\frac{\delta T_y^{\mu_y}}{T_y^{\mu_y}} = 0$:

$$\frac{\delta Q_x}{Q_x} = \frac{\delta T_x^{\mu_x}}{T_x^{\mu_x}} - \frac{\delta K_x^{\frac{(\lambda_x-\mu_x)(\lambda_y+\mu_y)-(\lambda_y-\mu_y)}{\lambda_y-\mu_y}}}{K_x^{\frac{(\lambda_x-\mu_x)(\lambda_y+\mu_y)-(\lambda_y-\mu_y)}{\lambda_y-\mu_y}}} \tag{81}$$

8. Growth based on three possibilities

Equation (81) implies that there are three possibilities for economic growth of one sector: Economic growth occurs in one sector with the help of: (1) an increase in the capital $\left(\frac{\delta T_x^{\mu_x}}{T_x^{\mu_x}} = 0\right)$; (2) an increase in the level of technology $\left(\frac{\delta K_x^{\frac{(\lambda_x-\mu_x)(\lambda_y+\mu_y)-(\lambda_y-\mu_y)}{\lambda_y-\mu_y}}}{K_x^{\frac{(\lambda_x-\mu_x)(\lambda_y+\mu_y)-(\lambda_y-\mu_y)}{\lambda_y-\mu_y}}} = 0\right)$; and (3) an increase in the capital and an increase in the level of technology, together.

Possibility 1: An increase in the capital results in economic growth

According to (81), if one sector grows positively with the help of an increase in the capital $\left(\frac{\delta T_x^{\mu_x}}{T_x^{\mu_x}} = 0\right)$, then the conditions are:

i. if $\mu_x > \mu_y$ then $\mu_y < \lambda_y$,
ii. if $\mu_x < \mu_y$ then $\mu_y > \lambda_y$.

Recognise that it is supposed $\lambda_x = \lambda_y$, so, the conditions are rewritten:

i. if $\mu_x > \mu_y$ then $\mu_y < \lambda$,
ii. if $\mu_x < \mu_y$ then $\mu_y > \lambda$.

As a consequence one can prove the following:

Proposition 1. *In an economy with two sectors, under Solow-neutrality and constant returns to scale conditions for the growing sector, and if labour elasticities are same,* *positive economic growth of one sector with the help of an increase in the capital of growing sector requires the followings:*

i. *if* $\mu_x > \mu_y$ *then* $\mu_y < \lambda$,
ii. *if* $\mu_x < \mu_y$ *then* $\mu_y > \lambda$.

Proposition 1 occurs for the bowed-out production possibility frontier. The other cases are stated in Table 1. As this study is an attempt to measure growth depending on an increase of capacity of x, it is supposed that constant returns to scale conditions are valid *only for* production of x $(\lambda + \mu_x = 1)$.

Thus, Proposition 1 will be rearranged:

Proposition 2. *In an economy with two sectors, under Solow-neutrality and constant returns to scale conditions for the growing sector, and if labour elasticities are same, positive economic growth of one sector with the help of an increase in the capital of growing sector requires the followings:*

i. *if* $\mu_x > \mu_y$ *then* $\mu_y < \lambda$ *and* $\mu_y + \lambda \neq 1$,
ii. *if* $\mu_x < \mu_y$ *then* $\mu_y > \lambda$ *and* $\mu_y + \lambda \neq 1$.

Possibility 2: An increase in the level of technology results in economic growth

According to (81) if one sector grows positively with the help of an increase in the level of technology $\left(\frac{\delta K_x^{\frac{(\lambda x - \mu_x)(\lambda y + \mu_y) - (\lambda y - \mu_y)}{\lambda y - \mu_y}}}{K_x^{\frac{(\lambda x - \mu_x)(\lambda y + \mu_y) - (\lambda y - \mu_y)}{\lambda y - \mu_y}}} = 0 \right)$, then the rate of growth becomes:

$$\frac{\delta Q_x}{Q_x} = \frac{\delta T_x^{\mu_x}}{T_x^{\mu_x}} \tag{82}$$

Possibility 3: An increase in the capital and the level of technology results in economic growth

According to (81), if one sector grows positively with the help of an increase in the capital and the level of technology, then the rate of growth becomes:

$$\frac{\delta Q_x}{Q_x} = \frac{\delta T_x^{\mu_x}}{T_x^{\mu_x}} - \frac{\delta K_x^{\frac{(\lambda x - \mu_x)(\lambda y + \mu_y) - (\lambda y - \mu_y)}{\lambda y - \mu_y}}}{K_x^{\frac{(\lambda x - \mu_x)(\lambda y + \mu_y) - (\lambda y - \mu_y)}{\lambda y - \mu_y}}} \tag{83}$$

Note that Proposition 2 should be valid.

9. Conclusion

The first aim of this study is to define the natural rate of growth in harmony with the nature of technological progress, steady-state and full capacity or full employment. It is strongly emphasised that assuming: (1) the attractor as the rate of growth of effective capital or capital–output ratio; (2) Solow-neutrality; (3) constant returns to scale; and (4) full capacity or full employment conditions are closely connected with each other for an economically true steady-state analysis. Thus, Solow-neutrality becomes compatible with the long-run equilibrium growth and the steady-state; i.e., the contradiction between Solow-neutrality and the long-run equilibrium growth and the steady-state conditions is solved. Therefore, economic growth models which investigate artificial intelligence, may consider starting from the Solow-neutrality assumption. This contribution seems to be very important for the near future, which will be characterised by extensive automation.

The second objective is to obtain the equation of production possibility frontier under Solow-neutrality. The third objective is to explain the conditions of economic growth under Solow-neutrality. The conditions of economic growth are shown by using the equation of concave production possibility frontier and by assuming the nature of technological progress as Solow-neutral. In order to show, first, the equation of the production possibility frontier is obtained. After that, concavity, convexity and linearity conditions are documented in Table 1.

According to these conditions, three possibilities are expressed to measure the rate of growth of one sector. First, economic growth occurs only with the help of an increase in the capital. Then, the condition is: If the nature of technological progress is Solow-neutral, if returns to scale conditions are constant for the growing sector, if labour elasticity is same between two sectors, then, growing positively with the help of an increase in the capital of growing sector requires: (1) if $\mu_x > \mu_y$ then$\mu_y < \lambda$ and $\mu_y + \lambda \neq 1$, ii) if $\mu_x < \mu_y$ then$\mu_y > \lambda$ and $\mu_y + \lambda \neq 1$ .This consequence points out that for an economy with two commodities; if there are constant returns to scale conditions for the growing sector, if the identifying assumption is Solow-neutral, and if elasticity of output with respect to labour is same between sectors, then for the non-growing sector it should *not* be constant returns to scale conditions $(\mu_y + \lambda \neq 1)$. Second, economic growth occurs only with the help of an increase in the level of technology. Then, the rate of economic growth equals to $\frac{\delta T_x^{\mu_x}}{T_x^{\mu_x}}$. Third, economic growth occurs with the help of an increase in the capital and the level of technology. Then, the rate of economic growth equals to $\frac{\delta T_x^{\mu_x}}{T_x^{\mu_x}} - \frac{\delta K_x^{\frac{(\lambda_x-\mu_x)(\lambda_y+\mu_y)-(\lambda_y-\mu_y)}{\lambda_y-\mu_y}}}{K_x^{\frac{(\lambda_x-\mu_x)(\lambda_y+\mu_y)-(\lambda_y-\mu_y)}{\lambda_y-\mu_y}}}$. This consequence implies that measurement of economic growth means to measure i) the rate of growth of the level of technology $\left(\frac{\delta T_x}{T_x}\right)$; (2) the rate of growth of the capital $\left(\frac{\delta K_x}{K_x}\right)$; (3) elasticity of output with respect to capital, for the growing sector (μ_x), iv) elasticity of output with respect to labour, for the other or growing sector $(\lambda_y = \lambda_x = \lambda)$, v) elasticity of output with respect to capital, for the other sector (μ_y).

The present work is the first attempt to measure growth depending on the equation of the concave production possibility frontier, under Solow-neutrality. For this purpose, first, the natural rate of growth has been explained in harmony with the economic concepts such as constant returns to scale, full capacity and steady-state. Besides, the equation of the concave production possibility frontier has been obtained under Solow-neutrality. Finally, economic growth has been explained based on the equation of the concave production possibility frontier, under Solow-neutrality. These are the main contributions of the study.

Our study points out further studies which attempt to calculate the contribution of productivity growth of capital. Based on growth accounting methodology, the contribution of productivity growth of capital can be calculated under Solow-neutrality. There are few studies on that subject. To make another empirical evaluation or illustration one can build a two-sector model. Using that model one can make an explanation beginning from the proposition 2. Note that proposition 2 implies the resource transfer from the non-growing sector to the growing sector. For example, if we transfer resources from the non-growing sector to the growing sector and if there are decreasing returns to scale or increasing returns to scale conditions for the non-growing sector, then, growth of one sector occurs thanks to that resource transfer. If there are decreasing returns to scale for the non-growing sector, then non-growing sector solves the problem of the overcapacity. If there are increasing returns to scale for the non-growing sector, then non-growing sector solves the problem of the idle capacity. Thus, for a growing economy, the constant returns to scale and non-constant returns to scale conditions occur together. Then, using an empirical model one can analyze which sectors have constant, increasing and decreasing returns to scale conditions. After that, economic growth occurs based on growth of the sector which has constant returns to scale conditions, by transferring resources from the other sectors which have non-constant returns to scale conditions.

Disclosure statement

No potential conflict of interest was reported by the authors.

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