A Multiple Case Study Examining How Third-Grade Students Who Struggle in Mathematics Make Sense of Fraction Concepts

2016

Rebecca Gault
University of Central Florida

A MULTIPLE CASE STUDY EXAMINING HOW THIRD-GRADE STUDENTS WHO STRUGGLE IN MATHEMATICS MAKE SENSE OF FRACTION CONCEPTS

by

REBECCA GAULT
B.S. Louisiana State University, 1992
M.A. University of Central Florida, 2006

A dissertation submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy
in the College of Education and Human Performance
at the University of Central Florida
Orlando, Florida

Summer Term
2016

Major Professor: Enrique Ortiz

ABSTRACT

A qualitative multiple case study was conducted to reveal the sense-making processes third-grade students who struggle in mathematics used to build an understanding of fraction concepts. Purposive sampling identified three participants who were struggling in a local school's third grade mathematics classes. This research describes how these participants made sense of fraction concepts through their strengths and struggles while engaged in 15 small-group intervention sessions. Vygotsky's (1934/1986/2012) theory that children's optimal learning is supported by teacher-student interactions was used as an interpretive framework. Tasks were developed over the course of the intervention sessions with consideration of a model developed by Lesh, Post, and Behr (1987) for connecting mathematical representations and the Common Core State Standards for Mathematics (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). Data, including transcripts, tapes, and artifacts, were analyzed using two frameworks. These were Geary's (2003) classification of three subtypes of learning disabilities in mathematics and Anghileri's (2006) descriptions of social-constructivist scaffolding techniques. The first analysis resulted in a description of each participant's strengths and struggles, including alignment with Geary's subtypes, and how these strengths and struggles interacted with participant's construction of knowledge about fractions. The second analysis described episodes of learning that were supported by social-constructivist scaffolding techniques and revealed how participants made sense of fractions through their interactions with each other, the researcher, and intervention tasks. The researcher found that each participant's learning process, including struggles, was unique, with each interacting in different ways with tasks, manipulatives, pictorial representations, and questioning. For each

participant, however, scaffolding techniques oriented around prompting and probing questions, participant verbalizations, and interactions with connected fraction representations were critical in their learning process.

This is dedicated to my amazing husband, Allen. You are my superhero. To my sons, Justin and Zachary, thank you for inspiring me to always seek to understand. To my mom and dad, Holly and Milford, thank you for showing me through your actions that I can find a solution to any problem life presents.

ACKNOWLEDGMENTS

Many have supported and encouraged me throughout this process. First, thank you to my committee chair, Dr. Enrique Ortiz, and the members of my committee: Dr. Juli Dixon, Dr. Megan Nickels, and Dr. Mary Little. Thank you to my fellow doctoral students: Heidi Eisenreich, Katie Harshman, Laura Tapp, Nesrin Sahin, Vernita Glenn-White, Edward Knote, and Makini Sutherland. I have learned much from each and every one of you. Finally, thank you to my friends and family who gave unwavering support to me as I chased this dream.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

CHAPTER 1: INTRODUCTION

The differences that people are born with are eclipsed by the learning opportunities they encounter throughout life, that, combined with the right messages, can propel children to the highest levels.

Jo Boaler, *The Elephant in the Classroom,* 2015

Research in mathematics education has long emphasized the need to teach mathematics from a conceptually-oriented methodology, particularly elementary school mathematics (Carpenter, Franke, Jacobs, Fennema, & Empson 1998; Cobb, Wood, Yackel, Nicholls, Wheatley, Trigatti, & Perlwitz, 1991; Cramer, Post, & del Mas 2002; Empson, 1999). Recently the widespread implementation of the Common Core State Standards for Mathematics (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) has also encouraged some movement toward conceptual learning in public schools (Larson, 2012). Larson (2012) asserted that to be successfully implemented this movement must become widespread.

As conceptual learning becomes more common, teachers, mathematics coaches, and administrators have begun to consider whether this approach is effective for students who struggle. For this study, conceptual learning is defined as learning that occurs when a student applies mathematical thinking that builds on his or her current mathematical knowledge, understands the methods he or she uses to solve problems, including problem types new to the student, and creates connections between different mathematical representations. Conceptual learning builds conceptual knowledge which Kilpatrick, Swafford, and Findell (2001) refer to as "knowledge that has been learned with understanding" (p. 119). Conceptual learning is supported by instruction that emphasizes students' thinking strategies, cognitively challenging tasks, and multiple mathematical representations (Kilpatrick et al., 2001).

For this study, the researcher used Geary's (2003) definition of a student who struggles as one who performs in the lowest 25th percentile on nationally normed achievement tests. Many researchers (Fuchs & Fuchs, 2003; Fuchs, Schumacher, Long, Nankung, Hamlett, Jordan, Gersten, Cirino, Siegler, & Changas, 2013; Hecht & Vagi, 2010; Jitendra, Dupuis, & Zaslofsky, 2014; Lewis, 2010; Mazzocco & Devlin, 2008) have used strategies similar to Geary's to define students who struggle as those who perform below a certain percentile on achievement tests. In studies about students who struggle in mathematics, the students may or may not be identified as having a learning disability. While some studies (Fuchs & Fuchs, 2003; Geary, 2003; Lewis, 2010) have used these criteria to assert that these students have a mathematical learning disability, others (Hecht & Vagi, 2010; Jitendra, Dupuis, & Zaslofsky, 2014) do not comment on learning disability status, instead focusing on the students' status as struggling or experiencing difficulties in mathematics. Mazzocco and Devlin (2008) distinguished between performance on the Woodcock-Johnson-Revised Calculation Subtest (Houghton Mifflin Harcourt, 1989) score at or below the 10th percentile, which they stated indicates a mathematical learning disability, and performance above the 10th but at or below 25th percentile, which they stated indicates that a student is struggling in mathematics. This study did not seek to determine whether participants had a mathematics learning disability, but instead included participants whose performance on state achievement tests and progress monitoring assessments indicated that they were struggling in mathematics. This is in line with other researchers (Fuchs et al., 2013; Hecht & Vagi, 2010; Jitendra, Dupuis, & Zaslofsky, 2014).

Some studies have focused on the types of scaffolding techniques that may support students who struggle in the process of learning (Anghileri, 2006; Broza & Kolikant, 2015; Dale & Scherrer, 2015; Moschkovich, 2015; Pfister, Moser Opitz, & Pauli, 2015; Putambekar & Hubscher, 2005; Van de Pole, Volman, & Beishuizen, 2010). For this study, descriptions of scaffolding techniques provided by Anghileri (2006) based on her own previous work (Anghileri, 1995; Anghileri & Baron, 1998; Coltman, Anghileri, & Petyaeva, 2002) and the work of other scholars (Tharpe & Gallimore, 1988; Wood, 1994; Wood, Bruner, & Ross, 1976) were used to identify a prevalent set of scaffolding techniques that were used in intervention sessions. These techniques are further described in Chapter 2 and include a) prompting and probing; b) looking, touching, and verbalizing; c) interpreting student work or talk; d) simplifying a problem; e) explaining and justifying; and f) negotiated meaning. Central to each of these scaffolding techniques is the intention that the student and teacher work to together to co-construct knowledge in a social setting (Anghileri, 2006). The use of these scaffolding techniques was originally intended to allow the researcher to gain a more thorough understanding of the participants' thinking, including their struggles and strengths. As study analysis progressed, it became clear that episodes using scaffolding techniques could be classified according to these descriptions, and that participants' processes of making sense of fraction concepts could be analyzed within these episodes.

The majority of research concerning students who struggle learning mathematics has focused on intervention and teaching methods based on procedural instruction that is explicit and direct (Flores & Kaylor, 2007; Fuchs et al., 2013; Gersten, Chard, Jayanthi, Baker, Morphy, & Flojo, 2009; Jitendra, Griffin, McGoey, Gardill, Bhat, & Riley, 1998; Joseph & Hunter, 2001;

Kroesbergen & Van Luit, 2003). Flores and Kaylor (2007) conducted a study with 30 at–risk middle school participants in which fraction concepts were taught using direct instruction methods that used scripted lessons, demonstration and modeling of tasks by the teacher followed by guided practice, and verbal responses in unison to teacher cues. Fuchs and colleagues (2013) studied the performance of 129 at-risk fourth-grade students participating in fraction lessons oriented around a measurement interpretation of fractions compared to a control group of 130 at-risk fourth-grade students whose lessons focused on a part-whole interpretation of fractions. Although the intervention focused on developing fraction understanding, the lessons were constructed to begin with teacher or tutor demonstration of skills, followed by teacher guided group work, and ending with independent student work. Joseph and Hunter (2001) conducted a qualitative study with three eighth-grade students receiving special education services in which a cue card strategy was employed to help the participants successfully complete fraction addition and subtraction problems. Cue cards contained examples of fraction addition and subtraction problems showing detailed steps in the solution process. Teachers initially instructed students how to solve fraction operations problems while showing students how to use the cue cards. Students were then expected to choose the appropriate cue card to assist in solving problems. The procedural instruction strategies used in these studies stands in contrast to the intent of this study, which was to provide an intervention environment focused on conceptual learning in the form of student-led construction of solution strategies.

Furthermore, few qualitative studies have been conducted that focus on how learners who struggle come to make sense of mathematics such as fraction concepts (Hunt & Empson, 2015; Joseph & Hunter, 2001; Lewis, 2010, 2014). Studies concerning learners who struggle have tended to employ assessment methods that focus on standardized test performance or curriculum-based test results (Flores & Kaylor, 2007; Fuchs & Fuchs, 2003; Fuchs et al., 2013; Gersten, Chard et al., 2009; Jitendra et al., 1998; Kroesbergen & Van Luit, 2003). Ginsburg (1997) suggested that measures of quantitative performance lack the ability to qualitatively shed light on the sense-making process that students who struggle in mathematics employ to construct conceptions of fractions. He further proposed that the complexity of mathematical thinking indicates that there is likely to be multiple cognitive processes that contribute to a student's difficulties in mathematics. Also, Ginsburg asserted that researchers need to use investigative techniques that involve close observation and participation in students' problem solving processes. Research methods that rely on standardized test results lack the ability to reveal student thinking (Ginsburg, 1997). As such, this study was designed to employ a set of 15 intervention sessions that provided opportunities for the researcher to understand how participant made sense of fraction concepts in an environment that supported participant co-construction of knowledge with the other participants and the researcher.

Without this type of research base to establish how students who struggle make sense of mathematical concepts during instruction, decisions may be made about instruction and learning environments based on assumptions that do not hold for these students. Additionally, fractions are considered to be a bridge between whole number mathematics and higher mathematics such as algebra, geometry, and calculus (Cramer et al., 2002; Mazzocco, Myers, Lewis, Hanich, & Murphy, 2013). A longitudinal study that analyzed data sets from the United States (n = 599) and the United Kingdom (n = 3677) was conducted that examined which types of mathematical knowledge best predicted later mathematical achievement (Siegler, Duncan, Davis-Kean, Duckworth, Claessens, Engel, Susperreguy, & Chen, 2012). Their analysis found that elementary school students' knowledge of fractions, along with the related concept of division, predicted their knowledge of algebra and their performance in mathematics in high school. The researchers proposed that a poor understanding of fractions might cause students to become reliant on poorly understood procedures and rote memorization, and that understanding of fractions are needed to build solution strategies for algebraic problems. Because of the implications of this link, it is of particular importance to address the needs of students who struggle to understand early fraction concepts in elementary school. This study was developed to answer questions about how three students who struggle in mathematics make sense of early fraction concepts during small group intervention sessions.

Statement of the Problem

Although many researchers in mathematics education have studied the conceptual thinking and reasoning of elementary mathematics students and the need for conceptually-based learning prior to instruction in procedures (Carpenter et al., 1998; Cobb et al., 1991; Cramer et al., 2002; Empson, 1999), few of these studies have specifically addressed the thinking or instructional needs of students who struggle in mathematics. At the same time, research about students who struggle in mathematics has tended to focus on procedural interventions and quantitative results of standardized tests or curriculum-based measures (Flores & Kaylor, 2007; Fuchs & Fuchs, 2003; Fuchs et al., 2013; Gersten, Chard et al., 2009; Jitendra et al., 1998; Kroesbergen & Van Luit, 2003).

Some researchers have sought to understand the sense-making process of students who struggle in mathematics (Broza & Kolikant, 2015; Hunt & Empson, 2015; Joseph & Hunter, 2001; Lewis, 2010, 2014). Broza and Kolikant (2015) examined the connections between meaningful thinking about mathematics and mathematics classroom activities, including scaffolded teaching and interactive computer work, for 11 fifth-grade students in an afterschool program focused on subtracting of decimal numbers. Lewis conducted a study (2010) focused on an eighth-grade student's conceptual understandings of simplifying fraction during weekly tutoring sessions conducted over the course of the academic year and another study (2014) focused on two college students' conceptual understanding of fractional magnitude and the relationship between numerator and denominator over a six-week one-to-one tutoring intervention. Joseph and Hunter (2001) examined how instruction on a self-regulating strategy, the use of cue cards, affected students' understanding of how to solve fraction addition and subtraction problems.

Few studies have attempted to qualitatively document how struggling third-grade students make sense of beginning fraction concepts such as equipartitioning and unit fractions, iteration of unit fractions, comparison, and equivalence. One study by Hunt and Empson (2015) used clinical interviews to examine how 10 third- through fifth-grade students applied strategies to solve equal-sharing problems. The study proceeded to qualitatively compare these students' strategies to strategies for solving equal-sharing problems used by typically-performing students. Although this current study was qualitative, it was not a replication of the work of Hunt and Empson (2015). Instead, the researcher conducted an exploratory examination of the struggles and strengths of three students, and how they made sense of mathematics with the support of social-constructivist scaffolding, while engaged with fraction concepts in intervention sessions.

Purpose of the Study

Students who struggle often do not have opportunities to engage in complex mathematical thinking throughout elementary and secondary school, and often do not attain mathematical proficiencies needed for college success (Boaler, 2015; Gamoran & Hannigan, 2000). In an economy increasingly dependent on knowledgeable workers (Carnevale & Desrochers, 2003), high school mathematics courses beyond the second algebra course are often viewed as gatekeepers to college (ACT, 2004; Carnevale & Desrochers, 2003). In fact, in 2004 a higher percentage of students (74%) who took higher-level mathematics courses in high school, such as trigonometry and calculus, met benchmarks indicating readiness for college algebra than students (13%) who completed the second algebra course as their final high school mathematics course (ACT, 2004). Students who struggle in mathematics in elementary school often do not enter the first algebra course before ninth grade, a circumstance which makes it difficult to take classes beyond the second algebra course before graduating high school (Gamoran & Hannigan, 2000). While success in algebra has been identified as a critical gateway to higher-level mathematics courses and college success (Gamoran & Hannigan, 2000), Mazzocco and colleagues (2013) identify success with fraction concepts as the most critical gateway along this path. Early fraction concepts first substantially appear in the third grade curriculum (NGACBP & CCSSO, 2010) indicating that the most critical point in mathematics education leading to the potential for college success may be earlier than once thought. Given that students who struggle in mathematics often have learning opportunities that are less conceptual and more focused on the rote use of procedures, opportunities to excel in challenging mathematics courses may not truly exist for these students (Boaler, 2015; Gamoran & Hannigan, 2000). Boaler (2015) explains that while students who struggle are denied opportunities to participate in challenging mathematics, it is precisely these opportunities to work on "complex mathematics that enables brain connections to develop" (p. *xviii*) and for the student to experience success in mathematics comparable to his or her peers.

Research on children's thinking (Carpenter et al., 1998; Cobb et al., 1991; Cramer et al., 2002; Empson, 1999) over the last several decades has established the abilities of elementary aged students to make sense of complex mathematics topics conceptually in learning environments where teachers are focused on children's thinking. Research that is specifically focused on how struggling mathematics students make sense of mathematics concepts has been less common. However this study attempted to contribute to the growing body of research (Hunt & Empson, 2015; Joseph & Hunter, 2001; Lewis, 2010, 2014) that has sought to qualitatively understand and describe how learners who struggle make sense of fraction concepts. This researcher did not find in her review of the literature a qualitative study that described the struggles and strengths of third-grade students who struggle in mathematics or that explored how these students interacted with social-constructivist scaffolding as they made sense of fraction concepts. The proposed qualitative multiple case study was intended to address this gap in the literature. Attention to this gap is important because a body of research, involving both qualitative and quantitative methods, concerning how learners who struggle make sense of early fraction concepts is needed to shed light on their needs as they are engaged in conceptually complex mathematics.

Research Questions

Two research questions for this study were designed to elicit a description of the participants' individual struggles and strengths, and to describe how the participants made sense of third-grade fraction concepts in intervention sessions using social-constructivist scaffolding techniques. Geary's (2003) classification of learning disabilities in mathematics was used as a lens to assist in description of struggles and strengths. Anghileri's (2006) descriptions of social-constructivist scaffolding techniques were used to identify and analyze participant interactions during episodes of scaffolding. According to the Common Core State Standards for Mathematics (NGACBP & CCSSO, 2010), fraction concepts covered in third grade include equipartitioning and unit fractions (3.G.2, 3.NF.2.a), iteration of unit fractions (3.NF.1, 3.NF.2.b), fraction equivalence (3.NF.3.a, 3.NF.3.b, 3.NF.3.c), and fraction comparison (3.NF.3.d). Both research questions for this study addresses these fraction concepts presented in the CCSSM standards covered in third grade. The research questions for this study are as follows:

1. What struggles and strengths of third-grade students are revealed in a small group intervention supported by social-constructivist scaffolding while focused on fraction concepts?
2. How do third-grade students who struggle in mathematics interact with social-constructivist scaffolding techniques as they make sense of fraction concepts?

Research Study Organization

This research study is presented in five chapters. This first chapter provides a rationale for the proposed study, a statement of the problem, the purpose of the proposed study, and the first presentation of the research questions. Chapter 2 presents a review of literature pertinent to this study, beginning with the establishment of an interpretive framework to guide the study based on Vygotsky's theories that children's optimal learning is supported by teacher-student interactions. Vygotsky (1934/1986/2012) put forth that a child can learn concepts that would otherwise be beyond his or her abilities if a teacher works in conjunction with the student to make sense of the concepts. Although Vygotsky (1930-1934/1978) viewed overly directed teaching as interfering with the child's ability to make sense of concepts, he believed the role of the teacher or tutor in supporting the student's development of conceptual understanding is crucial. Vygotsky (1930-1934/1978) proposed that the teacher or tutor should seek to co-construct knowledge with the child by asking probing and redirecting questions, and that only through this process is it possible to understand what the child is capable of learning and how the child develops an understanding of the concepts being considered (Vygotsky, 1930-1934/1978).

Chapter 3 reviews the research questions, explains the use of a case study research design, and details the procedures used for data collection and analysis including those intended to contribute to the trustworthiness of the study. Case study research design is well suited to answer research questions that seek to uncover how a process occurs, such as those posed in this study asking how students make sense of early fraction concepts. Furthermore, Butler (2006) suggests that the use of a case study research design can uncover links between an intervention and a student's learning while also acknowledging that case study research is often conducted under circumstances partially beyond the researcher's control.

Chapter 4 begins with a bracketing statement by the researcher concerning her personal experiences with students and family members who have had learning struggles and her attempt to set these experiences aside for the current study. Descriptions of participants and the intervention sessions are provided. Two frameworks for analysis are described and applied. Subtypes for classifying learning disabilities in mathematics, developed by Geary (2003), were used for the first framework to analyze the misconceptions and errors that influenced each participant's process of making sense of fraction concepts. A pattern of strengths for each participant also emerged and is presented as well. An analysis of how participants interacted with scaffolding techniques, as described by Anghileri (2006) and implemented during intervention sessions by the researcher, is presented as the second framework.

Chapter 5 begins with the presentation of findings for the study. First, findings associated with the first analysis framework, subtypes of mathematical difficulties, are presented followed by presentation of findings for the second analysis framework, scaffolding techniques. The implications of the findings are discussed next. Finally, recommendations for future research and a concluding statement for the manuscript are presented.

CHAPTER 2: LITERATURE REVIEW

This review of the literature begins with an overview of the interpretive framework used to guide the research study including Vygotsky's theories on children's optimal learning and teacher-student interactions that support this learning, continues with a discussion of the literature on instruction that supports children's learning of mathematics and children's learning about fraction concepts including a representational model (Lesh, Post, & Behr, 1987) and a learning progression based on the Common Core State Standards for Mathematics (NGACBP & CCSSO, 2010) which may support this learning. It concludes by addressing issues related to students who struggle, their identification, their thinking and learning, and interventions designed to improve their performance and understanding.

Interpretive Framework

With the work of Piaget in the 1960s, theories about how children learn moved from a cognitivist perspective to a more constructivist perspective. Constructivist theories were widened to include a social dimension when the research of Vygotsky, originally conducted in Russia in the 1920s and 1930s, became popular in the United States in the 1960s (Fosnot, 1996). Social-constructivist theories of learning posit that children actively construct information in their own minds while interacting with teachers, other students, and their environment, rather than receive information passively from adults (Fosnot, 1996). Vygotsky (1934/1986/2012) defined Piaget's ideas on the development of children's thinking as falling into two categories: informal, everyday learning and formal, school learning. According to Vygotsky (1934/1986/2012), Piaget's conceptions of children's learning require informal learning to be in conflict with formal learning, and true academic learning to only occur when formal learning gains primacy over informal learning.

Further, Vygotsky (1934/1986/2012) stated that Piaget created a separation between learning that involved creation of knowledge within the child's mind and learning that involved internalization of information obtained from the environment. Although Vygotsky used many of Piaget's early ideas on children's thinking and learning in developing his theories, he departed from Piaget in important ways. Vygotsky (1934/1986/2012) proposed that the separation between informal and formal learning is an artificial construct, that children process formal information in much the same way they process informal information, and that children bring to the class environment informal understandings of concepts that need to be integrated into formal understandings established by society. To accomplish this purpose a child must, in conjunction with the learning community including the teacher and classmates, construct meanings from classroom activities. Although Vygotsky (1934/1986/2012) admitted that there had been little focus on how children come to understand formal information until that time, he proposed that his observations of children learning suggested that a child will resist direct suggestions of teachers, and will only change in their ability to understand a concept gradually over time, with more experience working with a concept. Vygotsky (1934/1986/2012) posited that what appears to be learning of formal concepts on the part of the child is often no more than the child parroting the teacher, and that the child cannot reason about the concept unless there is a deeper exposure to the concept supported by the teacher's methods. Essentially, Vygotsky (1934/1986/2012) believed Piaget was shortsighted in denying a connection between a child's development of concepts and teacher instruction, and further stated that Piaget's view is one of conflict and antagonism between the formal and informal, rather than a productive connection between the two.

In extending Piaget's ideas on constructing knowledge, Vygotsky (1934/1986/2012) put forward that children's formal learning is dependent on their informal constructions of knowledge, and that it is possible to determine the level of formal learning a child is prepared to understand. Furthermore, a child can learn concepts that would otherwise be beyond his or her abilities if the teacher works in conjunction with the student to make sense of concepts. Although Vygotsky (1930-1934/1978) viewed overly-directed teaching as interfering with the child's ability to make sense of concepts, he believed the role of the teacher in supporting concept development was crucial. Vygotsky (1930-1934/1978) posited that it is not possible to ascertain what the child has learned or how it is learned by focusing only on the child's answers to problems. Instead, the adult should work with the child by asking probing and redirecting questions in an attempt to understand how the child develops an understanding of the concepts at hand (Vygotsky, 1930-1934/1978). In fact, Vygotsky proposed that academic testing or school tasks performed by children in isolation could reveal very little about a child's knowledge or capabilities. According to Fosnot (1996), Vygotsky argued "that the progress in concept formation achieved by the child in cooperation with an adult was a much more viable way to look at the capabilities of learners" (p. 19).

Optimal learning occurs, according to Vygotsky (1930-1934/1978), when a child works near the limits of his or her capabilities with the support of the teacher in developing concept knowledge. Vygotsky (1930-1934/1978) referred to such an event as a child working within his or her *zone of proximal development*. Although an untimely death prevented Vygotsky from extending his theories about this idea, using it in teaching became a focus of many educational researchers and theorists in western culture as translations of Vygotsky's work became available in English (Fosnot, 1996). Extending from Vygotsky's theories on expert-novice interactions within the *zone of proximal development*, scholars developed the metaphor of scaffolding (Fosnot, 1996).

The term scaffolding is attributed to Jerome Bruner and his colleagues (Bruner & Ratner, 1978; Ninio & Bruner, 1978; Wood, Bruner, & Ross, 1976) but was also further developed as a concept by others including Cazdan (1983), Cambourne (1988), and Graves (1983). Although scaffolding has become a term commonly used in mathematics education, early researchers (Bruner & Ratner, 1978; Cambourne, 1988; Cazdan, 1983; Graves, 1983; Ninio & Bruner, 1978; Wood, Bruner, & Ross, 1976) who developed descriptions and processes of scaffolding did so within the realms of literacy education and parenting. While some proponents (Cazdan, 1983) held to a description of scaffolding as a process using direct instruction and modeling at developmentally crucial moments, others (Cambourne, 1988; Graves, 1983) viewed scaffolding as the teacher's use of knowledge about the child's thinking to propose questions or suggest possibilities that the child is well poised to investigate.

According to Stone (1998a), some scholars have debated the usefulness of considering instruction with scaffolding as different from effective teaching, asserting that the elements of scaffolding merely match those of effective teaching. Other scholars (Broza & Kolikant, 2015; Moschkovich, 2015; Putambeker & Hubscher, 2003) point to issues that may be unique to learners who struggle, including tendencies to regress in understanding, unpredictability in constructing knowledge, and difficulties in assuming responsibility for learning, which necessitate a metaphor such as scaffolding to highlight the unique instructional needs of this group. Stone (1998a, 1998b) argued for continued use of the scaffolding metaphor and expressed concern that the roots of scaffolding might be poorly understood among practitioners, leading to scaffolding methods that abandoned the original social-constructivist intent of scholars. Stone (1998b) contended that the roots of scaffolding rest squarely in the work of Vygotsky and his theories on expert-novice interactions. As such, scaffolding should be understood as part of a social constructive process involving co-construction of meaning between the teacher and the student (Stone, 1998b). Although Bruner is credited with introducing the term in 1976, Cazdan was the first to explicitly link the term to Vygotsky's theories in her studies of language development and parent-child interactions (Stone, 1998b). However, Bruner later acknowledged the explicit connections between his metaphor of scaffolding and the theories of Vygotsky (Bruner, 1986; Stone, 1998b). Instruction that supports students' conceptual learning of mathematics aligns with Vygotsky's theories on social-constructivist learning. Specifically, instruction grounded in discourse in which students discuss, support, and compare their solution strategies for cognitively challenging tasks with teacher scaffolding encourages student learning. For this study, the researcher attempted to apply these ideas about children's learning of mathematics in the preparation for and conduction of the intervention sessions with the three participants.

Scaffolding

Although within mathematics education there is no generally agreed upon definition of scaffolding (Broza & Kolikant, 2015; Putambekar & Hubscher, 2005; Van de Pole et al., 2010), scholars point to some common elements of scaffolding (Anghileri, 2006; Broza & Kolikant, 2015; Dale & Scherrer, 2015; Moschkovich, 2015; Pfister et al., 2015; Putambekar & Hubscher, 2005; Van de Pole et al., 2010) including features such as: a) continual diagnosis during teaching and student work; b) intervention through questioning techniques; c) removal of scaffolding when possible; d) maintenance of student responsibility for thinking and learning; and e) student explication of mathematical thinking. In addition, Anghileri (2006) proposed that scaffolding should include concrete manipulatives and student-generated representations. Of particular importance to effective scaffolding are questioning techniques centered on the use of probing questions intended to push forward students' mathematical understandings and to remediate misunderstandings (Anghileri, 2006; Dale & Scherrer, 2015; Moschkovich, 2015). In a quasi-experimental video-observation study of 36 third-grade teachers' scaffolding practices in classrooms that included learners who struggle, Pfister and colleagues (2015) found that although a slight majority (54%) of teachers in the study were skilled at choosing tasks and using manipulatives, few (25%) were able to effectively use questioning techniques or conduct on-going evaluation to provide scaffolding. Wood (1994) proposed that interactions that teachers enact with students fall into two patterns, funneling and focusing. Funneling interaction techniques use questions that attempt to lead students through set of predetermined steps or procedures. Wood proposed that this pattern lead to superficial rather than meaningful mathematical understandings, while a focusing pattern would leave responsibility for mathematical thinking with the students. A focusing pattern uses questions that ask students to develop strategies to solve problems and to make sense of underlying concepts (Wood, 1994).

Anghileri further developed the ideas of scholars (Tharpe & Gallimore, 1988; Wood, 1994; Wood, Bruner, & Ross, 1976) who examined scaffolding and questioning techniques; she used these ideas in a series of observation and interview studies (Anghileri, 1995; Anghileri & Baron, 1998; Coltman, Anghileri, & Petyaeva, 2002) to develop a hierarchy of scaffolding activities including descriptions of scaffolding techniques. Tharpe and Gallimore (1988) used the term *assisted learning* to classify interactions between teachers and learners into six strategies as follows: a) modeling a process for imitation; b) rewarding or punishing based on desired behavior; c) giving feedback; d) instructing in specific actions; e) questioning that guides the student; and f) attending to cognition of student. Wood and colleagues (1976) also proposed a set of six scaffolding strategies as follows: a) enlisting student interest and engagement; b) simplifying a task; c) keeping the student focused on an objective; d) challenging and confirming student thinking; e) responding to the student's frustration level; and f) modeling solution strategies. Anghileri noted the common features of both sets of scaffolding techniques that focus on student cognition such as questioning that guides the student (Tharpe & Gallimore, 1988) and challenging and confirming student thinking (Wood et al., 1976).

In a set of case studies examining the interactions between nine to 13 year old students and their teachers about division problems, Anghileri (1995) found that students often connected language and strategies to mathematical contexts in ways that were not accurate. Observations of these cases showed that teachers who were able to bring listen carefully to students' reasoning then bring attention to the context of tasks encouraged students to interpret problems more accurately and find reasonable strategies to solve these problems (Anghileri, 1995). In a subsequent study, Anghileri and Baron (1998) observed 40 kindergarten students and 28 first-grade students in three schools who engaged with shape blocks during free-play time over eight weeks. They tested students before and after the intervention for success in five skills using the shape blocks including: a) matching two-dimensional shapes to three-dimensional shapes; b) measuring a tower of blocks; c) matching shapes using touch only; d) sorting blocks by shape; and e) reproducing a sequence of given blocks. Although students improved on some tasks after the intervention, the researchers found that kindergarteners had lower performance on matching shapes using touch and sorting blocks, while first-graders had lower performing on sorting blocks and reproducing a sequence. Anghileri and Baron asserted that free play was not adequate to help students in building an understanding of tasks involving classification of the blocks. Without opportunities to "describe and discuss their constructions with a teacher or peer whose mathematical understanding enables discussion," students were unable to progress in their understandings (Anghileri & Baron, 1998, p.63).

By examining the work of other scholars and her own research findings, Anghileri (2006) was able to develop an explanation of scaffolding on three levels with associated descriptions of specific scaffolding techniques within the three levels. The first level, which Anghileri identified as *the environment*, includes choices the teacher makes before instruction about grouping students, selecting tasks, and making manipulatives available. Also included in this level are statements made by the teacher during instruction to gain attention, encourage students, and validate student work. The second level is referred to as *explaining, reviewing, and restructuring*, and relates to the interactions between teachers and students that are specifically about mathematics. Within this level, Anghileri (2006) describes the following techniques: a) prompting and probing; b) looking, touching, and verbalizing; c) interpreting student work or talk; d) simplifying a problem; e) explaining and justifying; and f) negotiated meaning. Table 1 presents a summary of these scaffolding techniques as described by Anghileri.

Table 1

Level Two Scaffolding Techniques as Described by Anghileri (2006)

Technique	Description
Prompting and Probing	• Using questions to guide the student to a mathematical idea or solution • Using questions to guide the student to think more deeply
Looking, Touching, and Verbalizing	• Objects are manipulated or pictorials created by students • Students analyze and discuss what they see • Students talk about their mathematical ideas to others
Interpreting Student Work or Talk	• Clarifying student work • Clarifying student talk
Simplifying a Problem	• Creation of an intermediate task to shed light on concepts and strategies related to the original tasks • Cognitive complexity of the original task should be maintained
Explaining and Justifying	• Opportunities embedded within instruction for students to support or prove their work verbally • Opportunities embedded within instruction for students to support or prove their work in written format
Negotiated Meaning	• Development of mathematical ideas that are agreed upon by students • Supported by teacher guidance • Errors and misconceptions are addressed • Mathematical knowledge is created and shared within the group

The third level is identified as *developing conceptual thinking* and includes discussions generated and guided by teachers that generalize mathematical ideas and develop connections between these ideas. While aspects of each of the three levels of scaffolding as proposed by

Anghileri (2006) occurred during this study, the researcher found that scaffolding techniques described within the second level were most prevalent and, as such, were chosen as a framework for analysis for this study.

Instruction That Supports Children's Learning of Mathematics

According to Cobb, Yackel, and Wood (1992), the process of learning can be seen as unfolding simultaneously in the mind of the learner and in a social context. In their qualitative case study, Cobb and colleagues (1991) described learning opportunities that occurred in a second-grade year-long teaching experiment that used cooperative learning and discourse in ways not typical in traditional classrooms at that time. The teacher and 20 second-grade students participated in the study in which the researchers developed instructional activities, observed classroom activities, and video-taped class sessions for further analysis. Cobb and colleagues (1991) identified themes that emerged from the research, such as greater learning opportunities for the students generated by cooperative work, teacher-guided discourse used as a tool to construct meaning, and more focus on cognitively-challenging problem solving throughout the year. To expand on the themes generated in the work of Cobb and colleagues (1991), scholarly work concerning discourse in learning environments, the use of cognitively-challenging tasks, and the maintenance of cognitive challenge during implementation of tasks will be examined in greater detail in this discussion of the literature. For this study, this literature was used to inform the researcher's selection and implementation of intervention tasks, and attempts to guide discourse with the participants. It was the researcher's intention to ground the tasks and discourse of the intervention sessions in practices shown to support students in building understanding of mathematical ideas.

Discourse

Discourse in the learning environment is crucial to the development of students' understanding of mathematics concepts; however, the techniques a teacher uses to implement discourse can undermine, rather than support, student learning as a focus on low level discourse patterns has been shown to inhibit students from engaging in higher level thinking (Imm & Stylianou, 2012). Students engaged in discussion about a mathematics task with the teacher, other students, or the teacher and other students in conjunction, are participating in mathematical discourse (Hiebert & Wearne, 1993; Imm & Stylianou, 2012; Mehan, 1979; Nathan & Knuth, 2003; Stein, Engle, Smith, & Hughes, 2008). The following is a discussion of two patterns of discourse discussed in the literature: initiation, response, evaluation (IRE), and initiation, demonstration, evaluation and elaboration (IDE).

Initiation, response, evaluation (IRE). Mehan (1979) described a typical pattern of mathematical discourse that involved a teacher posing questions with one solution and that do not require a high level of thought. Mehan (1979) presented and analyzed three excerpts from first grade lessons on reading. After the teacher asked an initial question, students would attempt to provide short answers in succession until a correct answer was provided. The teacher would verify that the final answer given was correct and then would progress to another problem with little or no discussion of the meaning of the answer. If a correct answer was not obtained, the teacher would repeat questions or reduce the complexity of the questions until a correct answer was provided by a student. Mehan (1979) noted that this pattern of simplistic questions and short answers without elaboration would be repeated many times in a mathematics lesson and he labeled this pattern of discourse initiation, response, evaluation (IRE). He posited that instruction oriented around IRE discourse patterns limited teachers' ability to evaluate students' learning and students' opportunities to work with more challenging tasks. Several researchers have identified this IRE discourse pattern in learning environments in the United States (Hiebert & Wearne, 1993; Imm & Stylianou, 2012; Nathan & Knuth, 2003). Hiebert and Wearne (1993) proposed that "the kind of talk in which the teacher and students engage must have some effect on learning" (p. 396). They conducted a 12 week observation study of six second-grade classrooms in one school while instruction focused on multi-digit addition and subtraction. Two classrooms implemented instruction that emphasized an understanding of place value and gave students opportunities to connect different representations and solution strategies, while four classrooms continued with traditional instruction strategies that emphasized teacher-led development of procedures. The researchers found that in three of the four traditional classrooms, teachers asked fewer questions and these questions were predominantly about recall of facts and procedures. Extended descriptions and explanations were not called for by the teacher nor provided by students. Hiebert and Wearne (1993) found this was in contrast to discourse in the two treatment classrooms that emphasized an understanding of concepts where they observed questioning that called for students to describe their own and others' strategies, provide extended explanations, and pose new problems to the class. Within these classrooms, an understanding of concepts was described by the researchers as including student-generated strategies and application of these strategies and understandings to new tasks without a prescribed or teacher-guided procedure.

Initiation, demonstration, evaluation and elaboration (IDE). Nathan, Eilam, and Kim (2007) described an alternative discourse pattern based on their observation of a sixth-grade mathematics classroom, which they labeled initiation, demonstration, evaluation and elaboration (IDE). In the classroom, 20 sixth-grade students worked on one high level task for over an hour, cutting a pie into six equal-sized pieces with only three cuts, progressing from individual work to small group work and then to class discourse. The task, developed from a question proposed by a classmate, comprised the majority of the mathematics lesson and included students developing their own solution strategies independently and in conjunction with their small group before explaining their strategies to the class. Student-generated questions were the major vehicle used to make sense of one another's thinking during the explanatory portion of class discourse. Although Nathan and colleagues (2007) found that most discourse segments during the whole class portion of the lesson were initiated by the teacher, students provided the majority of the explanations and questions during discourse and were able to generate new mathematical questions that deepened thinking about the topic.

Researchers (Imm & Stylianou, 2012; Nathan & Knuth, 2003) have described learning environments that utilize IDE discourse patterns as including direct student-to-student talk, student questioning, and judgment of solutions by the class, rather than the teacher. Students are responsible for explaining their own mathematical reasoning as well as making sense of the mathematical reasoning of their classmates (Imm & Stylianou, 2012; Nathan & Knuth, 2003). The teacher's role becomes that of facilitator, as he or she guides students through mathematically rich discussion, rather than that of a conferring authority of mathematical knowledge (Kilpatrick et al., 2001; Stein et al., 2008). Five teacher practices that support implementation of high level mathematical discourse were proposed by Stein and colleagues (2008): a) anticipating common student answers to high-level tasks; b) observing student responses during individual and small-group work; c) carefully selecting student work to present for discussion; d) attending to the order of presentation of student work; and e) facilitating discussion so that students make mathematical connections between different students' solution strategies and between student solution strategies and mathematical ideas. For this study, anticipating the participants' answers and observing participants' responses were particularly relevant to the researcher, although it often proved difficult to anticipate answers.

Cognitively Challenging Tasks

Meaningful mathematical discourse occurs when students are presented with cognitively challenging mathematical tasks (Hiebert & Wearne, 1993; Imm & Stylianou, 2012). Mathematical tasks that elicit high-level discourse patterns require cognitively-deep student thinking, can be solved with multiple methods, and may result in more than one solution (Chapin, O'Connor, & Anderson, 2003; Nathan et al., 2007; Stein et al., 2008). According to Stigler and Hiebert (2004), the selection and implementation of cognitively-challenging tasks was the single factor found in common among countries with high achievement on the Trends in International Mathematics and Science Studies (TIMSS) of 1995 and 1999. Japan, the only high achieving country in both TIMSS studies, was found to have students spend an average of 15 minutes on each problem as compared to lower-performing countries, including the United

States, in which students averaged five minutes spent on each problem. Although lessons in Japan were found to consist almost entirely of challenging tasks in which students were often required to develop their own solution strategies, lessons in other high-performing countries, such as Hong Kong, were found to devote a large portion of class time to teacher-led instruction in procedures while maintaining a smaller portion of class time for solving cognitively-complex tasks. However, regardless of the amount of time in the lesson devoted to challenging tasks, Stigler and Hiebert (2004) observed that teachers in high-performing countries consistently approached cognitively-complex tasks as conceptual exercises for the students to perform rather than the more commonly observed technique in low-performing countries in which teachers converted problems into procedures for the students to imitate. Essentially, Stigler and Hiebert (2004) asserted that it was the implementation of cognitively-challenging tasks that separated high-performing countries from low-performing countries.

Implementation

Several researchers (for example, Boston, 2012; Hiebert & Wearne, 1993; Stein, Grover, & Henningsen, 1996; Stein, Remillard, & Smith, 2007; Wilhelm, 2014) have pointed to the importance of task selection that supports high-level cognitive reasoning while at the same time emphasizing the importance of the teacher's ability to maintain the task at a high cognitive level during implementation. Stein and colleagues (2007) noted that although curriculum materials that emphasized cognitively-complex tasks were critical in supporting teachers' enactment of students' high-level mathematical thinking, no curriculum is self-enacting. To support high-level cognitive reasoning, tasks need to be set up to encourage multiple solution strategies, to lend themselves to multiple representations, to require explanation and justification of students' solutions and strategies, and to involve group work as well as independent work. During implementation, multiple solution strategies and representations need to be used with explanations and justifications provided (Stein et al., 1996). Research on teaching (Boston, 2012; Boston & Smith, 2009; Stigler & Hiebert, 2004) has shown that teachers in the United States often lower the cognitive demand of tasks during implementation. During their observation study of twelve classrooms observed multiple times over a period of three years, Henningsen and Stein (1997) found that five factors dominated in supporting high-level implementation of cognitively-challenging tasks: a) building on students' prior knowledge; b) scaffolding that does not simplify the task; c) using the right amount of time for a task (not too little or too much); d) sustaining pressure for explanation and justification; and e) modeling of high-level performance by the teacher and students.

Wilhelm (2014) built on these factors by explaining that teachers should make sure students are familiar with contexts being used in tasks, use tasks with multiple entry points, and describe students' contributions as important. In a four year longitudinal study of 213 middle school teachers that sought to find correlations between various teacher factors and teachers' maintenance of cognitively challenging tasks, she found several significantly correlated factors. These were teachers' content knowledge for teaching mathematics ($B = 1.08$; $p < .05$), inquiry-oriented views of instruction ($B = 1.28$; $p < .05$), and productive views about instruction for students who were struggling ($B = 1.07$; $p < .05$). Wilhelm (2014) defined productive views about instruction for students who struggle as teacher beliefs which held that all students can be supported to successfully work with cognitively complex mathematics. Notably, Wilhelm (2014) found that a significant subset of teachers recognized the need for students to work on cognitively-challenging tasks, but indicated that they believed students who struggle needed more exposure to drill and procedure-oriented tasks before working at a higher level. These teachers often lowered the cognitive demand of the assigned tasks by eliminating the need for students to construct their own solution strategies. Instead, the teachers provided explicit instruction in a solution strategy through examples and removed prompts to explain and justify student thinking (Wilhelm, 2014). Contrary to this practice, Carpenter and colleagues (1998) found that students who solved and explained cognitively-complex problems with their own solution strategies before, or instead of, instruction in procedures demonstrated a deeper understanding of mathematics concepts. In keeping with Cambourne's (1988) interpretation of Vygotsky's ideas, Carpenter, Fennema, Peterson, Chiang, and Loef (1999) asserted the need for teachers to use their knowledge of children's mathematical thinking to guide instruction based on questioning and sense-making.

Carpenter and colleagues (1998) conducted a longitudinal study covering three years that involved 78 students who worked a variety of computation and problem-solving tasks in interview format during first grade, and then again in second and third grades. The researchers sought to investigate whether students who used strategies invented without or before explicit instruction in procedures ($n = 60$) performed differently than students who used standard algorithms ($n = 18$). Carpenter and colleagues (1998) found that the invented-strategies group had a significantly higher ($p < .05$) percentage of students who demonstrated base-ten knowledge (81%) than the standard-algorithm group (22%) at the beginning of second grade. Further interviews at the end of third grade revealed that students in the invented-strategy group performed significantly better ($p < .05$) on word problems that required transfer of known information to unique situations ($M = 1.37$) than students in the standard-algorithm group ($M = .44$). Carpenter and colleagues (1998) concluded that students who used invented strategies were better able to make sense of base-ten reasoning and were more successful at transferring their learning to new and unique problems. Further, the researchers proposed that early introduction of explicitly taught standard algorithm procedures may interfere with children's success in making sense of base ten reasoning.

Children's Learning about Fraction Concepts

The underpinnings of instruction that supports conceptual learning, high-level discourse and selection and enactment of cognitively challenging tasks are evident in the literature about fraction learning as well. However, ideas about how children come to make sense of fraction concepts differ. On the one hand, some who have studied children's thinking propose that understanding of fractions is best developed through tasks in contexts that are focused on equal-sharing and reasoning about fractional units relative to whole units (Empson & Levi, 2011). Other scholars contend that although these types of problems are one important aspect of building understanding of fraction concepts, manipulative models and representational drawings both in abstract tasks and tasks grounded in real-world contexts, play an equally important role (Cramer, Behr, Post, & Lesh, 2009; Cramer et al., 2002). Cramer and colleagues (2002) proposed that manipulatives in the form of fraction circles are the most important tool for helping students develop mental images of fractions while other researchers (Empson, 1999; Empson & Levi, 2011; Hunt & Empson, 2015) asserted that reasoning strategies about equal sharing are at the root of helping students to understand representations and to develop mental images of fractions. Still others (Mazzocco & Devlin, 2008) contend that tools, such as a number line, that emphasize models of a fraction as a unique number, rather than two numbers representing part and whole, are useful in connecting fraction learning to prior learning about whole numbers. Central to each of these approaches is the importance placed on student reasoning and student-generated solution strategies rather than emphasis on teacher-led procedural instruction (Cramer et al., 2009; Cramer et al., 2002; Empson, 1999; Empson & Levi, 2011; Hunt & Empson, 2015; Mazzocco & Devlin, 2008).

Teaching Strategies

Empson (1999) explored how first-grade students' thinking about fractions developed in a classroom where the teacher used the students' informal prior understandings of sharing to guide instruction in a discourse-rich environment. She conducted a case study of an instructional unit focused on fractions for four weeks in a first-grade class with 19 students. Using data obtained during clinical interviews prior to instruction, Empson (1999) and the classroom teacher planned instructional activities that incorporated students' knowledge. All class sessions in the instructional unit were observed, recorded, and transcribed by the researcher. Clinical interviews were also conducted with the students at the conclusion of the unit. Based on the pre- and post-instruction interviews, she was able to identify several results. She found that more students were able to use a valid partitioning strategy to solve an equal-sharing word problem after instruction (n = 14) than before instruction (n = 4). Prior to instruction, five students were able to correctly solve a proportional reasoning problem, whereas after instruction 10 students were able to correctly solve a similar problem. She also found that prior to instruction no students were able to identify a fraction correctly using part-whole conceptions of equal-sized pieces, referring instead to parts of a whole without regard to size. At the conclusion of the study, 13 students were able to use correct part-whole reasoning to identify fractions (Empson, 1999).

In a comparison study, Cramer and colleagues (2002) found that students taught using a conceptually-oriented initial fraction curriculum that emphasized the use of different representations of fractions and connections between those representations, outperformed students taught using traditional fraction curricula. Sixty-six fourth- and fifth-grade classes were randomly assigned to a treatment or control group. Nineteen fourth-grade classes with 470 students were assigned to the treatment group and 19 fourth-grade classes with 483 students were assigned to the control group. For fifth grade, 14 classes with 369 students were assigned to the treatment group and 14 classes with 344 students were assigned to the control group (Cramer et al., 2002). Treatment classes used materials from the Rational Number Project (RNP) developed by the researchers. Teachers in the treatment classes attended professional development, conducted by the researchers prior to the study, focused on conceptual learning and the use of the RNP materials. Classes in the control group used two commercially available traditional curricula from publishers Addison-Wesley and Harcourt Brace (Cramer et al., 2002). The researchers used a post-test, retention test experimental design with tests focused on evaluating student achievement in the following domains: a) fraction concepts; b) fraction equivalence; c) fraction order; d) operations; e) estimation; and f) transfer of concepts to unique situations. Multivariate analysis of variance was conducted with the class set as the experimental unit (Cramer et al., 2002). The research design also employed a qualitative component with 10 students each from treatment and control randomly selected to be interviewed three or four times over the course of the study (Cramer et al., 2002).

Cramer and colleagues (2002) found that students in the treatment group (n = 839) scored significantly higher than students in the control group (n = 827) for four of the six domains including fraction concepts ($F = 24.6$; $p < .0083$), fraction order ($F = 13.8$; $p < .0083$), estimation ($F = 10.3$; $p < .0083$), and transfer ($F = 18.9$; $p < .0083$). When all domains were combined for total test scores, a statistically significant difference ($F = 15.5$; $p < .0083$) was also found favoring performance of the treatment group over the control group (Cramer et al., 2002). Because no significant differences were found between grade levels within group, fourth- and fifth-grade results were combined by group. Large effect sizes were found for the total test (η^2 = .205) and four of the domains, fraction concepts (η^2 = .284), fraction order (η^2 = .955), estimation (η^2 = .182), and transfer (η^2 = .240), indicating that the differences between the treatment and control groups were of practical significance (Cramer et al., 2002). Results of interviews indicated students in the RNP treatment group used conceptual approaches to solve fraction problems 71% of the time while students in the traditional curriculum control group used conceptual approaches 15% of the time (Cramer et al., 2002). Additionally, students in the treatment interview group obtained correct answers 76% of the time while students in the control interview group obtained correct answers 47% of the time. They concluded that conceptually-oriented fraction learning using multiple and connected representations led to students who had higher performance on tests of fraction domains and more conceptually grounded understandings of fractions.

A Model for Connecting Representations

Students' work understanding and connecting multiple representations of fractions plays a vital role in developing their abilities to reason and generate solution strategies (Cramer et al., 2009). The *Principles and Standards for School Mathematics* (NCTM, 2000) proposed a link between mathematical representations of many types and students' abilities to make sense of mathematical concepts:

> Representations should be treated as essential elements in supporting students' understanding of mathematical concepts and relationships; in communicating mathematical approaches, arguments, and understandings to one's self and to others; in recognizing connections among related mathematical concepts; and in applying mathematics to realistic problem situations through modeling. (p. 67)

A model was proposed by Lesh and colleagues (1987) that reflects the use of representations to build student understanding by defining and connecting five different ways of representing mathematical ideas. An adaptation of that model is presented in Figure 1.

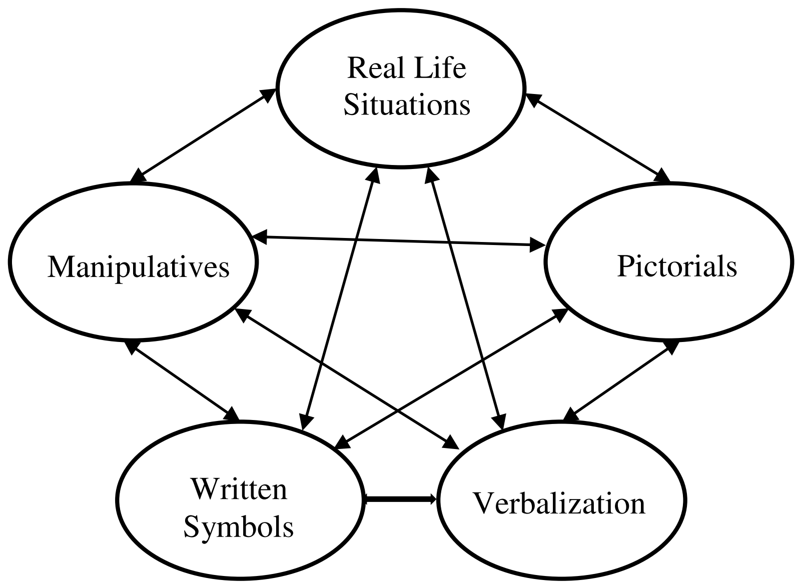


Figure 1: Adaptation of Model of Mathematical Representations and Connections

Lesh and colleagues (1987) described representations included in the model as follows. Manipulatives refer to concrete objects which students may touch, move, or otherwise physically control. For fractions, these may include fraction circles, rectangular fraction area models, fraction strips, paper strips for folding, and sets of objects. Pictures include those presented to students as part of instruction, such as a number line for representing the location of fraction values or a graphic showing several ways to depict a fraction, and pictures generated by students to interpret a context. Lesh and colleagues use the phrase *verbal symbols* to refer to the spoken language students or teachers use to interpret a problem context, explain reasoning and solution strategies, and communicate about connections between mathematical concepts and students' thinking. In this adaptation of the model the term *verbalization* is used in place of verbal symbols. Written symbols include both mathematical symbols and written words used in mathematical problems. Real-life situations refer to problems posed in contexts that are interesting, engaging, and relevant to students. Lesh and colleagues (1987) contended that representations, and translations among representations, play an important role in building understanding of fractions by connecting the multiple ways in which students' may think about fractions. Lesh's model was used as a guiding principle in choosing and developing tasks used in intervention sessions for this study. The researcher attempted to create opportunities for the participants to connect fraction representations in each of the ways presented above. Any of these representations could be presented as part of a problem or could be generated as part of a student's solution process and problems intended to offer opportunities to move from one type of representation to another were used (Lesh et al., 1987). For example, a problem was given in a real life context; then the student created pictures to solve the problem, and expressed an answer in written symbols. In this problem, the student connected a real life representation to pictorial and written symbol representations.

Fraction Learning Progression

In addition to providing multiple opportunities for students to make connections between the five representations of fractions, instruction in the Rational Number Project was sequenced to follow a logical order (Cramer et al., 2002). Meaning of symbols was first developed, followed by study of relationships between parts of fractions and relationships between different representations of fractions. Concepts of order and equivalence were introduced at that point, with estimation and computation strategies following (Cramer et al., 2002). While the Rational Number Project looked to the *Principles and Standards for School Mathematics* (NCTM, 2000) to inform the learning progression (Cramer et al., 2002), more recently the Common Core State Standards for Mathematics (CCSSM) has provided a similar learning progression for fraction learning (NGACBP & CCSSO, 2010). Prior to third grade, CCSSM introduces concepts of equal partitioning of shapes in the geometry strand. Starting in third grade, CCSSM calls for students to understand a unit fraction as one part of a whole partitioned into a number of equal-sized parts that can be iterated to produce non-unit fractions and to understand a fraction as a number that can be represented on a number line. Students are also expected to make sense of fraction equivalence and comparison (NGACBP & CCSSO, 2010). Moving into fourth grade, CCSSM calls for students to build fractions from unit fractions using addition and multiplication by incorporating prior knowledge about whole number operations, to use multiplication and division as a way of creating equivalent fractions, and to use more complex comparison strategies. In fifth grade, CCSSM has students focus on extending knowledge of fraction comparison and equivalence and incorporating this knowledge into operations with fractions. The CCSSM third-grade standards pertaining to fractions are presented in Table 2.

Tasks used during this study in the intervention sessions with the participants were developed to address the CCSSM third-grade fraction standards presented in Table 2. In keeping with the goals presented for fraction standards in the CCSSM and the researcher's goal to emphasize connections between fraction representations, tasks used during intervention sessions were classified according to these criteria. This information is presented in Chapter 3 in an abbreviated format and in detail in Appendices B and C.

Table 2

CCSSM Third Grade Fraction Standards

CCSSM Standard	Focus of Standard	CCSSM Description of Standard
3.G.2	Equipartitioning	Partition shapes into parts with equal areas. Express the area of each part as a unit fraction of the whole.
3.NF.1	Equipartitioning	Understand a fraction 1/*b* as the quantity formed by 1 part when a whole is partitioned into *b* equal parts; understand a fraction *a/b* as the quantity formed by *a* parts of size 1/*b*.
3.NF.2.a	Unit Fractions	Represent a fraction 1/*b* on a number line diagram by defining the interval from 0 to 1 as the whole and partitioning it into *b* equal parts. Recognize that each part has size 1/*b* and that the endpoint of the part based at 0 locates the number 1/*b* on the number line.
3.NF.2.b	Unit Fractions	Represent a fraction *a*/*b* on a number line diagram by marking off *a* lengths of 1/*b* from 0. Recognize that the resulting interval has size *a*/*b* and that its endpoint locates the number *a*/*b* on the number line.
3.NF.3.a	Equivalence & Comparison	Understand two fractions as equivalent (equal) if they are the same size or the same point on a number line.
3.NF.3.b	Equivalence & Comparison	Recognize and generate simple equivalent fractions, (e.g., $^1/_2 = {}^2/_4$, $^4/_6 = {}^2/_3$). Explain why the fractions are equivalent (e.g., by using a visual fraction model).
3.NF.3.c	Equivalence & Comparison	Express whole numbers as fractions, and recognize fractions that are equivalent to whole numbers.
3.NF.3.d	Equivalence & Comparison	Compare two fractions with the same numerator or the same denominator by reasoning about their size. Recognize that comparisons are valid only when the two fractions refer to the same whole. Record the results of comparisons with the symbols <, =, or >, and justify the conclusions, e.g., by using a visual fraction model.

Mathematics Students Who Struggle in the Literature

Prior to this section, literature reviewed has been primarily concerned with the learning of typically-achieving students. This section of the literature review presents studies focused on the learning of students who struggle. First, issues with the terminology, and associated working definitions, used to describe students who struggle are presented. Then studies that examine the mathematical thinking of students who struggle and studies that focus on mathematics interventions for these students are discussed.

Terminology

One challenge apparent in a review of studies focused on students who struggle in mathematics education is the inconsistent terminology used to denote students who have difficulty learning mathematics (Fletcher, Lyon, Fuchs, & Barnes, 2007). This appears to stem, in part, from difficulties identifying a population of students who can be considered to have mathematics learning disabilities. No generally agreed upon criteria or testing instrument exists to identify learning disabilities in mathematics and, unlike learning disabilities related to reading, there is yet to be a determination of the facets of learning disabilities related to mathematics (Fletcher et al., 2007; Geary, 2003). Whereas a learning disability in reading may be examined as specific weaknesses in fluency, comprehension, word recognition, or phonological processing (Fletcher et al., 2007), a learning disability in mathematics has typically been limited to difficulties with memorization and/or retrieval of number facts and difficulties with whole number operations (Fletcher et al., 2007; Fuchs & Fuchs, 2003; Geary, 2003). Some researchers (Fletcher et al., 2007; Geary, 2003) have suggested that learning disabilities in mathematics could be viewed in terms of difficulties with arithmetic, reasoning about rational numbers, algebraic thinking, geometric thinking, visualization, or interpretation of various types of mathematics representations including words and mathematics symbols. With no agreed upon criteria to identify learning disabilities in mathematics based on brain functioning or specific weaknesses, most researchers have relied upon various types of achievement data to identify students (Fuchs & Fuchs, 2003; Geary, 2003; Hecht & Vagi, 2010; Lewis, 2010; Mazzocco & Devlin, 2008) or previous identification by schools of students in need of remediation or classified as exceptional education students (Butler, Miller, Crehan, Babbitt, & Pierce, 2003; Gersten, Chard, et al., 2009; Hunt & Empson, 2015; Witzel, 2005; Zhang & Xin, 2012). Although Geary (2003) established a criterion of performance below the 25th percentile on nationally normed tests of mathematics achievement, it is not unusual to find other criteria prevalently used in the literature.

Arising from the issues of inconsistent identification are related issues of inconsistent terms. In some scholarly writings students are identified as having a learning disability in mathematics or a mathematics learning disability (Fuchs & Fuchs, 2003; Hunt & Empson, 2015; Lewis, 2010), while in others phrases such as ‘mathematics difficulties’ (Hecht & Vagi, 2010), ‘struggling in mathematics’ (Gersten, Chard, et al., 2009), or ‘at-risk’ (Fuchs et al., 2013) are employed. Table 3 presents the terms and operational definitions used in a selection of studies conducted over the past 12 years concerning mathematics and students who struggle. The researcher made a decision to limit the search for this information to the past 12 years to focus on the most recent developments in the use of these terms and operational definitions.

Table 3

Terms and Operational Definitions for Students Struggling in Mathematics

Author(s)	Year	Source & Topic	Type of Publication	Term	Operational Definition
Butler, Miller, Crehan, Babbitt, & Pierce	2003	Learning Disabilities Research & Practice Fraction instruction using CRA Grades: 6-8	Quantitative Research	Students with mathematics disabilities	Label of specific learning disability in mathematics
Fuchs & Fuchs	2003	Handbook of Learning Disabilities Whole number problem solving Grades: 2-6	Quantitative Research	Students with mathematics disabilities	Lowest performing 6-7% of student population
Geary	2003	Handbook of Learning Disabilities Whole number addition strategy choice Grades: 2, 4, 6	Quantitative Descriptive Research	Students with learning disabilities in arithmetic	Achievement test scores lower than the 25th percentile and low-average or higher IQ score
Butler, Buckingham, & Novak	2005	Learning Disabilities Research & Practice Use of Strategic Learning Strategies Grade: 8	Qualitative Research	Students struggling in mathematics	Learning disabilities in reading and mathematics or identified as underachieving
Witzel	2005	Learning Disabilities: A Contemporary Journal Using CRA to teach algebra Grades: 7, 8	Quantitative Research	Students with math difficulties	Label of specific learning disability in mathematics

Author(s)	Year	Source & Topic	Type of Publication	Term	Operational Definition
Louie, Brodesky, Brett, Yang, & Tan	2008	IES Report presenting case studies of in-depth practices at six elementary schools	Qualitative Review	Students with disabilities and other learners who struggle	IEP that lists a learning disability or poor performance
Mazzocco & Devlin	2008	Developmental Science Rational number reasoning Grades: 6-8	Quantitative Research	Children with mathematical learning disabilities versus children without who are struggling	MLD: Woodcock-Johnson Calculation subtest score at or below the 10th percentile on the; Struggling: between the 11th and 25th percentiles
Gersten, Beckman, Clarke, Foegen, Marsh, Star, & Witzel	2009	IES Report Practice guide to implement Response to Intervention in elementary and middle schools	Practice Guide	Students struggling with mathematics	Labeled learning disability or low performance
Gersten, Chard, Jayanthi, Morphy, & Flojo	2009	Review of Educational Research 42 intervention studies across mathematics topics	Meta-analysis	Students with learning disabilities or difficulty learning mathematics	IEP goals in mathematics and learning disability label
Hecht & Vagi	2010	Journal of Educational Psychology Emerging fraction skills Grades: 4, 5	Quantitative Research	Students having mathematical difficulties	Woodcock-Johnson III Calculation composite Score at or below the 25th percentile
Lewis	2010	Learning Disabilities: A Contemporary Journal Fraction equivalence Grade: 8	Qualitative Research	Mathematical learning disability	Test scores at or lower than the 25th percentile, no confounding factors, and lack of response to intervention

Author(s)	Year	Source & Topic	Type of Publication	Term	Operational Definition
Zhang & Xin	2012	The Journal of Educational Research 39 studies about word problem solving interventions	Meta-analysis	Students with mathematics difficulties	Learning disability label, or at risk for failure, or low achieving in mathematics
Fuchs, Schumacher, Long, Nankung, Hamlett, Jordan, Gersten, Cirino, Siegler, & Changas	2013	Journal of Educational Psychology Understanding of fraction concepts Grade: 4	Quantitative Research	At–risk learners	Scores below the 35th percentile on the Wide Range Achievement Test 4 Broad Based Calculation Assessment
Powell, Fuchs, & Fuchs	2013	Learning Disabilities Research & Practice Common Core Standards Grades: k-12	Literature Review for Practitioners	Students with mathematics difficulties	Estimates 3-6% of students have a mathematics learning disability and many more struggle with low performance
Hughes, Witzel, Riccomini, Fries, & Kanyongo	2014	The Journal of the International Association of Special Education 12 intervention studies about algebra	Meta-analysis	Learners with disabilities and learners who struggle	Labeled learning disabilities or at risk for having learning disabilities
Jitendra, Dupuis, & Zaslofsky	2014	Learning Disability Quarterly Arithmetic word problem solving Grade: 3	Quantitative Research	Students at risk for mathematics difficulties	Scores at or lower than the 40th percentile on the Measures of Academic Progress (MAP) mathematics subtest
Hunt & Empson	2015	Learning Disability Quarterly Equal-sharing strategies Grades: 3-5	Qualitative Research	Students with learning disabilities	Learning disabilities label and an IEP with mathematics goals

A review of Table 3 indicates that achievement test performance criteria, special education status, and low performance in classroom settings, are the primary criteria in identifying students for inclusion in studies. Some researchers (Lewis, 2010, 2014; Mazzocco & Devlin, 2008; Mazzocco, Myers, Lewis, Hanich, & Murphy, 2013) distinguish between those students known to have a learning disability and those students who exhibit poor performance for unknown reasons, while other researchers (Gersten, Chard et al., 2009; Hughes et al., 2014; Louie et al., 2008) regard students with or without a label as essentially a similar population on a continuum. Attending to the label used in a particular study is often inadequate to inform the reader of the author's intended population. The reader must also consider the inclusion criteria used to select participants for the study and the author's description of the participants.

For this study, the researcher made a decision to use the terminology 'students who struggle' to make it clear that it was unknown if the participants had learning disabilities. This is consistent with the work of some researchers (Butler et al., 2005; Hecht & Vagi, 2010; Jintendra et al., 2014; Louis et al., 2008; Zhang & Xin, 2012) who have referred to students as struggling or experiencing difficulties in mathematics. However, it should be noted that little consistency exists in terminology referring to these groups of students, as evidenced by the information on Table 3.

The Mathematical Thinking of Students Who Struggle

Although many studies (Carpenter et al., 1998; Cobb et al., 1991; Cramer et al., 2002; Empson, 1999) have addressed the thinking of students in the process of learning mathematics concepts, few studies have addressed the thinking of students who struggle while engaged in this process. Studies that have examined how children learn mathematics and the learning progressions they employ to learn have tended to focus on typically-achieving student populations with no delineation of those students who might be classified as struggling within the sample (Carpenter et al., 1998; Clements & Sarama, 2004; Cobb et al., 1991; Confrey et al., 2014; Cramer et al., 2002; Empson, 1999). A small number of studies have sought to examine how students who struggle think while making sense of arithmetic (Geary, 2003) or while making sense of fraction concepts (Hunt & Empson, 2015; Lewis, 2010, 2014; Mazzocco & Devlin, 2008; Mazzocco et al., 2013).

Geary (2003) emphasized that although performance on standardized achievement tests was commonly used to identify children with mathematics learning disabilities, these measures could not reveal anything about the cognitive strengths and weaknesses of these children. To address this gap, Geary proposed a model for understanding the thinking of students with mathematics learning disabilities by comparison with the thinking of typically-performing children. Because research providing insight into the thinking of children performing whole-number arithmetic was most developed, Geary (1990) had previously focused his research on the strategy choices and solution times of children identified as having learning disabilities in mathematics as they performed whole-number addition computations. Geary (1990) conducted a study with 52 first- and second-grade participants in which the participants were screened into the three groups. Participants were initially classified as normal (n = 23) or having a learning disability in mathematics (n = 29) based on whether or not they were receiving intervention services at school. With the learning disability group, a further distinction was made based on achievement test score improvement from the beginning to the end of the school year.

Participants (n = 13) receiving intervention services who improved to within the normal range of performance from the beginning to the end of the school year were classified as learning disabled-improved, while participants receiving intervention services who showed no improvement over the course of the year were classified as learning disabled-no change. As a result, Geary analyzed three groups for differences in strategy choice and solution times using 40 addition problem presented on a computer using time measurements. His findings indicated that there were no differences in strategy choice or solution times between the normal group and the learning disabled-improved group. Geary posited that the learning disabled-improved group was merely delayed in mathematical understanding rather than different when compared to the normal group. So although this group began the school year behind, with intervention they were able to improve to the level of their normal peers. The learning disabled-no change group were significantly different ($p < .05$) than the other two groups both in terms of strategy choice and solution times. According to Geary, differences in this group appeared to be related to issues with working memory, counting errors, retrieval errors, and longer or unpredictable retrieval times.

Based on his research, Geary (1990) concluded that while traditional measures allowed identification of students experiencing difficulty, they did not provide information about “factors underlying the academic deficit” (p. 364). Later, Geary (2003) proposed that learning disabilities in mathematics could be subcategorized as a procedural subtype and a semantic (working memory) subtype. Within the procedural subtype, Geary included issues related to poor conceptual understanding including inefficient and immature use of strategies and use of procedures without understanding associated concepts. Although combinations of the two subtypes were common, children who maintained use of less efficient solution strategies than their peers were classified as having the procedural subtype while students who primarily exhibited long reaction times for fact retrieval or calculation were classified as having the semantic memory subtype. A third subtype proposed by Geary was visuospatial deficits. Geary observed that disabilities associated with the procedural subtype tended to become less troublesome over time and may represent a developmental delay rather than a cognitive difference while those associated with working memory may be more entrenched and may indicate a true cognitive difference.

Within the mathematics education community, the terms *procedural* and *conceptual* are used to indicate different facets of mathematical understanding (Kilpatrick et al., 2001). Additionally, the word *procedural* is sometimes used to indicate a reliance on memorized procedures without conceptual understanding (Stein & Lane, 1996). In fact, Stein and Lane (1996) referred to two levels of procedure use, a lower level using procedures without connections to concepts and a higher level using procedures with connections to concepts. However, they also designated the highest level of mathematical thinking as "doing mathematics" (p. 58), a level in which deeply understood concepts guide student work with non-algorithmic thinking. On the other hand, Geary (2003) used the word *procedural* to define a subtype of learning disability in mathematics connected to lack of conceptual understanding. Because of these differences in word usage between Geary and the mathematics education community, this researcher will use *conceptual subtype* in place of *procedural subtype* from this point on in this manuscript.

Building on Geary's assertion that the thinking of students with difficulties learning mathematics is not well understood and the research of Mazzocco and colleagues (2013) that suggests students with learning disabilities in mathematics often begin to exhibit difficulties in mathematics when fractions enter the curriculum, Lewis (2014) sought to describe the mathematical understandings regarding fraction concepts of two adult college students, Lisa and Emily, believed to have mathematics learning disabilities. Lewis (2014) asserts that students who have a mathematics learning disability struggle with mathematics in different ways than their non-disabled peers and may not respond to instruction or intervention that is successful with their non-disabled peers who struggle in mathematics. Lewis (2014) collected pretest, posttest, and videotaped interview data during weekly tutoring sessions with the two adult college students over six weeks. After transcribing all videotaped sessions and scanning all documents, Lewis analyzed each session multiple times to find conceptual misunderstandings that persistently occurred multiple times across several tutoring sessions. These particular misunderstandings were identified as contributors to the students' mathematics difficulties.

For each of the two study subjects, Lewis (2014) identified six key perceptions that contributed to the students' mathematical misunderstandings. Persistent incorrect understandings about fractions were demonstrated by Lisa when she: a) viewed shaded regions of a picture of a fraction as deleted; b) viewed the line separating fractional parts as a representation of the fraction; c) ignored the different-sized parts of fractions represented pictorially when adding fractions without common denominators; d) compared only denominators when determining relative sizes of fractions; e) could not partition a fraction into an odd number of parts when drawing representations; and f) changed fractions to create easier manipulations without regard to maintaining the equivalence of the fractions. Emily demonstrated a different set of persistent incorrect understandings. She a) often arbitrarily chose the smaller of the shaded or non-shaded region to represent the numerator; b) could only view fractions such as ½ as an area partitioned into two parts; c) sometimes viewed numerators as the shaded part of the model and denominators as the unshaded part of the model; d) believed a fraction with more parts was larger than a fraction with less parts; e) believed parts that were visually close to ¼ were ¼; and f) understood her pictorial representations as answers with multiple possible interpretations rather than representations of a specific quantity.

In an earlier study, Lewis (2010) described the errors made by an eighth-grade student considered to have a mathematics learning disability based on her achievement test scores. As she solved problems involving simplifying fractions, the student had a 30% error rate. A detailed analysis of tutoring sessions revealed that the student used a multiples-list strategy for the numerator and the denominator to generate a simplified fraction. For example, if Emily was asked to simplify the fraction 8/12, she would first consider the lists of all multiples for all single digit whole numbers containing 8 and 12. She would mentally locate the list holding 8 and 12 adjacent to one another, in this case the list of multiples of 4. Upon recognizing that 8 is placed second and 12 is placed third on that list, she would use these ordinal placements to construct a new, simplified fraction, 2/3. However, this strategy used to compensate for a lack of memorized fact knowledge, required her to manipulate multiple multiplication sequences and created a heavy cognitive load for the student. An intervention designed to reduce the cognitive load associated with multiplication facts helped the student to reduce her error rate from 30% to 7% when simplifying fractions.

Lewis's (2010, 2014) studies provide a structure for understanding students with mathematics learning disabilities by focusing on their thinking and understanding of mathematics concepts rather than computational performance. For example, Lewis (2014) was able to detail how Lisa's understanding of the representation of a fraction, once drawn and shaded, led her to believe the shaded region of the fraction is taken away. Lisa's misunderstanding recurred on multiple occasions and did not resolve with explicit tutoring on the issue. Lewis (2014) suggested that instruction in fraction concepts using representational models may be ineffective for some students with mathematics learning disabilities, partly explaining why some students who struggle in mathematics do not respond to traditional tutoring interventions. Lewis (2010) was also able to highlight the connections between difficulties students may experience with whole number facts that create challenges when students are confronted with more complex mathematics, such as fraction concepts. In such cases, students may be able to understand the more complicated concepts underlying the new mathematics topic, but may be hindered by computational strategies that are inefficient or result in errors (Lewis, 2010).

Hunt and Empson (2015) assert that "little to no information exists explaining the nature of conceptual gaps in understanding fractions for students with learning disabilities" (p. 208). Without information of this type, it is difficult to develop effective interventions for learners who struggle (Hunt & Empson, 2015). The researchers conducted clinical interviews with 10 third-, fourth-, and fifth-graders as they solved equal-sharing problems. Equal-sharing problems which include scenarios in which the number of objects to be shared is greater than the number of people sharing, resulting in each person receiving a fractional amount greater than one, are particularly useful in developing students' thinking about fractions, according to Hunt and Empson (2015), because these problems link fractions with students' prior knowledge about division. For this reason, participants were presented with equal-sharing problems resulting in a solution greater than one and then were presented with equal-sharing problems resulting in a solution less than one. They used a framework that described how typically-achieving children approached solving equal-sharing problems. In this framework, students are classified as using one of four strategies: a) a no-coordination strategy in which shares are unequal or objects are not fully shared; b) a non-anticipatory strategy in which a trial and error approach to partitioning is used; c) an emergent-anticipatory strategy in which partitioning is anticipated based on the number of objects and people sharing; and d) an anticipatory strategy in which the relationship between a fraction's value and division of the numerator by the denominator is understood. Study results indicated that participants used the first three strategies but not the fourth, with 76% of all solution strategies falling into the non-anticipatory category. This finding suggests that students with learning disabilities may use similar strategies to their typically-performing peers; however, these students may retain use of less sophisticated strategies at an older age (Hunt & Empson, 2015), a result that aligns with the findings of Geary's (2003) study about addition strategies used by students with the conceptual subtype of mathematics learning disabilities. Additionally, Hunt and Empson (2015) reported that study participants often attempted to use poorly understood rote procedures in place of problem-solving strategies and asked for explicit teacher direction to solve problems. The researchers concluded that conceptually-based learning focused on students' current level of conceptual understanding, which may be at a different level than that of typically-achieving peers, is vitally important to their success in mathematics. A quantitative two-year longitudinal study with 55 fourth-graders experiencing mathematics difficulties and 126 typically-performing fourth-graders by Hecht and Vagi (2010) found that differences on fraction problem performance between the two groups could be attributed to deficits in conceptual knowledge about fractions rather than arithmetic fluency and working memory. This result supports the contention of Hunt and Empson (2015) that students who struggle need instruction focused on their current conceptual knowledge.

Researchers who have investigated the thinking of children who struggle in mathematics have reached different, sometimes conflicting and sometimes overlapping, conclusions. One school of thought holds that the thinking of children who may have mathematics learning disabilities is different from that of children, struggling or not struggling, who do not have mathematics learning disabilities in ways that may impact their ability to understand mathematics concepts in typical ways (Lewis, 2010, 2014; Mazzocco & Devlin, 2008; Mazzocco et al., 2013). An alternative viewpoint holds that children who struggle in mathematics, whether identified with a mathematics learning disability or not, think about mathematics concepts in similar ways and engage solution strategies that parallel those of students who do not struggle; however, students who struggle are delayed in developing more sophisticated strategies and conceptual thinking about mathematics when compared to their more typical peers (Hecht & Vagi, 2010; Hunt & Empson, 2015). Geary’s (2003) model of learning disability in mathematics offers a partial framework that may be employed to shed light on different interpretations of the causes of mathematics disabilities or difficulties in learning mathematics. Geary (2003) hypothesized that mathematics difficulties could be categorized as a) issues related to delayed use of efficient procedures including poor conceptual understanding; b) issues with semantic memory including long retrieval times and high error rates; and c) visuospatial deficits that result in difficulties connecting numerical information to spatial information. It may be that a student experiences one type of mathematical difficulty or a combination of two or three.

Commonly agreed upon by researchers (Fletcher et al., 2007; Fuchs & Fuchs, 2003; Geary, 2003; Lewis, 2010, 2014; Mazzocco & Devlin, 2008; Mazzocco et al., 2013) who study children who struggle in mathematics is the perspective that learning disabilities in mathematics are not well understood and may present in a variety of ways requiring differing treatments, and that the delineation between those who have mathematics learning disabilities and those who struggle in mathematics without a learning disability is not clear. Although this study did not seek to clarify the delineation between those who struggle and those who have a learning disability, it did seek to describe how the participants' struggles might relate to Geary's (2003) hypothesis. Of particular interest to this researcher were data that would support conflicting positions that children who struggle think about mathematics in similar, but immature, ways when compared to their typically-achieving peers or that these children think about mathematics in intrinsically different ways.

Mathematics Interventions for Students Who Struggle

Rather than focusing on student thinking, studies about learners who struggle in mathematics have tended to focus on intervention techniques and programs, with most of these studies using student performance outcomes via correct answers on assessments to evaluate the efficacy of interventions (Butler et al., 2003; Flores & Kaylor, 2007; Fuchs & Fuchs, 2003; Fuchs et al., 2013; Gersten, Chard et al., 2009; Jitendra et al., 1998; Kroesbergen, & Van Luit, 2003). In addition, these studies have often examined interventions and methods that emphasized explicit instruction in procedures. This is in contrast to studies that did not delineate between typical students and students who struggle, which have emphasized the need for conceptually-based learning prior to instruction in specific procedures (Carpenter et al., 1998; Cobb et al., 1991; Cramer et al., 2002; Empson, 1999).

Kroesbergen and van Luit (2003) conducted a meta-analysis of 58 studies of interventions for elementary mathematics students who were struggling. Studies were classified by intervention method as direct instruction (n = 35), self-instruction (n = 16), and mediation (n = 16). Direct instruction and self-instruction were largely procedural interventions whereas mediated intervention relied on teaching strategies designed to build conceptual understandings (Kroesbergen & van Luit, 2003). Kroesbergen and van Luit (2003) found that, across studies, direct instruction ($d = .91$) and self-instruction ($d = 1.45$) were more effective than mediation techniques ($d = .34$).

According to Gersten, Chard, and colleagues (2009), their meta-analysis of 42 studies on mathematics interventions for students with learning disabilities employed more stringent selection criteria than the meta-analysis conducted by Kroesbergen and van Luit (2003). Gersten and colleagues contended that because Kroesbergen and van Luit combined the analysis of single-subject and group designs, inflated effect sizes were produced for single-subject designs calling into question the results of the study. Gersten and colleagues limited studies selected for their meta-analysis to group design studies using randomized selection or quasi-experimental designs. In a result contrasting those found by Kroesbergen and van Luit (2003), Gersten, Chard, and colleagues (2009) found that the most effective method of instruction for students with learning disabilities across studies was conceptually-based learning that exposed students to multiple ways to solve problems, emphasized student-generated solution strategies, and included emphasis on discourse (*d* = 1.56). Explicit instruction, which included direct instruction methods, was found to be nearly as effective (*d* = 1.22). Other methods, which focused on visual aids, feedback, and peer tutoring, were found less effective with effect sizes ranging from .14 to 1.04. A summary of the effect sizes for studies grouped by instructional component found by Gersten, Chard, and colleagues is presented in Table 4.

Table 4

Effect Size Results Grouped by Instructional Component from Gersten, Chard, and Colleagues' Meta-Analysis

Instructional Component	Random Effect Sizes (d)	Significance
Conceptual Learning Focus	1.56	$p < .001$
Direct Instruction	1.22	$p < .001$
Student Verbalization of Mathematical Reasoning	1.04	$p < .001$
Sequencing of Examples	.82	$p < .001$
Visuals Used by Teachers and Students	.47	$p < .001$
Teacher to Student Feedback	.23	$p < .01$
Student to Student Feedback	.21	$p < .05$
Cross-age Tutoring	1.02	$p < .001$
Peer-assisted Learning Within a Class	.14	n.s.

Gersten, Chard, and colleagues (2009) concluded that although many studies and theories of learning within the exceptional education community point to the difficulty for students who struggle with learning disabilities to effectively learn in environments focused on conceptually-based learning, the results of their meta-analysis appeared “to be at odds with the notion that students with LD [learning disabilities] have difficulty with cognitively demanding routines” (p. 1232).

Butler and colleagues (2003) investigated the effects of using a concrete-representational-abstract (CRA) instructional method versus a representational-abstract (RA) instructional method to teach fraction equivalence to middle school students with mathematics learning disabilities. Students were separated into two treatment groups for 10 lessons. Both treatment groups ($n = 50$) were taught with pictorial and abstract methods, while one treatment group ($n = 26$) was taught using concrete manipulatives for the first three days of instruction (Butler et al., 2003). A control group ($n = 65$) consisting of middle school students without mathematics learning disabilities received instruction using only abstract methods (Butler et al., 2003). All groups were administered pre- and post-tests consisting of five subtests: area fractions, quantity fractions, abstract fractions, improper fractions, and word problems. Results showed that both treatment groups outperformed the control group in two out of five subtests: improper fractions ($p < .0005$) and word problems ($p < .01$) (Butler et al., 2003). Both the CRA and RA treatment groups showed significant improvements ($p < .05$) from pretest to post-test on all subtests. No significant differences were found between the CRA treatment group and the RA treatment group. Butler and colleagues (2003) concluded that instruction on fraction equivalency was more effective for students with mathematics learning disabilities when either concrete manipulatives and representational drawings or representational drawings alone were used rather than instruction focused on abstract methods.

Fuchs and colleagues (2013) conducted a twelve-week intervention study with fourth-graders considered to be at risk for mathematics learning disabilities. The study participants were assigned to a treatment group (n = 129) that included a focus on the measurement interpretation of fractions or a control group (n = 130) focused on learning fractions in more traditional ways using procedures and the part-whole interpretation of fractions. Additionally, Fuchs and colleagues (2013) reported that the intervention group did not receive instruction in procedures until approximately two-thirds of the way through the intervention while the control group received instruction in procedures throughout the study. According to Fuchs and colleagues (2013), part-whole interpretations of fractions are commonly found in mathematics classes in the United States and are supported by problems about equal sharing and the use of area models. Less common in mathematics classes across the United States are techniques that focus on the measurement interpretation of fractions such as representation on a number line and comparison strategies that occur when the numerators of two fractions are the same number (Fuchs et al., 2013). The researchers found that effect sizes, ranging from .29 to 2.50, were better for at-risk fourth grade students who participated in a fraction intervention that included a measurement interpretation of fractions than at-risk fourth graders who learned about fractions based on part-whole concepts with a procedural focus. It should be noted that although the intervention was organized around fraction concepts, learning activities were structured using techniques designed to reduce the challenges associated with working memory deficits, listening comprehension, attention, and low processing speed (Fuchs et al., 2013).

Conclusion

In this chapter, research and scholarly writing on methods that facilitate children's learning about mathematics in general and fractions specifically were considered along with learning progressions that support learning fraction concepts. Then, research concerning learners who were struggling in mathematics was reviewed, specifically a lack of a universally accepted label or labeling method for learners who struggle in mathematics, the thinking of children who struggle and how it might or might not differentiate from children who do not struggle, and current research on interventions for children who struggle in mathematics. Although there is a robust body of research concerning interventions, the body of research that seeks to describe the thinking of children who struggle in mathematics is much less prevalent. This study will seek to describe the thinking of three participants in a small-group intervention that focused on conceptual learning supported by socio-constructivist scaffolding rather than teacher-led instruction in procedures using guided examples.

CHAPTER 3: RESEARCH DESIGN AND METHODOLOGY

This chapter reviews the research questions for this study, presents a rationale for the use of case study research design, and describes the participants and setting, including the sampling selection criteria. Data collection and data analysis procedures are discussed, including the use of two analytical frameworks based on Geary's (2003) identifications of subtypes of mathematical learning disabilities and Anghileri's (2006) descriptions of socio-constructivist scaffolding techniques. Procedures to ensure the trustworthiness of the study are described. Finally, the potential strengths and limitations of the research design and the potential contributions of the study are discussed.

Research Questions

Two research questions for this study were designed to elicit a description of the participants' individual struggles and strengths, and to describe how the participants made sense of third-grade fraction concepts in intervention sessions using social-constructivist scaffolding techniques. Geary's (2003) classification of learning disabilities in mathematics was used as a lens to assist in description of struggles and strengths. Anghileri's (2006) descriptions of social-constructivist scaffolding techniques were used to identify and analyze participant interactions during episodes of scaffolding. According to the Common Core State Standards for Mathematics (NGACBP & CCSSO, 2010), fraction concepts covered in third grade include equipartitioning and unit fractions (3.G.2, 3.NF.2.a), iteration of unit fractions (3.NF.1, 3.NF.2.b), fraction equivalence (3.NF.3.a, 3.NF.3.b, 3.NF.3.c), and fraction comparison (3.NF.3.d). Both research questions for this study addresses these fraction concepts presented in the CCSSM standards covered in third grade. The research questions for this study are as follows:

1. What struggles and strengths of third-grade students are revealed in a small group intervention supported by social-constructivist scaffolding while focused on fraction concepts?
2. How do third-grade students who struggle in mathematics interact with social-constructivist scaffolding techniques as they make sense of fraction concepts?

Case Study Research Design

The research questions for this study, focused on the process of sense-making with questions of "what" and "how," were well situated to be addressed by qualitative research, and case study research in particular (Brantlinger, Jimenez, Klingner, Pugach, & Richardson, 2005). According to Butler (2006), qualitative studies tend to have one of six categories of focus, with process being one of these. Furthermore, studies of process align most closely with research questions seeking to understand how students reason, make sense of, or come to understand concepts or information (Butler, 2006). Brantlinger and colleagues (2005) define qualitative research as "a systematic approach to understanding qualities, or the essential nature, of a phenomenon within a particular context" (p. 195). According to Merriam (2009), qualitative case study research is an in-depth analysis of a bounded system in which the case is studied to "achieve as full an understanding of the phenomenon as possible" (p. 42). Case study has advantages as a research design when it is difficult to separate a phenomenon from the context in which it occurs and research questions are oriented toward how or why (Brantlinger et al., 2005;

Butler, 2006; Merriam, 2009; Yin, 2003). Case study research designs tend to employ the widest range of data collection approaches available, including interviews, observations, participants' verbal explanations of their strategies, and artifacts documenting student work or thinking, to facilitate triangulation of data in an attempt to generate a trustworthy result (Butler, 2006). Butler (2006) proposed that the case study research design is capable of revealing links between an intervention and student learning and that, while case study research cannot be conducted in a way that all conditions are controlled, it can "reveal complex embedded relationships that may be obscured within an experimental inquiry frame" (p. 921).

This particular study was focused on how students who struggle in mathematics make sense of fraction concepts covered in third grade; however, the researcher was not able to separate the sense-making process of the students from the context in which it occurred, including the intervention tasks and the researcher's scaffolding techniques. Case study research is not a system of hypothesis testing, but rather a process designed to provide qualitative insight, including rich description and development of themes (Creswell, 2007; Merriam, 2009; Yin, 2003). Case study research is generally characterized by the following features: a) a search for meaning and understanding; b) the researcher as the primary instrument for data collection and analysis; c) extensive data collection; d) an inductive investigation strategy; and e) a richly descriptive end product (Brantlinger et al., 2005; Creswell, 2007; Merriam, 2009).

Some researchers propose that case study is less a method of qualitative research than a choice of the phenomenon to be studied within a bounded system (Creswell, 2007; Merriam, 2009; Miles & Huberman, 1994). As such, the most important aspect of case study research is defining the case (Merriam, 2009). The case, however, exists within a bounded system which must also be defined in advance and in relation to some function which may provide enlightenment about the phenomenon (Merriam, 2009). Because three students participated in small group intervention sessions with the researcher, this was a multiple case study in which participants engaged in the function of making sense of fraction concepts. The third-grade participants were identified as struggling based on second-grade achievement test results, third-grade progress monitoring assessments, and teacher and principal input. The phenomenon of interest in this study was the sense-making processes employed by the third-grade students identified as struggling in mathematics while working on third-grade fraction concepts in a small group intervention setting. Table 5 shows the elements that were part of the bounded system for this study.

Table 5

Elements of the Bounded System for the Research Study

Elements of the Bounded System for the Research Study
• Each of three students in third grade who were identified as struggling in mathematics (the three cases)
• The researcher as instructor, data collection and data analysis instrument
• The half-hour intervention sessions conducted thrice-weekly over five weeks in a local school's afterschool program
• The intervention task items based on Lesh and colleagues' (1987) framework of five different ways of representing and connecting a mathematical concept
• Dialogue between the students and instructor including expert/novice interactions as described by Vygotsky (1934/1986/2012, 1930-1934/1978) including scaffolding techniques as suggested by Cambourne (1988) and Graves (1983) and elaborated by Anghileri (2006).
• The students' work artifacts
• The afterschool program location

Participants and Setting

Purposive, criterion-based sampling was employed to identify three third-grade students based on past scores on second-grade IOWA mathematics achievement tests and third-grade i-Ready progress monitoring assessments, and input from teachers and principal. The literature on case study research (Creswell, 2007; Merriam 2009) does not specify a number of participants for a multiple case study, stating instead that the number of cases studied should be adequate to answer the research questions. Yin (2003) suggests that a multiple case design, even a study of two cases, is stronger than a single case design in almost all cases, allowing the researcher to locate commonalities and differences between cases. The choice to study three cases in this study is based on limitations of time and resources, and the researcher acknowledges that although a larger number of cases might answer the research questions more adequately, three cases meets Yin's criteria to strengthen the current study. However, it is the intention of the researcher to establish a qualitative research design that could be replicated to build multiple data sets about elementary students who struggle in mathematics and how they make sense of mathematical concepts.

Purposive sampling was appropriate for this study because the research questions require participants that meet certain criteria (Brantlinger et al., 2005; Creswell, 2007). For the purposes of this study, participants were referred to as students who struggle in mathematics. However, in the literature, similar students have been referred to as students with difficulty in mathematics, students at risk for low performance in mathematics, or students with learning disabilities in mathematics (Fuchs et al., 2013; Gersten, Chard et al., 2009; Louie et al., 2008; Witzel, 2005). The lack of a consistent nomenclature and the lack of delineation between students with a learning disability in mathematics, either diagnosed or undiagnosed, and students with low performance in mathematics without a learning disability are complications that any researcher selecting participants for a study concerning struggling mathematics students will confront. The initial decision to seek participants who performed in the lowest 25% on second-grade achievement tests with future performance in the lowest 25% predicted by third-grade progress monitoring assessments was based on criteria set out by Geary (2003) to identify children who may have learning disabilities in mathematics. Researchers have cited Geary's identification techniques (Hecht & Vagi, 2010; Lewis, 2010) or used similar schemes to identify subsets of students (Fuchs & Fuchs, 2003; Fuchs et al., 2013; Jitendra et al., 2014; Lewis, 2010; Mazzocco & Devlin, 2008) that they considered to either have learning disabilities in mathematics or struggles in mathematics that may or may not have been related to learning disabilities.

For this study, locating participants who fit Geary's suggested criteria proved challenging. The researcher decided to widen the identification criteria to include students who scored in the lowest 40^{th} percentile on the second-grade achievement tests while still identified by progress monitoring assessments as at risk for performance in the lowest 25^{th} percentile for upcoming third-grade achievement tests. Using a higher percentile higher than the 25^{th} is an identification strategy that has been employed by other researchers such as Fuchs and colleagues (2013) and Jitendra and colleagues (2014). The principal contacted the eight third-grade teachers in the school by email with details about this intervention study to ask them to identify students meeting the original selection criteria. Of the eight teachers, two responded to the principal with names of five potential participants. The principal then contacted parents of these five students to provide information about the study and request permission for their children to participate.

Parents of two of the students gave permission for their children to participate, while parents of three of the students did not respond. At that point, the researcher made a decision to expand the selection criteria and one more student was identified by a teacher as meeting the criteria. The parents of this students agreed to his participation after being contacted by the principal.

None of the three participants in this study had an individual learning plan (IEP), and it is not known if any of the participants had specific learning disabilities in mathematics, reading, or both mathematics and reading. As part of this study, the researcher made no judgment about whether or not a participant had learning disabilities, regardless of IEP status, but sought to establish that the participant had current and past performance that indicated struggles to perform at an adequate level in mathematics. The criteria established for participant selection for this study included performance in the lowest 40% on second grade IOWA Mathematics test results and results on the third-grade i-Ready progress monitoring assessments that indicate a risk for low performance on end of third grade mathematics assessments. The principal, in conjunction with third-grade teachers, located six students who met the criteria for participation. Parents of these six students were given information about the study and a request for participation. Parents of three students returned the forms and agreed to allow their child to participate in the study. Because these three students met the selection criteria, no attempt was made to locate additional participants. Although intervention sessions were conducted after school, students were not required to be part of the afterschool program to participate in this study. Prior to the first intervention session, the researcher obtained verbal assent for study participation from each participant. In keeping with the need to protect the identity of participants, the names used in this study to represent the three students are pseudonyms.

These participants were drawn from a high performing, high socio-economic elementary school in central Florida. Results obtained from the Florida Department of Education (FLDOE) website (Florida Department of Education, n.d.a) indicate that the school has performed in the top 5% of all elementary schools in the state for third-grade reading and mathematics on the Florida Standards Assessments (FSA) for the last two years, while performing in the top 10% of all elementary schools in the state for fourth- and fifth-grade reading and mathematics on the FSA during the same time period. Statistics provided on the FLDOE website (FLDOE, n.d.b) indicate a free and reduced rate of 30.7% for the school as compared to rates of 66.4% and 53.3% for the county and state respectively. This researcher made a decision to locate students who were struggling in mathematics in a school where most students are not struggling. This decision was not based on any assumption about whether or not students who struggle in a high-performing schools are different from students who struggle in low-performing schools on the part of the researcher. Rather, this was a decision based on the researcher's assertion that students who struggle should be studied across the spectrum of schools, including high-performing and high socio-economic schools such as the one in this study. The researcher acknowledges that students who struggle may or may not have different experiences in a high-performing versus a low-performing school, or a high socio-economic versus low socio-economic school; however, this research study does not address these aspects.

Data Collection Procedures

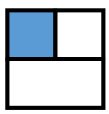
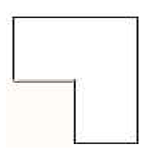
This study received approval by the Institutional Review Board (IRB) of the University of Central Florida in an expedited research review. IRB approval was provided before the commencement of data collection. As part of the IRB process, parents of participants were provided with an informed consent and consent signatures were obtained from parents who chose to allow their child to participate. Children gave verbal assent for participation in the study. The IRB approval letter, informed consent document, and verbal assent protocol are included in Appendix A.

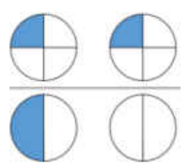
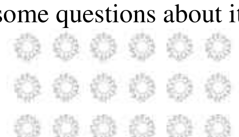
Intervention Sessions

The selected students participated in half-hour small group intervention sessions that occurred over five weeks, thrice-weekly, about fraction concepts. The intervention tasks consisted of problems and activities based on items available from the Rational Number Project (Cramer et al., 2009), *Making Sense of Mathematics for Teaching Grades 3-5* (Dixon, Nolan, Adams, Tobias, & Barmoha, 2016), and *Dimensions* (Ortiz, 2014), or developed based on guidance from commercially available books by Empson and Levi (2011), and Lamon (2010) designed to help teachers develop fraction tasks. Items used during the intervention sessions were developed to elaborate fraction concepts included in third grade according to the Common Core State Standards for Mathematics (NGACBP & CCSSO, 2010). Thirty-six activities that were used during the 15 intervention sessions are presented in Appendix B. These activities were classified according to the following criteria: a) the applicable Common Core State Standards for Mathematics (NGACBP & CCSSO, 2010); b) Lesh and colleagues' (1987) representational model; and c) type of fraction model or models used in the problem. Decisions on which items to use or new items to develop were made according to occurrences within the intervention sessions. Table 6 presents a representative sample of the material available in Appendix B. The intervention protocol, as enacted, is contained in Appendix C. Lesh and colleague's (1987) postulated that the learner has the greatest opportunity to make sense of mathematical concepts when a concept is worked by using multiple representations and that translating between different types of representations offers the most valuable learning experiences. For this study, these representations are labeled as real life contexts (RL), manipulatives (M), pictures (P), written symbols (W), and verbalization (V). Any of these representations may be presented as part of a problem or may be generated as part of a student's solution process and problems are intended to offer opportunities to move from one type of representation to another (Lesh et al., 1987). For example, a problem may be given in a real life context; then the student may create pictures to solve the problem and express an answer in written symbols. In Table 6 this sequence is shown as RL to P/W. Also important to note is that written symbol representation refers to both written language and written mathematical symbols while verbalization refers to all spoken language used by teacher and student while interpreting, solving, and answering a problem (Lesh et al., 1987).

Table 6

Classification of Intervention Activities

Activity Number and Example	CCSSM Focus	Representational Translation (Lesh et al., 1987)	Type of model(s)
Activity 3 What do you see? 	Equipartitioning 3.G.2 3.NF.1	P to W/V	Area Model
Activity 4 Jackie and Lianna have 13 cookies. If they share the cookies equally, how many cookies would each person get?	Equal sharing/Equal Partitioning 3.G.2 3.NF.1	RL to P/V	Set Model
Activity 6 Four children want to share 10 Publix sub sandwiches so that everyone gets the same. How much can each child have?	Unit Fractions 3.G.2 3.NF.1 3.NF.2.a 3.NF.2.b	RL/W to P/V	Set Model
Activity 13 Health First granola bars are square shaped, Lucius ate one piece of the granola bar and now it looks like this:  The piece that Janis ate is __ of a whole candy bar.	Equipartitioning and Unit Fractions 3.G.2 3.NF.1 3.NF.2.a 3.NF.2.b	RL/P/W to P/W/V	Area Models
Activity 18 Jordan said that 1 red piece is one-third. Andres said that 1 red piece is one-fourth.Who is correct?	Unit Fractions and Equivalence 3.G.2 3.NF.1 3.NF.3.c	M/RL/W to V/M	Area Model

Activity Number and Example	CCSSM Focus	Representational Translation (Lesh et al., 1987)	Type of model(s)
Activity 25 Dani wants to feed each of the children she babysits a half sandwich for lunch. If she babysits 8 children, how many sandwiches should she make?	Unit Fractions 3.G.2 3.NF.1 3.NF.2.a 3.NF.2.b	RL/W to P/V	Area and Linear Models
Activity 26 The cards for 0, ½, and 1 are placed on a table with space in between. Students place cards under the fraction cards in the correct location between 0, ½, and 1.	Comparison 3.NF.3.b 3.NF.3.a 3.NF.3.d 4.NF.2	M/W to M/V	Linear Model
Activity 29 Which set of circles has more shaded? 	Equivalence 3.G.2 3.NF.1 3.NF.3.b	P to V	Area Model
Activity 32 A group of 3 children are sharing 2 burritos so that each gets the same amount. How many burritos should 6 children share so that each child gets as much burrito as a child in the first group?	Equivalence 3.G.2 3.NF.1 3.NF.3.b	RL/W to P/V	Set Model
Activity 34 Look at this picture, then let's answer some questions about it:  Can you see thirds? How many suns are in 2/3 of the set?	Unit Fractions 3.G.2 3.NF.1	P/V to W/V	Set Model
Activity 36 Which fraction is larger? 3/4 or 2/6	Comparison 3.NF.3.d 4.NF.2	W/M to P(mental)/V	Used area model manipulatives

Note. RL=Real Life; M=Manipulatives; P=Pictures; W=Written Symbols; V=Verbalization.

All intervention sessions were audio- and video-taped and then transcribed shortly after the end of intervention sessions. In addition to maintaining observational notes during intervention sessions, the researcher reviewed audio- and video-tapes and transcriptions to create supplementary observational notes. Artifacts created by students while working with mathematics problems in the intervention sessions were collected and initial observational notes about these documents were generated. Most data collection activities occurred on-site during the intervention sessions with some activities, such as transcription of audio and visual files, along with additional observational notes, occurring off-site after intervention sessions.

Description of a Typical Intervention Session

For a typical intervention session, the researcher met the participants in the school's front office at the end of the school day. Then the group would move to the media center, which typically was lightly used at that time of day, to work at a table together. The participants might talk about the school day, or something like a video game, while the researcher talked with them and set up recorders and materials. Intervention typically started with the researcher saying, "I have something I want you guys to look at," or "I have something I want you guys to do," and then posing a question to the participants, or asking the participants to solve a problem. Although the researcher had at least one learning goal for each session, she did not talk about learning goals with the participants to start a session. Because the researcher wanted to create an environment where the participants would take the lead while supported by scaffolding in building mathematical ideas, she did not want to tell the students about the mathematical ideas they would be building.

In one session, the researcher's goal was to have the participants develop flexible thinking about defining one whole, particularly when using manipulatives. Participants' work in past sessions had indicated that they believed a whole circle was the only appropriate manipulative in a set of fraction circles to represent the whole. To begin the session, the researcher had sets of fraction circles available, but pulled out only one red piece, which is one-twelfth when a circle manipulative piece is defined as the whole. The researcher said, "I knew these two kids, Jordan and Andres. Jordan told me one red piece is one-third but Andres said one red piece is one-fourth. What do you guys think about what they said?" The discussion that followed lasted approximately 12 minutes and began with two participants, Clay and Daniel, asserting that both children were wrong. Marcos listened to the discussion for several minutes before saying he believed that Jordan and Andres could be right. As the discussion proceeded, the participants pulled out the fraction circle manipulatives and placed red pieces on other pieces until they discovered that three red pieces could fit on one tan piece and four red pieces could fit on one blue piece. After much discussion, Clay decided that four red pieces could be defined as the whole, while Marcos said that would be the same as saying the blue is the whole instead of the circle is the whole. The participants concluded that there was no rule saying a circle was the only piece in the manipulative set that could be a whole.

For the second part of the session, lasting approximately 20 minutes, the participants worked with written problems. When working from written problems, the researcher would typically ask for a volunteer to read the directions or the problem for the group. Often, Clay would volunteer to read, but Daniel also read for the group many times. Marcos did not volunteer to read for the group, but followed along with reading. In many cases, the researcher would reread directions or problems to the group or individual participants when asked, or when it became clear the participant did not remember or understand the reading. During this part of the session, the participants solved problems presented on a worksheet that defined different pieces of the manipulative set as the whole, then found the fraction name for given pieces. For example, in one problem the directions instructed the participants to define the whole as the yellow piece, a half circle, and then identify the fraction represented by one blue piece. One blue piece was one-fourth of a circle, but in this problem it was one-half of the defined whole. Like most activities involving written work, the participants worked on their own for about five minutes, then began talking about their ideas and the work of other participants. For this activity, after completing the second problem of six, the participants began talking with each other about their work. The researcher asked the participants to explain how they were using the manipulatives to find solutions. The remaining four problems were worked collaboratively. Then the participants, based on a suggestion by Marcos, began creating their own definitions of a whole and asking other participants to name given pieces. At this point, the time allotted for the session was ending. The researcher discussed with the participants the essential idea that fractions are meaningful in the context of a defined whole, and that the whole can be defined flexibly.

As the researcher stopped recording devices and collected participants' work, the participants picked up manipulatives and talked more about their interests. As with all sessions, the researcher walked the three participants to the front office where two were instructed to return to the afterschool program and one was instructed to wait for parent pickup. The researcher returned to the media center to write notes about the session and further plan for the next session. The next session was planned to include real-world problems in which the whole would be defined differently in each problem. For these problems, participants would construct pictorials to represent wholes and identify both unit and non-unit fractions.

Interviews

One goal of case study research is to produce a rich, thick description of the case within the bounded system (Creswell, 2007; Merriam, 2009). A semi-structured interview with each participant was conducted prior to the first intervention session to assist the researcher in building this type of description. Although these interviews were not expected to contribute directly to answering the research questions, they allowed the researcher to build a relationship with the participants, in addition to contributing to the descriptions of participants. For this interview, the researcher used a questioning frame suggested by Moustakas (1994) for phenomenological research (see Appendix C, Session 1) during which the researcher attempted to elicit information from the participant about his or her interests in general and feelings toward school, mathematics class, and mathematics. These questions were also designed to create a level of familiarity and comfort between each participant and the researcher. The discussion between each participant and the researcher lasted approximately 20 minutes each.

Data Collection during Intervention Sessions

The majority of data collection occurred during intervention sessions. These data were primarily analyzed to answer the research questions. The intervention sessions included aspects of clinical interviews intended to ascertain how the participants were building understanding about fraction concepts as they worked with items during the intervention sessions. Although interview techniques based on those suggested by Ginsburg (2009) were used, the researcher adjusted these techniques to draw upon Vygotsky's conceptions of the interaction between expert and novice through the use of scaffolding techniques. Specifically, the research focused on the use of probing questions and guiding statements that arose from interaction with the participants and attention to the participants' solution processes. This type of questioning technique, drawn from the literature on both clinical interviews and scaffolding, enabled the researcher to build an understanding of the participants' cognitive processes of making sense and the participants' learning potential (Ginsburg, 2009; Vygotsky, 1930-1934/1978). Ginsburg (2009) notes that clinical interviews originated from Piaget's attempts to combine the benefits of evaluating a student's task performance with those of observation techniques, and proceeds to state that Vygotsky's ideas on interviewing, although based on Piaget's clinical interview technique, included his belief that a student's construction of meaning may be created in conjunction with an expert tutor or teacher. Unlike Piaget's clinical interview technique, which would strictly prohibit conceptual input from a teacher, Vygotsky (1934/1986/2012) believed the clinical interview should be extended into the realm of a teaching tool by using questions designed to assist the student. In fact, Vygotsky (1934/1986/2012) believed that assessment of a student's true understanding and potential for understanding is not possible without techniques that allow for co-construction of knowledge between the novice and the expert.

When participants were presented with fraction items to work during intervention sessions, the researcher frequently asked participants to explain their work or justify their solution. A scaffolding orientation as proposed by Cambourne (1988) and Graves (1983) that involves careful observations of the student's work and vocalizations followed by questions intended to guide the student when needed was used by the researcher during intervention sessions. Cambourne (1988) proposed that scaffolding should not be viewed as an instructional process with predetermined steps, but rather a practice of formulating questions to redirect or deepen the student's conceptual thinking. Following Cambourne's (1988) conception of scaffolding, if the student appeared to be confused or asked for help, the researcher asked questions designed to assist the student in finding meaning in the underlying mathematical concepts. After the student solved the problem, if the student had errors or could not explain his reasoning, the researcher asked questions designed to elicit deeper thinking about the mathematical concepts. The researcher used questions that would support certain scaffolding techniques as described by Anghileri (2006), including prompting and probing, explaining and justifying, and negotiated meaning. Prompting and probing questions were intended to guide participants to think more deeply about fraction concepts and tasks while other questions were posed as requests for explanation and justification or negotiated meaning.

It was the intention of the researcher to avoid directly instructing the student in either procedures or concepts. In line with Broza and Kolikant's (2015) description of how scaffolding using questioning strategies interacted with meaningful learning of mathematics for students with low achievement in mathematics, the researcher attempted to use questions to encourage students to make explicit their mathematical thinking. For example, if the participant attended to the numbers in a word problem without considering the context, the researcher might ask the student, "What does the problem as you to do?" If the participant generated an answer to a problem but did not write or verbalize to explain or justify the answer, the researcher might ask the student, "Can you prove it?" or "How does your strategy prove it?" Participants were also questioned about their understanding of other participants' strategies and solutions including connections between different strategies. Table 7 presents examples of question stems the researcher frequently used to achieve this purpose along with the associated scaffolding technique.

Table 7

Question Stems Used in Scaffolding

Question Stems	Associated Scaffolding Technique
What does the problem ask you to do?	Prompting and Probing
Does your answer make sense?	Prompting and Probing
How do you know?	Prompting and Probing
Why did you decide to do this?	Prompting and Probing
Why does this matter?	Prompting and Probing
How did you get your answer?	Explaining and Justifying
Can you prove it?	Explaining and Justifying
Does your strategy prove it?	Explaining and Justifying
Can you tell me why?	Explaining and Justifying
Can you write something to show your solution strategy?	Explaining and Justifying
Do you agree with him?	Negotiated Meaning
Why do you agree or disagree?	Negotiated Meaning
Who do you think is right?	Negotiated Meaning
What do you think he did?	Negotiated Meaning
How is his strategy/drawing similar or different to yours?	Negotiated Meaning

To summarize, the researcher used questioning techniques based on student verbalizations and demonstrated work to help the participants extend their conceptual understanding of fraction concepts, to gage the participants' current and potential understandings of the concepts, and to uncover strengths and misunderstandings or errors that might have been preventing participants from successfully developing depth of understanding. In addition to information about the participants' thinking, using aspects of the clinical interview within the intervention sessions allowed the researcher to take note of affect, attention, and motivation, factors that may not be apparent in another data collection setting and were relevant to emerging codes and themes (Ginsburg, 2009).

Data Analysis Procedures

Procedures for data analysis outlined by Merriam (2009) and Creswell (2007) were the guides used for data analysis in this study. Additionally, qualitative coding relied on procedures outlined by Saldana (2013). Audio- and video-files of each intervention session were transcribed within several days of the last intervention session and observational notes were reviewed as supplementary notes were created. Transcripts, documents, and notes were analyzed to look for emerging codes and themes. The researcher generated a memo to herself at the conclusion of each intervention session to capture insights as close to real time as possible. Insights included strategies for upcoming intervention sessions, reflections, tentative themes, ideas, issues to pursue, and even hunches or instinct. These insights were used to inform and guide future data collection efforts. Decisions that resulted from these insights were recorded on a decision log that indicated how future sessions would be impacted. The decision log is contained in Appendix D. The data analysis was an ongoing process, which began with data collection and concluded only when a set of themes and answers to research questions could be formulated.

Coding of data began with recording of the participant's pseudonym, number of the intervention session, and classification of the tasks presented to students during the intervention sessions. Table 8 presents the codes used to classify tasks used during the intervention sessions.

Table 8

Codes for Classification of Tasks

Codes for Classification of Tasks
Type of fraction problem • Equipartitioning/Unit Fraction • Iteration of Unit Fractions • Equivalence • Comparison Lesh and colleagues' (1987) framework of five different ways of representing and connecting a mathematical concept • Real life situations • Manipulatives • Pictures • Written symbols • Verbalization Type of fraction model • Area model • Linear model – including the number line • Set model Material used to represent fractions • Fraction circle manipulatives • Paper strips, paper circles, paper squares intended for folding or cutting • Linear fraction manipulatives • Pre-drawn pictorial presentations • Student generated pictorial presentations

During data collection, the researcher broke down data into units of information. Merriam (2009) describes a unit of information as the smallest piece of data – whether one work or several pages of text – that can be interpreted with only a broad understanding of the goals of the research. For this study, the researcher found that the smallest units of data were episodes of conversation during work on tasks or during discussion of tasks subsequent to work. The researcher then compared these units of information to locate commonalities. The commonalities were used to generate first cycle codes to which future units of information were assigned. Emerging from first cycle coding were themes related to the nature of each participants' struggles. A pattern of misconceptions and errors emerged from the data collected from participants, which could be related to Geary's (2003) proposal that three subtypes of mathematical disability exist. At this point, Geary's subtypes of mathematical disabilities was chosen as an analytical framework for this study, with each misconception or error classified for each participant as fitting into one of the subtypes. Resulting from his work with elementary students who struggled with whole number operations, Geary proposed labeling three subtypes of mathematical disabilities as the conceptual subtype, the semantic memory subtype, and the visuospatial subtype. Characteristics of each subtype are listed in Table 9.

Table 9

Geary's Subtypes of Learning Disabilities in Mathematics

Subtype	Characteristics
Conceptual	• Use of inefficient or immature strategies • Mistakes made with use of procedures • Weak conceptual understanding • Difficulty executing multi-step strategies • Delay in development that often improves with time
Semantic Memory	• Struggle to retrieve mathematical facts • High error rate with fact retrieval • Time to retrieve facts is often longer than expected • Incorrect answers frequently related to numbers in the problem (for example, 3+4 might result in an answer of 5) • Cognitive difference rather than delay in development • Appears to be often associated with reading disabilities
Visuospatial	• Challenges in creating visual representations of mathematical information • Difficulties interpreting spatial information • Does not appear to be associated with reading disabilities

Implicit in this process is the commitment to recognizing new units of data which do not fit into any existing code and, thus, the necessity to generate new codes during analysis

(Merriam, 2009). During first cycle coding, a pattern of strengths began to emerge for each participant. Codes were created to describe these strengths. First cycle codes were combined or broken down into additional codes as second cycle coding focused on axial reorganization of first cycle codes. During axial reorganization of codes, a second line of themes emerged from the data concerning the interaction between participants' making sense of fraction concepts and scaffolding techniques employed. Although scaffolding techniques as described by Anghileri (2006) were always intended to be an integral part of the intervention sessions, it was not until the emergence of themes from second cycle coding that it became apparent that these scaffolding techniques should also be used as an analytical framework. Many researchers (Anghileri, 2006; Broza & Kolikant, 2015; Cambourne, 1988; Cazdan, 1983; Moschkovich, 2015; Putambeker & Hubscher, 2003) propose that scaffolding techniques are critical to success in mathematics for students who struggle. As such, episodes during intervention sessions were coded according to scaffolding techniques described by Anghileri (2006) and scaffolding episodes were analyzed for perceived success, partial success, or failure. The intent of this analysis was to describe instances in which the use of scaffolding was able to reveal participants' strengths or struggles and instances in which the techniques used enabled participants to make sense of fraction concepts. Although the researcher planned to use scaffolding techniques oriented around questioning strategies to support prompting and probing, specific decisions about appropriate questions were made in the moment during intervention sessions. The researcher determined which tasks would use manipulatives and pictorials in advance of intervention sessions; however, decisions were made to adjust use of these representations during intervention sessions as participants' needs became apparent. This approach to analyzing the instances of use and outcomes of scaffolding is in line with Vygotsky's (1934/1986/2012) assertion that assessment of a student's true understanding and potential for understanding is not possible without techniques that allow for co-construction of knowledge between the novice and the expert. The scaffolding techniques analyzed for this study are listed and described in Table 10.

Table 10

Scaffolding Techniques as Described by Anghileri (2006)

Technique	Description
Prompting and Probing	• Using questions to guide the student to a mathematical idea or solution • Using questions to guide the student to think more deeply
Looking, Touching, and Verbalizing	• Objects are manipulated or pictorials created by students • Students analyze and discuss what they see • Students talk about their mathematical ideas to others
Interpreting Student Work or Talk	• Clarifying student work • Clarifying student talk
Simplifying a Problem	• Creation of an intermediate task to shed light on concepts and strategies related to the original tasks • Cognitive complexity of the original task should be maintained
Explaining and Justifying	• Opportunities embedded within instruction for students to support or prove their work verbally • Opportunities embedded within instruction for students to support or prove their work in written format
Negotiated Meaning	• Development of mathematical ideas that are agreed upon by students • Supported by teacher guidance • Errors and misconceptions are addressed • Mathematical knowledge is created and shared within the group

While first and second cycle coding was proceeding, units of information were analyzed to determine relevancy to the research questions and purpose of this study. Units of information that did not shed light on these were sidelined for this study but may open a line of questioning valuable for future analysis or research. Merriam (2009) labels this type of qualitative data analysis as the constant-comparative method and posits that it is particularly suited to qualitative studies because it is inductive. The goal of the constant-comparative method is to make sense out of data by "consolidating, reducing, and interpreting" what the participants do and say, and what the researcher sees and hears (p. 175). For this study, coding and theme generation began during data collection, and continued after the conclusion of data collection leading to the eventual selection of the two data analysis frameworks and the results of the analysis.

Trustworthiness

Merriam (2009) posits that qualitative research can be judged to be trustworthy if the knowledge gained from the study is valid and reliable, and is obtained in an ethical manner, and further asserts that results are "trustworthy to the extent that there has been some rigor in carrying out the study" (p. 209). However, Merriam clarifies that the "standards for rigor in qualitative research necessarily differ from those of quantitative research" (p. 209). Throughout the course of this study, techniques proposed by many researchers as contributing to validity and reliability in qualitative studies will be used (Brantlinger et al., 2015; Butler, 2006; Creswell, 2007; Merriam, 2009; Moustakas, 1994; Saldana, 2013).

To assist in achieving trustworthiness in this research study, the researcher enacted the following strategies: a) bracketed her assumptions and positions; b) engaged in adequate data collection; c) employed triangulation of data; d) maintained records of collected data, analysis, and interpretation including the links between these; and e) created a thick, rich description of the results. The processes of bracketing and creating a thick, rich description of the results are intended to contribute to the validity of the study, while the other previously stated processes are expected to contribute to the reliability as well as the validity of the study.

Bracketing is described as a process qualitative researchers attempt to "set aside their experiences, as much as possible, to take a fresh perspective toward the phenomenon under examination" (Creswell, 2007, p. 59). Often researchers make a statement about their own experiences with the phenomenon in question including an explanation of their views (Brantlinger et al., 2005; Creswell, 2007; Moustakas, 1994). This researcher has included a bracketing statement as part of the data analysis that outlines the researcher's past experiences with students who struggle in mathematics, experiences teaching mathematics both from procedural and conceptual orientations, and experiences with family members who have reading and writing learning disabilities.

A thick, rich description is provided in the following chapter, which includes detailed portrayals of the participants, settings, and analytical findings with supporting evidence provided in the forms of transcript excerpts and student-created artifacts (Merriam, 2009). This type of description was intended to provide adequate detailed contexts of the study that readers may use to judge the transferability of the findings to their own circumstances (Merriam, 2009). For example, a reader may recognize similarities between the description of a participant's strengths and struggles and those of a student with whom he or she is working, or may seek to understand if a student in his or her class might build a deeper understanding of a given mathematics concept with scaffolding techniques similar to those detailed in episodes presented in the analysis.

The data collection process lasted for five weeks and occurred three times per week with each session lasting approximately one half hour. The length of researcher engagement with the participants should have allowed for adequate data collection in terms of opportunities to observe student work, to engage in questioning during student work, and to collect documents. The researcher made extensive adjustments to the tasks used in intervention sessions over the length of the study, thus changing the data analysis place to accommodate adjustments made to the data collected. Specifically, the original data collection plan was intended to include a variety of items addressing the concept of fraction comparison. Events during early sessions necessitated more focus on equipartitioning and unit fractions, thus limiting time available to focus on fraction comparison. With the exception of limited time devoted to fraction comparison, there was adequate data collection across topics though to make triangulation of data possible. The researcher was able to triangulate between real-time observational notes, documents produced by the participants, audio- and video-tapes of sessions, and transcripts of sessions. The researcher maintained and analyzed an audit trail of all data collected, memos generated, coding and analysis of data, and theme generation. Although a qualitative study of this type is not entirely reproducible, it was the intention of the researcher to provide an explanation to the reader that clarifies how the researcher arrived at the analytical findings of the study. The researcher also employed procedures suggested by Saldana (2013) specifically related to coding of data to contribute to the trustworthiness of a study. These procedures included line by line coding which reduced the likelihood of researcher bias, transcript creation as close to audio-recording as possible, initial coding occurred simultaneously with transcription, and analytic memos were used to track the evolution of the researcher's coding and thinking.

Brantlinger and colleagues (2005) posit that in addition to credibility measures, as previously described, the researcher should also attend to certain quality indicators that contribute to the trustworthiness of a study. As part of this study, the researcher has attempted to attend to quality indicators such as: a) appropriate participant selection criteria; b) sufficient time spent in sessions with participants; c) adequate recording, field notations, and timely transcriptions; d) meaningful artifact collection; e) systematic and meaningful coding; and f) conclusions substantiated by collected data.

Strengths and Limitations of the Research Design

For this study, the research questions, oriented around questions of what and how, led to the choice of case study research design. The research design, sampling procedure, and sampling size may limit generalization of study results beyond the participants studied (Creswell, 2007). In fact, Brantlinger and colleagues state that it is contrary to the "philosophies that ground qualitative scholarship to make authoritative pronouncements about what works for every person with disabilities in every social context" (p. 202). However, qualitative studies similar to this one have the potential to communicate valuable findings about the thinking of students who are struggling in school to readers (Brantlinger et al., 2005). It is the hope of this researcher that reading about how these learners make sense of early fraction concepts may help another researcher frame his or her research, or may help a practitioner view a learner who struggles in a different way leading to different teaching practices. The length of the study should also be considered. Although intervention sessions held thrice weekly over five weeks should have allowed enough time to strengthen data collection and analysis, a longer intervention period could further strengthen the researcher's ability to form a fuller picture of the participants' sense-making processes. In particular, the researcher's decision to focus more time within the study on equipartitioning and unit fraction concepts at the expense of fraction comparison is a limitation of the study. Also, the study focus on a narrow set of early fraction concepts, limits the potential to shed light on how struggling mathematics students might make sense of other mathematical concepts. Another limitation of the study is that the researcher served as the primary instrument of data collection and analysis. Attention to which data should be recorded and how that data should be analyzed depended upon the abilities, skills, and knowledge of the researcher. Themes that emerged from the data were only uncovered through the lens of the researcher, meaning that researcher bias was unlikely to be completely mitigated even with efforts to increase the trustworthiness of the study.

The identification of students to participate in this study was a potential limitation. The use of second-grade mathematics achievement tests and third-grade progress monitoring assessments to identify students who struggle with mathematics did not allow the researcher to know with any certainty why a participant may be struggling prior to the study. As such, it was not possible for the researcher to know in advance whether a student's struggles may relate to learning disabilities, which could be specific to reading or mathematics, or to other circumstances. However, this is a limitation common in both quantitative and qualitative research concerning students who struggle and studies about mathematics interventions for students who struggle. According to Ginsburg (1997) the research literature in mathematics education often fails to distinguish between the wider category of students who struggle and the subcategory within those who struggle who have learning disabilities. Often the identification of learning disabilities is based on an issue with reading performance or low performance across school subjects. Furthermore, scholars (Fletcher, 2007; Geary, 2003; Ginsburg, 1997) assert that current identification methods are inadequate to determine if the root cause for a student's struggles are learning disabilities in general, mathematics learning disabilities specifically, another cause, or a combination of causes. Geary (2003) posits that little has been learned about the root causes of children's difficulties in mathematics even when mathematics learning disabilities are assumed based on performance. A review of the literature reveals that studies that identified students as having a mathematical learning disability often used low performance or being at risk for failure as the only defining characteristic (Gersten, Chard et al., 2009; Jitendra et al., 1998). Other studies only identified students as struggling or having difficulty with mathematics (Flores & Kaylor, 2007; Fuchs & Fuchs, 2003; Fuchs et al., 2013; Kroesbergen, & Van Luit, 2003). These research studies make no attempt to define what would distinguish a student with a learning disability from a student who struggles. This is a limitation of this current study as well as, according to Ginsburg (1997), one common in the literature. Although Ginsburg made this statement in 1997, a review of more current research reveals this to still be the case (Butler et al, 2005; Fuchs et al, 2013; Gersten, Chard et al, 2009; Jitendra et al., 2014; Zhang & Xin, 2012). Future qualitative research that delineates between students who struggle, students who have general or reading learning disabilities, and students who have mathematics learning disabilities is needed. Investigations into the different needs and strengths of the three groups would be valuable in mathematics education practice. Before these types of studies can be conducted though, there is a need to establish common working definitions of terms through research and reflection on common practices.

CHAPTER 4: ANALYSIS

The presentation of the analysis begins with a statement by this researcher intended to describe the life experiences that the researcher has attempted to bracket out of the research process to reduce bias in the study. A description of the study participants and the intervention sessions as enacted follows. The analysis of the data is presented in two parts. The first part addresses the first analysis framework selected for this study, Geary's (2003) subtypes of learning difficulties. A description of Geary's subtypes is followed by an analysis of each participant's thinking and work related to fractions during the intervention sessions using the first framework. In the second part of the analysis presentation, the second analysis framework, the use of scaffolding techniques, is presented. Many researchers and educators (Bruner & Ratner, 1978; Cambourne, 1988; Cazdan, 1983; Ninio & Bruner, 1978; Wood, Bruner, & Ross, 1976) have proposed scaffolding techniques. However the analysis in the study is based on the use of scaffolding techniques proposed by Anghileri (2006) for use in the mathematics instruction learning process.

Bracketing Statement

Following Moustakas' (1994) phenomenological process, I have attempted to state and then bracket out my own experiences with students and family members who have experienced struggles in learning as I investigated the experiences of the participants in this study. As part of this attempt, I have included this statement regarding my personal experiences. It was my intent to reduce the influence of my past experiences while conducting this research, and to be transparent with the reader about these past experiences. I have had many experiences with children and adults who have learning difficulties, both as a teacher and a family member. As a middle school algebra teacher, I was struck by how often students who struggled to achieve an acceptable grade seemed to know more about algebra than their grades reflected. Some students could explain mathematical concepts at a high level but would struggle to complete work accurately or within typical time frames. These students frequently struggled with simple computation that made algebra difficult for them even though their conceptual understandings might be strong. Some students could explain or demonstrate procedures, but could not connect procedures to the deeper concepts. Often, these students would tell me that they used to be "good at math" but were not anymore and that they no longer liked mathematics, or believed they could learn it well. Within my own family, disabilities related to reading and writing are common. I have experienced first-hand how complications with reading can contribute to difficulties with mathematics task performance. My experiences have shown me that some people with reading disabilities struggle to work with tasks in a limited time frame or to express their mathematical ideas in words.

Description of Participants

Three students in third grade participated in the after-school intervention sessions. Pseudonyms for the students, used to protect the identities of the participants in this study, are Clay, Daniel, and Marcos. The researcher served as their mathematical guide during these sessions. Each of the students were identified by the principal using results from the second grade IOWA Mathematics Test from the previous year, fall and winter i-Ready progress monitoring assessments, and teacher input to the principal. Parent consent and student assent for participation and recording were obtained using IRB protocols prior to the first intervention session. Initially, the researcher had requested that each participant be identified as performing within the lowest 25% on a national normed measure. The IOWA Mathematics Test, administered to each participant in the spring of second grade, was used as this measure. Clay and Daniel fell within this guideline, performing at 20th and 25th percentiles respectively. In addition, i-Ready progress monitoring predicted that both Clay and Daniel were at risk of performing below grade level on the upcoming spring annual assessment. Marcos performed at the 40th percentile but was included in the study as well because his i-Ready progress monitoring predicted that he was at risk for below grade level performance on the upcoming spring annual assessment. Selection criteria to identify students struggling in mathematics have been extended beyond the 25th percentile in some studies. Using nationally normed tests of mathematics achievement, Fuchs and colleagues (2013) set selection criteria below the 35th percentile for fourth-graders and Jitendra and colleagues (2014) set selection criteria at or below the 40th percentile. Based on these past studies and Marcos' performance on third grade i-Ready assessments, a decision was made to include Marcos in the study. Marcos and Clay had the same classroom teacher, whereas Daniel was from another class in the same school. Marcos' teacher reported to the principal that Marcos was struggling to complete mathematics work in class and often performed poorly on class mathematics tests and quizzes. His teacher also noted that Marcos was prone to be quiet, but distracted, in class and he often did not finish his classwork. Although Clay's standardized test performance and progress monitoring fell below Marcos', his teacher reported that his classroom mathematics performance was sometimes good but uneven, with more struggle becoming apparent as fraction concepts were introduced. In class, Clay could be talkative and tended to finish his work quickly. Daniel's teacher reported that, although he sometimes worked slowly, he was a diligent and persistent worker in all of his classwork.

Description of Intervention Sessions

Fifteen intervention sessions were conducted in the school's media center during the after school program in which the researcher worked in a social environment simultaneously with the three participants. The duration of the sessions averaged 32 minutes per session, with the shortest session lasting 28 minutes and the longest 41 minutes. Intervention sessions were initially intended to focus on fraction topics of equipartitioning and unit fractions, iteration of unit fractions, fraction equivalence, and fraction comparison in relatively equal measures. As sessions progressed, data collected from each session influenced decisions about topics to cover in subsequent sessions. As a result, more time was devoted to developing understanding of equipartitioning and unit fractions, and iteration of unit fractions. Although the focus on fraction equivalence was reduced, tasks involving concepts of fraction equivalence were used in eight of 15 sessions. Tasks focused on fraction comparison occurred in two sessions. Topics covered during each session are shown on Table 11. A decision log, presented in Appendix D, was used to record decisions about topic focus, tasks, and materials made for subsequent sessions and, in three cases, during sessions.

Table 11

Fraction Topics Covered by Session

Session	Duration of Session in Minutes	Equipartitioning and Unit Fractions	Iteration of Unit Fractions	Fraction Equivalence	Fraction Comparison
1	36	✓			
2	30	✓	✓	✓	
3	32	✓	✓		
4	32	✓			
5	36	✓	✓	✓	
6	41		✓		
7	31	✓		✓	
8	28	✓	✓	✓	
9	29	✓	✓		
10	30	✓			✓
11	29			✓	
12	34			✓	
13	28	✓		✓	
14	28	✓	✓		
15	31	✓	✓	✓	✓

Analysis Framework 1: Geary's Subtypes of Learning Disabilities in Mathematics

As a result of his work with elementary students who exhibited difficulties in mathematics, specifically with whole number operations, Geary (2003) concluded that it may be possible to identify three subtypes of mathematical disability and labeled these as the conceptual subtype, the semantic memory subtype, and the visuospatial subtype. The conceptual subtype is characterized by a "poor understanding of the concepts underlying procedural use" (Geary, 2003, p. 205). The semantic memory subtype is associated with difficulties in retrieving mathematical information accurately and rapidly. The visuospatial subtype indicates difficulties in coordinating spatial information with abstract information. The defining characteristics of each subtype are outlines in Table 12.

Table 12

Geary's Subtypes of Learning Disabilities in Mathematics

Subtype	Characteristics
Conceptual	• Use of inefficient or immature strategies • Mistakes made with use of procedures • Weak conceptual understanding • Difficulty executing multi-step strategies • Delay in development that often improves with time
Semantic Memory	• Struggle to retrieve mathematical facts • High error rate with fact retrieval • Time to retrieve facts is often longer than expected • Incorrect answers frequently related to numbers in the problem (for example, 3+4 might result in an answer of 5) • Cognitive difference rather than delay in development • Appears to be often associated with reading disabilities
Visuospatial	• Challenges in creating visual representations of mathematical information • Difficulties interpreting spatial information • Does not appear to be associated with reading disabilities

Although Geary (2003) labeled these three subtypes as pertaining to mathematical disabilities, he is clear in advising that no measures exist that can specifically diagnose a learning disability in mathematics. As such, Geary used a criterion of performance in the lowest 25% on a mathematics achievement test in two consecutive years. Other researchers (Fuchs et al., 2013; Gersten, Chard et al., 2009; Hecht & Vagi, 2010; Jitendra et al., 2014; Zhang & Xin, 2012) have used similar criteria to identify students that they describe as struggling in mathematics, having mathematics difficulties, or at-risk for low performance in mathematics. For this study, the researcher has chosen to use similar performance criterion, along with performance on third-grade progress monitoring assessments, to identify students who are struggling in mathematics. The researcher will make no attempt to comment on the possibility that the students who participated in this study either do or do not have a learning disability in mathematics. However, what is clear is that the study participants can be considered to be struggling in mathematics according to their performance. Furthermore, work with these students across multiple sessions does seem to indicate that Geary's subtypes of mathematical disabilities is also a useful framework for analyzing the specific types of struggles these students confronted in their work with fractions. As such, in this study, the researcher referred to subtypes of mathematical difficulties rather than subtypes of mathematical disabilities. Table 13 presents the categorization of errors and misconceptions by subtype identified for each participant over the course of the 15 small group intervention sessions.

Table 13

Errors and Misconceptions by Subtype

Participant	Conceptual	Semantic Memory	Visuospatial	Total
Clay	25	8	9	42
Daniel	28	8	18	54
Marcos	9	19	5	33
Total	62	35	32	129

Each student exhibited errors and misconceptions that fit into each of the three subtypes. However, some patterns can be seen in the summary of the data. Clay clearly had more occurrences within the conceptual subtype than he had in the semantic memory or visuospatial subtypes, indicating that his conceptual understanding of fractions may be weak or immature. Marcos has few occurrences in either the conceptual or visuospatial subtypes, but relatively more in the sematic memory subtype. Marcos' struggles with accuracy and extended time to retrieve information were evident throughout the sessions. Although Daniel struggled most within the conceptual subtype, a comparison of his occurrences by subtype to those of the other two participants shows he was experiencing a struggle within the visuospatial subtype. Additionally, any analysis of difficulties for a particular student is incomplete without a corresponding analysis of strengths. An analysis of strengths and weakness, supported by transcript excerpts and student artifacts, for each participant follows. For Clay and Marcos, difficulties within the conceptual and semantic memory subtypes were respectively highlighted. For Daniel, the decision was made to focus on occurrences within the visuospatial subtype. The strengths discovered for each participant in this study do not always align with Geary's subtypes, but where alignment exists, the researcher described this alignment later in this manuscript.

Clay

Clay's struggles appeared to be generated by conceptual misunderstandings about the nature of fractions indicating struggles within the conceptual subtype. In this discussion, four issues that appeared to be related to Clay's weak conceptual understanding of fractions are covered: a) Clay's assertion that fractions do not have to have equal-sized pieces; b) Clay's practice of counting pieces to give a whole number answer to a problem that has a fractional answer; c) Clay's naming of unit fractions without attending to the defined whole; and d) Clay's use of whole number reasoning to compare fractions. In addition, episodes occurred which uncovered strengths related to Clay's reasoning abilities when he understood a concept and his ability to think flexibly about mathematics.

One clue as to the conceptual basis for Clay's struggles may be his correct interpretation of the need for equal-sized pieces in a set of pictorial representations of fractions while failing to attend to this requirement when problems were presented in contexts such as real world word problems. Figure 2 shows Clay's work with identification of two-fourths while attending to equal-sized pieces, which occurred during the first session. The directions for the problem set indicated that Clay should place an "x" beside each picture showing two-fourths shaded. Clay correctly identified pictorial representations of two-fourths using visual confirmation of equal-sized pieces.

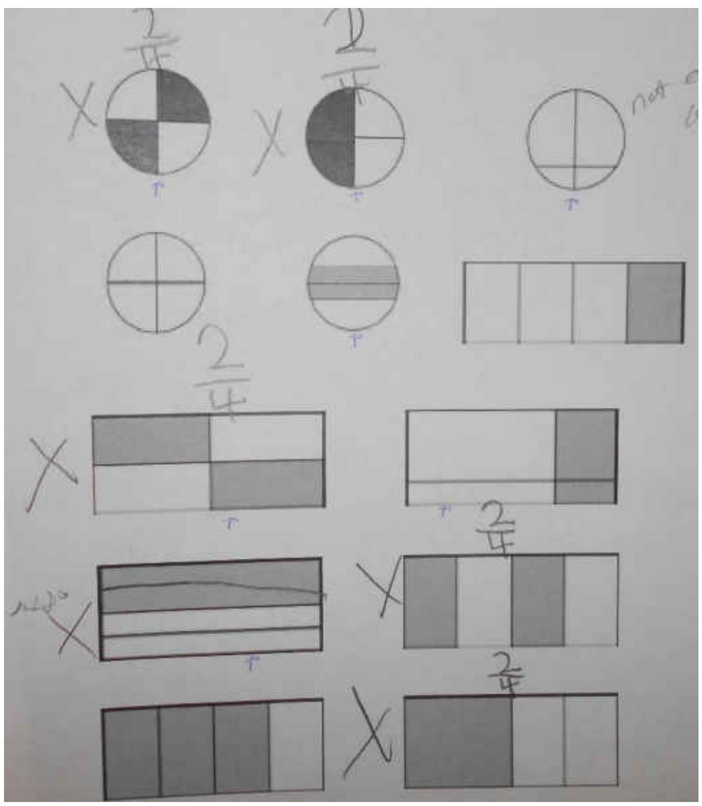


Figure 2: Clay's Identification of Two-Fourths While Attending to Equal-Sized Pieces

Later in the same session, Clay was presented with the following real-world context word problem and responded with the following answer in group discussion:

Jackie and Lianna have 13 cookies. If they share the cookies equally, how many cookies would each person get?

Clay: Hm! I think I've got it!
[about a minute later]
Clay: They both get seven.

Researcher: Ok, so tell me this. Let's look at yours, Daniel and Clay. You are telling me each person gets seven cookies? How many cookies is that?

Daniel: 14

Clay: You have to…. You cut the last thirteenth cookie in half and you get 14.

Researcher: Is that the same as having 14 cookies? If it's a different size?

Marcos: No!

Clay: Yes

Researcher: They're each getting six cookies and then there's one cookie left….and they have to share it somehow….are they each getting a whole cookie?

Clay: Yes!

Daniel: Um…. Split it in half.

Researcher: Ok, split the last cookie in half. Is that the same as getting whole cookie?

Daniel: No.

Clay: Yes.

During the fourth session, Clay communicated his misunderstanding about equal-sized pieces undergirding the concept of fractions.

Researcher: Are fractions always fair shares?

Clay: Well not all of the time. Some people get more.

Researcher: But to be a fraction, something we might call fourths or sixths…does it have to be a fair share?

Daniel: Yes.

Marcos: Yes.

Clay: Well those are fair…but a fraction might not be fair…or it could be…

Clay showed that he did not believe equal-sized pieces are a requirement of defining a fraction, although he attended to equal-sized pieces at times when he felt he might be expected to do so. Looking at Clay's work in Figure 2, one might assume that he has a grounded and accurate understanding of the relationship between fractions and equal-sized pieces. His discussions with the group during the intervention sessions, however, revealed a different picture. Clay was aware that fractions are sometimes presented in equal-sized pieces but his misconception led him to believe that this is not an essential part of establishing the meaning of fractions.

Clay consistently answered questions with a whole number answer derived by counting fractional pieces. In the third session, a problem about sharing peaches was presented and Clay engaged in the following discussion with the researcher:

> *4 children want to share 3 peaches so that everyone gets the same amount. How much peach can each child have?*
>
> Clay: So they each get three!
>
> Researcher: Three what?
>
> Clay: Three peaches.
>
> Researcher: What would you name each piece? As a fraction?
>
> Clay: Four fourths, a whole.
>
> Researcher: Okay, so what is the name of one piece? Just this one here?
>
> Clay: Fourths.
>
> Researcher: So your pieces are named fourths. How much did each child get?
>
> Clay: Three
>
> Researcher: Three what?
>
> Clay: Pieces of the peach.

Researcher: And what are those pieces named?

Clay: Fourths.

Researcher: So they got three ……

Clay: Pieces of the peach.

Researcher: Should we say they got three fourths of the peach?

Clay: Sure, you could.

During the fifth session, Clay demonstrated the misunderstanding again during a game with pieces that have fractional names and with a real world word problem.

Researcher (to Clay): What do you need to win?

Clay: Three more.

Researcher: Three more what?

Clay: Three more fractions.

Researcher: I see that, but three more of what kind of fraction?

[Clay is off-task, talking about something unrelated]

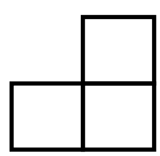
Daniel: Sixteenths! [Answering for Clay]

Researcher: Clay do you agree with him? He says you need three more sixteenths.

Clay: Yeah.

Later in this session, Clay has a reaction to the following word problem:

Healthfirst granola bars are shaped like a square. Lucius ate one piece of a granola bar and now it looks like this:



Lucius ate how much of the granola bar?

Clay: It tells you the answer!

Researcher: What? What answer?

Clay: It tells you in the problem. One piece. Lucius at one piece.

Clay attended to whole numbers within the fractional answers to a problem. This may be related to another misconception Clay demonstrated when working with fractions. Clay would name a unit fraction by counting the number of pieces available to him. Clay did not understand that he needed to attend to the definition of the whole pertaining to a particular problem. In the following exchange, the whole was defined as a circle manipulative. Clay had eight fourth-sized pieces.

Clay: 1, 2, 3, 4, 5, 6, 7, 8

Researcher: What are your pieces called Clay?

Clay: Eighths.

Researcher: Why are they eighths?

Clay: There are eight, so each one is an eighth….so eighths.

Researcher: Hmmm….do they fit on the whole?

Clay: [places four pieces on the whole] Oh, four of them do.

Researcher: There are eight of them. Are they eighths because there are eight of them?

Clay: Um….no…

Researcher: Not necessarily…what are they though?

Clay: They're fourths! Fourth!

Clay's challenge of using whole number knowledge to reason about fractions was apparent in later sessions as well when dealing with fraction equivalence and fraction comparison. When presented with the following graphic, shown in Figure 3, during session 11, the students were asked to state if the top set of circles or the bottom set of circles had more shaded. Clay's response follows in this transcript excerpt.

Marcos: They're both the same.

Researcher: What do you think Clay? You had something you really wanted to say. Tell us what you want to say.

Clay: Well, that's actually incorrect…..

Clay: Because this one…is not… This one is not shaded and this one is [referring to the bottom row]. And these two are shaded.

Researcher: So do you think one row has more shaded?

Clay: [points at the top row]

Clay: But they are the same….

Researcher: The top row is more, but they are the same?

Clay: So when these…when these two add to each other [pointing at the top row], then they become this one [pointing at the bottom row]. But…this one is bigger [pointing at the top row] because it has…both of them have one part shaded, and this one [pointing at the bottom row] has one part shaded.

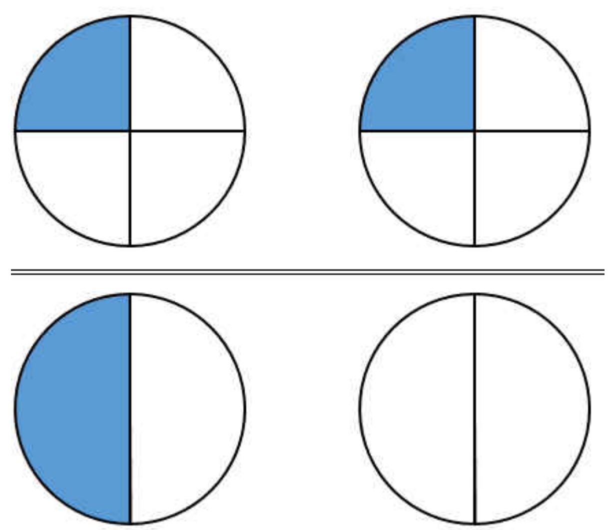


Figure 3: Which Set of Circles has More Shaded?

Clay demonstrated a visual sense that the same amount of area was covered in both sets of circles; however, he was entrenched in whole number reasoning that indicated to him that two pieces is still more than one piece. When working with fraction circle manipulatives in a subsequent session, Clay was able to successfully compare fractions. Later still in the intervention sessions, Clay worked with abstract fraction comparison tasks and reverted to whole number reasoning to explain his comparison of three-fourths and three-tenths. He also invoked a procedure for comparing fractions with the same denominator to support his answer in the discussion that follows. It was not until later in this session with the use of circle fraction manipulatives that Clay was able to correctly compare three-fourths to three-tenths.

> Researcher: See these two in the first row? These two fractions, three-fourths and three-tenths. Which one do you think is greater?
>
> Clay: Larger?
>
> Researcher: Yes, larger.
>
> Clay: This one [indicating three-tenths].
>
> Researcher: Why is this one larger?
>
> Clay: Because it has 10…10 groups and three shaded. That one just has four groups and three shaded.
>
> Researcher: Daniel said three-fourths was larger though…
>
> Clay: No…
>
> Researcher: Why not?
>
> Clay: Because if…the numerators are the same…the numerators are the same. But the denominators are not the same….so…you have to look at the denominators and see which one's bigger!

Clay's struggles with the conceptual underpinnings of fractions might lead one to assume that his mathematical reasoning skills are weak or his ability to work with complex mathematics is limited. However, Clay often demonstrated an ability to understand a mathematical situation quickly and accurately as demonstrated in the discussion excerpts below from sessions nine and 12.

> Researcher: Here is our question. Dani wants to feed each of the children she babysits a half sandwich for lunch. If she babysits eight children, how many sandwiches should she make?
>
> Clay: Okay! I think it's four!

It should be noted that although Clay had provided the correct answer immediately following the researcher's positing of the question, the group of three, including Clay, continued to work on this problem in mathematical depth for 20 minutes to prove that the answer of four was correct. During session 12 the following exchange occurred after working extensively with fraction circle manipulatives to discover fractions equivalent to one-half.

> Researcher: What did you come up with Clay? You had one more to do.
>
> Clay: Five-tenths!
>
> Researcher: Five-tenths? Write that one down! That's a great one! Can you show me where you made it with the circles?
>
> Clay: Oh…I didn't… I didn't make it. I didn't need to…
>
> Researcher: How'd you know it would work then?
>
> Clay: Well…five is half of ten…so…

Clay was also able to easily attend to flexible definitions of the whole, something both Daniel and Marco struggled with at first. This might have been related to Clay's disconnection of fractional pieces from a defined whole. However, when Clay was encouraged to coordinate naming unit fractions with a defined whole, his reasoning was remarkable flexible. In the following discussion, Daniel was struggling to understand that the circular manipulative piece is not the only potential definition for a whole in the set. The half circle or the quarter circle could be defined as the whole, resulting in the renaming of other pieces. In the case covered below, 12 red pieces are needed to cover a whole circle.

> Researcher: My question for you: I knew these two kids, Jordan and Andres. Jordan told me one red piece is one-third but Andres said one red piece is one-fourth. I want to know what you guys think about what they said.
>
> Daniel: All of them are wrong. No, they're both wrong.
>
> Researcher: Why are they both wrong?
>
> Daniel: It's not a third or a fourth.

Marcos: No, they are wrong because a whole then would just be three pieces, then it would be a third.

Clay: What? I give up!

Researcher: Clay what do you think about what they said?

Clay: Um….uh…

Researcher: What do you think about what they said? Could a red piece be one-third or one-fourth?

Clay: because….I mean yes it can.

Researcher: It could?

Clay: Yes.

Researcher: Why? How could it be one-fourth?

Clay: Because it could be four with one. Four could be the whole thing…then one is a fourth.

Clay's strong mathematical insight combined with his lack of conceptual understanding of fractions suggests that he is a fit in the conceptual subtype. Clay appeared to be capable of making sense of fraction given time to experience work with cognitively challenging tasks that address these concepts. However, Clay seemed to be prone to relying on procedures without understanding. This may be because Clay has not had enough time to engage with the concepts and he is searching for shortcuts to perform well in mathematics given the time constraints with which he is presented.

Daniel

Although occurrences of errors and misconceptions for Daniel indicated that he might fit into the conceptual subtype (25 occurrences) of mathematical difficulties as well as the visuospatial subtype (18 occurrences), he had more occurrences that were associated with the visuospatial subtype than Clay (9) or Marcos (5). In some cases, it seemed that his conceptual misunderstandings might be rooted in issues with visuospatial understanding. In particular, a review of Daniel's strengths show that these strengths tended to be connected to sound conceptual thinking about cognitively challenging problems. In this discussion, two issues that appeared to be related to Daniel's struggles within the visuospatial subtype are covered: a) difficulties discriminating between visual depictions of equal and non-equal areas; and b) difficulties constructing pictorials that showed equal piece sizes. Daniel's strengths related to sound conceptual thinking and his ability to challenge other participants to think critically about the validity of their answers are discussed as well.

One visuospatial issue that recurred throughout the intervention sessions related to Daniel's ability to correctly understand the need for equal-sized pieces, but not to construct or see equal sized pieces. The artifacts below were created during sessions one, four, and 10. The first artifact shown in Figure 4 shows Daniel's struggle to identify equal-sized pieces as a critical element in identifying pictorials showing two-fourths. The directions for the problem set indicated that Daniel should place an "x" beside each picture showing two-fourths shaded. It can be seen in the image of Daniel's work that he had initially places x's on two of the figures to show he believed these two represented two-fourths, and then he later erased the x marks when he heard the reasoning of his fellow student, Marcos.

> Researcher: Okay, so how about this one? [referring to rectangle shown on the right] Is that two-fourths?
>
> Clay: Nope.
>
> Marcos: No.
>
> Daniel: [after Clay and Marcos] Yes it is!
>
> Researcher: Why do you think so Daniel? Why is it two-fourths?

Daniel: Because this is supposed to be, like, one part of the shape and this is the other. And these two are…are…this is the rectangle, the rectangle, this shape…and they are both shaded.

Researcher: They are both shaded. So how many parts are there?

Daniel: Four.

Researcher: And how many parts are shaded?

Daniel: Two.

Researcher: Why would that not work? Marcos, you said that's not two-fourths? Why did you say that?

Marcos: Because it's not equal….the lines are not equal.

Researcher: What do you think Daniel? Does that make sense?

Daniel: Well…no…. Well, maybe.

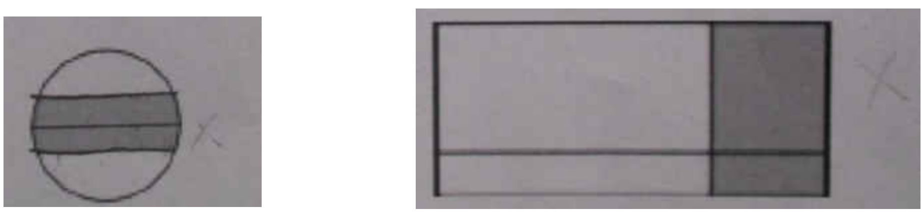


Figure 4: Daniel's Identification of Two-Fourths while Attending to Equal-Sized Pieces

Subsequent work in session three showed that Daniel continued to struggle with visualization of equal-sized pieces. In the artifact shown in Figure 5 and the discussion following, Daniel successfully found the solution to this problem.

4 children want to share 3 peaches so that everyone gets the same amount. How much peach can each child have?

Daniel's pictorial representation of the shared peaches showed circles divided in a way that was unlikely to generate equal-sized pieces. However, he talked about the need for fair shares among the four children and related the fractional value of the peach correctly.

Daniel: Okay…it's four so we split the peaches in four pieces.

Researcher: Okay.

Daniel: For example, this person gets this part of the peach. [He shades one part of the first peach as he explains]

Researcher: okay, all right.

Daniel: And this part of the peach. [As he shades one part of the second peach]

Daniel: And this part of the peach. [As he shades one part of the third peach]

Researcher: So how much of a whole peach does he have?

Daniel: Three-fourths!

Researcher: Awesome! How much will the other kids get?

Daniel: They'll get three-fourths too. They all have to get the same amount.

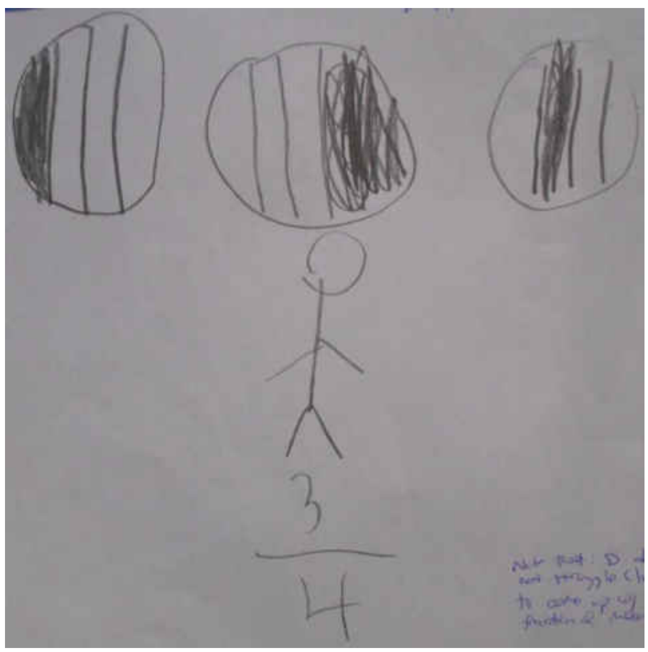


Figure 5: Daniel's Pictorial Showing Three Peaches Shared Among Four Children

Daniel's issues with visualizing equal-sized pieces are further complicated by the mathematical reality that equal-area pieces could be constructed with vertical lines if the widths of the slices were coordinated to produce equal areas. Helping Daniel to see when he had equal-sized pieces and when he did not was no trivial matter. Later in session 10, Daniel was still using his strategy to create equal-sized pieces. The problem is shown below along with Daniel's work in Figure 6. The artifact also shows a figure drawn by Marcos. Marcos had become concerned with Daniel's pictorial so he had attempted to show Daniel a way to divide the circle so that equal-sized pieces could be assured.

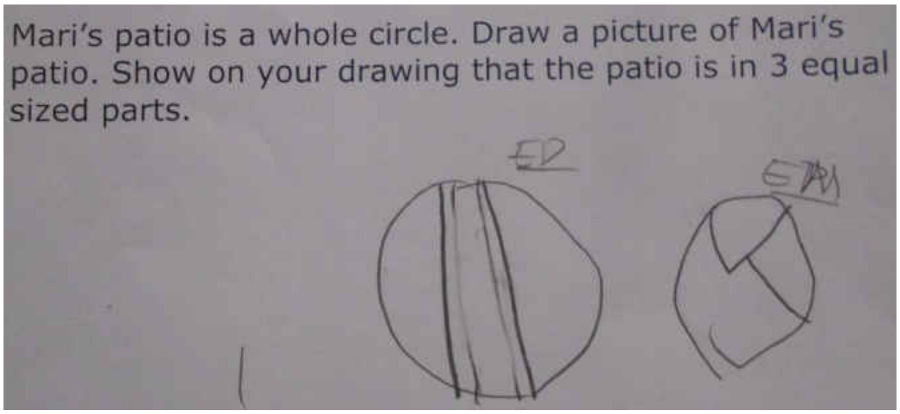


Figure 6: Daniel's Pictorial Showing Mari's Patio in Three Equal-Sized Parts

Although Marcos' drawing was far from an accurate depiction of equal-sized pieces, an issue covered subsequently in the section focused on Marcos, Daniel's work on the next problem, shown in Figure 7 and also during session 10, showed that Marcos' help influenced him. It can be seen that Daniel used his inaccurate dividing strategy on the half circle initially. However, with no prompting from this researcher or the other students, Daniel saw a problem with his drawing, erased the vertical lines, and constructed lines that divided the half circle into pie shaped regions.

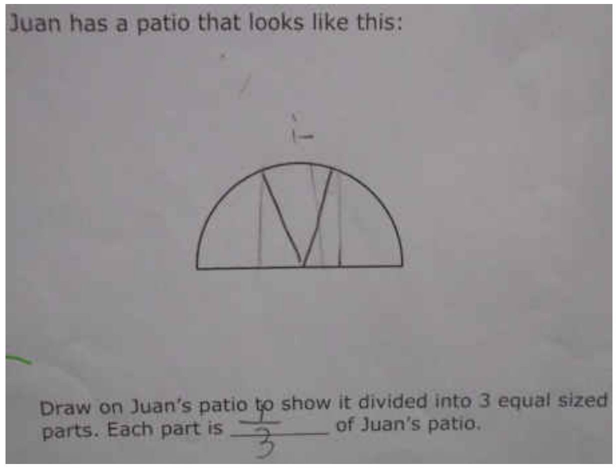


Figure 7: Daniel's Pictorial Showing Juan's Patio in Three Equal-Sized Parts

Perhaps because seeing or creating equal-sized pieces were challenges, Daniel would often regress to counting pieces rather than attending to fraction size when determining fraction equivalence or comparing fractions. When comparing the sets of circles shown in Figure 8 during session 11, Daniel interpreted the pictorial as presented in the transcript below.

> Researcher: I want to know what Daniel thinks. What do you think?
>
> Daniel: Well this one….well these are two [pointing at the top row] and this is one [pointing at the bottom row]
>
> Researcher: Is it the same amount of shading?
>
> Daniel: No…maybe…two is more than one.

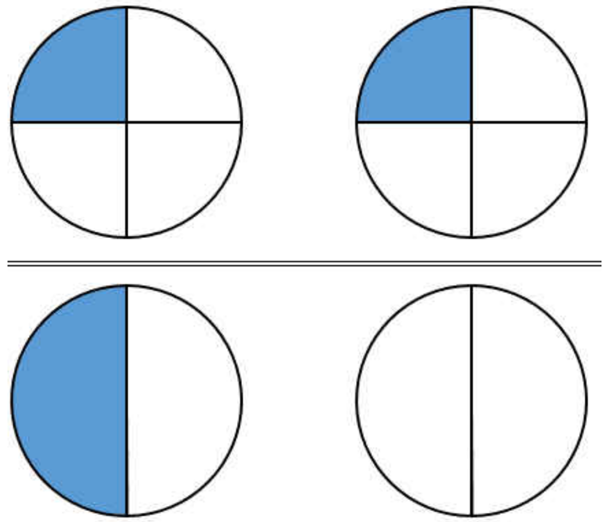


Figure 8: Which Set of Circles has More Shaded?

Then later in the same session, Daniel completed some work that provided insight as to how he saw areas when the directions instructed him to compare the figures on each row to determine if the shaded areas were equal. Daniel's work is shown in Figure 9. For this task, an x indicated that the two were not equal in area and a check indicated that the two were equal in area.

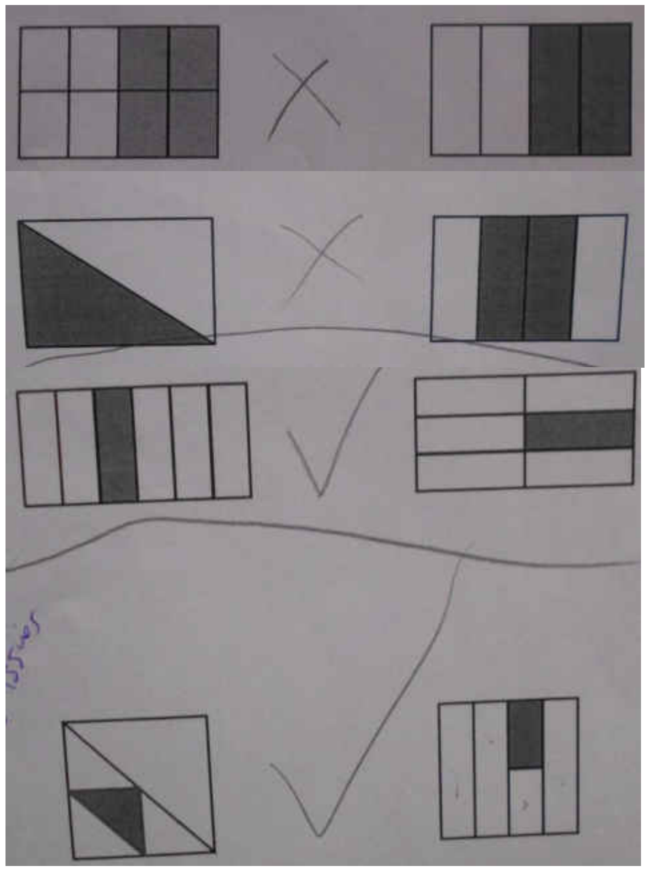


Figure 9: Are the Shaded Areas Equal?

In the first two comparisons, Daniel did not see the areas as equal. Particularly in the first pair, one might expect a student to be able to see the equality of the areas fairly easily, but this was not apparent to Daniel. In the second two pairs, Daniel was successful in determining the equivalency of area in the two figures, when it would seem visualization is more difficult. However, a close examination of the second two pairs reveals that each pair has an equal number of parts and an equal number of shaded parts. Pencil marks on the drawing indicate that Daniel counted pieces. For the first pair, Daniel counted one part shaded out of six for both figures. For the second pair, Daniel counted one part shaded out of five for both figures. It appears that, because Daniel struggled to visualize equality of area, he attended to counting parts, a strategy that can succeed at times even though conceptually inaccurate.

Inaccurate answers such as those shown in Daniel's work above may seem to indicate that Daniel struggles with understanding mathematics conceptually as well as visually. However, Daniel exhibited sound conceptual thinking throughout the sessions at times. In addition to applying rich conceptual thinking to make sense of problems, Daniel also exhibited strength with his ability to challenge other students to think critically about the validity of their reasoning and solutions. Examples of these strengths are provided below. On a real-world problem intended to connect reasoning about fraction equivalence to reasoning used in fair sharing problems, shown in Figure 10, Daniel invented a conceptually rich solution strategy, which was scribed.

> Daniel: Um…nine, it's nine dollars.
>
> Researcher: It's nine dollars?
>
> Daniel: Yes.
>
> Researcher: How did you get nine dollars?
>
> Daniel: Because two plus two equals four, and three plus…two plus four equals six and three plus three equals six…and three plus three [he means three plus six here] equals nine.

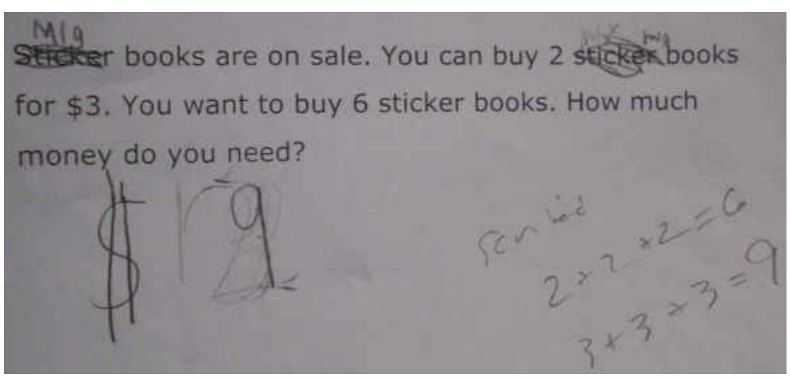


Figure 10: Daniel's Work with Equivalence

It should be noted that Daniel's insightful strategy only occurred after he had generated several incorrect solutions and I had used extended questioning to scaffold his persistence in finding a valid solution strategy. This episode is covered later under the second analysis framework, scaffolding techniques. However, the use of scaffolding does not negate the strong conceptual reasoning that Daniel was able to use and explain. In the following transcript excerpt from session eight Daniel demonstrated his ability to challenge other students to think critically. Seen another way, Daniel was capable of providing scaffolding to his fellow students. Before the start of this excerpt, we have defined a half circle as our whole.

> Clay: Why don't you agree with me? [To Daniel and Marcos] It's a seventh. There's seven of these! Oh no! Wait, eight of these. Eighths. So they're eighths!
>
> Daniel: He's probably confused that this is the whole. [Talking to me and holding a circle]
>
> Researcher: He thinks what is the whole?
>
> Daniel: The circle. He thinks the circle is the whole.
>
> Researcher: Oh…
>
> Clay: Look! I'm going to prove it! I'm going to prove it!
>
> Daniel: Okay, prove it.
>
> Clay: 1, 2, 3, 4, 5, 6, 7, 8 [Clay is counting pieces but not coordinating them with the half circle chosen to represent a whole]
>
> Daniel: [places four pieces on the half circle] Look Clay! How about this? Is this eighths?
>
> Clay: What?
>
> Daniel: But Clay, this was the whole. This is supposed to be the whole!
>
> Clay: Oh…

Daniel may very well have struggles related to conceptual misunderstandings in addition to struggles with visuospatial reasoning. However, Daniel exhibited strong conceptual reasoning strategies and understanding in certain circumstances. Although it seemed issues with visualization could complicate Daniel's thinking process, it was also evident that working in an environment where prompting for deeper thinking and expectations of high level work were the norm, Daniel often excelled.

Marcos

Marcos had relatively fewer occurrences in the conceptual or visuospatial subtypes than he had in the semantic memory subtype. Additionally, Marcos had fewer occurrences in these two subtypes than Clay or Daniel while having more occurrences in the semantic memory subtype than the other two participants. Marcos consistently struggled with accuracy in answers and strategies, often miscounting objects or misstating answers. Another issue for Marcos related to his struggle to create drawings that match his visualizations and, at times, Marcos struggled to record anything on paper that would represent his thinking and strategies. However, Marcos was generally able to provide strong verbal explanations of his strategies and thinking. This particular struggle with written expression was not one which I was able to categorize according to Geary's subtypes. On the other hand, Marcos' ability to understand concepts was one of his strengths along with his ability to interpret the mathematical thinking of others including misconceptions. Excerpts of transcripts and artifacts showing Marcos' work are used below to illuminate Marcos' struggles and strengths.

Struggles within the semantic memory subtype seem to be disconnected from abilities to think about mathematics in conceptually deep ways, at least for Marcos. Although Marcos was often working through a cloud of answers that were not what he meant to say, explanations that did not match his thought processes, and issues with counting and drawing, he engaged deeply with mathematics and generally appeared to enjoy doing so. I found that Marcos' struggles within the semantic memory subtype could be thought of as falling into four subcategories: a) drawing an unintended number of objects; b) misstating an answer or a strategy; c) incorrectly counting a number of objects; and d) retreating to a previously discarded incorrect strategy or solution.

Drawing an unintended number of objects impacted Marcos' ability to correctly work problem even when he had a strong understanding of how to work the problem. When asked to solve the problem below, Marcos produced the drawing shown in Figure 11. Although Marcos' work was logical, he drew 12 sandwiches rather than 10 initially causing him to find an incorrect solution. However, Marcos was able to understand how to approach and solve this problem much more quickly than the other participants. Marcos' needed help to see the discrepancy between the problem statement and his drawing. He then crossed out the two unneeded sandwiches and corrected his work.

> *4 children want to share 10 Publix sub sandwiches so that everyone gets the same amount. How much can each child have?*

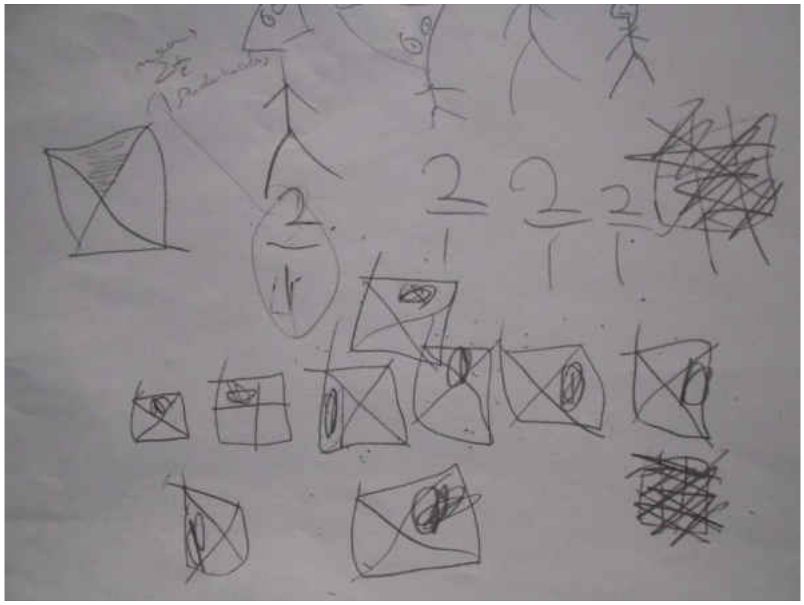


Figure 11: Marcos' Drawing of an Incorrect Number of Objects

In the next example of an occurrence within the semantic memory subtype, Marcos explained how he determined that his manipulative pieces are sixteenths. He covered half of the manipulative piece defined as the whole with eight of the sixteenth-sized pieces. Marcos understood how to mathematically prove that he knew the pieces were sixteenths but, as shown in the transcript excerpt below, he struggled to make his words match the mathematics in his mind.

> Researcher: What are those?
>
> Marcos: They're sixteenths.
>
> Researcher: Can you prove it?
>
> [Marcos lays out eight one-sixteenth size pieces on one-half of the rectangular game space]
>
> Researcher: I like the way you are doing that Marcos. I can really see…How many?
>
> Marcos: Eight times eight is two. Just imagine this is eight and this is eight.
>
> Researcher: Oh okay. Leave it the way it is… I'm not sure I understand what you meant by eight times eight is two…

Marcos: No, eight and eight is two eights, so sixteen…See sixteenths!

Researcher: Okay, I see now.

Miscounting of objects was also a consistent struggle for Marcos. Generally, Marcos understood that something had gone wrong with his counting and he corrected the error. This could be a time consuming process for Marcos but his frustration level with the effort required was generally low. Marcos did understand strategies to help with counting, such as moving objects as he counted them and being careful to attend to one-to-one counting. However, he still was more likely than the other participants to miscount a group of objects. This transcript excerpt contains an episode where Marcos needed to count his objects four times to be correct after knowing how many objects he had because another participant had counted them previously.

Researcher: Where's all the cookies? I thought there were 13 cookies? [Referring to the 13 counters that represent the 13 cookies in the problem statement]

Marcos: Uh…I had them….1, 2, 3, 4, 5, 6, 7, 8…Wait a minute [Marcos thinks he has miscounted so he starts over]
Marcos: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14 [He has counted one too many]

Researcher: How many?

Marcos: I'm counting them…. I got to count them again… 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12….
Marcos: 1, 2, 3, 4, 5, 6 , 7, 8, 9, 10, 11, 12, 13. Yes there are 13. There were always 13, I just kept counting them wrong.

Although occurring less often than other semantic memory issues, Marcos' occurrences of retreating to a formerly discarded answer or strategy impacted his ability to communicate his mathematical thinking. In the occurrence contained in the transcript below, Marcos struggles to match his verbal answer with the answer in his mind.

Researcher: What fraction of a peach is everyone going to get?

Marcos: Three fourths!

[Several minutes later when we are wrapping up the problem]
Researcher: What did you say they each got Marcos?

Marcos: Everybody gets thirds of the…

Researcher: They get some fourths? How many fourths do they get?

Marcos: Thirds!

Researcher: They get….

Marcos: Thirds!

Researcher: How many pieces do they each get?

Marcos: Three.

Researcher: And how many pieces are there in your peaches?

Marcos: Four

Researchers: So they get three…

Marcos: Thirds!

Clay: I decided to name the pieces fourths.

Researcher: Why did you name them that? [To Clay]

Marcos: Because there's four pieces. Because it was cut in four pieces.

Researcher: I thought you said it was thirds.

Marcos: No, I said fourths.

It seems clear that Marcos knew that he meant the answer to be fourths the entire time. But he substituted saying thirds when he meant fourths for a while, possibly because he was looking at three pieces for one person and the three crowded out thoughts of fourths for a while. Later when Clay said fourths, Marcos did not realize he had ever said anything different.

Marcos struggled in ways that could not easily be classified as belonging to any of Geary's (2003) subtypes. Although Marcos had strong conceptual thinking, he would struggled to record this thinking on paper. In the drawing presented in Figure 12 below, Marcos tried to draw a circle divided into three equal-sized pieces. Although, at the surface, this might appear to fit into the visuospatial subtype, Marcos' discussion about his drawings, presented in the transcript excerpt below, seemed to indicate that he had a clear visualization of how his drawing should look.

> Researcher: What do you think Marcos? Can this one be fixed so that they are all the same size?
>
> Marcos: It could be fixed like this. [He says this as he draws another one] …And I'm not good at drawing…It's upside down….So I apologize…That's not… [He continues trying to draw one that will work]
>
> Researcher: That wouldn't work out would it?
>
> Marcos: No. I'm bad at drawing.
>
> Researcher: No, you're not. You could fix it. Make it look right…
>
> Marcos: Equal! These are equal! They're supposed to look equal but they don't when I draw them.

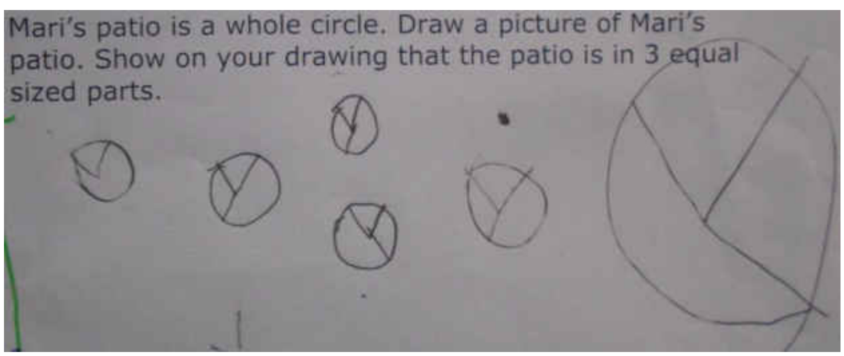


Figure 12: Marcos' Struggle to Draw Thirds as He Saw It in His Mind

In addition, writing seemed to be a struggle for Marcos. In Figure 13 below, work is presented that Marcos was able to competently accomplish, but only after writing his answers was no longer part of the task.

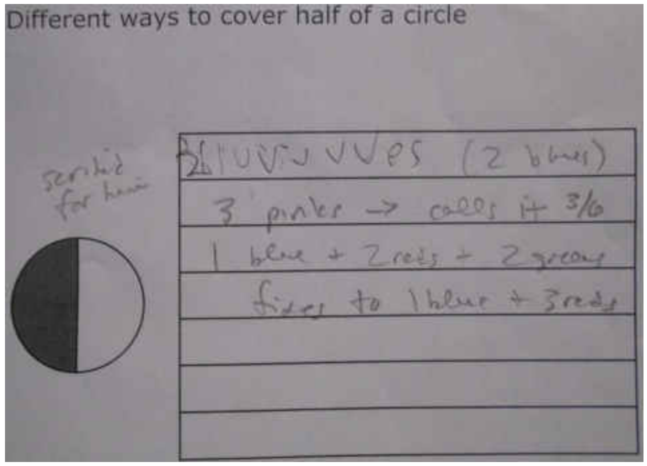


Figure 13: Marcos' Recorded and Scribed Work

Indeed there were times when Marcos was able to solve a problem with a thoughtful strategy while he talked, but was unable to record anything at all on paper. Figure 14 is included because it shows a problem that Marcos solved mentally and it includes my note that indicated no written work was completed even with prompting. The transcript excerpt contains Marcos' solution process for the problem.

> Researcher: There's two burritos for three children… but you want to get more burritos so six children can have the same amount as the first three children…How many burritos do you need to get?
>
> Marcos: [Quickly replies] Four burritos.
>
> Researcher: Four burritos? Why?
>
> Marcos: Because three plus three equals six so that means you doubled the burritos….
>
> Researcher: Show me, Marcos. Show me…Can you write something or draw something on your paper that would show what you just told me?
>
> Marcos: Okay… [He does not write anything on his paper but seems to think about it for a while]

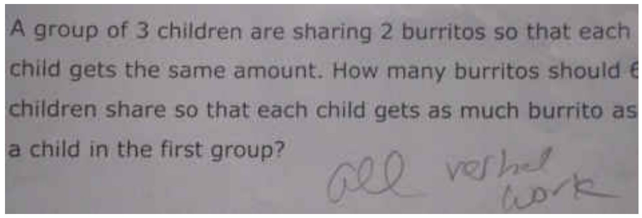


Figure 14: A Problem Marcos Worked Mentally and Explained Verbally

Marcos brought a useful strength to the group. He was consistently able to think about challenging problems in a conceptually rich way. In a variety of problem structures with a variety of representations, Marcos was able to connect mathematical ideas and stretch his own reasoning beyond his current experiences. During session five, Marcos played a game with Clay and Daniel using a rectangular area model. This model used one red piece of construction paper as the 'whole' and different colors of construction paper cut in pieces to represent halves, fourths, eighths, and sixteenths. A die, labeled with fractions on each side, was rolled to indicate the participant's next 'move'. The object of the game was to be the first player to cover the whole. Later, this game was also played with the objective to replace smaller pieces with larger pieces whenever possible; however, Marcos discovered this replacement possibility without instruction to do so. For a complete explanation of the game, along with explanations of other related games, see *Making Sense of Mathematics for Teaching Grades 3-5* (Dixon et al., 2016). In Figure 15 and the transcript excerpt, Marcos made a replacement move on his game board that required him to think flexibly about fraction equality.

> Researcher: So you rolled one-sixteenth but what kind of piece are you placing instead?
>
> Marcos: An eighth.
>
> Researcher: How could you do that? Why is that okay?
>
> Marcos: I took away this sixteenth so I could put an eighth instead.

Figure 15: Marcos' Move on the Game Board

In the transcript excerpt below, Marcos explained to the group why two different representations mean the same thing mathematically. The two representations are shown in Figure 16.

> Clay: Daniel made it like a square. And I made it like a rectangle. Like candy are shaped!
>
> Marcos: I did mine like Clay.
>
> Researcher: [To Clay] Okay, tell me this. Does yours mean the same thing as Daniel's?
>
> Clay: His is not right.
>
> Researcher: Why?
>
> Marcos: Well, look…No matter what it is you'll get the same thing. It's still four pieces no matter which way you make it.

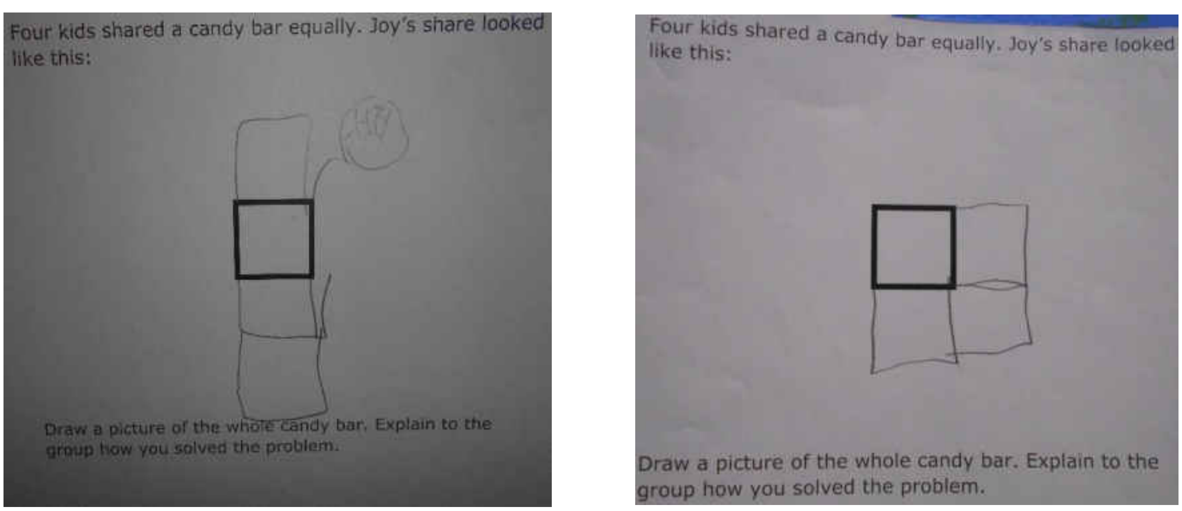


Figure 16: Comparing Two Ways to Draw a Candy Bar

In a set of tasks where students were expected to attend to unit fractions when I was changing the definition of the whole between tasks, Marcos was quick to understand not only the concept that the whole could be defined differently, but that he could define the whole for the group. Marcos was able to do this with no prompting as shown in the transcript below. This may indicate that Marcos understood that he did not need the researcher to continue to define the whole. Marcos might have realized that he could create the task as well as the solution.

> Researcher: Is it okay to decide the whole is something other than the circle Clay?
>
> Clay: Um…Yeah?
>
> Marcos: What if I made this grey one the whole?
>
> Daniel: No…
>
> Researcher: Why not?
> [Both Clay and Daniel are struggling to see that a manipulative other than the circle can be the whole]
>
> Marcos: Can't I make this one the whole?
>
> Researcher: You could make the grey one a whole…Give it a try. See what you find out.

Marcos was quick to connect representations of mathematical information. When playing a game with fraction cards designed to be similar to a number line, see Figure 17, Marcos made the connection between the layout of the cards and a ruler, if the ruler is showing eighths. His explanation of his connection is shown in the transcript excerpt below.

> Marcos: Um… I think the one-eighth should be here…
>
> Clay: Yeah…
>
> Researcher: Okay Marcos, and Clay agrees. Why do you think that Marcos?
>
> Marcos: Because it's a ruler and a ruler…But, but the ruler is like one-eighth, two-eighths, three-eighths, four-eighths…
>
> Researcher: Oh, I see!

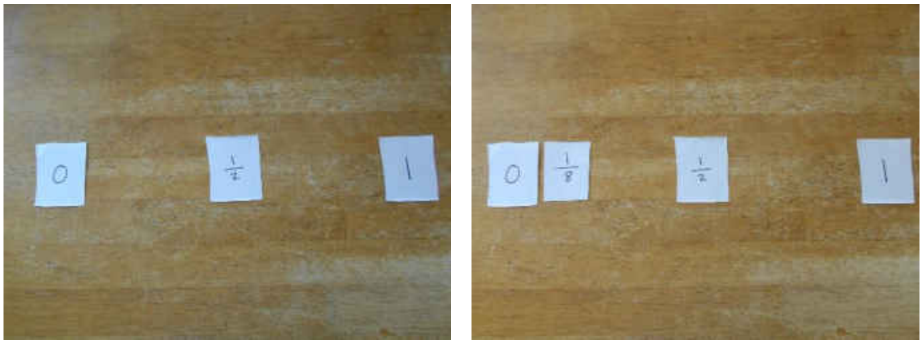


Figure 17: Fraction Cards Set Up Like a Number Line

When working with the task shown in Figure 18, Clay and Daniel created groups of three objects when asked to make thirds with a group of 18 objects. However, Marcos quickly identified how to create thirds and explained his reasoning to the group, as shown in the transcript below. It should be noted that the researcher did not instruct the participants to make groups. Marcos was the first person to associate creating thirds with making groups at all, and he correctly identified the number of groups needed as three.

> Marcos: Wait I know how to make it into thirds. I know how to make it into three groups.
>
> Researcher: What did you just say?
>
> Marcos: You want me to make it into three groups and I know how to make it into three groups.
>
> Researcher: Why would you want to make it into three groups?
>
> Marcos: Because you said you wanted it to be in thirds not sixths.
>
> Researcher: What is it right now?
>
> Marcos: Sixths.
>
> Researcher: Oh. What do you think Clay? If you have six groups is it sixths?
>
> Clay: Yes.
>
> *Look at this picture, and then let's answer some questions about it.*

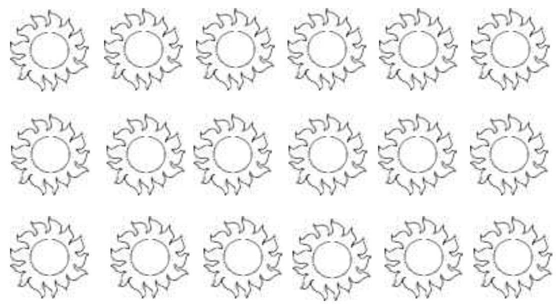
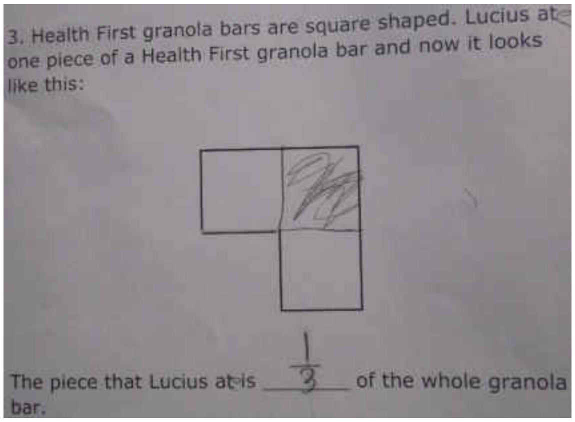


Figure 18: Working with Fractions in a Set Model

In addition to demonstrating his own strong conceptual understanding of fractions, Marcos was often able to understand the mathematical thinking of others. In the task discussed in the transcript excerpt below and in Figure 19, Marcos identified Clay's misconception about the problem and provided a correct interpretation intended to help Clay correct his misconception.

> Researcher [to Clay]: How did you decide he ate one-third?
>
> Clay: I shaded one and there's three…So he ate one-third of a granola bar.
>
> Marcos: Wait, did the granola bar look like that before he ate it?
>
> Researcher: Clay says Lucius ate one-third of it…What's your question Marcos?
>
> Marcos [reading out loud to himself]: Of the Health First granola bar…Health First granola bar...Now it looks like this…
>
> Researcher: What does that mean to you?
>
> Marcos: It means it used to look like this. [Marcos draws in the missing fourth-sized piece]
>
> Researcher: Why did you have to change the shape Marcos?
>
> Marcos: I didn't change it. I did it like it was, put that part in.



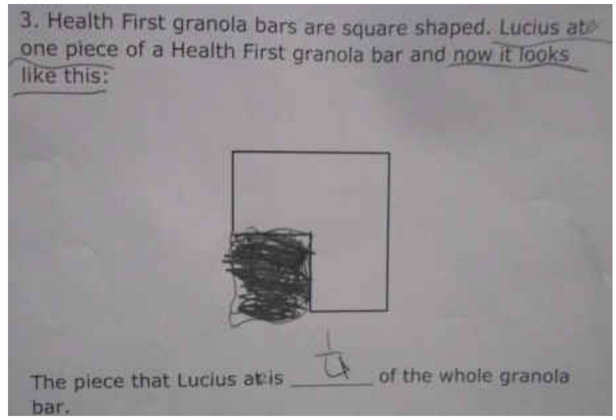


Figure 19: Marcos' Interpretation of Clay's Misconception

Marcos also saw that Daniel's drawing of a circle would not result in three equal-sized pieces. As shown in Figure 20, Marcos attempted to provide Daniel with a guide as to how the circle could be divided into three equal-sized pieces. As discussed before, Marcos struggled to create a picture of a circle divided in three equal-sized pieces as he visualized it in his mind.

> Researcher: What do you think Marcos? Does Daniel's drawing work?
>
> Marcos: No… Because this one's the same as this one…But this one's different.
>
> Researcher: How could he fix it?
>
> Marcos: He could draw it like this…

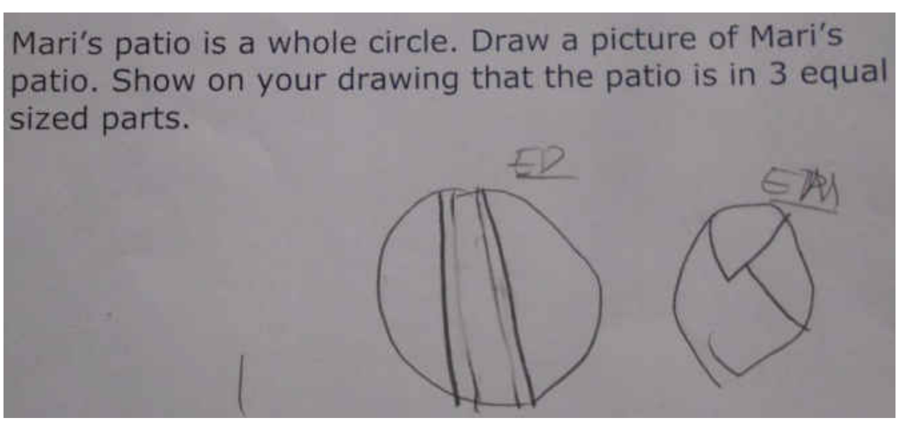


Figure 20: Marcos' Attempt to Help Daniel Construct a Circle with Three Equal-Sized Pieces

When Daniel used whole number reasoning to compare unit fractions, Marcos observed carefully what Daniel was trying to accomplish. Marcos disagreed with Daniel's answer but searched for a way to understand Daniel's reasoning. Marcos correctly arrived at the conclusion that Daniel was using whole number reasoning even though he did not have the language to fully express his interpretation of Daniel's thinking. The next transcript excerpt contains this part of the discussion.

> Researcher: This is in four pieces and this is in eight pieces…What are these?
>
> Daniel: These are fourths and these are eighths.
>
> Researcher: Okay, what's bigger fourths or eighths?
>
> Daniel: Eighths?
>
> Researcher: How are eighths bigger than fourths?
>
> Daniel: Um…
>
> Marcos: It's bigger in math!
>
> Researcher: Bigger in math? Do you mean bigger in whole numbers if you count how many?
>
> Marcos: Yes…That's what he means…But one-fourth is bigger than one-eighth.

Marcos proved to be a capable student of mathematics, although this result is not in line with his class performance, annual test results, or progress monitoring. In fact, Marcos was a student who had been often placed in remediation for help with mathematics. However, Marcos brought understandings of whole number concepts and fraction concepts to the intervention sessions. Despite often demonstrating conceptual understanding of fractions, his ability to communicate about mathematics was often compromised, with issues of written expression. Also, as shown by occurrences in the semantic memory subgroup (19), Marcos often knew the correct solution in his mind but would verbally communicate a different answer. It is possible that Marcos' mathematical thinking abilities were often obscured by his struggles with memory and writing.

Analysis Framework 2: Scaffolding Techniques

The intervention sessions in this study were built on the use of socio-constructivist scaffolding techniques as outlined by Anghileri (2006). The use of scaffolding allowed a lens into the thinking of students that might have been missing in another approach. The researcher was able to gain a deeper revelation of the participants' understandings including the individual strengths and misconceptions each brought to the intervention sessions. The analysis of these insights has been presented and now the impact of specific scaffolding techniques is also discussed. Scaffolding may be particularly important for students who are struggling as a way to ascertain their current depth of knowledge, to reveal patterns of strengths and weaknesses, and to support their potential for mathematical thinking (Anghileri, 2006; Broza & Kolikant, 2015; Cambourne, 1988; Cazdan, 1983; Moschkovich, 2015; Putambeker & Hubscher, 2003).

The researcher makes no claims that the scaffolding techniques employed in this study rose to a high level consistently. At times, scaffolding techniques were not well implemented and did not result in the desired result. However, there were many instances during the intervention sessions that demonstrated the power of social-constructivist scaffolding techniques as originally described by Bruner and his colleagues (Bruner & Ratner, 1978; Ninio & Bruner, 1978; Wood, Bruner, & Ross, 1976) and further developed by others (Anghileri, 2006; Cambourne, 1988; Cazdan, 1983) to diagnose students' struggles, locate students' strengths, and guide students to making sense of fraction concepts. This occurred even in scaffolding episodes that might be judged partially successful. Within this section is a review of the major scaffolding techniques that the researcher attempted to implement as well as a discussion of the outcomes of these techniques supported by transcript excerpts and student artifacts.

Anghileri (2006) proposed a three level hierarchy of scaffolding strategies that support learning. Within the second level, Anghileri (2006) describes the following techniques: a) prompting and probing; b) looking, touching, and verbalizing; c) interpreting student work or talk; d) simplifying a problem; e) explaining and justifying; and f) negotiated meaning. After working with the participants in the intervention sessions, the researcher analyzed the episodes of scaffolding that occurred during the sessions to categorize into these six techniques. This is not an inclusive list of scaffolding techniques described by Anghileri, but instead a subset of those described that were the most prevalent during this study. For each technique described below, an episode or episodes from the intervention sessions and an analysis of the relative success or failure of the technique in that instance is supported by transcript excerpts and student artifacts.

Prompting and Probing

Prompting and probing involves using questions to guide the student to a solution or mathematical idea and questioning that asks the student to think more deeply about the work he or she is doing (Anghileri, 2006). This scaffolding technique is supported by an understanding of Nathan and colleagues (2007) discourse pattern of initiation, demonstration, and evaluation and elaboration (IDE) as well as the second practice, observation of student responses, described by Stein and colleagues (2008) as one of their five practices that support implementation of high level discourse. This was the scaffolding technique most extensively used throughout the intervention sessions. The next example, which occurred during session 13, contained in the transcript excerpt, relates Daniel's use of a conceptually sound strategy to solve the problem below. The transcript indicates that this was not likely a strategy Daniel would have arrived at without extended prompting and probing. However, the thinking was clearly Daniel's original ideas.

> *Sticker books are on sale. You can buy 2 sticker books for $3. You want to buy 6 sticker books. How much money do you need?*
>
> Daniel: Twelve.
>
> Researcher: You need twelve dollars?
>
> Daniel: Yeah, twelve.
>
> Researcher: Why do you need twelve dollars?
>
> Daniel: Because it costs two. So six times two equals twelve.
>
> Researcher: Does it say each sticker book costs two dollars?
>
> Daniel: It says you can buy….two sticker books for three dollars… Oh…
>
> Researcher: It's okay, just keep thinking about it.
> Daniel: Let me change it. Eighteen? Eighteen…
>
> Researcher: How is it eighteen dollars?
>
> Daniel: Because three times six is eighteen. So you need….oh wait never mind…. I got confused! I keep getting confused!!
>
> Researcher: It's okay to be confused.
>
> Marcos: If you're going to buy six sticker books, how much money do you need?
>
> Researcher: That's what I want to know. How much money do you need?
>
> Daniel: Okay I'm done…
>
> Researcher: So you say you need two dollars?
>
> Daniel: Yes.
>
> Researcher: Okay…So before you needed three dollars to buy two sticker books, now you need two dollars to buy six sticker books? It's even less?

Daniel: Ummmm…..Ahhh….. Yes…Wait!

Researcher: Keep thinking about that.

[After a few minutes]

Daniel: Ok ,ok, three.

Researcher: Well before you bought two sticker books for three dollars so you are saying you want to buy six sticker books and it's still three dollars?

Daniel: Ehhhhh…..

Researcher: You are saying you'll buy six books so you are buying more books right?

Daniel: Uhhh…Um…Nine, it's nine dollars.

Researcher: It's nine dollars?

Daniel: Yes!

Researcher: How did you get nine dollars?

Ed – because 2 + 2 equals 4, and 3 plus…… 2 + 4 equals 6 and 3 + 3 equals 6, and 3 + 3 [he means 3 + 6 here] equals 9

Looking, Touching, and Verbalizing

The scaffolding strategy of looking, touching, and verbalizing is enacted when students use manipulatives or create pictorials, analyze what they see, and talk about their mathematical ideas to others (Anghileri, 2006). There is a relationship between this scaffolding technique and Lesh and colleagues' (1987) model of connection representations including manipulatives, pictorials, written symbols, verbalization, and real world problems. Although it is difficult to connect all of the six models in one task, throughout a series of tasks over weeks or months as many connections should be made as possible. During session 10, the participants attempted to place unit fraction cards correctly between cards showing zero, one-half, and one. Cards for zero, one-half, and one were placed with equal spacing on the table. The participants were then given a card with one-fourth on it and asked to place it, followed by a request to place a card with one-eighth on it. As shown in Figure 21, the participants initially placed one-fourth between one-half and one, then one-eighth between one-fourth and one. This activity was intended to connect abstract representation of fractions to a linear model similar to a number line with the cards acting as manipulatives. As the participants placed the cards and talked about their reasoning, it became apparent that they were not attending to the size of unit fractions but rather the size of the whole numbers in the denominator of the fractions.

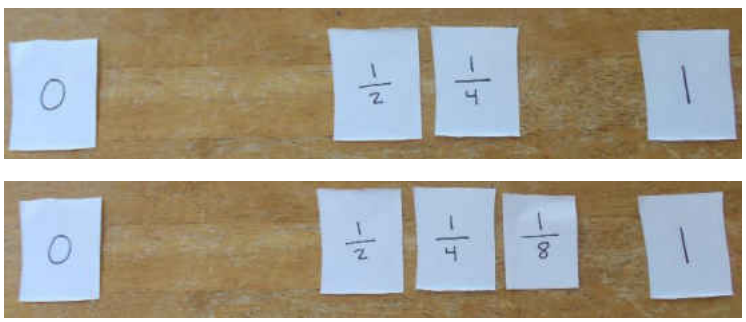


Figure 21: Fraction Card Initial Placement by Participants

Because the researcher believed the participants were struggling to compare the sizes of the unit fractions, she moved the participants to a different manipulative that used a linear model structure. The participants cut halves, fourths, and eighths from fraction strips, as shown in Figure 22, then compared the sizes of the unit fractions.

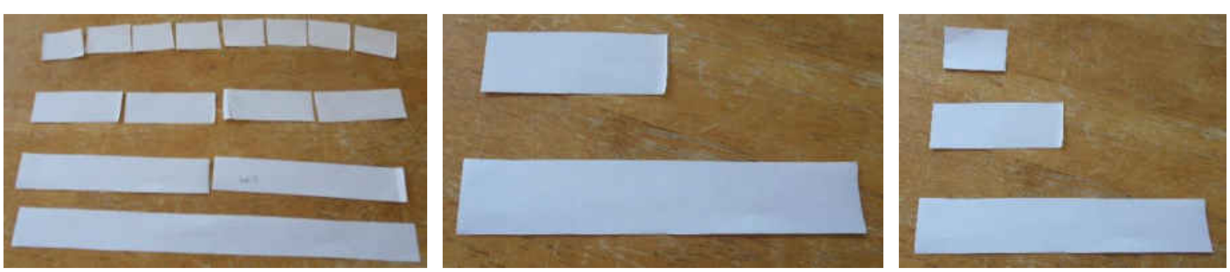


Figure 22: Fraction Strip Cuttings

After much surprise and discussion about the relative sizes of the fraction strips, the participants realized that a whole cut into more pieces would generate a smaller unit fraction than a whole cut into fewer pieces. At this point, the participants were able to move back to the fraction card game and place the unit fractions according to size, as shown in Figure 23.

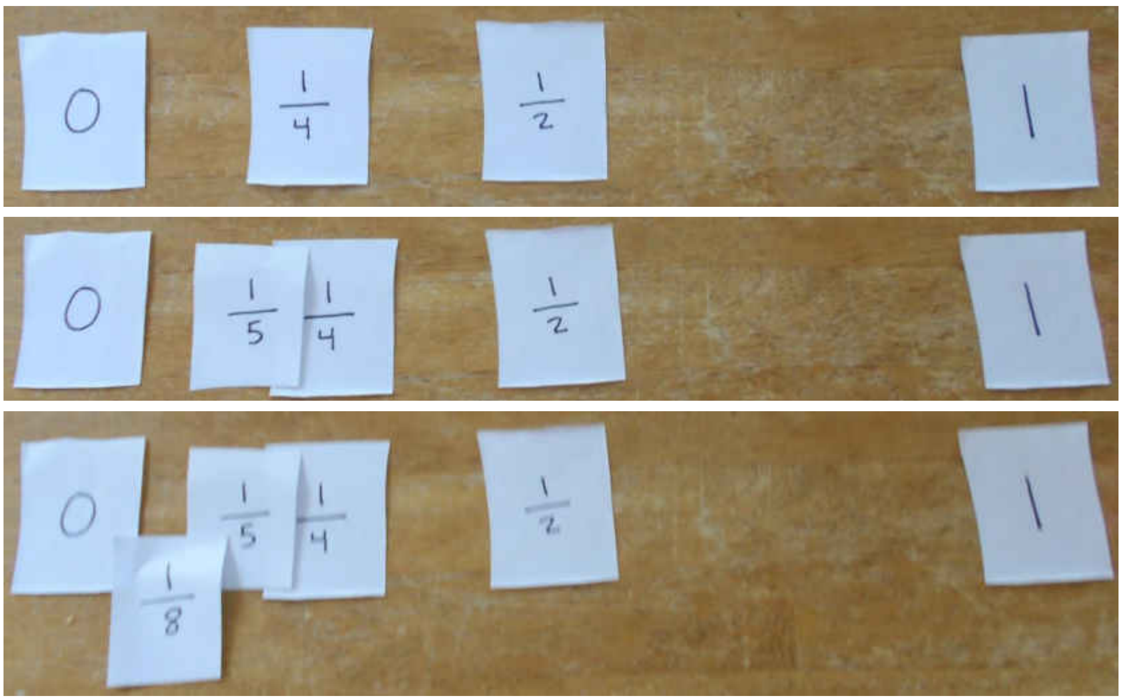


Figure 23: Fraction Card Successful Placement by Participants

The researcher would not classify this as a wholly successful attempt at scaffolding, however, because in later sessions in work with fraction comparison it was apparent that the participants were still struggling with concepts of fraction comparison related to the relative sizes of unit fractions.

Interpreting Student Work or Talk

Interpreting student work or talk occurs when the teacher explains the work or rephrases the verbalizations of students to make mathematical ideas clear and accurate for the benefit of the student and others in that class (Anghileri, 2006). Especially for students struggling in mathematics, this is a valuable scaffolding technique; however, it is a technique that should be used only when it is truly needed (Dale & Scherrer, 2015; Moschkovich, 2015; Pfister et al., 2015). A student may well be able to interpret his or her own work, or make his or her verbalizations more clear, with the use of other scaffolding techniques, such as prompting and probing (Dale & Scherrer, 2015; Moschkovich, 2015; Pfister et al., 2015). In the intervention sessions, there were times when it was clear that a participant had a mathematically accurate and conceptually rich thought that he was struggling to verbalize. In the following exchange, Daniel was trying to find a way to explain his correct answer. The researcher rephrased his verbalization to fit his work.

> Daniel: So we are splitting them in half…The seven one…half goes to Liana and the other half goes to Jackie.
>
> Researcher: Okay Daniel. I like the way you labeled it. So they each have six cookies and then that last cookie goes half to Liana and half to Jackie?
>
> Daniel: Yes.

In the following exchange, the researcher prompted Clay to correct his inaccurate use of vocabulary.

> Researcher: You're going to give five pieces to each person? Will you use each of your ten subs like that?
>
> Clay: To make them even…Make them even.
>
> Researcher: To make them equal? To make them the same size?
>
> Clay: Yes, the same size.

Researcher: Okay, that would be equal, not even. What's even?

Clay: Oh!... 2, 4, 6, 8, 10

For students who are struggling, it may be especially important for a teacher to interpret the student's work or verbalizations. In Daniel's case, he clearly understood the answer but needed help constructing a verbalization that was mathematically accurate. In Clay's case, the inaccurate use of mathematics vocabulary could further deepen conceptual struggles for Clay in the future if not addressed with appropriate learning experiences.

Simplifying a Problem

The goal of simplifying a problem is not to reduce the cognitive complexity of the task, but to create an intermediate task that will help the student build an understanding of what is required to complete the original task (Anghileri, 2006). In practice, simplifying a problem without reducing the cognitive complexity is challenging. When teachers are too quick to simplify a task or do so in a way that the complexity is lost, that teacher may narrow students' opportunities to learn (Henningsen & Stein, 1997). Because the researcher attempted to avoid simplifying a task until it seemed to be necessary, occurrences of the use of this scaffolding technique were uncommon in the intervention sessions conducted for this study. When all participants seemed unable to move forward with a task in a logical way, the researcher presented a simplified version of the problem that maintained the cognitive challenge of the task. During session nine, the participants were presented with the following problem:

Dani wants to feed each of the children she babysits a half sandwich for lunch. If she babysits 8 children, how many sandwiches should she make?

After several minutes working with the problem, all three students were pursuing unproductive strategies. Daniel believed he needed to give each child one whole sandwich. Meanwhile, although Clay had initially solved the problem correctly, as he worked he became convinced that he should divide one sandwich among the eight children. Marcos had decided to draw a set of sandwiches each divided into four equal-sized pieces and see how many sandwiches he would need to give each child a fourth of a sandwich, a strategy that would have worked had Marcos divided the sandwiches into halves. Because Daniel and Clay were not focused on how to iterate the share of a sandwich and Marcos was iterating an incorrect share of the sandwich, the researcher decided to take another approach. She asked the participants to make two halves from a paper strip. The discussion that followed is contained in the next transcript excerpt.

> Researcher: So I want you guys to work something out. Let's put Dani aside for a minute. Okay, Clay help me out. Let's say this is one sandwich. [I give him a paper strip] How many children can you give this sandwich to if we say every child has to get half of a sandwich?
>
> Clay: 1, 1, 1, 1, 1, 1, 1, 1 [He is counting out portions for eight children]
>
> Researcher: Are you giving each a half of a sandwich?
>
> Clay: Um…No…
>
> Researcher: [To Daniel, who is holding a paper strip] How many half sandwiches are you holding there Daniel?
>
> Daniel: Two
>
> Researcher: So how many people can get half sandwiches from the sandwich you're holding Daniel?
>
> Daniel: Two.
>
> Clay: Oh! Two!
>
> Researcher: Do you need more sandwiches to feed all eight kids half sandwiches?
>
> Marcos: Yes!

Researcher: How many sandwiches would you need Marcos?

Marcos: Um…. Three more!

Researcher: What do you say Clay?

Clay: I have eight sandwiches.

Researcher: There's eight kids…Do you need eight sandwiches to give eight kids a half of a sandwich?

Clay: Yes.

Researcher: So how many kids could share this one sandwich here? [Referring to a paper strip that he has folded in half]

Clay: Two.

Researcher: And how many kids could share this sandwich? {Pointing at another paper strip]
Clay: Two….Oh….

Researcher: How many kids have you given half sandwiches to?

Clay: Four…I see, I see… Dani needs four sandwiches [Clay shows us with his paper strips].

The participants were able to successfully solve the problem, but it was clear that strategizing about a problem that reversed the equal-sharing process was challenging for them, particularly when the problem resulted in more than one object being shared.

Explaining and Justifying

Although participants struggled at times with talking about their mathematical thinking and work, each participant successfully explained and justified his solution strategy several times throughout the sessions. Opportunities to provide verbal explanations and justifications seemed to assist the participants in grounding their emerging conceptual understanding of fractions. For the participants, providing written explanations and justifications was very challenging although Clay and Daniel often produced pictorials or used manipulatives to support verbal justifications, while Marcos relied on his ability to verbalize. In the transcript below, Daniel explained as he worked with manipulatives how he had identified a fractional piece. At first, it was not clear that Daniel understood because he agreed with an answer already given. However, his explanation and justification made it clear that he did understand the concept.

> Researcher: What is the name of that piece?
>
> Clay: It's a fourth.
>
> Researcher: Marcos do you agree that these are fourths in this game? What do you think Daniel?
>
> Marcos: Yes!
>
> Daniel: Um…Yes.
>
> Researcher: How do you know that Daniel?
>
> Daniel: Because like…Because like if you put it here…You need two to complete this side…So on the other side there's two…So there's four.

Marcos was generally the participant who used verbalization the most. Often Marcos seemed to work through his mathematical thinking by talking. In the next transcript excerpt, Marcos explained his method, justified his assertion that he had identified ninths, and then went on to justify his solution in two more ways.

> Researcher: Where are your ninths?
>
> Clay: No I don't see ninths…
>
> Marcos: Do you want me to help you?
>
> [Meanwhile, Daniel circles suns in sets of two to make nine groups]
> Researcher: What did you just do Daniel? What did you figure out?
>
> Daniel: By circling two, two of the suns…I don't know how to say it!
>
> [Marcos starts talking. He is very excited about how he solved the problem]
> Researcher: Marcos do you want to explain it?
>
> Marcos: I made two, four, six, eight…There's two in each.
>
> Researcher: Why are there two in each?
>
> Marcos: Because I know that two times nine equals 18 and there's 18 in all. One, two...Three, four…Five, six…Seven, eight…Nine, ten…Eleven, twelve…Thirteen, fourteen…Fifteen,

sixteen…Seventeen, eighteen. So I put like…One, two…One, two…One, two… Show one group equals two, two groups equals four [he tracks the number of groups on his fingers, ending with nine fingers] six, eight, ten, twelve, fourteen, sixteen, eighteen…See! Nine groups. It's ninths!

For the task shown in Figure 24, Clay was partially successful in providing the explanation and justification for his work, as shown in the transcript excerpt below. He connected the words in the problem statement to the drawing he made but was not able to verbalize that his pictorial represented the condition of the lemon bar before the party. At the end of the transcript it was clear that this is a process that was very challenging for him. Clay did not give a complete explanation and justification but he was able to link his pictorial to the problem as given.

Researcher: Okay, let's look at Clays drawing. Can you tell us how you decided to draw what you drew?

Clay: Well…Umm… I just did that because it was a lemon bar. So I…so this…so this is what it looked like before.

Researcher: How did you know what it looked like before?

Clay: Because somebody ate half of it.

Researcher: How did you know that?

Clay: Because the problem…the problem…

Researcher: What does it say?

Clay: It says this was after the party…after…this picture it shows.

Researcher: Okay then you decided to add on a piece. Why'd you add on that piece?

Clay: Because…Because, because…Because…

Researcher: Do you want me to come back to you in a minute? [He nods yes] Because you did something very cool here.

;

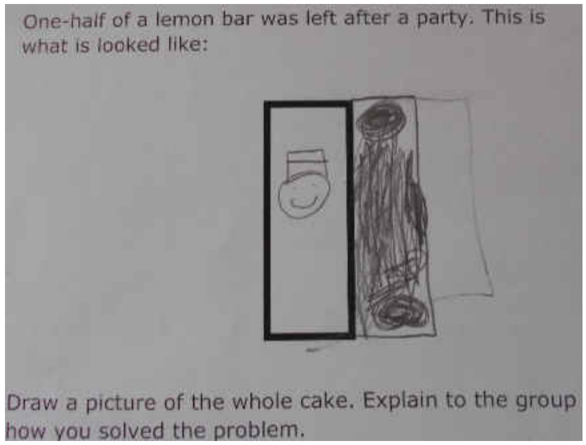


Figure 24: Clay's Work on the Lemon Bar Problem

All of the participants were able to generate explanations and justifications for their work when asked to do so by the researcher. Clay and Daniel were also successful in producing pictorials with written work that demonstrated their reasoning and supported their verbal explanations and justifications. For Marcos, verbalization with manipulative demonstration was his primary method of explanation and justification.

Negotiated Meaning

Students need opportunities to discuss their mathematical ideas, and this includes those ideas that are incorrect or not logical. When students develop mathematical ideas together with a teacher seeking to guide the group to mathematically logical conclusions, they develop an ownership of these ideas as a group (Anghileri, 2006). Once a mathematical idea becomes accepted and understood by the group, it can be used to advance further learning. The negotiation and acceptance of a mathematical truth by a group is referred to as knowledge that is *taken-as-shared* by Cobb, Yackel, and Wood (1992). Discussing mathematical ideas that include errors and misconceptions can feel like a risky endeavor, particularly if students are struggling to understand the concepts. What might happen if a misconception becomes accepted by the group? In practice, discussion of errors and misconceptions has the power to improve learning and understanding (Anghileri, 2006).

In an example of negotiated meaning that included misconceptions, the participants discussed the meaning of a fraction during session one, based on the representation shown in Figure 25. The discussion contained in the transcript excerpt that follows highlights a conversation that seemed risky because Daniel's misconception was entrenched and it seemed he might be able to convince the other two participants to conceive of it in his way. In the end, Marcos agreed with Clay about the nature of the fraction but Daniel was still somewhat unsure. The group had partially negotiated equal-sized parts for fractions, but this was a concept that reemerged many times over the course of the intervention sessions and in the end this negotiated meaning was only a partial success.

> Researcher: All right. Here's my question for you guys. What do you see?
>
> Daniel: One three…One three
>
> Clay: One-fourth!
>
> Researcher: Well, when I was teaching kids last year, Janelle said she saw one-third and Lui said he saw one-fourth…Who do you think was right?
>
> Marcos and Clay: Lui!
>
> Researcher: Who do you think was right Daniel?
>
> Daniel: The person who said one-third…
>
> Researcher: Okay. Do you guys understand how he is getting one-third?
>
> [Clay and Marcos nod yes]
>
> Researcher: How is he getting one-third Clay?
>
> Clay: Um… It's because they're …uh…It's one-third…But I think it's one-fourth because when you do this [he draws a line to continue the vertical line in the middle of the square] it becomes one-fourth.

Marcos: And they're not equal!

Researcher: What's not equal?

Marcos: The parts.

Researcher: So...What do you think Daniel? Marcos says they're not equal in the original form…

Daniel: It's because this one's the big chunk and these two are the small ones.

Researcher: Okay.

Daniel: That would be thirds and originally they didn't have that line so I thought it was thirds.

Researcher: Okay, what are you thinking now?

Daniel: It's still one-third.

Researcher: It's still one-third? Could it be one-third and one-fourth at the same time if the square is the whole?

Daniel: Maybe…

Researcher: What do you guys think?

Clay: Well…No.

Marcos: No…But I say yes.

Researcher: So it could be one-third and one-fourth? How could it be one-third?

Marcos: Because if they…Because if they explain the thirds…And it could also be fourths…Just imagine this line here [referring to the one Clay drew in]. But it's thirds.

Researcher: Oh…Okay. Is it okay for it to be like this? One big piece and two small pieces?

Clay: It's not equal!

Researcher: Is it ever okay if our pieces are not the same size in fractions?

Clay: No, it's not…that's not.

Marcos: I think I agree with Clay.

Clay: It's not really…fair…for other people…because a person gets the bigger piece and another person gets a small piece.

Researcher: Oh, okay.

Marcos: Um…Yeah, I agree with Clay.

Researcher: What do you think Daniel?

Daniel: Well I think a fraction could be like different shapes…because it could be like a circle having bigger chunks and small chunks of the circle.

Researcher: Would that be a fair share though?

Daniel: No. If it has to be a fair share then it has to be the same size…

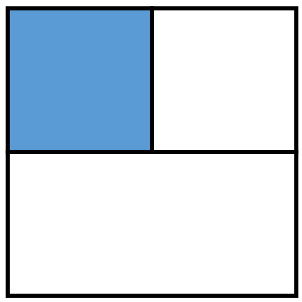


Figure 25: Representation of One-Fourth Shaded

Meanings can become negotiated in an incorrect way unintentionally. Clay’s work with fractions up until this point, possibly within class and within this intervention, had left him with an understanding that a shaded region in a pictorial was necessary to allow identification of a fraction. This misconception was uncovered during an exchange about the problem below. The following transcript excerpt contains the participants’ attempt to determine if Clay’s idea about shading and fractions was correct or incorrect. This might not be a fully accepted concept for Clay until he has had reasons to confront his misunderstanding several times in the future. However, this discussion provided a beginning to Clay’s new understanding and represented an incidence of negotiated meaning among the participants.

Lucy’s garden is a square. Draw a picture of Lucy’s garden that shows it is 3 equal-sized parts. How much is each part?

Marcos: I did it! It’s one-third!

Researcher: Marcos how can you prove that one part is one-third?

Marcos: Because this is three parts…Because this is three thirds in the whole.

Researcher: What do you think Daniel? Does his explanation tell why it's one-third?

Daniel: Um hum. [Yes]

Researcher: How would you explain it?

Daniel: By shading in one of the rectangles in the square.

Researcher: Does that prove it's one-third? Clay he's shading one part in the square. How would you explain that to someone?

Clay: Because when you shade one part, you know it's one something.

Researcher: Well let me ask you a question then. If you don't shade in one part can you still prove it's one-third?

Marcos: Yes.

Researcher: How would you prove one part is one-third without shading it?

Marcos: Because it's…[He draws a circle then divides in three. Marcos struggles to draw this correctly but it is clear he intends to make equal-sized pieces]

Researcher: Can you tell me in words?

Marcos: No.

Researcher: Could you tell me in words Daniel?

Daniel: Well…

Researcher: What about you Clay? Could you tell me in words why this is one-third [pointing at Marcos' drawing]

Clay: Okay. It's one-third because one-third…

Researcher: Because?

Clay: One-third is a shaded part of a third…

Researcher: Does it have to be shaded to be one-third?

Clay: Um…Yes!

Marcos: No it doesn't!

Researcher: Why not?

Marcos: Because if you don't shade it…Because each of those is one of them.

Researcher: One of how many?

Marcos: Three!

> Researcher: Okay. What do you think Daniel? Do you agree with Marcos? That it can show one-third if it's shaded but that one part is still one-third if it's not shaded.
>
> Daniel: Um…Yeah.
>
> Researcher: Why do you agree Daniel?
>
> Daniel: Because…Ah…Well, every one of them…Every one of them is a third…one-third. No matter if it's shaded or not shaded…
>
> Researcher: Clay what do you think? Do you agree with Daniel?
>
> Clay: Uh…
>
> Researcher: What did Daniel say? Tell me in your own words.
>
> Clay: Well…He said…that…it is one-third…because…because…because…
>
> Researcher: Clay do you want to ask Daniel to say it again so you can hear him?
>
> Daniel: Because…Um…It's like every one…It's still one-third if it's…like…not shaded.
>
> Researcher: Why?
>
> Daniel: Because every one is a third.
>
> Researcher: Clay what did he say?
>
> Clay: He said that it's one-third but it doesn't have to be shaded because…uh…because shapes don't have to be shaded to be a fraction.
>
> Researcher: Does that make sense?
>
> Clay: Yes…Yeah…Because…because it's still one part and there's still three parts. You can shade it to show it but it's still one-third if it's not.

Although the students were not always able to arrive at a negotiated meaning that was agreed upon by all three members of the group, when they did find agreement they were successful in building correct understandings of the fraction concepts being considered. Ideas arrived at through negotiation could have been used to support reasoning about subsequent fraction concepts. However, this did not happen during this intervention study. Instead the participants tended to renegotiate concepts each time they arose. This may have indicated that understandings of these fraction concepts were still in the process of development.

Conclusion

This chapter began with a bracketing statement followed by descriptions of the study participants and the intervention sessions as enacted. The analysis of data was organized in two parts to address the two research questions of the study. The first part of the analysis used Geary's (2003) proposed subtypes of learning disabilities in mathematics as a framework to develop descriptions of each participants' struggles and strengths with evidence provided in the forms of transcript excerpts and artifacts. Analysis of the three participants indicates that illustrated different patterns of struggles and strengths, with one participant's struggles placing him in the conceptual subtype and another's placing him in the semantic memory subtype. The third participant presented struggles that were largely balanced between two subtypes but he was the only participant to have struggles associated with the visuospatial subtype. The second part of the analysis used scaffolding techniques described by Anghileri (2006) as a lens to describe how the participants made sense of fraction concepts in an intervention setting intended to support socio-constructivist learning. Episodes highlighting the scaffolding techniques of *prompting and probing* and *looking, touching, and verbalizing* supported and revealed participants' processes of sense-making. Episodes highlighting the scaffolding technique of *explanation and justification* demonstrated participants' abilities to verbally support and defend their solutions and strategies with their own mathematical reasoning, while episodes highlighting the scaffolding technique of *negotiated meaning* showed the difficulty these participants had in reaching meanings that were agreed upon.

CHAPTER 5: CONCLUSION

This study was intended to address a gap in the literature by using a qualitative research approach to understand and describe how third-grade students who struggle in mathematics make sense of fraction concepts. Vygotsky's (1934/1986/2012) proposal that an understanding of how a student makes sense of a concept can be best ascertained by a teacher working in conjunction with that student to make sense of that concept was used as an interpretive framework to guide this research. As such, a set of intervention sessions were conducted in which the researcher and the participants worked together to make sense of fraction concepts. In keeping with Vygotsky's (1930-1934/1978) social-constructivist view on expert-novice interactions, scaffolding techniques were employed to support participants' reasoning about fractions. Ultimately, these scaffolding techniques allowed the researcher to build a description of each participant's strengths and struggles, and became a lens through which to view the participants' processes of making sense of fraction concepts.

Two analysis frameworks were selected to answer the research questions for this study. For the first framework a classification system for subtypes of learning disabilities in mathematics developed by Geary (2003) was chosen. Using Geary's descriptions of three subtypes, conceptual, semantic memory, and visuospatial, the participants occurrences of errors and misconceptions were classified. For presentation in this study, each subtype was illustrated for one participant. For Clay and Marcos, numbers of occurrences within the conceptual and semantic memory subtypes respectively were dominant. Analysis for Clay focused on describing struggles within the conceptual subtype while analysis for Marcos focused on defining struggles within the semantic memory subtype and with written expression. Daniel's occurrences were greatest in the conceptual subtype, but his occurrences in the visuospatial subtype exceeded those of either Clay or Marcos. For this reason, and because Daniel often exhibited strength with conceptual thinking, the researcher decided to focus analysis of Daniel's struggles within the visuospatial subtype. The second analysis framework utilized Anghileri's (2006) descriptions of socio-constructivist scaffolding techniques to identify episodes within the intervention sessions where these techniques revealed how the participants made sense of fraction concepts with each other and the researcher. These scaffolding techniques included: a) prompting and probing; b) looking, touching, and verbalizing; c) interpreting student work or talk; d) simplifying a problem; e) explaining and justifying; and f) negotiated meaning.

The participants' strengths and struggles are essential to how they make sense of fraction concepts and a set of scaffolding techniques grounded in social-constructivist learning theory is essential to revealing their sense making processes, including their misconceptions and how these contribute to their understandings. Additionally, the use of scaffolding techniques provides a way to address each student's struggle, both within the conceptual, semantic memory, and visuospatial subtypes as described by Geary (2003) and learning issues that extend beyond Geary's framework. In an environment that allows extended exploration using multiple connected representations, supported by a teacher's skilled questioning and opportunities for student verbalization, strengths can be used to address struggles. Geary (2003) suggests that students who struggle within the conceptual and visuospatial subgroups are likely to see these struggles decrease over time while students who struggle within the semantic memory subtype may continue to struggle. Marcos' struggles were primarily in the semantic memory subtype. However, he proved to be conceptually sound in his thinking and eloquent in expressing his ideas. Although Marcos may continue to struggle with fact recall, accurate counting, and written expression, future opportunities to demonstrate his conceptual strengths in mathematics may be crucial to his own understanding of himself as a strong student in mathematics. Clay and Daniel also need opportunities to reason about mathematics in environments that emphasize conceptual focus and make use of connections between representations. It is this type of environment that helps them to uncover their own misunderstandings and to use both their own strengths and the strengths of other students to build mathematical understandings.

This concluding chapter presents a discussion of the findings of the study, in the form of a description of each participant's strengths and struggles followed by a discussion of the participants' sense making process as revealed by scaffolding techniques. A discussion of the implications of the findings for research and practice follows. Lastly, recommendations for future research are outlined.

Findings

During the intervention sessions, the use of scaffolding techniques with participants revealed a pattern of struggles and strengths that were unique for, although at times overlapping among, the three participants. Additionally, it became clear that the scaffolding techniques implemented for their value to allow co-construction of knowledge within the group, provided a second lens through which the participants' processes of making sense of fraction concepts could be considered.

Analysis Framework 1: Geary's Subtypes of Learning Disabilities in Mathematics

The first analysis framework was used to answer the first research question: What struggles and strengths of third-grade students are revealed in a small group intervention supported by social-constructivist scaffolding while focused on fraction concepts?

Although Geary (2003) labeled his subtypes as pertaining to learning disabilities in mathematics, the decision was made for this study to use these subtypes as a way of examining the thinking of students who struggle in mathematics. Supporting this decision is Geary's (2003) assertion that no process exists which can definitively distinguish between a student with a learning disability in mathematics and a student who struggles in mathematics. The participants in this study were known to struggle in mathematics, but it is unknown if any have a learning disability that is affecting their challenges in mathematics learning. The following discussion includes a description of each participant's struggles related to one of Geary's subtypes (conceptual, semantic memory, and visuospatial), a description of strengths revealed during intervention for each participant, and a discussion of their strengths and struggles considered together. Analyses for Clay, Daniel, and Marcos were focused on occurrences in the conceptual, visuospatial, and semantic memory subtypes respectively.

Clay's struggles were primarily grounded in the conceptual subtype with 25 occurrences versus eight and nine respectively in the semantic memory and visuospatial subtypes as described by Geary (2003). In the earliest intervention sessions, Clay attended to the need for equal-sized pieces in pictorial representations of fractions. However, Clay struggled with translating this reasoning to work with real-world context problems. In considering the work of Lesh and colleagues (1987), which proposed that student's need to make connections between different representation of fractions to build understanding of fraction concepts, it is possible that Clay's conceptual understandings of fractions were weak because he had not had opportunities to make these connections. Clay may have worked with different representations of fractions in the past, but he needed opportunities to work with tasks in situations where immediate connections were made, and he needed prompting and probing scaffolding techniques that helped him confront his misconceptions and generate more logical connections. This was evidenced when paper strip cuttings were used to make connections to abstract reasoning of the comparative sizes of unit fractions and also when reasoning about a real-world context problem was connected to manipulatives when it was clear Clay was struggling to generate correct pictorials. Clay worked with fractions using whole number reasoning when he stated solutions in number of pieces without reference to a denominator, identified unit fractions without attending to the whole, and compared fractions by choosing as the greatest fraction the one with the largest denominator. Clay's struggles indicated that he often did not understand the concepts that were underlying the fraction tasks used in the intervention sessions. Despite his conceptual struggles, Clay demonstrated that at times he could insightfully give a solution to a complex problem before working with the problem. When asked to state how many sandwiches would be needed if each person was to receive a half sandwich, Clay immediately knew the answer would be four. Clay struggled to provide a solution strategy that would support his insightful answer, but this does not diminish the strength of his insight. Clay was able to demonstrate this strength on several occasions. It is possible that Clay had information mathematical knowledge which he has been unable to connect to his formal learning in school. Additionally, when Clay was encouraged to consider the whole when working on fraction tasks, he showed that he could be flexible in his thinking about defining that whole. Like the participants Hunt and Empson (2015) studied, Clay's work with fraction concepts was delayed, rather than different, when compared to that of typically achieving third-graders.

Daniel's struggles occurred most often within the conceptual (28) and visuospatial (18) subtypes. Because Daniel was the only participant of the three to demonstrate a great number of struggles within the visuospatial subtype, the researcher made a decision to examine his struggles in this area in-depth. Daniel's interactions with the group indicated that he understood the need for equal-sized pieces, but often misinterpreted the size of pieces presented in pictorial representations of fractions. Potentially related to Daniel's difficulties with being able to judge the relative size of pieces, he also divided his own pictorial representations in ways that were unlikely to result in equal-sized pieces. Daniel's thought process about visual representations had some parallels with the thinking of participants described by Lewis (2010, 2014) in her studies with students who struggled to understand fractions. Specifically, in line with Lewis' (2014) findings, Daniel struggled to see whether or not a pictorial presented equal-sized pieces, to create pictorial's showing equal-sized pieces, and ignored different-sized parts of fractions when comparing fractions. Lewis posited that this type of struggle represents an intrinsically different way of thinking, rather than an immature but typical way of thinking about mathematics. Although Geary (2003) states that the conceptual subtype represents immature, but typical, reasoning, while the semantic memory subtype is indicative of cognitive differences, he states that it is unclear if the visuospatial subtype is associated with immature or atypical reasoning. Although struggles within the conceptual subtype were evident, there were also many occurrences when Daniel demonstrated his ability to use reasoning to create strategies, solve problems, and support his solutions. Daniel was often able to use his knowledge of whole numbers to create solution strategies for equal sharing and fraction equivalence problems that he could explain and support. Daniel's strengths were most evident when he was working with real-world context problems. Occasionally Daniel needed scaffolding support to connect various models of representation as described by Lesh and colleagues (1987), especially when pictorials were involved. However, Daniel was able to make connections between manipulatives and abstract representations of fractions with minimal scaffolding. Daniel's own reasoning often led him to engage with other participants in productive discussions when he did not understand or agree with their solution strategies, particularly when those students worked with manipulatives in ways Daniel did not agree with.

Occurrences within the semantic memory subtype (19) were greater for Marcos than occurrences within the conceptual (9) or visuospatial (5) subtypes. These occurrences generally included errors of miscounting objects and misstating answers. Marcos also consistently demonstrated a struggle to express his thinking in written format. This may be a struggle related to issues in the semantic memory subtype or it may represent a different subtype not defined by Geary (2003). It became clear over the course of the intervention sessions that Marcos was often unable to accurately communicate his sound reasoning about fraction concepts in a written format and that Marcos also struggled to accurately report numbers in his mind when he was verbalizing his strategies. However, Marcos verbalizations about his strategies were conceptually sound and thorough. Marcos was the most successful of the three participants in connecting different representations of fractions as proposed by Lesh and colleagues (1987). In fact, Marcos often used a manipulative model or another participant's drawing to support his verbalizations of his solution strategies indicating that he able to successfully connect visual models to his reasoning even when writing or drawing to support his solution strategy was difficult. Marcos' strength was his ability to apply sound reasoning to mathematical situations as well as his ability to understand how the other participants thought about fraction concepts and Marcos' struggles with written format and accurate attention to numbers did not negate these strengths. Geary (2003) states that the semantic memory subtype represents a cognitive difference in the way a student's mind works from the majority of his or her peers, not in terms of conceptual competence but rather in terms of memory access. This statement holds relevance for Marcos. Because his mind makes sense of fractions in ways that are actually typical and accurate, he should be a candidate for high performance. However, because his mind does not allow him to remember facts quickly or easily coordinate mental processes with physical processes, he struggles to perform to his potential in school. Interacting with teachers who can see to it that Marcos has opportunities to apply his sound conceptual thinking while supporting his struggles will be critical to Marcos' future success in mathematics.

Taken together, the pattern of misconceptions and errors for Clay, Daniel, and Marcos demonstrates that identifying a student as struggling in mathematics, in and of itself, is not adequate. Referring to Clay as a student who is struggling in mathematics means something different than referring to Marcos or Daniel as a student who is struggling in mathematics. Clay demonstrated a struggle to make sense of fraction concepts, whereas Marcos demonstrated a struggle to express his thinking about fraction concepts. Daniel's struggle to make sense of fraction concepts was complicated by his issues connecting visual representations with mathematical ideas. Only with a deeper analysis of each participant's struggles does it become clear that each struggle is unique. The three participants in this study support Geary's (1990, 2003) position that students who struggle in mathematics may have different cognitive sources that underlie their struggles.

Clay's struggles within the conceptual subtype were intrinsically different from Marcos' struggles within the semantic memory subtype. As Hunt and Empson (2015) suggested of their study participants, Clay may need instruction focused on concepts that meet him at his current level of understanding. For Clay, additional time spent on equipartitioning, unit fractions, and iteration during the intervention sessions was required to build his understanding of fraction meaning. Additionally, Clay's interactions with fraction concepts during the intervention sessions suggests that he needed extended opportunities to make connections between manipulatives, pictorial representations, and real-world problems. On the other hand, Marcos generally exhibited sound abilities to understand fraction concepts, a result not in line with his performance on assessments in class. Marcos' struggles to accurately count, to remember facts, to accurately report his own solutions, and to express himself in a written format made it challenging for him to demonstrate his reasoning. However, because the intervention environment was oriented around verbalization, and prompting and probing from the researcher, he was able to do so. Opportunities for Marcos' to talk about his thinking with the other participants and the researcher supported Clay and Daniel in their endeavors to make sense of fraction concepts as well.

Daniel, like Clay, demonstrated struggles within the conceptual subtype. However, unlike either Clay or Marcos, Daniel also struggled within the visuospatial subtype. Daniel's struggles were more complicated to address, as he demonstrated sound conceptual reasoning at times and misconceptions at others. Because Daniel struggled to interpret and create visual information related to fractions, it is possible that some of his occurrences within the conceptual subtype were related to occurrences in the visuospatial subtype. For Daniel, opportunities to work in conjunction with the other participants revealed to him other ways of interpreting or creating a pictorial. In the earliest intervention sessions, Daniel did not seem to see differences in sizes of pieces. Therefore, the need for equal-sized pieces was not something to which Daniel attended. Daniel struggled throughout the first half of the intervention sessions with the necessity of creating equal-sized pieces to represent a fraction. However, with repeated interactions with the other participants, which included looking at their work and being guided in his creations by them, Daniel did eventually attend to the need to create equal-sized pieces in his drawings. Like Clay, Daniel may also need instruction geared to his current demonstration of knowledge about fraction concepts. It may also be that Daniel needs instruction focused on connecting abstract mathematical information to pictorial representations of that information.

Analysis Framework 2: Scaffolding Techniques

The second analysis framework was used to answer the second research question: How do third-grade students who struggle in mathematics interact with social-constructivist scaffolding techniques as they make sense of fraction concepts?

Scaffolding was initially intended to be used in this study as a way to support and reveal the participants' reasoning about fraction concepts. During analysis of the participants' struggles and strengths, it became clear that specific scaffolding techniques could be used as a framework to analyze how the participants made sense of fraction concepts through interaction with each other and the researcher. To enact this analysis, Anghileri's (2006) description of scaffolding techniques was used to identify and classify episodes of scaffolding. The six scaffolding techniques used for analysis in this study were a) prompting and probing; b) looking, touching, and verbalizing; c) simplifying a problem; d) explaining and justifying; and e) negotiated meaning. The following discussion presents insights about student reasoning found during scaffolding episodes.

Prompting and probing was the scaffolding technique used most often over the course of the intervention sessions. This is in part due to the fact that prompting and probing was used to support every other scaffolding technique employed as well as used on its own. Through use of this technique, the researcher found that Daniel and Clay were prone to initially taking a shallow view of some tasks. Rather than making sense of a problem statement, Daniel at times resorted to using numbers in the problem to generate a quick answer. Extended probing about the reasonableness of Daniel's solution and the strategy used caused Daniel to look for new ways to use the information in the problem statement to reason about an answer. Daniel made sense of fraction concepts by discussing his strategies with others, participants and researcher, who were prepared to challenge his assumptions, and then revising his strategy and his work. Daniel often repeated this process several times with a problem eventually constructing a workable solution strategy supported by his written work and verbal explanations. Clay was more entrenched in his thinking than Daniel. Clay continued throughout the intervention sessions to conceive of fractions as numbers of pieces where the amount of pieces in the whole was irrelevant. For this reason, Clay struggled to construct workable strategies to identify the name of a unit fraction, to iterate a unit fraction, and to determine equality or relative size of fractions. Marcos made sense of fraction concepts in more successful ways, but prompting and probing were especially central to his reasoning. Marcos appeared to use prompting and probing from the researcher, along with his own verbalization and those of other participants, as his primary vehicle to reason. It was also often the only way Marcos' demonstrated his reasoning. Nathan and colleagues (2007) found that sixth-grade students could make sense of mathematical tasks by developing their own solution strategies and participating in discourse oriented around extended questioning and explanation. This study extends the findings of Nathan and colleagues by demonstrating that students who struggle were also successful working with their own solution strategies in a discourse-rich environment focused on extended questioning, by both the researcher and the participants, with explanations of strategies.

Looking, touching, and verbalizing is a scaffolding technique that takes into account the use of manipulatives, interpretation and creation of pictorials, and student verbalizations about their reasoning using manipulatives and pictorials. Clay and Daniel in particular struggled to relate the number of pieces in a whole to the size of that unit fraction. When working with abstract fractions, such as in the fraction card activity or comparing fractions directly, Clay and Daniel needed to work with a manipulative to make sense of the relative sizes of unit fractions. Although pictorials were helpful to Clay, Daniel needed to place fraction strip cuttings next to each other or stack fraction circle pieces to be able to verbalize his understanding about the links between number of pieces in a whole and fraction size. As they worked with manipulatives in tasks, the researcher expected that Clay and Daniel would learn to reason about fractions without manipulatives or pictorials. In reality, Daniel needed to use manipulatives to make sense of most fraction concepts throughout the intervention sessions and Clay needed manipulatives to reason about fraction comparison. Although Marcos could reason verbally about fraction concepts without manipulatives, he often used manipulatives to demonstrate his reasoning to the other participants and the researcher. Other researchers (Butler et al., 2003; Cramer et al., 2002) have found that students improve their understandings of fraction concepts with the use of manipulatives and pictorials. Specifically, Cramer and colleagues (2002) found that connecting fraction tasks using manipulative or pictorial representations to real-world or abstract contexts assisted students in building understandings of the concepts underlying work with fractions. Butler and colleagues (2003) focused on the needs of students who struggled in mathematics and found that use of manipulatives and pictorials significantly improved these students' performance on fraction tasks when compared to instruction that focused on instruction in fractions that did not use these tools. This study extends the findings of those studies by describing specific conditions which supported the participants' in developing their reasoning about fraction concepts with the integration of manipulatives and pictorials. This researcher found that, although manipulatives and pictorials could be powerful tools in building understandings, they were most useful when embedded within a context problem and used to address a specific misconception arising in work with that problem.

Two scaffolding techniques, interpreting student work or talk and simplifying a problem, were rarely used during the course of the intervention sessions. The researcher sought to scaffold participants in their attempts to make sense of fraction concepts primarily by prompting and probing, and the use of activities which encouraged looking, touching, and verbalizing. Although at times necessary and useful, interpreting student work or talk might have circumvented participants' opportunities to reason about a problem and simplifying a problem might have reduced the cognitive complexity of tasks. In this study, the participants sometimes made sense of a concept in a logical way but struggled to verbalize their meaning accurately or correctly. Marcos was prone to giving a correct solution and then using a different number as he talked about his solution process. The researcher would repeat Marcos' original answer back to him and, in most cases, he would realize his error and continue with his correct original answer. In the episode highlighted as one of Marcos' struggles in Chapter 4, Marcos struggled to understand that he had changed his answer. A critical part of Marcos' sense-making process in this intervention involved similar redirections when needed. Clay and Daniel occasionally struggled to verbalize a strategy accurately or use vocabulary accurately. In these cases, the researcher interpreted the participant's meaning to clarify both for the group and the individual.

Simplifying the problem was a scaffolding technique employed only when attempts to use the two primary scaffolding strategies, prompting and probing and looking, touching, and verbalizing, resulted in no mathematically relevant reasoning about the problem. In the episode highlighted in Chapter 4 for this technique, Clay and Daniel were not able to conceive of a logical way to iterate half sandwiches to give to eight children. Instead, they were approaching the problem as one of sharing a sandwich with eight children. This is a strategy that can be successful with this particular problem, but after several minutes with other scaffolding techniques, Clay and Daniel were making no progress. Once presented with one sandwich to share, and knowing that children would be given half sandwiches, both were able to reason that one sandwich could feed two children. This simplified problem gave them a way to link their knowledge of sharing problems to a situation that required iteration. They were able to iterate the half sandwiches in pairs to arrive at the conclusion that four sandwiches would feed the children.

Explaining and justifying and negotiated meaning are two scaffolding techniques that are only fully enacted when students can support their reasoning verbally and in written format and when students can build reasoning about a mathematical concept that becomes accepted by the group. Although Daniel and Marcos demonstrated an ability to verbalize support for their reasoning about fraction concepts on many occasions, Clay often struggled to explain his reasoning. Daniel was able to generate pictorials that supported his reasoning on a regular basis, as was Clay at times. Marcos succeeded in providing sound verbal explanations and justifications, often supported by using manipulatives to demonstrate his thinking, but struggled to write explanations and justifications. Findings related to Marcos supported Broza and Kolikant's (2015) assertion that some students need verbal opportunities to demonstrate understandings that they cannot demonstrate in written format. Questioning by the researcher that asked a participant why a solution strategy worked or how he used his strategy to arrive at a solution often elicited responses from Daniel or Marcos that caused them to think more deeply about supporting their solutions and strategies. Clay struggled more than Daniel or Marcos to explain and justify his work even when he generated correct solutions, possibly because Clay often made sense of fraction tasks in conceptually inaccurate ways. Clay's struggles suggested that he needed more opportunities to make sense of fraction concepts before he would be able to support his reasoning.

In general, the participants did not reach a point where a mathematical truth was *taken-as-shared* (Cobb et al., 1992) by the group and used as to support an argument about a subsequent concept. The participants in this study often struggled to reach a negotiated meaning, with many incorrect mathematical ideas discussed and supported throughout the intervention sessions. However, instances of negotiated meaning did not result in misconceptions being spread through the group. In each case of negotiated meaning, correct mathematical understandings eventually prevailed. As Anghileri (2006) proposed discussion of these incorrect mathematical ideas, including misconceptions and errors, created a powerful learning environment and improved the understandings of the participants.

Implications

Over the course of this study, implications from the findings about student learning with teacher support have emerged. The findings of this study supported Geary's (2003) proposal that learning difficulties in mathematics differ among students and may be classified based on the types of misconceptions and errors a student demonstrates. Implied by this finding is a need for teachers of students who struggle to become familiar these subtypes so that they can tailor instruction to the needs of the individual learner. Also apparent in the findings is that students who struggle bring strengths to their work with mathematics as well. A learning environment can be designed to uncover and develop these strengths, or it can be designed in ways that overlook the mathematical strengths of students who struggle. For students who struggle, a supportive environment is critical because missed learning opportunities and unrecognized abilities may limit future opportunities to engage deeply with mathematics (Boaler, 2015). Vygotsky (1934/1986/2012) believed that a teacher can only understand the thinking of the student when they engage together in the construction of knowledge in an expert-novice relationship. This researcher attempted to use scaffolding techniques in line with Vygotsky's theories to support student learning in a small group intervention.

The findings indicate that the value of knowledge co-construction is two-fold. First, it provides the avenue through which teachers can more thoroughly understand the mathematical thinking of their students, including struggles and strengths each student experiences as he or she makes sense of mathematical concepts. Second, a co-constructivist environment has the power to support the deep mathematical learning of students, particularly those who struggle. Students who struggle need support from their teachers, but the form this support takes is crucial. Often, support proposed for students who struggle in mathematics is direct instruction and instruction that makes mathematical ideas explicit for the student (Flores & Kaylor, 2007; Fuchs et al., 2013; Jitendra et al., 1998; Joseph & Hunter, 2001. However, researchers (Carpenter et al., 1998; Cobb et al., 1991; Cramer et al., 2002; Empson, 1999) have found that students need opportunities to explicate mathematical ideas in their own minds to build understanding of concepts. The findings of this study are in line with these researchers in the field of mathematics education and also align with the findings of Gersten and colleagues' (2009) meta-analysis indicating that students who struggle learn most effectively in conceptually oriented learning environments. A small group intervention oriented around social-constructivist learning and support can provide these opportunities. Some students may think about mathematics in conceptually deep ways, but may be unable to demonstrate their understandings on worksheets or written assessments (Anghileri, 2006; Broza & Kolikant, 2015). Indeed Broza and Kolikant (2015) proposed that some students who struggle are best able to demonstrate "mathematical reasoning orally when placed in intimate and supportive learning environments, such as small groups tutoring" (p. 1095). Experiences in this study also support the assertion that some students who struggle need opportunities to verbalize their thinking so that they can demonstrate and further develop their understandings of mathematics. Other students need a teacher as expert to work with them to uncover current levels of conceptual understanding, and then to design instruction and provide scaffolding that extends current levels of conceptual understanding, an implication in line with the research findings of Hunt and Empson (2015) and Wilhelm (2014). In a social-constructivist learning environment, these needs can be met as students have opportunities to build understandings together and with the teacher.

By engaging in social-constructivist oriented scaffolding, teachers can use cognitively challenging tasks to support student learning while maintaining a focus on students' thinking. Among scaffolding techniques found to be most useful in activating student thinking about mathematics were two named and described by Anghileri (2006): a) prompting and probing; and b) looking, touching, and verbalizing. Prompting and probing as a scaffolding technique has the ability to replace overly-directed teaching practices thus encouraging students to develop their own mathematical ideas and connections. This study found that students who struggle had a tendency to turn to poorly understood procedures, to use strategies that only partially addressed a problem, and to attend to surface features of a problem. For example, one participant invoked a procedure for comparing fractions with the same denominator when comparing fractions with the same numerator. Also, when working with real-world context problems, participants often began by working arithmetic equations without attending to the problem context. Prompting and probing from the researcher helped these participants to move past their initial ways of thinking about problems. The findings showed that questioning asking students to reconsider problem contexts, to think about missing parts of solution strategies, and to explain thinking and strategies, can assist students who struggle in making sense of the conceptual underpinnings of mathematics.

The scaffolding technique of looking, touching, and verbalizing encompasses the use of both manipulatives and pictorials. This study was able to extend findings of research studies (Butler et al., 2003; Cramer et al., 2002) about the use of manipulatives and pictorials to help students build understanding of fraction concepts. Experiences in this study indicated that when manipulatives were used to explore a fraction concept, such as comparing sizes of unit fractions, participants further built understandings of these concepts by verbalizing their thinking during explorations. However, these understandings were not easily transferred to problems set in either real-world or abstract contexts. More valuable than attempts to make connections between representations presented in different tasks, was the use of manipulatives as the need became apparent during real-world or abstract tasks. The findings of this study with regard to manipulative use suggest teachers need to attend carefully to student thinking during tasks. Teachers may expect that previous work with manipulatives may support student thinking in subsequent tasks performed without manipulatives. While this may be the case for some students, others may not be successful in making these connections. By working with a student who is struggling to build mathematical meaning, a teacher can diagnose the need to work with manipulatives at the moment it has the most potential to help the student make sense of concepts. For example, within a class some students may be successfully reasoning about fraction sizes while others need to work with fraction circles or paper strip cuttings to understand the mathematical idea. Pictorials present a similar dilemma. Students working with pictorials given in a task may not transfer concepts about fractions to problems that call for student-generated pictorials. By providing opportunities for students to create pictorials and verbalize their thinking about the process of creation, students can strengthen their understandings of given pictorials as well.

This study suggests that teachers need a deep understanding of how students use manipulatives and pictorials to make sense of mathematics. Specifically, teachers need to realize that work with manipulatives may not transfer to work with real-world or abstract problems, unless the teacher provides opportunities for students to integrate manipulative work into these contexts. Also, teachers need to understand that students may make sense of pictorials given in a problem differently from pictorials they generate. Finally, during this study manipulatives and pictorials were more productive in when students discussed their thinking about these representations with each other and the teacher. Furthermore, this study holds important implications for students who struggle in mathematics and the teachers who work with them. Teachers need to co-construct knowledge with students to better understand the students' struggles, strengths, current level of understanding, and potential for mathematical learning. Students who struggle in mathematics need opportunities to co-construct knowledge with teachers and other students to demonstrate their abilities, to better use their current level of understanding, and to work in environments that require deep mathematical thinking. Without these opportunities, mathematics instruction for students who struggle may not support students in reaching their potential and may limit their lifetime opportunities.

Recommendations for Future Research

The current study may suggest directions for future research. First, more qualitative studies need to be conducted to build a better understanding of how students may fit into Geary's (2003) subtypes of learning disabilities in mathematics and these studies need to be extended into other mathematical topics, such as algebra and geometry. Additionally, studies that address the potential need for differing interventions based on subtype would be a further step in this research agenda. Lesh and colleagues (1987) have suggested that connecting representations is critical to students' abilities to make sense of fraction concepts. Given that the findings in this study suggested that students who struggled within the conceptual and visuospatial subtypes had difficulty connecting work with manipulatives to real-world context problems, future studies that investigated this potential issue more deeply could be beneficial. Future research could also be focused on examination of which scaffolding techniques hold the most promise for students who struggle. Investigations into professional development that help teachers learn how to enact socio-constructivist scaffolding techniques would benefit learners who struggle as well. To further address the needs of students who struggle in all facets of mathematics, ethnographies that seek to understand the learning communities these students and their teachers participate in and create could also be valuable.

Conclusion

This chapter discussed findings from the study, implications, and recommendations for future research. The findings of this study suggest that these participants who struggled in mathematics made sense of fraction concepts involving equipartitioning, identification and iteration of unit fractions, fraction equivalence, and to a lesser degree fraction comparison, through responding to prompting and probing questions from the researcher and through interactions with manipulatives, interpreting and creating pictorials, and verbalizing their reasoning with each other and the researcher. Further, opportunities to connect different representations of fractions, such as connecting abstract reasoning about unit fraction size to paper strip manipulatives, played a crucial role in the participants' processes of making sense. At the same time, the study revealed that each participant's struggle was unique and not always grounded in conceptual misunderstandings. For each participant, regardless of the subtype of learning difficulty identified by the researcher, prompting and probing as a primary scaffolding technique created an environment where the participants could co-construct understandings of fraction concepts in conjunction with each other and the researcher. As Siegler and colleagues (2012) found, mastery of fraction concepts may be crucial to students' understandings of later mathematical concepts, such as those found in algebra. For students who struggle in mathematics, it may be crucial that their teachers can understand and address their misconceptions about fraction concepts in elementary school, uncover their strengths, and provide scaffolding in line with their needs to ensure that sound conceptual understandings are built. This study was intended to as an initial foray into understanding the struggles and strengths of these students and how they interacted with socio-constructivist scaffolding as they learned about fraction concepts during a set of intervention sessions.

APPENDIX A:
IRB APPROVAL, INFORMED CONSENT, VERBAL ASSENT

IRB Approval Letter, Informed Consent, and Verbal Assent Protocol

University of Central Florida

University of Central Florida Institutional Review Board
Office of Research & Commercialization
12201 Research Parkway, Suite 501
Orlando, Florida 32826-3246
Telephone: 407-823-2901 or 407-882-2276
www.research.ucf.edu/compliance/irb.html

Approval of Human Research

From: **UCF Institutional Review Board #1**
FWA00000351, IRB00001138

To: **Rebecca Grice Gault**

Date: **December 03, 2015**

Dear Researcher:

On 12/03/2015, the IRB approved the following human participant research until 12/02/2016 inclusive:

Type of Review:	UCF Initial Review Submission Form
Project Title:	A multiple case study: How do third grade students who struggle in mathematics make sense of fraction concepts involving representation, comparison, and equivalence?
Investigator:	Rebecca Grice Gault
IRB Number:	SBE-15-11774
Funding Agency:	
Grant Title:	
Research ID:	N/A

The scientific merit of the research was considered during the IRB review. The Continuing Review Application must be submitted 30days prior to the expiration date for studies that were previously expedited, and 60 days prior to the expiration date for research that was previously reviewed at a convened meeting. Do not make changes to the study (i.e., protocol, methodology, consent form, personnel, site, etc.) before obtaining IRB approval. A Modification Form **cannot** be used to extend the approval period of a study. All forms may be completed and submitted online at https://iris.research.ucf.edu .

If continuing review approval is not granted before the expiration date of 12/02/2016, approval of this research expires on that date. When you have completed your research, please submit a Study Closure request in iRIS so that IRB records will be accurate.

Use of the approved, stamped consent document(s) is required. The new form supersedes all previous versions, which are now invalid for further use. Only approved investigators (or other approved key study personnel) may solicit consent for research participation. Participants or their representatives must receive a copy of the consent form(s).

All data, including signed consent forms if applicable, must be retained and secured per protocol for a minimum of five years (six if HIPAA applies) past the completion of this research. Any links to the identification of participants should be maintained and secured per protocol. Additional requirements may be imposed by your funding agency, your department, or other entities. Access to data is limited to authorized individuals listed as key study personnel.

In the conduct of this research, you are responsible to follow the requirements of the Investigator Manual.

On behalf of Sophia Dziegielewski, Ph.D., L.C.S.W., UCF IRB Chair, this letter is signed by:

Signature applied by Joanne Muratori on 12/03/2015 04:39:49 PM EST

IRB Manager

University of Central Florida

A multiple case study: How do third grade students who struggle in mathematics make sense of fraction concepts involving representation, comparison, and equivalence?

Informed Consent

Principal Investigator:	Rebecca Gault, Doctoral Candidate
Faculty Advisor:	Enrique Ortiz, Ed.D.
Investigational Site(s):	An elementary school in central Florida

How to Return this Consent Form:
You are provided with two copies of this consent form. If you give consent for your child to participate in the research, please sign one copy and return it to the front desk at your child's school, sealed in the envelope provided, and keep the other copy for your records. As an alternative, if you would like to meet with me to discuss the study you could return this form to me in person if you decide to have your child participate.

Introduction:

Researchers at the University of Central Florida (UCF) study many topics. To do this we need the help of people who agree to take part in a research study. You are being asked to allow your child to take part in a research study which will include about four people at your child's school. Your child is being invited to take part in this research study because he or she is a third grade student and may benefit from additional instruction in fractions.

The person doing this research is Rebecca Gault of the University of Central Florida. Because the researcher is a doctoral student, she is being guided by Dr. Enrique Ortiz, a UCF faculty advisor in Mathematics Education.

What you should know about a research study:

- Someone will explain this research study to you.
- A research study is something you volunteer for.
- Whether or not you take part is up to you.
- You should allow your child to take part in this study only because you want to.
- You can choose not to take part in the research study.
- You can agree to take part now and later change your mind.
- Whatever you decide it will not be held against you or your child.
- Feel free to ask all the questions you want before you decide.

Purpose of the research study:

The purpose of the tutoring intervention and research study is to help third graders struggling in mathematics better understand fraction concepts and to understand how students who are struggling best make sense of mathematics concepts such as fractions.

What your child will be asked to do in the study:

Your child will attend tutoring on Tuesdays, Wednesdays, and Thursdays, from February 16 to March 24 after school from 3:15 to 4:00 at your child's elementary school. This tutoring will be about fraction concepts covered in third grade mathematics. During the first and last tutoring sessions, the researcher will also ask your child to solve a few fraction problems to better understand how your child is thinking about fractions. During the first and last sessions, the researcher may also ask your child questions about how he or she feels about mathematics and doing work with mathematics. Your child does not have to answer every question or complete every task. You or your child will not lose any benefits if your child skips questions or tasks.

Location:

Tutoring sessions will occur at your child's elementary school.

Time required:

We expect that your child will be in this research study for five weeks, three afternoons each week.

Audio or video taping:

Your child will be audio taped during this study. If you do not want your child to be audio taped, your child will not be able to be in the study. Please discuss this with the researcher if you have any concerns. If your child is audio taped, the tape will be kept in a locked, safe place. The tape will be erased or destroyed when the study is completed. Transcriptions will be kept for research purposes but will not include your child's name or any identifying markers.

If you give your permission, your child will be video-taped during this study. If you do not want your child to be video-taped, your child can still be in the study. If you do not want your child videotaped, only audio recording will occur. Please discuss this with the researcher if you have any concerns. If your child is video-taped, the tape will be kept in a locked, safe place. The tape will be erased or destroyed when the study is completed.

Risks:

Your child may feel some anxiety or frustration from working with challenging mathematical tasks or being video-taped while working on tasks. Every effort will be made by the researcher to ensure that any anxiety or frustration will be minimal. If your child becomes anxious because of videotaping, the videotape will be turned off. If your child becomes anxious because of mathematical tasks, the researcher will help the child with the task or move on to another task.

Benefits:

We cannot promise any benefits to you, your child, or others from your child taking part in this research. However, possible benefits include that your child may develop a deeper understanding of fraction concepts and a foundation for future learning in mathematics. Also, your child may experience a lowered feeling of anxiety about mathematics and a greater sense of confidence in his or her ability to learn about mathematics.

Compensation or payment:

There is no monetary compensation or other payment to you or your child for your child's part in this study. Your child will receive a set of fraction manipulatives and a fraction game set.

Confidentiality:

We will limit personal data collected in this study to people who have a need to review this information. We cannot promise complete secrecy.

Study contact for questions about the study or to report a problem:

If you have questions, concerns, or complaints, or think the research has hurt your child talk to Rebecca Gault, doctoral candidate, University of Central Florida at (321) 202-5087 or rebecca.gault@knights.ucf.edu, or Dr. Enrique Ortiz, Faculty Supervisor, University of Central Florida at enrique.ortiz@ucf.edu.

IRB contact about you and your child's rights in the study or to report a complaint:
Research at the University of Central Florida involving human participants is carried out under the

oversight of the Institutional Review Board (UCF IRB). This research has been reviewed and approved by the IRB. For information about the rights of people who take part in research, please contact: Institutional Review Board, University of Central Florida, Office of Research & Commercialization, 12201 Research Parkway, Suite 501, Orlando, FL 32826-3246 or by telephone at (407) 823-2901. You may also talk to them for any of the following:

- Your questions, concerns, or complaints are not being answered by the research team.
- You cannot reach the research team.
- You want to talk to someone besides the research team.
- You want to get information or provide input about this research.

Withdrawing from the study:

You may decide not to have your child continue in the research study at any time without it being held against you or your child. If you decide to have your child leave the research, contact the researcher so that the researcher can remove your information from the study. You can email or call using the information above, or speak to her before or after a tutoring session.

Your signature below indicates your permission for the child named below to take part in this research.

DO NOT SIGN THIS FORM AFTER THE IRB EXPIRATION DATE BELOW

____
Name of participant

______________________________ _______________
Signature of parent or guardian Date

- ❑ Parent
- ❑ Guardian (See note below)

____
Printed name of parent or guardian

Note on permission by guardians: An individual may provide permission for a child only if that individual can provide a written document indicating that he or she is legally authorized to consent to the child's general medical care. Attach the documentation to the signed document.

Protocol for Verbal Student Assent

At the beginning of the first meeting with student/participant:

Hi, my name Ms. Rebecca and I am from UCF. I'd like to talk with you today about math and I'd especially like to talk with you about fractions. I am very interested in your ideas about math and fractions and you can help me understand how kids think about these things. Would you be willing to talk with me about math and fractions?

Sometimes I forget things when I talk to people. Would it be okay if I record our conversation [indicate the audio/video recorder] so I can listen to it later? No one besides me will hear or see it, but it you don't want me to, that's okay, and I won't record our conversation. If you change your mind about being recorded at any time, just let me know and I'll turn it off. [Turn on audio/video recorder *only* if student assents.]

Also, if you change your mind at any time about talking to me about math and fractions, just let me know and we'll stop. Okay? [Proceed with initial interview if student assents.]

At the beginning of each subsequent meeting with student/participant:

Hi ________. It's very nice to see you today. Can we work some fraction math problems together today?

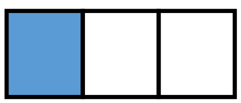
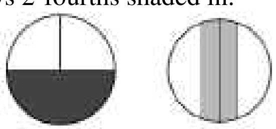
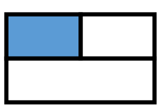
Would it be okay if I record our work together today? No one besides me will hear or see it, but if you don't want me to, that's okay, and I won't record our work. If you change your mind about being recorded at any time, just let me know and I'll turn it off. [Turn on audio/video recorder *only* if student assents.]

Also, if you change your mind at any time about working fraction math problems with me, just let me know and we'll stop. Okay? [Proceed with tutoring session if student assents.]

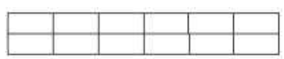
APPENDIX B:
CLASSIFICATION OF INTERVENTION ACTIVITIES

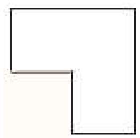
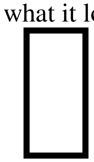
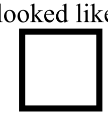
Table 14

Classification of Intervention Activities

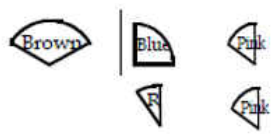
Activity Number and Example	CCSSM Focus	Representational Translation (Lesh et al., 1987)	Type of model(s)	Activity Source
Activity 1 Identify and iterate the unit fraction. 	Unit Fractions 3.G.2 3.NF.1 3.NF.2.a 3.NF.2.b	P to V	Linear Model	
Activity 2 Place an "X" beside each picture that shows 2-fourths shaded in. 	Equipartitioning 3.G.2 3.NF.1	P to W/V	Area Model	Cramer et al., 2009, Rational Number Project Lesson 4 Student Page G
Activity 3 What do you see? 	Equipartitioning 3.G.2 3.NF.1	P to W/V	Area Model	
Activity 4 Jackie and Lianna have 13 cookies. If they share the cookies equally, how many cookies would each person get?	Equal sharing/Equal Partitioning 3.G.2 3.NF.1	RL to P/V	Set Model	Empson & Levi, 2011 p. 29-31

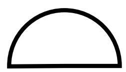
Activity Number and Example	CCSSM Focus	Representational Translation (Lesh et al., 1987)	Type of model(s)	Activity Source
Activity 5 Use area fraction kits to play "Race to a Whole" game.	Unit Fractions and Equivalence 3.G.2 3.NF.1 3.NF.3.c 3.NF.3.b	M to M/V	Area Model	Dixon et al., 2016, p. 80-83
Activity 6 Four children want to share 10 Publix sub sandwiches so that everyone gets the same. How much can each child have?	Unit Fractions 3.G.2 3.NF.1 3.NF.2.a 3.NF.2.b	RL/W to P/V	Area and Linear Models	Empson & Levi, 2011, p. 65
Activity 7 Four children want to share 3 peaches so that everyone gets the same. How much peach can each child have?	Unit Fractions 3.G.2 3.NF.1 3.NF.2.a 3.NF.2.b	RL/W to P/V	Area and Linear Models	Empson & Levi, 2011, p. 65
Activity 8 Draw lines then cut the paper circle to make equal-sized pieces.	Unit Fractions 3.G.2	M/P to M/V	Area Model	
Activity 9 Draw lines then cut the paper square to make equal-sized pieces.	Unit Fractions 3.G.2	M/P to M/V	Area Model	

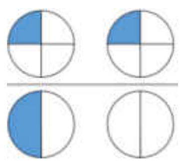
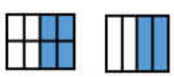
Activity Number and Example	CCSSM Focus	Representational Translation (Lesh et al., 1987)	Type of model(s)	Activity Source
Activity 10 This is a candy bar. Draw to show it divided into 5 equal-sized pieces. 	Equipartitioning and Unit Fractions 3.G.2 3.NF.1 3.NF.2.a 3.NF.2.b	RL/P/W to P/W/V	Area and Linear Models	Cramer et al., 2009, Rational Number Project Lesson 4 Student Pages A and B
Activity 11 This is a picture of a pan of brownies:  The pan is cut into how many equal-sized brownies? Each brownie piece is __ of the whole pan.	Equipartitioning and Unit Fractions 3.G.2 3.NF.1	RL/P/W to P/W/V	Area Model	Cramer et al., 2009, Rational Number Project Lesson 4 Student Pages A and B
Activity 12 Use area fraction kits to play “Race to a Whole” game.	Unit Fractions and Equivalence 3.G.2 3.NF.1 3.NF.3.c 3.NF.3.b	M to M/V	Area Model	

Activity Number and Example	CCSSM Focus	Representational Translation (Lesh et al., 1987)	Type of model(s)	Activity Source
Activity 13 Health First granola bars are square shaped, Lucius ate one piece of the granola bar and now it looks like this:  The piece that Janis ate is __ of a whole candy bar.	Equipartitioning and Unit Fractions 3.G.2 3.NF.1 3.NF.2.a 3.NF.2.b	RL/P/W to P/W/V	Area and Linear Models	Cramer et al., 2009, Rational Number Project Lesson 4 Student Pages A and B
Activity 14 One-half of a lemon bar was left after a party. This is what it looked like:  Draw a picture of the whole cake.	Equipartitioning and Unit Fractions 3.G.2 3.NF.1 3.NF.2.a 3.NF.2.b	RL/P/W to P/W/V	Area and Linear Models	Cramer et al., 2009, Rational Number Project Lesson 4 Student Pages A and B
Activity 15 Four kids shared a candy bar equally. Joy's share looked like this:  Draw a picture of the whole candy bar.	Equipartitioning and Unit Fractions 3.G.2 3.NF.1 3.NF.2.a 3.NF.2.b	RL/P/W to P/W/V	Area and Linear Models	Cramer et al., 2009, Rational Number Project Lesson 4 Student Pages A and B

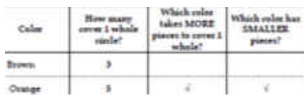
Activity Number and Example	CCSSM Focus	Representational Translation (Lesh et al., 1987)	Type of model(s)	Activity Source
Activity 16 Oscar had a garden shaped like a rectangle. Draw a picture of Oscar's garden and show that the garden is in 9 equal-sized parts	Equipartitioning and Unit Fractions 3.G.2 3.NF.1	RL//W to P/W/V	Area Models	Cramer et al., 2009, Rational Number Project Lesson 4 Student Pages A and B
Activity 17 Lucy has a garden shaped like a square. Draw a picture of Lucy's garden and show that the garden is in 3 equal-sized parts	Equipartitioning and Unit Fractions 3.G.2 3.NF.1	RL//W to P/W/V	Area Models	Cramer et al., 2009, Rational Number Project Lesson 4 Student Pages A and B
Activity 18 Jordan said that 1 red piece is one-third. Andres said that 1 red piece is one-fourth.Who is correct?	Unit Fractions 3.G.2 3.NF.1	M/RL/W to V/M	Area Model	Cramer et al., 2009, Rational Number Project Lesson 2 Wrap-Up Question
Activity 19 Change the unit to 1 blue? What fraction name can you give these pieces? 1 grey? 1 red?	Unit Fractions 3.G.2	M/W to V/W	Area Model	Cramer et al., 2009, Rational Number Project Lesson 3 Student Page A

Activity Number and Example	CCSSM Focus	Representational Translation (Lesh et al., 1987)	Type of model(s)	Activity Source
Activity 20 The yellow piece is the unit. How many blues cover the yellow piece? 1 blue is __ of the yellow.	Unit Fractions and Equivalence 3.G.2 3.NF.3.c	V to M	Area Model	Cramer et al., 2009, Rational Number Project Lesson 2 Large Group Instruction
Activity 21 Fully cover the shape on the left with any combination of shapes on the right that will work. 	Equivalence 3.NF.3.c 3.NF.3.b	P to M/V	Area Model	Cramer et al., 2009, Rational Number Project Lesson 1 Transparency 1
Activity 22 Draw a picture of a pizza. Show on your drawing the pizza cut in 2 fair shares. Each fair share is __ of a whole pizza.	Equipartitioning and Unit Fractions 3.G.2 3.NF.1	RW/V to P/V/W	Area Model	Cramer RNP Lesson 2 Student Pages A & B
Activity 23 Mari's patio is a whole circle. Draw a picture of Mari's patio. Show on your drawing that the patio is in 3 equal-sized parts. Each part is __ of Mari's patio.	Equipartitioning and Unit Fractions 3.G.2 3.NF.1	RW/V to P/V/W	Area Model	Cramer RNP Lesson 2 Student Pages A & B

Activity Number and Example	CCSSM Focus	Representational Translation (Lesh et al., 1987)	Type of model(s)	Activity Source
Activity 24 Juan has a patio that looks like this. Draw on Juan's patio to show it divided into 3 equal sized parts. Each part is ____of Juan's patio. 	Equipartitioning and Unit Fractions 3.G.2 3.NF.1	RL/P/W to P/W/V	Area Model	Cramer et al., 2009, Rational Number Project Lesson 2 Student Pages A and B
Activity 25 Dani wants to feed each of the children she babysits a half sandwich for lunch. If she babysits 8 children, how many sandwiches should she make?	Unit Fractions 3.G.2 3.NF.1 3.NF.2.a 3.NF.2.b	RL/W to P/V	Area and Linear Models	Empson & Levi, 2011, p. 65
Activity 26 The cards for 0, ½, and 1 are placed on a table with space in between. Students place cards under the fraction cards in the correct location between 0, ½, and 1.	Comparison 3.NF.3.b 3.NF.3.a 3.NF.3.d 4.NF.2	M to M/V	Linear Model	
Activity 27 Students fold paper strips to explore 2, 4, 8 equal parts.	Unit Fractions and Equivalence 3.G.2 3.NF.1 3.NF.2.a 3.NF.2.b 3.NF.3.c 3.NF.3.b 3.NF.3.a	M/V to M/V	Linear Model	Cramer et al., 2009, Rational Number Project Lesson 4 Large Group Instruction

Activity Number and Example	CCSSM Focus	Representational Translation (Lesh et al., 1987)	Type of model(s)	Activity Source
Activity 28 Use linear fraction tile kits to play “Race to a Whole” game.	Unit Fractions and Equivalence 3.G.2 3.NF.1 3.NF.2.a 3.NF.2.b 3.NF.3.c	M to V	Linear Model	Ortiz, 2014, *Dimensions*
Activity 29 Which set of circles has more shaded? 	Equivalence 3.G.2 3.NF.1 3.NF.3.b	P to V	Area Model	
Activity 30 Are the shaded areas equal? 	Equivalence 3.G.2 3.NF.1 3.NF.3.b	P/W to V	Area Model	Lamon, 2010, p. 96
Activity 31 How many different ways can you cover a half circle manipulative?	Equivalence 3.NF.3.c 3.NF.3.b	M/V to M/W	Area Model	Cramer et al., 2009, Rational Number Lesson 8 Large Group Instruction

Activity Number and Example	CCSSM Focus	Representational Translation (Lesh et al., 1987)	Type of model(s)	Activity Source
Activity 32 A group of 3 children are sharing 2 burritos so that each gets the same amount. How many burritos should 6 children share so that each child gets as much burrito as a child in the first group?	Equivalence 3.G.2 3.NF.1 3.NF.3.b	RL/W to P/V	Set Model	Empson & Levi, 2011, p. 140
Activity 33 Sticker books are on sale. You can buy 2 sticker books for $3. You want to buy 6 sticker books. How much money do you need?	Equivalence 3.G.2 3.NF.1 3.NF.3.b	RL/W to P/V	Set Model	Empson & Levi, 2011, p. 140
Activity 34 Look at this picture, then let's answer some questions about it:  Can you see thirds? How many suns are in 2/3 of the set?	Unit Fractions 3.G.2 3.NF.1	P/V to W/V	Set Model	Lamon, 2010, p. 135

Activity Number and Example	CCSSM Focus	Representational Translation (Lesh et al., 1987)	Type of model(s)	Activity Source
Activity 35 Fraction circles are used to complete a table about piece sizes.  Color | How many cover 1 whole circle? | Which color takes MORE pieces to cover 1 whole? | Which color has SMALLER pieces? Brown | 3 | | Orange | 5 | √ | √	Equivalence 3.G.2 3.NF.1 3.NF.3.c 3.NF.3.d	M/W to M/V/W	Area Model	Cramer et al., 2009, Rational Number Lesson 6 Student Pages A and B
Activity 36 Which fraction is larger? 3/4 or 2/6	Comparison 3.NF.3.d 4.NF.2	W to P(mental)/V	Model will exist in the mind of the child and is likely to be an area model	Cramer et al., 2009, Rational Number Lesson 7 Warm-Up

Note. RL=Real Life; M=Manipulatives; P=Pictures; W=Written Symbols; V=Verbalization.

APPENDIX C:
ENACTED INTERVENTION PROTOCOL

Enacted Intervention Protocol

<u>Prior to Session 1</u>

Goals:

- Establish a rapport with the student
- Gain an understanding of student's attitudes, motivation, and experiences with mathematics and mathematics class.

Materials/Source:

- Unstructured interview protocol including questions developed based on Moustakas' (1994) interview technique for phenomenological research.

Unstructured Interview:

- Unstructured interview – the researcher will ask the participants the questions verbally and will use prompts as needed.

Table 15

Unstructured Interview

Data/Rationale	Question	Prompts
Icebreaker; background	Would you tell me a little about yourself and your school?	What has third grade been like so far? What was second grade like? What is your favorite thing about school and why? What has been your favorite subject and why?
To gain a description of the student's experiences in mathematics class.	Can you tell me what math class is like for you? *[What have you experienced in math class? – Creswell, 2007, p61 citing Moustakas, 1994, my adaption for third-graders]*	What kinds of things have you learned in math class so far this year? What do you do in math class when something doesn't make sense to you? *(If further prompting is needed: What helps you the most when you don't understand something in math class?)* What is it like when you work with a partner (another student) in math class? What does your teacher do when she's teaching math? (avoid phrasing like "to help you learn math" because it is leading)
To find out what kinds of beliefs the student has about mathematics class.	What kinds of things can happen in math class that make you like it? Or not like it? [*What contexts or situations have typically influenced or affected your experiences of math class? – Creswell, 2007, p. 61 citing Moustakas, 1994, my adaption for third graders]*	Do you think math is easy, hard, or sometimes both? (this one-word-answer is intended to set up the next question) What do you think makes it easy (hard) for you? What do you think about if you get to do math work with a partner (another student)? *(if further prompting is needed continue with these potential questions: Does it make math easier or harder to understand? Are some partners super helpful, or not so helpful? How do you help your partner?)* What do you think can make math class fun? What do you think can make math class not fun? What do you think about when it's time to pay attention to the teacher? If you could tell your teacher to do one thing differently what would you tell her?

Session 1

Focus: Equipartitioning, identification of unit fraction, fraction equivalence

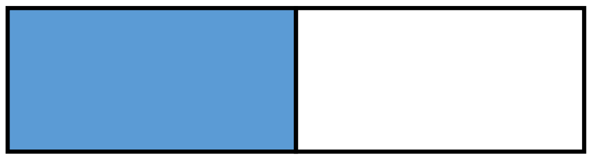
Materials:

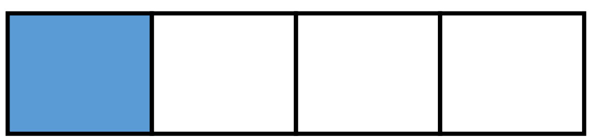
- Cramer RNP Lesson 4 Student Page G
- 5 drawn same-size rectangle “brownie pans” each showing a different unit fraction “eaten” (shaded) for ½, 1/3, ¼, 1/6, 1/8
- Rectangular drawing showing a ½ and two 1/4s, with a ¼ shaded, with 3 hypothetical student interpretations of above rectangular
- Equal-sharing word problem

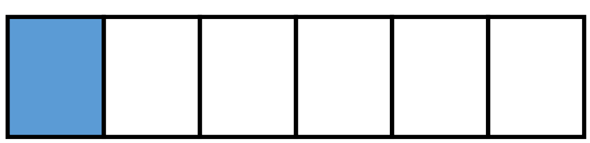
Activities:

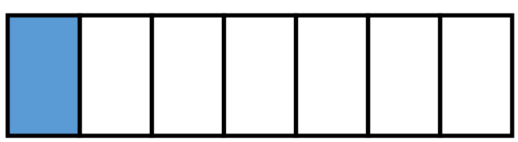
- Activity 1 - Use “brownie pan” rectangles to review naming conventions for unit fractions.
- Activity 2 - Present students with 12 different pictorial models (circles and rectangles) with partitions and shaded parts. Ask students which models show 2-fourths shaded. Source: Cramer RNP Lesson 4 Student Page G
- Activity 3 - Have students consider other students’ interpretations of rectangular drawing with shading. Ask students if they can come up with any other alternative interpretations of the rectangle and its shaded area. Students should come to the conclusion that, depending on what is considered to be the whole, several interpretations can be correct. However, part of the discussion will include establishing the convention that shaded parts of a whole *usually* indicate the part to be considered.
- Activity 4 - Word problem about an equipartitioning (equal sharing) situation that which students may be familiar.

Activity 1



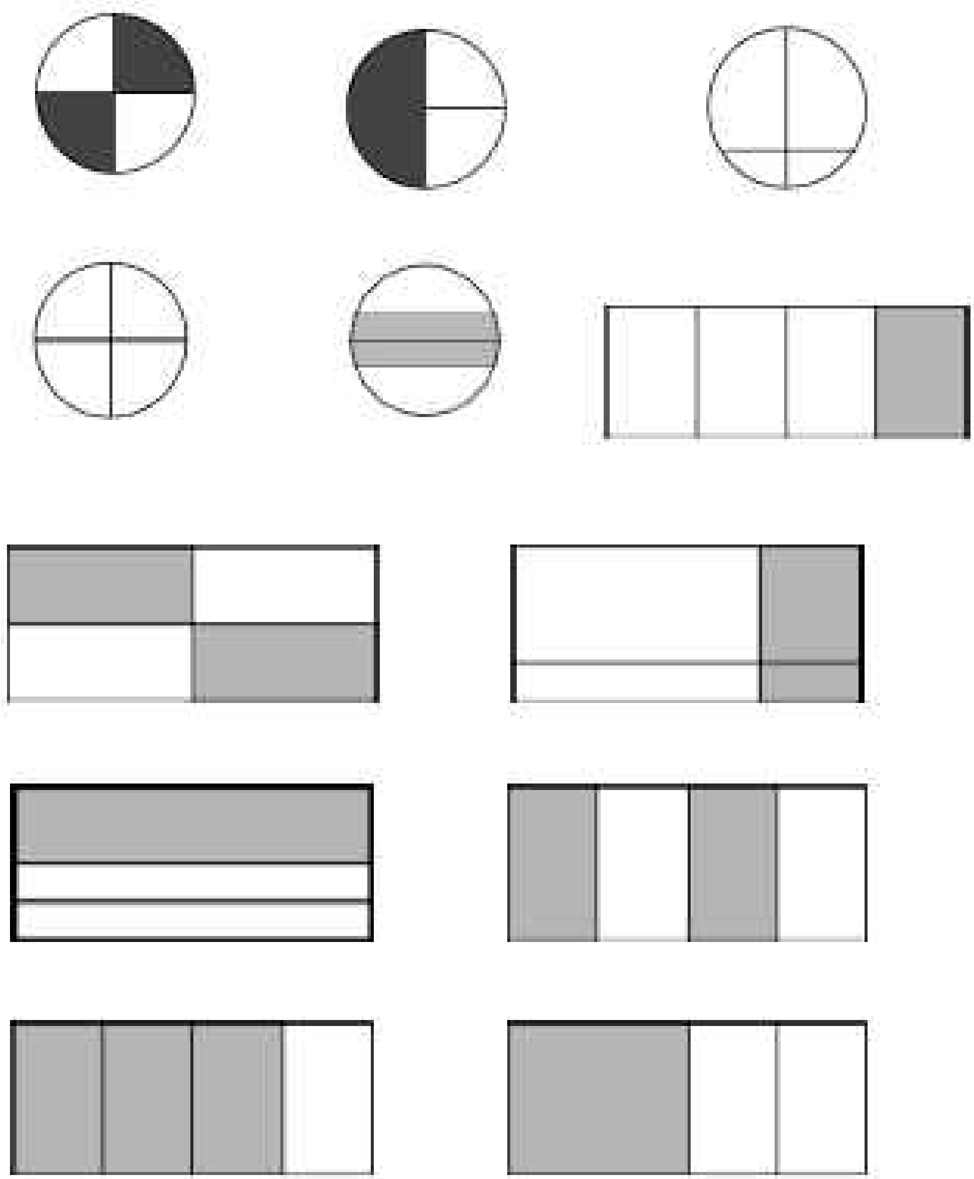






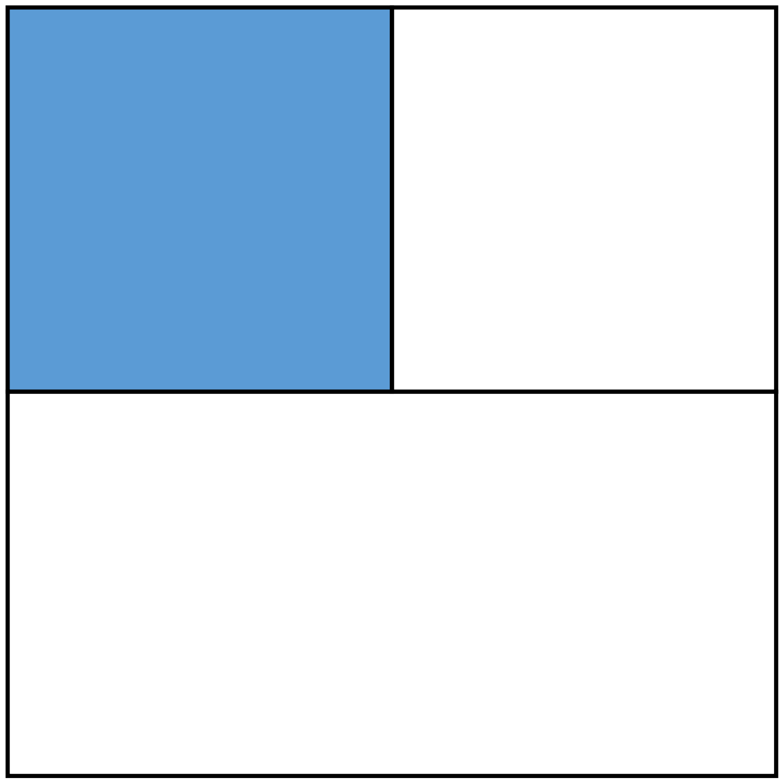
Activity 2

Look at each picture carefully. Place an “X” beside each picture that shows 2-fourths shaded in. You may need to draw in lines to determine if 2-fourths are shaded.



Activity 3

What do you see?



Janelle said that she sees one-third.

Lui said that he sees one-fourth.

Emily said that she sees one and a half. Could she be right?

Activity 4

Jackie and Lianna have 13 cookies. If they share the cookies equally, how many cookies would each person get?

Session 2

Focus: Equipartitioning, identification of unit fraction, iteration of unit fraction, fraction equivalence

Materials:

- Equal-sharing word problem
- Area Fraction Kit

Activities:

- Activity 4 (continued) - Word problem about an equipartitioning (equal sharing) situation that which students may be familiar.
- Activity 5 - Have students work with Rectangular Fraction Area Model Kits to play the game “Race to a Whole”. Students play in pairs or threes. Each student rolls the die then covers part of the “whole” area with the fraction piece indicated by the die. Students take turns rolling their die until someone completes the whole.. The game was played three times.

<u>**Session 3**</u>

Focus: Equipartitioning, identification of unit fraction, iteration of unit fraction

Materials:

- Equal-sharing word problems

Activities:

- Activity 6 - Word problem about an equipartitioning (equal sharing) situation that which students may be familiar.
- Activity 7 - Word problem about an equipartitioning (equal sharing) situation that which students may be familiar.

Activity 6

4 children want to share 10 Publix subs sandwiches so that everyone gets the same amount. How much can each child have?

Activity 7

4 children want to share 3 peaches so that everyone gets the same amount. How much peach can each child have?

Session 4

Focus: Equipartitioning

Materials:

- Paper circles and squares
- Problems from RNP Lesson 4 Student Pages A & B

Activities:

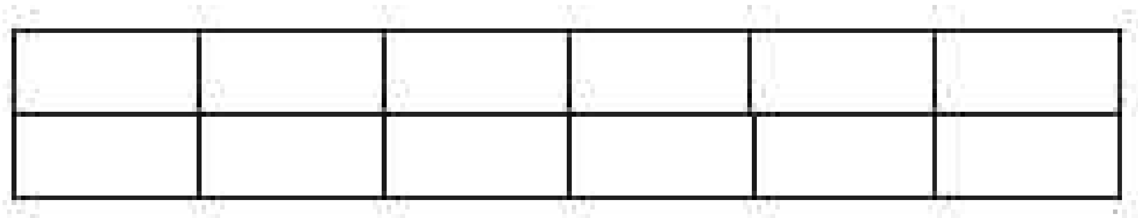
- Activity 8 – Students used paper circles to draw lines and cut along these lines to find strategies that created equal-sized pieces
- Activity 9 – Students used paper squares to draw lines and cut along these lines to find strategies that created equal-sized pieces
- Activity 10 – Students divided a rectangle representing a candy bar into five equal-sized pieces. Source: Cramer RNP Lesson 4 Student Pages A & B
- Activity 11 – Students work with a rectangle showing 12 equal-sized pieces to identify the unit fraction. Source: Cramer RNP Lesson 4 Student Pages A & B

Activity 10

This is a candy bar. Draw to show it divided into 5 equal-sized pieces.

Activity 11

This is a picture of a pan of brownies:



The pan is cut into how many equal-sized brownies?

Each brownie piece is ______ of the whole pan.

Session 5

Focus: Equipartitioning, identification of unit fraction, iteration of unit fraction, fraction equivalence

Materials:

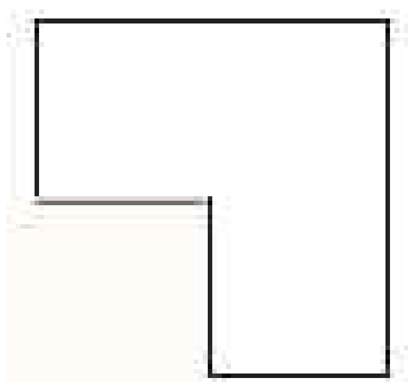
- Area Fraction Kit
- Problems from RNP Lesson 4 Student Pages A & B

Activities:

- Activity 12 - Activity 5 - Have students work with Rectangular Fraction Area Model Kits to play the game "Race to a Whole". Students play in pairs or threes. Each student rolls the die then covers part of the "whole" area with the fraction piece indicated by the die. Students take turns rolling their die until someone completes the whole. The game was played once with the rule that after each turn pieces needed to be converted to the smallest piece possible. Then the game was played a second time with the rule that after each turn pieces needed to be converted to the largest piece possible.
- Activity 13 - Students work with a graphic showing three-fourths of a rectangle given in a word problem. Source: Cramer RNP Lesson 4 Student Pages A & B

Activity 13

Health First granola bars are square shaped. Lucius at one piece of a Health First granola bar and now it looks like this:



The piece that Lucius at is _______ of the whole granola bar.

Session 6

Focus: Equipartitioning, teration of unit fraction

Materials:

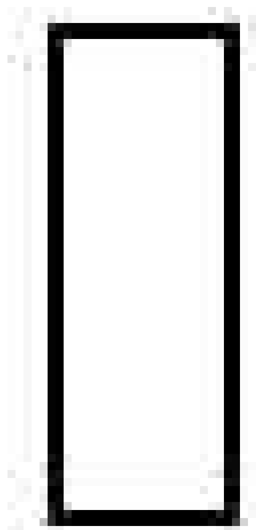
- Problems from RNP Lesson 4 Student Pages A & B

Activities:

- Activity 14 – Have students work with a word problem including graphic showing one-half of an object.
- Activity 15 - Have students work with a word problem including graphic showing one-fourth of an object.
- Activity 16 – Word problem asking students to generate a pictorial showing a rectangle in nine equal-sized pieces.
- Activity 17 – Word problem asking students to generate a pictorial showing a square in three equal-sized pieces

Activity 14

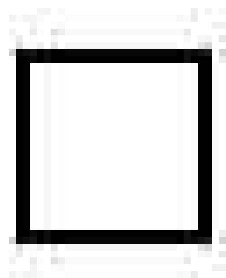
One-half of a lemon bar was left after a party. This is what is looked like:



Draw a picture of the whole cake. Explain to the group how you solved the problem.

Activity 15

Four kids shared a candy bar equally. Joy's share looked like this:



Draw a picture of the whole candy bar. Explain to the group how you solved the problem.

Activity 16

Oscar has a garden shaped like a rectangle. Draw a picture of Oscar's garden and show that the garden is in 9 equal-sized parts.

Activity 17

Lucy has a garden shaped like a square. Draw a picture of Lucy's garden and show that the garden is in 3 equal-sized parts.

Session 7

Focus: Equipartitioning, fraction equivalence

Materials:

- Cramer RNP Lesson 3 Student Pages A
- Fraction circles

Activities:

- Activity 18 – Ask students: "Jordan said 1 red is one-third, Andres said 1 red is one-fourth. Who is correct?" Both are correct: 1 red is one-third of blue and 1 red is one-fourth of brown. If students struggle to understand how both could be right, ask students "You have called these pieces (while showing yellow, blue, pink, & red) 1/2, yet they are all different sizes. How is this possible?"
- Activity 19 – Develop naming of fractions in a context that emphasizes the flexibility of the unit. Source: Cramer RNP Lesson 3 Student Page A

Activity 18

Jordan said that 1 red piece is one-third.

Andres said that 1 red piece is one-fourth.

Who is correct?

Activity 19

Naming Fraction Amounts
Using Circles

Use fraction circles to find the names of the different fraction pieces.

I. The black circle is the unit. What fraction name can you give these pieces?

1 yellow ___1-half___ 1 brown ________________

1 blue ________________ 1 gray ________________

1 white ________________ 1 green ________________

1 red ________________ 1 pink ________________

II. Now make 1 yellow unit. What fraction name can you give these pieces?

1 blue ________________ 1 gray ________________

1 pink ________________ 1 red ________________

III. Change the unit to 1 blue. What fraction name can you give these pieces?

1 gray ________________ 1 red ________________

IV. Change the unit to 1 orange. What fraction name can you give these pieces?

1 purple ________________ 1 green ________________

Session 8

Focus: Equipartitioning, identification of unit fraction, iteration of unit fraction, fraction equivalence

Materials:

- Cramer RNP Lesson 3 Student Pages A
- Fraction circles
- Cramer RNP Lesson 2 Large Group Instruction
- Cramer RNP Lesson 1 Transparency 1

Activities:

- Activity 19 (continued) – Develop naming of fractions in a context that emphasizes the flexibility of the unit. Source: Cramer RNP Lesson 3 Student Page A
- Activity 20 - Use fraction circles to have students work flexibly with differently defined units.
- Activity 21 - Introduce fraction circle manipulatives by exploring and comparing pieces and their sizes. The student will fully cover the shape on the left with any combination of shapes on the right that will work. Source: Cramer RNP Lesson 1 Transparency 1

The yellow piece is the unit.

How many blues cover the yellow piece?

1 blue is _________ of the yellow.

The blue piece is the unit.

How many reds cover the blue piece?

1 red is _________ of the blue.

The brown piece is the unit.

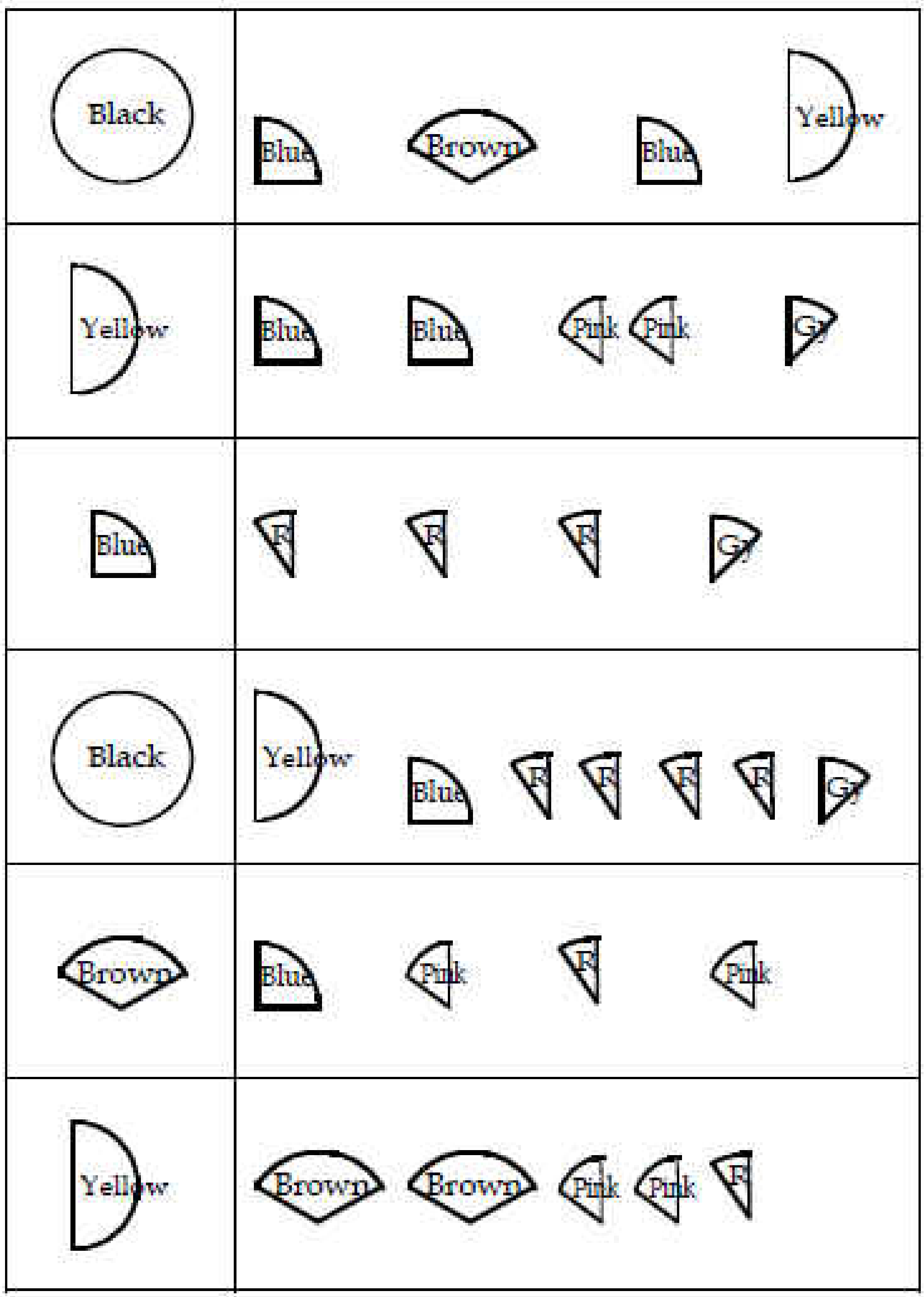
How many reds cover the brown piece?

1 red is _________ of the brown.

What color is 1-half of the blue?

What color is 1-third of the yellow?

Activity 21


Black
Brown
Yellow
Blue
Pink
Black
Yellow
Brown
Yellow
Brown
Brown
Pink
Pink

Session 9

Focus: Equipartitioning, unit fractions, iteration of unit fractions

Materials:

- Cramer RNP Lesson 2 Student Pages A & B
- Iteration of one-half word problem

Activities:

- Activity 22 - Students will work flexibly with differently defined units. The focus is on developing the idea that the definition of the unit is flexible (ie the circle may be the whole, or the half-circle may be the whole, or any piece or combination of pieces may be the whole). Students draw a pizza cut into two fair shares. Source: Cramer RNP Lesson 2 Student Pages A & B.
- Activity 23 - Students will work flexibly with differently defined units. The focus is on developing the idea that the definition of the unit is flexible (ie the circle may be the whole, or the half-circle may be the whole, or any piece or combination of pieces may be the whole). Students draw a circular patio cut into three equal parts. Source: Cramer RNP Lesson 2 Student Pages A & B.
- Activity 24 - Students will work flexibly with differently defined units. The focus is on developing the idea that the definition of the unit is flexible (ie the circle may be the whole, or the half-circle may be the whole, or any piece or combination of pieces may be the whole). Students draw a half-circular patio cut into three equal parts. Source: Cramer RNP Lesson 2 Student Pages A & B.
- Activity 25 - Have students solve a problem required iteration of one-half.

Activity 22

Draw a picture of a pizza. Show on your drawing the pizza cut into 2 fair shares.

Each fair share is _______ of the whole pizza.

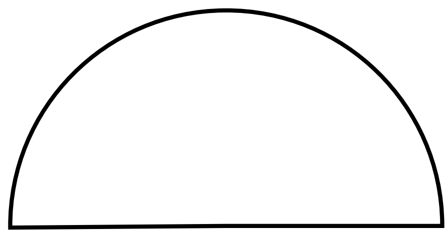
Activity 23

Mari's patio is a whole circle. Draw a picture of Mari's patio. Show on your drawing that the patio is in 3 equal sized parts.

Each part is ______ of Mari's patio.

Activity 24

Juan has a patio that looks like this:



Draw on Juan's patio to show it divided into 3 equal sized parts. Each part is ________ of Juan's patio.

Activity 25

Dani wants to feed each of the children she babysits a half sandwich for lunch. If she babysits 8 children, how many sandwiches should she make?

Session 10

Focus: Equipartitioning, fraction comparison

Materials:

- Cards with fractions written on the front
- Paper fraction strips
- Linear Fraction Kits

Activities:

- Activity 26 - Have students place cards with fractions written on them in the correct location between the cards for 0, ½, and 1, placed on a table with space in between.
    - Students place the cards for one-fourth, one-eighth, and one-fifth
- Activity 27 – Based on lack of success with card placement, students were instructed to use paper strips to fold then cut pieces to represent one-half, one-fourth, and one-eighth. These paper strip cuttings were compared to support reasoning about the relative sizes of the unit fractions.
- Activity 28 - Have students work with Linear Fraction Kits. Start in the fully covered configuration and have students remove pieces to discover which sets of removed pieces reveal a unit fraction and/or an equivalent fraction. Have students work with these manipulatives to play the game "Race to a Whole". Each student rolls the die then covers part of the "whole" strip with the fraction piece indicated by the die. Students take turns rolling their die until someone completes the whole. Source: Ortiz, Dimensions 34(2)

Session 11

Focus: Fraction equivalence

Materials:

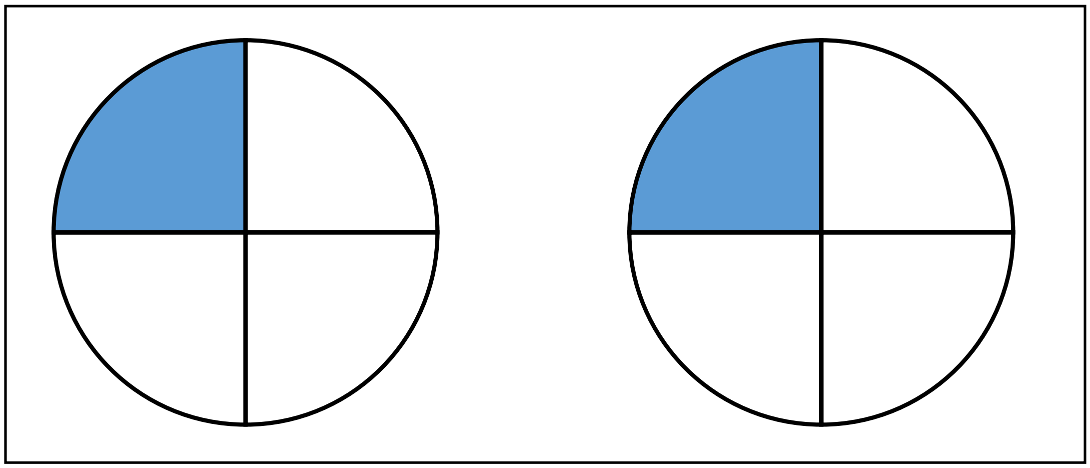
- Linear Fraction Kits
- 2 sets of double circle drawings where the first set shows each circle partitioned into quarters with lines and each circle has one part shaded, the second set shows each circle partitioned into halves with lines and the first circle has one part shaded
- Pairs of rectangles with shading

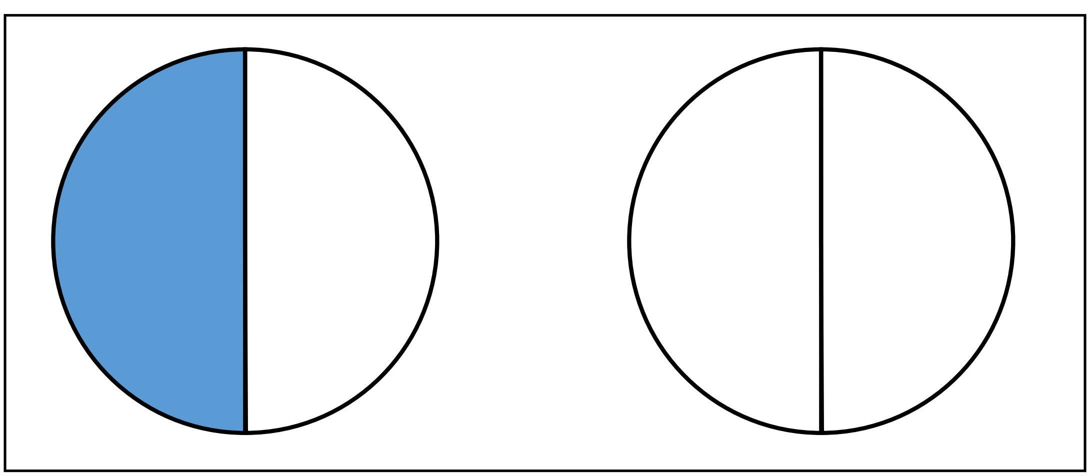
Activities:

- Activity 28 (continued) - Have students work with Linear Fraction Kits. Start in the fully covered configuration and have students remove pieces to discover which sets of removed pieces reveal a unit fraction and/or an equivalent fraction. Have students work with these manipulatives to play the game “Race to a Whole”. Each student rolls the die then covers part of the “whole” strip with the fraction piece indicated by the die. Students take turns rolling their die until someone completes the whole. Source: Ortiz, Dimensions 34(2)
- Activity 29 - Ask students to consider the 2 sets of double circles. How are they alike? How are they different? How are the shaded portions of each alike and different? Ask students to think about the circles as cookies which are cut in pieces where the shading represents the portion you may have. Which set would you choose to get your portion of cookie from and why?
- Activity 30 – Students compared pairs of rectangular figures with shading to determine if the pairs had equal areas shaded.

Activity 29

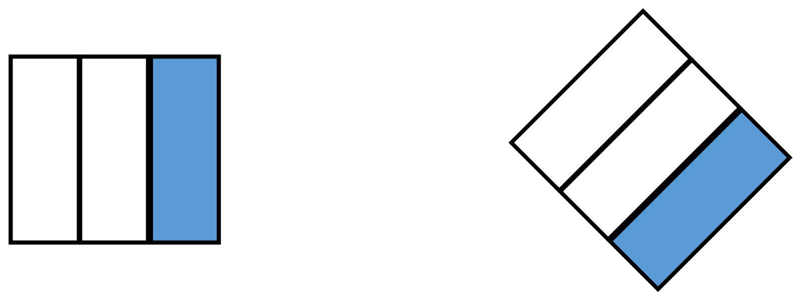
Which set of circles has more shaded?



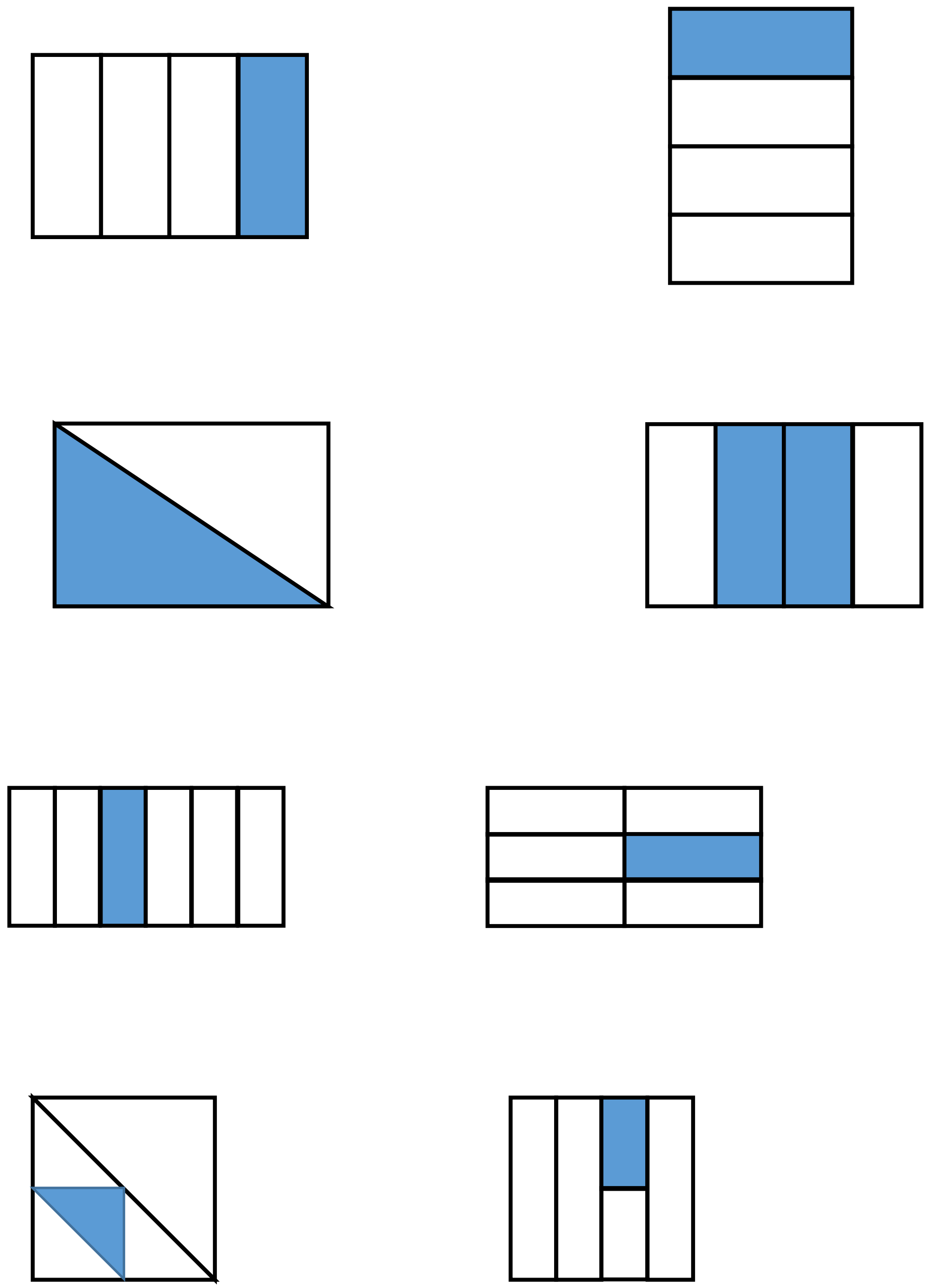


Activity 30

Are the shaded areas equal? Justify your answer.



Are the shaded areas equal? Justify your answer.



Session 12

Focus: Fraction equivalence

Materials:

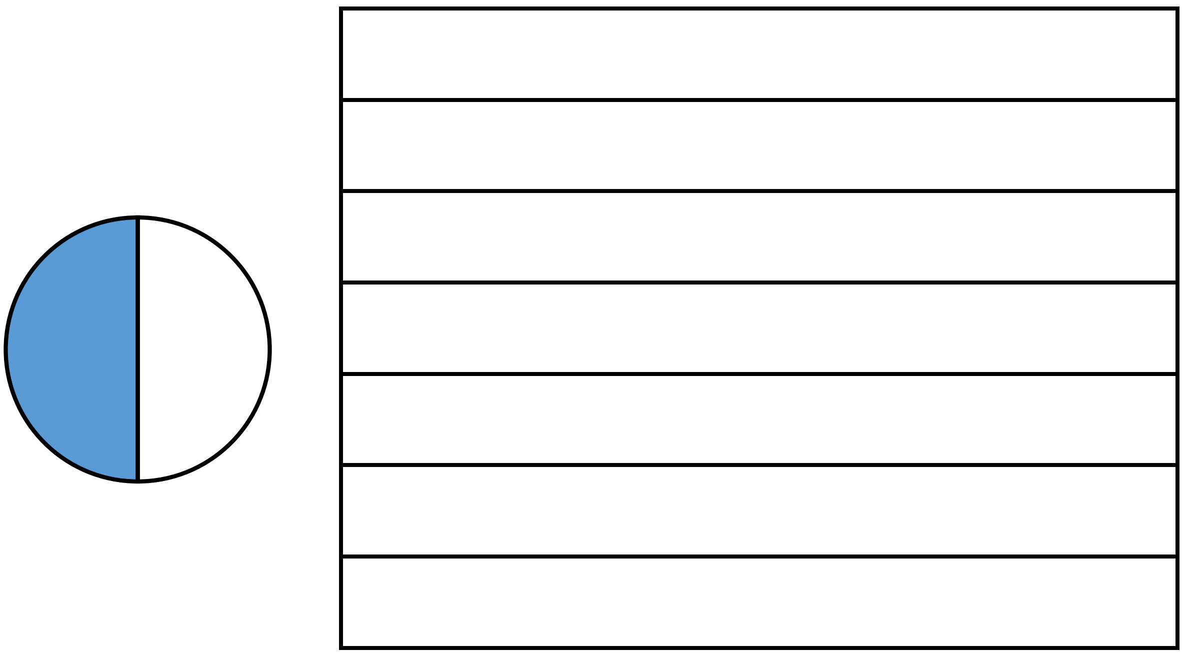
- Fraction circles
- Informal record sheet to record work with fraction circles

Activities:

- Activity 31 –
    - Cover 1 whole circle with a ½ circle and ask students to find ways to cover the remaining ½ of the circle. Record the students' answers by color. Source: Cramer RNP Lesson 9 large group instruction
    - Repeat the activity with the specification that the remaining ½ circle has to be covered with the same color. Record answers like this: 1 yellow is the same as 2 blues, 1 yellow is the same as 3 pinks. Ask students what each arrangement has in common. Look for responses along the lines of "they are all the same." Source: Cramer RNP Lesson 9 large group instruction

Activity 31

Let’s find different ways to cover half of a circle:



Session 13

Focus: Equipartitioning, unit fractions, fraction equivalence

Materials:

- Equivalent share word problems

Activities:

- Activity 32 - Have students solve an equivalency word problem where the equal share is less than one
- Activity 33 - Have students solve an equivalency word problem where the equal share is greater than one

Activity 32

A group of 3 children are sharing 2 burritos so that each child gets the same amount. How many burritos should 6 children share so that each child gets as much burrito as a child in the first group?

Activity 33

Sticker books are on sale. You can buy 2 sticker books for $3. You want to buy 6 sticker books. How much money do you need?

Session 14

Focus: Equipartitioning, unit fractions, fraction equivalence

Materials:

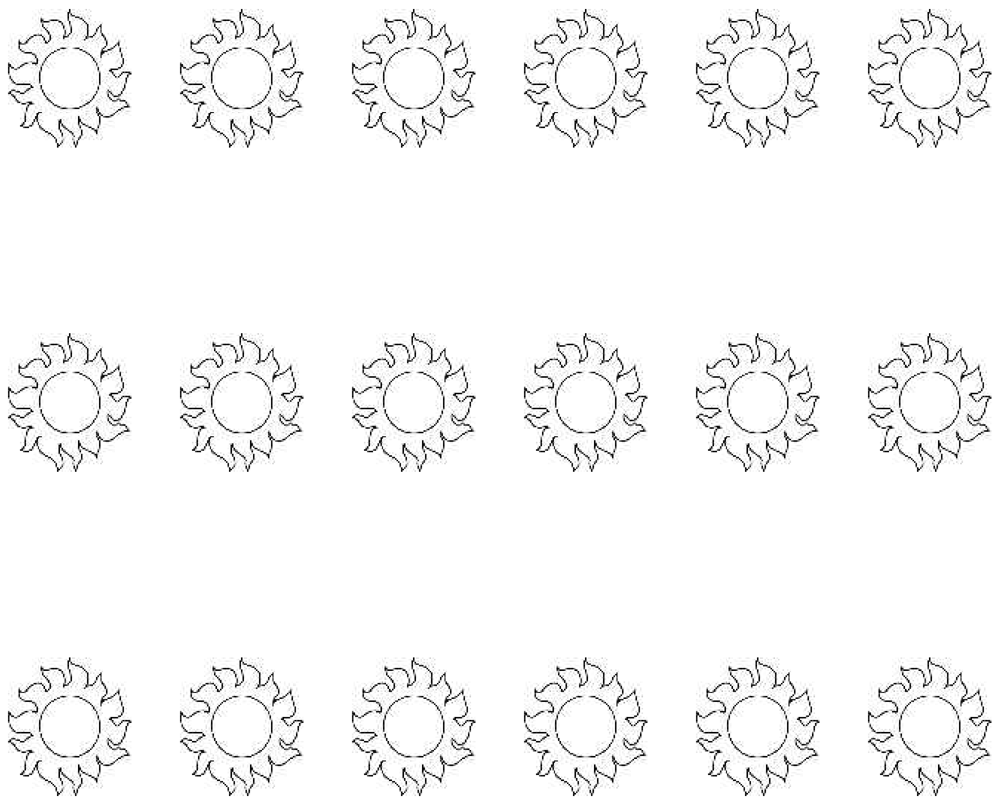
- Pictorial of 18 objects

Activities:

- Activity 34 - Have students examine a set of 18 objects and ask students what they see. Move on to asking specific fraction questions about the set models such as “Can you see thirds? (If needed, “can you break this set of objects into 3 groups?”) How many objects are in 2 groups of thirds, how many objects are 2/3 of the whole set? Focus on thirds, sixths, ninths, twelfths, and eighteenths.

Activity 34

Look at this picture, then let's answer some questions about it:



Session 15

Focus: Equipartitioning, unit fractions, iteration of unit fractions, fraction equivalence, fraction comparison

Materials:

- Cramer RNP Lesson 6 Student Page A
- Pairs of fractions for fraction comparison activity
- Fraction circles

Activities:

- Activity 35 - Have students compare 2 different fraction circle pieces to decide how many each requires to cover a whole circle, which color takes more pieces to cover the circle, and which color has smaller pieces. Source: Cramer RNP Lesson 6 Student Page A
- Activity 36 - Have students compare pairs of fractions to identify the larger fraction. Students should try to use reasoning to justify their selections. Reasoning proved challenging so fraction circles were used to provide a manipulative connection students could use to support their reasoning.

Activity 35

<u>Directions:</u> Use fraction circles to fill in the table.

Color	How many cover 1 whole circle?	Which color takes MORE pieces to cover 1 whole?	Which color has SMALLER pieces?
1. Brown	3		
Orange	5	√	√
2. Orange			
White			
3. Purple			
White			
4. Gray			
Green			
5. White			
Green			
6. Orange			
Purple			
7. Gray			
Brown			
8. Brown			
Green			

Activity 36

Which is fraction is larger?

$$\frac{3}{4} \quad \text{or} \quad \frac{3}{10}.$$

$$\frac{5}{7} \quad \text{or} \quad \frac{3}{7}$$

$$\frac{7}{8} \quad \text{or} \quad \frac{4}{5}$$

APPENDIX D:
DECISION LOG

Table 16

Decision Log

Sessions	Notes and Decisions
1-2	Redo Lianna and Jackie sharing problem; Use counters and have students cut a counter to share the last cookie.
2-3	More work is needed on equipartioning.
3-4	Students need to work on creating equal sized pieces. Some see need for equal sized pieces in pictorials and some do not. It's not clear if they will attend to equal sizes in drawing.
4-5	Need to move into iteration. Area fraction game can help with this while also bringing fraction equivalence into it.
5-6	Clay has problem connecting given pictorials to given context (Healthfirst problem). A simplified version of this problem might help (try the lemon bar problem).
6-7	Issues with seeing the whole inflexibly. Students need to understand that the whole can be defined differently and they need to look for that in every new task. Naming a unit fraction in conjunction with a defined whole is also an issue, especially for Clay.
7-8	Continue with activities coordinating the naming of the unit fraction to the definition of the whole. Fraction equivalence will also be addressed.
8-9	Work still needed on defining the whole. Also need to see if they will address equal sized pieces in their own drawings. Further develop iteration of unit fractions in real world context problems.
9	During session 9, Daniel did not use the vertical line cuts on the paper circles. During the session I decided to present the vertical lines on a circle so that the group could cut and discuss whether this generates equal size pieces.
9-10	Daniel still sees vertical lines as a way to create equal sized pieces in a circle. I think he knows he needs equal sized pieces but he doesn't see that this is not creating them. We need to cut paper circles and possibly squares too.
10	During session 10, using the card game that works like a number line was intriguing to the students but it was clear they were not thinking about sizes of unit fractions to compare fractions for placement on the line. So I decided to take a break with the card number line activity to work with cutting paper strips into unit fraction sizes so the students could see how more pieces cut from a whole made smaller pieces.
10-11	Unit fraction knowledge and piece size knowledge is at a point where we can move the focus to fraction equivalence although these concepts will continue to come up with tasks and be addressed.
11-12	Reasoning and discussion about equivalent fractions occurred in session 11. We will move to using manipulatives to further develop this concept in the next session. We will try to connect work with manipulatives to abstract concepts involving fraction names as well. We also worked with a linear model that can relate to number lines. This was very challenging work for the students and I need to find more ways to have them think about comparison of fractions.

Sessions	Notes and Decisions
12-13	Attempt to move on to real world contexts making use of fraction equivalence.
13-14	The students used primarily whole number knowledge to work context problems. Strategies were good but it might be that these tasks should have been used earlier. Attempt to extend fraction equivalence with work on set model tasks.
14-15	Work with set models was very successful. Defining the unit fraction in the set and iteration became a major focus of the task, so fraction equivalence was emphasized less than originally intended. We will not be able to continue to develop fraction equivalence with sets as we are nearing the end of our sessions. Also fraction comparison has been barely addressed. Will use the last session to work on fraction comparison with manipulatives and abstract problems.
15	The last task was designed to focus on abstract reasoning about fraction comparison by using the connection between the number of pieces in a whole and the size of the pieces. The students were struggling to construct this type of reasoning with the abstract tasks. So I decided manipulatives were needed as a visual aid to help students construct this reasoning.

REFERENCES

ACT, Inc. (2004). Crisis at the core: Preparing all students for college and work. Iowa City, IA: ACT Research Report Series.

Anghileri, J. (2005). Language, arithmetic, and the negotiation of meaning. *For the Learning of Mathematics, 21*(3), 10-14.

Anghileri, J. (2006). Scaffolding practices that enhance mathematics learning. *Journal of Mathematics Teacher Education, 9,* 33-52.

Anghileri, J. & Baron, S. (1998). Playing with the materials of study: Poleidoblocs. *Education 3-13, 27*(2), 57-64.

Boaler, J. (2015). *The elephant in the classroom: Helping children learn and love mathematics.* London: Souvenir Press.

Boston, M.D. (2012). Assessing instructional quality in mathematics. *The Elementary School Journal, 113*(1), 76-104.

Boston, M.D. & Smith, M.S. (2009). Transforming secondary mathematics teaching: Increasing the cognitive demands of instructional tasks used in teachers' classrooms. *Journal for Research in Mathematics Education, 40*(2), 119-156.

Brantlinger, E., Jimenez, R., Klinger, J., Pugach, M., & Richardson, V. (2005). Qualitative studies in special education. *Exceptional Children, 71*(2), 195-207.

Broza, O. & Kolikant, Y.B. (2015). Contingent teaching to low-achieving students in mathematics: Challenges and potential for scaffolding meaningful learning. *ZDM: The International Journal of Mathematics Education, 47*(7), 1093-1105.

Bruner, J. (1986). *Actual minds, possible worlds.* Cambridge, MA: Harvard University Press.

Bruner, J. & Ratner, N. (1978). Games, social exchange and the acquisition of language. *Journal of Child Language, 5*(1), 391-401.

Butler, D.L. (2006). Frames inquiry in educational psychology: Beyond the quantitative-qualitative divide. In P.A. Alexander & P.H. Winne (Eds.), *Handbook of educational psychology* (pp. 903-927). Mahwah, NJ: Lawrence Erlbaum Associates, Inc.

Butler, D.L., Buckingham, B. & Novak Lauscher, H.J. (2005). Promoting strategic learning by eighth-grade students struggling in mathematics: A report of three case studies. *Learning Disabilities Research & Practice, 20*(3), 156-174.

Butler, F.M., Miller, S.P., Crehan, K., Babbitt, B., & Pierce, T. (2003). Fraction instruction for students with mathematics disabilities: Comparing two teaching sequences. *Learning Disabilities Practice, 18*(2), 99-111.

Cambourne, B. (1988). *The whole story: Natural learning and the acquisition of literacy in the classroom.* Gosford, New Zealand: Ashton Scholastic.

Carnevale, A.P. & Desrochers, D.M. (2003). Standards for what?: The economic roots of k-16 reform. Princeton, NJ: Educational Testing Service.

Carpenter, T.P., Fennema, E., Peterson, P.L., Chiang, C. & Loef, M. (1989). Using knowledge of children's mathematics thinking in classroom teaching: An experimental study. *American Educational Research Journal, 26*(4), 499-531.

Carpenter, T.P., Franke, M.L., Jacobs, V.R., Fennema, E., & Empson, S.B. (1998). A longitudinal study of intervention and understanding in children's multi-digit addition and subtraction. *Journal for Research in Mathematics Education, 29*(1), 3-20.

Cazdan, C. (1983). Adult assistance to language development: Scaffolds, models, and direct instruction. In R. Parker & F.A. Davis (Eds.), *Developing literacy: Young children's use of language* (pp. 3-18). Newark, DE: International Reading Association.

Chapin, S.H., O'Connor, C. & Anderson, C.C. (2003). *Classroom discussions: Using math talk to help students learn: Grades 1-6.* Sausalito, CA: Math Solutions Publications.

Clements, D. & Sarama, J. (2004). Learning trajectories in mathematics education. *Mathematical Thinking and Learning, 6*(2), 81-89.

Cobb, P., Wood, T., Yackel, E., Nicholls, J., Wheatley, G., Trigatti, B., & Perlwitz, M. (1991). Assessment of a problem-centered, second-grade mathematics project. *Journal for Research in Mathematics Education, 22*(1), 3-29.

Cobb, P., Yackel, E., & Wood, T. (1992). A constructivist alternative to the representational view of mind in mathematics education. *Journal for Research in Mathematics Education, 23*(1), 2-33.

Coltman, P., Anghileri, J., & Petyaeva, D. (2002). Scaffolding learning through meaningful tasks and adult interaction. *Early Years, 22*(1), 39-49.

Confrey, J., Maloney, A.P., & Corley, A.K. (2014). Learning trajectories: A framework for connecting standards with curriculum. *ZDM Mathematics Education, 46,* 719-733.

Cramer, K.A., Behr, M., Post, T.R., & Lesh, R. (2009). *Rational number project: Initial fraction ideas* [PDF document]. Retrieved from The Rational Number Project site: http://www.cehd.umn.edu/ci/rationalnumberproject

Cramer, K.A., Post, T.R., & del Mas, R.C. (2002). Initial fraction learning by fourth- and fifth-grade students: A comparison of the effects of using commercial curricula with the effects of using the Rational Number Project curriculum. *Journal for Research in Mathematics Education, 33*(2), 111-144.

Creswell, J.W. (2007). *Qualitative inquiry and research design: Choosing among five approaches* (2nd ed.). Thousand Oaks, CA: Sage Publications.

Dale, R. & Scherrer, J. (2015). Goldilocks discourse: Math scaffolding that's just right. *Phi Delta Kappan, 97*(2), 58-61.

Dixon, J.K., Nolan, E.C., Adams, T.L., Tobias, J.M., & Barmoha, G. (2016). *Making sense of mathematics for teaching grades 3-5.* Bloomington, IN: Solution Tree Press.

Empson, S.B. (1999). Equal sharing and shared meaning: The development of fraction concepts in a first-grade classroom. *Cognition and Instruction, 17*(3), 283-342.

Empson, S.B. & Levi, L. (2011). *Extending children's mathematics: Fractions and decimals: Innovations in Cognitively Guided Instruction.* Portsmouth, NH: Heinemann.

Fletcher, J.M., Lyon, G.R., Fuchs, L.S., & Barnes, M.A. (2007). *Learning disabilities: From identification to intervention.* New York: The Guilford Press.

Flores, M.M. & Kaylor, M. (2007). The effects of a direct instruction program on the fraction performance of middle school students at-risk for failure in mathematics. *Journal of Instructional Psychology, 34*(2), 84-94.

Florida Department of Education. (n.d.a). Pk-12 public school data publications and reports. Retrieved from www.fldoe.org/accountability/data-sys/edu-info-accountability-services/pk-12-public-schools-data-pubs-reports/students.stml

Florida Department of Education. (n.d.b). Florida standards assessments results. Retrieved from www.fldoe.org/accountability/assessments/d-12-student-assessments/results

Fosnot, F.J. (1996). Constructivism: A psychological theory of learning. In C.J. Fosnot (Ed.), *Constructivism: Theory, perspectives, and practice* (pp. 8-33). New York: Teachers College Press.

Fuchs, L.S. & Fuchs, D. (2003). Enhancing the mathematical problem solving of students with mathematics disabilities. In H.L. Swanson, K.R. Harris, & S. Graham (Eds.), *Handbook of learning disabilities* (pp.306-322). New York: The Guilford Press.

Fuchs, L.S., Schumacher, R.F., Long, J., Nankung, J., Hamlett, C.L., Jordan, N.C., Gersten, R., Cirino, P.T., Siegler, R., & Changas, P. (2013). Improving at-risk learners' understanding of fractions. *Journal of Educational Psychology, 105*(3), 683-700.

Gamoran, A. & Hannigan, E.C. (2000). Algebra for everyone: Benefits of college-preparatory mathematics for students with diverse abilities in early secondary school. *Educational Evaluation and Policy Analysis, 22*(3), 241-254.

Geary, D.C. (1990). A componential analysis of an early learning deficit in mathematics. *Journal of Experimental Child Psychology, 49*(3), 363-383.

Geary, D.C. (2003). Learning disabilities in arithmetic: Problem-solving differences and cognitive deficits. In H.L. Swanson, K.R. Harris, & S. Graham (Eds.), *Handbook of learning disabilities* (pp.199-212). New York: The Guilford Press.

Gersten, R., Beckman, S., Clarke, B., Foegen, A., Marsh, L., Star, J., & Witzel, B. (2009). *Assisting students struggling with mathematics: Response to intervention (RtI) for elementary and middle schools* (NCEE 2009-4060). Washington, DC: National Center for Education Evaluation and Regional Assistance, Institute of Education Sciences, U.S.

Department of Education. Retrieved from: http://ies.ed.gov/ncee/wwc/publications/practiceguides/

Gersten, R., Chard, D.J., Jayanthi, M., Baker, S., Morphy, P., & Flojo, J. (2009). Mathematics instruction for students with learning disabilities: A meta-analysis of instructional components. *Review of Educational Research, 79*(3), 1202-1242.

Ginsburg, H.P. (1997). Mathematics learning disabilities: A view from developmental psychology. *Journal of Learning Disabilities, 30*(1), 20-33.

Ginsburg, H.P. (2009). The challenge of formative assessment in mathematics education: Children's minds, teachers' minds. *Human Development, 52,* 109-128.

Graves, D.H. (1983). *Writing: Teachers and children at work.* Portsmouth, NH: Heinemann Educational Books.

Hecht, S.A. & Vagi, K.J. (2010). Sources of group and individual differences in emerging fraction skills. *Journal of Educational Psychology, 102*(4), 843-859.

Henningsen, M. & Stein, M.K. (1997). Mathematical tasks and student cognition: Classroom-based factors that support and inhibit high-level mathematical thinking and reasoning. *Journal for Research in Mathematics Education, 28*(5), 524-549.

Hiebert, J. & Wearne, D. (1993). Instructional tasks, classroom discourse, and students' learning in second-grade arithmetic. *American Educational Research Journal, 30*(2), 393-425.

Houghton Mifflin Harcourt. (1989). *Woodcock Johnson-Revised Calculations Subtest.* Boston: Houghton Mifflin Harcourt.

Hughes, E.M., Witzel, B.S., Ricomini, P.J., Fries, K.M., & Kanyongo, G.Y. (2014). A meta-analysis of algebra interventions for learners with disabilities and struggling learners. *The Journal of the International Association of Special Education, 15*(1), 36-47.

Hunt, J.H. & Empson, S.B. (2015). Exploratory study of informal strategies for equal sharing problems of students with learning disabilities. *Learning Disability Quarterly, 38*(4), 208-220.

Imm, K. & Stylianou, D.A. (2012). Talking mathematically: An analysis of discourse communities. *The Journal of Mathematical Behavior, 31,* 130-148.

Jitendra, A.K., Dupuis, D.N., & Zaslofsky, A.F. (2014). Curriculum-based measurement and standards-based mathematics: Monitoring the arithmetic word problem-solving performance of third-grade students at risk for mathematics difficulties. *Learning Disability Quarterly, 37*(4), 241-251.

Jitendra, A.K., Griffin, C.C., McGoey, K., Gardill, M.C., Bhat, P., & Riley, T. (1998). Effects of mathematical word problem solving by students at risk or with mild disabilities. *The Journal of Educational Research, 91*(6), 345-355.

Joseph, L.M. & Hunter, A.D. (2001). Differential application of a cue card strategy for solving fraction problems: Exploring instructional utility of the cognitive assessment system. *Child Study Journal, 31*(2), 123-136.

Kroesbergen, E.H. & Van Luit, J.E.H. (2003). Mathematics interventions for children with special needs: A meta-analysis. *Remedial and Special Education, 24*(2), 97-114.

Lamon, S.J. (2010). *Teaching fractions and ratios for understanding: Essential content knowledge and instructional strategies for teachers* (2nd Ed.). New York: Routledge.

Larson, M.R. (2012). Will CCSSM matter in ten years?: Reflect and discuss. *Teaching Children Mathematics, 19*(2), 108-115.

Lesh, R., Post, T., & Behr, M. (1987). Representations and translations among representations in mathematics learning and problem solving. In C Janvier (Ed.), *Problems of representations in the teaching and learning of mathematics* (pp. 33-40). Hillsdale, NJ: Lawrence Erlbaum Associates, Inc.

Lewis, K.E. (2010). Understanding mathematical learning disabilities: A case study of errors and explanations. *Learning Disabilities: A Contemporary Journal, 8*(2), 9-18.

Lewis, K.E. (2014). Difference not deficit: Reconceptualizing mathematical learning disabilities. *Journal for Research in Mathematics Education, 45*(3), 351-396.

Louie, J., Brodesky, A., Brett, J., Yang, L., & Tan, Y. (2008). *Math education practices for students with disabilities and other struggling learners: Case studies of six schools in two Northeast and Islands Region States* (REL 2008 – No. 053). Washington, DC: U.S. Department of Education, Institute of Education Sciences, National Center for Education Evaluation and Regional Assistance, Regional Educational Laboratory Northeast and Islands. Retrieved from: http://ies.ed.gov/ncee/edlabs

Mazzocco, M.M.M. & Devlin, K.T. (2008). Parts and 'holes': Gaps in rational number sense among children with vs. without mathematical learning disabilities. *Developmental Science, 11*(5), 681-691.

Mazzocco, M.M.M., Myers, G.F., Lewis, K.E., Hanich, L.B., & Murphy, M.M. (2013). Limited knowledge of fraction representations differentiates middle school students with mathematics learning disabilities (dyscalculia) versus low mathematics achievement. *Journal of Experimental Child Psychology, 115,* 371-387.

Mehan,H. (1979). What time is it Denise? Asking known information questions in classroom discourse. *Theory into Practice, 28*(4), 285-294.

Merriam, S.B. (2009). *Qualitative research: A guide to design and implementation.* San Francisco, CA: Jossey-Bass.

Miles, M.B. & Huberman, A.M. (1994). *Qualitative data analysis: An expanded sourcebook* (2nd Ed.). Thousand Oaks, CA: Sage.

Moschkovich, J.N. (2015). Scaffolding student participation in mathematical practices. *ZDM: The International Journal on Mathematics Education, 47*(7), 1067-1078.

Moustakas, C. (1994). *Phenomenological research methods.* Thousand Oaks, CA: Sage.

Nathan, M.J., Eilam, B., & Kim, S. (2007). To disagree, we must also agree: How intersubjectivity structures and perpetuates discourse in a mathematical classroom. (Working Paper No. 2006-6). Retrieved from Wisconsin Center for Education Research site: http://www.wcer.wisc.edu/

Nathan, M.J. & Knuth, E.J. (2003). A study of whole classroom mathematical discourse and teacher change. *Cognition and Instruction, 21*(2), 175-207.

National Council of Teachers of Mathematics (NCTM). (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM.

National Governors Association Center for Best Practices, & Council of Chief State School Officers. (2010). Common core state standards for mathematics. Washington, DC: Retrieved from http://www.corestandards.org/Math/

National Research Council. (2001). *Adding it up: Helping children learn mathematics.* J. Kilpatrick, J. Swafford, & B. Findell (Eds.). Mathematics Learning Study Committee, Center for Education, Division of Behavioral and Social Sciences and Education. Washington, DC: National Academy Press.

Ninio, A. & Bruner, J. (1978). The achievement and antecedents of labelling. *Journal of Child Language, 5*(1), 1-15.

Ortiz, E. (2014). The development of fraction ideas and the use of learning levels (part 1). *Dimensions, 34*(2), 17-26.

Pfister, M., Moser Opitz, E., & Pauli, C. (2015). Scaffolding for mathematics teaching in inclusive primary classrooms: A video study. *ZDM: The International Journal of Mathematics Education, 47*(7), 1079-1092.

Powell, S.R., Fuchs, L.S., & Fuchs, D. (2013). Reaching the mountaintop: Addressing the common core standards in mathematics for students with mathematics difficulties. *Learning Disabilities Research & Practice, 28*(1), 38-48.

Putambekar, S. & Hubscher, R. (2005). Tools for scaffolding students in a complex learning environment: What have we gained and what have we missed? *Educational Psychologist, 40*(1), 1-12.

Saldana, J. (2013). *The coding manual for qualitative researchers* (2nd Ed.). Los Angeles, CA: Sage.

Siegler, R.S., Duncan, G.J., Davis-Kean, P.E., Duckworth, K., Claessens, A., Engel, M., Susperreguy, M.E., & Chen, M. (2012). Early predictors of high school mathematics achievement. *Association for Psychological Science, 23*(7), 691-697.

Stein, M.K., Engle, R.A., Smith, M.S., & Hughes, E.K. (2008). Orchestrating productive mathematical discussions: Five practices for helping teacher move beyond show and tell. *Mathematical Thinking and Learning, 10*(4), 313-340.

Stein, M.K., Grover, B.W., & Henningsen, M. (1996). Building student capacity for mathematical thinking and reasoning: An analysis of mathematical tasks used in reform classrooms. *American Educational Research Journal, 33*(2), 455-488.

Stein, M.K., Remillard, J., & Smith, M.S. (2007). How curriculum influences student learning. In F.K. Lester, Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 319-369). Charlotte, NC: Information Age Publishing.

Stigler, J.W. & Hiebert, J. (2004). Improving mathematics teaching. *Educational Leadership, 61*(5), 12-17.

Stone, C.A. (1998a). Should we salvage the scaffolding metaphor? *Journal of Learning Disabilities, 31*(4), 409-413.

Stone, C.A. (1998b). The metaphor of scaffolding: Its utility for the field of learning disabilities. *Journal of Learning Disabilities, 31*(4), 344-364.

Tharpe, R., & Gallimore, R. (1988). *Rousing minds to life: Teaching, learning, and schooling in social context.* Cambridge: Cambridge University Press.

Van de Pol, J., Volman, M., & Beishuizen, J. (2010). Scaffolding in a teacher-student interaction: A decade of research. *Educational Psychology Review, 22*(3), 271-296.

Vygotsky, L. (1934/1986/2012). *Thought and language.* (E. Hanfmann, G. Vakar, & A. Kozulin, eds. and trans.). Cambridge, MA: The MIT Press.

Vygotsky, L. S. (1930-1934/1978). *Mind in society: The development of higher psychological processes* (M. Cole, V. John-Steiner, S. Scribner & E. Souberman., Eds.) (A. R. Luria, M. Lopez-Morillas & M. Cole [with J. V. Wertsch], Trans.) Cambridge, Mass.: Harvard University Press. (Original manuscripts [ca. 1930-1934])

Wilhelm, A.G. (2014). Mathematics teachers' enactment of cognitively demanding tasks: Investigating links to teachers' knowledge and conceptions. *Journal for Research in Mathematics Education, 45*(5), 636-674.

Witzel, B.S. (2005). Using CRA to teach algebra to students with math difficulties in inclusive settings. *Learning Disabilities: A Contemporary Journal, 3*(2), 49-60.

Wood, D., Bruner, J., & Ross, G. (1976). The role of tutoring in problem solving. *Journal of Child Psychology and Psychiatry, 17,* 89-100.

Wood, T. (1994). Patterns of interaction and culture of mathematics classrooms. In S. Lerman (Ed.), *Cultural perspectives on the mathematics classroom* (pp. 149-168). Boston: Kluwer.

Yin, R.K. (2003). *Case study research: Design and methods* (3rd ed.). Thousand Oaks, CA: Sage.

Zhang, D. & Xin, X.P. (2012). A follow-up meta-analysis for word-problem-solving interventions for students with mathematics difficulties. *The Journal of Educational Research, 105,* 303-318.