2019

Hadwiger Numbers and Gallai-Ramsey Numbers of Special Graphs

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by

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A disseration submitted in partial fulfilment of the requirements
for the degree of Doctor of Philosophy
in the Department of Mathematics
in the College of Sciences
at the University of Central Florida
Orlando, Florida

Summer Term
2019

Major Professor: Zi-Xia Song

ABSTRACT

This dissertation explores two separate topics on graphs.

We first study a far-reaching generalization of the Four Color Theorem. Given a graph G, we use $\chi(G)$ to denote the chromatic number; $\alpha(G)$ the independence number; and $h(G)$ the *Hadwiger number*, which is the largest integer t such that the complete graph K_t can be obtained from a subgraph of G by contracting edges. Hadwiger's conjecture from 1943 states that for every graph G, $h(G) \geq \chi(G)$. This is perhaps the most famous conjecture in Graph Theory and remains open even for graphs G with $\alpha(G) \leq 2$. Let W_5 denote the wheel on six vertices. We establish more evidence for Hadwiger's conjecture by proving that $h(G) \geq \chi(G)$ for all graphs G such that $\alpha(G) \leq 2$ and G does not contain W_5 as an induced subgraph.

Our second topic is related to Ramsey theory, a field that has intrigued those who study combinatorics for many decades. Computing the classical Ramsey numbers is a notoriously difficult problem, leaving many basic questions unanswered even after more than 80 years. We study Ramsey numbers under Gallai-colorings. A *Gallai-coloring* of a complete graph is an edge-coloring such that no triangle is colored with three distinct colors. Given a graph H and an integer $k \geq 1$, the Gallai-Ramsey number, denoted $GR_k(H)$, is the least positive integer n such that every Gallai-coloring of K_n with at most k colors contains a monochromatic copy of H. It turns out that $GR_k(H)$ is more well-behaved than the classical Ramsey number $R_k(H)$, though finding exact values of $GR_k(H)$ is far from trivial. We show that for all $k \geq 3$, $GR_k(C_{2n+1}) = n \cdot 2^k + 1$ for $n \in \{4, 5, 6, 7\}$, and $GR_k(C_{2n+1}) \leq (n \ln n) \cdot 2^k - (k+1)n + 1$ for all $n \geq 8$, where C_{2n+1} denotes a cycle on $2n+1$ vertices.

To Casey and Rowan

ACKNOWLEDGMENTS

Any accomplishment I have achieved throughout the course of my life is not due solely to my own efforts. The completion of this degree is no different. None of the best things in my life would be possible were it not for the support of others. In my own small way I want to express my gratitude to the many people who shared this experience with me and gave me the mental fortitude and encouragement to keep going.

To my closest friends in the department: Nathaniel, Jingmei, Fangfang, Ramchandra, Ted, Dave, Elliot, Cheng, Wei, Blake and Majid, thank you. Thanks for the many jokes, academic help, political discussions, and for commiserating with me throughout this entire process. You kept me sane.

To Dr. Gurel, thank you. I deeply appreciate the funding you allocated to me for two summers, which made it financially possible for me to remain focused on my studies.

To my committee members, Dr. Zhao, Dr. Brennan and Dr. DeMara, thank you for your service on my dissertation committee.

To Cory, Jen, Mark, Neena, Shay, Sonny, Dave and Sue thank you for your love and support, and for poking fun at me all the time and distracting me from my worries with this degree. To Bob, Charel, Eric, Holly, Scot and Patti, thank you for the immense support and interest you have invested in my education over the years. To my dear friends Xousaen, Mary, Jay, Kathy, Mario and Kori, thanks for the emotional support throughout this program, and for your curiosity about my progress. To the rest of my very large family, please know that I also deeply appreciate your love and support. To Dr. Vanderwerff, Dr. Howe and Dr. Clarke, thank you for teaching me to love mathematics as an undergraduate.

To my parents, Blayne and Laurin, I am deeply indebted. I still remember when I first started school, you told me to simply do my best. I always felt that advice released me of an obligation to perform at any particular level, and that you were implicitly giving me permission to make mistakes so long as I try. I have certainly made uncountably many mistakes but I always try remember those words. Without your love and support from the beginning I would not be here, so thank you, from the bottom of my heart.

To my advisor Dr. Song, thank you. I have always been amazed that you agreed to take me on as a student in the first place, especially because I knew precisely nothing about Graph Theory prior to taking your course. From the beginning you were encouraging, and always have pushed me to do more, write as nicely as possible, clarify arguments, and in general think deeply about mathematics. Suffice it to say I have not been a perfect student, but without your patience through my many, many mistakes I would not have made it to this point. On a personal level, I also profoundly appreciate your support while I went through the process of becoming a new father. Pursuing a degree is hard enough without also having to care for an infant, so your understanding of my personal circumstance is the only reason I was able to complete the program. There is no way I can think of to truly express the gratitude I feel for your guidance over the past few years, but I hope I can convey here, even in the smallest sense, my appreciation for your efforts to mould me into a better person.

Without a doubt, most importantly and above all, I am grateful to my wife, Casey. You have seen me through sleepless and anxious nights, encouraged me hundreds of times when I was sure I was going to fail, congratulated me on my successes, and believed in me even when I did not believe in myself. There is nothing I can do to repay you for your love, selflessness and support while I pursued a dream. To you I owe any good thing that comes of this degree, because you have sacrificed right along with me throughout this process. Thank you, and I love you and Rowan with all my heart.

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CHAPTER 1: INTRODUCTION

The first part of this dissertation explores a deep conjecture attributed to Swiss mathematician Hugo Hadwiger. Motivated by what was then the Four Color Conjecture, Hadwiger posed his famous conjecture at a colloquim at Eidgenössiche Technische Hochschule on December 15, 1942 [84]. This conjecture is regarded by many as one of the most profound in Graph Theory because of its relationship to what is now the Four Color Theorem (4CT) (see [84] or [77]). Specifically, Hadwiger's Conjecture (HC) implies 4CT, and in two cases is actually equivalent to 4CT. As a result, HC is viewed as a far-reaching generalization of 4CT. To date, only five cases are known, but many partial results have been subsequently shown. In this dissertation, we develop more partial results by restricting the maximum order of an independent set in a graph to two and forbidding certain induced subgraphs.

The second area of focus in this dissertation concerns Ramsey theory, named after British mathematician Frank Ramsey [45]. Problems in Ramsey theory can typically be simply stated and easily understood even by those without much formal mathematical training. However, this subject is quite profound despite its apparent simplicity. Ramsey theory asserts that "complete disorder is an impossibility," a characterization often attributed to mathematician Theodore Motzkin [45]. Due to the incredible level of difficulty, many basic problems in Ramsey theory remain unsolved despite being nearly a century old. Often the best information we have on a classical Ramsey number is a relatively poor bound. Motivated by this, we study *Gallai-Ramsey numbers*, a topic which falls under the umbrella of Ramsey theory, but whose computations usually prove to be more tractable due to a structural result of Hungarian mathematician Tibor Gallai [44, 65]. However, Gallai-Ramsey numbers are still far from trivial to compute.

We begin this dissertation by providing a review of the relevant graph-theoretic definitions and then move on to supply historical context and motivation for the problems we study. The remainder of the dissertation is organized as follows. First, we provide our partial results concerning HC. Next, we supply the proofs for the Gallai-Ramsey numbers of four odd cycles, and then establish an improved upper bound on the Gallai-Ramsey numbers of all odd cycles. Finally, we conclude this dissertation with a discussion of possible avenues for future research.

1.1 Preliminary Definitions and Results

Following the conventions set out in [26], a *graph* $G = (V, E)$ is a pair such that $E \subseteq [V]^2$, where the notation $[A]^r$ denotes the set of r-element subsets of a set A. The elements of V represent the *vertices* and the elements of E the *edges* of a graph G. The notation $V(G)$ and $E(G)$ is commonly used to denote the *vertex set* and *edge set*, respectively, of a graph G. A *loop* is an edge such that both ends are the same vertex. A graph G has *multiple edges* if there are at least two edges sharing the same ends. A graph G is *simple* if it contains neither loops nor multiple edges. The number of vertices in a graph G is its *order*, commonly denoted either as $|G|$ or $|V(G)|$. Similarly, the number of edges in a graph G is its *size*, denoted either $\|G\|$ or $|E(G)|$. A graph G is *finite* if $|G|$ is finite; otherwise it is *infinite*. For the purposes of this dissertation, we shall assume that all graphs here and henceforth are finite and simple.

If the 2-element set defining $e \in E(G)$ contains $v \in V(G)$, we say the vertex v is *incident* with the edge e. Two vertices $u, v \in V(G)$ are *adjacent* in G if both $u \in e$ and $v \in e$ for some $e \in E(G)$, and we say that u and v are the *ends* of e. Similarly, $e, f \in E(G)$ are said to be *adjacent* if $v \in e \cap f$ for some $v \in V(G)$. If $u, v \in V(G)$ are adjacent, we will use the notation uv to denote the edge containing them; additionally, we will say u and v

are *neighbors* and call the set of vertices adjacent to u its *neighborhood*, denoted by $N(u)$. Similarly, we define $N[u] := N(u) \cup \{u\}$ to be the *closed neighborhood* of the vertex u. More generally, let $U, W \subseteq V(G)$. We say that U is *complete* to W if for every $u \in U$ and $w \in W$ we have $uw \in E(G)$. Likewise, U is *anticomplete* to W if for every $u \in U$ and $w \in W$ we have $uw \notin E(G)$. The *degree* of a vertex v, denoted $d_G(v)$ or simply $d(v)$ if the graph G is understood, is the number of edges incident with v, or equivalently, the number of neighbors of v. A *matching* M is a set of independent edges in a graph G. The *complement* of the graph G is denoted $\overline{G}$, where $\overline{G}$ has vertex set V and edge set $[V]^2 \setminus E$. In other words, for $u, v \in V(G)$, $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$.

A graph $H = (V', E')$ is a *subgraph* of $G = (V, E)$, denoted by $H \subseteq G$, if both $V' \subseteq V$ and $E' \subseteq E$. Given $A \subseteq V(G)$, let $G[A]$ denote the subgraph of G obtained from G by deleting all vertices in $V(G) \setminus A$. A graph H is an *induced subgraph* of G if $H = G[A]$ for some $A \subseteq V(G)$. We say that two graphs G and H are *isomorphic*, denoted $G \simeq H$, if there exists a bijection $\varphi : V(G) \to V(H)$ such that $xy \in E(G)$ if and only if $\varphi(x)\varphi(y) \in E(H)$. We say the graph G is *H-free* if G contains no induced subgraph isomorphic to the graph H. If $e = uv \in E(G)$, we denote by G/e the graph obtained from G by *contracting* the edge e into a new vertex, say w, which is adjacent to all the former neighbors of u and v. In particular, we have $G/e = (V', E')$, where $V' := V(G \setminus \{u, v\}) \cup \{w\}$ and $E' := E(G \setminus \{u, v\}) \cup \{wz : uz \in E(G) \setminus uv \text{ or } vz \in E(G) \setminus uv\}$. Therefore, we say a graph G contains an H *minor*, denoted $G \succcurlyeq H$, if H can be obtained from a subgraph of G through a sequence of (possibly empty) edge contractions. A graph G is said to be H *minor-free* if G does not contain the graph H as a minor. See Figure 1.1 for examples of these definitions.

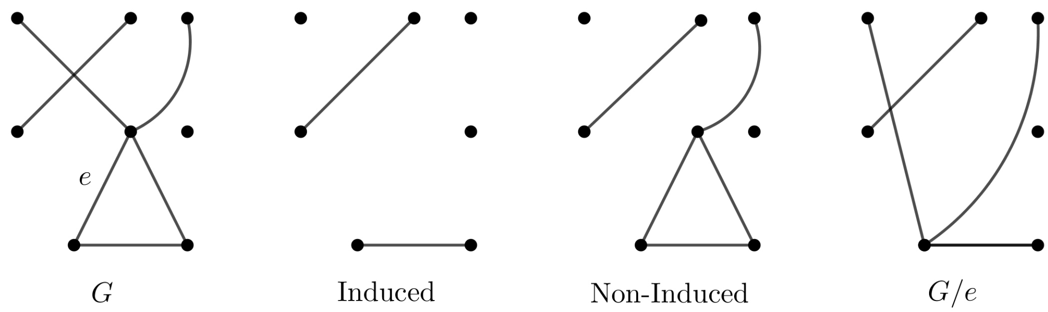


Figure 1.1: A graph G with example subgraphs and minor

Let us now describe some frequently used graphs. A *path* $P = (V, E)$ of order n, denoted P_n, is a graph of the form $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n\}$. If $n \geq 3$, the graph $P_n \cup \{v_nv_1\}$ is called a *cycle* of order n, denoted C_n. A cycle C_n (resp., path P_n) is *odd* if n is odd, and *even* if n is even. If G is a simple graph with $|G| = n$ and all vertices in G are pairwise adjacent, we say G is a *complete graph on n vertices*, or more simply a *complete graph*, denoted by K_n. A set of pairwise adjacent vertices is called a *clique*. A *complete bipartite* graph G admits a partition of $V(G)$ into two sets, called *partite sets*, such that two vertices are adjacent if and only if they belong to different partite sets. A complete bipartite graph G with partite sets A and B where $|A| = n$ and $|B| = m$ is denoted by $K_{n,m}$. A *star* is a complete bipartite graph of the form $K_{1,n}$, with the singleton vertex set in the vertex partition being the *center* of the star. The notation $G + H = (V, E)$ to denotes the *join* of two vertex disjoint graphs G and H, where $V := V(G) \cup V(H)$ and $E = E(G) \cup E(H) \cup \{uv : u \in V(G) \text{ and } v \in V(H)\}$. Thus using this notation we see that $K_{1,n} \simeq K_1 + \overline{K_n}$. Finally, we use K_n^- or $K_n^=$ to represent the graph obtained by deleting one edge or two edges from K_n, respectively. Examples of these graphs are depicted in Figure 1.2.

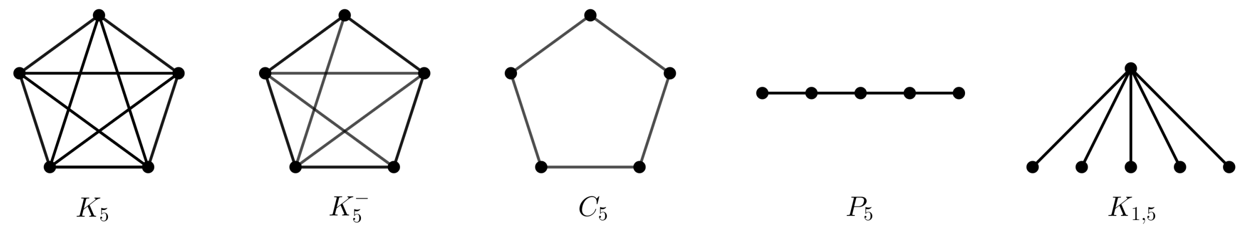


Figure 1.2: Examples of common graphs

Some important graph invariants we shall need are as follows. Define the *minimum degree* of a graph G to be $\delta(G) := \min\{d(v) : v \in V(G)\}$. Similarly, $\Delta(G) := \max\{d(v) : v \in V(G)\}$ denotes the *maximum degree*. Next, the *clique number* of a graph G, denoted $\omega(G)$, is the largest integer t such that $K_t \subseteq G$. A set of pairwise non-adjacent vertices in G is called an *independent set*, or sometimes a *stable set*. Thus, the *independence number*, denoted $\alpha(G)$, is the largest integer t such that $K_t \subseteq \overline{G}$. In other words, the independence number is simply the order of the largest independent set in G. From here one readily observes that $\alpha(G) = \omega(\overline{G})$ and $\alpha(\overline{G}) = \omega(G)$.

With the definition of independent sets in mind, let us discuss general bipartite graphs. A graph G is said to be *bipartite* if $V(G)$ can be partitioned into two (possibly empty) independent sets. It is easy to see that every odd cycle is not bipartite, and so every bipartite graph contains no odd cycles. A well-known result of König from 1936 [60] states that this obvious necessary condition is also sufficient.

Theorem 1.1.1 ([60]) *A graph is bipartite if and only if it contains no odd cycle.*

The notion of bipartite graphs can be generalized. An *r-partite graph* G admits a partition of $V(G)$ into r independent sets such that every edge in G has its ends in distinct sets in the partition. A *complete r-partite graph* (or a *complete multipartite graph*) is an r-

partite graph such that *every* pair of vertices belonging to distinct sets in the partition are adjacent. A complete r-partite graph with partite sets $A_1, A_2, \ldots, A_r$, where $n_i := |A_i|$ for all $i \in \{1, 2, \ldots, r\}$ is denoted $K_{n_1,\ldots,n_r}$.

A *Hamilton cycle* contains all vertices of the graph. If a graph G contains a Hamilton cycle, we say G is *Hamiltonian.* There are many well-known sufficient conditions which guarantee a graph G to be Hamiltonian. We state a particularly useful one here, due to Dirac in 1952 [28].

Theorem 1.1.2 ([28]) *Every graph G with $|G| \geq 3$ and $\delta(G) \geq |G|/2$ is Hamiltonian.*

Define $[k] := \{1, 2, \ldots k\}$ for any positive integer k. A *k-coloring* of the vertices of a graph G is a function $c : V(G) \to [k]$ such that $c(u) \neq c(v)$ for all $uv \in E(G)$. If a graph G admits a k-coloring, we say that G is *k-colorable.* The minimum value of k for which the graph G is k-colorable is the *chromatic number* of G, denoted $\chi(G)$. For all $i \in [k]$, we say $V_i := \{v \in V(G) : c(v) = i\}$ is the *vertex color class* (or just *color class*) associated with color i. Similarly, a *k-edge coloring* of a graph G is a function $c : E(G) \to [k]$, and a *proper edge coloring* is one in which $c(e) \neq c(f)$ for any pair of adjacent edges in G. Likewise, we say $E_i := \{e \in E(G) : c(e) = i\}$ is the *edge color class* associated with color i for all $i \in [k]$. For two disjoint sets $U, W \subseteq V(G)$, we say U is *mc-complete* to W under the edge coloring c if all the edges between U and W in G are colored the same color under c. In particular, we say U is *j-complete* to W if all the edges between U and W in G are colored by color $j \in [k]$ under c. Thus, for example, we will often say U is *blue-complete* to W if all the edges between U and W in G are colored blue under c. On occasion we shall wish to focus on the graph induced by a particular edge color. We will use the notation $G_i[U]$ to denote the graph induced by all edges with both ends in the vertex set U having the color i under the k-edge coloring c. In particular, we will use the notation $G_b[U]$ (resp., $G_r[U]$) when i is blue (resp., i is red).

Finally, we shall require the following simple but well-known result, often referred to as the *Pigeonhole Principle*. It is stated here in the mould of [87].

Theorem 1.1.3 *Let k and n be positive integers. If a set consisting of more than kn elements is partitioned into n subsets, then some subset contains more than k elements.*

1.2 Hadwiger's Conjecture

The story of Hadwiger's Conjecture begins with a well-known problem in Graph Theory that traces its origins back to 1852. According to the account by Maritz and Mouton [67], a young South African lawyer at University College London by the name of Francis Guthrie had been coloring the counties on a map of England when he noticed he never needed more than four colors to ensure no two counties with a common border would share a color. Though at the time he was studying law, Francis had previously been a student of Augustus De Morgan, and by this time his brother Frederick Guthrie was studying under De Morgan. Francis asked Fredrick to relay this problem to De Morgan, who ultimately passed it along to Sir William Rowan Hamilton for insight, though Hamilton declined to consider it further. De Morgan would spend the remainder of his life looking for a solution to this problem, which came to be known as the Four Color Conjecture.

We need a definition to formally state this problem in graph-theoretic terms. A graph G is *planar* if it can be drawn in such a way that no two edges intersect, except for the possibility that they share a common end. Francis Guthrie's original observation can then be reduced to a graph theory problem by replacing each county in England with a vertex and drawing edges to represent their border relationships. Naturally, such a graph is planar. Thus Guthrie's question can be generalized and restated as follows: is it true that every planar graph G is 4-colorable?

As Thomas points out in [83], two failed attempts to prove this conjecture arose in 1879 and 1880 by Kempe and Tait, respectively. Both proofs stood intact for 11 years, with Kempe's finally being disproven by Heawood in 1890, and Tait's falling one year later in 1891 due to Petersen. Even though both Kempe and Tait had incorrect proofs, neither one was completely without merit. Kempe managed to show that all planar graphs are 5-colorable in addition to developing a still-useful tool known as Kempe chains. Similarly, Tait showed the conjecture is actually equivalent to a cubic (meaning the degree of every vertex is three) planar graph having a proper 3-edge coloring. Finally in 1977, albeit with the help of computers, Appel and Haken proved what is now known as the Four Color Theorem (4CT).

Theorem 1.2.1 ([3, 4]) *Every planar graph is 4-colorable.*

However, this proof was not entirely clear, so Robertson, Sanders, Seymour and Thomas deduced a much shorter proof (see [72]), though still computer-assisted.

As many learn early on in any traditional course in Graph Theory, neither K_5 nor $K_{3,3}$ are planar. These two graphs turn out to be a certificate of planarity of sorts, as discovered by Kuritowski and Wagner in the 1930's. Before stating the formal results, we shall need a definition. We say two paths are *independent* if they do not share an inner vertex. Suppose now that given a graph H, we replace all the edges of H with independent paths to obtain a graph H'. Then H' is a *subdivision* of H. If H' is a subgraph of another graph G, we then say that H is a *topological minor* of G. As a note, a graph H may be a minor of a graph G, but not necessarily a topological minor. To see a demonstration of this, consider the Petersen graph G. Upon making 5 edge contractions, we find that $G \succcurlyeq K_5$ (see Figure 1.3). However, because there is no vertex of degree 4 in G, we see that K_5 is not a topological minor of G. Thus every topological minor is a minor, but in general the converse is not true.

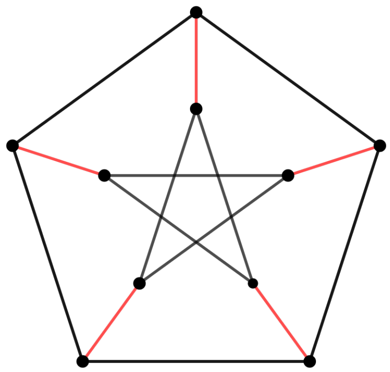


Figure 1.3: Contracting the red edges of the Petersen graph gives a K_5 minor

We now state the well-known results of Kuratowski and Wagner, given in 1930 and 1937, respectively.

Theorem 1.2.2 ([62]) *A graph G is a planar graph if and only if neither K_5 nor $K_{3,3}$ is a topological minor of G.*

Theorem 1.2.3 ([86]) *A graph G is a planar graph if and only if neither K_5 nor $K_{3,3}$ is a minor of G.*

As Seymour points out in his survey [77], it seems completely reasonable on some level that K_5 should be excluded as a minor, because K_5 is not 4-colorable. But one may wonder: why is it necessary to also exclude $K_{3,3}$? Suppose we relaxed these restrictions and *only* excluded K_5 as a minor. Is the resulting class of such graphs still 4-colorable? More generally, one may ask: if K_5 is changed to K_{t+1}, and 4-colorable to t-colorable, is the result true?

This is exactly what Swiss mathematician Hugo Hadwiger first proposed during a colloquium at Eidgenössiche Technische Hochschule on December 15, 1942 [84], which has since come to be known as Hadwiger's Conjecture (HC), and first appeared in print the following year.

Conjecture 1.2.4 ([50]) *For all $t \geq 0$, every K_{t+1} minor-free graph is t-colorable.*

An equivalent formulation of this conjecture is frequently stated as follows: every t-chromatic graph has a K_t minor. As Toft points out in his survey [84], Hadwiger had actually been originally inspired by Wagner's proof in 1937 [86] that HC and 4CT were logically equivalent. Interestingly, Thomas [83] and many others have noted that for all $t \geq 4$, HC implies 4CT. This fact can be seen by starting with any planar graph G, then adding a K_{t-4} to obtain a new graph $H := G + K_{t-4}$. Since G is planar, then by Theorem 1.2.3, $G \not\succeq K_5$, meaning that H must be K_{t+1} minor-free. Assuming HC is true, we have $\chi(H) \leq t$. Since no vertex in G can share the same color as any vertex in K_{t-4} under a proper coloring, $\chi(G) + (t-4) = \chi(G) + \chi(K_{t-4}) = \chi(H) \leq t$, giving $\chi(G) \leq 4$ as desired. Therefore, HC can be viewed as a generalization of 4CT.

Hadwiger's original presentation of the conjecture [50] contains proofs for the cases $t \leq 3$. Dirac [27] also independently supplied a proof for these cases in 1952. Wagner [86] proved that 4CT is equivalent to HC, establishing the case $t = 4$. It was not until 1993 that Robertson, Seymour and Thomas [73] proved the case $t = 5$, a result which earned them the 1994 Fulkerson Prize. The cases $t \geq 6$ remain open as of this writing.

Further historical explanation of the development of HC can be found in [84]. We now move on to set the stage for our particular results concerning HC.

Given the considerable effort expended by many to show even the first six cases of HC, and with no real promise of generalization, researchers began to apply restrictions on the graphs being investigated in a hopes of discovering either further confirmation of the conjecture or possibly a counterexample. We presently survey some of these results for the purposes of this dissertation, though a fairly recent and comprehensive collection of partial results can be found in [77].

Although HC appears too difficult to prove in general, some have managed to obtain partially affirming results, summarized in Table 1.1. We also note in particular the proof of the result mentioned in the table due to [1] is computer-assisted. In 2016, Rolek and Song supplied a much shorter, computer-free version in addition to the results listed in the table.

Table 1.1: Partial Results for HC

Excluded Minor(s)	$\chi(G) \leq t$	Reference
K_7, $K_{4,4}$	$t = 6$	[57]
$K_7^{=}$	$t = 6$	[52]
K_7^{-}	$t = 8$	[53]
K_7	$t = 8$	[1]
K_8	$t = 10$	
$K_8^{=}$	$t = 8$	[75]
K_8^{-}	$t = 9$	
K_9	$t = 12$	
$K_9^{=}$	$t = 10$	[74]

Rather than look at general graphs with excluded minors, some have instead chosen to restrict the class of graph in question. Some promising and affirming results have consequently arisen. First, let us briefly mention a special class of graphs. A graph G is said to be *perfect* if every induced subgraph H of G satisfies $\chi(H) = \omega(H)$. Hence, every perfect graph satisfies HC. Far less trivial classes of graphs have of course been studied. We say a graph G is *claw-free* if G is $K_{1,3}$-free. A graph G is *quasi-line* if for every vertex $v \in V(G)$, the neighborhood $N(v)$ can be partitioned into two cliques. Finally, given a graph G, its *line graph* $L(G)$ is defined such that each vertex of $L(G)$ is an edge in G, and two vertices in $L(G)$ are adjacent if and only if their corresponding edges in G share an end vertex. As pointed out in [21], line graphs are a proper subclass of quasi-line graphs, which in turn are a proper subclass of claw-free graphs. The results for these classes are summarized in Table 1.2.

Table 1.2: HC Results for Graph Classes

Graph Class	$G \succcurlyeq K_t$	Reference
Line Graphs	$t = \chi(G)$	[71]
Quasi-line Graphs	$t = \chi(G)$	[19]
Claw-free Graphs	$t = \lceil \frac{2}{3}\chi(G) \rceil$	[20]

Given a graph G, let $c : V(G) \to [k]$ be a k-coloring of $V(G)$, with color classes V_i, for all $i \in [k]$. Since $|V_i| \leq \alpha(G)$ for all $i \in [k]$, we have the following fact.

Fact 1.2.5 $|G| \leq \chi(G)\alpha(G)$ for any graph G.

In other words, Fact 1.2.5 gives that $\chi(G) \geq |G|/\alpha(G)$ for any graph G.

Let us now introduce some new notation which we shall use frequently. Define the *Hadwiger number* to be $h(G) := \max\{t : G \succcurlyeq K_t\}$. Conjecture 1.2.4 can now be restated as follows: for every graph G, $h(G) \geq \chi(G)$. Motivated by this observation, one direction is to try to prove (or disprove) that $h(G) = \lceil |G|/\alpha(G) \rceil$, as this would be the minimum-order minor one should now expect. One of the earliest results in this direction, due to Duchet and Meyniel in 1982, is as follows.

Theorem 1.2.6 ([29]) $h(G) \geq |G|/(2\alpha(G) - 1)$.

This is not ideal, of course, primarily because of the factor of two. In 2010, Fox [40] improved this to the following.

Theorem 1.2.7 ([40]) $h(G) \geq |G|/(1.983\alpha(G))$.

One year later, Balogh and Kostochka [5] proved a slightly better result.

Theorem 1.2.8 ([5]) $h(G) \geq |G|/(1.94792\alpha(G))$.

Still other work was done by Kawarabayashi and Song [58] to improve the previous results for smaller values of $\alpha(G)$.

Theorem 1.2.9 ([58]) *If* $\alpha(G) \geq 3$*, then* $h(G) \geq |G|/(2\alpha(G) - 2)$.

Additionally, B. Thomas and Song [82] showed that upon forbidding certain induced subgraphs, HC can be verified outright, by way of quasi-line graphs.

Theorem 1.2.10 ([82]) *If* $\alpha(G) \geq 3$ *and* G *is* $\{C_4, C_5, C_6, \ldots, C_{2\alpha(G)-1}\}$*-free, then* $h(G) \geq \chi(G)$.

The case of verifying HC when $\alpha(G) = 2$ is of particular interest. This may seem to be quite a substantial limitation at first, but this restriction means that $\overline{G}$ is triangle-free. As Plummer, Stiebitz and Toft observe in [68], a vast number of triangle-free graphs exist so this limitation is not as restrictive as one may initially think. In his survey, Seymour says the following about the case $\alpha(G) = 2$, which we quote directly (pp. 424–425, [77]).

> "This seems to me to be an excellent place to look for a counterexample. My own belief is, if it is true for graphs with stability number two then it is probably true in general, so it would be very nice to decide this case."

We first mention a very useful result of Plummer, Stiebitz and Toft [68] that establishes an equivalence of Hadwiger's conjecture in this context.

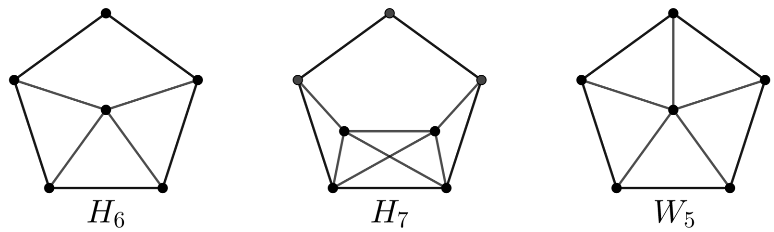


Figure 1.4: The graphs H_6, H_7 and W_5

Theorem 1.2.11 ([68]) *Let G be a graph with $\alpha(G) = 2$. Then $h(G) \geq \chi(G)$ if and only if $h(G) \geq \lceil |G|/2 \rceil$.*

In the same paper, Plummer, Stiebitz and Toft [68] proved the following.

Theorem 1.2.12 ([68]) *Let G be a graph with $\alpha(G) \leq 2$. If G is H-free, where H is a graph with $|H| = 4$ and $\alpha(H) \leq 2$, or $H = C_5$, or $H = H_7$ (see Figure 1.4), then $h(G) \geq \chi(G)$.*

In 2010, Kriesell [61] further augmented this list of forbidden subgraphs to include all cases of graphs with independence number at most two on five vertices.

Theorem 1.2.13 ([61]) *Let G be a graph with $\alpha(G) \leq 2$. If G is H-free, where H is a graph with $|H| = 5$ and $\alpha(H) \leq 2$, or $H = H_6$ (see Figure 1.4), then $h(G) \geq \chi(G)$.*

Let W_5 denote the wheel on six vertices (see Figure 1.4). We study Conjecture 1.2.4 for W_5-free graphs with independence number at most two. Our main result is stated as follows.

Theorem 1.2.14 ([8]) *Let G be a graph with $\alpha(G) \leq 2$. If G is W_5-free, then $h(G) \geq \chi(G)$.*

The proof of Theorem 1.2.14, given in Chapter 2, relies only on Theorem 1.2.11, Theorem 1.2.12 when $H = C_5$ and the following result of Chudnovsky and Seymour [22].

Theorem 1.2.15 ([22]) *Let G be a graph with $\alpha(G) \leq 2$. If*

$$\omega(G) \geq \begin{cases} |G|/4, & \text{if } |G| \text{ is even} \\ (|G|+3)/4, & \text{if } |G| \text{ is odd,} \end{cases}$$

then $h(G) \geq \chi(G)$.

Before we continue, let us recall several useful results.

Theorem 1.2.16 ([75]) *For $7 \leq t \leq 9$, let G be a graph with $2t-5$ vertices and $\alpha(G) = 2$. Then $h(G) \geq t-2$.*

Theorem 1.2.17 ([55]) *Let G be a graph on n vertices with $e(G) \geq 6n-19$. Then $h(G) \geq 8$.*

Theorem 1.2.18 ([78]) *Let G be a graph on n vertices with $e(G) \geq 7n-26$. Then $h(G) \geq 9$.*

Using the above theorems we can prove a similar result for $\overline{K_{1,5}}$-free graphs.

Corollary 1.2.19 ([8]) *Let G be a graph with $\alpha(G) \leq 2$. If G is $\overline{K_{1,5}}$-free, then $h(G) \geq \chi(G)$.*

Proof. Let G be a $\overline{K_{1,5}}$-free graph on n vertices with $\alpha(G) \leq 2$. By Theorem 1.2.11, it suffices to show that $h(G) \geq \lceil n/2 \rceil$. Suppose that $h(G) < \lceil n/2 \rceil$, where G is chosen with n to be minimum. By the minimality of n, G has no dominating edges. By Theorem 1.2.14, G must contain an induced W_5. Since $h(W_5) \geq 4$, we see that $n \geq 9$. We next claim that $n \leq 17$. Let $v \in V(G)$ be a vertex of minimum degree. Then $d(v) \geq n-5$ because G

is $\overline{K_{1,5}}$-free. Let $A := V(G) \setminus N[v]$ and $B := N(v)$. Then $|A| \leq 4$ and $G[A]$ is a clique because $\alpha(G) \leq 2$. Note that every vertex in A has at most three non-neighbors in B. Additionally, every vertex $b \in B$ must have a non-neighbor in A, else bv is a dominating edge. Hence, by counting the number of edges between A and B in $\overline{G}$, $|B| \leq 3|A| \leq 12$. Then $n = |G| \leq |A| + |B| + |\{v\}| \leq 4 + 12 + 1 = 17$. Since $e(G) \geq (n-5)n/2$, from Theorems 1.2.16, 1.2.17 and 1.2.18, it is straightforward to check that $h(G) \geq \lceil n/2 \rceil$ for all $9 \leq n \leq 17$, a contradiction.

This completes the proof of Corollary 1.2.19. ■

It is worth noting that if G is a K_6-free graph on n vertices with $\alpha(G) \leq 2$ but does not satisfy Conjecture 1.2.4, then G contains a K_5 subgraph by Theorem 1.2.13, and $n \leq 17$ because $R(K_3, K_6) = 18$ (see [59], and Section 1.3 for a discussion of Ramsey numbers). But then by Theorem 1.2.15, $h(G) \geq \chi(G)$, a contradiction. Thus Conjecture 1.2.4 holds for K_6-free graphs G with $\alpha(G) \leq 2$.

Similarly, if G is a K_7-free graph on n vertices with $\alpha(G) \leq 2$ but does not satisfy Conjecture 1.2.4, then G contains a K_6 subgraph from the previous paragraph, and $n \leq 22$ because $R(K_3, K_7) = 23$ (see [46] and [56]). But then by Theorem 1.2.15, $h(G) \geq \chi(G)$, a contradiction. Thus Conjecture 1.2.4 holds for K_7-free graphs G with $\alpha(G) \leq 2$. We summarize these results in the following remark.

Remark 1.2.20 ([8]) Let G be a K_t-free graph with $\alpha(G) \leq 2$, where $t \leq 7$. Then $h(G) \geq \chi(G)$.

We now spend the remainder of this chapter introducing our primary area of study in this dissertation.

1.3 Ramsey Theory

In 1930, Frank Ramsey tragically passed away at the young age of 26 from complications of abdominal surgery, but in the same year a paper of his appeared posthumously, entitled *On a Problem of Formal Logic*. His main goal was to address "the problem of finding a regular procedure to determine the truth or falsity of any given logical formula [70]," but along the way proved two famous results which now bear his name, and consequently founded an entire branch of combinatorics.

1.3.1 Classical Ramsey Numbers

Consider any k-edge coloring (not a proper edge coloring) of the complete graph K_n. Then $H \subseteq K_n$ is *monochromatic* if all edges of the subgraph H are colored the same.

Let $G, H_1, \dots, H_k$ be graphs. A common and useful notational convention used in this area, as observed in [7], is as follows. We write $G \longrightarrow (H_1, \dots, H_k)$ if every k-edge coloring of G contains a monochromatic copy of H_i for some color $i \in [k]$. For a given collection of graphs we write $R(H_1, \dots, H_k) = \min\{n : K_n \longrightarrow (H_1, \dots, H_k)\}$. In particular, if H_i is isomorphic to the graph H for all $i \in [k]$, we write $R_k(H)$. If $R(H_1, \dots, H_k) = R$ for some collection of graphs, then there is some k-edge coloring of K_{R-1} such that for all $i \in [k]$, no monochromatic copy of H_i in color i appears. To indicate this, we will write $K_{R-1} \not\longrightarrow (H_1, \dots, H_k)$, and we call such a k-edge coloring of K_{R-1} *bad*.

We now state Ramsey's theorem below with modern phrasing and notation.

Theorem 1.3.1 ([70]) *For any $k \geq 1$, let $H_1, \dots, H_k$ be any collection of graphs. Then there exists a number $R(H_1, \dots, H_k)$ such that for any k-edge coloring of K_n with $n \geq$*

$R(H_1, \ldots, H_k)$, *K_n contains a monochromatic copy of H_i in color i for some $i \in [k]$.*

One of the most attractive aspects of Ramsey-related problems is the simplicity in which they can be stated. Often encountered early in one's exposure to Graph Theory is the so-called "Party Problem." Here, it is stated as presented in [12].

> "Prove that at a gathering of any six people, some three of them are either mutual acquaintances or complete strangers to each other."

Interestingly, this problem also appeared on a Putnam exam in 1953 [15], phrased slightly differently.

> "Six points are in general position in space (no three in a line, no four in a plane). The fifteen line segments joining them in pairs are drawn and then painted, some segments red, some blue. Prove that some triangle has all its sides the same color."

Restated in graph-theoretic terms, the above asks for proof that $R(K_3, K_3) = 6$. As it turns out, this proof is quite elegant and straightforward, and as such is quite commonly assigned as a "homework problem" for those learning Ramsey theory. Many versions of the solution exist, see for example [12]. For completeness, we supply a proof here.

Proof. To see that $K_5 \not\longrightarrow (K_3, K_3)$, consider the bad coloring of K_5 depicted in Figure 1.5.

To see that $K_6 \longrightarrow (K_3, K_3)$, choose one vertex v. Label its neighbors $u_1, \ldots, u_5$. By the Pigeonhole Principle (Theorem 1.1.3), at least three of the edges vu_i, $i \in [5]$ must be the same color, say blue. In particular, we may assume that vu_1, vu_2 and vu_3 are blue.

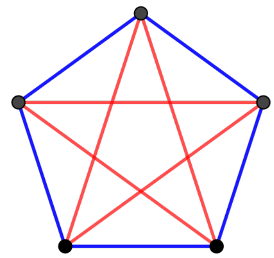


Figure 1.5: A bad coloring of K_5

If any of u_1u_2, u_2u_3 or u_1u_3 are blue, then we find a blue triangle with vertices v, u_1, u_2, or v, u_2, u_3, or v, u_1, u_3. Thus, each of u_1u_2, u_2u_3 or u_1u_3 are red, giving a red triangle with vertices u_1, u_2, u_3.

Taken together, the above shows that $R(K_3, K_3) \geq 6$ and $R(K_3, K_3) \leq 6$, from which we conclude that $R(K_3, K_3) = 6$. ■

After witnessing the beauty of the proof for $R(K_3, K_3) = 6$, one might have hope that a general formula for $R(K_t, K_t)$ may exist for all $t \geq 1$. In 1955, Greenwood and Gleason [47] provided a proof for $R(K_4, K_4) = 18$. However, it was beyond evident by this point in time that calculating exact Ramsey numbers is, to say the least, a nontrivial task. As of this writing, even the exact value of $R(K_5, K_5)$ remains unknown. Graham and Spencer [45] shared the following anecdote of Erdős to convey the true difficulty of calculating Ramsey numbers.

> "Aliens invade the earth and threaten to obliterate it in a year's time unless human beings can find the Ramsey number for red five and blue five. We could marshall the world's best minds and fastest computers, and within a year we could probably calculate the value. If the aliens demanded the Ramsey number

for red six and blue six, however, we would have no choice but to launch a preemptive attack."

A natural follow-up question is: what general bounds can be achieved if the exact values are too difficult to compute? Ramsey [70] was able to show in his original paper that $R(K_t, K_t) \leq 2^{2t-3}$. In 1947, Erdős [33] proved that $R(K_t, K_t) > 2^{t/2}$, and thus according to these bounds, $6 \leq R(K_5, K_5) \leq 128$. Since that time, these bounds have been improved considerably. The best known lower bound of 43 was provided by Exoo [36] in 1989. More recently, in 2017 Angeltveit and McKay [2] improved the upper bound to 48 with the assistance of a computer program. More diagonal and off-diagonal Ramsey numbers and bounds of complete graphs can be found in [69].

Generalizations of the two-edge coloring complete graph case explored above have also been studied extensively. One naturally can extend the problem to the case of multiple colors and other collections of graphs. In fact, Greenwood and Gleason also proved [47] that $R_3(K_3) = 17$. In the early 1970's, other collections of graphs began to be examined in more depth, including cycles, paths and much more (see for example [16] and [24]). The list of known results pertaining to classical Ramsey numbers has vastly grown throughout the years, with perhaps the most complete and up-to-date collection appearing in the dynamic survey by Radziszowski [69].

Since this dissertation focuses primarily on the Ramsey-type results for cycles, we highlight in the below theorem a particularly useful set of results. These and other known diagonal Ramsey numbers are summarized also in Table 1.3.

Theorem 1.3.2 ([39, 76]) *For all $n \geq 4$, $R(C_{2n}, C_{2n}) = 3n - 1$. Moreover, for all $n \geq 2$, $R(C_{2n+1}, C_{2n+1}) = 4n + 1$.*

Table 1.3: Diagonal Ramsey Numbers of Cycles

Ramsey Number	Reference(s)
$R(C_4, C_4) = 6$	[16]
$R(C_6, C_6) = 8$	[16]
$RC_{2n}, C_{2n}) = 3n - 1,\ n \geq 4$	[39, 76]
$R(C_{2n+1}, C_{2n+1}) = 4n + 1,\ n \geq 2$	[39, 76]
$R_3(C_3) = 17$	[47]
$R_3(C_4) = 11$	[6]
$R_3(C_5) = 17$	[89]
$R_3(C_6) = 12$	[90]
$R_3(C_7) = 25$	[37]
$R_3(C_8) = 16$	[80]
$R_4(C_4) = 18$	[35, 81]

To the author's knowledge, this is the most up-to-date list of known diagonal results. In particular, the two-color Ramsey numbers for cycles were completely solved independently by Faudree and Schelp [39] and Rosta [76] in the early 1970's. Additionally, mixed parity cycles for the two-color case were considered in the same papers.

Theorem 1.3.3 ([39, 76]) *For $4 \leq m < \ell$ with m even and ℓ odd, $R(C_m, C_\ell) = \max\{\ell - 1 + m/2,\ 2m - 1\}$.*

Permitting additional edge colors naturally complicates the computation of Ramsey numbers. As of this writing, $R_3(C_n)$ remains open for all $n \geq 9$. With regard to odd cycles, Bondy and Erdős [7] are often credited with making the following conjecture for the three-color case, sometimes called the Triple Odd Cycle Conjecture. However, it should be noted that although researchers frequently point to [7] as the source of this conjecture, it does not explicitly appear there.

Conjecture 1.3.4 *$R_3(C_{2n+1}) = 8n + 1$ for all $n \geq 2$.*

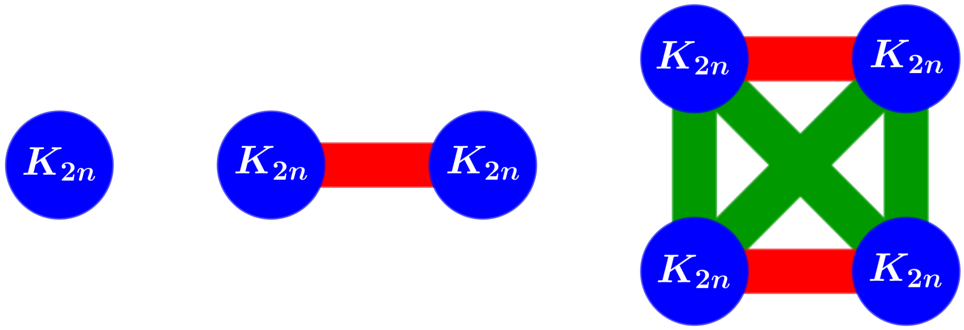


Figure 1.6: A lower bound construction showing $R_3(C_{2n+1}) \geq 8n+1$

It is not difficult to show that $R_3(C_{2n+1}) \geq 8n+1$ for all $n \geq 2$. Following the construction of a bad coloring outlined in [34], begin with a K_{2n} and color all edges the same color, say blue. Since $|K_{2n}| = 2n$, there is no possibility of a blue C_{2n+1} appearing as a subgraph. For the second step, create two copies of the K_{2n} with all edges colored blue and insert all edges between these copies colored the same color, say red. The graph induced on the blue edges certainly contains no blue C_{2n+1} because both cliques have order $2n$. Moreover, there is no red C_{2n+1} because the graph induced on the red edges is bipartite and thus contains no odd cycle by Theorem 1.1.1. Finally, create two copies of the graph in step two, this time joined together by all edges of a new color, say green. Again, there is no blue or red C_{2n+1} for the same reasons as above, and no green C_{2n+1} because the graph induced on the green edges is again bipartite. This process is illustrated in the Figure 1.6. Note that the rightmost graph in Figure 1.6 has exactly $8n$ vertices and no monochromatic C_{2n+1}.

Although Conjecture 1.3.4 remains open, there are asymptotic results. The following was proved by Łuczak in 1999.

Theorem 1.3.5 ([64]) $R_3(C_{2n+1}) \leq 8n + o(n)$.

In 2016, Jenssen and Skokan [54] proved that the conjecture holds for sufficiently large n.

Theorem 1.3.6 ([54]) $R_3(C_{2n+1}) = 8n + 1$ *for sufficiently large* n*.*

An analogous situation exists for even cycles. In 2005, Dzido, Nowik and Szuca [31] supplied a lower bound construction.

Theorem 1.3.7 ([31]) $R_3(C_{2n}) \geq 4n$ *for all* $n \geq 2$*.*

As such, a similar "Triple Even Cycle Conjecture" exists, where $R_3(C_{2n})$ is conjectured to be $4n$ for all $n \geq 3$ due to Dzido [30] in his Ph.D. thesis. Related asymptotic results have been similarly shown, but since this dissertation focuses on examining odd cycles, they are omitted here. We do wish to point out, however, that as of this writing $R_3(C_{10})$ remains the first open case for even cycles, but by Theorem 1.3.7 we know that $R_3(C_{10}) \geq 20$.

Interestingly, the lower bound construction given above for odd cycles extends to k colors by simply continuing the aforementioned process. For example, a bad coloring with four colors would be achieved by creating two copies of the rightmost graph in Figure 1.6 and joining these copies with all edges between them colored by a new color, say yellow. Again no yellow C_{2n+1} occurs because the graph induced on the yellow edges is bipartite. In general, if G_{k-1} is the graph formed with the bad coloring as described above using $k-1$ colors, we find a bad coloring with k colors by creating two copies of G_{k-1} and coloring all edges between them with color k, forming G_k. By construction, $|G_k| = n \cdot 2^k$, from which the following conjecture arises. Bondy and Erdős are likewise credited with this conjecture in [7], although the explicit statement of it does not appear there.

Conjecture 1.3.8 $R_k(C_{2n+1}) = n \cdot 2^k + 1$ *for all* $n \geq 2$*.*

When $k \geq 2$ is fixed and n is sufficiently large, Jenssen and Skokan [54] proved that the conjecture holds.

Theorem 1.3.9 ([54]) *For any fixed $k \geq 2$ and n sufficiently large, $R_k(C_{2n+1}) = n \cdot 2^k + 1$.*

Curiously, however, Day and Johnson disproved Conjecture 1.3.8 when n is fixed and k is sufficiently large.

Theorem 1.3.10 ([25]) *For all n there exists a constant $\epsilon = \epsilon(n) > 0$ such that, for all sufficiently large k, $R_k(C_{2n+1}) > 2n \cdot (2+\epsilon)^{k-1}$.*

As one may imagine, many variants of the problem discussed above exist. For example, researchers have extended this problem to include hypergraphs, where edges may contain more than two vertices. Another avenue of research involves coloring graphs which are not complete. For instance, what is the smallest value of n such that $K_{n,n} \longrightarrow (K_{t,t}, K_{t,t})$ for some t? Yet another is the size Ramsey number. Let $\mathcal{G}$ denote the set of all graphs G such that $G \longrightarrow (H_1, H_2)$. The *size Ramsey number* is defined as $\hat{R}(H_1, H_2) = \min\{|E(G)| : G \in \mathcal{G}\}$. In this way we study particular variant which is computationally more feasible though still far from trivial.

1.3.2 Gallai-Ramsey Numbers

A *rainbow triangle* is a copy of K_3 with all edges colored differently. A *Gallai coloring* of a complete graph is an edge-coloring that contains no rainbow triangle. A *Gallai k-coloring* is a Gallai coloring that uses at most k colors. Let $G, H_1, H_2, \ldots, H_k$ be a collection of graphs. Following the notational convention of the previous section, we write $G \xrightarrow{Gallai} (H_1, \ldots, H_k)$ if every Gallai k-coloring of G contains a monochromatic copy of H_i for some color $i \in [k]$. We can therefore define the *Gallai-Ramsey number* to be $GR(H_1, \ldots, H_k) := \min\{n : K_n \xrightarrow{Gallai} (H_1, \ldots, H_k)\}$. If H_i is isomorphic to the graph H for all $i \in [k]$, we simply write $GR_k(H)$.

Because we must have $k \geq 3$ for a rainbow triangle to occur, we note the following.

Fact 1.3.11 Let H_1, H_2 be any graphs. Then $GR(H_1, H_2) = R(H_1, H_2)$.

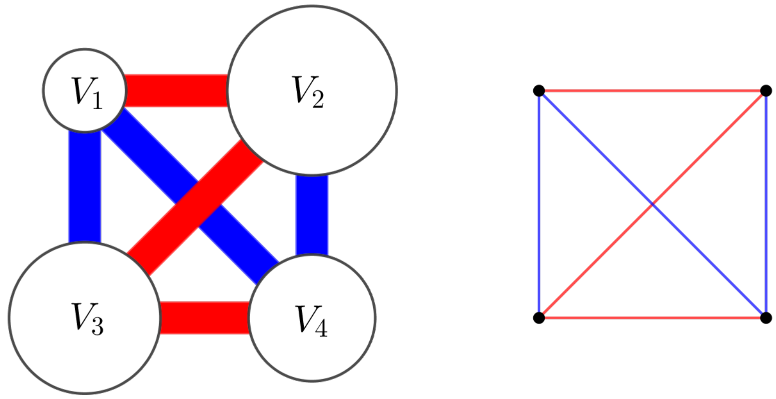
Alternatively, one may define the Gallai-Ramsey number as the least integer n such that every k-edge coloring of K_n contains either a rainbow triangle or a monochromatic copy of the graph H_i for some color $i \in [k]$. Therefore, intuitively one expects n to be smaller when searching for a rainbow triangle or a monochromatic copy of H_i for some color i in a given k-edge coloring, as opposed to searching for *only* a monochromatic copy of H_i for some color i. Because of this, we have the following fact.

Fact 1.3.12 $GR(H_1, \ldots, H_k) \leq R(H_1, \ldots H_k)$.

Therefore the Gallai-Ramsey number provides a natural lower bound to the classical Ramsey number. In particular, $GR_k(H) \leq R_k(H)$ for any graph H.

Central to the theory behind Gallai-Ramsey numbers is a structural result due to Tibor Gallai in 1967 [44]. Originally intended to discuss the properties of "transitively orientable graphs," Gallai's paper also included some structural results that happen hold for all graphs, which have found a variety of other applications. It should be noted that Gallai's original paper appeared in German but an English translation of it was published in 2001 by Maffray and Preissmann [65].

Theorem 1.3.13 ([44, 65]) *For any Gallai k-coloring c of a complete graph G with $|G| \geq 2$, $V(G)$ can be partitioned into nonempty sets $V_1, V_2, \ldots, V_p$ with $p > 1$ so that at most two colors are used on the edges in $E(G) \setminus (E(G[V_1]) \cup \cdots \cup E(G[V_p]))$ and only one color is used on the edges between any fixed pair (V_i, V_j) under c, for all $i \neq j$.*



Gallai Partition Reduced Graph

Figure 1.7: A Gallai partition and the corresponding reduced graph

We call the partition of $V(G)$ into $V_1, V_2, \ldots, V_p$ described in Theorem 1.3.13 a *Gallai partition* of the vertices. Often we will refer to V_i, where $i \in [p]$, as the *parts* of the Gallai partition. Although Theorem 1.3.13 is indeed a powerful structural result, we are careful to note that no information is provided regarding $|G[V_i]|$, nor about the colors appearing on the edges in $G[V_i]$, for all $i \in [p]$. Given a Gallai partition, we can define a new graph. Let $v_i \in V_i$ for each $i \in [p]$. Define the *reduced graph* $\mathcal{R}$ to be the graph $G[\{v_1, v_2, \ldots, v_p\}]$. In other words, contracting each part of the Gallai partition to one vertex produces a new complete graph of smaller order, the reduced graph, with at most two colors appearing on its edges. An example of a Gallai partition and its corresponding reduced graph is shown in Figure 1.7. If H is any graph, we therefore see that a monochromatic copy of H appearing as a subgraph of $\mathcal{R}$ must also appear as a subgraph of G. For this reason, $R(H, H)$ closely relates to $GR_k(H)$.

Fortunately, the behavior of the Gallai-Ramsey number is more predictable than that of the classical Ramsey number. In 2010, Gyárfás, Sárközy, Sebő, and Selkow [49] classified the nature of $GR_k(H)$ depending on whether or not H is bipartite.

Theorem 1.3.14 ([49]) *Let H be a fixed graph with no isolated vertices and let $k \geq 1$ be an integer. Then $GR_k(H)$ is exponential in k if H is not bipartite, linear in k if H is bipartite but not a star, and constant (does not depend on k) when H is a star.*

We now survey some of the relevant known results for Gallai-Ramsey numbers. In 1983, Chung and Graham [23] contributed what is possibly the first result for Gallai-Ramsey numbers. Their research was motivated by an earlier question attributed to T. A. Brown in their paper, which we mention here.

> "What is the largest number $f(k)$ of vertices a complete graph can have such that it is possible to k-color its edges so that every triangle has edges of *exactly* two colors?"

Fascinatingly, their proof did not rely on Gallai's structural result in [44].

Theorem 1.3.15 ([23]) *For all $k \geq 1$,* $GR_k(K_3) = \begin{cases} 5^{k/2} + 1, & \text{if } k \text{ is even} \\ 2 \cdot 5^{(k-1)/2} + 1, & \text{if } k \text{ is odd.} \end{cases}$

Gyárfás, Sárközy, Sebő, and Selkow [49] provided an alternative proof to Theorem 1.3.15 in 2010 which does use Gallai's result, and is therefore somewhat shorter. Liu, Magnant, Saito, Schiermeyer, and Shi [63] established the next open case in 2017.

Theorem 1.3.16 ([63]) *For all $k \geq 1$,* $GR_k(K_4) = \begin{cases} 17^{k/2} + 1, & \text{if } k \text{ is even} \\ 3 \cdot 17^{(k-1)/2} + 1, & \text{if } k \text{ is odd.} \end{cases}$

Magnant and Schiermeyer [66] announced a proof of the analogous result for K_5 in 2019, though the author of this dissertation has not personally verified it. We state it here for completeness.

Theorem 1.3.17 ([66]) *For all $k \geq 2$,*

$$GR_k(K_5) = \begin{cases} (R(K_5, K_5) - 1)^{k/2} + 1, & \text{if } k \text{ is even} \\ 4(R(K_5, K_5) - 1)^{(k-1)/2} + 1, & \text{if } k \text{ is odd} \end{cases}$$

unless $R(K_5, K_5) = 43$, in which case

$$\begin{cases} GR(K_5) = 43, & \text{if } k = 2 \\ 42^{k/2} + 1 \leq GR_k(K_5) \leq 43^{k/2} + 1, & \text{if } k \geq 4 \text{ is even} \\ 169 \cdot 42^{(k-3)/2} + 1 \leq GR_k(K_5) \leq 4 \cdot 43^{(k-1)/2} + 1, & \text{if } k \geq 3 \text{ is odd.} \end{cases}$$

In 2015, Fox, Grinshpun and Pach [41] posed a conjecture to describe this apparent pattern.

Conjecture 1.3.18 ([41]) *For all $k \geq 1$ and $t \geq 3$,*

$$GR_k(K_t) = \begin{cases} (R(K_t, K_t) - 1)^{k/2} + 1, & \text{if } k \text{ is even} \\ (t-1)(R(K_t, K_t) - 1)^{(k-1)/2} + 1, & \text{if } k \text{ is odd.} \end{cases}$$

As we have seen, Conjecture 1.3.18 has been verified for $t = 3$, 4 and 5. Interestingly, Magnant and Schiermeyer [66] also constructed a three-edge-colored K_{169} that contains neither a rainbow K_3 nor a monochromatic K_5. If $R(K_5, K_5) = 43$ (the current best-known lower bound from [36]), then the above conjecture is false because the formula would give $GR_3(K_5) = 4 \cdot 42 + 1 = 169$.

Naturally, some have conducted research on the Gallai-Ramsey numbers of other graphs, including stars, books, paths, and cycles among others. In the same 2010 paper mentioned previously, Gyárfás, Sárközy, Sebő, and Selkow [49] established the Gallai-Ramsey number of stars.

Theorem 1.3.19 ([49]) *For all $t \geq 2$ and $k \geq 3$,* $GR_k(K_{1,t}) = \begin{cases} (5t-3)/2, & \text{if } t \text{ is odd} \\ 5t/2 - 3, & \text{if } t \text{ is even.} \end{cases}$

As a minor note regarding Theorem 1.3.19, the formula is not true for $t = 2$ as originally stated. Clearly, one requires at least three vertices to obtain a monochromatic $K_{1,2}$ or a rainbow K_3, so $GR_k(K_{1,2}) \geq 3$. On the other hand, we see that $GR_k(K_{1,2}) \leq 3$ because with any k-coloring of the edges of a K_3, there are exactly two choices: either there is a monochromatic $K_{1,2}$ subgraph; or all edges of the K_3 are colored differently, yielding a rainbow K_3. However, the formula for Theorem 1.3.19 gives $(5 \cdot 2)/2 - 3 = 2$.

Noting that $K_{1,2} \simeq P_3$, this result was proven in 2010 by Faudree, et. al. [38] along with the Gallai-Ramsey number for other paths.

Theorem 1.3.20 ([38]) *For all $k \geq 1$ and $n \in \{3, 4, 5, 6\}$, $GR_k(P_n) = \lfloor (n-2)/2 \rfloor k + \lceil n/2 \rceil + 1$.*

This list was subsequently expanded by J. Zhang, Lei, Shi and Song [91].

Theorem 1.3.21 ([91]) *For all $k \geq 1$ and $n \in \{7, 9, 10, 11\}$, $GR_k(P_n) = \lfloor (n-2)/2 \rfloor k + \lceil n/2 \rceil + 1$.*

The bounds for all paths were also given in [38] and improved upon by Hall, Magnant, Ozeki and Tsugaki [51] in 2014.

Theorem 1.3.22 ([51]) *For all integers $k \geq 1$ and $n \geq 3$,*

$$\left\lfloor \frac{n-2}{2} \right\rfloor k + \left\lceil \frac{n}{2} \right\rceil + 1 \leq GR_k(P_n) \leq \left\lfloor \frac{n-2}{2} \right\rfloor + 3 \left\lfloor \frac{n}{2} \right\rfloor$$

A list of the Gallai-Ramsey numbers for a variety of other graphs, as well as some mixed and off-diagonal cases, can be found in the dynamic survey by Fujita, Magnant and Ozeki [43]. However, as these other cases are not the primary focus of this dissertation, we now turn our attention to the known results for the diagonal Gallai-Ramsey numbers of cycles. In Table 1.4, we provide a list of the diagonal Gallai-Ramsey numbers and their references for small even and odd cycles, including those proven in this dissertation.

We note that although Song and J. Zhang [79] are not credited with the original proofs of $GR_k(C_6)$ and $GR_k(C_8)$, we cite them in Table 1.4 because their proof was different and substantially shorter than those appearing in [42] and [48], respectively. We also wish to point out that the new proof by Song and J. Zhang both fixes the incomplete proof for $GR_k(C_8)$ originally provided in [48] and handles some mixed Gallai-Ramsey numbers of even cycles and paths.

When the original work was done for this dissertation, there were some best known general bounds at the time, which we summarize in the following theorem. The lower bounds were provided in 1976 by Erdős et. al. [34], whereas the upper bounds are found in the 2014 paper by Hall et. al. [51].

Theorem 1.3.23 ([34, 51]) *For all $k \geq 1$ and $n \geq 2$,*

(i) $(n-1)k + n + 1 \leq GR_k(C_{2n}) \leq (n-1)k + 3n$,

(ii) $n \cdot 2^k + 1 \leq GR_k(C_{2k+1}) \leq (2^{k+3} - 3)n \ln n$.

Table 1.4: Diagonal Gallai-Ramsey Numbers of Small Cycles

Gallai-Ramsey Number	Reference(s)
$GR_k(C_3) = \begin{cases} 5^{k/2}+1, & \text{if } k \text{ is even} \\ 2 \cdot 5^{(k-1)/2}+1, & \text{if } k \text{ is odd} \end{cases}$	[23]
$GR_k(C_4) = k+4$	[38]
$GR_k(C_5) = 2 \cdot 2^k + 1$	[42]
$GR_k(C_6) = 2k+4$	[42, 79]
$GR_k(C_7) = 3 \cdot 2^k + 1$	[13]
$GR_k(C_8) = 3k+5$	[48, 79]
$GR_k(C_9) = 4 \cdot 2^k + 1$	[10, 9]
$GR_k(C_{10}) = 4k+6$	[91]
$GR_k(C_{11}) = 5 \cdot 2^k + 1$	[10]
$GR_k(C_{12}) = 5k+7$	[91]
$GR_k(C_{13}) = 6 \cdot 2^k + 1$	[11]
$GR_k(C_{15}) = 7 \cdot 2^k + 1$	[11]

Recently, the Gallai-Ramsey numbers for all even cycles were settled by Chen, Song and F. Zhang [18].

Theorem 1.3.24 ([18]) *For all $k \geq 2$ and $n \geq 2$,*

$$GR_k(C_{2n}) = \begin{cases} (n-1)k+n+1 & \text{if } n \geq 3 \\ (n-1)k+n+2 & \text{if } n = 2. \end{cases}$$

The upper bound for odd cycles was subsequently improved in 2018 by Chen, Li and Pei.

Theorem 1.3.25 ([17]) *For all $k \geq 2$, $GR_k(C_{2n+1}) \leq (4n + n\log_2 n) \cdot 2^k$.*

With the exceptions of C_3 and C_4, it is believed that the lower bounds are likely the true values of the Gallai-Ramsey numbers for all cycles. We also note here that $GR_3(C_{10}) = 18$ [91], but by Theorem 1.3.7, $R_3(C_{10}) \geq 20$. Thus, in contrast with the odd cycle case, the Gallai-Ramsey numbers for even cycles do not provide strong partial evidence for the "Triple Even Cycle Conjecture."

The first segment of our work in this area concerns $GR_k(C_{2n+1})$ for $n \in \{4, 5, 6, 7\}$. Due to the nature of their proofs, we now state our results in the following separate theorems to be proven in Chapter 3.

Theorem 1.3.26 ([9, 10]) *For all $k \geq 1$ and $n \in \{4, 5\}$, $GR_k(C_{2n+1}) = n \cdot 2^k + 1$.*

Theorem 1.3.27 ([11]) *For all $k \geq 1$ and $n \in \{6, 7\}$, $GR_k(C_{2n+1}) = n \cdot 2^k + 1$.*

After the above work was completed, we managed to improve Theorem 1.3.25 which was at the time the best-known general upper bound. We supply the proof of Theorem 1.3.28 in Chapter 4.

Theorem 1.3.28 ([11]) *For all $k \geq 1$ and $n \geq 8$, $GR_k(C_{2n+1}) \leq (n \ln n) \cdot 2^k - (k+1)n + 1$.*

Since the time we completed our main work (see Chapters 3 and 4), Chen, Song and F. Zhang [18] announced a generalization of our results which confirms the long-held belief that the Gallai-Ramsey number for all odd cycles (with the exception of C_3) should match the lower bound. The results of this dissertation are indeed cited in [18].

Theorem 1.3.29 ([18]) *For all $k \geq 1$ and $n \geq 3$, $GR_k(C_{2n+1}) = n \cdot 2^k + 1$.*

Let us then return to our main goal. In Chapter 2 we supply the proof of Theorem 1.2.14. In Chapter 3 we prove Theorem 1.3.26 in Section 3.2 and Theorem 1.3.27 in Section 3.3. We then prove Theorem 1.3.28 in Chapter 4. Finally, we conclude this dissertation with a discussion of future work in Chapter 5.

CHAPTER 2: HADWIGER'S CONJECTURE FOR W_5-FREE GRAPHS

2.1 Proof of Theorem 1.2.14

Let G be a W_5-free graph on n vertices with $\alpha(G) \leq 2$. By Theorem 1.2.11, it suffices to show that $h(G) \geq \lceil n/2 \rceil$. Suppose $h(G) < \lceil n/2 \rceil$. We choose such a graph G with n minimum. By Theorem 1.2.12, G must contain an induced C_5. Then $\alpha := \alpha(G) = 2$. Note that $(n+3)/4 \leq \lceil (n+2)/4 \rceil$ for odd n. By Theorem 1.2.15, $\omega(G) < \lceil (n+2)/4 \rceil$ when n is odd, and $\omega(G) < \lceil n/4 \rceil$ when n is even.

Since G has an induced C_5, let $X := \bigcup_{i=1}^5 X_i$ be a maximal inflation of C_5 in G such that for all $i \in [5]$, $G[X_i]$ is a clique, X_i is complete to $X_{i-1} \cup X_{i+1}$ and anticomplete to $X_{i-2} \cup X_{i+2}$, where all arithmetic on indices here and henceforth is done modulo 5. Then $X_i \neq \emptyset$ for all $i \in [5]$. Since $\alpha = 2$ and G is W_5-free, no vertex in $V(G) \setminus X$ is complete to X and every vertex in $V(G) \setminus X$ must be complete to at least three consecutive X_i's on the maximal inflation of C_5. For each $i \in [5]$, let

$Y_i := \{v \in V(G) \setminus X \mid v \text{ is complete to } X \setminus X_i \text{ and has a non-neighbor in } X_i\}$

$Z_i := \{v \in V(G) \setminus X \mid v \text{ is complete to } X \setminus (X_i \cup X_{i+1}) \text{ and has a non-neighbor in } X_i \text{ and in } X_{i+1}\}.$

Let $Y := \bigcup_{i=1}^5 Y_i$ and $Z := \bigcup_{i=1}^5 Z_i$. By definition, $Y \cap Z = \emptyset$ and $Y \cup Z = V(G) \setminus X$. By the maximality of $|X|$, no vertex in Z_i is anticomplete to $X_i \cup X_{i+1}$ in G, else, such a vertex can be placed in X_{i+3} to obtain a larger inflation of C_5.

Claim 2.1.1 *For all $i \in [5]$, $G[Z_i]$ is a clique.*

Proof. Suppose some $G[Z_i]$, say $G[Z_1]$, is not a clique. Then there exist $z_1, z_1' \in Z_1$ such that $z_1z_1' \notin E(G)$. By definition of Z_1, there exist $x_1 \in X_1$ and $x_2 \in X_2$ such that $z_1x_1, z_1x_2 \notin E(G)$. Since $\alpha = 2$, we see that $z_1'x_1, z_1'x_2 \in E(G)$. But then $G[\{z_1', x_1, x_2, x_3, x_4, x_5\}] = W_5$, where $x_i \in X_i$ for all $i \in \{3, 4, 5\}$, a contradiction. ■

We can use similar reasoning to deduce an analogous statement for $G[Y_i]$ for all $i \in [5]$.

Claim 2.1.2 *For all $i \in [5]$, Y_i is anticomplete to X_i, and so $G[Y_i]$ is a clique.*

With the following observation, we can partition the sets Z_i for all $i \in [5]$.

Claim 2.1.3 *For all $i \in [5]$, every vertex in Z_i is either anticomplete to X_i, or anticomplete to X_{i+1}, but not both.*

Proof. As observed earlier, for all $i \in [5]$, no vertex in Z_i is anticomplete to $X_i \cup X_{i+1}$. Suppose there exists some $i \in [5]$, say $i = 1$, such that some vertex, say $z \in Z_1$ is neither anticomplete to X_i nor anticomplete to X_{i+1}. Then there exist $x_1 \in X_1$ and $x_2, \in X_2$ such that $zx_1, zx_2 \notin E(G)$. Let $x_i \in X_i$ for all $i \in \{3, 4, 5\}$. By definition of Z_1, z is complete to $\{x_3, x_4, x_5\}$. But then $G[\{z, x_1, x_2, x_3, x_4, x_5\}] = W_5$, a contradiction. ■

For each $i \in [5]$, let

$$Z_i^i := \{z \in Z_i \mid z \text{ is anticomplete to } X_i\}$$

$$Z_i^{i+1} := \{z \in Z_i \mid z \text{ is anticomplete to } X_{i+1}\}.$$

By Claim 2.1.3, $Z_i = Z_i^i \cup Z_i^{i+1}$ and $Z_i^i \cap Z_i^{i+1} = \emptyset$ for all $i \in [5]$.

Since $\alpha = 2$, by the choice of $Y_i, Z_i, Z_i^i, Z_i^{i+1}$, we see that

Claim 2.1.4 *For all $i \in [5]$, both $G[Z_{i-1}^i \cup Y_i \cup Z_i]$ and $G[Z_{i-1} \cup Y_i \cup Z_i^i]$ are cliques.*

We next show that

Claim 2.1.5 *For all $i \in [5]$, every vertex in Z_i^i is complete to Y_{i-1} or complete to Z_{i+1}^{i+2}.*

Proof. Suppose the statement is false. We may assume that there exists some vertex $z \in Z_1^1$ such that $zy_5, zz_2 \notin E(G)$, where $y_5 \in Y_5$ and $z_2 \in Z_2^3$. Since $\alpha = 2$, we see that $y_5z_2 \in E(G)$. Then $G[\{x_4, y_5, z_2, x_5, z, x_3\}] = W_5$, where $x_i \in X_i$ for all $i \in \{3, 4, 5\}$, a contradiction. ■

Claim 2.1.6 *For all $i \in [5]$, every vertex in Y_i is either complete to Y_{i-1} or complete to Y_{i+2}.*

Proof. Suppose not. We may assume there exist vertices $y_1 \in Y_1$, $y_3 \in Y_3$ and $y_5 \in Y_5$ such that $y_1y_3, y_1y_5 \notin E(G)$. Then $y_3y_5 \in E(G)$ because $\alpha = 2$. Then $G[\{x_4, y_5, y_3, x_5, y_1, x_3\}] = W_5$, where $x_i \in X_i$ for all $i \in \{3, 4, 5\}$, a contradiction. ■

By Claim 2.1.5, $Z_1^1 = A_1 \cup B_1$ and $Z_3^3 = A_3 \cup B_3$, where $A_i \cap B_i = \emptyset$, and

$$A_i := \{v \in Z_i^i \mid v \text{ is complete to } Y_{i-1}\}$$

$$B_i := \{v \in Z_i^i \mid v \text{ is complete to } Z_{i+1}^{i+2} \text{ and has a non-neighbor in } Y_{i-1}\}.$$

for $i \in \{1, 3\}$. By Claim 2.1.6, $Y_1 = Y_1' \cup Y_1''$, where

$$Y_1' := \{v \in Y_1 \mid v \text{ is complete to } Y_5\}$$

$$Y_1'' := \{v \in Y_1 \mid v \text{ is complete to } Y_3 \text{ and has a non-neighbor in } Y_5\}.$$

Then $Y_1' \cap Y_1'' = \emptyset$. We claim that A_3 is complete to Y_1'' in G. To see this, suppose there exist vertices $z \in A_3$ and $y_1 \in Y_1''$ such that $zy_1 \notin E(G)$. By the choice of Y_1'', there exists a vertex $y_5 \in Y_5$ such that $y_1y_5 \notin E(G)$. Then $zy_5 \in E(G)$ because $\alpha = 2$. Since $z \in Z_3^3$, there exists some vertex $x_4 \in X_4$ such that $zx_4 \in E(G)$. But then $G[\{x_4, z, y_5, x_3, y_1, x_5\}] = W_5$, where $x_3 \in X_3$ and $x_5 \in X_5$, a contradiction. This proves that A_3 is complete to Y_1'' in G, as claimed. Let

$$H_1 := G[X_3 \cup X_4 \cup Y_5 \cup Z_5 \cup Y_1' \cup A_1]$$

$$H_2 := G[X_4 \cup X_5 \cup B_1 \cup Z_1^2 \cup Y_2 \cup Z_2]$$

$$H_3 := G[X_1 \cup X_2 \cup B_3 \cup Z_3^4 \cup Y_4 \cup Z_4]$$

$$H_4 := G[X_5 \cup Y_1'' \cup Y_3 \cup A_3]$$

Note that each of H_1, H_2, H_3 and H_4 is a clique in G, and $|H_1| + |H_2| + |H_3| + |H_4| = |G| + |X_4| + |X_5| \geq n + 2$. It follows that $\omega(G) \geq \max\{|H_1|, |H_2|, |H_3|, |H_4|\} \geq \lceil (n + 2)/4 \rceil$, a contradiction.

This completes the proof of Theorem 1.2.14. ■

CHAPTER 3: GALLAI-RAMSEY NUMBERS OF SMALL ODD CYCLES

In this chapter, our goal is to prove our results concerning Gallai-Ramsey numbers of odd cycles. First, we introduce some very useful lemmas in Section 3.1 which we shall require at various points later on. We then prove Theorem 1.3.26 in Section 3.2, followed by the proof of Theorem 1.3.27 in Section 3.3 because its proof is substantially more complicated.

3.1 Preliminaries

Lemma 3.1.1 ([9, 10]) *For all $n \geq 3$ and $k \geq 1$, let c be a k-coloring of the edges of a complete graph G on at least $2n+1$ vertices. Let $U, W \subseteq V(G)$ be two disjoint sets with $|U| \geq n$ and $|W| \geq n$. If U is mc-complete, say blue-complete, to W under the coloring c, then no vertex in $V(G) \setminus (U \cup W)$ is blue-complete to $U \cup W$ in G. Moreover, if $|W| \geq n+1$ (resp. $|U| \geq n+1$), then $G[W]$ (resp. $G[U]$) has no blue edges.*

Proof. For the first case, suppose there exists a vertex $x \in V(G) \setminus (U \cup W)$ such that x is blue-complete to $U \cup W$ in G. Let $U = \{u_1, \ldots, u_{|U|}\}$ and $W = \{w_1, \ldots, w_{|W|}\}$. We then obtain a blue C_{2n+1} with vertices $u_1, x, w_1, u_2, w_2, \ldots, u_n, w_n$ in order when $|U| \geq n, |W| \geq n$. For the second case, assume $|W| \geq n+1$ and w_1w_2 is colored blue under c. Then we find a blue C_{2n+1} with the vertices $u_1, w_1, w_2, u_2, w_3, \ldots, u_n, w_{n+1}$ in order, a contradiction. If $|U| \geq n+1$, the proof is identical. ■

Lemma 3.1.2 ([9, 10]) *For all $\ell \geq 3$ and $n \geq 1$, let $n_1, n_2, \ldots, n_\ell$ be positive integers such that $n_i \leq n$ for all $i \in [\ell]$ and $\sum\limits_{i=1}^{\ell} n_i \geq 2n+1$. Then the complete multipartite graph $K_{n_1,n_2,\ldots,n_\ell}$*

has a cycle of length $2n+1$*.*

Proof. Let $G := K_{n'_1,n'_2,\ldots,n'_{\ell}}$ be an induced subgraph of $K_{n_1,n_2,\ldots,n_\ell}$ such that: $\ell' \geq 3$; $\sum_{i=1}^{\ell'} n'_i = 2n+1$; and for all $i \in [\ell']$, $1 \leq n'_i \leq n$. Then $\delta(G) \geq n+1 \geq |G|/2$. By Theorem 1.1.2, G has a Hamilton cycle, and so $K_{n_1,n_2,\ldots,n_\ell}$ has a cycle of length $2n+1$. ■

The final lemma due to Hall et. al. will be used in the proof of Theorem 1.3.28.

Lemma 3.1.3 ([51]) *For* $1 \leq t \leq n$*, any Gallai-colored complete graph having a Gallai partition with at least* $4\lceil n/t\rceil + 1$ *parts each of order at least* t *contains a monochromatic* C_{2n+1}*.*

3.2 Proof of Theorem 1.3.26

Let $n \in \{4,5\}$. As mentioned in Section 1.3.1, Erdős, Faudree, Rousseau and Schelp [34] gave a construction for the lower bound of $R_k(C_{2n+1})$, illustrated in Figure 1.6. Because this construction is also rainbow triangle-free, we have $GR_k(C_{2n+1}) \geq n \cdot 2^k + 1$ for all $k \geq 1$. Therefore, the remainder of the proof will show that $GR_k(C_{2n+1}) \leq n \cdot 2^k + 1$ for all $k \geq 1$.

First, note that the case $k=1$ is trivial. Combining the result $R(C_{2n+1}, C_{2n+1}) = 4n+1$ for all $n \geq 2$ [39, 76] with Fact 1.3.11, we may assume that $k \geq 3$. Let $G := K_{n\cdot 2^k+1}$ and let c be any Gallai k-coloring of G.

Suppose that G does not contain any monochromatic C_{2n+1} under c, so that the coloring c is bad. Among all complete graphs on $n \cdot 2^k + 1$ vertices with a bad Gallai k-coloring, we choose G with k minimum; that is, G is the minimum-order counterexample to the desired result. We next show such a graph G cannot exist through a series of claims, therefore concluding that G must contain a monochromatic copy of C_{2n+1} under the coloring c.

The first claim asserts that provided enough vertices are mc-complete to certain vertex sets, the order of these sets can be controlled.

Claim 3.2.1 ([9, 10]) *Let $W \subseteq V(G)$ and let $\ell \geq 3$ be an integer. Let $x_1, \ldots, x_\ell \in V(G) \setminus W$ such that $\{x_1, \ldots, x_\ell\}$ is mc-complete, say blue-complete, to W under c. Let $q \in \{0, 1, \ldots, k-1\}$ be the number of colors, other than blue, missing on $G[W]$ under c.*

(i) If $\ell \geq n$, then $|W| \leq n \cdot 2^{k-1-q}$.

(ii) If $\ell = n-1$, then $|W| \leq n \cdot 2^{k-1-q} + 2$.

(iii) If $\ell = n-2$, then $n = 5$ and $|W| \leq 8 \cdot 2^{k-1-q} - 1$.

Proof. If $|W| < \max\{2n+1-\ell, n+1\}$, then the above statements hold trivially, so we may assume that $|W| \geq \max\{2n+1-\ell, n+1\}$. We may further assume that $G[W]$ contains at least one blue edge, otherwise by the minimality of k, $|W| \leq n \cdot 2^{k-1-q}$, giving the result. Note that $q \leq k-1$. If $q = k-1$, then all the edges of $G[W]$ are colored only blue. Since $\{x_1, \ldots, x_\ell\}$ is blue-complete to W and $|W| \geq \max\{2n+1-\ell, n+1\}$, we see that $G[W \cup \{x_1, \ldots, x_\ell\}]$ contains a blue C_{2n+1}, a contradiction. Thus $q \leq k-2$. Since $|W| \geq n+1$ and $G[W]$ contains at least one blue edge, by Lemma 3.1.1, $\ell \leq n-1$. Let W^* be a minimal set of vertices in W such that $G[W \setminus W^*]$ has no blue edges. By minimality of k, $|W \setminus W^*| \leq n \cdot 2^{k-1-q}$. Our strategy now is to examine the possible longest blue paths that can occur in $G[W]$.

We now consider the case when $\ell = n-1$. As there are exactly three possible ways to create longest blue paths using three blue edges, define $\mathcal{F} := \{3P_2, P_3 \cup P_2, P_4\}$. Given $F \in \mathcal{F}$, enumerate its vertices with $v_1, v_2, \ldots, v_j$, where $4 \leq j \leq 6$, and enumerate the remaining vertices of W with $v_{j+1}, v_{j+2}, \ldots, v_{|W|}$, noting that $|W| \geq 2n+1-\ell = n+2$. If $F \subseteq G_b[W]$ for some $F \in \mathcal{F}$, we obtain a blue C_{2n+1} in one of the following ways.

$$C_{2n+1} = \begin{cases} x_1P_2x_2P_2x_3P_2x_4v_7 \cdots x_{n-1}v_{n+2}x_1, & \text{if } F = 3P_2 \\ x_1P_3x_2P_2x_3v_6 \cdots x_{n-1}v_{n+2}x_1, & \text{if } F = P_3 \cup P_2 \\ x_1P_4x_2v_5 \cdots x_{n-1}v_{n+2}x_1, & \text{if } F = P_4 \end{cases}$$

a contradiction. Thus $|W^*| \leq 2$, and so $|W| \leq n \cdot 2^{k-1-q} + 2$. This establishes (ii).

Finally, let $\ell = n-2$. Since $3 \leq \ell$, then $n = 5$ and thus $\ell = 3$. Note that $|W| \geq 2n+1-\ell \geq 8$. Let P be a longest blue path in $G[W]$ with vertices $v_1, \ldots, v_{|P|}$ in order. Since $\{x_1, x_2, x_3\}$ is blue-complete to W, we see that $|P| \leq 5$, else we obtain a blue C_{11} with vertices $x_1, v_1, \ldots, v_6, x_2, v_7, x_3, v_8$ in order, where $v_7, v_8 \in W \setminus \{v_1, \ldots, v_6\}$, a contradiction. If $|W^*| \leq 4$, then

$$|W| = |W \setminus W^*| + |W^*| \leq n \cdot 2^{k-1-q} + 4 < 8 \cdot 2^{k-1-q} - 1,$$

because $q \leq k-2$ and $k \geq 3$. Thus we may assume that $|W^*| \geq 5$. By the choice of W^*, we see that $|P| \in \{2, 3\}$, else we obtain a blue C_{11}. Furthermore, if $|P| = 3$, then $G[W \setminus V(P)]$ has no blue path on three vertices. Thus all the blue edges in $G[W \setminus V(P)]$ induce a blue matching. Let $m := |W^* \setminus V(P)|$ and let $u_2w_2, \ldots, u_{m+1}w_{m+1}$ be all the blue edges in $G[W \setminus V(P)]$, where $u_2, \ldots, u_{m+1}, w_2, \ldots, w_{m+1}$ are all distinct. By the choice of W^*, we may assume that $u_2, \ldots, u_{m+1} \in W^*$. Let $u_1 = v_1$ and $w_1 = v_2$, $A := W \setminus (V(P) \cup$

$\{u_2, \ldots, u_{m+1}, w_2, \ldots, w_{m+1}\}$), and

$$B := \begin{cases} \{u_1, u_2, \ldots, u_{m+1}\}, & \text{if } |A| \leq 1 \\ \{u_1, u_2, \ldots, u_{m+1}\} \cup \{a_1, a_2\}, & \text{if } |A| \geq 2 \end{cases}$$

where $a_1, a_2 \in A$ with $a_1 \neq a_2$. We claim that $|B| \leq 3 \cdot 2^{k-1-q}$. Suppose $|B| \geq 3 \cdot 2^{k-1-q} + 1$. By the main result of [13], $G[B]$ has a monochromatic, say green, C_7. Then $|V(C_7) \cap \{u_1, u_2, \ldots, u_{m+1}\}| \geq 5$ and so $C_7 \setminus \{a_1, a_2\}$ has a matching of size two. We may assume that $u_2u_3, u_4u_5 \in E(C_7)$. Since G has no rainbow triangles under the coloring c, we see that for any $i \in \{2, 4\}$, $\{u_i, w_i\}$ is green-complete to $\{u_{i+1}, w_{i+1}\}$. Thus we obtain a green C_{11} from the C_7 by replacing the edge u_2u_3 with the path $u_2w_3w_2u_3$ and edge u_4u_5 with the path $u_4w_5w_4u_5$, a contradiction (see Figure 3.1). Thus $|B| \leq 3 \cdot 2^{k-1-q}$, as claimed.

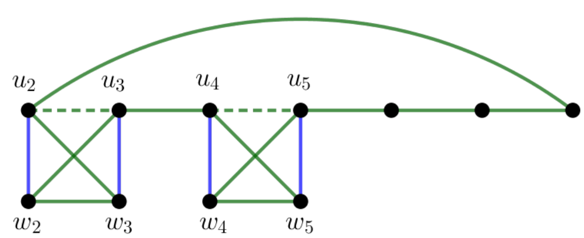


Figure 3.1: An example of a green C_{11} arising from a green C_7

When $|A| \leq 1$, we have $|W| = |A| + 2|B| + |V(P) \setminus \{v_1, v_2\}| \leq 1 + 6 \cdot 2^{k-1-q} + 1 < 8 \cdot 2^{k-1-q} - 1$ because $q \leq k-2$ and $k \geq 3$. When $|A| \geq 2$, since $G[A \cup \{w_1, w_2, \ldots, w_{m+1}\}]$ has no blue edges, by minimality of k, $|A \cup \{w_1, w_2, \ldots, w_{m+1}\}| \leq 5 \cdot 2^{k-1-q}$. Hence,

$$\begin{aligned}|W| &= |A \cup \{w_1, w_2, \ldots, w_{m+1}\}| + |B \setminus \{a_1, a_2\}| + |V(P) \setminus \{v_1, v_2\}| \\ &\leq 5 \cdot 2^{k-1-q} + (3 \cdot 2^{k-1-q} - 2) + 1 \\ &= 8 \cdot 2^{k-1-q} - 1.\end{aligned}$$

This completes the proof of Claim 3.2.1. ■

As it turns out, although we are guaranteed a Gallai partition when c is bad, we have no control over the partition itself. In particular, as mentioned in Section 1.3.2, the order of the parts cannot be controlled. Parts that are too small are difficult to deal with, so we put them aside for later use. We formally define this process below. An illustration of this can be found in Figure 3.2.

Let $X_1, \ldots, X_m$ be a disjoint subsets of $V(G)$ such that m is maximum and for all $j \in [m]$, one of the following holds.

(a) $1 \leq |X_j| \leq 2$, and X_j is mc-complete to $V(G) \setminus \bigcup_{i \in [j]} X_i$ under c, or

(b) $3 \leq |X_j| \leq 4$, and X_j can be partitioned into two non-empty sets X_{j_1} and X_{j_2}, where $j_1, j_2 \in [k]$ are two distinct colors, such that for each $t \in \{1, 2\}$, $1 \leq |X_{j_t}| \leq 2$, X_{j_t} is j_t-complete to $V(G) \setminus \bigcup_{i \in [j]} X_i$ but not j_t-complete to $X_{j_{3-t}}$, and all the edges between X_{j_1} and X_{j_2} in G are colored using only the colors j_1 and j_2.

With the above in mind, define the set $X := \bigcup_{j \in [m]} X_j$. We point out that such a sequence $X_1, \ldots, X_m$ may not exist. For each $x \in X$, let $c(x)$ be the unique color on the edges between x and $V(G) \setminus X$ under c. For all $i \in [k]$, let $X_i^* := \{x \in X : c(x) = \text{color } i\}$. Then $X = \bigcup_{i \in [k]} X_i^*$. Note that X_i^* is possibly empty for all $i \in [k]$. In line with our notation thus far, we write X_b^* (resp., X_r^*) to denote X_i^* when $i = $ blue (resp., $i = $ red).

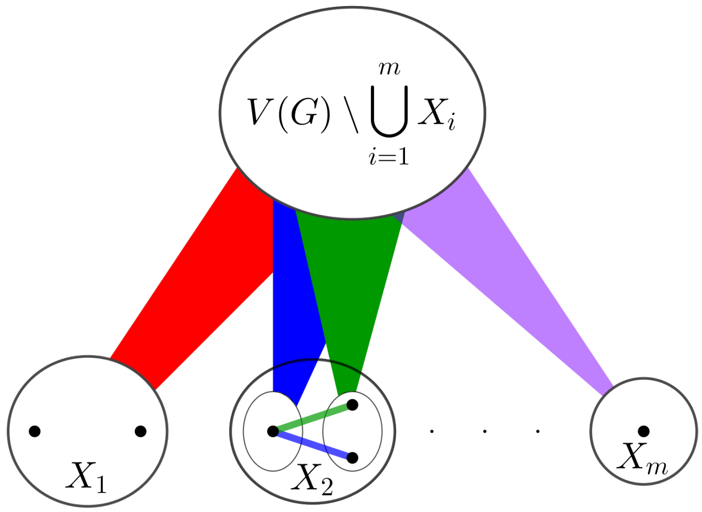


Figure 3.2: Constructing the set X

Claim 3.2.2 ([9, 10]) *For all $i \in [k]$, $|X_i^*| \leq 2$.*

Proof. Suppose the statement is false. Then $m \geq 2$. When choosing $X_1, X_2, \ldots, X_m$, let $j \in [m-1]$ be the largest index such that $|X_p^* \cap (X_1 \cup X_2 \cup \cdots \cup X_j)| \leq 2$ for all $p \in [k]$. Then $3 \leq |X_i^* \cap (X_1 \cup X_2 \cup \cdots \cup X_j \cup X_{j+1})| \leq 4$ for some color $i \in [k]$ by the choice of j, where the indices i and and j exist from our assumption that the statement is false. Let $A := X_1 \cup X_2 \cup \cdots \cup X_j \cup X_{j+1}$. By the choice of $X_1, X_2, \ldots, X_m$, there are at most two colors $i \in [k]$ such that $3 \leq |X_i^* \cap A| \leq 4$. We may assume that such a color i is either blue or red. Let $A_b := \{x \in A : c(x) = \text{blue}\}$ and $A_r := \{x \in A : c(x) = \text{red}\}$. It suffices to consider the worst case, namely when $3 \leq |A_b| \leq 4$ and $3 \leq |A_r| \leq 4$. For any color $p \in [k]$ other than red and blue, $|X_p^* \cap A| \leq 2$. Then by the choice of j, $|A \setminus (A_b \cup A_r)| \leq 2(k-2)$. We may assume that $|A_b| \geq |A_r|$. Suppose $|A_b| \geq n-1$. By Claim 3.2.1(ii) applied to any

$n-1$ vertices in A_b and $V(G)\setminus A$, then $|V(G)\setminus A| \le n\cdot 2^{k-1}+2$. Since $|A_b|\le 4\le n$,

$$|G| = |A\setminus(A_b\cup A_r)| + |A_b| + |A_r| + |V(G)\setminus A| \le 2(k-2)+n+n+(n\cdot 2^{k-1}+2) < n\cdot 2^k+1$$

for all $k\ge 3$ and $n\in\{4,5\}$, a contradiction. Finally, if $3\le |A_b|\le n-2$, then $|A_b|=3$ and $n=5$. By Claim 3.2.1(iii) applied to A_b and $V(G)\setminus A$, we see that $|V(G)\setminus A|\le 8\cdot 2^{k-1}-1$. Thus,

$$|G| = |A\setminus(A_b\cup A_r)| + |A_b| + |A_r| + |V(G)\setminus A| \le 2(k-2)+3+3+(8\cdot 2^{k-1}-1) < 5\cdot 2^k+1$$

for all $k\ge 3$, a contradiction. ■

It immediately follows that $|X|\le 2k$ from Claim 3.2.2. We now define a useful partition of X. Let $X'\subseteq X$ be such that for all $i\in[k]$, $|X'\cap X_i^*|=1$ when $X_i^*\ne\emptyset$. Let $X'':=X\setminus X'$.

Now, consider a Gallai partition $A_1,\ldots,A_p$ of $G\setminus X$ with $p\ge 2$. We may assume that $1\le |A_1|\le\cdots\le|A_s|<3\le|A_{s+1}|\le\cdots\le|A_p|$, where $0\le s\le p$. Let $\mathcal{R}$ be the reduced graph of $G\setminus X$ with vertices $a_1,a_2,\ldots,a_p$, where $a_i\in A_i$ for all $i\in[p]$. By Theorem 1.3.13, we may assume that the edges of $\mathcal{R}$ are colored red and blue. As pointed out in Section 1.3.2, any monochromatic C_{2n+1} in $\mathcal{R}$ would yield a monochromatic C_{2n+1} in G, so $\mathcal{R}$ has neither a red nor a blue C_{2n+1}. By Theorem 1.3.2, $p\le 4n$. Then $|A_p|\ge 2$ because $|G\setminus X|\ge n\cdot 2^k+1-2k\ge 8n-5$, and further, if $|A_p|=2$, then $k=3$. Thus

$|A_{p-4n+8}| = 2$, otherwise

$$\begin{aligned}
|G| &= \sum_{i=0}^{4n-7} |A_{p-i}| + \sum_{i=4n-8}^{p-1} |A_{p-i}| + |X| \\
&\leq 2(4n-8) + [p-(4n-8)] + 6 \\
&\leq 8n-2 \\
&< n \cdot 2^3 + 1,
\end{aligned}$$

a contradiction. By Theorem 1.3.2, we have $R(C_{2n-3}, C_{2n-3}) = 4n-7$. Thus $\mathcal{R}[\{a_{p-4n+8}, a_{p-4n+9}, \ldots, a_p\}]$ has a monochromatic, say blue, C_{2n-3}, and so $G[A_{p-4n+8} \cup A_{p-4n+9} \cup \cdots \cup A_p]$ has a blue C_{2n+1}, a contradiction. Therefore we conclude $|A_p| \geq 3$, giving $p - s \geq 1$. Let

$$\begin{aligned}
B &:= \{a_i \in \{a_1, \ldots, a_{p-1}\} \mid a_i a_p \text{ is colored blue in } \mathcal{R}\} \\
R &:= \{a_j \in \{a_1, \ldots, a_{p-1}\} \mid a_j a_p \text{ is colored red in } \mathcal{R}\}
\end{aligned}$$

Then $|B| + |R| = p - 1$. Let $B_G := \bigcup_{a_i \in B} A_i$ and $R_G := \bigcup_{a_j \in R} A_j$. We illustrate the above ideas in Figure 3.3.

Claim 3.2.3 ([9, 10]) *If $|A_p| \geq n$ and $|B| \geq 3$ (resp. $|R| \geq 3$), then $|B_G| \leq 2n$ (resp. $|R_G| \leq 2n$).*

Proof. Suppose $|A_p| \geq n$ and $|B| \geq 3$ but $|B_G| \geq 2n+1$. By Claim 3.1.1, $G[B_G]$ has no blue edges and no vertex in X is blue-complete to $V(G) \setminus X$. Thus all the edges of $\mathcal{R}[B]$ are colored red in $\mathcal{R}$. Let $q := |B|$ and let $B := \{a_{i_1}, a_{i_2}, \ldots, a_{i_q}\}$ with $|A_{i_1}| \geq |A_{i_2}| \geq \cdots \geq |A_{i_q}|$. Then $G[B_G] \setminus \bigcup_{j=1}^{q} E(G[A_{i_j}])$ is a complete multipartite graph with at least three parts. If $|A_{i_1}| \leq n$, then by Lemma 3.1.2 applied to $G[B_G] \setminus \bigcup_{j=1}^{q} E(G[A_{i_j}])$, $G[B_G]$ has a red C_{2n+1}, a contradiction. Thus $|A_{i_1}| \geq n+1$.

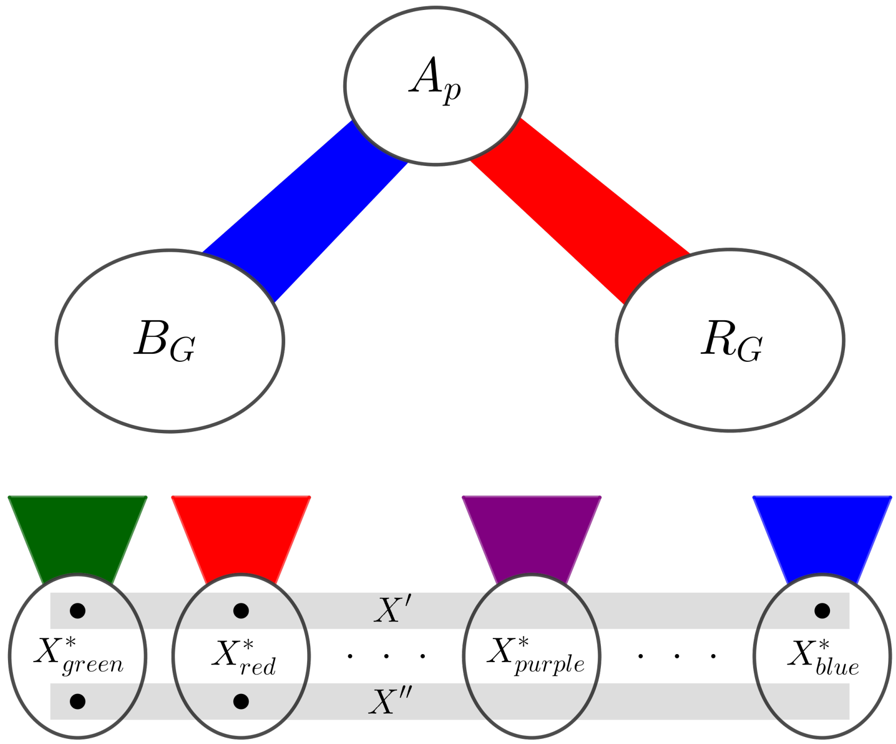


Figure 3.3: Gallai partition of $G \setminus X$ and a partition of X

Let $Q_b := \{v \in R_G : v \text{ is blue-complete to } A_{i_1}\}$, and $Q_r := \{v \in R_G : v \text{ is red-complete to } A_{i_1}\}$. Then $Q_b \cup Q_r = R_G$. Let $Q := (B_G \setminus A_{i_1}) \cup Q_r \cup X^*_r$. Then Q is red-complete to A_{i_1} and $G[Q]$ must contain red edges, because $|B| \geq 3$ and all the edges of $\mathcal{R}[B]$ are colored red. By Claim 3.1.1 applied to A_{i_1} and Q, $|Q| \leq n$. Note that $|A_p \cup Q_b| \geq |A_p| \geq |A_{i_1}| \geq n+1$ and $A_p \cup Q_b$ is blue-complete to A_{i_1}. By Claim 3.1.1 applied to A_{i_1} and $A_p \cup Q_b$, $G[A_p \cup Q_b]$ has no blue edges. Since no vertex in X is blue-complete to $V(G) \setminus X$, we see that $G[A_p \cup Q_b \cup (X' \setminus X^*_r)]$ has no blue edges. By minimality of k, $|A_p \cup Q_b \cup (X' \setminus X^*_r)| \leq n \cdot 2^{k-1}$. Suppose first that $Q_r \cup X^*_r = \emptyset$. Then $Q_b = R_G$ and $G[B_G \cup X'']$ has no blue edges. By minimality of k,

$|B_G \cup X''| \leq n \cdot 2^{k-1}$. But then

$$|G| = |B_G \cup X''| + |A_p \cup Q_b \cup X'| \leq n \cdot 2^{k-1} + n \cdot 2^{k-1} < n \cdot 2^k + 1,$$

a contradiction. Thus $Q_r \cup X_r^* \neq \emptyset$. Since $|B| \geq 3$, we see that $|B_G \setminus A_{i_1}| \geq 2$. Thus $n \geq |Q| \geq 3$.

Note that $G[A_{i_1}]$ has no blue edges and $|X'' \setminus X_r^*| \leq k-2$. By Claim 3.2.1 applied to Q and A_{i_1}, we see that

$$|A_{i_1}| \leq \begin{cases} n \cdot 2^{k-2} & \text{if } |Q| = n \\ n \cdot 2^{k-2} + 2 & \text{if } |Q| = n-1 \\ 8 \cdot 2^{k-2} - 1 & \text{if } |Q| = n-2 \text{ and } n = 5. \end{cases}$$

But then

$$\begin{aligned} |G| &= |Q| + |A_{i_1}| + |A_p \cup Q_b \cup (X' \setminus X_r^*)| + |X'' \setminus X_r^*| \\ &\leq \begin{cases} n + n \cdot 2^{k-2} + n \cdot 2^{k-1} + (k-2) & \text{if } |Q| = n \\ (n-1) + (n \cdot 2^{k-2} + 2) + n \cdot 2^{k-1} + (k-2) & \text{if } |Q| = n-1 \\ (n-2) + (8 \cdot 2^{k-2} - 1) + n \cdot 2^{k-1} + (k-2) & \text{if } |Q| = n-2 \text{ and } n = 5 \end{cases} \\ &< n \cdot 2^k + 1 \end{aligned}$$

for all $k \geq 3$, a contradiction. This proves that if $|A_p| \geq n$ and $|B| \geq 3$, then $|B_G| \leq 2n$. Similarly, one can prove that if $|A_p| \geq n$ and $|R| \geq 3$, then $|R_G| \leq 2n$. ■

Claim 3.2.4 $p \leq 2n - 1$.

Proof. Suppose $p \geq 2n$. Then $|B| + |R| = p - 1 \geq 2n - 1$. We claim that $|A_p| \leq n - 1$. Suppose $|A_p| \geq n$. We may assume that $|B| \geq |R|$. Then $|B_G| \geq |B| \geq n > 3$. By Claim 3.2.3, $|B_G| \leq 2n$. If $|R_G| \geq n + 1$, then applying Lemma 3.1.1 to A_p and R_G, $G[R_G]$ has no red edges, and $X_r^* = \emptyset$. Then $|X''| \leq k - 1$ and $G[R_G \cup X']$ has no red edges so that by minimality of k, $|R_G \cup X'| \leq n \cdot 2^{k-1}$. Then

$$|A_p| = |G| - |B_G| - |R_G \cup X'| - |X''| \geq n \cdot 2^k + 1 - 2n - n \cdot 2^{k-1} - (k - 1) \geq 2n - 1,$$

for all $k \geq 3$. By Lemma 3.1.1 applied to A_p and B_G, $G[A_p]$ has no blue edges and no vertex in X is blue-complete to $V(G) \setminus X$. Thus $G[A_p \cup X'']$ has neither red nor blue edges, and so $|A_p \cup X''| \leq n \cdot 2^{k-2}$ by the choice of k. But then

$$|B_G| = |G| - |R_G \cup X'| - |A_p \cup X''| \geq n \cdot 2^k + 1 - n \cdot 2^{k-1} - n \cdot 2^{k-2} \geq 2n + 1,$$

for all $k \geq 3$, contrary to Claim 3.2.3. This proves that $|R_G| \leq n$. Then

$$|A_p \cup X'| = |G| - |B_G| - |R_G| - |X''| \geq (n \cdot 2^k + 1) - 2n - n - k > n \cdot 2^{k-1} + 1.$$

By minimality of k, $G[A_p \cup X']$ must have blue edges. Since $|A_p| \geq n$ and $|B_G| \geq n$, by Lemma 3.1.1 applied to A_p and B_G, $|A_p| = n$ and $X_b^* = \emptyset$. Thus $|X| \leq 2(k - 1)$. But then

$$|G| = |B_G| + |R_G| + |A_p| + |X| \leq 2n + n + n + 2(k - 1) < n \cdot 2^k + 1,$$

for all $k \geq 3$, a contradiction. This proves that $|A_p| \leq n - 1$, as claimed.

Since $|A_p| \geq 3$, we have $3 \leq |A_p| \leq n - 1$. Then $k = 3$ because $n \in \{4, 5\}$ and $|G| = n \cdot 2^k + 1$, and so in particular, $|G| = 8n + 1$ and $|X| \le 6$. Therefore,

$$|B_G| + |R_G| = |G| - |A_p| - |X| \ge (8n + 1) - (n - 1) - 6 = 7n - 4.$$

We may thus assume that $|B_G| \ge 2n + 3$. We next prove that $|A_p| \le n - 2$. Suppose $|A_p| = n - 1$. Let $B^* \subseteq B_G$ with $|B^*|$ minimal such that $G[B_G \setminus B^*]$ has no blue edges. By the proof of Claim 3.2.1(ii), $|B^*| \le 2$. Then $|B_G \setminus B^*| \ge 2n+1$, and so $|B \setminus B^*| \ge 3$ because $|A_i| \le n - 1$ for all $i \in [p]$. By the choice of B^*, all the edges in $\mathcal{R}[B \setminus B^*]$ are colored red. But then by Lemma 3.1.2, $G[B_G \setminus B^*]$ has a red C_{2n+1}, a contradiction.

From the above argument, $3 \le |A_p| \le n-2$, and thus $|A_p| = 3$, $n = 5$, $|G| = 41$, and $p \le 20$. If $|A_{p-7}| = 3$ or $|A_{p-12}| \ge 2$, then $\mathcal{R}[\{a_{p-8}, a_{p-7}, \ldots, a_p\}]$ has a monochromatic C_5, or $\mathcal{R}[\{a_{p-12}, a_{p-11}, \ldots, a_p\}]$ has a monochromatic C_7 because $R(C_5, C_5) = 9$ and $R(C_7, C_7) = 13$. In either case, G has a monochromatic C_{11}, a contradiction. Thus $|A_{p-7}| \le 2$ and $|A_{p-12}| \le 1$. Then $|A_{p-7}| = 2$, otherwise $|G| \le 7 \cdot 3 + 13 \cdot 1 + 6 < 41$, a contradiction. Since $R(C_6, C_6) = 8$ (see Table 1.3) we see that $\mathcal{R}[\{a_{p-7}, a_{p-6}, \ldots, a_p\}]$ has a monochromatic, say blue, C_6, and so $G \setminus X$ has a blue C_{10}. Thus $X_b^* = \emptyset$, so $|X| \le 2(k - 1) = 4$. Furthermore, if $|A_{p-8}| = 2$, then $|A_{p-4}| = 2$, else $\mathcal{R}[\{a_{p-8}, a_{p-7}, \ldots, a_p\}]$ has a monochromatic C_5, and so G has a monochromatic C_{11}, a contradiction. But then

$$|G| = \sum_{i=0}^{p-1} |A_{p-i}| + |X| \le \begin{cases} [4 \cdot 3 + 8 \cdot 2 + (p - 12) \cdot 1] + 4 \le 40, & \text{if } |A_{p-8}| = 2 \\ [7 \cdot 3 + 2 + (p - 8) \cdot 1] + 4 \le 39, & \text{if } |A_{p-8}| \le 1. \end{cases}$$

In both cases, $|G| < 41$, a contradiction. ■

Claim 3.2.5 ([9, 10]) $|A_p| \ge n + 1$.

Proof. For a contradiction, suppose $|A_p| \leq n$. Then $|A_{p-1}| \leq n$. By Claim 3.2.4, $p \leq 2n-1$. We may assume that a_pa_{p-1} is colored blue in $\mathcal{R}$. Then $|A_p \cup A_{p-1} \cup X_b^*| \leq 2n$, otherwise $|A_p| = n$ and $|A_{p-1} \cup X_b^*| \geq n+1$ so that by Lemma 3.1.1 we create a blue C_{2n+1}, a contradiction. If $|A_{p-4}| \geq n-1$, then $\mathcal{R}[\{a_{p-4}, a_{p-3}, \ldots, a_p\}]$ has a monochromatic C_3 or C_5, and so G contains a monochromatic C_{2n+1}, a contradiction. Thus $|A_{p-4}| \leq n-2$. But then

$$\begin{aligned}|G| &= |A_p \cup A_{p-1} \cup X_b^*| + (|A_{p-2}| + |A_{p-3}|) + \sum_{i=4}^{p-1} |A_{p-i}| + |X \setminus X_b^*| \\ &\leq 2n + 2n + (p-4)(n-2) + 2(k-1) \\ &\leq 4n + (2n-5)(n-2) + 2k - 2 \\ &< n \cdot 2^k + 1\end{aligned}$$

for all $n \in \{4, 5\}$ and $k \geq 3$, a contradiction. ■

Let us now introduce some notation which we will employ for the rest of the proof.

Definition 3.2.6 $B_G^* := B_G \cup X_b^*$ and $R_G^* := R_G \cup X_r^*$.

Claim 3.2.7 ([9, 10]) $2 \leq p - s \leq 3n - 7$.

Proof. To see why the upper bound is true, suppose that $p - s \geq 3n - 6$. Then $\mathcal{R}[\{a_{p-3n+7}, a_{p-3n+8}, \ldots, a_p\}]$ has a monochromatic C_{2n-5} because $R(C_{2n-5}, C_{2n-5}) = 3n - 6$ when $n \in \{4, 5\}$. But then G contains a monochromatic C_{2n+1}, giving the desired contradiction.

To see why the lower bound holds, now suppose $p - s \leq 1$. As noted earlier, $p - s \geq 1$. Therefore, $p - s = 1$, and so $|A_i| \leq 2$ for all $i \in [p-1]$. By Claim 3.2.4, $p \leq 2n - 1$. Then $|B_G \cup R_G| \leq 2(p-1)$ and so $|B_G^* \cup R_G^*| \leq 2(p-1) + 2 + 2 = 2(p+1) \leq 4n$. We may assume that $|B_G^*| \geq |R_G^*|$. Suppose first that $|R_G^*| \geq n$. Then $|B_G^*| \geq n$. By Claim 3.2.5 $|A_p| \geq n+1$, so by Lemma 3.1.1, $G[A_p]$ has neither blue nor red edges. Then $|A_p| \leq n \cdot 2^{k-2}$ from the minimality of k. However,

$$|G| = |B_G^*| + |R_G^*| + |A_p| + |X \setminus (B_G^* \cup R_G^*)| \leq 4n + n \cdot 2^{k-2} + 2(k-2) < n \cdot 2^k + 1$$

for all $k \geq 3$, a contradiction. Thus $|R_G^*| \leq n-1$. If $|B| \geq n+1$, then $|B_G| \leq 2n$ by Claim 3.2.3; otherwise, $|B| \leq n$. In any case, $|B_G^*| \leq 2n+2$. If $|B_G^*| \geq n-1$, then applying Claim 3.2.1(i,ii) to B_G^* and A_p implies that

$$|B_G^*| + |A_p| \leq \begin{cases} (n-1) + (n \cdot 2^{k-1} + 2), & \text{if } |B_G^*| = n-1 \\ (2n+2) + n \cdot 2^{k-1}, & \text{if } |B_G^*| \geq n. \end{cases}$$

Either way, $|B_G^*| + |A_p| \leq 2n + n \cdot 2^{k-1} + 2$. But then

$$|G| = |R_G^*| + (|B_G^*| + |A_p|) + |X \setminus (B_G^* \cup R_G^*)| \leq (n-1) + (2n + n \cdot 2^{k-1} + 2) + 2(k-2) < n \cdot 2^k + 1$$

for all $k \geq 3$ and $n \in \{4, 5\}$, a contradiction. Thus $|R_G^*| \leq |B_G^*| \leq n-2$. If $|B_G^*| = 3$, then $n = 5$. By Claim 3.2.1(iii) applied to B_G^* and A_p, $|A_p| \leq 8 \cdot 2^{k-1} - 1$. But then,

$$|G| = |B_G^*| + |R_G^*| + |A_p| + |X \setminus (B_G^* \cup R_G^*)| \leq 3 + 3 + (8 \cdot 2^{k-1} - 1) + 2(k-2) < 5 \cdot 2^k + 1$$

for all $k \geq 3$, a contradiction. Thus $|R_G^*| \leq |B_G^*| \leq 2$. Note that $B \neq \emptyset$ or $R \neq \emptyset$ because $p \geq 2$. The maximality of m when choosing $X_1, \ldots, X_m$ by condition (a) implies $B^* \neq \emptyset$, $R^* \neq \emptyset$, and B_G^* is neither blue- nor red-complete to R_G^* in G. On the other hand, the maximality of m again implies by condition (b) that $B_G^* = \emptyset$ and $R_G^* = \emptyset$, contrary to $p \geq 2$, therefore yielding the desired result. ■

Claim 3.2.8 ([9, 10]) $|A_{p-2}| \leq n-1$.

Proof. We will prove this claim by contradiction. Suppose $|A_{p-2}| \geq n$, so that $n \leq |A_{p-2}| \leq |A_{p-1}| \leq |A_p|$. As is our custom, we may assume that $\mathcal{R}$ contains blue and red edges. Now, $\mathcal{R}[\{a_{p-2}, a_{p-1}, a_p\}]$ is not a monochromatic triangle in $\mathcal{R}$, for otherwise we find a monochromatic C_{2n+1}. Without loss of generality, let B_1, B_2, B_3 be a permutation of A_{p-2}, A_{p-1}, A_p such that B_2 is blue-complete, to $B_1 \cup B_3$ in G. Then B_1 must be red-complete to B_3 in G. We may assume that $|B_1| \geq |B_3|$. By Lemma 3.1.1, $X_b^* = \emptyset$ and $X_r^* = \emptyset$. Let $A := V(G) \setminus (B_1 \cup B_2 \cup B_3 \cup X)$. Then no vertex in A is red-complete to $B_1 \cup B_3$ in G by Lemma 3.1.1, and no vertex in A is blue-complete to $B_1 \cup B_2$ or $B_2 \cup B_3$ in G. Together, these conditions imply that A is red-complete to B_2 in G; otherwise we find a blue C_{2n+1} by way of Lemma 3.1.1. Next, let us define the following sets:

$$B_1^* := \{b \in A \mid b \text{ is blue-complete to } B_1 \text{ only in } G\}$$

$$B_2^* := \{b \in A \mid b \text{ is blue-complete to both } B_1 \text{ and } B_3 \text{ in } G\}$$

$$B_3^* := \{b \in A \mid b \text{ is blue-complete to } B_3 \text{ only in } G\}.$$

Note that B_1^*, B_2^*, B_3^* are pairwise disjoint and possibly empty. Then $A = B_1^* \cup B_2^* \cup B_3^*$. An illustration of this entire configuration is depicted in Figure 3.4.

We now claim that $G[A]$ has no blue edges. Suppose that $G[A]$ has a blue edge, say, uv. Let $b_1, \ldots, b_{n-1} \in B_1$, $b_n, \ldots, b_{2n-2} \in B_2$, and $b_{2n-1} \in B_3$. If uv is an edge in $G[B_1^* \cup B_2^*]$, then we obtain a blue C_{2n+1} with vertices $b_1, u, v, b_2, b_n, b_{2n-1}, b_{n+1}, b_3, b_{n+2}, \ldots, b_{n-1}, b_{2n-2}$ in order, a contradiction. Similarly, uv is not an edge in $G[B_2^* \cup B_3^*]$. Thus uv must be an edge in $G[B_1^* \cup B_3^*]$ with one end in B_1^* and the other in B_3^*. We may assume that $u \in B_1^*$ and $v \in B_3^*$. Then we obtain a blue C_{2n+1} with vertices $b_1, u, v, b_{2n-1}, b_n, b_2, b_{n+1}, \ldots, b_{n-1}, b_{2n-2}$ in order, a contradiction. Therefore $G[A]$ has no blue edges, so that $|A| \leq n \cdot 2^{k-1}$ by minimality of k.

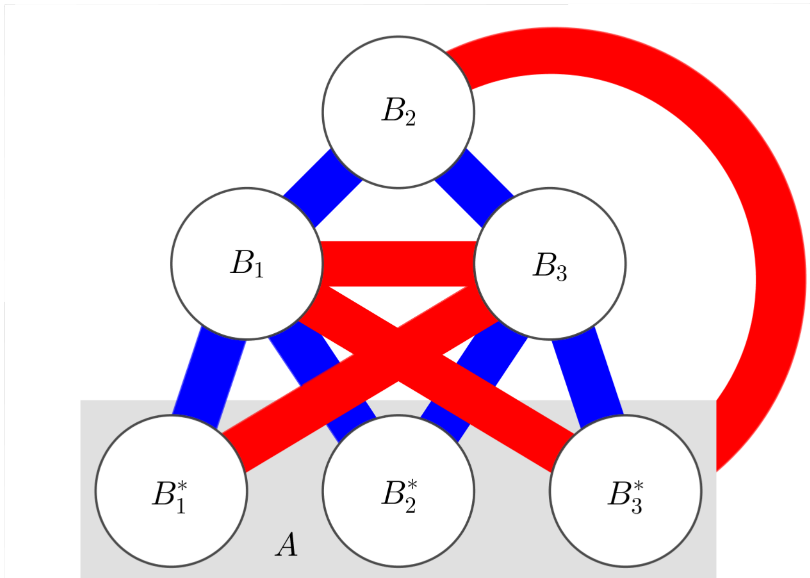


Figure 3.4: Three large parts in the Gallai partition

Next, we claim that $|B_2 \cup A \cup X'| \leq n \cdot 2^{k-1}$. If not, then $|B_2 \cup A \cup X'| \geq n \cdot 2^{k-1} + 1$ so that by minimality of k, $G[B_2 \cup A \cup X']$ must contain blue edges. Since $G[A]$ has no blue edges, A is red-complete to B_2, and $X_b^* = \emptyset$, it follows $G[B_2]$ must contain blue edges. By Lemma 3.1.1, $|B_2| = n$, and so $B_2 \neq A_p$ by Claim 3.2.5. Without loss of generality, assume $B_1 = A_p$. Then $G[B_1]$ has neither red nor blue edges by Lemma 3.1.1, and thus $G[B_1 \cup X']$ has neither red nor blue edges. By minimality of k, $|B_1 \cup X'| \leq n \cdot 2^{k-2}$ and so $|B_3 \cup X''| \leq |B_1 \cup X'| \leq n \cdot 2^{k-2}$. Note that $A = \emptyset$. If not, then for any $v \in A$, $G[B_2 \cup \{v\}]$ has blue edges and $B_2 \cup \{v\}$ is blue-complete to either B_1 or B_3, contrary to Lemma 3.1.1. But then

$$|G| = |B_1 \cup X'| + |B_2| + |B_3 \cup X''| \leq n \cdot 2^{k-2} + n + n \cdot 2^{k-2} < n \cdot 2^k + 1,$$

for all $k \geq 3$, a contradiction. This proves that $|B_2 \cup A \cup X'| \leq n \cdot 2^{k-1}$.

Now, $|B_1| \geq |B_3|$ and $|B_1|+|B_3| = |G|-|B_2 \cup A \cup X'|-|X''| \geq n \cdot 2^{k-1}+1-(k-2) \geq 2n+1$, so $|B_1| \geq n+1$. Since $|B_2| \geq n$ and $|B_3| \geq n$, by Lemma 3.1.1 $G[B_1]$ has neither red nor blue edges. Further, since $X_r^* = \emptyset$ and $X_b^* = \emptyset$, $G[B_1 \cup X'']$ has neither red nor blue edges, so that $|B_3| \leq |B_1 \cup X''| \leq n \cdot 2^{k-2}$ by minimality of k. But then

$$|G| = |B_2 \cup A \cup X'| + |B_1 \cup X''| + |B_3| \leq n \cdot 2^{k-1} + n \cdot 2^{k-2} + n \cdot 2^{k-2} = n \cdot 2^k,$$

a contradiction. ■

By Claim 3.2.7, $2 \leq p-s \leq 3n-7$ and so $|A_{p-1}| \geq 3$. We may now assume that $a_p a_{p-1}$ is colored blue in $\mathcal{R}$. Then $a_{p-1} \in B$ and so $A_{p-1} \subseteq B_G$. Thus $|B_G| \geq |A_{p-1}| \geq 3$.

Claim 3.2.9 ([9, 10]) *$|R_G^*| \leq 2n$.*

Proof. Suppose $|R_G^*| \geq 2n+1$. By Claim 3.2.5, $|A_p| \geq n+1$. Further, $G[R_G^*]$ has no red edges by Lemma 3.1.1, so $X_r^* = \emptyset$, $|R_G^*| = |R_G|$, and all the edges in $\mathcal{R}[R]$ are colored blue. Note that by Claim 3.2.3, $|R| \leq 2$. Additionally, $|A_{p-2}| \leq n-1$ by Claim 3.2.8. Since $A_{p-1} \cap R_G = \emptyset$ and $|R_G| \geq 2n+1$, then $|R| \geq 3$, a contradiction. ■

Claim 3.2.10 ([9, 10]) *$|A_{p-1}| \leq n$.*

Proof. For the sake of contradiction, suppose $|A_{p-1}| \geq n+1$. Then from our above assumption, $|B_G| \geq |A_{p-1}| \geq n+1$. Lemma 3.1.1 guarantees that neither $G[A_p]$ nor $G[B_G]$ has blue edges, and $X_b^* = \emptyset$, giving $|X| \leq 2(k-1)$. Again from the choice of k, $|B_G \cup X''| \leq n \cdot 2^{k-1}$ and $|A_p \cup X'| \leq n \cdot 2^{k-1}$.

We claim that $G[R_G]$ has blue edges. Suppose not. Then $G[A_p \cup R_G \cup X']$ has no blue edges. By the choice of k, $|A_p \cup R_G \cup X'| \leq n \cdot 2^{k-1}$. But then $|B_G \cup X''| = |G| - |A_p \cup R_G \cup X'| \geq$

$n \cdot 2^{k-1}+1$, a contradiction. Thus $G[R_G]$ has blue edges, as claimed. Combining this with Claim 3.2.9, $2 \leq |R_G| \leq |R_G^*| \leq 2n$. We complete this claim in two cases.

First, consider the case when $|R_G^*| \geq n-1$. We will show that $|A_p \cup (X' \setminus X_r^*)| + |R_G^*| \leq n \cdot 2^{k-2} + \max\{2n, k+n-1\}$ (recall by Definition 3.2.6, $R_G^* = R_G \cup X_r^*$). If $|R_G^*| \geq n$, then Lemma 3.1.1 implies $G[A_p]$ has no red edges, so $G[A_p \cup (X' \setminus X_r^*)]$ has no red edges and thus $|A_p \cup (X' \setminus X_r^*)| \leq n \cdot 2^{k-2}$ by the minimality of k. Therefore, $|A_p \cup (X' \setminus X_r^*)| + |R_G^*| \leq n \cdot 2^{k-2} + 2n$. If $|R_G^*| = n-1$, then applying Claim 3.2.1(ii) to R_G^* and A_p, $|A_p| \leq n \cdot 2^{k-2}+2$. Thus $|A_p \cup (X' \setminus X_r^*)| + |R_G^*| \leq n \cdot 2^{k-2} + 2 + (k-2) + (n-1) = n \cdot 2^{k-2} + k + n - 1$, giving the desired result. But then

$$|G| = (|A_p \cup (X' \setminus X_r^*)| + |R_G^*|) + |B_G \cup (X'' \setminus X_r^*)| \leq (n \cdot 2^{k-2} + \max\{2n, k+n-1\}) + n \cdot 2^{k-1} < n \cdot 2^k + 1,$$

for all $k \geq 3$, a contradiction.

Finally, we examine the case when $2 \leq |R_G| \leq |R_G^*| \leq n-2$. If $|R_G^*| = 3$, then $n = 5$. By applying Claim 3.2.1(iii) to R_G^* and A_p, $|A_p| \leq 8 \cdot 2^{k-2} - 1$. However,

$$|G| \leq |A_p| + |B_G \cup X''| + |R_G^*| + |X' \setminus X_r^*| \leq (8 \cdot 2^{k-2} - 1) + 5 \cdot 2^{k-1} + 3 + (k-2) < 5 \cdot 2^k + 1,$$

for all $k \geq 3$, a contradiction. Thus, because $G[R_G]$ contains a blue edge, $|R_G^*| = |R_G| = 2$, and so in particular, $X_r^* = \emptyset$ and $|X''| \leq k-2$. Let $R_G = \{a, b\}$. Then ab must be colored blue under c. Without loss of generality, suppose b is red-complete to B_G in G. Then neither $G[A_p \cup \{a\} \cup X']$ nor $G[B_G \cup \{b\} \cup X'']$ has blue edges. By minimality of k, $|A_p \cup \{a\} \cup X'| \leq n \cdot 2^{k-1}$ and $|B_G \cup \{b\} \cup X''| \leq n \cdot 2^{k-1}$. But then $|G| = |A_p \cup \{a\} \cup X'| + |B_G \cup \{b\} \cup X''| \leq n \cdot 2^{k-1} + n \cdot 2^{k-1} < n \cdot 2^k + 1$ for all $k \geq 3$, a contradiction. Thus neither a nor b is red-complete to B_G in G. Let $a', b' \in B_G$ be such that aa' and bb' are colored blue under c. Then $a' = b'$, or else we obtain a blue C_{2n+1} in G with vertices $a', a, b, b', x_1, y_1, x_2, \ldots, y_{n-2}, x_{n-1}$ in order, where $x_1, \ldots, x_{n-1} \in A_p$ and $y_1, \ldots, y_{n-2} \in B_G \setminus \{a', b'\}$, a contradiction. Thus $\{a, b\}$ is red-complete to $B_G \setminus a'$ in G. Then there exists $i \in [s]$ such that $A_i = \{a'\}$. Since $G[B_G]$ has no blue edges, we see that $\{a, b, a'\}$ must be red-complete to $B_G \setminus a'$ in G. By Claim 3.2.1(ii,iii) applied to $\{a, b, a'\}$ and $B_G \setminus a'$, we have $|B_G \setminus a'| \leq (4n-12) \cdot 2^{k-2} + (14-3n)$. But then

$$\begin{aligned}|G| &= |A_p \cup X'| + |B_G \setminus a'| + |\{a, b, a'\}| + |X''| \\ &\leq n \cdot 2^{k-1} + [(4n-12) \cdot 2^{k-2} + (14-3n)] + 3 + (k-2) \\ &< n \cdot 2^k + 1\end{aligned}$$

for all $k \geq 3$, a contradiction. Hence, $|A_{p-1}| \leq n$. ■

With these claims in mind, we are now ready to complete the proof. From Claim 3.2.9, $|R_G| \leq |R_G^*| \leq 2n$. This allows us to divide the remaining proof into two cases.

First consider the case when $|R_G| \geq n$. Recall that $|A_p| \geq n+1$ by Claim 3.2.5. By Lemma 3.1.1, $G[A_p]$ has no red edges and $X_r^* = \emptyset$, so $|X| \leq 2(k-1)$. We assert that in this case, $|B_G| \geq n$. To see why, suppose $|B_G| \leq n-1$. If $|B_G| = n-1$, then $|A_p| \leq n \cdot 2^{k-2} + 2$ by Claim 3.2.1(ii) applied to B_G and A_p. However, then

$$|G| = |A_p| + |B_G| + |R_G| + |X| \leq (n \cdot 2^{k-2} + 2) + (n-1) + 2n + 2(k-1) < n \cdot 2^k + 1$$

for all $k \geq 3$, a contradiction. Hence we consider when $3 \leq |B_G| \leq n-2$. Then $n = 5$ and $|B_G| = 3$, so by Claim 3.2.1(iii) applied to B_G and A_p, $|A_p| \leq 8 \cdot 2^{k-2} - 1$. Adding back together,

$$|G| = |A_p| + |B_G| + |R_G| + |X| \leq (8 \cdot 2^{k-2} - 1) + 3 + 10 + 2(k-1) < 5 \cdot 2^k + 1$$

for all $k \geq 3$, a contradiction, so $|B_G| \geq n$, as claimed. Therefore, $G[A_p]$ has no blue edges and $X_b^* = \emptyset$ from Lemma 3.1.1. Since $G[A_p \cup X']$ has neither red nor blue edges, and both X_r^* and X_b^* are empty, it follows that $|X''| \leq k-2$ and $|A_p \cup X'| \leq n \cdot 2^{k-2}$ by minimality of k. If $|B_G| = n$, then

$$|G| = |A_p \cup X'| + |X''| + (|B_G| + |R_G|) \leq n \cdot 2^{k-2} + (k-2) + (n+2n) < n \cdot 2^k + 1,$$

for all $k \geq 3$, a contradiction, so we conclude $|B_G| \geq n+1$. Invoking Lemma 3.1.1 again, $G[B_G]$ has no blue edges and so $G[B_G \cup X'']$ has no blue edges, giving again that $|B_G \cup X''| \leq n \cdot 2^{k-1}$ by minimality of k. But then

$$|G| = |A_p \cup X'| + |B_G \cup X''| + |R_G| \leq n \cdot 2^{k-2} + n \cdot 2^{k-1} + 2n < n \cdot 2^k + 1,$$

for all $k \geq 3$, a contradiction.

Finally, we consider the case when $|R_G| \leq n-1$. If $|B_G| \geq 2n+1$, then by Lemma 3.1.1, $G[B_G]$ has no blue edges, so all the edges in $\mathcal{R}[B]$ are colored red. However, $|A_{p-1}| \leq n$ by Claim 3.2.10, and consequently $|B| \geq 3$, contrary to Claim 3.2.3. Thus $3 \leq |A_{p-1}| \leq |B_G| \leq 2n$. As we have done above, if $|B_G| = n-1$, we apply Claim 3.2.1(ii) to B_G and A_p. If $|B_G| \geq n$, apply Claim 3.2.1(i) to B_G and A_p, and Lemma 3.1.1 to X. Putting these results together,

$$|A_p| + |B_G| + |X| \leq \begin{cases} (n \cdot 2^{k-1} + 2) + (n-1) + 2k, & \text{if } |B_G| = n-1 \\ n \cdot 2^{k-1} + 2n + 2(k-1), & \text{if } |B_G| \geq n. \end{cases}$$

Regardless, $|A_p| + |B_G| + |X| \leq n \cdot 2^{k-1} + 2n + 2k - 2$. But then

$$|G| = (|A_p| + |B_G| + |X|) + |R_G| \leq (n \cdot 2^{k-1} + 2n + 2k - 2) + (n-1) < n \cdot 2^k + 1,$$

for all $k \geq 3$, a contradiction, so it must be that $3 \leq |B_G| \leq n-2$, meaning $|B_G| = 3$ and $n = 5$. If $|R_G^*| \geq 4$ or $|B_G^*| \geq 4$, applying Claim 3.2.1(ii) to any four vertices in R_G^* or B_G^* and A_p yields $|A_p| \leq 5 \cdot 2^{k-1} + 2$. Consequently,

$$|G| = |A_p| + |B_G| + |R_G| + |X| \leq (5 \cdot 2^{k-1} + 2) + 3 + 4 + 2k < 5 \cdot 2^k + 1,$$

for all $k \geq 3$, a contradiction. Therefore $|B_G| = |B_G^*| = 3$, whence $X_b^* = \emptyset$, and $|R_G| \leq |R_G^*| \leq 3$. Thus $|X \setminus X_r^*| \leq 2(k-2)$. Moreover, $|A_p| \leq 8 \cdot 2^{k-1} - 1$ by Claim 3.2.1(iii) applied to B_G and A_p. As a result,

$$|G| = |A_p| + |B_G| + |R_G^*| + |X \setminus X_r^*| \leq (8 \cdot 2^{k-1} - 1) + 3 + 3 + 2(k-2) < 5 \cdot 2^k + 1,$$

for all $k \geq 3$, a contradiction.

This completes the proof of Theorem 1.3.26. ■

3.3 Proof of Theorem 1.3.27

Let $n \in \{6, 7\}$. It suffices to show that $GR_k(C_{2n+1}) \leq n \cdot 2^k + 1$ for all $k \geq 1$. This is trivially true for $k = 1$. By Theorem 1.3.2 and the fact that $GR(C_{2n+1}, C_{2n+1}) = R(C_{2n+1}, C_{2n+1})$, we may assume that $k \geq 3$. Let $G := K_{n \cdot 2^k + 1}$ and let $c : E(G) \to [k]$ be any Gallai coloring of G. We next show that G contains a monochromatic copy of C_{2n+1} under the coloring c.

Suppose that G does not contain any monochromatic copy of C_{2n+1} under c. Then c is bad. Among all complete graphs on $n \cdot 2^k + 1$ vertices with a bad k-edge-coloring, we again choose G to be a minimum-order counterexample with respect to k.

Claim 3.3.1 *Let $W \subseteq V(G)$ and let $\ell \geq 3$ be an integer. Let $x_1, \ldots, x_\ell \in V(G) \setminus W$ such* *that $\{x_1, \ldots, x_\ell\}$ is mc-complete, say blue-complete, to W under c. Let $q \in \{0, 1, \ldots, k-1\}$ be the number of colors, other than blue, missing on $G[W]$ under c.*

(i) If $\ell \geq n$, then $|W| \leq n \cdot 2^{k-1-q}$.

(ii) If $\ell = n-1$, then $|W| \leq n \cdot 2^{k-1-q} + 2$.

(iii) If $\ell = n-2$, then $|W| \leq (21-2n) \cdot 2^{k-1-q} + (5n-31)$

(iv) If $\ell = n-3$, then $|W| \leq 11 \cdot 2^{k-1-q} + (n-7)$

(v) If $\ell = n-4$, then $n = 7$ and $|W| \leq 13 \cdot 2^{k-1-q}$.

Put another way, Claim 3.3.1 asserts that

$$|W| \leq \begin{cases} (2n-1) \cdot 2^{k-1-q} + (n-7), & \text{if } \ell \geq 3 \\ (2n-3) \cdot 2^{k-1-q} + (n-7), & \text{if } \ell \geq 4. \end{cases}$$

Proof. Each statement (i)-(v) is trivially true if $|W| < \max\{2n+1-\ell, n+1\}$. Thus, we may assume that $|W| \geq \max\{2n+1-\ell, n+1\}$. Note that $q \leq k-1$. If $q = k-1$, then all the edges of $G[W]$ are colored only blue. Since $\{x_1, \ldots, x_\ell\}$ is blue-complete to W and $|W| \geq \max\{2n+1-\ell, n+1\}$, we see that $G[W \cup \{x_1, \ldots, x_\ell\}]$ contains a blue C_{2n+1}, a contradiction. Thus $q \leq k-2$.

First, assume $\ell \geq n$. Since $|W| \geq n+1$, by Lemma 3.1.1, $G[W]$ contains no blue edges. By minimality of k, $|W| \leq n \cdot 2^{k-1-q}$, establishing (i).

For the remainder of the proof, we may assume that $\ell \leq n-1$, and that $G[W]$ contains at least one blue edge, otherwise $|W| \leq n \cdot 2^{k-1-q}$ by minimality of k, giving the result. Let W^*

be a minimal set of vertices in W such that $G[W \setminus W^*]$ has no blue edges. By minimality of k, $|W \setminus W^*| \leq n \cdot 2^{k-1-q}$.

Let P be a longest blue path in $G[W]$ with vertices $v_1, \ldots, v_{|P|}$ in order, where $|P| \geq 2$. It can be easily checked that if $|P| \geq 2(n-\ell)+2$, or $|P| = 2(n-\ell)+1$ along with a blue edge in $G[W \setminus V(P)]$, then $G[W \cup \{x_1, \ldots, x_\ell\}]$ has a blue C_{2n+1}, a contradiction. Thus $|P| \leq 2(n-\ell)+1$. Assume $|P| = 2(n-\ell)+1$. Then $G[W \setminus \{v_2, \ldots, v_{|P|}\}]$ has no blue edges. By minimality of k, $|W| = |W \setminus \{v_2, \ldots, v_{|P|}\}| + |P \setminus \{v_1\}| \leq n \cdot 2^{k-1-q} + (|P|-1) = n \cdot 2^{k-1-q} + 2(n-\ell)$, as desired for each $\ell \in \{n-1, n-2, n-3, n-4\}$. Thus $2 \leq |P| \leq 2(n-\ell)$.

We now consider the case $\ell = n-1$. Then $|P| = 2$. If $G[W]$ contains three blue edges, say u_1w_1, u_2w_2, u_3w_3, such that $u_1, u_2, u_3, w_1, w_2, w_3$ are all distinct, then we obtain a blue C_{2n+1} with vertices

$$\begin{cases} v_1, u_1, w_1, v_2, u_2, w_2, v_3, u_3, w_3, v_4, u_4, v_5, u_5, & \text{if } n = 6 \\ v_1, u_1, w_1, v_2, u_2, w_2, v_3, u_3, w_3, v_4, u_4, v_5, u_5, v_6, u_6, & \text{if } n = 7 \end{cases}$$

in order, where $u_4, u_5, u_6 \in W \setminus \{u_1, u_2, u_3, w_1, w_2, w_3\}$, a contradiction. Thus $|W^*| \leq 2$ because $|P| = 2$. Hence, $|W| = |W \setminus W^*| + |W^*| \leq n \cdot 2^{k-1-q} + 2$. This establishes (ii).

Thus $\ell \in \{n-2, n-3, n-4\}$. Assume first that $|P| = 2$. Then all the blue edges of $G[W]$ form a matching. Let $u_1w_1, \ldots, u_mw_m$ be all the blue edges of $G[W]$. Let

$$A := \begin{cases} \{u_1, \ldots, u_m\}, & \text{if } |W| = |\{u_1, \ldots, u_m, w_1, \ldots, w_m\}| = 2m \\ \{u_1, \ldots, u_m\} \cup \{a\}, & \text{if } |W| - 2m \geq 1 \text{ and } a \in W \setminus \{u_1, \ldots, u_m, w_1, \ldots, w_m\} \\ \{u_1, \ldots, u_m\} \cup \{a_1, a_2\}, & \text{if } n = 7,\ |W| - 2m \geq 2 \text{ and } a_1, a_2 \in W \setminus \{u_1, \ldots, u_m, w_1, \ldots, w_m\}. \end{cases}$$

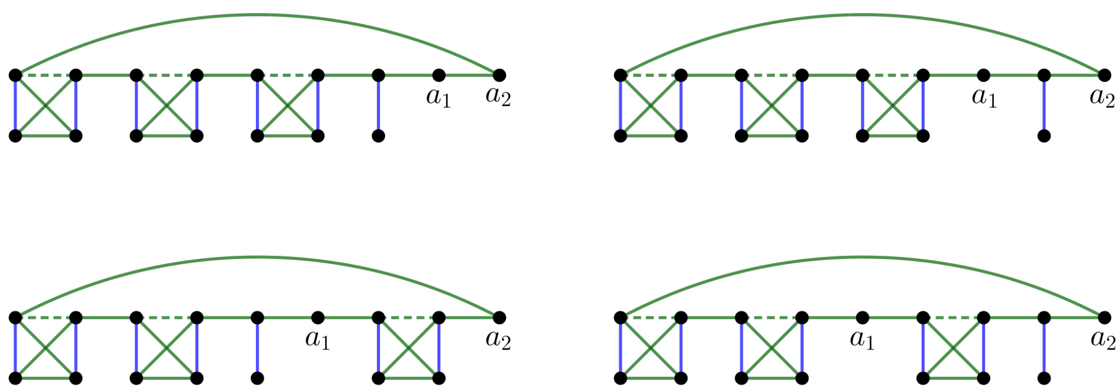


Figure 3.5: Possible ways a green C_{15} arises from a green C_9

Suppose $|A| \geq (n-3) \cdot 2^{k-1-q} + 1$. By [13] and Theorem 1.3.26, $G[A]$ has a monochromatic, say green, C_{2n-5} with $|V(C_{2n-5}) \cap \{u_1, \ldots, u_m\}| \geq 12-n$. If $n = 6$, then we may assume that $E(C_7) = \{u_1u_2, u_2u_3, \ldots, u_6u_7, u_7u_1\}$. Since G has no rainbow triangles under the coloring c, then for any $i \in \{1,3,5\}$, $\{u_i, w_i\}$ is green-complete to $\{u_{i+1}, w_{i+1}\}$. Thus we obtain a green C_{13} from the green C_7 by replacing the edge u_iu_{i+1} with the path $u_iw_{i+1}w_iu_{i+1}$ for each $i \in \{1,3,5\}$. If $n = 7$, there are four possible ways up to permutation that a_1 and a_2 can be arranged on the green C_9 (see Figure 3.5). Similar to the case for $n = 6$, we therefore obtain a green C_{15}, a contradiction. Thus, $|A| \leq (n-3) \cdot 2^{k-1-q}$. Therefore,

$$
\begin{aligned}
|W| &= |W \setminus A| + |A| \\
&\leq \begin{cases} 2[(n-3) \cdot 2^{k-1-q}], & \text{if } |W| = 2m \\ (7 \cdot 2^{k-1-q} - 1) + 4 \cdot 2^{k-1-q}, & \text{if } |W| \geq 2m+1 \\ (n \cdot 2^{k-1-q} - 2) + (n-3) \cdot 2^{k-1-q}, & \text{if } n = 7 \text{ and } |W| \geq 2m+2 \end{cases}
\end{aligned}
$$

$$= \begin{cases} (n-3)\cdot 2^{k-q}, & \text{if } |W| = 2m \\ (2n-3)\cdot 2^{k-1-q} - 1, & \text{if } |W| \geq 2m+1 \\ 11\cdot 2^{k-1-q} - 2, & \text{if } n = 7 \text{ and } |W| \geq 2m+2, \end{cases} \tag{3.1}$$

as desired for each $\ell \in \{n-2, n-3, n-4\}$. So we may assume that $3 \leq |P| \leq 2(n-\ell)$.

Next suppose $\ell = n-2$. Then $|P| \in \{3,4\}$. Thus $|W^*| \leq 4$, else we obtain a blue C_{2n+1}. Hence, $|W| = |W \setminus W^*| + |W^*| \leq n \cdot 2^{k-1-q} + 4$, establishing (iii).

By the above arguments, we may now assume that $\ell \in \{n-3, n-4\}$. Suppose $|P| = 3$. Then each component of the subgraph of $G[W]$ induced by all its blue edges is isomorphic to a K_3, a star, or a P_2. Partition W into the sets W_1, W_2 and W_3, described below.

W_1 : Select one vertex from each blue K_3

W_2 : Select one vertex from each blue K_3 not in W_1, the center vertex in each blue star, and one vertex from each blue P_2

$W_3 := W \setminus (W_1 \cup W_2)$

Then $W = W_1 \cup W_2 \cup W_3$ with $|W| = |W_1| + |W_2| + |W_3|$. This partition is illustrated in Figure 3.6.

Note $|W_3| \leq n \cdot 2^{k-1-q}$ due to the minimality of k. By an argument similar to the case $|P| = 2$, we have $|W_2| \leq (n-3)\cdot 2^{k-1-q}$. Therefore, our task is to appropriately bound W_1.

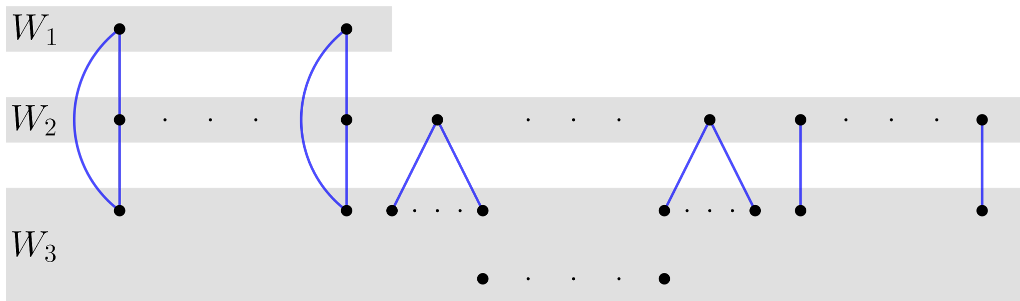


Figure 3.6: Partition of W

Define

$$A := \begin{cases} W_1, & \text{if } n = 6,\ \ell = n-3 \text{ and } |W| = 3|W_1| \\ W_1 \cup \{a\}, & \text{if } n = 6,\ \ell = n-3 \text{ and } |W| > 3|W_1| \\ W_1, & \text{if } n = 7 \text{ and } \ell \in \{n-3, n-4\}, \end{cases}$$

where $a \in W$ does not belong to a blue K_3.

We claim that $|A| \le 2 \cdot 2^{k-1-q}$. First note if $n = 7$ and $\ell = n - 3$, then $|A| \le 3$ otherwise we find a blue C_{15}, giving the result. Now, suppose $|A| \ge 2 \cdot 2^{k-1-q} + 1$. Then $G[A]$ has a monochromatic, say green, C_5 with vertices u_1, u_2, u_3, u_4, u_5 in order. We may assume that $a \notin \{u_1, u_2, u_3, u_4\}$. Enumerate the vertices of the corresponding blue K_3's in $G[W]$ as u_i, y_i, z_i for all $i \in [n-2]$. Since G has no rainbow triangles under the coloring c, then for any $i \in [n-3]$, $\{u_i, y_i, z_i\}$ is green-complete to $\{u_{i+1}, y_{i+1}, z_{i+1}\}$. Additionally, we note $\{u_5, y_5, z_5\}$ is green-complete to $\{u_1, y_1, z_1\}$ when $n = 7$. Then we obtain a green C_{2n+1} with vertices

$$\begin{cases} u_1, u_2, y_1, y_2, z_1, z_2, z_3, z_4, y_3, y_4, u_3, u_4, u_5, & \text{if } n = 6 \\ u_1, y_2, y_3, y_4, y_5, y_1, z_2, z_3, z_4, z_5, z_1, z_2, u_3, u_4, u_5, & \text{if } n = 7 \end{cases}$$

in order, a contradiction (see Figure 3.7). Thus $|A| \le 2 \cdot 2^{k-1-q}$.

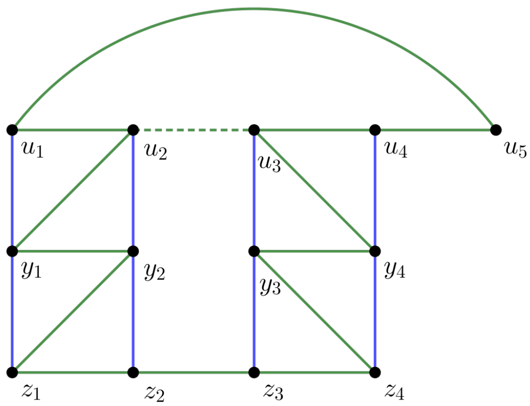


Figure 3.7: A green C_{13} arising from a green C_5

Therefore,

$$
\begin{aligned}
|W| &= |W_1| + |W_2| + |W_3| \\
&\le \begin{cases} 3(2 \cdot 2^{k-1-q}) & \text{if } n = 6,\ \ell = n-3 \text{ and } |W| = 3|W_1| \\ (2 \cdot 2^{k-1-q} - 1) + 3 \cdot 2^{k-1-q} + 6 \cdot 2^{k-1-q} & \text{if } n = 6,\ \ell = n-3 \text{ and } |W| > 3|W_1| \\ 3 + 3 \cdot 2^{k-1-q} + 6 \cdot 2^{k-1-q} & \text{if } n = 7 \text{ and } \ell = n-3 \\ 2 \cdot 2^{k-1-q} + 4 \cdot 2^{k-1-q} + 7 \cdot 2^{k-1-q} & \text{if } n = 7 \text{ and } \ell = n-4 \end{cases} \\
&= \begin{cases} 6 \cdot 2^{k-1-q} & \text{if } n = 6,\ \ell = n-3 \text{ and } |W| = 3|W_1| \\ 11 \cdot 2^{k-1-q} - 1 & \text{if } n = 6,\ \ell = n-3 \text{ and } |W| > 3|W_1| \\ 9 \cdot 2^{k-1-q} + 3 & \text{if } n = 7 \text{ and } \ell = n-3 \\ 13 \cdot 2^{k-1-q} & \text{if } n = 7 \text{ and } \ell = n-4 \end{cases}
\end{aligned}
$$

as desired. So we may assume that $4 \le |P| \le 2(n - \ell)$.

Now suppose $\ell = n-3$. Then $4 \le |P| \le 6$. Assume first that $|W^*| \le 7$. Then

$$|W| = |W \setminus W^*| + |W^*| \le n \cdot 2^{k-1-q} + 7 < 11 \cdot 2^{k-1-q} + n - 7$$

because $q \le k-2$ and $k \ge 3$. So we may assume that $|W^*| \ge 8$. Let P' be a longest blue path in $G[W \setminus V(P)]$.

Let us first handle the case when $n = 6$. Because $|W^*| \ge 8$, we have $|P| \in \{4, 5\}$, and

$$|P'| \le \begin{cases} 3, & \text{if } |P| = 4 \\ 2, & \text{if } |P| = 5. \end{cases}$$

Moreover, when $|P| = 4$, there is at most one P' such that $|P'| = 3$, otherwise we obtain a blue C_{13}. When $|P| = 4$, it suffices to consider the worst-case scenario, namely when $|P'| = 3$, with vertices y_1, y_2, y_3 in order. Define

$$A := \begin{cases} \{v_2, v_3, v_4, y_3\}, & \text{if } |P| = 4 \\ \{v_2, v_3, v_4, v_5\}, & \text{if } |P| = 5 \end{cases}$$

Then the blue edges of $G[W \setminus A]$ induce a matching. Similar to the above case when $|P| = 2$, we obtain $|W \setminus A| \le 9 \cdot 2^{k-1-q} - 1$. Hence,

$$|W| = |W \setminus A| + |A| \le 9 \cdot 2^{k-1-q} - 1 + 3 = 9 \cdot 2^{k-1-q} + 2,$$

which is less than the desired bound.

Now we consider when $n = 7$. Again because $|W^*| \ge 8$, we have $|P| = 4$ and $|P'| = 2$, else we obtain a blue C_{15}. Thus the blue edges in $G[W \setminus V(P)]$ form an induced matching. Let

$u_1w_1, \ldots, u_mw_m$ comprise the blue edges of $G[W \setminus V(P)]$. Define

$$A := W \setminus \{v_1, v_2, v_3, v_4, u_1, \ldots, u_m, w_1, \ldots, w_m\}$$
$$B := \begin{cases} \{v_1, u_1, \ldots, u_m\} \cup A, & \text{if } |A| \leq 1 \\ \{v_1, u_1, \ldots, u_m\} \cup \{a_1, a_2\}, & \text{if } |A| \geq 2, \text{ where } a_1, a_2 \in A \end{cases}$$

By similar reasoning to the case when $|P| = 2$, we have $|B| \leq 4 \cdot 2^{k-1-q}$. Note that when $|A| \geq 2$, $|W \setminus \{v_1, v_2, v_3, u_1, \ldots, u_m\}| \leq 7 \cdot 2^{k-1-q}$ by minimality of k. Therefore,

$$|W| = \begin{cases} 2|B \setminus (A \cup \{v_1\})| + |A| + |P| & |A| \leq 1 \\ |W \setminus \{v_1, v_2, v_3, u_1, \ldots, u_m\}| + |B \setminus \{a_1, a_2\}| + |\{v_2, v_3\}| & |A| \geq 2, \end{cases}$$
$$\leq \begin{cases} 2(4 \cdot 2^{k-1-q} - 1) + 1 + 4 & \text{if } |A| \leq 1 \\ 7 \cdot 2^{k-1-q} + (4 \cdot 2^{k-1-q} - 2) + 2 & \text{if } |A| \geq 2, \end{cases}$$

yielding the desired bound because $q \leq k - 2$ and $k \geq 3$. This establishes case (iv).

Finally, we prove case (v), when $\ell = n - 4$ and $n = 7$. Assume first that $|W^*| \leq 12$. Then

$$|W| = |W \setminus W^*| + |W^*| \leq 7 \cdot 2^{k-1-q} + 12 \leq 13 \cdot 2^{k-1-q}$$

because $q \leq k - 2$ and $k \geq 3$. Thus we may assume that $|W^*| \geq 13$. Hence, $4 \leq |P| \leq 7$, else we obtain a blue C_{15}. Again we will let P' be a longest blue path in $G[W \setminus V(P)]$.

Let us first handle the cases when $|P| \in \{6, 7\}$. Then there exists a subset $A \subseteq W$ such that $|A| \leq 5$ and all the blue edges in $G[W \setminus A]$ form a matching. By similar reasoning to the case $|P| = 2$, we have $|W \setminus A| \leq 11 \cdot 2^{k-1-q} - 1$, which yields

$$|W| = |W \setminus A| + |A| \leq 11 \cdot 2^{k-1-q} - 1 + 5 \leq 13 \cdot 2^{k-1-q},$$

because $q \leq k-2$ and $k \geq 3$.

Now suppose $|P| = 5$. Except for one case, we may apply identical reasoning as when $|P| \in \{6, 7\}$. The only case we need to consider is when $|P'| = 3$ for possibly many disjoint longest blue paths in $G[W \setminus V(P)]$. Apply the partition on $G[W \setminus \{v_1, v_2\}]$ used to derive the case when $|P| = 3$, to obtain corresponding parts W_1', W_2' and W_3' (see Figure 3.6). By similar reasoning, we find $|W_1'| \leq 2 \cdot 2^{k-1-q}$, and $|W_3'| \leq 7 \cdot 2^{k-1-q}$. From an argument similar to the case $|P| = 2$ used to obtain (3.1), $|W_2'| \leq 4 \cdot 2^{k-1-q} - 2$. Adding the parts together,

$$\begin{aligned}
|W| &= |W_1'| + |W_2'| + |W_3'| + |\{v_1, v_2\}| \\
&\leq 2 \cdot 2^{k-1-q} + (4 \cdot 2^{k-1-q} - 2) + 7 \cdot 2^{k-1-q} + 2 \\
&= 13 \cdot 2^{k-1-q}
\end{aligned}$$

since $q \leq k-2$ and $k \geq 3$, as desired.

Thus $|P| = 4$. Then $G[W \setminus V(P)]$ has at most one blue P_4, else we obtain a blue C_{15}. It suffices to consider the worst-case scenario when $G[W \setminus V(P)]$ has exactly one blue P_4, with vertices y_1, y_2, y_3, y_4 in order. Then each component of the subgraph of $G[W \setminus \{v_4, y_4\}]$

induced by all its blue edges is isomorphic to a K_3, a star, or a P_2. Define the following sets:

A_0 : All vertices $v \in W$ such that v is not incident with any blue edge in $G[W]$

A_1 : Select one vertex from each blue K_3

A_2 : Select one vertex from each blue K_3 not in W_1, the center vertex in each blue star, and one vertex from each blue P_2

We next choose W_1 and W_2 judiciously. If $G[W \setminus \{v_4, y_4\}]$ has no blue star, let

$$W_1 := A_1$$

$$W_2 := \begin{cases} A_2 \cup A_0, & \text{if } |A_0| \leq 1 \\ A_2 \cup \{a_1, a_2\}, & \text{if } |A_0| \geq 2, \text{ where } a_1, a_2 \in A_0 \end{cases}$$

$$W_3 := W \setminus (A_1 \cup A_2) \cup \{v_4, y_4\}.$$

If on the other hand $G[W \setminus \{v_4, y_4\}]$ has at least one blue star with center vertex x and two leaves x_1, x_2, let

$$W_1 := A_1 \cup \{x_1\}$$

$$W_2 := (A_2 \setminus \{x\}) \cup \{x_1, x_2\}$$

$$W_3 := W \setminus (A_1 \cup A_2 \cup \{v_4, y_4\}).$$

By a similar argument to that given for the case $|P| = 3$ (with $\ell = n - 4$ and $n = 7$),

$|W_1| \leq 2 \cdot 2^{k-1-q}$, $|W_2| \leq 4 \cdot 2^{k-1-q}$ and $|W_3| \leq 7 \cdot 2^{k-1-q}$. Therefore,

$$|W| = \begin{cases} |W_1| + 2|W_2 \setminus A_0| + |A_0| + |\{v_4, y_4\}| & |A_0| \leq 1 \text{ and } G[W] \text{ has no blue star} \\ |W_1| + |W_2 \setminus \{a_1, a_2\}| + |\{v_4, y_4\}| + |W_3| & |A_0| \geq 2 \text{ and } G[W] \text{ has no blue star} \\ |W_1 \setminus \{y_1\}| + |W_2 \setminus \{y_1, y_2\}| + |\{x, v_4, y_4\}| + |W_3| & G[W] \text{ has a blue star,} \end{cases}$$

$$\leq \begin{cases} 2 \cdot 2^{k-1-q} + 8 \cdot 2^{k-1-q} + 1 + 2 & |A_0| \leq 1 \text{ and } G[W] \text{ has no blue star} \\ 2 \cdot 2^{k-1-q} + (4 \cdot 2^{k-1-q} - 2) + 2 + 7 \cdot 2^{k-1-q} & |A_0| \geq 2 \text{ and } G[W] \text{ has no blue star} \\ (2 \cdot 2^{k-1-q} - 1) + (4 \cdot 2^{k-1-q} - 2) + 3 + 7 \cdot 2^{k-1-q} & G[W] \text{ has a blue star,} \end{cases}$$

yielding the desired bound because $q \leq k-2$ and $k \geq 3$.

This completes the proof of Claim 3.3.1. ■

Let $X_1, \ldots, X_m$ be a maximum sequence of disjoint subsets of $V(G)$ such that, for all $j \in [m]$, one of the following holds:

(a) $1 \leq |X_j| \leq 3$, and X_j is mc-complete to $V(G) \setminus \bigcup_{i \in [j]} X_i$ under c, or

(b) $4 \leq |X_j| \leq 6$, and X_j can be partitioned into two non-empty sets X_{j_1} and X_{j_2}, where $j_1, j_2 \in [k]$ are two distinct colors, such that for each $t \in \{1, 2\}$, $1 \leq |X_{j_t}| \leq 3$, X_{j_t} is j_t-complete to $V(G) \setminus \bigcup_{i \in [j]} X_i$ but not j_t-complete to $X_{j_{3-t}}$, and all the edges between X_{j_1} and X_{j_2} in G are colored using only the colors j_1 and j_2.

Note that such a sequence $X_1, \ldots, X_m$ may not exist. Let $X := \bigcup_{j \in [m]} X_j$. For each $x \in X$, let $c(x)$ be the unique color on the edges between x and $V(G) \setminus X$ under c. For all $i \in [k]$, let $X_i^* := \{x \in X : c(x) \text{ is color } i\}$. Then $X = \bigcup_{i \in [k]} X_i^*$. It is worth noting that for all $i \in [k]$, X_i^* is possibly empty. By abusing the notation, we use X_b^*, X_r^* and X_g^* to denote X_i^* when i is blue, red or green, respectively.

Claim 3.3.2 *For all $i \in [k]$, $|X_i^*| \leq 3$. Hence, $|X| \leq 3k$.*

Proof. Suppose the statement is false. Then $m \geq 2$. When choosing $X_1, X_2, \ldots, X_m$, let $j \in [m-1]$ be the largest index such that $|X_p^* \cap (X_1 \cup X_2 \cup \cdots \cup X_j)| \leq 3$ for all colors $p \in [k]$. Then $4 \leq |X_i^* \cap (X_1 \cup X_2 \cup \cdots \cup X_j \cup X_{j+1})| \leq 6$ for some color $i \in [k]$ by the choice of j. Such a color i and an index j exist due to the assumption that the statement of Claim 3.3.2 is false. Let $A := X_1 \cup X_2 \cup \cdots \cup X_j \cup X_{j+1}$. By the choice of $X_1, X_2, \ldots, X_m$, there are at most two colors $i \in [k]$ such that $4 \leq |X_i^* \cap A| \leq 6$. We may assume that such a color i is red or blue. Let $A_b := \{x \in A : c(x) \text{ is color blue}\}$ and $A_r := \{x \in A : c(x) \text{ is color red}\}$. It suffices to consider the worst-case scenario when $4 \leq |A_b| \leq 6$ and $4 \leq |A_r| \leq 6$. Then for any color $p \in [k]$ other than red and blue, $|X_p^* \cap A| \leq 3$. Thus by the choice of j, $|A \setminus (A_b \cup A_r)| \leq 3(k-2)$. We may assume that $|A_b| \geq |A_r|$. Note that $4 \leq |A_b| \leq 6 \leq n$. By Claim 3.3.1 applied to A_b and $V(G) \setminus A$, we see that

$$|V(G) \setminus A| \leq \begin{cases} (2n-3) \cdot 2^{k-1} + (n-7), & \text{if } |A_b| = 4 \\ n \cdot 2^{k-1} + (2n-10), & \text{if } |A_b| = 5 \\ n \cdot 2^{k-1} + (2n-12), & \text{if } |A_b| = 6. \end{cases}$$

But then,

$$\begin{aligned} |G| &= |A \setminus (A_b \cup A_r)| + |A_b| + |A_r| + |V(G) \setminus A| \\ &\leq 3(k-2) + \begin{cases} 4 + 4 + [(2n-3) \cdot 2^{k-1} + (n-7)], & \text{if } |A_b| = 4 \\ 5 + 5 + [n \cdot 2^{k-1} + (2n-10)], & \text{if } |A_b| = 5 \\ 6 + 6 + [n \cdot 2^{k-1} + (2n-12)], & \text{if } |A_b| = 6 \end{cases} \\ &< n \cdot 2^k + 1 \end{aligned}$$

for all $k \geq 3$ and $n \in \{6, 7\}$, a contradiction. ■

By Claim 3.3.2, $|X| \leq 3k$. Let $X' \subseteq X$ be such that for all $i \in [k]$, $|X' \cap X_i^*| = 1$ when $X_i^* \neq \emptyset$. Similarly, define $X'' \subseteq X$ such that for all $i \in [k]$, $|X'' \cap (X_i^* \setminus X')| = 1$ when $X_i^* \setminus X' \neq \emptyset$. Finally, let $X''' := X \setminus (X' \cup X'')$. Now consider a Gallai partition $A_1, \ldots, A_p$ of $G \setminus X$ with $p \geq 2$. We may assume that $1 \leq |A_1| \leq \cdots \leq |A_s| < 3 \leq |A_{s+1}| \leq \cdots \leq |A_p|$, where $0 \leq s \leq p$. Let $\mathcal{R}$ be the reduced graph of $G \setminus X$ with vertices $a_1, a_2, \ldots, a_p$, where $a_i \in A_i$ for all $i \in [p]$. By Theorem 1.3.13, we may assume that every edge of $\mathcal{R}$ is colored either red or blue. Note that any monochromatic C_{2n+1} in $\mathcal{R}$ would yield a monochromatic C_{2n+1} in G. Thus $\mathcal{R}$ has neither a red nor a blue C_{2n+1}. By Theorem 1.3.2, $p \leq 4n$. Then $|A_p| \geq 2$ because $|G \setminus X| \geq n \cdot 2^k + 1 - 3k \geq 8n - 8$ and $n \in \{6, 7\}$.

Claim 3.3.3 $|A_{p-8}| \leq 2$ *and* $|A_{p-4n+12}| \leq 1$*. Moreover, if* $|A_{p-7}| \geq 3$*, then* $|A_{p-4n+16}| \leq n - 6$*. Similarly, if* $|A_{p-4n+13}| \geq 2$*, then* $p \leq 4n - 12$*.*

Proof. Suppose $|A_{p-8}| \geq 3$ or $|A_{p-7}| \geq 3$ and $|A_{p-4n+16}| \geq n - 5$. By Theorem 1.3.2, $R(C_{2n-7}, C_{2n-7}) = 4n - 15$. We see that either $\mathcal{R}[\{a_{p-8}, a_{p-7}, \ldots, a_p\}]$ has a monochromatic C_5 that gives a monochromatic C_{2n+1} in G, or $\mathcal{R}[\{a_{p-4n+16}, a_{p-4n+15}, \ldots, a_p\}]$ has a monochromatic C_{2n-7} which again yields a monochromatic C_{2n+1} in G, a contradiction. Similarly, suppose $|A_{p-4n+12}| \geq 2$ or $|A_{p-4n+13}| \geq 2$ and $p \geq 4n - 11$ (and so $|A_{p-4n+12}| \geq 1$). By Theorem 1.3.2, $R(C_{2n-5}, C_{2n-5}) = 4n - 11$. Thus $\mathcal{R}[\{a_{p-4n+12}, a_{p-4n+13}, \ldots, a_p\}]$ has a monochromatic C_{2n-5}, again yielding a monochromatic C_{2n+1} in G, a contradiction. ■

Claim 3.3.4 $|A_p| \geq 4$.

Proof. Suppose $|A_p| \leq 3$. Then $n \cdot 2^k + 1 - 3k \leq |G \setminus X| \leq p|A_p| = 12n$ because $p \leq 4n$ and $|X| \leq 3k$. It follows that $k = 3$ and so $|X| \leq 3k = 9$ and $|G| = 8n + 1$. Thus $p \geq 2n + 1$

because $|A_p| \leq 3$. Let green be the third color. Since $|A_p| \leq 3$, we see that G has no green C_{2n} under the coloring c. We claim that either $|X_r^*| = 0$ or $|X_b^*| = 0$. Suppose $|X_r^*| \geq 1$ and $|X_b^*| \geq 1$. Since G has no green C_{2n} and

$$|G| - |A_p \cup X| \geq (8n+1) - 3 - 9 \geq 8n - 11 > 6n - 3 \geq GR_3(C_{2n}),$$

by Theorem 1.3.23 (i), there is either a red or a blue C_{2n} in $G \setminus (A_p \cup X)$. Thus $G \setminus (A_p \cup X_g^*)$ has either a red or a blue C_{2n+1} under c, a contradiction. This proves that either $|X_r^*| = 0$ or $|X_b^*| = 0$. We may assume that $|X_b^*| = 0$. Then $|X'| \leq 2$ and so $|X| = |X_r^* \cup X_g^*| \leq 6$. By Claim 3.3.3, $|A_{p-8}| \leq 2$ and $|A_{p-4n+12}| \leq 1$. If $p \leq 4n - 6$ or $|X| \leq 4$, then

$$|G| = \sum_{i=1}^{p} |A_i| + |X| \leq 3 \cdot 8 + 2(4n - 20) + (p - 4n + 12) + |X| \leq 8n < 8n + 1,$$

a contradiction. Thus $p \geq 4n - 5$ and $|X| \geq 5$. Since $|X'| \leq 2$ and $|X| \geq 5$, by Claim 3.3.2, $|X'| = 2$ and $|X_r^*| \geq 2$. By Theorem 1.3.3, $R(C_{2n-2}, C_{2n+1}) = 4n - 5$. It follows that $\mathcal{R}[\{a_1, \ldots, a_{4n-5}\}]$ has either a red C_{2n-2} or a blue C_{2n+1}. Since c is bad, we see that $\mathcal{R}[\{a_1, \ldots, a_{4n-5}\}]$ has a red C_{2n-2}. But then $G[V(C_{2n-2}) \cup X_r^* \cup \{v\}]$, where $v \in A_p$, has a red C_{2n+1}, a contradiction. ■

Claim 3.3.5 *If $|A_p| \leq n$, then $|A_{p-2}| \leq 3$.*

Proof. Suppose $|A_p| \leq n$ but $|A_{p-2}| \geq 4$. Since $|A_p| \leq n$, we have $|G| - |A_p \cup A_{p-1} \cup A_{p-2}| - |X| \geq n \cdot 2^k + 1 - 3n - 3k \geq 5n - 8$. Let B_1, B_2, B_3 be a permutation of A_{p-2}, A_{p-1}, A_p such that B_2 is, say, blue-complete to $B_1 \cup B_3$ in G. This is possible due to Theorem 1.3.13. Let $b_1, \ldots, b_4 \in B_1$, $b_5, \ldots, b_8 \in B_2$, and $b_9, \ldots, b_{12} \in B_3$. Let $A := V(G) \setminus (B_1 \cup B_2 \cup B_3 \cup X)$, and define

$$B_1^* := \{v \in A \mid v \text{ is blue-complete to } B_1 \text{ and red-complete to } B_3 \text{ in } G\}$$

$$B_2^* := \{v \in A \mid v \text{ is blue-complete to } B_1 \cup B_3 \text{ in } G\}$$

$$B_3^* := \{v \in A \mid v \text{ is red-complete to } B_1 \cup B_3 \text{ in } G\}$$

$$B_4^* := \{v \in A \mid v \text{ is red-complete to } B_1 \text{ and blue-complete to } B_3 \text{ in } G\}.$$

Then $A = B_1^* \cup B_2^* \cup B_3^* \cup B_4^*$ and so $|B_1^* \cup B_2^* \cup B_3^* \cup B_4^*| \geq 5n-8$. Note that $B_1^*, B_2^*, B_3^*, B_4^*$ are pairwise disjoint. Suppose first that B_1 is red-complete to B_3 in G. By Lemma 3.1.1 applied to B_3^* and $B_1 \cup B_3$, $|B_3^*| \leq n-1$. Thus $|B_1^*|+|B_2^*|+|B_4^*| \geq 5n-8-(n-1) = 4n-7 \geq 2n+5$ because $n \in \{6, 7\}$. By symmetry, we may assume that $|B_1^*| + |B_2^*| \geq n+3$. We claim that $G[B_1^* \cup B_2^* \cup B_4^*]$ has no blue edges. Suppose not. Let uv be a blue edge in $G[B_1^* \cup B_2^* \cup B_4^*]$. Since $|B_1^*| + |B_2^*| \geq n+3$, let $x, y \in B_1^* \cup B_2^*$ be two distinct vertices that are different from u and v. If $u, v \in B_1^* \cup B_2^*$, then we find a blue C_{2n+1} with vertices

$$\begin{cases} u, v, b_1, b_5, b_9, b_6, b_{10}, b_7, b_{11}, b_8, b_2, x, b_3, & \text{if } n = 6 \\ u, v, b_1, b_5, b_9, b_6, b_{10}, b_7, b_{11}, b_8, b_2, x, b_3, y, b_4, & \text{if } n = 7 \end{cases}$$

in order, a contradiction. Thus we may assume that $v \in B_4^*$. If $u \in B_1^* \cup B_2^*$, then we find a blue C_{2n+1} with vertices

$$\begin{cases} u, v, b_9, b_5, b_{10}, b_6, b_{11}, b_7, b_{12}, b_8, b_1, x, b_2, & \text{if } n = 6 \\ u, v, b_9, b_5, b_{10}, b_6, b_{11}, b_7, b_{12}, b_8, b_1, x, b_2, y, b_3, & \text{if } n = 7 \end{cases}$$

in order, a contradiction. Thus $u, v \in B_4^*$. But similarly, we obtain a blue C_{2n+1} with vertices

$$\begin{cases} u, v, b_9, b_5, b_1, x, b_2, b_6, b_3, y, b_4, b_7, b_{10}, & \text{if } n = 6 \\ u, v, b_9, b_5, b_1, x, b_2, b_6, b_{10}, b_7, b_3, y, b_4, b_8, b_{11}, & \text{if } n = 7 \end{cases}$$

in order, a contradiction. This proves that $G[B_1^* \cup B_2^* \cup B_4^*]$ contains no blue edges. Since $|B_1^*| + |B_2^*| + |B_4^*| \geq 2n + 5$ and $|A_p| \leq n$, by Lemma 3.1.2, $G[B_1^* \cup B_2^* \cup B_4^*]$ has a red C_{2n+1}, a contradiction. Thus B_1 must be blue-complete to B_3. Then $|B_1 \cup B_2 \cup B_3| \leq 2n$, else we obtain a blue C_{2n+1} in $G[B_1 \cup B_2 \cup B_3]$. By Lemma 3.1.1 applied to $B_2 \cup B_2^*$ and $B_1 \cup B_3$, we see that $|B_2^*| \leq n - 5$. If $|B_1^*| \geq 3$, let $x, y, z \in B_1^*$ be distinct vertices. Then we find a blue C_{2n+1} with vertices

$$\begin{cases} b_1, b_5, b_9, b_6, b_{10}, b_7, b_{11}, b_8, b_{12}, b_2, x, b_3, y, & \text{if } n = 6 \\ b_1, b_5, b_9, b_6, b_{10}, b_7, b_{11}, b_8, b_{12}, b_2, x, b_3, y, b_4, z, & \text{if } n = 7 \end{cases}$$

in order, a contradiction. Thus $|B_1^*| \leq 2$. Similarly, $|B_4^*| \leq 2$. Therefore,

$$|B_3^*| = |G| - |X| - |B_1 \cup B_2 \cup B_3| - |B_1^* \cup B_2^* \cup B_4^*| \geq n \cdot 2^k + 1 - 3k - 2n - (n - 5 + 2 + 2) \geq 5n - 7.$$

By Lemma 3.1.1 applied to B_3^* and $B_1 \cup B_3$, $G[B_3^*]$ contains no red edges. But then by Lemma 3.1.2 and the fact that $|A_p| \leq n$ and $|B_3^*| \geq 5n - 7$, $G[B_3^*]$ must contain a blue C_{2n+1}, a contradiction. This proves that if $|A_p| \leq n$, then $|A_{p-2}| \leq 3$. ■

By Claim 3.3.4, $|A_p| \geq 4$ and so $p - s \geq 1$. Let

$$B := \{a_i \in \{a_1, \ldots, a_{p-1}\} \mid a_i a_p \text{ is colored blue in } \mathcal{R}\}$$

$$R := \{a_j \in \{a_1, \ldots, a_{p-1}\} \mid a_j a_p \text{ is colored red in } \mathcal{R}\}$$

Then $|B| + |R| = p - 1$. Let $B_G := \bigcup_{a_i \in B} A_i$ and $R_G := \bigcup_{a_j \in R} A_j$.

Claim 3.3.6 *If every vertex in X is neither i- nor j-complete to $V(G) \setminus X$ for two distinct colors $i, j \in [k]$, then $X''' = \emptyset$.*

Proof. Suppose $X''' \neq \emptyset$. We may assume that every vertex in X is neither red- nor blue-complete to $V(G) \setminus X$. Then there exists at least one color $\ell \in [k]$ other than red and blue such that $|X^*_\ell| = 3$. We claim that $k \geq 4$. Suppose $k = 3$. Then $|G| = 8n + 1$. We may assume the third color is green. Then $|X| = |X^*_g| = 3$. By Claim 3.3.1 applied to X^*_g and $V(G) \setminus X^*_g$, $|V(G) \setminus X^*_g| \leq 4(2n - 1) + (n - 7)$. But then

$$|G| = |X^*_g| + |V(G) \setminus X^*_g| \leq 3 + 4(2n - 1) + (n - 7) < 8n + 1,$$

because $n \in \{6, 7\}$, a contradiction. Thus $k \geq 4$, as claimed. When choosing $X_1, X_2, \ldots, X_m$, let $q \in [m]$ be the smallest index such that for some color $\ell' \in [k]$ other than red and blue, $|X^*_{\ell'} \cap (X_1 \cup \cdots \cup X_q)| = 3$. By the choice of q, $|X^*_j \cap (X_1 \cup \cdots \cup X_{q-1})| \leq 2$ for all $j \in [k]$. By the property (b) when choosing $X_1, X_2, \ldots, X_m$, there are possibly two colors $q_1 := \ell'$ and $q_2 \in [k]$ such that $|X^*_{q_1}| = 3$ and $|X^*_{q_2}| \leq 3$. Since no vertex in X is red- or blue-complete to $V(G) \setminus X$, we see that $|(X_1 \cup \cdots \cup X_q) \setminus (X^*_{q_1} \cup X^*_{q_2})| \leq 2(k - 4)$. By Claim 3.3.1 applied to $X^*_{q_1}$ and $V(G) \setminus (X_1 \cup \cdots \cup X_q)$, $|V(G) \setminus (X_1 \cup \cdots \cup X_q)| \leq (2n - 1) \cdot 2^{k-1} + (n - 7)$. But then

$$\begin{aligned}
|G| &= |(X_1 \cup \cdots \cup X_q) \setminus (X^*_{q_1} \cup X^*_{q_2})| + |X^*_{q_1} \cup X^*_{q_2}| + |V(G) \setminus (X_1 \cup \cdots \cup X_q)| \\
&\leq 2(k - 4) + 6 + [(2n - 1) \cdot 2^{k-1} + (n - 7)] \\
&< n \cdot 2^k + 1
\end{aligned}$$

for all $k \geq 4$, a contradiction. ∎

Claim 3.3.7 *If $|A_p| \geq n$ and $|B| \geq 3$ (resp. $|R| \geq 3$), then $|B_G| \leq 2n$ (resp. $|R_G| \leq 2n$).*

Proof. Suppose $|A_p| \geq n$ and $|B| \geq 3$ but $|B_G| \geq 2n+1$. By Claim 3.1.1, $G[B_G]$ has no blue edges and no vertex in X is blue-complete to $V(G) \setminus X$. Thus all the edges of $\mathcal{R}[B]$ are colored red in $\mathcal{R}$. Let $q := |B|$ and let $B := \{a_{i_1}, a_{i_2}, \ldots, a_{i_q}\}$ with $|A_{i_1}| \geq |A_{i_2}| \geq \cdots \geq |A_{i_q}|$. Then $G[B_G] \setminus \bigcup_{j=1}^{q} E(G[A_{i_j}])$ is a complete multipartite graph with at least three parts. If $|A_{i_1}| \leq n$, then by Lemma 3.1.2 applied to $G[B_G] \setminus \bigcup_{j=1}^{q} E(G[A_{i_j}])$, $G[B_G]$ has a red C_{2n+1}, a contradiction. Thus $|A_{i_1}| \geq n+1$. Let $Q_b := \{v \in R_G : v \text{ is blue-complete to } A_{i_1}\}$, and $Q_r := \{v \in R_G : v \text{ is red-complete to } A_{i_1}\}$. Then $Q_b \cup Q_r = R_G$. Let $Q := (B_G \setminus A_{i_1}) \cup Q_r \cup X_r^*$. Then Q is red-complete to A_{i_1} and $G[Q]$ must contain red edges, because $|B| \geq 3$ and all the edges of $\mathcal{R}[B]$ are colored red. By Claim 3.1.1 applied to A_{i_1} and Q, $|Q| \leq n$. Note that $|A_p \cup Q_b| \geq |A_p| \geq |A_{i_1}| \geq n+1$ and $A_p \cup Q_b$ is blue-complete to A_{i_1}. By Claim 3.1.1 applied to A_{i_1} and $A_p \cup Q_b$, $G[A_p \cup Q_b]$ has no blue edges. Since no vertex in X is blue-complete to $V(G) \setminus X$, we see that $G[A_p \cup Q_b \cup (X' \setminus X_r^*)]$ has no blue edges. By minimality of k, $|A_p \cup Q_b \cup (X' \setminus X_r^*)| \leq n \cdot 2^{k-1}$. Suppose first that $Q_r \cup X_r^* = \emptyset$. Then $Q_b = R_G$ and $G[B_G \cup X'']$ has no blue edges. By minimality of k, $|B_G \cup X''| \leq n \cdot 2^{k-1}$. Since no vertex in X is red- or blue-complete to $V(G) \setminus X$, by Claim 3.3.6, $X''' = \emptyset$. But then

$$|G| = |B_G \cup X''| + |A_p \cup Q_b \cup X'| \leq n \cdot 2^{k-1} + n \cdot 2^{k-1} < n \cdot 2^k + 1,$$

a contradiction. Thus $Q_r \cup X_r^* \neq \emptyset$. Since $|B| \geq 3$, we see that $|B_G \setminus A_{i_1}| \geq 2$. Thus $n \geq |Q| \geq 3$.

We next claim that either $|Q| \geq 4$ or $k \geq 6$. Suppose $|Q| = 3$ and $k \leq 5$. Then $|Q_r \cup X_r^*| = 1$ and $|B_G \setminus A_{i_1}| = 2$. Suppose $k = 3$. We may assume that the third color is green. Since Q is red-complete to A_{i_1}, we see that $G[A_{i_1}]$ has neither red C_{2n-2} nor a green C_{2n+1}. By

Theorem 1.3.3, $|A_{i_1}| \leq R(C_{2n-2}, C_{2n+1}) - 1 = 4n - 6$. But then

$$|G| = |Q| + |A_{i_1}| + |A_p \cup Q_b| + |X_g^*| \leq 3 + (4n-6) + n \cdot 2^{3-1} + 3 = 8n < 8n+1,$$

a contradiction. Thus $k \in \{4, 5\}$. Then $|X' \setminus X_r^*| \leq k - 3$, else, by Theorem 1.3.23 (i), $|A_{i_1}| \leq GR_{k-1}(C_{2n}) - 1 \leq (k-1)(n-1) + 3n - 1$. But then

$$\begin{aligned} |G| &= |Q| + |A_{i_1}| + |A_p \cup Q_b \cup (X' \setminus X_r^*)| + |(X'' \cup X''') \setminus X_r^*| \\ &\leq 3 + [(k-1)(n-1) + 3n - 1] + n \cdot 2^{k-1} + 2(k-2) \\ &< n \cdot 2^k + 1 \end{aligned}$$

for all $k \in \{4, 5\}$ and $n \in \{6, 7\}$, a contradiction. Thus $|X' \setminus X_r^*| \leq k-3$, and so $|X'' \setminus X_r^*| \leq k - 3$. In particular, by Claim 3.3.6, this implies $X''' = \emptyset$. By Claim 3.3.1 applied to Q and A_{i_1}, $|A_{i_1}| \leq (2n-1) \cdot 2^{k-2} + (n-7)$. But then

$$\begin{aligned} |G| &= |Q| + |A_{i_1}| + |A_p \cup Q_b \cup (X' \setminus X_r^*)| + |X'' \setminus X_r^*| \\ &\leq 3 + [(2n-1) \cdot 2^{k-2} + (n-7)] + n \cdot 2^{k-1} + (k-3) \\ &< n \cdot 2^k + 1 \end{aligned}$$

for $k \in \{4, 5\}$, a contradiction. This proves that either $|Q| \geq 4$ or $k \geq 6$, as claimed.

Note that $G[A_{i_1}]$ has no blue edges and $|(X'' \cup X''') \setminus X_r^*| \leq 2(k-2)$. By Claim 3.3.1 applied to Q and A_{i_1}, we see that

$$|A_{i_1}| \leq \begin{cases} n \cdot 2^{k-2} & \text{if } |Q| = n \\ n \cdot 2^{k-2} + 2 & \text{if } |Q| = n-1 \\ (21-2n) \cdot 2^{k-2} + (5n-31) & \text{if } |Q| = n-2 \\ 11 \cdot 2^{k-2} + (n-7) & \text{if } |Q| = n-3 \\ 13 \cdot 2^{k-2} & \text{if } |Q| = n-4 \text{ and } n = 7. \end{cases}$$

But then

$$\begin{aligned} |G| &= |Q| + |A_{i_1}| + |A_p \cup Q_b \cup (X' \setminus X_r^*)| + |(X'' \cup X''') \setminus X_r^*| \\ &\leq \begin{cases} n + n \cdot 2^{k-2} + n \cdot 2^{k-1} + 2(k-2) & \text{if } |Q| = n \\ (n-1) + (n \cdot 2^{k-2} + 2) + n \cdot 2^{k-1} + 2(k-2) & \text{if } |Q| = n-1 \\ (n-2) + [(21-2n) \cdot 2^{k-2} + (5n-31)] + n \cdot 2^{k-1} + 2(k-2) & \text{if } |Q| = n-2 \\ (n-3) + [11 \cdot 2^{k-2} + (n-7)] + n \cdot 2^{k-1} + 2(k-2) & \text{if } |Q| = n-3 \text{ and } n = 7 \\ 3 + [(2n-1) \cdot 2^{k-2} + (n-7)] + n \cdot 2^{k-1} + 2(k-2) & \text{if } |Q| = 3 \text{ and } k \geq 6. \end{cases} \end{aligned}$$

In each case, we have $|G| < n \cdot 2^k + 1$, a contradiction. This proves that if $|A_p| \geq n$ and $|B| \geq 3$, then $|B_G| \leq 2n$. Similarly, one can prove that if $|A_p| \geq n$ and $|R| \geq 3$, then $|R_G| \leq 2n$. ∎

Claim 3.3.8 $p \leq 2n + 1$.

Proof. Suppose $p \geq 2n + 2$. Then $|B| + |R| = p - 1 \geq 2n + 1$. We claim that $|A_p| \leq n$. Suppose $|A_p| \geq n + 1$. We may assume that $|B| \geq |R|$. Then $|B_G| \geq |B| \geq n + 1$. By

Claim 3.3.7, $|B_G| \leq 2n$, and by Claim 3.1.1, $G[A_p]$ has no blue edges and $X_b^* = \emptyset$. Then $|X'' \cup X'''| \leq 2(k-1)$. If $|R_G| \geq n+1$, then by Claim 3.1.1, neither $G[R_G]$ nor $G[A_p]$ has red edges and $X_r^* = \emptyset$. By Claim 3.3.6, $X''' = \emptyset$. Note that $G[A_p \cup X']$ has neither red nor blue edges, and $G[R_G \cup X'']$ has no red edges. Then by minimality of k,

$$|G| = |A_p \cup X'| + |B_G| + |R_G \cup X''| + |X'''| \leq n \cdot 2^{k-2} + 2n + n \cdot 2^{k-1} < n \cdot 2^k + 1$$

for all $k \geq 3$, a contradiction. Thus, $|R_G| \leq n$. Then for all $k \geq 3$,

$$|A_p \cup X'| = |G| - |B_G| - |R_G| - |X'' \cup X'''| \geq n \cdot 2^k + 1 - 2n - n - 2(k-1) > n \cdot 2^{k-1} + 1.$$

Since $G[A_p \cup X']$ has no blue edges, by the choice of k, $G[A_p \cup X']$ has a monochromatic C_{2n+1}, a contradiction. This proves that $|A_p| \leq n$, as claimed.

Note that by Claim 3.3.4, $|A_p| \geq 4$. Additionally, Claims 3.3.5 and 3.3.3 give $|A_{p-2}| \leq 3$ and $|A_{p-8}| \leq 2$ with $|A_{p-4n+12}| \leq 1$, respectively. Therefore, $k = 3$, $|G| = 8n+1$ and $|X| \leq 9$. Because $n \in \{6, 7\}$,

$$|B_G| + |R_G| = |G| - |A_p| - |X| \geq (8n+1) - n - 9 = 7n - 8 > 6n - 3 \geq GR_3(C_{2n})$$

by Theorem 1.3.23 (i). Therefore, $|X| \leq 6$, otherwise we find a monochromatic C_{2n+1}. Recalculating the above inequality with this fact, we obtain

$$|B_G| + |R_G| = |G| - |A_p| - |X| \geq (8n+1) - n - 6 = 7n - 5.$$

Thus at least one of $|B_G| \geq 3n+1$ or $|R_G| \geq 3n+1$, so we may assume $|B_G| \geq 3n+1$. We next prove that $4 \leq |A_p| \leq n$ is impossible.

Suppose first that $5 \leq |A_p| \leq n$ and let $B^* \subseteq B_G$ be a minimal set such that $G[B_G \setminus B^*]$ has no blue edges. If $|B^*| \leq 2n-10$, then $|B_G \setminus B^*| \geq (3n+1)-(2n-10) = n+11 \geq 2n+4$, because $n \in \{6,7\}$. Therefore, $|B \setminus B^*| \geq 3$, and all edges of $\mathcal{R}[B \setminus B^*]$ are colored red, so that by Lemma 3.1.2 we find a monochromatic C_{2n+1}, a contradiction. Thus, $|B^*| \geq 2n-9$. Define the family of graphs

$$\mathcal{H}_1 := \{(2n-9)K_2, (2n-11)K_2 \cup P_3, (15-2n)K_2 \cup 2P_{2n-11}, \\ 2P_{n-5} \cup P_4, P_{2n-11} \cup P_4, (15-2n)K_2 \cup P_{4n-23}, P_{2n-8}\}$$

It follows that $G[B_G]$ contains a blue $H \in \mathcal{H}_1$, so that along with the vertices in A_p, we find a blue C_{2n+1}, a contradiction.

Therefore suppose $|A_p| = 4$. Then $|B_G| + |R_G| = |G| - |A_p| - |X| \geq 8n+1-4-6 \geq 8n-9$, so that $|B_G| \geq 4n-4$. Let B^* be defined as above. If $|B^*| \leq 2n-5$, then $|B_G \setminus B^*| \geq (4n-4)-(2n-5) = 2n+1$, and thus $|B \setminus B^*| \geq 3$. Since $\mathcal{R}[B \setminus B^*]$ contains only red edges, by Lemma 3.1.2, there is a red C_{2n+1}, a contradiction. Thus, $|B^*| \geq 2n-4$. Define the family of graphs

$$\mathcal{H}_2 := \{(2n-4)K_2, (14-n)K_2 \cup P_{3n-17}, (20-2n)K_2 \cup 2P_{2n-11}, 8K_2 \cup P_{2n-11}\}.$$

Let M denote a matching of size $m \geq 0$. For any $H \in \mathcal{H}_2$, let $H' := H \cup M$. It follows that $G[B_G]$ contains a blue H', where m is chosen to be as large as possible. Then removing at most two vertices, say $x, y \in V(H)$ from the longest blue subpaths in H, we obtain $M' := H' \setminus \{x, y\}$, which is a matching of size $m' \geq 6$. Denote the edges in M' by u_iv_i, for all $i \in [m']$. Put another way, this means the blue edges in $G[B_G \setminus \{x, y\}]$ induce a blue matching. Let us define a new Gallai partition of $G[B_G \setminus \{x, y\}]$ in the following manner. If $|A_{i_j}| = |A_{i_\ell}| = 1$ for some pair $j, \ell \in [q]$, and if A_{i_j} is blue-complete to A_{i_ℓ}, then create the new part $A_{i_s} := A_{i_j} \cup A_{i_\ell}$, so that $|A_{i_s}| = 2$, where $s \in [q']$ and $q' \leq q$; otherwise, define A_{i_j} to be the same. By construction, only red edges appear between any two parts of this modified partition. We may assume $A_{i_1}, \ldots, A_{i_t}$ are all parts of the modified Gallai partition of $B_G \setminus \{x, y\}$ containing blue edges. Because $m' \geq 6$, we see that $\bigcup_{j=1}^{t} |A_{i_j}| \geq 12$, and because $|A_p| = 4$, we also have $t \geq 3$. In particular, if $\bigcup_{j=1}^{t} |A_{i_j}| \geq 2n + 1$, we are done by Lemma 3.1.2 because $G\left[\bigcup_{j=1}^{t} A_{i_j}\right] - \bigcup_{j=1}^{t} E(A_{i_j})$ is a complete multipartite graph containing only red edges. Thus we may assume $12 \leq \sum_{j=1}^{t} |A_{i_j}| \leq 2n$. Note that $|B_G \setminus \{x, y\}| - \sum_{j=1}^{t} |A_{i_j}| \geq (4n - 4) - 2 - 2n = 2n - 6$. Define $r := 2n + 1 - \sum_{j=1}^{t} |A_{i_j}|$, and choose distinct vertices $v_1, \ldots, v_r \in B_G \setminus \left(\{x, y\} \cup \bigcup_{j=1}^{t} A_{i_j}\right)$. Because $v_1, \ldots, v_r \notin \bigcup_{j=1}^{t} A_{i_j}$, we see that $\{v_1, \ldots, v_r\}$ is red-complete to $\bigcup_{j=1}^{t} A_{i_j}$, again yielding a red C_{2n+1} by Lemma 3.1.2, again forcing a contradiction. ■

Claim 3.3.9 *$|A_p| \geq n + 1$.*

Proof. Suppose $|A_p| \leq n$. Then $p \geq 9$ because $|G| \geq 8n + 1$. By Claim 3.3.8, we have $9 \leq p \leq 2n+1$. We may assume that $a_p a_{p-1}$ is colored blue in $\mathcal{R}$. Then $|A_p \cup A_{p-1} \cup X_b^*| \leq 2n$, else $|X_b^*| \geq 1$ and so $G[A_p \cup A_{p-1} \cup X_b^*]$ has a blue C_{2n+1}, a contradiction. It follows that $|A_p \cup A_{p-1} \cup X| = |A_p \cup A_{p-1} \cup X_b^*| + |X \setminus X_b^*| \leq 2n + 3(k - 1)$. By Claim 3.3.5 and Claim 3.3.3, $|A_{p-2}| \leq 3$ and $|A_{p-8}| \leq 2$. But then

$$
\begin{aligned}
|G| &= |A_p \cup A_{p-1} \cup X| + \sum_{i=p-7}^{p-2} |A_i| + \sum_{i=1}^{p-8} |A_i| \\
&\leq [2n + 3(k - 1)] + 18 + 2(2n + 1 - 8) \\
&= 6n + 3k + 1 \\
&< n \cdot 2^k + 1,
\end{aligned}
$$

for $n \in \{6, 7\}$ and all $k \geq 3$, a contradiction. ■

Claim 3.3.10 $|A_{p-2}| \leq n$.

Proof. Suppose $|A_{p-2}| \geq n+1$. Then $n+1 \leq |A_{p-2}| \leq |A_{p-1}| \leq |A_p|$ and so $\mathcal{R}[\{a_{p-2}, a_{p-1}, a_p\}]$ is not a monochromatic triangle in $\mathcal{R}$ (else $G[A_p \cup A_{p-1} \cup A_{p-2}]$ has a a monochromatic C_{2n+1}). Let B_1, B_2, B_3 be a permutation of A_{p-2}, A_{p-1}, A_p such that B_2 is, say blue-complete, to $B_1 \cup B_3$ in G. Then B_1 must be red-complete to B_3 in G. By Claim 3.1.1, $X_r^* = \emptyset$ and $X_b^* = \emptyset$. By Claim 3.3.6, $X''' = \emptyset$. Let $A := V(G) \setminus (B_1 \cup B_2 \cup B_3 \cup X' \cup X'')$. By Claim 3.1.1 again, $G[B_2]$ has no blue edges, and neither $G[B_1 \cup X']$ nor $G[B_3 \cup X'']$ has red or blue edges. By minimality of k, $|B_1 \cup X'| \leq n \cdot 2^{k-2}$ and $|B_3 \cup X''| \leq n \cdot 2^{k-2}$. It follows that $|A \cup B_2| = |G| - |B_1 \cup X'| - |B_3 \cup X''| \geq n \cdot 2^{k-1} + 1$. By minimality of k, $G[A \cup B_2]$ must have blue edges. By Claim 3.1.1, no vertex in A is red-complete to $B_1 \cup B_3$ in G, and no vertex in A is blue-complete to $B_1 \cup B_2$ or $B_2 \cup B_3$ in G. This implies that A must be red-complete to B_2 in G. It follows that $G[A]$ must contain a blue edge, say uv. Let $b_1, \ldots, b_{n-1} \in B_1$, $b_n, \ldots, b_{2n-2} \in B_2$, and $b_{2n-1} \in B_3$. If $\{u, v\}$ is blue-complete to B_1, then we obtain a blue C_{2n+1} with vertices $b_1, u, v, b_2, b_n, b_{2n-1}, b_{n+1}, b_3, b_{n+2}, \ldots, b_{n-1}, b_{2n-2}$ in order, a contradiction. Thus $\{u, v\}$ is not blue-complete to B_1. Similarly, $\{u, v\}$ is not blue-complete to B_3. Since no vertex in A is red-complete to $B_1 \cup B_3$, we may assume that u is blue-complete to B_1 and v is blue-complete to B_3. But then we obtain a blue C_{2n+1} with vertices $b_1, u, v, b_{2n-1}, b_n, b_2, b_{n+1}, \ldots, b_{n-1}, b_{2n-2}$ in order. ■

Claim 3.3.11 $|B_G| \geq 4$ *or* $|R_G| \geq 4$.

Proof. Suppose $|B_G| \leq 3$ and $|R_G| \leq 3$. Since $p \geq 2$, we see that $B_G \neq \emptyset$ or $R_G \neq \emptyset$. By maximality of m (see condition (a) when choosing $X_1, X_2, \ldots, X_m$), $B_G \neq \emptyset$, $R_G \neq \emptyset$, and B_G is neither red- nor blue-complete to R_G in G. But then, since $|B_G| \leq 3$ and $|R_G| \leq 3$, by maximality of m again (see condition (b) when choosing $X_1, X_2, \ldots, X_m$), $B_G = \emptyset$ and

$R_G = \emptyset$, a contradiction. ■

Claim 3.3.12 $2 \leq p - s \leq 8$.

Proof. By Claim 3.3.3, $|A_{p-8}| \leq 2$ and so $p - s \leq 8$. Suppose $p - s \leq 1$. Then $p - s = 1$ because $p - s \geq 1$. Thus $|A_i| \leq 2$ for all $i \in [p-1]$ by the choice of p and s. By Claim 3.3.8, $p \leq 2n+1$. Then $|B_G \cup R_G| \leq 2(p-1)$ and so $|B_G^* \cup R_G^*| \leq 2(p-1)+3+3 = 2(p+2) \leq 4n+6$. We may assume that $|B_G^*| \geq |R_G^*|$. If $|R_G^*| \geq n$, then $|B_G^*| \geq n$. By Claim 3.3.9 and Claim 3.1.1, $G[A_p]$ has neither blue nor red edges. By minimality of k, $|A_p| \leq n \cdot 2^{k-2}$. But then

$$|G| = |B_G^* \cup R_G^*| + |A_p| + |X \setminus (B_G^* \cup R_G^*)| \leq (4n+6) + n \cdot 2^{k-2} + 3(k-2) < n \cdot 2^k + 1$$

for all $k \geq 3$, a contradiction. Thus $|R_G^*| \leq n - 1$. We claim that $|B_G^*| \leq 2n + 3$. This is trivially true if $|B| \leq n$. If $|B| \geq n+1$, then $|B_G| \leq 2n$ by Claim 3.3.7. Thus $|B_G^*| \leq 2n+3$, as claimed. If $|B_G^*| \geq n-1$, then applying Claim 3.3.1 to B_G^* and A_p implies that

$$|B_G^*| + |A_p| \leq \begin{cases} (n-1) + (n \cdot 2^{k-1} + 2), & \text{if } |B_G^*| = n-1 \\ (2n+3) + n \cdot 2^{k-1}, & \text{if } |B_G^*| \geq n. \end{cases}$$

In either case, $|B_G^*| + |A_p| \leq (2n+3) + n \cdot 2^{k-1}$. But then

$$|G| = |R_G^*| + |B_G^*| + |A_p| + |X \setminus (B_G^* \cup R_G^*)| \leq (n-1) + [(2n+3) + n \cdot 2^{k-1}] + 3(k-2) < n \cdot 2^k + 1,$$

for all $k \geq 3$ and $n \in \{6, 7\}$, a contradiction. Thus $|R_G^*| \leq |B_G^*| \leq n - 2$. If $|B_G^*| = n - 2$, then by Claim 3.3.1, $|A_p| \leq (21 - 2n) \cdot 2^{k-1} + (5n - 31)$. But then

$$|G| = |R_G^*| + |B_G^*| + |A_p| + |X \setminus (B_G^* \cup R_G^*)| \leq 2(n-2) + [(21-2n) \cdot 2^{k-1} + (5n-31)] + 3(k-2) < n \cdot 2^k + 1,$$

for all $k \geq 3$ and $n \in \{6, 7\}$, a contradiction. Thus $|R_G^*| \leq |B_G^*| \leq n - 3$. By Claim 3.3.11, $|R_G^*| \leq |B_G^*| = |B_G| = 4$ and $n = 7$. By Claim 3.3.1, $|A_p| \leq 11 \cdot 2^{k-1}$. But then

$$|G| = |R_G^*| + |B_G^*| + |A_p| + |X \setminus (B_G^* \cup R_G^*)| \leq 4 + 4 + 11 \cdot 2^{k-1} + 3(k-2) < 7 \cdot 2^k + 1,$$

for all $k \geq 3$, a contradiction. ■

By Claim 3.3.12, $2 \leq p - s \leq 8$ and so $|A_{p-1}| \geq 3$. We may now assume that $a_p a_{p-1}$ is colored blue in $\mathcal{R}$. Then $a_{p-1} \in B$ and so $A_{p-1} \subseteq B_G$. Thus $|B_G^*| \geq |B_G| \geq |A_{p-1}| \geq 3$.

Claim 3.3.13 *$|R_G^*| \leq 2n$.*

Proof. Suppose $|R_G^*| \geq 2n + 1$. By Claim 3.3.9, $|A_p| \geq n + 1$. By Claim 3.1.1, $G[R_G^*]$ has no red edges. Thus $|R_G^*| = |R_G|$ and so $X_r^* = \emptyset$. In particular, all the edges in $\mathcal{R}[R]$ are colored blue. By Claim 3.3.7, $|R| \leq 2$. By Claim 3.3.10, $|A_{p-2}| \leq n$. Since $A_{p-1} \cap R_G = \emptyset$ and $|R_G| \geq 2n + 1$, we see that $|R| \geq 3$, a contradiction. ■

Claim 3.3.14 *$|A_{p-1}| \leq n$.*

Proof. Suppose $|A_{p-1}| \geq n + 1$. Then $|B_G| \geq |A_{p-1}| \geq n + 1$. By Claim 3.1.1, neither $G[A_p]$ nor $G[B_G]$ has blue edges, and $X_b^* = \emptyset$. Thus $|X| \leq 3(k - 1)$. We claim that $X''' = \emptyset$. Suppose $X''' \neq \emptyset$. By Claim 3.3.6, $|X_i^*| \geq 1$ for every color $i \in [k]$ other than blue, and $|X_j^*| = 3$ for some color $j \in [k]$ other than blue. Then by Claim 3.3.1 applied to X_j^* and $V(G) \setminus X$, $|V(G) \setminus X| \leq (2n - 1) \cdot 2^{k-1} + n - 7$. Thus $|X| \geq 3k - 4$, else,

$$|G| = |V(G) \setminus X| + |X| \leq [(2n - 1) \cdot 2^{k-1} + n - 7] + 3k - 5 < n \cdot 2^k + 1$$

for all $k \geq 3$, a contradiction. We claim that $k \geq 4$. Suppose $k = 3$. We may assume that the third color is green. Since $|X| \geq 3k-4=5$, we have $|X_r^*| \geq 2$ and $|X_g^*| \geq 2$. By Claim 3.1.1 applied to A_p and R_G^*, $|R_G^*| \leq n$. Thus $|A_p|+|B_G| = |G|-|R_G^*|-|X_g^*| \geq 8n+1-n-3 = 7n-2$. Thus either $|A_p| \geq 3n+2$ or $|B_G| \geq 3n+2$. We may assume that $|A_p| \geq 3n+2$. By Theorem 1.3.2, $G[A_p]$ has either a red or a green C_{2n}. Thus either $G[A_p \cup X_r^*]$ has a red C_{2n+1} or $G[A_p \cup X_g^*]$ has a green C_{2n+1}, a contradiction, meaning that $k \geq 4$, as claimed.

Since $|X| \geq 3k-4$, by Claim 3.3.2, we may assume that $2 \leq |X_g^*| \leq 3$, and $|X_i^*| = 3$ for every color $i \in [k]$ other than blue and green. When choosing $X_1, X_2, \ldots, X_m$, let $q \in [m]$ be the smallest index such that for some color $\ell \in [k]$ other than blue, $|X_\ell^* \cap (X_1 \cup \cdots \cup X_q)| = 3$. By the choice of q, $|X_j^* \cap (X_1 \cup \cdots \cup X_{q-1})| \leq 2$ for all $j \in [k]$. By property (b) when choosing $X_1, X_2, \ldots, X_m$, there are possibly two colors $q_1, q_2 \in [k]$ such that $q_1 = \ell$, $|X_{q_1}^* \cap (X_1 \cup \cdots \cup X_q)| = 3$ and $|X_{q_2}^* \cap (X_1 \cup \cdots \cup X_q)| \leq 3$. Since $X_b^* = \emptyset$, $k \geq 4$ and $|X_i^*| = 3$ for every color $i \in [k]$ other than blue and green, we see that $q < m$ and so $|(X_1 \cup \cdots \cup X_q) \setminus (X_{q_1}^* \cup X_{q_2}^*)| \leq 2(k-4)$. By Claim 3.3.1 applied to $X_{q_1}^*$ and $V(G) \setminus (X_1 \cup \cdots \cup X_q)$, $|V(G) \setminus (X_1 \cup \cdots \cup X_q)| \leq (2n-1) \cdot 2^{k-1} + (n-7)$. But then

$$\begin{aligned}
|G| &= |(X_1 \cup \cdots \cup X_q) \setminus (X_{q_1}^* \cup X_{q_2}^*)| + |X_{q_1}^* \cup X_{q_2}^*| + |V(G) \setminus (X_1 \cup \cdots \cup X_q)| \\
&\leq 2(k-4) + 6 + [(2n-1) \cdot 2^{k-1} + (n-7)] \\
&< n \cdot 2^k + 1
\end{aligned}$$

for all $k \geq 4$, a contradiction. This proves that $X''' = \emptyset$, as claimed. Thus $|X| \leq 2(k-1)$.

Since neither $G[A_p]$ nor $G[B_G]$ has blue edges and $X_b^* = \emptyset$, we see that neither $G[A_p \cup X']$ nor $G[B_G \cup X'']$ has blue edges. By the choice of k, $|A_p \cup X'| \leq n \cdot 2^{k-1}$ and $|B_G \cup X''| \leq n \cdot 2^{k-1}$. We claim that $G[R_G]$ has blue edges.

Suppose $G[R_G]$ has no blue edges. Then $G[A_p \cup R_G \cup X']$ has no blue edges. By the choice of k, $|A_p \cup R_G \cup X'| \le n \cdot 2^{k-1}$. But then $|B_G \cup X''| = |G| - |A_p \cup R_G \cup X'| \ge n \cdot 2^{k-1} + 1$, a contradiction. Thus $G[R_G]$ has blue edges, as claimed. Then $|R_G| \ge 2$. By Claim 3.3.13, $2 \le |R_G| \le |R^*_G| \le 2n$. Suppose $|R^*_G| \ge n-1$. We claim that $|A_p \cup (X' \setminus X^*_r)| + |R^*_G| \le n \cdot 2^{k-2} + \max\{2n, k+n-1\}$. If $|R^*_G| \ge n$, then by Claim 3.1.1, $G[A_p]$ has no red edges and so $G[A_p \cup (X' \setminus X^*_r)]$ has no red edges. By the choice of k, $|A_p \cup (X' \setminus X^*_r)| \le n \cdot 2^{k-2}$ and so $|A_p \cup (X' \setminus X^*_r)| + |R^*_G| \le n \cdot 2^{k-2} + 2n$. If $|R^*_G| = n-1$, then applying Claim 3.3.1 to R^*_G and A_p, $|A_p| \le n \cdot 2^{k-2} + 2$. Thus $|A_p \cup (X' \setminus X^*_r)| + |R^*_G| \le (n \cdot 2^{k-2} + 2) + (k-2) + (n-1) = n \cdot 2^{k-2} + k + n - 1$, and so $|A_p \cup (X' \setminus X^*_r)| + |R^*_G| \le n \cdot 2^{k-2} + \max\{2n, k+n-1\}$, as claimed. But then

$$|G| = |A_p \cup (X' \setminus X^*_r)| + |R^*_G| + |B_G \cup (X'' \setminus X^*_r)| \le (n \cdot 2^{k-2} + \max\{2n, k+n-1\}) + n \cdot 2^{k-1} < n \cdot 2^k + 1,$$

for all $k \ge 3$, a contradiction.

Next, suppose $|R^*_G| = n-2$. By applying Claim 3.3.1 to R^*_G and A_p we see that $|A_p| \le (21-2n) \cdot 2^{k-2} + (5n-31)$. But then

$$\begin{aligned}
|G| &\le |A_p| + |B_G \cup X''| + |R^*_G| + |X' \setminus X^*_r| \\
&\le [(21-2n) \cdot 2^{k-2} + (5n-31)] + n \cdot 2^{k-1} + (n-2) + (k-2) \\
&< n \cdot 2^k + 1,
\end{aligned}$$

for all $k \ge 3$, a contradiction. Thus $|R^*_G| \le n-3$. If $|R^*_G| = 4$, then $n = 7$ and so by Claim 3.3.1 applied to R^*_G and A_p, $|A_p| \le 11 \cdot 2^{k-2}$. However,

$$\begin{aligned}
|G| &\le |A_p| + |B_G \cup X''| + |R^*_G| + |X' \setminus R^*_G| \\
&\le 11 \cdot 2^{k-2} + 7 \cdot 2^{k-1} + 4 + (k-2) \\
&< 7 \cdot 2^k + 1,
\end{aligned}$$

for all $k \geq 3$, a contradiction. Therefore, $|R^*_G| \leq 3$.

Let xy be a blue edge in $G[R_G]$. This is possible because $G[R_G]$ has blue edges. We claim that either x or y is red-complete to B_G. Suppose there exist $x', y' \in B_G$ such that xx' and yy' are colored blue. Then $x' = y'$, else we obtain a blue C_{2n+1} by Claim 3.1.1 applied to B_G and $A_p \cup \{x, y\}$. Thus x' is the unique vertex in B_G such that $\{x, y\}$ is red-complete to $B_G \setminus x'$ in G and xx', yx' are colored blue. Then there exists $i \in [s]$ such that $A_i = \{x'\}$. Since $G[B_G]$ has no blue edges, we see that $\{x, y, x'\}$ must be red-complete to $B_G \setminus x'$ in G.

Now, if $|R^*_G| = 3$, let $R^*_G = \{x, y, z\}$. If either zx or zy is blue, then $X^*_r = \emptyset$ and by the above reasoning, z is also red-complete to $B_G \setminus x'$. The same is true if $z \in X^*_r$. By Claim 3.3.1, $|B_G \setminus x'| \leq (2n-3) \cdot 2^{k-2} + (n-7)$ and $|A_p| \leq (2n-1) \cdot 2^{k-2} + (n-7)$. But then

$$\begin{aligned}
|G| &= |A_p| + |B_G \setminus x'| + |R^*_G \cup x'| + |X| \\
&\leq [(2n-1) \cdot 2^{k-2} + (n-7)] + [(2n-3) \cdot 2^{k-2} + (n-7)] + 4 + 2(k-2) \\
&< n \cdot 2^k + 1
\end{aligned}$$

for all $k \geq 3$, a contradiction. Therefore, we may assume both zx and zy are red, but that $z \notin X^*_r$.

In what follows, we now assume $2 \leq |R_G| \leq |R^*_G| \leq 3$. By Claim 3.3.1 applied to $\{x, y, x'\}$ and $B_G \setminus x'$, $|B_G \setminus x'| \leq (2n-1) \cdot 2^{k-2} + n - 7$. Note that $G[A_p \cup X' \cup \{x, z\}]$ has no blue edges if $|R^*_G| = 3$, and similarly $G[A_p \cup X' \cup \{x\}]$ if $|R_G| = 2$. Then $|X''| \geq k - 2$, else,

$$\begin{aligned}
|G| &= |A_p \cup X' \cup \{x, z\}| + |B_G \setminus x'| + |\{y, x'\}| + |X''| \\
&\leq n \cdot 2^{k-1} + [(2n-1) \cdot 2^{k-2} + n - 7] + 2 + (k-3) \\
&< n \cdot 2^k + 1,
\end{aligned}$$

for all $k \geq 3$, a contradiction. Since $2 \leq |R_G| \leq |R_G^*| \leq 3$, we see that $|X_r^*| \leq 1$. It follows that $|X_i^*| = 2$ for all colors $i \in [k]$ other than red and blue. Then neither $G[A_p]$ nor $G[B_G \setminus \{x'\}]$ has a monochromatic C_{2n-1} in any color $i \in [k]$ other than red and blue. Clearly, neither $G[A_p]$ nor $G[B_G \setminus \{x'\}]$ has red C_{2n-1} because $\{x, y\}$ is red-complete to both A_p and $B_G \setminus \{x'\}$. By Theorem 1.3.26 for $n = 6$ and Theorem 1.3.27 for $n = 7$, $|B_G \setminus x'| \leq (n-1) \cdot 2^{k-1}$ and $|A_p| \leq (n-1) \cdot 2^{k-1}$ (note that although proved simultaneously here, the proof for $GR_k(C_{13})$ is independent of the proof for $GR_k(C_{15})$, so we may use that $GR_k(C_{13}) = 6 \cdot 2^k + 1$). But then

$$|G| = |A_p| + |B_G \setminus x'| + |R_G^* \cup \{x'\}| + |X \setminus X_r^*| \leq (n-1) \cdot 2^{k-1} + (n-1) \cdot 2^{k-1} + 4 + 2(k-2) < n \cdot 2^k + 1,$$

for all $k \geq 3$, a contradiction. This proves that either x or y is red-complete to B_G. We may assume that x is red-complete to B_G.

Suppose $|R_G| = 2$. Then $R_G = \{x, y\}$ and $|X_r^*| \leq 1$. It follows that neither $G[A_p \cup \{y\} \cup X']$ nor $G[B_G \cup \{x\} \cup X'']$ has blue edges. By minimality of k, $|A_p \cup \{y\} \cup X'| \leq n \cdot 2^{k-1}$ and $|B_G \cup \{x\} \cup X''| \leq n \cdot 2^{k-1}$. But then $|G| = |A_p \cup \{y\} \cup X'| + |B_G \cup \{x\} \cup X''| \leq n \cdot 2^{k-1} + n \cdot 2^{k-1} < n \cdot 2^k + 1$ for all $k \geq 3$, a contradiction. Thus $|R_G| = |R_G^*| = 3$. Then $X_r^* = \emptyset$ and $G[A_p]$ has no red C_{2n}. Clearly, $|X'| \leq k - 2$. We claim that $|X'| \leq k - 3$. Suppose $|X'| = k - 2$. Then $|X_i^*| \geq 1$ for all colors $i \in [k]$ other than red and blue. Thus $G[A_p]$ has no monochromatic C_{2n} in any colors $i \in [k]$ other than blue. Since $G[A_p]$ has no blue edges, by Theorem 1.3.23, $|A_p| \leq (n-1)(k-1) + 3n - 1$. Then $k = 3$, else,

$$|G| = |A_p| + |B_G \cup X''| + |R_G| + |X'| \leq [(n-1)(k-1) + 3n - 1] + n \cdot 2^{k-1} + 3 + (k-2) < n \cdot 2^k + 1$$

for all $k \geq 4$. By Theorem 1.3.2, we see that $|A_p| \leq 3n-2$ if $k=3$. But then

$$|G| = |A_p| + |B_G \cup X''| + |R_G| + |X'| \leq (3n-2) + 4n + 3 + 1 = 7n + 2 < 8n + 1.$$

Thus $|X''| \leq |X'| \leq k-3$, as claimed. Since x is red-complete to B_G, it follows that $G[B_G \cup \{x\} \cup X'']$ has no blue edges. By minimality of k, $|B_G \cup \{x\} \cup X''| \leq n \cdot 2^{k-1}$. By Claim 3.3.1 applied to R_G and A_p, $|A_p| \leq (2n-1) \cdot 2^{k-2} + n - 7$. But then

$$|G| = |A_p| + |B_G \cup \{x\} \cup X''| + |R_G \backslash x| + |X'| \leq [(2n-1) \cdot 2^{k-2} + n - 7] + n \cdot 2^{k-1} + 2 + (k-3) < n \cdot 2^k + 1$$

for all $k \geq 3$, a contradiction. Hence, $|A_{p-1}| \leq n$. ■

By Claim 3.3.14, $|A_{p-2}| \leq |A_{p-1}| \leq n$. Then $|B_G| \leq 2n$, because this is trivially true when $|B| \leq 2$, and follows from Claim 3.3.7 when $|B| \geq 3$. By Claim 3.3.13, $|R_G| \leq |R_G^*| \leq 2n$. Then $|B_G| + |R_G| \leq 4n$. Finally, recall that $|B_G| \geq |A_{p-1}| \geq 3$ because $A_{p-1} \subseteq B_G$. We first consider the case when $|R_G^*| \geq n$. Since $|A_p| \geq n+1$, by Claim 3.1.1, $G[A_p]$ has no red edges. We claim that $|B_G| \geq n$. Suppose $3 \leq |B_G| \leq n-1$. Then $|A_p| \leq (2n-1) \cdot 2^{k-2} + n - 7$ by Claim 3.3.1 applied to B_G and A_p. But then

$$|G| = |A_p| + |B_G| + |R_G^*| + |X \setminus X_r^*| \leq [(2n-1) \cdot 2^{k-2} + n - 7] + (n-1) + 2n + 3(k-1) < n \cdot 2^k + 1,$$

for all $k \geq 3$, a contradiction. Thus $|B_G| \geq n$, as claimed. By Claim 3.1.1, $G[A_p]$ has no blue edges and $X_b^* = \emptyset$, so $|X'''| \leq |X''| \leq k-1$. Since $G[A_p \cup X']$ has neither red nor blue edges, it follows that $|A_p \cup X'| \leq n \cdot 2^{k-2}$ by minimality of k. But then

$$|G| = |A_p \cup X'| + |X'' \cup X'''| + (|B_G| + |R_G|) \leq n \cdot 2^{k-2} + 2(k-1) + 4n < n \cdot 2^k + 1,$$

for all $k \geq 3$, a contradiction.

It remains to consider the case when $|R_G^*| \leq n-1$. If $|B_G^*| \geq n-1$, by Claim 3.3.1 applied to B_G^* and A_p, we have

$$|A_p| + |B_G^*| + |X \setminus (X_r^* \cup X_b^*)| \leq \begin{cases} (n \cdot 2^{k-1} + 2) + (n-1) + 3(k-2), & \text{if } |B_G^*| = n-1 \\ n \cdot 2^{k-1} + (2n+3) + 3(k-2), & \text{if } |B_G^*| \geq n. \end{cases}$$

Thus in either case, $|A_p| + |B_G^*| + |X \setminus (X_r^* \cup X_b^*)| \leq n \cdot 2^{k-1} + 2n + 3k - 3$. But then

$$|G| = (|A_p| + |B_G^*| + |X \setminus (X_r^* \cup X_b^*)|) + |R_G^*| \leq (n \cdot 2^{k-1} + 2n + 3k - 3) + (n-1) < n \cdot 2^k + 1,$$

for all $k \geq 3$, a contradiction. Thus $3 \leq |B_G^*| \leq n-2$. By Claim 3.3.11, either $|B_G^*| \geq 4$ or $|R_G^*| \geq 4$. By applying Claim 3.3.1 to B_G^* when $|B_G^*| \geq 4$ (or R_G^* when $|R_G^*| \geq 4$) and A_p, we have $|A_p| \leq (2n-3) \cdot 2^{k-1} + n - 7$. Then $|R_G^*| \geq n-2$, else

$$\begin{aligned} |G| &= |A_p| + |B_G^*| + |R_G^*| + |X \setminus (X_r^* \cup X_b^*)| \\ &\leq [(2n-3) \cdot 2^{k-1} + (n-7)] + (n-2) + (n-3) + 3(k-2) \\ &< n \cdot 2^k + 1, \end{aligned}$$

for all $k \geq 3$ and $n \in \{6, 7\}$, a contradiction. Thus $n-2 \leq |R_G^*| \leq n-1$. By Claim 3.3.1 applied to R_G^* and A_p, $|A_p| \leq (21-2n) \cdot 2^{k-1-q} + (5n-31)$. But then

$$\begin{aligned} |G| &= |A_p| + |B_G^*| + |R_G^*| + |X \setminus (X_r^* \cup X_b^*)| \\ &\leq [(21-2n) \cdot 2^{k-1} + (5n-31)] + (n-2) + (n-1) + 3(k-2) \\ &< n \cdot 2^k + 1, \end{aligned}$$

for all $k \geq 3$ and $n \in \{6, 7\}$, a contradiction.

This completes the proof of Theorem 1.3.27. $\blacksquare$

CHAPTER 4: IMPROVED UPPER BOUND FOR $GR_k(C_{2n+1})$

4.1 Proof of Theorem 1.3.28

Let $n \geq 8$ be given as in the statement. For all $k \geq 1$, define the function

$$f(k,n) := \begin{cases} 2n+1 & \text{if } k=1 \\ 4n+1 & \text{if } k=2 \\ (n \ln n) \cdot 2^k - (k+1)n + 1 & \text{if } k \geq 3. \end{cases}$$

Clearly, $GR_1(C_{2n+1}) \leq f(1,n)$ and by Theorem 1.3.2, $GR(C_{2n+1}, C_{2n+1}) \leq f(2,n)$. It suffices to show that $GR_k(C_{2n+1}) \leq f(k,n)$ for all $k \geq 3$. Let $G := K_{f(k,n)}$ and let $c : E(G) \to [k]$ be any Gallai-coloring of G. Suppose that G does not contain any monochromatic copy of C_{2n+1} under c. Then c is bad. Among all complete graphs on $f(k,n)$ vertices with a bad Gallai k-coloring, we choose G with k minimum. Let $X_1, \ldots, X_k$ be disjoint subsets of $V(G)$ such that for each $i \in [k]$, X_i (possibly empty) is mc-complete in color i to $V(G) \setminus \bigcup_{i=1}^k X_i$. Choose $X_1, \ldots, X_k$ so that $\sum_{i=1}^k |X_i| \leq (k+1)n$ is as large as possible. Denote $X := \bigcup_{i=1}^k X_i$. Then $|X| \leq (k+1)n$. We next prove several claims.

Claim 4.1.1 *For all $i \in [k]$, $|X_i| \leq n-3$.*

Proof. Suppose $|X_i| \geq n-2$ for some color $i \in [k]$. We may assume that color i is blue. We next show that $|G \setminus X| \leq f(k-1,n)+3$. Suppose $|G \setminus X| \geq f(k-1,n)+4$. Let A be a minimal set of vertices of $G \setminus X$ such that $G \setminus (X \cup A)$ has no blue edges. By minimality of k, $|G \setminus (X \cup A)| \leq f(k-1,n)-1$. Then $|A| \geq 5$ and so $G \setminus X$ must contain blue edges. Thus $|X_i| \leq n-1$, otherwise for any blue edge uv in $G \setminus X$, we obtain a blue C_{2n+1} by Lemma 3.1.1.

Let $t := n-|X_i|$. Then $t \in \{1,2\}$ because $n-2 \leq |X_i| \leq n-1$. It follows that $G\backslash X$ has a blue $H \in \{(2t+1)K_2, (2t-1)K_2 \cup P_3, K_2 \cup 2P_{t+1}, tK_2 \cup P_{t+2}, P_4 \cup (t-1)P_3, K_2 \cup P_{2t+1}, P_{2t+2}\}$. But then we obtain a blue C_{2n+1} using $n-t$ vertices in X_i, all vertices and edges of H, and $n+t+1-|H|$ vertices in $V(G) \setminus (X \cup V(H))$, a contradiction. This proves that $|G \setminus X| \leq f(k-1,n)+3$. Thus

$$\begin{aligned}|G| &= |X| + |G \setminus X| \leq (k+1)n + f(k-1,n) + 3 \\ &= \begin{cases} 4n + (4n+1) + 3, & \text{if } k = 3 \\ (k+1)n + [(n \ln n) \cdot 2^{k-1} - kn + 1] + 3, & \text{if } k \geq 4 \end{cases}\end{aligned}$$

so that in any case, $|G| < f(k,n)$ for all $k \geq 3$ and $n \geq 8$, a contradiction. ■

Claim 4.1.2 *$X_i = \emptyset$ for some $i \in [k]$.*

Proof. Suppose $X_i \neq \emptyset$ for every $i \in [k]$. By Claim 4.1.1, $|X| \leq k(n-3)$. Then

$$|G \setminus X| \geq f(k,n) - k(n-3) = (n \ln n) \cdot 2^k - (k+1)n + 1 - k(n-3) \geq (n-1)k + 3n,$$

for all $k \geq 3$ and $n \geq 8$. By Theorem 1.3.23, $G \setminus X$ contains a monochromatic C_{2n}, and thus G contains a monochromatic C_{2n+1}, since $X_i \neq \emptyset$ for all $i \in [k]$, a contradiction. ■

By Claims 4.1.1 and 4.1.2, $|X| \leq (k-1)(n-3)$. Consider now a Gallai partition of $G \setminus X$ with parts $A_1, \ldots, A_p$, where $p \geq 2$ and $|A_1| \leq |A_2| \leq \cdots \leq |A_p|$. By Theorem 1.3.2, $p \leq 4n$. Additionally, let us define the sets

$$B := \{a_i \in \{a_1, \ldots, a_{p-1}\} \mid a_i a_p \text{ is colored blue in } \mathcal{R}\}$$

$$R := \{a_j \in \{a_1, \ldots, a_{p-1}\} \mid a_j a_p \text{ is colored red in } \mathcal{R}\}$$

This motivates us to define the related sets in G as $B_G := \bigcup_{a_i \in B} A_i$ and $R_G := \bigcup_{a_j \in R} A_j$. Moreover, we employ the notation X_r to indicate X_i when $i =$ red, and likewise X_b when $i =$ blue.

Claim 4.1.3 $|B_G \cup R_G| \geq 2n+1$.

Proof. Suppose $|B_G \cup R_G| \leq 2n$. Then every vertex in $B_G \cup R_G$ is either red- or blue-complete to A_p. We may assume that X_1 is red-complete to $V(G) \setminus X$ and X_2 is blue-complete to $V(G) \setminus X$. Let $X_1' := X_1 \cup R_G$, $X_2' := X_2 \cup B_G$, and $X_i' := X_i$ for all $i \in \{3, \ldots, k\}$. But then

$$\left| \bigcup_{i=1}^{k} X_i' \right| = |X \cup B_G \cup R_G| \leq (k-1)(n-3) + 2n = (k+1)n - 3(k-1) < (k+1)n,$$

contrary to the choice of $X_1, \ldots, X_k$. Thus $|B_G \cup R_G| \geq 2n+1$. ■

Claim 4.1.4 *If* $|A_p| \leq n$, *then* $|A_{p-2}| \leq \lfloor n/2 \rfloor$.

Proof. Let $q := \lfloor n/2 \rfloor$. Suppose $|A_p| \leq n$ but $|A_{p-2}| \geq q+1$. Then $|G| - |A_p \cup A_{p-1} \cup A_{p-2}| - |X| \geq f(k,n) - 3n - (n-3)(k-1) \geq 4n$ for all $k \geq 3$ and $n \geq 8$. Let B_1, B_2, B_3 be a permutation of A_{p-2}, A_{p-1}, A_p such that B_2 is, say, blue-complete to $B_1 \cup B_3$ in G. Let $b_1, \ldots, b_{q+1} \in B_1$, $b_{q+2}, \ldots, b_{2q+2} \in B_2$, and $b_{2q+3}, \ldots, b_{3q+3} \in B_3$. Let $A := V(G) \setminus (B_1 \cup B_2 \cup B_3 \cup X)$, and define

$$B_1^* := \{v \in A \mid v \text{ is blue-complete to } B_1 \text{ and red-complete to } B_3 \text{ in } G\}$$

$$B_2^* := \{v \in A \mid v \text{ is blue-complete to } B_1 \cup B_3 \text{ in } G\}$$

$$B_3^* := \{v \in A \mid v \text{ is red-complete to } B_1 \cup B_3 \text{ in } G\}$$

$$B_4^* := \{v \in A \mid v \text{ is red-complete to } B_1 \text{ and blue-complete to } B_3 \text{ in } G\}.$$

Then $A = B_1^* \cup B_2^* \cup B_3^* \cup B_4^*$ and so $|A| = |G| - |A_p \cup A_{p-1} \cup A_{p-2}| - |X| \geq 3n$. Note that $B_1^*, B_2^*, B_3^*, B_4^*$ are pairwise disjoint. Suppose first that B_1 is red-complete to B_3 in G. By Lemma 3.1.1 applied to B_3^* and $B_1 \cup B_3$, $|B_3^*| \leq n-1$. Thus $|B_1^*| + |B_2^*| + |B_4^*| \geq 3n-(n-1) = 2n+1$. By symmetry, we may assume that $|B_1^*|+|B_2^*| \geq n+1$. We claim that $G[B_1^* \cup B_2^* \cup B_4^*]$ has no blue edges. Suppose not. Let uv be a blue edge in $G[B_1^* \cup B_2^* \cup B_4^*]$. Since $|B_1^*| + |B_2^*| \geq n+1$, let $x_1, \ldots, x_{q-1} \in B_1^* \cup B_2^*$ be distinct vertices that are different from u and v. If $u, v \in B_1^* \cup B_2^*$, then we find a blue C_{2n+1} with vertices

$$\begin{cases} u, v, b_1, b_{q+2}, b_{2q+3}, b_{q+3}, \ldots, b_{3q+3}, b_{2q+2}, b_2, x_1, b_3, \ldots, x_{q-2}, b_q, & \text{if } n \text{ is even} \\ u, v, b_1, b_{q+2}, b_{2q+3}, b_{q+3}, \ldots, b_{3q+3}, b_{2q+2}, b_2, x_1, b_3, \ldots, x_{q-2}, b_q, x_{q-1}, b_{q+1}, & \text{if } n \text{ is odd} \end{cases}$$

a contradiction. Thus we may assume that $v \in B_4^*$. If $u \in B_1^* \cup B_2^*$, then we find a blue C_{2n+1} with vertices

$$\begin{cases} u, v, b_{2q+3}, b_{q+2}, b_{2q+4}, \ldots, b_{3q+3}, b_{2q+2}, b_1, x_1, \ldots, x_{q-2}, b_{q-1}, & \text{if } n \text{ is even} \\ u, v, b_{2q+3}, b_{q+2}, b_{2q+4}, \ldots, b_{3q+3}, b_{2q+2}, b_1, x_1, \ldots, b_{q-1}, x_{q-1}, b_q, & \text{if } n \text{ is odd} \end{cases}$$

a contradiction. Thus $u, v \in B_4^*$. But then we obtain a blue C_{2n+1} with vertices

$$\begin{cases} u, v, b_{2q+3}, b_{q+2}, b_1, x_1, b_2, \ldots, x_{q-1}, b_q, b_{q+3}, b_{2q+4}, b_{q+4}, \ldots, b_{2q+1}, b_{3q+2}, & \text{if } n \text{ is even} \\ u, v, b_{2q+3}, b_{q+2}, b_1, x_1, b_2, \ldots, x_{q-1}, b_q, b_{q+3}, b_{2q+4}, b_{q+4}, \ldots, b_{2q+2}, b_{3q+3}, & \text{if } n \text{ is odd} \end{cases}$$

a contradiction. This proves that $G[B_1^* \cup B_2^* \cup B_4^*]$ contains no blue edges.

Since $|B_1^*| + |B_2^*| + |B_4^*| \geq 2n+1$ and $|A_p| \leq n$, by Lemma 3.1.2, $G[B_1^* \cup B_2^* \cup B_4^*]$ has a red C_{2n+1}, a contradiction. Thus B_1 must be blue-complete to B_3. Then $|B_1 \cup B_2 \cup B_3| \leq 2n$, else we obtain a blue C_{2n+1} in $G[B_1 \cup B_2 \cup B_3]$. By Lemma 3.1.1 applied to $B_2 \cup B_2^*$ and

$B_1 \cup B_3$, we see that $|B_2^*| \leq q-1$. If $|B_1^*| \geq q$, let $x_1, \ldots, x_q \in B_1^*$ be distinct vertices. Then we find a blue C_{2n+1} with vertices

$$\begin{cases} b_1, b_{q+2}, b_{2q+3}, b_{q+3}, \ldots, b_{3q+3}, b_2, x_1, \ldots, b_q, x_{q-1}, & \text{if } n \text{ is even} \\ b_1, b_{q+2}, b_{2q+3}, b_{q+3}, \ldots, b_{3q+3}, b_2, x_1, \ldots, b_q, x_{q-1}, b_{q+1}, x_q, & \text{if } n \text{ is odd} \end{cases}$$

a contradiction. Thus $|B_1^*| \leq q-1$, and similarly, $|B_4^*| \leq q-1$. Therefore,

$$\begin{aligned} |B_3^*| &= |G| - |X| - |B_1 \cup B_2 \cup B_3| - |B_1^*| - |B_2^*| - |B_4^*| \\ &\geq f(k,n) - (k-1)(n-3) - 2n - (q-1) - (q-1) - (q-1) \\ &\geq 2n+1. \end{aligned}$$

By Lemma 3.1.1 applied to B_3^* and $B_1 \cup B_3$, $G[B_3^*]$ contains no red edges. But then by Lemma 3.1.2 and the fact that $|A_p| \leq n$ and $|B_3^*| \geq 2n+1$, $G[B_3^*]$ must contain a blue C_{2n+1}, a contradiction. ∎

Claim 4.1.5 $|A_p| \geq n+1$.

Proof. Suppose $|A_p| \leq n$. Let $r_i := |\{j \in [p] : |A_j| \geq i\}|$. Then $|G \setminus X| = \sum_{i=1}^{n} r_i$. Let

$q := |A_{p-2}|$. By Lemma 3.1.3 and Claim 4.1.4,

$$
\begin{aligned}
|G| &= |X| + (|A_p| - q) + (|A_{p-1}| - q) + \sum_{i=1}^{q} r_i \\
&\le (k-1)(n-3) + (2n-2q) + \sum_{i=1}^{q} 4\left\lceil \frac{n}{i} \right\rceil \\
&\le (k-1)(n-3) + (2n-2q) + \sum_{i=1}^{q} 4\left(\frac{n}{i} + 1\right) \\
&= (k-1)(n-3) + 2n + 2q + 4n\sum_{i=1}^{q} \frac{1}{i} \\
&\le \begin{cases}
(k-1)(n-3) + 2n + 8 + 4n\displaystyle\sum_{i=1}^{4} \frac{1}{i}, & n \in \{8,9\},\ q = \left\lfloor \frac{n}{2} \right\rfloor = 4 \\
(k-1)(n-3) + 2n + 10 + 4n\displaystyle\sum_{i=1}^{5} \frac{1}{i}, & n = 10,\ q = \left\lfloor \frac{n}{2} \right\rfloor = 5 \\
(k-1)(n-3) + 2n + 2\left\lfloor \dfrac{n}{2} \right\rfloor + 4n\left(1 + \displaystyle\int_1^{\lfloor \frac{n}{2} \rfloor} \frac{1}{x}\,dx\right), & n \ge 11,\ q = \left\lfloor \frac{n}{2} \right\rfloor \\
(k-1)(n-3) + 2n + 2\left(\left\lfloor \dfrac{n}{2} \right\rfloor - 1\right) + 4n\left(1 + \displaystyle\int_1^{\lfloor \frac{n}{2} \rfloor - 1} \frac{1}{x}\,dx\right), & n \ge 8,\ q \le \left\lfloor \frac{n}{2} \right\rfloor - 1.
\end{cases} \\
&\le \begin{cases}
(k-1)(n-3) + 2n + 8 + \dfrac{25n}{3}, & n \in \{8,9\},\ q = \left\lfloor \frac{n}{2} \right\rfloor = 4 \\
(k-1)(n-3) + 2n + 10 + \dfrac{137n}{15}, & n = 10,\ q = \left\lfloor \frac{n}{2} \right\rfloor = 5 \\
(k-1)(n-3) + 2n + 2\left\lfloor \dfrac{n}{2} \right\rfloor + 4n\left(1 + \ln\dfrac{n}{2}\right), & n \ge 11,\ q = \left\lfloor \frac{n}{2} \right\rfloor \\
(k-1)(n-3) + 2n + 2\left(\left\lfloor \dfrac{n}{2} \right\rfloor - 1\right) + 4n\left[1 + \ln\left(\dfrac{n}{2} - 1\right)\right], & n \ge 8,\ q \le \left\lfloor \frac{n}{2} \right\rfloor - 1.
\end{cases} \\
&< f(k,n),
\end{aligned}
$$

for all $k \ge 3$ and $n \ge 8$, a contradiction. ∎

Claim 4.1.6 *$|B_G| \geq n+1$ and $|R_G| \geq n+1$. Moreover, both X_r and X_b are empty, giving $|X| \leq (k-2)(n-3)$.*

Proof. We may assume that $|B_G| \geq |R_G|$. By Claim 4.1.3, $|B_G| \geq n+1$. Suppose for a contradiction that $|R_G| \leq n$. Since $|A_p| \geq n+1$ by Claim 4.1.5, then by Lemma 3.1.1 $X_b = \emptyset$ and neither $G[A_p]$ nor $G[B_G]$ has blue edges. By minimality of k, $|A_p| \leq f(k-1,n)-1$ and $|B_G| \leq f(k-1,n)-1$. Note that $|X| > (k-2)(n-3)$, otherwise

$$|G| = |A_p| + |B_G| + |R_G| + |X| \leq 2[f(k-1,n)-1] + n + (k-2)(n-3) < f(k,n)$$

for all $k \geq 3$ and $n \geq 8$, a contradiction. By Claim 4.1.1, $X_i \neq \emptyset$ for all $i \in [k]$ other than blue. Thus neither $G[A_p]$ nor $G[B_G]$ has monochromatic C_{2n}. By Theorem 1.3.23, $|A_p| \leq (k-1)(n-1)+3n-1$ and $|B_G| \leq (k-1)(n-1)+3n-1$. But then

$$|G| = |A_p| + |B_G| + |R_G| + |X| \leq 2[(k-1)(n-1)+3n-1] + n + (k-1)(n-3) < f(k,n)$$

for all $k \geq 3$ and $n \geq 8$, a contradiction. Thus, $|B_G| \geq n+1$ and $|R_G| \geq n+1$. Therefore, Lemma 3.1.1 implies $X_r = \emptyset$, and thus we have $|X| \leq (k-2)(n-3)$. ■

Claim 4.1.7 *$|A_{p-2}| \leq n$.*

Proof. Suppose $|A_{p-2}| \geq n+1$. Then $n+1 \leq |A_{p-2}| \leq |A_{p-1}| \leq |A_p|$ and so $\mathcal{R}[\{a_{p-2}, a_{p-1}, a_p\}]$ is not a monochromatic triangle in $\mathcal{R}$ (else $G[A_p \cup A_{p-1} \cup A_{p-2}]$ has a monochromatic C_{2n+1}). Let B_1, B_2, B_3 be a permutation of A_{p-2}, A_{p-1}, A_p such that B_2 is, say blue-complete, to $B_1 \cup B_3$ in G. Then B_1 must be red-complete to B_3 in G. By Claim 4.1.6, $|X| \leq (k-2)(n-3)$. Let $A := V(G) \setminus (B_1 \cup B_2 \cup B_3 \cup X)$. By Claim 3.1.1 again, $G[B_2]$ has no blue edges, and neither $G[B_1]$ nor $G[B_3]$ has red or blue edges. By minimality of k, $|B_1| \leq f(k-2,n)-1$ and $|B_3| \leq f(k-2,n)-1$. Observe that

$$\begin{aligned}
|A \cup B_2| &= |G| - |B_1| - |B_3| - |X| \\
&= f(k,n) - 2[f(k-2,n)-1] - (k-2)(n-3) \\
&= \begin{cases} (8n\ln n - 4n + 1) - 2(2n) - (n-3), & k=3 \\ (16n\ln n - 5n + 1) - 2(4n) - 2(n-3), & k=4 \\ [(n\ln n)\cdot 2^k - (k+1)n + 1] - 2[(n\ln n)\cdot 2^{k-2} - (k-1)n] - (k-2)(n-3), & k \geq 5. \end{cases}
\end{aligned}$$

In any case, we see that $|A \cup B_2| \geq f(k-1,n)$. By minimality of k, $G[A \cup B_2]$ must have blue edges. By Claim 3.1.1, no vertex in A is red-complete to $B_1 \cup B_3$ in G, and no vertex in A is blue-complete to $B_1 \cup B_2$ or $B_2 \cup B_3$ in G. This implies that A must be red-complete to B_2 in G. It follows that $G[A]$ must contain a blue edge, say uv. Let $b_1, \ldots, b_{n-1} \in B_1$, $b_n, \ldots, b_{2n-2} \in B_2$, and $b_{2n-1} \in B_3$. If $\{u,v\}$ is blue-complete to B_1, then we obtain a blue C_{2n+1} with vertices $b_1, u, v, b_2, b_n, b_{2n-1}, b_{n+1}, b_3, b_{n+2}, \ldots, b_{n-1}, b_{2n-2}$ in order, a contradiction. Thus $\{u,v\}$ is not blue-complete to B_1. Similarly, $\{u,v\}$ is not blue-complete to B_3. Since no vertex in A is red-complete to $B_1 \cup B_3$, we may assume that u is blue-complete to B_1 and v is blue-complete to B_3. But then we obtain a blue C_{2n+1} with vertices $b_1, u, v, b_{2n-1}, b_n, b_2, b_{n+1}, \ldots, b_{n-1}, b_{2n-2}$ in order. ■

We may assume that $A_{p-1} \subseteq B_G$. By Claim 4.1.7, $|A_{p-2}| \leq n$. By Lemma 3.1.2, $|R_G| \leq 2n$ and $|B_G \setminus A_{p-1}| \leq 2n$. By Claim 4.1.6, $|X| \leq (k-2)(n-3)$. By minimality of k, $|A_p| \leq$

$f(k-2,n)-1$. Then

$$\begin{aligned}
|G| &= |A_p| + |A_{p-1}| + |B_G \setminus A_{p-1}| + |R_G| + |X| \\
&\leq 2[f(k-2,n)-1] + 2n + 2n + (k-2)(n-3) \\
&= \begin{cases} 2(2n) + 2n + 2n + (n-3), & k=3 \\ 2(4n) + 2n + 2n + 2(n-3), & k=4 \\ 2[(n\ln n)\cdot 2^{k-2} - (k-1)n] + 2n + 2n + (k-2)(n-3), & k \geq 5. \end{cases}
\end{aligned}$$

In any case, we see that $|G| < f(k,n)$ for all $k \geq 3$ and $n \geq 8$, a contradiction.

This completes the proof of Theorem 1.3.28. ■

CHAPTER 5: FUTURE WORK

In this chapter we discuss some further possible research areas related to the work in this dissertation.

5.1 Hadwiger Numbers

Several different avenues of study come to mind branching off of the work done in this dissertation. Certainly the most obvious project, but perhaps the least helpful, would entail exploring other forbidden subgraphs when $\alpha(G) = 2$ to verify HC. To date, only six forbidden subgraphs H such that $\alpha(H) \leq 2$ and $|H| \geq 6$ are known, which are W_5, $\overline{K_{1,5}}$, K_6, K_7, H_6, and H_7 (four of these were proven in our work). One could then complete the list of six vertex graphs (of which there are more than 30), but this seems a tedious task which may not be very instructive.

A more interesting question is the following. Let us consider a notably weaker conjecture that HC, attributed independently to Woodall, and Duchet and Meyniel (see [84]).

Conjecture 5.1.1 *For any graph G, $h(G) \geq |G|/\alpha(G)$.*

Coupled with Fact 1.2.5, HC implies Conjecture 5.1.1. Proving this conjecture when $\alpha(G) \leq 2$ would of course establish HC by the equivalence given in Theorem 1.2.11. But what if $\alpha(G) \geq 3$? This conjecture seems too difficult to prove in full generality, so one could certainly attempt a similar approach of forbidding certain subgraphs when, say, $\alpha(G) = 3$ in order to obtain partial evidence. The author tried in several ways to forbid C_5 and prove the conjecture holds for $\alpha(G) = 3$. However, this situation is far more complicated because of the increased difficulty in forcing cliques of the desired order. For $\alpha(G) \geq 3$ we are also without an analogue to Theorem 1.2.15 which was extremely helpful in our work. Even if one manages to show Conjecture 5.1.1 holds for $\alpha(G) = 3$ with certain forbidden subgraphs, such a result is still not general enough to satisfy most researchers interested in this topic.

Recall that if HC is true, then there must exist a partition of $V(G)$ into t independent sets for any K_{t+1} minor-free graph. This condition on the partition can be relaxed in the following ways. First, we can instead look for a partition of $V(G)$ into t (not necessarily independent) sets $V_1, \ldots, V_t$, called a *defective coloring*, such that $\Delta(G[V_i]) \leq d$ for all $i \in [t]$ for some $d \geq 0$, called the *defect*. Showing that such a partition exists with defect zero is exactly HC. Some promising results have been obtained to this effect (see the dynamic survey by Wood [88]). An initial result of Edwards, Kang, Kim, Oum, and Seymour [32] in 2015 showed that every K_{t+1} minor-free graph G is t-colorable with defect $O((t+1)^2 \log(t+1))$. Van den Heuvel and Wood [85] improved this result in 2018 to show that G is t-colorable with defect $t-1$. Recall that HC remains open for all $t \geq 6$. If considering all such values of t is too hard, an interesting problem would then be to consider the case when $t = 6$ and improve the defect to be as close to zero as possible. A second relaxation of HC comes about by asking the same question, but instead of bounding the maximum degree we restrict the order of the components found in $V_1, \ldots, V_t$. If the maximum order of any monochromatic component is c, then we say G is t-colorable with *clustering* c. Several results have been obtained in this direction as well but we omit them here (see [88] for more information).

5.2 Ramsey Theory and Gallai-Ramsey Numbers

As mentioned in Chapter 1, our result concerning the Gallai-Ramsey numbers of odd cycles was generalized shortly after we completed our project by [18], though our work was cited by that group. It would be interesting if one could find general bounds on $GR_k(K_n)$ for all $n \geq 6$ because this could potentially help to improve bounds on $R(K_n, K_n)$ in those cases. For the case when $n = 5$, it would be extremely helpful to settle this case independent of $R(K_5, K_5)$. We ideally would like the Gallai-Ramsey number to indicate something about the classical Ramsey number because the latter is typically so difficult to compute. Having a proof of $GR_k(K_5)$ independent of $R(K_5, K_5)$ would either provide more evidence for or disprove Conjecture 1.3.18.

Because $GR_k(C_{2n+1})$ has been settled for all n, another natural area of exploration is the classical three-color Ramsey number. Currently, $R_3(C_n)$ is known only for $3 \leq n \leq 7$ (see the survey by Radziszowski [69]). Conjecture 1.3.4 (the Triple Odd Cycle conjecture) mentioned in Chapter 1 remains open and so verifying that $R_3(C_9) = 33$ would provide more evidence of the conjecture. However, the proof that $R_3(C_7) = 25$ is long and difficult (see [37]), so a new technique must be introduced in order to make further progress in this area.

Recent work has also been conducted on Gallai-colorings of hypergraphs (see [14]). This seems an interesting area of study since even fewer Ramsey-type results are known for hypergraphs. Therefore, one could study the odd cycle Gallai-Ramsey problem in the context of 3-uniform hypergraphs to further generalize the results in this dissertation.

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