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ON THE THEORY OF ZETA-FUNCTIONS AND L -FUNCTIONS

by

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B.S. University of Central Florida, 2012

A thesis submitted in partial fulfillment of the requirements
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ABSTRACT

In this thesis we provide a body of knowledge that concerns Riemann zeta-function and its generalizations in a cohesive manner. In particular, we have studied and mentioned some recent results regarding Hurwitz and Lerch functions, as well as Dirichlet's L -function. We have also investigated some fundamental concepts related to these functions and their universality properties. In addition, we also discuss different formulations and approaches to the proof of the Prime Number Theorem and the Riemann Hypothesis. These two topics constitute the main theme of this thesis. For the Prime Number Theorem, we provide a thorough discussion that compares and contrasts Norbert Wiener's proof with that of Newman's short proof. We have also related them to Hadamard's and de la Vallee Poussin's original proofs written in 1896. As far as the Riemann Hypothesis is concerned, we discuss some recent results related to equivalent formulations of the Riemann Hypothesis as well as the Generalized Riemann Hypothesis.

keywords: Riemann zeta function, Hurwitz zeta function, L -functions, Dedekind zeta function, Universality, Prime Number Theorem, Riemann Hypothesis, Generalized Riemann Hypothesis, Analytic Number Theory, Special Functions.

This thesis is lovingly dedicated to my parents and family. Their support, encouragement, and constant love have sustained me throughout my life.

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TABLE OF CONTENTS

LIST OF FIGURES	ix
NOTATION AND CONVENTIONS	x
CHAPTER 1: INTRODUCTION	1
1.1 Historical Background	1
1.2 Generalizations of the Riemann Zeta-Function	4
1.2.1 Hurwitz Zeta-Function	4
1.2.2 Lerch Zeta-Function	6
1.2.3 Dirichlet L -Functions	11
1.2.4 Dedekind Zeta Function	13
1.2.5 Euler's Product	15
CHAPTER 2: THE UNIVERSALITY PROPERTY	20
2.1 Introduction	20
2.2 Universality Theorem for the Riemann Zeta-Function	20
2.3 Universality Theorem for the Hurwitz Zeta-Function and Lerch Zeta-Function	21

2.4	Universality Theorem for L -Functions	22
2.5	Discussion	28
CHAPTER 3: THE PRIME NUMBER THEOREM		29
3.1	Historical Background	29
3.2	Equivalent Formulations of the Prime Number Theorem	31
3.3	An Analytic Proof of the Prime Number Theorem	34
3.3.1	Preliminary Results	35
3.3.2	Contour Integral Representation of $\Phi(x)/x^2$	37
3.3.3	Bounds on $ \zeta(s) $ and $ \zeta'(s) $ near $\Re(s) = 1$	41
3.3.4	Non-vanishing of $\zeta(s)$ on the Line $\Re(s) = 1$	42
3.3.5	Finalizing the Proof	45
3.4	Tauberian Proofs of the Prime Number Theorem	49
3.4.1	Introduction	49
3.4.2	Wiener's Proof of the Prime Number Theorem	50
3.4.3	Newman's Proof of the Prime Number Theorem	55
3.5	Discussion	59
CHAPTER 4: THE RIEMANN HYPOTHESIS		61

4.1	Introduction	61
4.2	Equivalent Formulations of the Riemann Hypothesis	64
4.2.1	Formulation in Terms of the Divisor Function	64
4.2.2	Derivative of the Riemann Zeta-Function	64
4.2.3	An Integral Equation Related to the Riemann Hypothesis	65
4.3	Equivalent Formulations of the Generalized Riemann Hypothesis	67
4.4	Discussion	68
CHAPTER 5: Future Work		69
LIST OF REFERENCES		71

LIST OF FIGURES

Figure 1.1: (a) Initial contour at angle $\varphi = 0$. Hankel's contour: (b) The case when $\varphi \in \{\alpha \in (-\pi/2, \pi/2), \text{Arg}(a) + \alpha < \pi/2\}$	7
Figure 3.1: Contour Integral	38
Figure 3.2: Contour Integral for \mathbf{R}	47
Figure 3.3: Newman's Contour Integral	56

NOTATION AND CONVENTIONS

$\Re(s)$	Real part of a complex number $s = \sigma + it$, here $\sigma = \Re(s)$ unless stated otherwise.
$\Im(s)$	Imaginary part of a complex number $s = \sigma + it$ here $t = \Im(s)$ unless stated otherwise.
$\text{Res}(f, c)$	The residue of the function f at c .
\sum	The infinite summation $\sum_{n=0}^{\infty}$, otherwise the limits will be specified.
\prod	The infinite product $\prod_{n=0}^{\infty}$, otherwise the limits will be specified.
$\Gamma(z)$	The gamma function.
$\chi(n)$	Dirichlet character.
$L(s, \chi)$	Dirichlet L -function.
$Li_s(z)$	The Polylogarithm function.
$\zeta(s)$	Riemann zeta-function.
$\zeta(s, a)$	Hurwitz zeta-function.
$\zeta_K(s)$	Dedekind zeta-function.
$\Phi(z, s, a)$	Lerch Transcendent.
$\phi(z, s, a)$	Lerch zeta-function.
$\chi(n)$	Dirichlet character.
$\pi(x)$	The prime counting function.
$\mu(n)$	Möbius function.
$\Lambda(n)$	von Mangoldt function.
$\vartheta(n)$	Chebychev's ϑ -function.
$\psi(n)$	Chebychev's ψ -function.
$\phi(n)$	Euler's totient function.
$\omega(n)$	Number of distinct prime factors of n .
$\Omega(n)$	Total number of prime factors of n .
$\sigma_x(n)$	Sum of the x th powers of the positive divisors of n .

$d(n)$	Number of divisors of n , and is related to the sum of divisor function by $d(n) = \sigma_0(n)$.
$\sigma(n)$	Sum of divisors of n , commonly written without subscript $\sigma(n) = \sigma_1(n)$.
$\vartheta(\tau, y)$	Jacobi Theta Function.
$\tau(n)$	Ramanujan tau function.
$f \sim g$	f, g are asymptotically equal or $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = 1$.
$f(x) = O(g(x))$	Big O notation.
$N(T)$	Number of zeros of the $\zeta(s)$ function in the region $0 \leq \sigma \leq 1, 0 < t \leq T$.

CHAPTER 1: INTRODUCTION

1.1 Historical Background

Number Theory is one of the oldest branches of mathematics. Gauss, one of the greatest mathematicians of all times, referred to it as the queen of mathematics. In the past decade, many mathematicians have realized the power of complex analysis and its applications to number theory, the two fields united into what is now known as Analytic Number Theory, which among other things, is the study of the distribution of prime numbers.

Prime numbers are exotic in nature, and had battled the brightest mathematicians for a very long time. One of whom is Riemann, who studied the distribution of primes using his zeta function of a complex variable. Recently, important results have been developed from analytic and geometric points of view. These advances brought new breakthroughs, solve longstanding problems, and raised new questions in number theory.

With the recent solution of the Fermat's Last Theorem, which involved collaborations from different branches of mathematics, many mathematicians appreciated the fascinating mathematics that have been developed in the process. Yet today, there are still many unsolved problems in number theory. One of which is the Riemann Hypothesis. Mathematicians have speculated that the mathematics and the tools which are needed to rigorously prove it has not been developed yet. In what follows, we examine some results related to the zeta functions and its generalizations in order to understand the theory better.

Although the connection between complex analysis and number theory at first may not appear very obvious, the full force of the machinery and tools provided became evident specifically in the past two decades. Thanks the work of Riemann, who bridged the gap by first looking at the Euler's

formula and considering complex arguments. He then forced the function to be holomorphic by defining it as a contour integral then deforming this contour to allow the function to attain further complex values using the process known as *analytic continuation*. After showing that ζ can be defined over all \mathbb{C} except for a simple pole at $s = 1$, many results can be derived from the formula; For instance, one can see that different constructions of ratios and multiples of Riemann zeta-function constitute the formal generating functions for most *Arithmetic Functions* [21] (Pg. 171). This can be seen from the following famous identities:

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}, \quad \Re(s) > 1 \quad (1.1.1)$$

$$\frac{\zeta'(s)}{\zeta(s)} = - \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}, \quad \Re(s) > 1 \quad (1.1.2)$$

$$\frac{\zeta(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^s}, \quad \Re(s) > 1 \quad (1.1.3)$$

$$\frac{\zeta(2s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s}, \quad \Re(s) > 1 \quad (1.1.4)$$

$$\zeta^2(s) = \sum_{n=1}^{\infty} \frac{d(n)}{n^s}, \quad \Re(s) > 1 \quad (1.1.5)$$

$$\zeta^k(s) = \sum_{n=1}^{\infty} \frac{d_k(n)}{n^s}, \quad \Re(s) > 1 (k \geq 2) \quad (1.1.6)$$

$$\frac{\zeta^2(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{2^{\omega(n)}}{n^s}, \quad \Re(s) > 1 \quad (1.1.7)$$

$$\frac{\zeta^3(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{d(n^2)}{n^s}, \quad \Re(s) > 1 \quad (1.1.8)$$

$$\frac{\zeta^4(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{d^2(n)}{n^s}, \quad \Re(s) > 1 \quad (1.1.9)$$

$$\frac{\zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)}{\zeta(2s-a-b)} = \sum_{n=1}^{\infty} \frac{\sigma_a(n)\sigma_b(n)}{n^s}, \quad (1.1.10)$$

$$\Re(s) > \max\{1, \Re(a) + 1, \Re(b) + 1, \Re(a + b) + 1\}$$

$$\frac{\zeta(s-1)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s}, \quad \Re(s) > 2 \quad (1.1.11)$$

$$\zeta(s)\zeta(s-k) = \sum_{n=1}^{\infty} \frac{\sigma_k(n)}{n^s}, \quad \Re(s) > \max\{1, \Re(k) + 1\} \quad (1.1.12)$$

$$\frac{1 - 2^{1-s}}{1 - 2^{-s}} \zeta(s-1) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}, \quad \Re(s) > 2 \quad (1.1.13)$$

It is worth mentioning that identities (1.1.9) and (1.1.10) are discovered by Ramanujan. For the proofs and derivations, please refer to Hardy and Wright [27] (Pg. 318-341). Now we shall begin studying some preliminary results regarding the theory of Riemann zeta-function and its generalizations.

1.2 Generalizations of the Riemann Zeta-Function

1.2.1 Hurwitz Zeta-Function

The Zeta function proposed by Riemann was studied intensively in Number Theory. For the avid reader who is interested in a through study of the Riemann zeta-function we refer to Titchmarsh's *The Theory of the Riemann zeta-Function* [53] and Apostol's *Introduction to Analytic Number Theory* [3]. Here we give a brief introduction discussing the Hurwitz zeta-function, named after the German mathematician Adolf Hurwitz (1859 - 1919), who studied it intensively. It is defined by:

$$\zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} \quad (\Re(s) > 1, \Re(a) > 0) \quad (1.2.1)$$

One reason to study this *shifted* form is that it simplifies studying L -functions as we shall see later.

Fortunately, an analytic continuation exists for $\zeta(s, a)$ and takes the following form [53]

$$\zeta(s, a) = \frac{e^{-i\pi s} \Gamma(1-s)}{2\pi i} \int_C \frac{z^{s-1} e^{-az}}{1-e^{-z}} dz \quad (1.2.2)$$

this analytic continuation is valid for $\mathbb{C} \setminus \{1\}$ where $s = 1$ is a simple pole and residue 1. This fact is reflected in the following Laurent expansion [15]:

$$\zeta(s, a) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma_n(a) (s-1)^n \quad (1.2.3)$$

Where $\gamma_k(a)$ are Stieltjes constants and $\gamma_0(a) = -\psi(a)$ where $\psi(z) = \Gamma'(z)/\Gamma(z)$. The proof of the analytic continuation property follows the same steps as in the proof of analytic continuation of the Riemann zeta-function and can be found in Titchmarsh [53] (Pg. 37). The conclusion that the Hurwitz zeta-function is analytic in the whole complex s -plane with the exception of a simple pole

at $s = 1$ provides an important step in studying its properties and gives rise to many derivations, which include the following Hermite integral representation as a direct application of Abel-Plana formula [1]:

$$\zeta(s, a) = \frac{a^{-s}}{2} + \frac{a^{1-s}}{s-1} + 2 \int_0^\infty \frac{\sin(s \arctan(z/a))}{(z^2 + a^2)^{s/2}(e^{2\pi z} - 1)} dz, \quad s \neq 1, \quad \Re(a) > 0 \quad (1.2.4)$$

This can be used for numerical evaluations of the function. Moreover, from the original definition and Dirichlet series one can find:

$$\zeta(s, a) = \frac{\Gamma(1-s)i}{(2\pi)^{1-s}} \left(e^{-\pi is/2} Li_{s-1}(e^{2\pi ia}) - e^{\pi is/2} Li_{s-1}(e^{-2\pi ia}) \right) \quad (1.2.5)$$

Recall that the polylogarithm function is defined by:

$$Li_s(z) = \sum_{n=0}^{\infty} \frac{z^n}{n^s} \quad (1.2.6)$$

Assuming a rational argument $a = p/q$ and rearranging terms, one can show that the polylogarithm can be decomposed into a linear combination of Hurwitz zeta-functions as follows:

$$Li_{s-1}(e^{2\pi ip/q}) = q^{-s} \sum_{n=1}^q e^{2\pi inp/q} \zeta(s, n/q) \quad (1.2.7)$$

And therefore a multiplication formula analogous to that of Riemann zeta-function can be obtained:

$$\zeta(1-s, p/q) = \frac{2\Gamma(s)}{(2\pi q)^s} \sum_{k=1}^q \cos\left(\frac{\pi s}{2} - \frac{2\pi kp}{q}\right) \zeta(s, p/q), \quad (1 \leq p \leq q, s \neq 0) \quad (1.2.8)$$

Using this, V.S. Adamchik [1] (2006) was able to derive some special values of the form $\zeta(2n + 1, p/q)$ which he used to give formulas for Glaisher's numbers (which are analogous to Eulerian numbers for counting the number of permutations). For more properties and identities refer to Cohen [16].

1.2.2 Lerch Zeta-Function

First we discuss the *Lerch transcendent* function, which is named after the Czech mathematician Mathias Lerch (1860 - 1922). This function is a generalization of both the Hurwitz zeta function and the polylogarithm function $Li_s(z)$. The Lerch transcendent is defined by:

$$\Phi(z, s, a) = \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s} \quad (|z| < 1, \Re(s) > 1 \text{ and } \Re(a) > 0) \quad (1.2.9)$$

The Lerch transcendent function reduces to the Hurwitz zeta function when $z = 1$ and the polylogarithm functions which can be demonstrated in the following relations:

$$\begin{aligned} \Phi(1, s, a) &= \zeta(s, a) \\ z \Phi(z, s, 1) &= Li_s(z) \\ \Phi(1, s, 1) &= \zeta(s, 1) = \zeta(s) \end{aligned} \quad (1.2.10)$$

The Lerch transcendent function appears occasionally in particle physics and thermodynamics to represent certain distributions such as the Bose-Einstein Distribution. It satisfies the following identity [20] (Pg. 27):

$$\Phi(z, s, a) = z^n \Phi(z, s, a+n) + \sum_{k=0}^{n-1} \frac{z^k}{(k+a)^s} \quad (1.2.11)$$

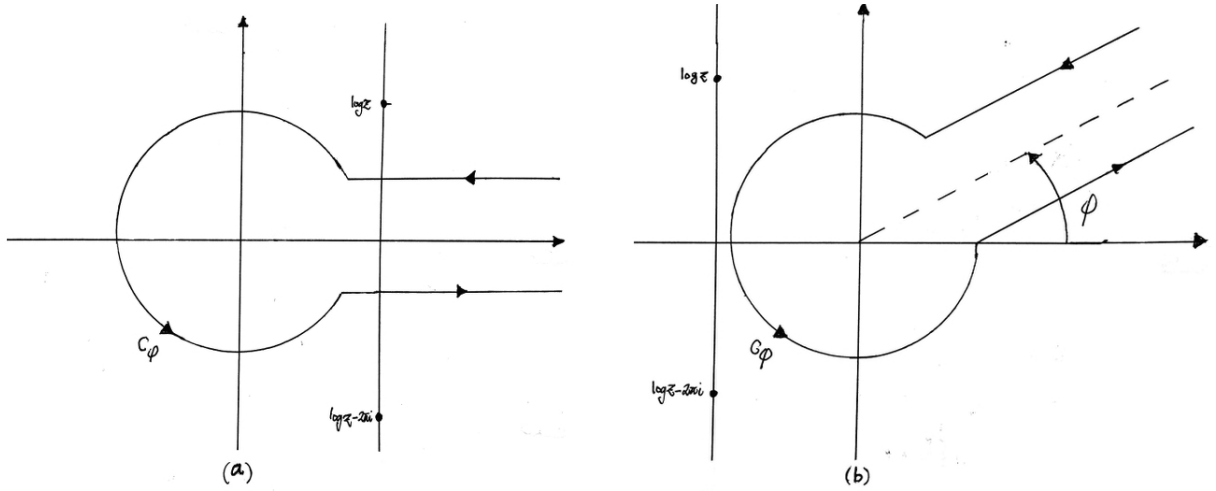


Figure 1.1: (a) Initial contour at angle $\varphi = 0$. Hankel's contour: (b) The case when $\varphi \in \{\alpha \in (-\pi/2, \pi/2), |\text{Arg}(a) + \alpha| < \pi/2\}$

An analytic continuation for z is given by [20] (Pg. 27) :

$$\Phi(z, s, a) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1} e^{-at}}{1 - ze^{-t}} dt \quad (\Re(a) > 0 \text{ for } |z| \leq 1, \Re(s) > 0, z \notin [0, \infty)) \quad (1.2.12)$$

This can be extended to an analytic continuation in both s and a ;

Let

$$\Omega_a = \begin{cases} \mathbb{C} \setminus [1, \infty) & \text{if } \Re(a) > 0 \\ \{z \in \mathbb{C}, |z| < 1\} & \text{if } \Re(a) \leq 0 \end{cases}$$

And let $z \in \Omega_a$, $a \in \mathbb{C} \setminus \mathbb{R}^-$ and $s \in \mathbb{C}$ hence

$$\Phi(z, s, a) = G(s) \int_{C_\varphi} \frac{t^{s-1} e^{-at}}{1 - ze^{-t}} dt \quad (1.2.13)$$

where:

$$\begin{aligned} \text{(i). } G(s) &:= \begin{cases} \frac{1}{\Gamma(s)} & \text{if } \Re(s) > 0 \\ \frac{i\Gamma(1-s)e^{i\pi(1-s)}}{2\pi} & \text{if } s \notin \mathbb{N} \end{cases} \\ \text{(ii). } C_\varphi &:= \begin{cases} [0, \infty e^{i\varphi}) & \text{if } \Re(s) > 0 \\ \mathcal{L}_\varphi & \text{if } s \notin \mathbb{N} \end{cases} \\ \text{(iii). } &\begin{cases} \varphi = 0 & \text{if } \Re(a) > 0 \\ \varphi \in \{\alpha \in (-\pi/2, \pi/2), |\text{Arg}(a) + \alpha| < \pi/2\} & \text{if } \Re(a) \leq 0 \end{cases} \end{aligned}$$

where \mathcal{L}_φ is the Hankel contour defined in Figure 1.1. The proof is similar to Hurwitz zeta-function's analytic continuation which deploys Hankel contour, See [24] (2002) for details.

A series involving the Lerch transcendent is given by:

$$\sum_{n=2}^{\infty} \frac{(s)_k}{k!} \Phi(z, s+k, a) t^k = \Phi(z, s, a-t) \quad (|t| < |a|, s \neq 1) \quad (1.2.14)$$

Where $(s)_k$ is the Pochhammer symbol. Choi and Sirvastava [14] (2005) published the following

identity as a generalization:

$$\begin{aligned}
\sum_{n=2}^{\infty} \Phi(z, k, a) \frac{t^{n+k}}{(k)_{n+1}} &= \frac{(-1)^n}{n!} \left[\Phi'(z, -n, a-t) - \Phi'(z, -n, a) \right] \\
&+ \sum_{k=1}^n \frac{(-1)^{n+k}}{n!} \binom{n}{k} \left[(H_n - H_{n-k}) \Phi(z, k-n, a) - \Phi'(z, k-n, a) \right] t^k \\
&+ \left[H_n V_1(z, a) - V_2(z, a) \right] \frac{t^{n+1}}{(n+1)!} \quad (|t| < |a|, |z| < 1)
\end{aligned} \tag{1.2.15}$$

Where H_n is the harmonic number and $V_1(z, a)$ and $V_2(z, a)$ are defined by:

$$V_1(z, a) := \lim_{s \rightarrow -n} (s+n) \Phi(z, s+n+1, a)$$

and

$$V_2(z, a) := \lim_{s \rightarrow -n} \left[\Phi(z, s+n+1, a) + (s+n) \Phi'(z, s+n+1, a) \right]$$

Now let $z \mapsto e^{2n\pi iz}$ then define:

$$\phi(z, s, a) = \sum_{n=0}^{\infty} \frac{e^{2n\pi iz}}{(n+a)^s} \quad (|z| < 1, \Re(s) > 1 \text{ and } \Re(a) > 0) \tag{1.2.16}$$

This function is the *Lerch zeta-Function* which was studied by Lipschitz and Lerch [20]. Not to be confused with the previous form. Sometimes referred to in the literature as the *Hurwitz-Lerch zeta-function*. One thing to note here is that it is periodic. Lerch gave the following functional equation based on one of Riemann's proofs of his zeta functional equation:

$$\phi(z, 1-s, a) = \frac{\Gamma(s)}{(2\pi)^s} \left[e^{\pi i(s/2-2az)} \phi(z, s, -a) + e^{\pi i(-s/2+2a(1-z))} \phi(1-z, s, a) \right] \tag{1.2.17}$$

For $a \in (0, 1)$ and $z \in (0, 1)$. The derivation is technical. However, an elegant proof of the

functional equation was given later by Apostol in his paper using Jacobi Theta functions [2] [13].

Namely:

$$\vartheta_3(y, \tau) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau + 2i n y} \quad (1.2.18)$$

Which satisfies the functional equation:

$$\vartheta_3(y, \tau) = (-i\tau)^{-1/2} e^{y^2/\pi i \tau} \vartheta_3(y/\tau, -1/\tau) \quad (1.2.19)$$

He exploited an identity from another Riemann's proof of his zeta functional equation [53] (Pg. 21

Third Method):

$$(\pi)^{-s/2} \Gamma\left(\frac{s}{2}\right) \sum_{n=1}^{\infty} a_n f_n^{-s/2} = \int_0^{\infty} z^{s/2-1} \sum_{n=1}^{\infty} a_n e^{-\pi z f_n} dz \quad (1.2.20)$$

Along with the differential-difference equations

$$\frac{\partial \phi(z, s, a)}{\partial a} = -s \phi(z, s + 1, a) \quad (1.2.21)$$

and

$$\frac{\partial \phi(z, s, a)}{\partial z} + 2\pi i a \phi(z, s, a) = 2\pi i \phi(z, s - 1, a) \quad (1.2.22)$$

In order to arrive at (1.2.17).

1.2.3 Dirichlet L-Functions

First we start with a two definitions:

Definition 1.2.1. *In the theory of Multiplicative Functions, we say that complex-valued arithmetic function f defined on the positive integers is **Multiplicative** iff*

$$f(nm) = f(n)f(m) \quad \text{whenever} \quad \gcd(n, m) = 1 \quad (1.2.23)$$

*In the case where (1.2.23) is true for all n and m we say that f is **Completely Multiplicative**.*

Definition 1.2.2. *Let k be an integer, then the **Dirichlet character modulo k** is a complex-valued function on the integers $\chi : \mathbb{Z} \setminus n\mathbb{Z} \rightarrow \mathbb{C}$ which possess the following properties:*

- (i). $\chi(mn) = \chi(m)\chi(n)$, for all $m, n \in \mathbb{Z}$ (completely multiplicative).
- (ii). $\chi(n + k) = \chi(n)$ for all $n \in \mathbb{Z}$ (periodic with period k).
- (iii). $\chi(n) = 0$ if $\gcd(n, k) > 1$.

Thus we can define the Dirichlet L -function by:

$$L(s, \chi) = \sum_{n=0}^{\infty} \frac{\chi(n)}{n^s} \quad (\Re(s) > 1) \quad (1.2.24)$$

The first property implies that Dirichlet characters are *Completely Multiplicative* by nature, a property which plays an important role later in the Euler Product section. L -functions play an important role in analytic number theory since they were utilized by Dirichlet to prove his famous theorem regarding *Primes in Arithmetic Progression* in 1837, one of the main breakthroughs in Number Theory. Moreover, there exists $\varphi(k)$ Dirichlet characters modulo k because $\chi(n)^{\varphi(k)} = 1$ whenever $\gcd(n, k) = 1$ hence $\chi(n)$ is a root of unity, where φ is Euler's totient function. Consequently, this also implies that there exists $\varphi(k)$ corresponding L -functions for each character. Finally, if χ

is a Dirichlet character, then so does the complex conjugate $\bar{\chi}$.

Next, we analyze the following less obvious relation between Dirichlet L -functions and Hurwitz zeta-function. Suppose that χ is a character of modulo k , we can rearrange the terms in $L(s, \chi)$ based of their residue class mod k . Hence we can write: $n = qk + r$. Where $1 \leq r \leq k$ and $q \in \mathbb{N}$. As follows [3] (Pg. 249):

$$\begin{aligned}
 L(s, \chi) &= \sum_{n=0}^{\infty} \frac{\chi(n)}{n^s} \\
 &= \sum_{r=1}^k \sum_{q=0}^{\infty} \frac{\chi(qk + r)}{(qk + r)^s} \\
 &= k^{-s} \sum_{r=1}^k \chi(r) \sum_{q=0}^{\infty} \left(q + \frac{r}{k}\right)^{-s} \\
 &= k^{-s} \sum_{r=1}^k \chi(r) \zeta\left(s, \frac{r}{k}\right)
 \end{aligned} \tag{1.2.25}$$

This shows that any L -function can be written as a finite sum of Hurwitz zeta-functions, which can be very useful in studying L -function since we already have established results for the Hurwitz zeta-function. This result is also referred to as "universality property" which will be discussed in chapter two. Moreover, the inversion formula is given by:

$$\zeta\left(s, \frac{r}{k}\right) = \frac{k^s}{\varphi(r)} \sum_{\chi} \bar{\chi}(r) L(s, \chi) \quad \frac{r}{k} \neq 1, \frac{1}{2} \tag{1.2.26}$$

Where χ runs over all Dirichlet characters modulo k and φ is Euler's totient function. The functional equation is given in the next theorem:

Theorem 1.2.1. *For a Dirichlet character χ modulo k we have the following functional equation:*

$$L(1-s, \chi) = \frac{k^{s-1} \Gamma(s) \tau(\chi)}{(2\pi)^s} \left\{ e^{-\pi i s/2} + \chi(-1) e^{\pi i s/2} \right\} L(s, \bar{\chi}) \quad (1.2.27)$$

Where $\tau(\chi) = \sum_{k \bmod n} \chi(k) e^{2i\pi k/n}$ is Gauss Sum.

This relation can be derived directly from (1.2.25). Alternatively, it can be derived the same way as in Apostol's derivation of Lerch zeta-function by applying Poisson summation formula to the theta function.

As for the analytic continuation, the matter is far more delicate and requires deeper results that we will not cover here. For full exposition, please refer to Cohen [16] (Pg. 163-165).

1.2.4 Dedekind Zeta Function

The *Dedekind zeta-function* is defined as follows:

$$\zeta_K(s) = \sum_{\mathfrak{a} \subset \mathbb{Z}_K} \frac{1}{\mathcal{N}(\mathfrak{a})^s} \quad (1.2.28)$$

Where \mathfrak{a} runs through all integral ideals of \mathbb{Z}_K . \mathcal{N} denotes the absolute norm. Suppose K is a general number field and let \mathbb{Z}_K be its ring of integers, then the existence and uniqueness of prime decomposition hold for ideals, which is the case for Dedekind domains. Therefore, this gives an Euler product expansion:

$$\zeta_K(s) = \prod_{\mathfrak{p}} \frac{1}{1 - \mathcal{N}(\mathfrak{p})^{-s}} \quad (1.2.29)$$

Where \mathfrak{p} runs through all prime ideals of \mathbb{Z}_K . The product formula holds if and only if there exists a unique prime ideal decomposition. A functional equation for Dedekind zeta-function is known; before we discuss the main result. We shall go over some preliminary definitions:

Definition 1.2.3. (Index of a Subgroup $[* : *]$) For a subgroup H of a group G . The index of H , $[G : H]$ is the cardinal number of the left cosets of H in G .

Definition 1.2.4. (Fundamental Discriminant $d(K)$) An integer $d(K)$ is the discriminant of the extension $K \setminus \mathbb{Q}$.

Definition 1.2.5. (Class Number $h(K)$) For any ideal I in a Dedekind ring K , there is an ideal I_i such that $II_i = z$ where z is a principle ideal (ideal of rank 1). In the case of a finite ideal class group, there is a finite list of ideals I_i such that the equation $II_i = z$ is satisfied for some I . The size of this list is the class number.

Definition 1.2.6. (Regulator $R(K)$) Let $\epsilon_1, \epsilon_2, \dots, \epsilon_n$ be generators of a free abelian subgroup. The regulator is given by $|\det A_i|$ where A_i is a sub-matrix of A with entries of the form $\log \|\epsilon_j\|$.

Definition 1.2.7. A number z is a root of unity if it satisfies $z^n = 1$ for $n \in \mathbb{N}$. $w(K)$ is the number of roots of unity in K .

The following theorem provides us with some beautiful results [16]:

Theorem 1.2.2. Let K be a number field of degree $n = [K : \mathbb{Q}]$ and signature (r_1, r_2) . Denote by $d(K), h(K), R(K), w(K)$ the discriminant, class number, regulator, and number of roots of unity in K [16]. Then:

(1) The function $\zeta_K(s)$ has an analytic continuation to a meromorphic function with a simple pole at $s = 1$.

(2) We have the functional equation $\Lambda_K(1 - s) = \Lambda_K(s)$, where:

$$\Lambda_K(s) = |d(k)|^{s/2} \gamma(s)^{r_1+r_2} \gamma(s+1)^{r_2} \zeta_K(s) \quad (1.2.30)$$

(3) If we set $r = r_1 + r_2 - 1$, which is the rank of the unit group of K , then $\zeta_K(s)$ has a zero at $s = 0$ of order r thus:

$$\lim_{s \rightarrow 0} s^{-r} \zeta_K(s) = -\frac{h(K)R(K)}{w(K)} \quad (1.2.31)$$

(4) We apply (2) in (3), we get the Class Number Formula:

$$\lim_{s \rightarrow 1} (s-1) \zeta_K(s) = 2^{r_1} (2\pi)^{r_2} \frac{h(K)R(K)}{w(K)|d(k)|^{1/2}} \quad (1.2.32)$$

The *Class Number Formula* relates all these important invariants, which were defined earlier, of a number field to a special value of its Dedekind zeta function. Thus, it can provide us with valuable information about the field. It is also trivial case to see that when $K = \mathbb{Q}$ we have $\zeta_{\mathbb{Q}}(s) = \zeta(s)$.

We omit the proof of the theorem. the first proof was given by Hecke [29]), who proved this result using Theta functions of n variables and generalized Poisson summation formula. A more recent proof is due to J.T. Tate's thesis [52].

1.2.5 Euler's Product

One way to realize the connection between the Riemann zeta-function and number theory is to consider the following expansion expressing the function as an infinite product over the primes:

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \quad (\Re(s) > 1) \quad (1.2.33)$$

Where p is a prime. Euler used this identity to give a direct proof of the infinitude of primes. This follows immediately from the Fundamental Theorem of Arithmetic. A rigorous proof of this result can be found in Titchmarsh [53] (Pg.1-2). And [61] (Pg.1). One natural question to ask is if all zeta-functions or even L -functions possess this nice property, and the short answer is no. Next we

ask, under what suitable assumptions we have an *Euler Product* expansion. Which is the statement of the following theorem.

Theorem 1.2.3. : *Let f be a multiplicative arithmetical function such that the series $\sum_{n=1}^{\infty} f(n)$ is absolutely convergent. Thus, this series can be expressed as an absolutely convergent infinite product:*

$$\sum_{n=1}^{\infty} f(n) = \prod_p [1 + f(p) + f(p^2) + \dots] \quad (1.2.34)$$

And in the case f is completely multiplicative, we have the following:

$$\sum_{n=1}^{\infty} f(n) = \prod_p \left(1 - \frac{1}{f(p)}\right)^{-1} \quad (1.2.35)$$

Proof. The proof is due to Apostol [3], and we include it for the sake of completeness. Define the finite product:

$$P(x) = \prod_{p \leq x} [1 + f(p) + f(p^2) + \dots] \quad (1.2.36)$$

Since this is by definition finite hence convergent, we can multiply or rearrange the product in any way we desire. From the Fundamental Theorem of Arithmetics we have the unique prime factorization as $n = p_1^{a_1} p_2^{a_2} p_3^{a_3} \dots p_r^{a_r}$. Since f is multiplicative we have:

$$f(p_1^{a_1} p_2^{a_2} p_3^{a_3} \dots p_r^{a_r}) = f(p_1^{a_1}) f(p_2^{a_2}) f(p_3^{a_3}) \dots f(p_r^{a_r}) \quad (1.2.37)$$

And hence we have:

$$P(x) = \sum_{n \in A} f(n) \quad (1.2.38)$$

Where A is the set consisting of all the integers having a prime factor $\leq x$. Therefore:

$$\sum_{n=1}^{\infty} f(n) - P(x) = \sum_{n \in B} f(n) \quad (1.2.39)$$

Where B is the set consisting of all the integers having at least one prime factor $> x$. Thus:

$$\left| \sum_{n=1}^{\infty} f(n) - P(x) \right| \leq \sum_{n \in B} |f(n)| \leq \sum_{n > x} |f(n)| \quad (1.2.40)$$

By assumption the last sum tends to 0 as $x \rightarrow \infty$ since $\sum |f(n)|$ is absolutely convergent. Therefore:

$$P(x) \rightarrow \sum |f(n)| \quad \text{as } x \rightarrow \infty \quad (1.2.41)$$

To show absolute convergence of the infinite product we note that $\prod(1 + a_n)$ converges absolutely iff $\sum a_n$ converges absolutely thus we have:

$$\sum_{p \leq x} |f(p) + f(p^2) + \dots| \leq \sum_{p \leq x} (|f(p)| + |f(p^2)| + \dots) \leq \sum_{n=2}^{\infty} |f(n)| \quad (1.2.42)$$

Note that partial sums are bounded hence $\sum_{p \leq x} |f(p) + f(p^2) + \dots|$ is convergent therefore absolute convergence of the product in (1.2.34) is implied. To prove the formula (1.2.35), we note first that if f is multiplicative, then we have: $f(p^n) = f(p)^n$ hence each term of the infinite product of (1.2.34) is a geometric series of the form: $\frac{1}{1-f(p)}$ therefore (1.2.34) is absolutely convergent. ■

Theorem 1.2.4. Assume $\sum_{n=1}^{\infty} f(n)n^{-s}$ is absolutely convergent for $\Re(s) > \sigma_0$, then

$$\sum_{n=1}^{\infty} f(n)n^{-s} = \prod_p \frac{1}{1 - f(p)p^{-s}} \quad \Re(s) > \sigma_0 \quad (1.2.43)$$

Where σ_0 is some positive number.

Proof. A direct application of Theorem 1.2.3. ■

Now since the Dirichlet character χ is completely multiplicative by definition, we obtain:

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \frac{1}{1 - \chi(p)p^{-s}} \quad (1.2.44)$$

Which is an Euler product representation of the Dirichlet L -Function. Moreover, we can have the following representations for identities (1.1.1), (1.1.3) and (1.1.4)

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \prod_p (1 - p^{-s}), \quad \Re(s) > 1 \quad (1.2.45)$$

$$\frac{\zeta(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^s} = \prod_p (1 + p^{-s}), \quad \Re(s) > 1 \quad (1.2.46)$$

$$\frac{\zeta(2s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \prod_p \frac{1}{1 + p^{-s}}, \quad \Re(s) > 1 \quad (1.2.47)$$

Since these products are absolutely and uniformly convergent for $\Re(s) > 1$, we can multiply them or take ratios to obtain new Euler product representations, for example, we may obtain the following identity:

$$\zeta(s) \cdot \frac{\zeta(s)}{\zeta(2s)} = \prod_p \frac{1 + p^{-s}}{1 - p^{-s}}, \quad \Re(s) > 1 \quad (1.2.48)$$

Comparing this with (1.1.7), we see that:

$$\sum_{n=1}^{\infty} \frac{2^{\omega(n)}}{n^s} = \prod_p \frac{1 + p^{-s}}{1 - p^{-s}}, \quad \Re(s) > 1 \quad (1.2.49)$$

Remark. *In the case of Hurwitz zeta-function, the Euler product expansion is valid if and only if $a = \frac{1}{2}$ and $a = 1$ since they trivially correspond to the special values $\zeta(s, 1) = \zeta(s)$ and $\zeta(s, 1/2) = (2^s - 1)\zeta(s)$, respectively. See Titchmarsh ([53] pg. 36). As for the existence of a general Euler Product of an abstract field, we have already demonstrated the case for Dedekind domain where the uniqueness of prime factorization of the ideals is the necessary condition.*

*A more general form of the L-functions is that of Artin's L-function: Let $K \setminus Q$ be a Galois extension and $\rho : Gal(K \setminus Q) \rightarrow GL_n(\mathbb{C})$ be a nontrivial, irreducible continuous representation of its Galois group. Artin and R. Brauer had investigated the existence of an analytic continuation in the case of L-functions defined with Galois representations ρ instead of Dirichlet characters [4] [5]. Namely $L(s, \rho)$. Brauer proved that ρ can be continued to a meromorphic representation [11]. Artin then conjectured that it can be continued to an entire function and satisfy a functional equation. The **Artin's Conjecture** asserts that $L(s, \rho)$ is in fact, entire, with the exception of a simple pole at $s = 1$ hence dealing with the location of poles. On the other hand, the Generalized Riemann Hypothesis, deals with the locations of the zeros of certain L-series. Making the conjecture one of the important unsolved problems in Number Theory. For an excellent treatment of this topic, please refer to Bernstein [8].*

CHAPTER 2: THE UNIVERSALITY PROPERTY

2.1 Introduction

Universality is the ability of Riemann zeta-function and its other generalizations to approximate arbitrary non-vanishing holomorphic functions well. The chapter discusses the universality of generalizations of the zeta function where some exhibit a *Joint Universality* allowing a collection of analytic functions to be approximated by pairing each function with different zeta or L -functions at the same value. We shall first discuss the universality of Riemann zeta-function and then proceed with its generalizations.

2.2 Universality Theorem for the Riemann Zeta-Function

The first denseness result regarding the Riemann zeta-function $\zeta(\sigma + it)$ was discovered by H. Bohr's and Courant in 1914 [10]. He proved the following theorem:

Theorem 2.2.1. *For a fixed σ in $1/2 < \sigma < 1$, the set:*

$$\{\zeta(\sigma + it) : t \in \mathbb{R}\} \tag{2.2.1}$$

is dense in \mathbb{C} .

S. M. Voronin's generalized this result in 1972. The following three theorems are due to him:

Theorem 2.2.2. *Let s_0, \dots, s_n be a set of distinct complex numbers inside the strip $1/2 < \sigma < 1$,*

then the sequence:

$$\{(\zeta(s_0 + imt), \zeta(s_1 + imt), \dots, \zeta(s_n + imt))\} \quad (m \in \mathbb{N}, t > 0) \quad (2.2.2)$$

is dense in \mathbb{C}^n .

Theorem 2.2.3. *Let s_0 be fixed point inside the strip $1/2 < \sigma < 1$, then the set:*

$$\{(\zeta(s_0 + imt), \zeta'(s_0 + imt), \dots, \zeta^{(n-1)}(s_0 + imt))\} \quad (m \in \mathbb{N}, t > 0) \quad (2.2.3)$$

is dense in \mathbb{C}^n .

Since the two previous results are valid for finite dimensional spaces, Voronin considered infinite dimensional spaces, that is, function spaces, and derived the stronger form of his theorem, which is stated in the following result:

Theorem 2.2.4. *Given a compact set $K \subset D = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\}$ with connected complement and let the continuous function $f(s)$ defined on K be analytic in $\text{Int}(K)$. Then: for all $\epsilon > 0$:*

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \text{meas} \{ \tau \in [0, T] : \sup_{s \in K} |\zeta(s + i\tau) - f(s)| < \epsilon \} > 0 \quad (2.2.4)$$

where *meas* is Lebesgue measure of a measurable set.

For a proofs of the aforementioned results please refer to [58], [32] and [57].

2.3 Universality Theorem for the Hurwitz Zeta-Function and Lerch Zeta-Function

For the case of Hurwitz zeta-function and Lerch zeta-functions, a joint universality result had been published in (2012) [39]. In the case of Hurwtiz zeta-function, the shifts of $\zeta(s + i\tau)$, $s \in$

\mathbb{C} , $\tau \in \mathbb{R}$ approximate uniformly a wide class of analytic functions. This is also the case for Lerch zeta-function, which follows closely the same lines of proof of the universality of Hurwitz zeta-function (the proof is of a probabilistic nature). We shall omit the proof here and give a references to Laurincikas's publications [38], [39] and [40]

Theorem 2.3.1. (*Joint Universality of Hurwitz Zeta-Function*) Suppose that α rational or transcendental, not equal to 1 or 1/2. Given a compact set $K \subset D = \{s \in \mathbb{C} : \frac{1}{2} < \Re(s) < 1\}$ with connected complement and let $f(s)$ be a continuous function on K and analytic in $\text{Int}(K)$ then for all $\epsilon > 0$,

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \text{meas} \{ \tau \in [0, T] : \sup_{s \in K} |\zeta(s + i\tau, \alpha) - f(s)| < \epsilon \} > 0 \quad (2.3.1)$$

Where *meas* is Lebesgue measure of a measurable set. [39]

Theorem 2.3.2. (*Joint Universality of Lerch Zeta-Function*) Suppose that α is rational or transcendental, not equal to 1 or 1/2. Given a compact set $K \subset D = \{s \in \mathbb{C} : \frac{1}{2} < \Re(s) < 1\}$ with connected complement and let $f(s)$ be a continuous function on K and analytic in $\text{Int}(K)$ then for all $\epsilon > 0$,

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \text{meas} \{ \tau \in [0, T] : \sup_{s \in K} |\Phi(z, s + i\tau, a) - f(s)| < \epsilon \} > 0 \quad (2.3.2)$$

Where *meas* is Lebesgue measure of a measurable set. [38]

2.4 Universality Theorem for L -Functions

Since this is the most general case, we shall discuss the proof here, this is due to the fact that most zeta-functions can be represented one way or another by L -functions. The proof is due to

Bjaskar Bagchi [6]. Before we discuss the proof, we shall provide some preliminary lemmas and definitions which are needed for the proof.

Definition 2.4.1. *Let A be a subset of the topological space X and denote by I_A the indicator function of A . That is $I_A : X \rightarrow \{0, 1\}$ defined by*

$$I_A = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases} \quad (2.4.1)$$

For Borel subsets of A of the real line, denote by $\underline{d}(A)$, $\bar{d}(A)$ and $d(A)$ respectively the lower density, the upper density and the density. That is:

$$\underline{d}(A) = \liminf_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T I_A(t) dt \quad (2.4.2)$$

$$\bar{d}(A) = \limsup_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T I_A(t) dt \quad (2.4.3)$$

$$d(A) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T I_A(t) dt \quad (2.4.4)$$

Before we prove the Main Theorem we need the following lemma

Lemma 2.4.1. *Let $k \geq 1$, and let $\chi_1, \chi_2, \dots, \chi_n$ be distinct Dirichlet characters modulo k . For $j = 1, \dots, n$. Define K to be simply connected compact subsets of the sub-critical strip $\Omega = \{z \in \mathbb{C} : 1/2 < \Re(z) < 1\}$. $H(\Omega)$ is the the set of analytic functions defined on the strip Ω , equipped with the topology of uniform convergence. Let $f = (f_1, f_2, \dots, f_n) \in S^n$ where*

$S = \{f \in H(\Omega) : f \equiv 0 \text{ or } \frac{1}{f} \in H(\Omega)\}$. Then the set of all $t \in \mathbb{R}$ for which:

$$\sup_{1 \leq j \leq n} \sup_{z \in K} |L(z + it, \chi_j) - f_j(z)| < \epsilon \quad (2.4.5)$$

Has a positive lower density for every $\epsilon > 0$.

Proof. Define $\gamma = \{z \in \mathbb{C} : |z| = 1\}$ to be the unit circle in the complex plane. Let $F \in H^n(\Omega)$ be a sequence defined by $F = (F_1, F_2, \dots, F_n)$ where $F_j = L(*, \chi_j)$ for each $j = 1, 2, \dots, n$. it suffices to show that for all $t \in \mathbb{R}$ we have:

$$\sup_{z \in K} |F(z + it) - f(z)| < \epsilon \quad (2.4.6)$$

has positive lower density for every $\epsilon > 0$. For positive integers m , and $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m) \in \gamma_m$ define the following two finite products:

$$F_{j,m}(z) = \prod_{i=1}^m (1 - \chi_j(p_i) p_i^{-z})^{-1}, \quad z \in \Omega, \quad 1 \leq j \leq n \quad (2.4.7)$$

$$F_{j,m}(z, \alpha) = \prod_{i=1}^m (1 - \alpha_i \chi_j(p_i) p_i^{-z})^{-1}, \quad z \in \Omega, \quad 1 \leq j \leq n \quad (2.4.8)$$

Let E be a compact subset of Ω such that K is contained in the interior of E . Let $g \in H^n(\Omega)$ and $\int_E |g(z)|^2 dz < \delta$ for $\delta > 0$. It follows that $\sup_{z \in K} |g(z)| < \epsilon/2$. There exists an $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_l) \in \gamma_m$ such that:

$$\sup_{z \in E} |F_m(z, \alpha) - f(z)| < \epsilon/2 \quad (2.4.9)$$

Since $(z, \alpha) \mapsto |F_m(z, \alpha) - f(z)|$ is uniformly continuous, there exists an open set $U \subseteq \gamma_m$ such that (2.4.9) is true for all $\alpha \in U$. Choose U to such that it has the property $\mu(\partial U) = 0$ where μ is Haar Measure on γ_m . Since $\log p_1, \log p_2, \dots, \log p_m$ are linearly independent over \mathbb{Q} . By Weyl's

Criterion [35], the set $\{p_1^{-it}, p_2^{-it}, \dots, p_m^{-it} : t \in \mathbb{R}\}$ is uniformly distributed over γ_m . Let V be the set of all $t \in \mathbb{R}$ such that $\alpha_t = (p_1^{-it}, p_2^{-it}, \dots, p_m^{-it})$ is in U which implies that V has positive density $\mu(V) = \mu(U) > 0$ also known as μ -continuous. Thus:

$$\sup_{z \in E} \left| F_m(z, \alpha_t) - f(z) \right| < \epsilon/2, \quad t \in V, \alpha_t \in U \implies \sup_{z \in E} \left| F_m(z + it) - f(z) \right| < \epsilon/2 \quad t \in V \quad (2.4.10)$$

Let $W \equiv \{t \in V : \int_E |F(z + it) - F_m(z + it)|^2 dz < \delta\}$. Hence we have:

$$\sup_{z \in K} \left| F(z + it) - F_m(z + it) \right| < \delta/2 \quad t \in W \quad (2.4.11)$$

Combining (2.4.10) and (2.4.11) means that (2.4.6) holds for $t \in W$. Hence it suffices to show that $\underline{d}(W) > 0$. For the sake of contradiction, assume that this is not the case. Then we have:

$$I = \liminf_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T I_v(t) (1 - I_w(t)) \int_E |F(z + it) - F_m(z + it)|^2 dz dt \geq \delta d(V) \quad (2.4.12)$$

Which implies that $I \geq \delta \mu(U)$. On the other hand:

$$I \leq c_0 \sup_{z \in E} \liminf_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T I_v(t) |F(z + it) - F_m(z + it)|^2 dt \quad (2.4.13)$$

Where c_0 is the Lebesgue measure of E , so that $0 < c_0 < \infty$. If we choose $c_1 = (\epsilon/2 + \sup_{z \in E} |f(z)|)^2$ then from (2.4.10) we have

$$\sup_{z \in E} \left| F_m(z + it) - f(z) \right|^2 \leq c_1 < \infty \quad t \in V \quad (2.4.14)$$

Hence

$$I \leq c_0 c_1 \sup_{z \in E} \limsup_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T I_v(t) \left| 1 - \frac{F(z + it)}{F_m(z + it)} \right|^2 dt \quad (2.4.15)$$

Let m be large so that all the prime divisors of k occur amongst the primes p_1, p_2, \dots, p_m . Since

U is μ -continuous and that α_t is uniformly distributed on γ_m . Following the same lines of proof in Titchmarsh [54] (Pg. 304-306) one can show:

$$\limsup_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T I_v(t) \left| 1 - \frac{F(z+it)}{F_m(z+it)} \right|^2 dt = \mu(U) n \sum_{(p,q)=1, q>1} |q^{-z}|^2 \quad (2.4.16)$$

Which converges uniformly for $z \in E$. Choose $x_0 = \min_{z \in E} \{\Re(z)\}$ such that $x_0 > 1/2$. Then we obtain:

$$I \leq c_0 c_1 \mu(U) \sum_{q=m+1}^{\infty} n^{-2x_0} \quad (2.4.17)$$

Combine (2.4.12) and (2.4.17) we get

$$0 < \delta \mu(U) \leq c_0 c_1 \mu(U) \sum_{q=m+1}^{\infty} n^{-2x_0} \quad (2.4.18)$$

Let $c = \frac{\delta}{nc_0c_1}$ which is independent of m and note that $0 < \mu(U) \leq 1$ Thus we have:

$$\sum_{q=m+1}^{\infty} n^{-2x_0} \geq c > 0 \quad (2.4.19)$$

if we let $m \rightarrow \infty$ then 2.4.19 is false. Hence we must have $\underline{d}(W) > 0$. ■

Theorem 2.4.1. (*Joint Universality of Dirichlet L-function*) Let $k \geq 1$, and let $\chi_1, \chi_2, \dots, \chi_n$ be distinct Dirichlet characters modulo k . For $j = 1, \dots, n$. Define K_j to be simply connected compact subsets of the sub-critical strip $\Omega = \{z \in \mathbb{C} : 1/2 < \Re(z) < 1\}$. Let $f_j(s)$ be a non-vanishing continuous function on K_j which is analytic in the interior of each K_j . Then the set of all $t \in \mathbb{R}$ for which:

$$\sup_{1 \leq j \leq n} \sup_{z \in K_j} |L(z+it, \chi_j) - f_j(z)| < \epsilon \quad (2.4.20)$$

Has a positive lower density for every $\epsilon > 0$.

Proof. From Mergelyan's theorem [48] there is a sequence of polynomials $\{P_{m,j}\}$ such that $P_{m,j}(z) \rightarrow f_j(z)$ uniformly for $z \in K_j$. Let $g_j = P_{m,j}$, then $g_j \neq 0$ for $z \in K_j$ and

$$\sup_{z \in K_j} |f_j(z) - g_j(z)| < \epsilon/4 \quad 1 \leq j \leq n \quad (2.4.21)$$

g_j has only finitely many zeros, we choose simply connected regions E_j containing K_j such that $g_j \neq 0$ for $z \in E_j$. Hence we can define the logarithms $\log g_j$ in E_j which are holomorphic in the interior of each K_j . There is a sequence $Q_{m,j}(z) \rightarrow \log g_j$ uniformly on K_j . Define: $h_j = e^{Q_{m,j}(z)}$
Then

$$\sup_{z \in K_j} |g_j(z) - h_j(z)| < \epsilon/4 \quad 1 \leq j \leq n \quad (2.4.22)$$

Combining (2.4.21) and (2.4.22) yields:

$$\sup_{1 \leq j \leq n} \sup_{z \in K_j} |f_j(z) - h_j(z)| < \epsilon/2 \quad (2.4.23)$$

where $h = (h_1, h_2, \dots, h_n) \in S^n$ From Lemma (2.4.1) we have for all $t \in \mathbb{R}$:

$$\sup_{1 \leq j \leq n} \sup_{z \in K} |L(z + it, \chi_j) - h_j(z)| < \epsilon/2 \quad (2.4.24)$$

which has a positive lower density for every $\epsilon > 0$. Combining (2.4.23) and (2.4.24) yields the desired result. ■

The next two important corollaries are also due Bagchi [6]:

Corollary 2.4.1. *Let $k \geq 1$, and let $\chi_1, \chi_2, \dots, \chi_n$ be distinct Dirichlet characters modulo k . Let K be a simply connected compact subset of the sub-critical strip $\Omega = \{z \in \mathbb{C} : 1/2 < \Re(z) < 1\}$. Let f be a non-vanishing continuous function on K which is analytic in the interior of each K . Then the set of all $t \in \mathbb{R}$:*

$$\sup_{z \in K} |L(z + it, \chi) - f(z)| < \epsilon \quad (2.4.25)$$

Has a positive lower density for every $\epsilon > 0$.

Corollary 2.4.2. *Let $k \geq 1$, and let $\chi_1, \chi_2, \dots, \chi_n$ be distinct Dirichlet characters modulo k . Then the set:*

$$\{L(*, \chi_1), L(*, \chi_2), \dots, L(*, \chi_n)\} \quad (2.4.26)$$

does not satisfy any nontrivial algebraic-differential equation.

2.5 Discussion

It is important to note here that the universality of Hurwitz and Lerch zeta-functions hold only for the case when α is rational and transcendental [39]. The question of whether this statement holds for irrational α still remains open. Another thing to note here is that Bagchi avoided probabilistic methods as much as possible, and only borrowed some concepts needed in establishing Haar measure. On the other hand, Laurincikas's and Bohr's proofs rely on many concepts from probability theory and the conclusion is formulated in terms of positive lower density. The reader who is interested in further probabilistic approaches may refer to Laurincikas [37] and Harald Bohr [9] publications. Finally, there are some results related to the universality of Dedekind zeta-function, the reader may refer to Reich's publication [45].

CHAPTER 3: THE PRIME NUMBER THEOREM

3.1 Historical Background

It is no secret that primes have had fascinated mathematicians since antiquity. As much as they were obsessed with patterns, they were anxious to find patterns involving prime numbers so that one can anticipate the next prime. Alternatively one could look at the primes less than or equal to a given magnitude x . Call such function $\pi(x)$. Around the eighteenth century, mathematicians had struggled to find an explicit formula for $\pi(x)$ or at least an asymptotic relation. Fortunately, today we have the following asymptotic law: The **Prime Number Theorem** asserts that:

$$\pi(x) \sim \frac{x}{\log x} \quad \text{as } x \rightarrow \infty$$

Around 1798, the first published statement regarding the prime number theorem was due to Adrien-Marie Legendre [25], the statement is that $\pi(x)$ is of the form $\frac{x}{A \ln x + B}$ where A and B are some constants; He later refined his conjecture to give $A = 1$ and $B = 1.08366$. But it turned out that the estimated value for B was false. Later on, Peter Dirichlet found a better numerical approximation to $\pi(x)$ using the logarithmic integral $\text{li}(x) = \int_0^x \frac{dt}{\ln t}$ and communicated his idea to Carl Friedrich Gauss. In fact, Gauss suspected long before, just by studying tables of the primes, that the probability for which a prime occurs within a neighborhood of a number n has the density function $1/\ln n$, which is exactly the aforementioned logarithmic integral if we considered any prime in the interval $[a, b)$. However, Gauss never published his findings and considered the subject more of a hobby of his. Eventually, analysis of both Legendre and Dirichlet's formulas led to the asymptotic relation $\pi(x) \sim x/\ln(x)$. Many mathematicians attempted to prove this result and it became known historically as the prime number theorem.

In 1848, a Russian mathematician named Chebyshev attacked the problem but was only able to succeed in proving a weaker form of the theorem; that if the limit of ratio $\pi(x)/(x \ln x)$ exists at all, then that limit has to be 1, he provided upper and lower bounds for this ratio. Even though he did not prove the prime number theorem, the estimates on the bounds were good enough to make him solve the Bertrand's postulate; That there is a prime number between n and $2n$ for any given $n \geq 2$.

In 1859, Bernhard Riemann published his famous memoir *On the Number of Primes Less Than a Given Magnitude* which is also his only paper written on the subject. In this paper, he introduced an explicit formula for $\pi(x)$ and explained how $\pi(x)$ is related to the zeros of the analytically continued version of the zeta function $\zeta(s)$. Moreover, he suggested that all such zeros must lie on the critical line $\Re(s) = 1/2$, which later became his famous millennium problem: The *Riemann Hypothesis*.

Not until 1896, Hadamard and de la Vallée Poussin were able to prove the prime number theorem independently [26] [18]. Their proofs exploited nontrivial properties of the Riemann zeta Function $\zeta(s)$. Moreover, in 1980, D. J. Newman published his paper *Simple Proof of the Prime Number Theorem* [61] in the American Mathematical Monthly and provided an "elementary" proof that was shorter and simplified through the use of Tauberian theorems. For further historical background the reader may refer to [25] and [7]. Before we discuss the proofs, we start with a useful lemma.

Lemma 3.1.1. *For any arithmetic function $a(n)$ let: $A(x) = \sum_{n \leq x} a(n)$ where $A(x) = 0$ if $x < 1$. Assume $f \in C^1[y, x]$, where $0 < y < x$ then the following relation (also known as **Abel's Identity**) holds:*

$$\sum_{y < n \leq x} a(n)f(n) = A(x)f(x) - A(y)f(y) - \int_y^x A(t)f'(t)dt \quad (3.1.1)$$

Proof. Since $A(x)$ is a step function with jump $a(n)$ at each integer n ; the sum can be expressed

as a Riemann-Stieltjes integral [48]: $\sum_{y < n \leq x} a(n)f(n) = \int_y^x f(t) dA(t)$. The result follows from applying integration by parts. ■

3.2 Equivalent Formulations of the Prime Number Theorem

The following three theorems are in fact equivalent [3] (Pg. 78-80).

Theorem 3.2.1.

$$\pi(x) \sim \frac{x}{\log x} \tag{3.2.1}$$

As $x \rightarrow \infty$ where $\pi(x)$ is the prime counting function defined by $\pi(x) = \sum_{p \leq x} 1$.

Theorem 3.2.2.

$$\vartheta(x) \sim x \tag{3.2.2}$$

As $x \rightarrow \infty$

where $\vartheta(x)$ is Chebyshev's ϑ -function defined by $\vartheta(x) = \sum_{p \leq x} \log p$.

Theorem 3.2.3.

$$\psi(x) \sim x \tag{3.2.3}$$

As $x \rightarrow \infty$ where $\psi(x)$ is Chebyshev's ψ -function defined by $\psi(x) = \sum_{n \leq x} \Lambda(n)$. And:

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^m \text{ for some } m \\ 0 & \text{otherwise} \end{cases}$$

Theorem 3.2.4. All forms of the Prime Number Theorem are equivalent.

Proof. (3.2.1) \Rightarrow (3.2.2)

Define the arithmetic function:

$$a(n) = \begin{cases} 1 & \text{if } n \text{ is a prime} \\ 0 & \text{otherwise} \end{cases} \quad (3.2.4)$$

Therefore we can rewrite: $\pi(x) = \sum_{1 < n \leq x} a(n)$ and $\vartheta(x) = \sum_{1 < n \leq x} a(n) \log n$. Applying Abel's identity (3.1.1) with $f(x) = \log x$, $y = 1$ yields:

$$\begin{aligned} \vartheta(x) &= \sum_{y < n \leq x} a(n) f(n) = \pi(x) \log x - \pi(1) \log 1 - \int_1^x \frac{\pi(t)}{t} dt \\ &= \pi(x) \log x - \int_2^x \frac{\pi(t)}{t} dt \end{aligned} \quad (3.2.5)$$

since $\pi(t) = 0$ for $t < 2$.

On the other hand, let $b(n) = a(n) \log n$ hence we have:

$$\pi(x) = \sum_{1 < n \leq x} b(n) \frac{1}{\log n} \quad \vartheta(x) = \sum_{1 < n \leq x} b(n)$$

Applying Abel's identity (3.1.1) with $f(x) = \frac{1}{\log x}$, $y = 3/2$ we get:

$$\begin{aligned} \pi(x) &= \frac{\vartheta(x)}{\log x} - \frac{\vartheta(3/2)}{\log 3/2} + \int_1^x \frac{\vartheta(t)}{t \log^2 t} dt \\ \pi(x) &= \frac{\vartheta(x)}{\log x} + \int_2^x \frac{\vartheta(t)}{t \log^2 t} dt \end{aligned} \quad (3.2.6)$$

since $\vartheta(t) = 0$ for $t < 2$. Dividing (3.2.5) by x and (3.2.6) by $\frac{x}{\log x}$ yields :

$$\frac{\vartheta(x)}{x} = \frac{\pi(x) \log x}{x} - \frac{1}{x} \int_2^x \frac{\pi(t)}{t} dt \quad (3.2.7)$$

$$\frac{\pi(x) \log x}{x} = \frac{\vartheta(x)}{x} + \frac{\log x}{x} \int_2^x \frac{\vartheta(t)}{t \log^2 t} dt \quad (3.2.8)$$

For formula (3.2.7) It suffices to show: $\lim_{x \rightarrow \infty} \frac{1}{x} \int_2^x \frac{\pi(t)}{t} dt = 0$. Indeed; by assumption $\pi(x) \sim x/\log x$ Hence

$$\frac{\pi(t)}{t} = O\left(\frac{1}{\log t}\right) \implies \frac{1}{x} \int_2^x \frac{\pi(t)}{t} dt = O\left(\frac{1}{x} \int_2^x \frac{1}{\log t} dt\right) \quad (3.2.9)$$

Moreover

$$\int_2^x \frac{1}{\log t} dt = \int_2^{\sqrt{x}} \frac{1}{\log t} dt + \int_{\sqrt{x}}^x \frac{1}{\log t} dt \leq \frac{\sqrt{x}}{\log 2} + \frac{x - \sqrt{x}}{\log \sqrt{x}} \quad (3.2.10)$$

Therefore

$$\lim_{x \rightarrow \infty} \frac{1}{x} \int_2^x \frac{1}{\log t} dt = 0 \implies \vartheta(x) \sim x \quad (3.2.11)$$

(3.2.2) \implies (3.2.1)

For formula (3.2.8) It suffices to show: $\lim_{x \rightarrow \infty} \frac{\log x}{x} \int_2^x \frac{\vartheta(t)}{t \log^2 t} dt = 0$. Indeed; by assumption $\vartheta(x) \sim x$ and we have

$$\frac{\log x}{x} \int_2^x \frac{\vartheta(t)}{t \log^2 t} dt = O\left(\frac{\log x}{x} \int_2^x \frac{1}{\log^2 t} dt\right) \quad (3.2.12)$$

Moreover

$$\int_2^x \frac{1}{\log^2 t} dt = \int_2^{\sqrt{x}} \frac{1}{\log^2 t} dt + \int_{\sqrt{x}}^x \frac{1}{\log^2 t} dt \leq \frac{\sqrt{x}}{\log^2 2} + \frac{x - \sqrt{x}}{\log^2 \sqrt{x}} \quad (3.2.13)$$

Therefore

$$\lim_{x \rightarrow \infty} \frac{\log x}{x} \int_2^x \frac{1}{\log^2 t} dt = 0 \implies \pi(x) \sim x/\log x \quad (3.2.14)$$

(3.2.2) \iff (3.2.3)

First we observe that

$$\psi(x) = \sum_{n \leq \log_2 x} \vartheta(x^{1/n}) \implies 0 \leq \psi(x) - \vartheta(x) = \sum_{2 \leq n \leq \log_2 x} \vartheta(x^{1/n}) \quad (3.2.15)$$

Moreover we have,

$$\vartheta(x) \leq \sum_{p \leq x} \log x \leq x \log x \quad (3.2.16)$$

Combining the last two relations yield:

$$0 \leq \psi(x) - \vartheta(x) \leq \sum_{2 \leq n \leq \log_2 x} x^{1/n} \log x^{1/n} \leq (\log_2 x) \sqrt{x} \log \sqrt{x} = \frac{\sqrt{x}(\log x)^2}{2 \log 2} \quad (3.2.17)$$

Divide by x (since $x > 0$) we get the inequality:

$$0 \leq \frac{\psi(x)}{x} - \frac{\vartheta(x)}{x} \leq \frac{\log^2 x}{2\sqrt{x} \log 2} \implies \lim_{x \rightarrow \infty} \left(\frac{\psi(x)}{x} - \frac{\vartheta(x)}{x} \right) = 0 \implies \frac{\psi(x)}{x} \quad (3.2.18)$$

Therefore $\frac{\vartheta(x)}{x}$ must tend to the same limit. ■

3.3 An Analytic Proof of the Prime Number Theorem

The following is Hadamard and De La Vallée Poussin's proof of the prime number theorem that uses some techniques from complex analysis. Our main objective is to prove the equivalent statement of the Prime Number Theorem (3.2.3). Since $\psi(x)$ is a step function; it is convenient to consider the integral $\Phi(x) = \int_1^x \psi(t) dt$ which is a continuous piecewise function. It

would be nice if we can find a result showing that formal differentiation of the asymptotic relation $\Phi(x) \sim \frac{1}{2}x^2 \implies \psi(x) \sim x$ as $x \rightarrow \infty$; Which is goal of the next lemmas.

3.3.1 Preliminary Results

Lemma 3.3.1. Suppose $a(n) \geq 0 \forall n$, let $A(x) = \sum_{n \leq x} a(n)$ and let $B(x) = \int_1^x A(t)dt$ then:

$$B(x) \sim Lx^c \quad \text{as } x \rightarrow \infty \implies A(x) \sim cLx^{c-1} \quad \text{as } x \rightarrow \infty \quad (3.3.1)$$

for some $c > 0$ and $L > 0$.

Proof. $A(x)$ is increasing function since $a(n) \geq 0 \forall n$. Choose $\varepsilon > 1$ and note that:

$$B(\varepsilon x) - B(x) = \int_x^{\varepsilon x} A(t)dt \geq \int_x^{\varepsilon x} A(x)dt = x(\varepsilon - 1)A(x) \quad (3.3.2)$$

Hence: $x A(x) \leq \frac{1}{\varepsilon - 1}(B(\varepsilon x) - B(x))$ dividing by x^c (since $x^c > 0$)

$$\frac{A(x)}{x^{c-1}} \leq \frac{1}{\varepsilon - 1} \left(\frac{B(\varepsilon x)}{(\varepsilon x)^c} \varepsilon^c - \frac{B(x)}{x^c} \right) \quad (3.3.3)$$

Taking the limit $x \rightarrow \infty$ while keeping ε fixed we get:

$$\limsup_{x \rightarrow \infty} \frac{A(x)}{x^{c-1}} \leq \frac{1}{\varepsilon - 1} (L\varepsilon^c - L) = \frac{\varepsilon^c - 1}{\varepsilon - 1} \quad (3.3.4)$$

Now we have:

$$\limsup_{x \rightarrow \infty} \frac{A(x)}{x^{c-1}} \leq L \lim_{\varepsilon \rightarrow 1^+} \frac{\varepsilon^c - 1^c}{\varepsilon - 1} = L(cx^{c-1})|_{x=1} = cL \quad (3.3.5)$$

A similar argument with $B(x) - B(\delta x)$ and $0 < \delta < 1$ yields:

$$\liminf_{x \rightarrow \infty} \frac{A(x)}{x^{c-1}} \geq L \lim_{\delta \rightarrow 1^-} \frac{1^c - \delta^c}{1 - \delta} = L(cx^{c-1})|_{x=1} = cL \quad (3.3.6)$$

The last two inequalities prove the result. ■

Lemma 3.3.2. *Let $a(n)$ be an arithmetical function and $A(x) = \sum_{n \leq x} a(n)$ where $A(x) = 0$ if $x < 1$ then:*

$$\sum_{n \leq x} (x - n)a(n) = \int_1^x A(t)dt \quad (3.3.7)$$

Proof. Applying Abel's Identity (3.1.1) with $f(x) = x$ which has a continuous derivative on $[1, x]$ and noting:

$$\sum_{n \leq x} a(n)f(n) = \sum_{n \leq x} na(n) \quad (3.3.8)$$

$$A(x)f(x) = x \sum_{n \leq x} a(n) \quad (3.3.9)$$

we have:

$$x \sum_{n \leq x} a(n) - \sum_{n \leq x} na(n) = \int_1^x A(t) \cdot 1 dt \quad (3.3.10)$$

Which is exactly what we want. ■

Theorem 3.3.1.

$$\Phi(x) \sim \frac{x^2}{2} \implies \psi(x) \sim x \quad \text{as } x \rightarrow \infty \quad (3.3.11)$$

With

$$\Phi(x) = \sum_{n \leq x} (x - n)\Lambda(n) \quad (3.3.12)$$

Proof. Set $a(n) = \Lambda(n)$, $\Lambda(n) \geq 0$, $A(x) = \psi(x)$ and $B(x) = \Phi(x)$ and apply lemmas (3.3.1) and (3.3.2). ■

3.3.2 Contour Integral Representation of $\Phi(x)/x^2$

We shall next represent $\Phi(x)$ in terms of a contour integral.

Lemma 3.3.3. *if $c > 0$ and $u > 0$, then for every integer $n \geq 0$ we have:*

$$\frac{1}{2\pi i} \int_{c-\infty i}^{c+\infty i} \frac{u^{-z}}{z(z+1)\dots(z+n)} dz = \begin{cases} \frac{(1-u)^n}{n!} & \text{if } 0 < u \leq 1 \\ 0 & \text{if } u > 1 \end{cases} \quad (3.3.13)$$

Proof. Note that the gamma function is a meromorphic function that has poles at the negative integers, namely $z = 0, -1, \dots, -n$. Repeated application of the functional equation $\Gamma(z+1) = z\Gamma(z)$ yields

$$\frac{1}{2\pi i} \int_{c-\infty i}^{c+\infty i} \frac{u^{-z} dz}{z(z+1)\dots(z+n)} = \frac{1}{2\pi i} \oint_{C(R)} \frac{u^{-z}}{\Gamma(z+n+1)} dz \quad (3.3.14)$$

Where the contour $C(R)$ is the circle shown in (a) and (b) having a radius greater than $2n + c$ so that all poles of the gamma function lie within the contour, and then show that the integral vanishes along these circular paths. As demonstrated in figure 3.1:

Let $|z| = R$ and note that:

$$\left| \frac{u^{-z}}{z(z+1)\dots(z+n)} \right| = \frac{u^{-\Re z}}{|z||z+1|\dots|z+n|} \leq \frac{u^{-c}}{R|z+1|\dots|z+n|} \quad (3.3.15)$$

The inequality follows from the fact that $u^{-\Re z}$ is an increasing function if $0 < u \leq 1$ and a decreasing function if $u > 1$

Now if $1 \leq k \leq n$ we have: $|z+k| \geq |z| - k = R - k \geq R - n \geq R/2$ but $R > 2n$ therefore the integral along each circular arc is dominated by: $\frac{2\pi R u^{-c}}{R(\frac{1}{2}R)^n} = O(R^{-n})$ and this tends to 0 as $R \rightarrow \infty$

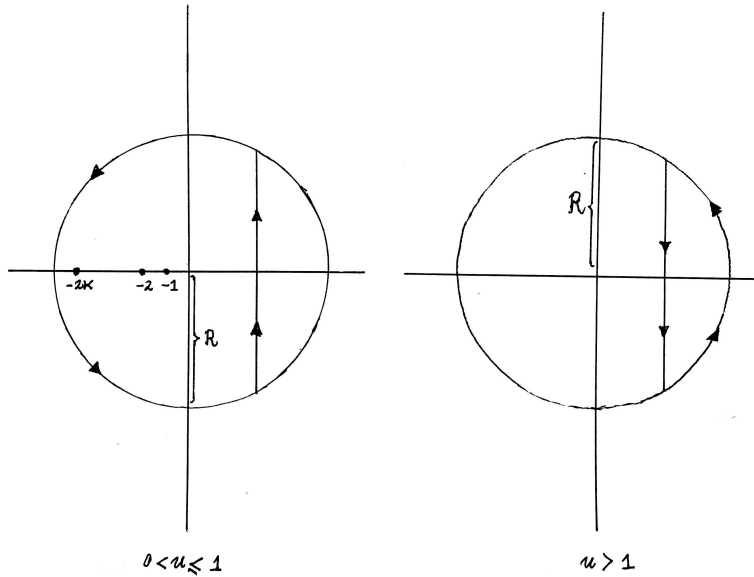


Figure 3.1: Contour Integral

since $n \geq 1$. If $u > 1$ the integrand is analytic inside $C(R)$ and hence by Cauchy's Theorem is zero. Letting $R \rightarrow \infty$ we prove the lemma for this case.

For the case where $0 < u \leq 1$ we apply Cauchy's Residue Theorem since the integrand has poles at the negative integers:

$$\begin{aligned}
 \frac{1}{2\pi i} \oint_{C(R)} \frac{u^{-z}}{\Gamma(z+n+1)} dz &= \sum_{k=0}^n \text{Res}\left(\frac{u^{-z}}{\Gamma(z+n+1)}; z = -k\right) \\
 &= \sum_{k=0}^n \frac{u^k}{\Gamma(n+1-k)} \text{Res}(\Gamma(z); z = -k) \\
 &= \sum_{k=0}^n \frac{u^k (-1)^k}{(n-k)! k!} = \frac{1}{n!} \sum_{k=0}^n \binom{n}{k} (-u)^k = \frac{(1-u)^n}{n!}
 \end{aligned}$$

Letting $R \rightarrow \infty$ we get the lemma. ■

Theorem 3.3.2. if $c > 1$ and $x \geq 1$ we have:

$$\frac{\Phi(x)}{x^2} = \frac{1}{2\pi i} \int_{c-\infty i}^{c+\infty i} \frac{x^{s-1}}{s(s+1)} \left(-\frac{\zeta'(s)}{\zeta(s)} \right) ds \quad (3.3.16)$$

Proof. Using equation (3.3.12) we get $\frac{\Phi(x)}{x} = \sum_{n \leq x} (1 - \frac{n}{x}) \Lambda(n)$. Now we applying lemma (3.3.3) with $n = 1$ and $u = k/x$. If $n \leq x$ yields:

$$1 - \frac{n}{x} = \frac{1}{2\pi i} \int_{c-\infty i}^{c+\infty i} \frac{(x/n)^s}{s(s+1)} ds \quad (3.3.17)$$

Multiplying the relation by $\Lambda(n)$ and summing over $n \leq x$ yields:

$$\frac{\Phi(x)}{x} = \sum_{n \leq x} (1 - \frac{n}{x}) \Lambda(n) = \sum_{n=1}^{\infty} \frac{1}{2\pi i} \int_{c-\infty i}^{c+\infty i} \frac{\Lambda(n)(x/n)^s}{s(s+1)} ds \quad (3.3.18)$$

Since the integral vanishes if $n > x$. Moreover the partial sums possess the inequality:

$$\sum_{n=1}^N \int_{c-\infty i}^{c+\infty i} \frac{\Lambda(n)(x/n)^c}{|s||s+1|} ds = \sum_{n=1}^N \frac{\Lambda(n)}{n^c} \int_{c-\infty i}^{c+\infty i} \frac{x^c}{|s||s+1|} ds \leq M \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^c} \quad (3.3.19)$$

where M is a constant; Which imply that the integrand is convergent and we can integrate (3.3.18) term by term to obtain:

$$\frac{\Phi(x)}{x} = \frac{1}{2\pi i} \int_{c-\infty i}^{c+\infty i} \frac{x^s}{s(s+1)} \left(\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} \right) ds = \frac{1}{2\pi i} \int_{c-\infty i}^{c+\infty i} \frac{x^s}{s(s+1)} \left(-\frac{\zeta'(s)}{\zeta(s)} \right) ds \quad (3.3.20)$$

Now we divide by x to obtain the result. ■

Theorem 3.3.3. if $c > 1$ and $x \geq 1$ we have:

$$\frac{\Phi(x)}{x^2} - \frac{1}{2} \left(1 - \frac{1}{x}\right)^2 = \frac{1}{2\pi i} \int_{c-\infty i}^{c+\infty i} x^{s-1} h(s) ds \quad (3.3.21)$$

Where:

$$h(s) = \frac{1}{s(s+1)} \left(-\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1} \right) \quad (3.3.22)$$

Proof. Applying lemma (3.3.3) with $n = 2$ yields:

$$\frac{1}{2} \left(1 - \frac{1}{x}\right)^2 = \frac{1}{2\pi i} \int_{c-\infty i}^{c+\infty i} \frac{x^s}{s(s+1)(s+2)} ds \quad \text{where } c > 0 \quad (3.3.23)$$

Now substitute $s - 1$ for s and subtract the result from (3.3.16) to prove the theorem. ■

Remark. Note that the path of integration of the integral (3.3.21) can be parametrized if we write $s = c + it$ and observe that $x^{s-1} = x^{c-1} e^{it \log x}$. Thus the integral can be rewritten as:

$$\frac{\Phi(x)}{x^2} - \frac{1}{2} \left(1 - \frac{1}{x}\right)^2 = \frac{x^{c-1}}{2\pi} \int_{-\infty}^{+\infty} h(c + it) e^{it \log x} dt \quad (3.3.24)$$

The reason why we want it in this form is because we want to deploy the **Riemann-Lebesgue lemma** which is a major result in Fourier Analysis to conclude that:

$$\lim_{x \rightarrow \infty} \int_{-\infty}^{+\infty} h(c + it) e^{it \log x} dt = 0 \quad (3.3.25)$$

Which is the case here since $\int_{-\infty}^{+\infty} |h(c + it)| dt$ converges if $c > 1$, however, the term x^{s-1} tends to ∞ as $c > 1$ and we arrive at the indeterminate form $\infty \cdot 0$. If we could only allow the value $c = 1$ the term that is causing the trouble shall disappear. Therefore, what we really need to do is study the behavior of $h(1 + it)$ which involves studying $\zeta'(s)/\zeta(s)$ on the neighborhood of the line $\Re(s) = 1$.

3.3.3 Bounds on $|\zeta(s)|$ and $|\zeta'(s)|$ near $\Re(s) = 1$

We shall now use a representation of $\zeta(s)$ where $s = \sigma + it$ valid for $\sigma > 0$ (which can be obtained by Euler's summation formula) see [3]:

$$\zeta(s) = \sum_{n=1}^N \frac{1}{n^s} - s \int_N^{\infty} \frac{x - [x]}{x^{s+1}} dx + \frac{N^{1-s}}{s-1} \quad (3.3.26)$$

Differentiating this expression yields:

$$\zeta'(s) = - \sum_{n=1}^N \frac{\log n}{n^s} - s \int_N^{\infty} \frac{(x - [x]) \log x}{x^{s+1}} dx - \int_N^{\infty} \frac{(x - [x]) \log x}{x^{s+1}} dx - \frac{N^{1-s} \log N}{s-1} + \frac{N^{1-s}}{(s-1)^2} \quad (3.3.27)$$

Theorem 3.3.4. For all $\epsilon > 0$ there exists $M(\epsilon)$ such that: $|\zeta(s)| \leq M \log t$ and $|\zeta'(s)| \leq M \log^2 t$ $\forall s$ with $\sigma \geq 1/2$ satisfying:

$$\frac{\epsilon}{\log t} > 1 - \sigma \quad \text{and} \quad t \geq e \quad (3.3.28)$$

Proof. If $\sigma \geq 2$ we have: $|\zeta(s)| \leq \zeta(2)$ and $|\zeta'(s)| \leq \zeta'(2)$ and the inequalities are satisfied. Now suppose that: $\sigma < 2$ then: $|s| \leq \sigma + t \leq 2 + t < 2t$ and $t \geq |s-1| \implies \frac{1}{|s-1|} \leq \frac{1}{t}$ Using equation (3.3.26) to estimate $|\zeta(s)|$ we get:

$$|\zeta(s)| \leq \sum_{n=1}^N \frac{1}{n^\sigma} + 2t \int_N^{\infty} \frac{1}{x^{\sigma+1}} dx + \frac{N^{1-\sigma}}{t} = \sum_{n=1}^N n^{-\sigma} + \frac{2t}{\sigma N^\sigma} + \frac{N^{1-\sigma}}{t} \quad (3.3.29)$$

Let N depend on t by choosing $N = [t]$; Then $N \leq t < N + 1$ and $\log n \leq \log t$ if $n \leq N$. Now we apply assumption (3.3.28) and note that the exponential function is monotonically increasing, hence we have:

$$n^{-\sigma} = \frac{n^{1-\sigma}}{n} = \frac{e^{(1-\sigma) \log n}}{n} < \frac{e^{\epsilon(\frac{\log n}{\log t})}}{n} \leq \frac{e^\epsilon}{n} = O\left(\frac{1}{n}\right) \quad (3.3.30)$$

Moreover since $n^{-\sigma} \geq N^{-\sigma}$ we have $\frac{2t}{\sigma N^\sigma} = O\left(\frac{N+1}{N}\right) = O(1)$ and $\frac{N^{1-\sigma}}{t} = \frac{N}{tN^\sigma} = O\left(\frac{1}{n}\right) = O(1)$;

Therefore we conclude that $|\zeta(s)| = O\left(\sum_{n=1}^N \frac{1}{n}\right) + O(1) = O(\log N) + O(1) = O(\log t)$

We apply the same type of argument to (3.3.27) the only difference is that an extra factor of $\log N$ will appear on the right but since $\log N = O(\log t)$ so we have: $|\zeta'(s)| = O(\log^2 t)$

■

3.3.4 Non-vanishing of $\zeta(s)$ on the Line $\Re(s) = 1$

This is the key ingredient, which was proved independently by de la Vallée Poussin's and Hadamard in 1896. The elegant proof which is detailed below is due to de la Vallée Poussin [31] [53] (Pg. 41). For Hadamard's alternative proof, please refer to [26]. The proof of this nontrivial property of the Riemann zeta-function allowed both de la Vallée Poussin and Hadamard to deduce the Prime Number Theorem. De la Vallée Poussin's proof involves exploiting a simple trigonometric identity as we shall see.

Theorem 3.3.5. *If suppose $s = \sigma + it$ and $\sigma > 1$ we have:*

$$\zeta(\sigma)^3 |\zeta(\sigma + it)|^4 |\zeta(\sigma + 2it)| \geq 1 \quad (3.3.31)$$

Proof. First, we will need to deploy the following identity (Proof can be found in Apostol - Chapter 11.9 [3]):

$$\zeta(s) = e^{G(s)} \quad \text{where} \quad G(s) = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log n} n^{-s} = \sum_p \sum_{m=1}^{\infty} \frac{1}{mp^{m\sigma}} \quad (\sigma > 1) \quad (3.3.32)$$

Which can be written as:

$$\zeta(s) = \exp\left\{\sum_p \sum_{m=1}^{\infty} \frac{e^{-imt \log p}}{mp^{m\sigma}}\right\} \implies |\zeta(s)| = \exp\left\{\sum_p \sum_{m=1}^{\infty} \frac{\cos mt \log p}{mp^{m\sigma}}\right\} \quad (3.3.33)$$

Second, we may apply this for the cases $s = \sigma$, $s = \sigma + it$ and $s = \sigma + 2it$, to obtain:

$$\zeta(\sigma)^3 |\zeta(\sigma + it)|^4 |\zeta(\sigma + 2it)| = \exp\left\{\sum_p \sum_{m=1}^{\infty} \frac{3 + 4 \cos mt \log p + \cos 2mt \log p}{mp^{m\sigma}}\right\} \quad (3.3.34)$$

But from trigonometry we have $3 + 4 \cos \theta + \cos 2\theta = 3 + 4 \cos \theta + 2 \cos^2 \theta - 1 = 2(1 + \cos \theta)^2 \geq 0$.

The series is nonnegative and we get the result. ■

Theorem 3.3.6. $\zeta(1 + it) \neq 0 \quad \forall t \in \mathbb{R}$.

Proof. Let $s = \sigma + it$, it suffices to look at the case $t \neq 0$; We rewrite the inequality (3.3.31) to get:

$$\{(\sigma - 1)\zeta(\sigma)\}^3 \left| \frac{\zeta(\sigma + it)}{\sigma - 1} \right|^4 |\zeta(\sigma + 2it)| \geq \frac{1}{\sigma - 1} \quad (3.3.35)$$

Now observe that $\lim_{\sigma \rightarrow 1^+} (\sigma - 1)\zeta(\sigma) = 1$ since $\zeta(s)$ has a residue 1 at pole $s = 1$.

Moreover: $\lim_{\sigma \rightarrow 1^+} |\zeta(\sigma + 2it)| = |\zeta(1 + 2it)|$.

Next, assume to the contrary that $\zeta(1 + it) = 0$; We can then write the quotient:

$$\lim_{\sigma \rightarrow 1^+} \left| \frac{\zeta(\sigma + it) - \zeta(1 + it)}{\sigma - 1} \right|^4 = |\zeta'(\sigma + it)|^4 \quad (3.3.36)$$

So the left side of the inequality tends to $|\zeta'(\sigma + it)|^4 |\zeta(1 + 2it)|$ as $\sigma \rightarrow 1^+$ while the right side tends to ∞ as $\sigma \rightarrow 1^+$. A contradiction. ■

Theorem 3.3.7. For all $\epsilon > 0$ there exists $M(\epsilon)$ such that $|\frac{1}{\zeta(s)}| \leq M \log^7 t$ and $|\frac{\zeta'(s)}{\zeta(s)}| \leq M \log^9 t$ $\forall s$ with $\sigma \geq 1$ and $t \geq e$

Proof. The inequality hold for the case $\sigma \geq 2$ since:

$$\left| \frac{1}{\zeta(s)} \right| = \left| \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \right| \leq \sum_{n=1}^{\infty} \frac{1}{n^2} \leq \zeta(2) \quad (3.3.37)$$

Moreover:

$$\left| \frac{\zeta'(s)}{\zeta(s)} \right| \leq \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^2} \quad (3.3.38)$$

Now suppose that $1 \leq \sigma \leq 2$ and rewrite the inequality (3.3.31) as follows:

$$\zeta(\sigma)^{3/4} |\zeta(\sigma + 2it)|^{1/4} \geq \frac{1}{|\zeta(\sigma + it)|} \quad (3.3.39)$$

Now the quantity $(\sigma - 1)\zeta(\sigma)$ is bounded in the interval say by some constant M ; Hence: $\zeta(\sigma) \leq \frac{M}{\sigma - 1}$ if $1 < \sigma \leq 2$ Also $\zeta(\sigma + 2it) = O(\log t)$ by (3.3.4); Substituting the bounds derived into the inequality yields:

$$\frac{1}{|\zeta(\sigma + it)|} \leq \frac{M^{3/4}(\log t)^{1/4}}{(\sigma - 1)^{3/4}} = \frac{A(\log t)^{1/4}}{(\sigma - 1)^{3/4}} \quad (3.3.40)$$

Where A is an absolute constant. Hence for some constant $B > 0$ we have:

$$\frac{1}{|\zeta(\sigma + it)|} > \frac{B(\sigma - 1)^{3/4}}{(\log t)^{1/4}} \quad \text{if } 1 < \sigma \leq 2 \quad \text{and } t \geq e \quad (3.3.41)$$

Which holds for $\sigma = 1$. Now let α be any number satisfying $1 < \alpha < 2$, then $1 \leq \sigma \leq \alpha, t \geq e$.

Using (3.3.4) we get

$$\begin{aligned} |\zeta(\sigma + it) - \zeta(\alpha + it)| &\leq \int_{\sigma}^{\alpha} |\zeta'(u + it)| du \\ &\leq (\alpha - \sigma)M \log^2 t \\ &\leq (\alpha - 1)M \log^2 t \end{aligned} \quad (3.3.42)$$

Applying the triangle inequality yields

$$\begin{aligned}
|\zeta(\sigma + it)| &\geq |\zeta(\alpha + it)| - |\zeta(\sigma + it) - \zeta(\alpha + it)| \\
&\geq |\zeta(\alpha + it)| - (\alpha - 1)M \log^2 t \\
&\geq \frac{B(\alpha - 1)^{3/4}}{(\log t)^{1/4}} - (\alpha - 1)M \log^2 t
\end{aligned} \tag{3.3.43}$$

Which holds for any α satisfying $1 < \alpha < 2$

Choose $\alpha = 1 + \left(\frac{B}{2M}\right)^4 \frac{1}{(\log t)^9}$, then we get:

$$|\zeta(\sigma + it)| \geq (\alpha - 1)M \log^2 t = \frac{C}{(\log t)^7} \tag{3.3.44}$$

This proves that $|\zeta(s)| \geq C \log^{-7} t$ where $\sigma \geq 1$ and $t \geq e$ giving us a corresponding upper bound for $\left|\frac{1}{\zeta(s)}\right|$. To estimate the bound for $\left|\frac{\zeta'(s)}{\zeta(s)}\right|$ we apply (3.3.4) and follow the same steps to get an extra factor of $\log^2 t$

■

3.3.5 Finalizing the Proof

Lemma 3.3.4. *If $f(s)$ has a pole of order k at $s = \alpha$ then the quotient $\frac{f'(s)}{f(s)}$ has a first order pole at $s = \alpha$ with residue $-k$.*

Proof. Let $g(s) = f(s)(s - \alpha)^k$ where g is analytic at α and $g(\alpha) \neq 0$; Then we have:

$$\begin{aligned}
f'(s) &= \frac{g'(s)}{(s - \alpha)^k} - \frac{kg(s)}{(s - \alpha)^{k+1}} = \frac{g(s)}{(s - \alpha)^k} \left\{ \frac{-k}{s - \alpha} + \frac{g'(s)}{g(s)} \right\} \\
\implies \frac{f'(s)}{f(s)} &= \frac{-k}{s - \alpha} + \frac{g'(s)}{g(s)}
\end{aligned}$$

■

Theorem 3.3.8. *The following statement holds:*

$$h(s) = -\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1} \text{ is analytic at } s = 1 \quad (3.3.45)$$

Proof. It is known that $\zeta(s)$ has a pole of order 1 at $s = 1$; Applying lemma 3.3.4 we conclude that $-\frac{\zeta'(s)}{\zeta(s)}$ has a first order pole at $s = 1$, it is also clear that $\frac{1}{s-1}$ has a pole at $s = 1$ of order 1, hence their difference is analytic at $s = 1$.

■

Theorem 3.3.9. *For $x \geq 1$ we have:*

$$\frac{\Phi(x)}{x^2} - \frac{1}{2}\left(1 - \frac{1}{x}\right)^2 = \frac{1}{2\pi} \int_{-\infty}^{+\infty} h(1+it)e^{it \log x} dt \quad (3.3.46)$$

Where

$$\int_{-\infty}^{+\infty} |h(1+it)| < \infty$$

Hence by **Riemann-Lebesgue lemma**: $\Phi(x) \sim x^2/2$ as $x \rightarrow \infty$.

Proof. Following the result of theorem 3.3.3. First we are going to need shift the path of integration to the line $\Re(s) = 1$ to fix c at 1. Fortunately, this can be achieved by showing that contour integral taken counter clockwise around the loop is independent of the path, as shown in figure 3.2. Indeed this is the case here since the integrand $x^{s-1}h(s)$ is analytic inside and on R by the previous theorem 3.3.8. Next, we show that the integral along horizontal line segments tend to 0 as $T \rightarrow \infty$, since the integrand has the same absolute value at conjugate points, without loss of generality, it suffices to consider the upper segment, namely $t = T$. On this line segment, we have the following estimates

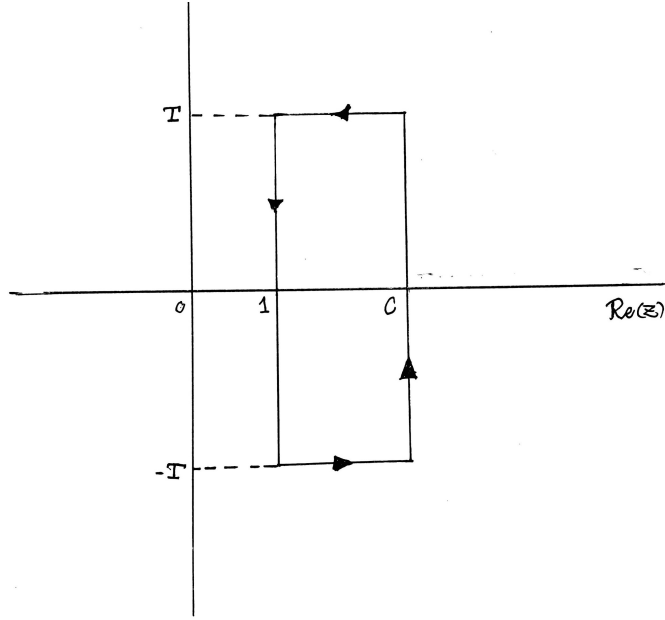


Figure 3.2: Contour Integral for R

$$\left| \frac{1}{s(s+1)} \right| \leq \frac{1}{T^2} \quad \text{and} \quad \left| \frac{1}{s(s+1)(s+2)} \right| \leq \frac{1}{T^3} \leq \frac{1}{T^2} \quad (3.3.47)$$

Moreover, by theorem 3.3.7 there is a constant M such that $\left| \frac{\zeta'(s)}{\zeta(s)} \right| \leq M \log^9 t$ if $\sigma \geq 1$ and $t \geq e$ hence if $T \geq e$. Applying these we can estimate an upper bound for $|h(s)|$

$$|h(s)| \leq \frac{M \log^9 T}{T^2} \quad (3.3.48)$$

Therefore the integral along the upper line segment can be estimated as follows

$$\left| \int_{1+iT}^{c+iT} x^{s-1} h(s) ds \right| \leq \int_1^c x^{c-1} \frac{M \log^9 T}{T^2} d\sigma = M x^{c-1} \frac{\log^9 T}{T^2} (c-1) \quad (3.3.49)$$

Which tends 0 as $T \rightarrow \infty$ and thus we have:

$$\int_{c-\infty i}^{c+\infty i} x^{s-1} h(s) ds = \int_{1-\infty i}^{1+\infty i} x^{s-1} h(s) ds \quad (3.3.50)$$

And on the line $\Re(s) = 1$ we obtain:

$$\frac{1}{2\pi i} \int_{1-\infty i}^{1+\infty i} x^{s-1} h(s) ds = \frac{1}{2\pi} \int_{-\infty}^{+\infty} h(1+it) e^{it \log x} dt \quad (3.3.51)$$

This proves the theorem. Now observe that

$$\int_{-\infty}^{+\infty} |h(1+it)| dt = \int_{-e}^e |h(1+it)| dt + \int_e^{\infty} |h(1+it)| dt + \int_{-\infty}^{-e} |h(1+it)| dt \quad (3.3.52)$$

On the integral from e to ∞ we have:

$$|h(1+it)| \leq \frac{M \log^9 t}{t^2} \quad (3.3.53)$$

Therefore $\int_e^{\infty} |h(1+it)| dt$ Converges, similarly $\int_{-\infty}^{-e} |h(1+it)| dt$ converges; Therefore $\int_{-\infty}^{+\infty} |h(1+it)| dt$ converges. By the **Riemann-Lebesgue lemma** we conclude that:

$$\lim_{x \rightarrow \infty} \int_{-\infty}^{+\infty} h(1+it) e^{it \log x} dt = 0 \quad (3.3.54)$$

Hence the right side (3.3.46) of vanishes as $x \rightarrow \infty$ while the left side gives us $\Phi(x) \sim x^2/2$. And we deduce that $\psi(x) \sim x$ as $x \rightarrow \infty$ by theorem 3.3.1. This concludes the analytic proof of the prime number theorem.



3.4 Tauberian Proofs of the Prime Number Theorem

3.4.1 Introduction

It is a well-known that if we have a summation of the form:

$$\sum_{n=0}^{\infty} a_n = S \quad (3.4.1)$$

Then we can say that the following holds:

$$\lim_{x \rightarrow 0^-} \frac{1}{1-x} \sum_{n=0}^{\infty} a_n x^n = S \quad (3.4.2)$$

This result is due to Abel, and these theorems are grouped in what is known as "Abelian Theorems". The converse of this result is not always true. One has to impose certain conditions in order for it to hold. In 1897, A. Tauber [59] [44] established that the result holds if $a_n = o(1/n)$. Hardy was first to coin the term "Tauberian Theorems". Hardy and Littlewood tackled this problem and derived the refined condition: $na_n > -K$ for some constant K . One special series of this type is the following Lambert series:

$$\sum_{n=0}^{\infty} a_n \frac{x^n}{1-x^n} \quad (3.4.3)$$

Which is closely related to the distribution of primes. Many earlier attempts at attacking the Prime Number Theorem using Lambert Series had failed. It wasn't until Hardy and Littlewood finally have showed that the Prime Number Theorem is equivalent to a Tauberian theorem concerning Lambert series that mathematicians were able to successfully give a rigorous proof. There are

many proofs of the Prime Number Theorem. In what follows, we shall demonstrate two proofs concerning Tauberian approaches to the Prime Number Theorem:

3.4.2 Wiener's Proof of the Prime Number Theorem

In his paper *Tauberian Theorems* [59], Norbert Wiener proved many results that provide necessary and sufficient conditions for when a function in L_1 or L_2 can be approximated by translations of a given function. Equivalently, the set of linear combinations of translations of f are dense if and only if the zero set of the Fourier transform of f has Lebesgue measure equal to zero. He then reformulated his result as a Tauberian theorem concerning a Lambert Series and showed that the Prime Number Theorem follows as a result. In fact, Wiener refined the proof of Landu of the Prime Number Theorem which deployed the following Lemma.

Theorem 3.4.1. (Landu's Lemma) *Let $f(z) = \sum_{n=1}^{\infty} a_n n^z$ be a convergent Dirichlet series with $a_n \geq 0$. Moreover, suppose that*

$$F(z) = f(z) - \frac{A}{z-1} \quad (\Re(z) > 1) \quad (3.4.4)$$

is an analytic continuation of $f(z)$ at $\Re(z) = 1$ except for a pole at $z = 1$ of order one and has a principle part $\frac{A}{z-1}$ where A is some constant. And suppose that:

$$F(z) = O(|z|^\alpha) \quad (3.4.5)$$

Then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n a_k = A \quad (3.4.6)$$

It was Landu [34] who deployed this lemma for the proof of the Prime Number Theorem using

$a_n = \Lambda(n)$, $F(z) = -\frac{\zeta'(z)}{\zeta(z)}$, and the condition that $\zeta'(s)/\zeta(s)$ is of order $O(|z|^\alpha)$. Hardy and Littlewood relaxed this condition to $O(e^{\alpha|z|})$. Later, Wiener showed with his student Ikehara, that this requirement is not necessary, as we shall see in the following theorem. Before we do that we shall state the following theorem (Proof can be found in Wiener [59] Page 30).

Theorem 3.4.2. *Assuming the following set of hypotheses:*

(I). $\varphi(\lambda)$ is a function of bounded total variation over any interval $(\epsilon, 1/\epsilon)$ where $\epsilon \in (0, 1)$ and $\varphi(0) = 0$.

(II).

$$\int_u^{2u} \frac{1}{\lambda} |d\varphi(\lambda)| - \int_u^{2u} \frac{1}{\lambda} d\varphi(\lambda) \leq N \quad (0 < u < \infty) \quad (3.4.7)$$

(III). Σ is a class of continuous functions $M(\lambda)$ where $\sum_{k=-\infty}^{\infty} \max_{2^k \leq \lambda \leq 2^{k+1}} \lambda |M(\lambda)|$ converges. Let $N_1, N_2 \in \Sigma$.

(IV).

$$\lim_{\lambda \rightarrow 0^+} \frac{1}{\lambda} \int_0^\infty N_1\left(\frac{\mu}{\lambda}\right) d\varphi(\mu) = A \int_0^\infty N_1(\mu) d\mu \quad (3.4.8)$$

(V).

$$\int_0^\infty N_1(\lambda) \lambda^{iu} d\lambda \neq 0 \quad (\forall u \in \mathbb{R}) \quad (3.4.9)$$

(VI).

$$M(\lambda) \geq 0, \left| \frac{1}{\lambda} \int_0^\infty M\left(\frac{\mu}{\lambda}\right) d\varphi(\mu) \right| \leq \text{const} \quad (0 \leq \lambda < \infty, \text{ where } M(\lambda) \in \Sigma) \quad (3.4.10)$$

Then

$$\lim_{\lambda \rightarrow 0^+} \frac{1}{\lambda} \int_0^\infty N_2\left(\frac{\mu}{\lambda}\right) d\varphi(\mu) = A \int_0^\infty N_2(\mu) d\mu \quad (3.4.11)$$

Theorem 3.4.3. (Wiener-Ikehara's Tauberian Theorem) Let $\alpha(x)$ be a monotone increasing function, and let:

$$f(u) = \int_{0^+}^{\infty} x^{-u} d\alpha(x) \quad (\Re(u) > 1) \quad (3.4.12)$$

Let

$$g(u) = f(u) - \frac{A}{u-1} \quad (3.4.13)$$

converges to a finite limit as $\Re(u) \rightarrow 1$ over any finite interval of the line $\Re(u) = 1$ Then:

$$\lim_{n \rightarrow \infty} \frac{\alpha(n)}{n} = A \quad (3.4.14)$$

Where the integral is Riemann-Stieltjes integral.

Proof. Let $\beta(\xi) = \alpha(e^\xi)e^{-\xi} + \int_0^\xi e^{-\xi}\alpha(e^\xi)d\xi$ - At hence implies $d\beta(\xi) = e^{-\xi}d\alpha(e^\xi) - A d\xi$

Assume that

$$\beta(x) = \beta(0^+) \quad (-\infty \leq x \leq 0) \quad (3.4.15)$$

then (3.4.13) becomes:

$$g(u) = \int_{-\infty}^{\infty} e^{(1-u)\xi} d\beta(\xi) \quad (\Re(u) > 1) \quad (3.4.16)$$

We need to show that

$$\lim_{\eta \rightarrow \infty} \int_{-\infty}^{\eta} e^{(\xi-\eta)\xi} d\beta(\xi) = 0 \quad (3.4.17)$$

Wiener showed that this is equivalent to (3.4.14) [59]. If $\epsilon > 0$ and η , and taking into account that

the double integral is absolutely convergent we have:

$$\begin{aligned}
\int_{-B}^B \left(1 - \frac{|u|}{B}\right) g(iu + \epsilon + 1) e^{iu\eta} du &= - \int_{-B}^B \left(1 - \frac{|u|}{B}\right) du \int_{-\infty}^{\infty} e^{iu(\eta-\xi)} e^{-\epsilon\xi} d\beta(\xi) \\
&= - \int_{-\infty}^{\infty} e^{-\epsilon\xi} d\beta(\xi) \int_{-B}^B \left(1 - \frac{|u|}{B}\right) e^{iu(\eta-\xi)} du \quad (3.4.18) \\
&= - \int_{-\infty}^{\infty} \frac{2(\cos(B(\eta-\xi)) - 1)}{B(\eta-\xi)^2} e^{-\epsilon\xi} d\beta(\xi)
\end{aligned}$$

Taking limits as $\epsilon \rightarrow 0$, we have by a theorem due to Bray [12] related to Stieltjes integrals:

$$\int_{-B}^B \left(1 - \frac{|u|}{B}\right) g(iu + 1) e^{iu\eta} d\eta = - \int_{-\infty}^{\infty} \frac{2(\cos(B(\eta-\xi)) - 1)}{B(\eta-\xi)^2} d\beta(\xi) \quad (3.4.19)$$

Taking limits as $\eta \rightarrow \infty$, and noting that $\left(1 - \frac{|u|}{B}\right) g(iu + 1) e^{iu\eta}$ is summable over $(-B, B)$ we have:

$$0 = \lim_{\eta \rightarrow \infty} \int_{-B}^B \left(1 - \frac{|u|}{B}\right) g(iu + 1) e^{iu\eta} du = - \lim_{\eta \rightarrow \infty} \int_{-\infty}^{\infty} \frac{2(\cos(B(\eta-\xi)) - 1)}{B(\eta-\xi)^2} d\beta(\xi) \quad (3.4.20)$$

Due to this and (3.4.15) we find that $\int_{-\infty}^{\infty} \frac{2(\cos(B(\eta-\xi)) - 1)}{B(\eta-\xi)^2} d\beta(\xi)$ is bounded. Moreover, we have

$\int_n^{n+1} d\beta(\xi) > -A$. Finally, we show that condition (V) of Theorem 3.4.1 is satisfied since:

$$\int_{-\infty}^{\infty} \frac{2(\cos(B(\eta)) - 1)}{B(\eta)^2} e^{iu\eta} d\eta = \frac{1}{2\pi} \left(1 - \frac{|u|}{B}\right) \quad (|u| < B) \quad (3.4.21)$$

Since all conditions of Theorem 3.4.1 are satisfied, the result holds. ■

Corollary 3.4.1. *Let $\alpha(x)$ be a monotone increasing function, and let:*

$$f(u) = \int_{0+}^{\infty} x^{-u} d\alpha(x) \quad (\Re(u) > 1) \quad (3.4.22)$$

Let

$$g(u) = e^{f(u)}(u-1)^A \quad (0 < A < 4/3) \quad (3.4.23)$$

when continued analytically and is regular for $\Re(u) = 1$, and does not vanish at $u = 1$ Then:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \int_{0+}^n \log x \, d\alpha(x) = A \quad (3.4.24)$$

Where the integral is Riemann-Stieltjes integral. (proof is given by Wiener [59] page 47)

Before applying this corollary, we note first that we have the well known relation:

$$\Pi(x) = \pi(x) + \frac{1}{2}\pi(x^{1/2}) + \frac{1}{3}\pi(x^{1/3}) + \dots \quad (3.4.25)$$

Then

$$|\Pi(x) - \pi(x)| = \left| \frac{1}{2}\pi(x^{1/2}) + \dots + \frac{\pi\left(x^{\frac{1}{(\log_2 x + 1)}}\right)}{\log_2 x + 1} \right| \leq x^{1/2}(\log_2 x + 1) = O(x^{1/2} \log x) \quad (3.4.26)$$

Hence

$$\lim_{x \rightarrow \infty} \frac{\Pi(x)}{x} = 0 \implies \lim_{n \rightarrow \infty} \frac{1}{n} \int_0^n \frac{\Pi(x)}{x} dx = 0 \quad (3.4.27)$$

Now we apply the corollary with $\alpha(x) = \Pi(x)$ and $e^{f(u)} = \zeta(u)$, which satisfy the hypotheses and

therefore we get:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \int_{0+}^n \log x \, d\Pi(x) = 1 \quad (3.4.28)$$

Integrating this expression by parts yields:

$$\lim_{n \rightarrow \infty} \left[\frac{\Pi(n) \log n}{n} - \frac{1}{n} \int_0^n \frac{\Pi(x)}{x} dx \right] = 1 \quad (3.4.29)$$

From (3.4.27) we see that $\lim_{n \rightarrow \infty} \frac{\Pi(n) \log n}{n} = 1 \implies \Pi(n) \sim \frac{n}{\log n}$. Therefore, from the inequality (3.4.26) we get:

$$\pi(n) \sim \frac{n}{\log n} \quad (3.4.30)$$

Remark. We note that relation (3.4.28) in the last proof was the missing piece of the puzzle that was needed to imply the Prime Number Theorem. The theory of Tauberian theorems developed by Wiener and Ikehara gave a shortcut to the proof. Alternatively, the same conclusion can be achieved if we consider the following limit of averages of von Mangoldt function:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \Lambda(k) = 1 \quad (3.4.31)$$

Which was originally considered by Hadamard and de la Vallée Poussin's. However, this later relation was derived by looking at nontrivial properties of the Lambert series $\sum_{n=1}^{\infty} \Lambda(n) \frac{x^n}{1-x^n}$.

3.4.3 Newman's Proof of the Prime Number Theorem

Newman published his paper "Short Proof of the Prime Number Theorem" [43] [61] [42] which is much shorter than its analytical counterpart by deploying a particular form of a Tauberian theorem. Since the proof shares some elements with the analytical case, we will just refer to the results here.

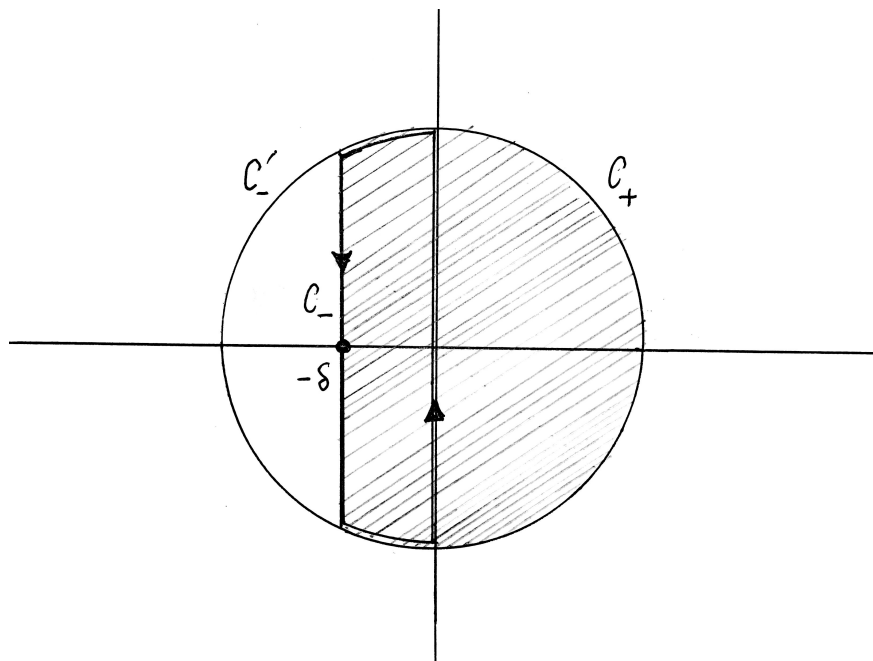


Figure 3.3: Newman's Contour Integral

Newman technically deduced that Chebyshev's $\vartheta(x) \sim x$ from his Tauberian theorem (which was as we showed earlier to be equivalent to the main PNT I (refer to (3.2.2) and (3.2.4)). Before we proceed with the proof we shall prove his Tauberian theorem:

Theorem 3.4.4. (Newman's Tauberian Theorem) *Let $f(t)$ ($t \geq 0$) be bounded and locally integrable function and suppose that the function $g(z) = \int_0^\infty f(t)e^{-zt}dt$ ($\Re(z) > 0$) extends holomorphically to $\Re(z) \geq 0$. Then $\int_0^\infty f(t)dt$ converges (and is equal to $g(0)$).*

Proof. For $T > 0$ let $g_T(z) = \int_0^T f(t)e^{-zt}dt$ and notice that this is an entire function. Next we need to show that $\lim_{T \rightarrow \infty} g_T(0) = g(0)$. Let R be a radius and $\{z \in \mathbb{C} : |z| \leq R, \Re(z) \geq -\delta\}$ be a region where C (figure 3.3) is the boundary where $\delta_R > 0$ is small enough for $g(z)$ to be holomorphic in and on the region. Then by Cauchy's Theorem:

$$g(0) - g_T(0) = \frac{1}{2\pi i} \int_C \left(g(z) - g_T(z) \right) e^{zT} \left(1 + \frac{z^2}{R^2} \right) \frac{dz}{z} \quad (3.4.32)$$

Note that on the semicircle $C_+ = C \cap \{\Re(z) > 0\}$ we have:

$$|g(z) - g_T(z)| = \left| \int_T^\infty f(t) e^{-zt} dt \right| \leq \max_{t \geq 0} |f(t)| \cdot \int_T^\infty |e^{-zt}| dt \leq \frac{M e^{-\Re(z)T}}{\Re(z)} \quad (\Re(z) > 0) \quad (3.4.33)$$

Where $M = \max_{t \geq 0} |f(t)|$ and note that since $z = Re^{i\theta}$ hence

$$\frac{z^2}{R^2} = \frac{z^2}{z\bar{z}} = \left(1 + \frac{z}{\bar{z}} \right) = \left(\frac{z + \bar{z}}{\bar{z}} \right) \cdot \frac{1}{z} = \frac{2\Re(z)}{R^2} \quad (3.4.34)$$

Therefore

$$\left| e^{zT} \left(1 + \frac{z^2}{R^2} \right) \frac{1}{z} \right| = e^{\Re(z)T} \cdot \frac{2\Re(z)}{R^2} \quad (3.4.35)$$

Combining these estimates we get that the integrand in (3.4.32) is bounded by $\frac{2M}{R^2}$. since:

$$|g(z) - g_T(z)| \cdot \left| e^{zT} \left(1 + \frac{z^2}{R^2} \right) \frac{1}{z} \right| \leq \frac{M e^{-\Re(z)T} \cdot e^{\Re(z)T}}{\Re(z)} \cdot \frac{2\Re(z)}{R^2} \quad (3.4.36)$$

Thus we have:

$$|g(0) - g_T(0)| = \left| \frac{1}{2\pi i} \int_C (g(z) - g_T(z)) e^{zT} \left(1 + \frac{z^2}{R^2} \right) \frac{dz}{z} \right| \leq \frac{1}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2M\pi R}{R^2} d\theta = \frac{M}{R} \quad (3.4.37)$$

For the integral over $C_- = C \cap \{\Re(z) < 0\}$ we look at $g(z)$ and $g_T(z)$ separately. Since g_T is an entire function, the path of integration involving the term g_T can be replaced by the semicircle $C'_- = \{z \in \mathbb{C}, |z| = R, \Re(z) < 0\}$ and the contribution to $g(0) - g_T(0)$ of the integral along this semicircle becomes bounded since:

$$|g_T(z)| = \left| \int_0^T f(t)e^{-z\theta} dt \right| \leq M \int_0^T |e^{-z\theta}| d\theta = \frac{Me^{-\Re(z)T}}{|\Re(z)|} \quad (\Re(z) < 0) \quad (3.4.38)$$

The integral over $C''_- = \{z \in \mathbb{C}, |z| = R, \Re(z) > 0\}$ tends to 0 as $T \rightarrow \infty$ since the integrand consists of $g(z)(1+z^2/R^2)/z$, which is independent of T and the function e^{zT} goes to 0 uniformly. Hence $\limsup_{T \rightarrow \infty} |g(0) - g_T(0)| \leq \frac{2B}{R}$, since R is arbitrary, we are done. ■

Now we proceed with an outline of the proof:

(I). Newman introduced the following function: $\Phi(s) = \sum_{p \text{ prime}} \frac{\log p}{p^s} = s \int_0^\infty e^{-st} \vartheta(e^t) dt$ (not to be confused with $\Phi(s)$ used in the analytic proof). Which is easily seen to be absolutely and locally uniformly convergent for $\sigma > 1$ (II). Establishing Euler's product which follows directly from (1.2.44) with the Trivial Dirichlet character $\chi(n) = 1$:

$$\zeta(s) = L(s, 1) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1 - p^{-s}} \quad \Re(s) > 1 \quad (3.4.39)$$

(III). Showing that $\vartheta(x) = O(x)$.

Proof. This follows from the identity:

$$4^n = (1+1)^{2n} \geq \binom{2n}{n} \geq \prod_{n < p \leq 2n} p = e^{\vartheta(2n) - \vartheta(n)} \quad (3.4.40)$$

hence $\vartheta(x)$ changes by $O(\log x)$ if x changes by $O(1)$, $\vartheta(x) - \vartheta(x/2) \leq Cx$ for any $C > \log 2$ for

all $x \geq x_0 = x_0(C)$. Summing over $x, x/2, \dots, x/2^r$ we obtain $\vartheta(x) \leq 2Cx + O(1)$. ■

(IV). Establishing holomorphicity of $\Phi(s) - \frac{1}{s-1}$ for $\Re(s) > 1$ which follows from (3.3.8) using the relation:

$$h(s) = -\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1} = \left(\Phi(s) - \frac{1}{s-1} \right) + \sum_{p \text{ prime}} \frac{\log p}{p^s(p^s-1)} \quad (3.4.41)$$

by applying Lemma (3.3.4) and noting that the final sum $\sum_{p \text{ prime}} \frac{\log p}{p^s(p^s-1)}$ converges for $\sigma > 1/2$ (V). $\zeta(1+iy) \neq 0$. Proved in Theorem 3.3.6. (VI). $\int_1^\infty \frac{\vartheta(x)-x}{x^2} dx$ converges.

Proof. Since $\frac{\Phi(z+1)}{(z+1)} = \int_0^\infty e^{-(z+1)t} \vartheta(e^t) dt$ and $\frac{1}{z} = \int_0^\infty e^{-zt} dt$, a direct application of Theorem (3.4.4) with the functions:

$$f(t) = \vartheta(e^t)e^{-t} - 1 \quad g(z) = \frac{\Phi(z+1)}{(z+1)} - \frac{1}{z} \quad (3.4.42)$$

which are satisfied by (III) (3.4.3) and (IV) (3.4.3) yields the result. ■

(VII). Establishing the asymptotic relation $\vartheta(x) \sim x$ by showing that (VI) converge to zero. Consequently, proving the Prime Number Theorem. For great remarks regarding the proof we refer the reader to J. Korevaar's article [33].

3.5 Discussion

We note that the proofs discussed thus far have a mixture of both analytical and elementary elements [19]. Some authors refer to a proof as being elementary if most of the methods used are derived from elementary techniques. It is important not to confuse elementary here as being easy.

For an excellent exposition of a proof of this type, we refer to Erdős's paper "*On a New Method in Elementary Number Theory Which Leads to An Elementary Proof of the Prime Number Theorem*" [22]. On the other hand, authors refer to a proof as being analytic if most of the methods deployed come from complex analysis. Analytic methods organize arithmetic data through use of complex analytic functions such as Riemann zeta-function. It is worth mentioning that the central theme of all these proofs that we showcased thus far required the nontrivial property: that $\zeta(x + iy) \neq 0$ on the real line $x = 1$, which turned out to be the necessary condition to prove the Prime Number Theorem. The power in Weiner's proof came from simplifying Hadamard and de la Valle-Poussin's complicated proof by means of Tauberian theorems and illuminating the fact that the converse of the PNT is in fact true, that is given that the PNT holds, we can deduce that $\zeta(x + iy) \neq 0$ on the real line $x = 1$. Finally, we demonstrate two important implications of the Prime Number Theorem (which are in fact, equivalent) in the following beautiful asymptotic relations:

$$\lim_{x \rightarrow \infty} \frac{\pi(x) \log \pi(x)}{x} = 1 \quad (3.5.1)$$

$$\lim_{x \rightarrow \infty} \frac{p_n}{n \log n} = 1 \quad \text{where } p \text{ is a prime number} \quad (3.5.2)$$

Therefore one can see that the asymptotic relation $p_n \sim n \log n$ tells us that the sequence of primes behave like $n \log n$. The proof of these implications can be found in [3] (Pg. 80-82).

CHAPTER 4: THE RIEMANN HYPOTHESIS

4.1 Introduction

We have explored in the previous chapter how to get a handle on the behavior of prime numbers using complex analysis, by calculating the number of primes less than or equal to a certain magnitude. In 1859, Bernhard Riemann introduced his zeta function and extensively studied its properties. He used the functional equation he derived along with some basic properties of the Gamma function to conclude that $\zeta(s)$ vanishes exactly whenever $s = -2n$ where $n \in \mathbb{N}$. He called these the *trivial zeros*. Although, he could not prove it, he noticed that all the other *non-trivial zeros* lie along the *critical line* $\Re z = \frac{1}{2}$. This became his famous conjecture; The *Riemann Hypothesis* [46].

Today, this problem is still unsettled. It is the Hilbert's eighth problem in David Hilbert's list of 23 unsolved problems as well as one of the Clay Mathematics Institute's Millennium Prize Problems. There has been many breakthroughs in the field of analytical number theory to support the truth of this claim. We will discuss in this chapter some of the main results, study equivalent formulations to the Riemann Hypothesis, and examine some of its consequences.

One consequence of proving the RH (Riemann Hypothesis) can be demonstrated in the following example. In the previous chapter we proved the following relation in the analytical approach to the Prime Number Theorem, which relates Riemann zeta function to Chebychev's ψ -function:

$$-\frac{\zeta'(s)}{\zeta(s)} = s \int_0^\infty \psi(x)x^{-(s+1)}dx \implies \psi(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left(-\frac{\zeta'(s)}{\zeta(s)} \right) x^s \frac{ds}{s} \quad (4.1.1)$$

Here the later formula is simply obtained by taking the inverse Mellin transform. With clever analysis of this relation using Perron's formula, von Mangoldt was able to derive the following relation:

$$\psi(x) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} - \log(2\pi) - \frac{1}{2} \log(1 - x^{-2}) \quad (4.1.2)$$

Where ρ denotes the zeros of the of Riemann zeta-function. In fact, one can deduce the Prime Number Theorem by showing:

$$\lim_{x \rightarrow \infty} \sum_{\rho} \frac{x^{\rho-1}}{\rho} = 0 \quad (4.1.3)$$

which follows from the result of non vanishing of $\zeta(s)$ on $\Re(s) = 1$. On the other hand, define the following function

$$J(x) = \pi(x) + \frac{1}{2}\pi(x^2) + \frac{1}{3}\pi(x^3) + \dots \quad (4.1.4)$$

From the Möbius inversion formula we get

$$\pi(x) = \sum \frac{\mu(n)}{n} J(x^{1/n}) \quad (4.1.5)$$

Riemann used this to derive the following relation (also known as *Riemann's Explicit Formula*) [46] [31]:

$$J(x) = \text{Li}(x) - \sum_{\rho} \text{Li}(x^{\rho}) - \log 2 + \int_x^{\infty} \frac{dt}{t(t^2 - 1) \log t} \quad x > 1 \quad (4.1.6)$$

Where

$$J(x) = -\frac{1}{2\pi i \log x} \int_{c-i\infty}^{c+i\infty} \frac{d}{ds} \left(\frac{\log \zeta}{s} \right) x^s ds \quad (4.1.7)$$

The inverse Mellin transform is used here again. We can see that the Prime Number Theorem can be obtained from Riemann's explicit formula by assuming the asymptotic laws $\pi(x) \sim J(x)$ and $x/\log x \sim \text{Li}(x)$ and showing:

$$\lim_{x \rightarrow \infty} \frac{\sum \text{Li}(x^\rho)}{(x \log x)} = 0 \quad (4.1.8)$$

Which is much more difficult to prove. Hence if we assume the RH, we get the *stronger* form of the prime number theorem:

$$\pi(x) = \text{Li}(x) + O(\sqrt{x} \log x) \quad (4.1.9)$$

The proof of this implication is found in Helege von Koch, *Sur La Distribution Des Nombres Premiers* [56]. By examining (4.1.5) and Riemann's Explicit Formula (4.1.6) one can see how the distribution of the nontrivial zeros ρ of $\zeta(s)$ could fully determine the distribution of the primes. Therefore it is important to study the behavior of the zeros of the Riemann zeta-function.

Definition 4.1.1. Define $N(T)$ to be the number of zeros of the $\zeta(s)$ function in the region $0 \leq \sigma \leq 1, 0 < t \leq T$.

Riemann conjectured that $N(T)$ of the non-trivial zeros $\rho = \beta + i\gamma$ with $0 < \gamma \leq T$ satisfies the following asymptotic formula [46]:

$$N(T) \sim \frac{T}{2\pi} \log \frac{T}{2\pi e} \quad (4.1.10)$$

Which later proved by Von Mangoldt, who rectified it to:

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e} + O(\log T) \quad (4.1.11)$$

Which is a direct application of the Argument Principle.

4.2 Equivalent Formulations of the Riemann Hypothesis

The beauty of mathematics lie in the fact that one can transform a difficult problem to another problem which can be easier to tackle. Consequently, it is natural to ask whether there exists some alternative formulations of the RH. We shall showcase some of these results here.

4.2.1 Formulation in Terms of the Divisor Function

In 1984, Robin [47] derived an inequality which he proved is equivalent to the Riemann hypothesis, namely:

$$\sigma(n) < e^\gamma n \log \log n \quad (n \geq 5041) \quad (4.2.1)$$

Where γ is the Euler-Mascheroni constant. Another equivalent formulation is due to Lagarias [36].

Which takes the form:

$$\sigma(n) \leq H_n + e^{H_n} \log(H_n) \quad (n \geq 1) \quad (4.2.2)$$

Where H_n is the Harmonic Number. In a recent publication [23], Eum et al (2015) improved upon Robin's bound on the divisor function $\sigma(n)$. Their statement satisfies the same inequality but with the added restriction $n \equiv 0 \pmod{6}$ and $\gcd(6, n/6) \neq 1$.

4.2.2 Derivative of the Riemann Zeta-Function

Can we relate the behavior of zeros of $\zeta'(s)$ to that of $\zeta(s)$? The affirmation is the statement of Speiser's Theorem[51]. In 1934 he showed that the RH is equivalent to the statement that there are

no zeros of $\zeta'(s)$ in the strip $0 < \Re(s) < 1/2$. Thus, the statement that $\zeta(s)$ only has nontrivial zeros on the critical line is equivalent to the statement that $\zeta'(s)$ does not have *any* zeros on the critical line.

4.2.3 An Integral Equation Related to the Riemann Hypothesis

In 1953, Salem showed that the RH is equivalent to the following theorem:

Theorem 4.2.1. *Suppose that f is a bounded measurable function on \mathbb{R} , then the RH is equivalent to the statement that the integral equation:*

$$\int_{-\infty}^{\infty} \frac{e^{-\sigma f(y)}}{e^{e^{x-y}} + 1} dy = 0 \quad (4.2.3)$$

has no bounded solution other than the trivial case $f(y) \equiv 0$ for $\frac{1}{2} < \sigma < 1$.

The proof can be found in [49]. Recently, Semyon Yakubovich published an equivalent formulation of the Riemann Hypothesis based on Salem's work [60] (2013). The following is the statement of their result:

Theorem 4.2.2. *Suppose that for a bounded measurable function f on \mathbb{R} we can define the Meijer transformation:*

$$(\mathcal{K}_n f)(x) = \int_{-\infty}^{+\infty} e^{-\delta u} K_0(2\sqrt{n}e^{(x-u)/2}) f(u) du \quad 1/2 < \delta < 1 \quad (4.2.4)$$

Where $K_0(z)$ is the modified Bessel function of the second kind, then the Riemann Hypothesis is

equivalent to the assertion that the following equation:

$$\sum_{n=1}^{\infty} d(n)[(\mathcal{K}_n f)(x) - 4(\mathcal{K}_{2n} f)(x) + 4(\mathcal{K}_{4n} f)(x)] = 0 \quad (4.2.5)$$

Has no non-trivial solutions.

Proof. Using properties of Mellin Transform and its convolution [55] on the zeta function series representation:

$$(1 - 2^{1-s})\zeta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} \quad (\Re(s) > 0) \quad (4.2.6)$$

yields:

$$\begin{aligned} \left[(1 - 2^{1-s})\zeta(s)\Gamma(s) \right]^2 &= \int_0^{\infty} t^{s-1} \int_0^{\infty} \frac{du}{u(e^{t/u} + 1)(e^u + 1)} \quad (\Re(s) > 0) \\ &= \int_0^{\infty} t^{s-1} \frac{1}{2\pi i} \int_{\mu-i\infty}^{\mu+i\infty} [(1 - 2^{1-s})\zeta(s)\Gamma(s)]^2 x^{-s} ds dt \\ &= \int_0^{\infty} \int_0^{\infty} \frac{t^s du dt}{u(e^{t/u} + 1)(e^u + 1)} > 0 \end{aligned} \quad (4.2.7)$$

Applying the identity $\zeta^2(s) = \sum_{n=1}^{\infty} \frac{d(n)}{n^s}$ and using the integral representations of the modified Bessel Function:

$$\begin{aligned} K_v(2\sqrt{x}) &= \frac{1}{4\pi i} \int_{a-i\infty}^{a+i\infty} \Gamma\left(s + \frac{v}{2}\right)\Gamma\left(s - \frac{v}{2}\right)x^{-s} ds, a > |\Re(v)| \\ K_v(x) &= \int_0^{\infty} e^{-x \cosh u} \cosh vu du \end{aligned} \quad (4.2.8)$$

yields:

$$\int_0^{\infty} \int_0^{\infty} \frac{t^s du dt}{u(e^{t/u} + 1)(e^u + 1)} = 2 \int_0^{\infty} t^{s-1} \left(\sum_{n=1}^{\infty} d(n) \left[K_0(2\sqrt{nx}) - 4K_0(2\sqrt{2nx}) + 4K_0(4\sqrt{nx}) \right] \right) dt \quad (4.2.9)$$

Integrating term by term here is allowed since the series is absolutely and uniformly convergent since $d(n) = O(n^\epsilon)$ where $\epsilon > 0$ as $n \rightarrow \infty$. The LHS expression can be transformed into salem's integral type by simple change of variables. ■

4.3 Equivalent Formulations of the Generalized Riemann Hypothesis

The classical generalization of the RH is in terms of Dirichlet L -Functions which states that all nontrivial zeros of $L(\chi, s)$ lie on the critical line $\Re(s) = 1/2$. It is easy to see that GRH implies RH. We shall note next another version of the GRH namely the one in terms of Dedekind Zeta Function, sometimes referred to in the Literature as the *Extended Riemann Hypothesis* (ERH), which states that all nontrivial zeros of $\zeta_K(s)$ lie on the critical line $\Re(s) = 1/2$.

The importance of the Dedekind zeta-function comes from the fact that it can be factored into L -functions having a simpler functional equation, as in the case for quadratic fields of the form $K = \mathbb{Q}(\sqrt{D})$, namely:

$$\zeta_K(s) = \zeta(s)L(\chi_D, s) \tag{4.3.1}$$

Where $\chi_D(n) = \left(\frac{D}{n}\right)$ is the Legendre-Kronecker character. We have discussed properties of the Dedekind Zeta Function in the first chapter. Let K be a number field of degree $n = r_1 + 2r_2$. Then the functional equation of $\zeta_K(s)$ satisfies:

$$\zeta_K(1 - s) = A(s)\zeta_K(s) \tag{4.3.2}$$

Where

$$A(s) = |D_{K/\mathbb{Q}}|^{s-\frac{1}{2}} \left(\cos \frac{\pi s}{2}\right)^{r_1+r_2} \left(\sin \frac{\pi s}{2}\right)^{r_2} 2^{(1-s)n} \pi^{-sn} \Gamma^n(s) \tag{4.3.3}$$

In 2012, Hu and Ye found an equivalent formulation of the GRH:

Theorem 4.3.1. *Let $\zeta_K(s) = \alpha_k(s - \frac{1}{2})^\mu + \beta_k(s - \frac{1}{2})^{\mu+1} + \dots$ be the Taylor expansion at $s = 1/2$ where $\alpha_k \neq 0$ and β_k are real numbers and μ is a non-negative even integer, then the GRH is equivalent to:*

$$\log |D_{K \setminus \mathbb{Q}}| = -8 + n(\gamma + \log(8\pi)) + \frac{\pi}{2}r_1 - \frac{2}{\pi} \int_0^\infty t^{-2} \log \frac{|\zeta_K(\frac{1}{2} + it)|^2}{|\alpha_k|^2 t^{2\mu}} dt \quad (4.3.4)$$

Where γ is Euler's constant.

For the proof and other related results, the reader can consult [30].

4.4 Discussion

There are many results in favor of the Riemann Hypothesis. One of which is **Hardy's Theorem**; It asserts that there are infinitely many zeros of $\zeta(s) = \sigma + it$ on the critical line $\sigma = 1/2$. The proof of Hardy's Theorem can be found in [28]. His theorem establishes a necessary condition for the truth of the Riemann Hypothesis. Mathematicians have continued to attack the problem from this point of view by showing that the RH holds for proportions of the critical line. The first result is due to Selberg, who showed that a positive proportion of the zeros of $\zeta(s)$ lie on the critical line [50]. Next it was improved by Levinson to $\frac{1}{3}$ [41]. The best and most recent result is due to Conrey who proved that $\frac{2}{5}$ of the zeros lie on the critical line [17]. It is worth noting that there are other generalizations of the RH which are not discussed here, namely the Generalized Riemann Hypothesis of automorphic forms and L -functions based on Hecke characters.

CHAPTER 5: Future Work

With the recent proof of *Fermat's Last Theorem*, it is no secret that the theory of zeta and L -functions had sparked particular interest amongst number theorists. The main merits of the results discussed thus far illuminate important aspects of other types of zeta-functions may not be well-known amongst mathematicians, or rather have not been investigated in full details. For instance, we have seen how the introduction of a *perturbed* version of the Riemann zeta-function, had major consequences in shedding light on its *Universality Property*. Namely, how L -functions can be represented in terms of a finite sums of *shifted* zeta-functions.

The other aspect of the theory is *Euler Product's* representation. This indispensable tool is very useful since it constructs the bridge between these types of functions and arithmetic functions, in other words, Dirichlet series are exactly the generating functions of the arithmetic functions. We noted that this is not always possible and can be guaranteed only when a character in question is completely multiplicative. The study of the properties attached to zeta-functions defined on abstract fields as well as questions about convergence had been developed into a full theory, namely *Multiplicative Number Theory* with the aid of fundamental results from Fourier Analysis and Algebraic Number Theory.

We also discussed other non-analytic approaches to the PNT utilizing *Tauberian Theorems*. One of the crowning achievements of Number Theory. Finally, we have discussed new formulations of the *Riemann Hypothesis* and witnessed how these equivalent formulations are presented in unusual settings, for instance, in terms of Integral Equations. These new-found results may spark curiosity amongst those who are interested in undertaking research and tackling open questions. This hopefully will constitute the driving force of a rigorous proof (or a counterexample) the *Riemann Hypothesis*.

The author wishes to pursue further research in the area of analytic number theory. Particularly in the study of L -functions defined over Elliptic Curves. As well as Artin's L -functions and the Artin Conjecture, which are very active and attractive areas of research.

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