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Weierstrass Vertices and Divisor Theory of Graphs

Ajani Ruwandhika Chulangi De Vas Gunasekara
University of Central Florida

WEIERSTRASS VERTICES AND DIVISOR THEORY OF GRAPHS

by

AJANI RUWANDHIKA CHULANGI DE VAS GUNASEKARA
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ABSTRACT

Chip-firing games and divisor theory on finite, connected, undirected and unweighted graphs have been studied as analogs of divisor theory on Riemann Surfaces. As part of this theory, a version of the one-dimensional Riemann-Roch theorem was introduced for graphs by Matt Baker in 2007. Properties of algebraic curves that have been studied can be applied to study graphs by means of the divisor theory of graphs. In this research, we investigate the property of a vertex of a graph having the Weierstrass property in analogy to the theory of Weierstrass points on algebraic curves. The weight of the Weierstrass vertices is then calculated in a manner analogous to the algebraic curve case. Although there are many graphs for which all vertices are Weierstrass vertices, there are bounds on the total weight of the Weierstrass vertices as a function of the arithmetic genus. For complete graphs, all of the vertices are Weierstrass when the number of vertices (n) is greater than or equals to 4 and no vertex is Weierstrass for n strictly less than 4. We study the complete graphs on 4, 5 and 6 vertices and reveal a pattern in the gap sequence for higher cases of n. Furthermore, we introduce a formula to calculate the Weierstrass weight of a vertex of the complete graph on n vertices. Additionally, we prove that Weierstrass semigroup of complete graphs is 2 - generated. Moreover, we show that there are no graphs of genus 2 and 6 vertices with all the vertices being normal Weierstrass vertices and generalize this result to any graph with genus g.

Keywords: Divisor of a graph, Dhar's burning algorithm, gap sequence, Weierstrass semigroup, Weierstrass vertex, Weierstrass weight.

To my parents and aunts for their continuous support, and to my fiancé Shasheeka for being the wings to my dreams.

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CHAPTER 1: THE CHIP FIRING GAME

From this point, we will be examining concepts from the graph - theoretic viewpoint. This chapter reviews the chip firing game and its variations, focusing on Matt Baker's version. This is one of the main concepts in this thesis. We are defining the rank of a divisor and linear equivalence of two or more divisors based on the chip firing game. Also we interpret the Weierstrass vertices which we will build upon in the subsequent chapters.

1.1 Evolution of Chip Firing Game

Chip firing game was first introduced by Spencer in 1986 when he was studying the concept of a "balancing game", [21]. From there onwards different versions of chip firing games were introduced. Here mainly concentrated on the chip firing game defined by M.Baker. [2], Sections 1.5, 5.5.

Spencer's original paper from 1986 addresses the following 3 questions;
Let $||.||$ be the max norm and k be a large positive constant;

1. Let $v_1, v_2, ..., v_n \in \mathbb{R}^n$, $||v_i|| \leq 1$. Do there exist $e_1, e_2, ..., e_n \in \{+1, -1\}$ so that $||e_1v_1 + e_2v_2 + ... + e_nv_n|| \leq kn^{\frac{1}{2}}$?

2. Let $v_1, v_2, ..., v_n \in \mathbb{R}^n$, $||v_i|| \leq 1$. Do there exist $e_1, e_2, ..., e_n \in \{+1, -1\}$ so that $||e_1v_1 + e_2v_2 + ... + e_nv_t|| \leq kn^{\frac{1}{2}}$ for all $t, 1 \leq t \leq n$?

   In seeking a solution to these problems a third question is present itself in the form of a strategic game which also can be called as a balancing game.

3. Consider the following n - round perfect information game between two players; Pusher and

Chooser. A vector $w \in \mathbb{R}^n$, called the position vector, is set at the start of the game to 0. In the i^{th} round Pusher selects a vector $v = v_i \in \mathbb{R}^n$ with $||v|| \leq 1$. We call v the move. Chooser then resets the position w to either $w+v$ or $w-v$. With perfect play can Chooser assure that at the end of the n^{th} round, $||w|| \leq kn^{\frac{1}{2}}$?

Inspired by this result, in 1991, Björner, generalized this result to graphs and that game is known to be the chip-firing game, see [6].

Björner's version defined as follows; let $G = (V, E)$ be a finite, undirected, unweighted and connected graph. Initially $N (\in \mathbb{N})$ is the number of chips distributed among the vertices of G such that, c_i chips are at vertex v_i for $i \in [n]$. That is $\sum_{i=1}^{n} c_i = N$. Then a vertex v_i with degree d_i is randomly choosed when $c_i \geq d_i$ and d_i chips were distributed along its incidence edges to its neighboring vertices one per each neighbor and remaining $c_i - d_i$ will stay at v_i. This process is known as firing. Firing the vertices of G is continued until there is no such possibility.

Example 1.1: Firing the House - X graph with a $[3,2,3,1,0]$ initial configuration.

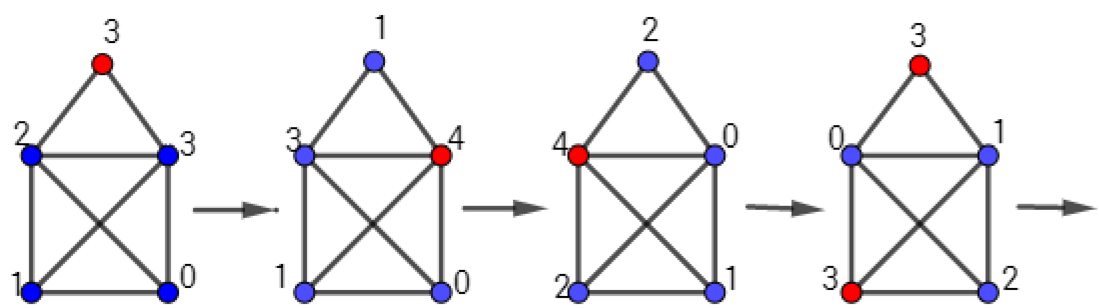

Figure 1.1: Firing sequence of the House - X graph with firing vertices marked in red.

This game can be finite or infinite because one specific vertex can be fired infinitely many times as long as it satisfies the firing requirement. But the finiteness of the game can be measured.

Theorem 1.2 [6, Theorem 2.3]

(1) If $N > 2|E(G)| - |V(G)|$, then game is infinite whatever the initial configuration is.

(2) If $N < |E(G)|$, then game is finite whatever the initial configuration is.

(3) If $|E(G)| \leq N \leq 2|E(G)| - |V(G)|$, then there exist an initial configuration which can be terminated finitely and also another initial configuration which can't be terminated finitely.

1.1.1 Chip firing game: Matt Baker's Version

In this game there are two legal moves called chip firing and reverse chip firing. Firing has the same meaning as in Björner's game. Reverse chip firing relates to receiving a chip from each adjacent vertex via it's incidence edges. The chip configuration of a graph can be identified as a particular divisor corresponding to that position. So the degree of the divisor is the total number of chips distributed in the graph, and this value can be a negative integer too. There are no restrictions in this game, a vertex can have a negative number of chips, however divisor degree will stay the same. Here the objective is to find a configuration where no vertex has a negative number of chips. In the language of divisors, this is equivalent to getting an effective divisor.

This game can be thought of as a kind of dollar game too. Chips can be thought of as dollars and a vertex assigned with a negative number of chips said to be in debt. So a vertex which is in debt can borrow dollars from its adjacent vertices. Then the number of dollars in that vertex will increase by its vertex degree and the number of dollars of its adjacent vertices will decrease by 1, because they have lent a dollar to the vertex in debt. To end the game all vertices must be out of debt. Regardless of how many firing or reverse firing moves have been made, total number of dollars or chips will remain unchanged.

Example 1.3: Firing and reverse firing of the House - X graph with $[-2, 2, 1, -1, 3]$ initial configuration.

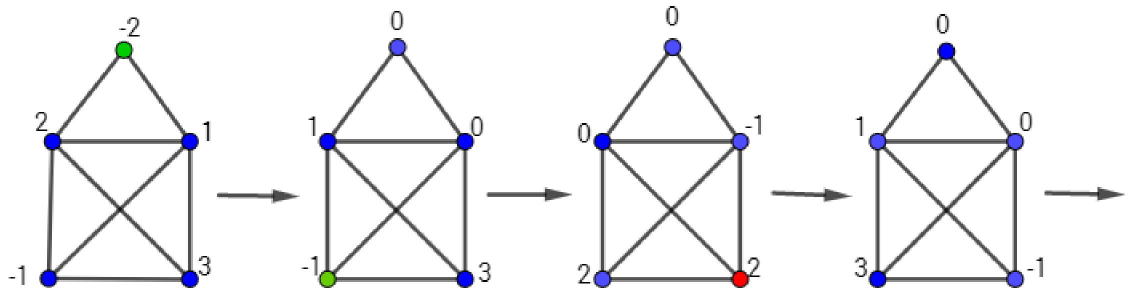

Figure 1.2: Firing and reverse firing sequence of the House - X graph with firing vertices marked in red and reverse firing vertices marked in green.

Theorem 1.4 [2, Theorem 1.9]

Let $G = (V, E)$ be a graph satisfying the conditions mentioned in section 1.2 and N be the total number of chips/dollars present at any stage. Let g be the arithmetic genus of G.

(1) If $N \geq g$, then there is always a winning strategy.

(2) If $N \leq g - 1$, then there is always an initial configuration for which no winning strategy exists.

1.1.2 Linear Equivalence of Divisors

Definition 1.5

Let $\{v_1, v_2, \ldots, v_n\}$ be the vertex set of G. Then a divisor D on G is an integer linear combination of vertices of G.

$$D = \sum_{v_i \in V(G)} D(v_i) v_i \tag{1.1}$$

Where $D(v_i) \in \mathbb{Z}$.

Remark 1.6: The set of all divisors on G, $Div(G)$ forms a free abelian group over $V(G)$ with the basis $\{v_1, v_2, \ldots, v_n\}$ under the additive operator.

Definition 1.7

A divisor D is said to be *effective* if and only if all $D(v_i) \geq 0$, $(D \geq 0)$. The set of effective divisors on G denoted by $Div_+(G)$. D is said to be the zero divisor when $D(v_i) = 0$, for all $i = 1, 2, \ldots, n$. The zero divisor acts as the zero element of the free abelian group.

Definition 1.8

The degree of a divisor D on G is the finite sum of $D(v_i), \forall i = 1, 2, \ldots, n$.

$$deg(D) = \sum_{v_i \in V(G)} D(v_i) \tag{1.2}$$

Remark 1.9: For each $k \in \mathbb{Z}$, $Div_+^k(G) = \{D \in Div(G) | deg(D) = k, D \geq 0\}$.

Definition 1.10

The canonical divisor K_G of G is defined as follows.

$$K_G = \sum_{v \in V(G)} (d_G(v) - 2)(v) \tag{1.3}$$

Proposition 1.11 [13, page 6]

Let G be a graph with arithmetical genus g on G. The degree of the Canonical divisor of G is $2g - 2$.

Definition 1.12 (*Principal divisor*)

The set of Principal divisors of G, $Prin(G)$ is defined as $Prin(G) = \{div(f) \mid f : V(G) \to \mathbb{Z}\}$.

$$div(f) = \sum_{v_i \in V(G)} \sum_{e=v_i v_j} (f(v_i) - f(v_j))(v_i) \tag{1.4}$$

where $i \neq j$.

Proposition 1.13

The degree of a Principal divisor on G equals to zero.

Proof.

$$Deg(div(f)) = \sum_{v_i \in V(G)} \sum_{e=v_i v_j} (f(v_i) - f(v_j))$$

This counts each $f(v_i)$ twice as $+$ and $-$ for each vertex $v_i \in V(G)$, implies summation equals to zero. □

The equivalence relation can be defined on the divisor group $Div(G)$. Two divisors D and D' on G are said to be linearly equivalent $(D \equiv D')$ whenever $D - D'$ or $D' - D \in Prin(G)$. Since the degree of a principal divisor is zero, if two divisors are equivalent, their degrees must be the same. In terms of chip firing D and D' on G are linearly equivalent if and only if one can get from the other by finite sequence of firings and reverse firings, as described in Baker's chip firing game.

For example consider all the divisors obtained in the process of firing and reverse firing in figure 1.2. According to this concept, those divisors are linearly equivalent to each other.

That is $[-2, 2, 1, -1, 3] \equiv [0, 1, 0, -1, 3] \equiv [0, 0, -1, 2, 2] \equiv [0, 1, 0, 3, -1]$. And the degree is fixed, which is equal to 3.

Remark 1.14: To indicate the linear equivalence, "$\equiv$" have been used throughout this thesis.

1.2 Rank of a Divisor

Definition 1.15 (*Linear System of a divisor - D*)

The set of all effective divisors equivalent to D is said to be the linear system of D and denoted by $|D|$; $|D| = \{E \in Div(G) | E \geq 0, E \equiv D\}$.

Definition 1.16 (*Rank of a divisor - r*)

Rank of a divisor D is the maximum number of chips that can be subtracted from that divisor while maintaining it's linear equivalence to an effective divisor.

$r(D) = -1$ if $|D| = \emptyset$

otherwise, $r(D) = max\{k \in \mathbb{Z}_0^+ \quad \text{such that} \quad |D - E| \neq \emptyset, \quad \text{for all} \quad E \in Div_+^k(G)\}$

From the point of view of either the Dollar game or the Chip firing game, $r(D) \geq 0$ implies that each vertex can be brought out of debt, obtaining a debt free graph.

Proposition 1.17

Let D be a divisor of G.

(1) If $deg(D) < 0$, then $r(D) = -1$.

(2) If $deg(D) = 0$, then $r(D) = 0$ whenever $D \equiv 0$ and r(D) = -1 otherwise.

Proof.

(1) If $deg(D) < 0$ then D can't be equivalent to an effective divisor. Hence $|D| = \emptyset$. Therefore, $r(D) = -1$ by the definition.

(2) If $deg(D) = 0$, maximum possibility is D can be linearly equivalent to zero divisor and hence $r(D) = 0$. If $D \not\equiv [0]$, then again $|D| = \emptyset$ and therefore r(D) = -1.

□

We now look at the most important theorem in this subject, the Riemann - Roch theorem for graphs which was proved by Baker and Norine [2].

Theorem 1.18 (*Riemann-Roch for graphs*)

Let D be a divisor of G, K_G be the canonical divisor of G and g be the genus.

Then,

$$r(D) - r(K_G - D) = deg(D) - g + 1 \tag{1.5}$$

We now are able to introduce the objects of interest in this study.

Definition 1.19 (*Weierstrass vertices on a graph*)

By the analogy with algebraic curves, a Weierstrass vertex on a graph can be defined as follows. $v \in V(G)$ is said to be a Weierstrass vertex when $r(g(v)) \geq 1$.

Lemma 1.20

The following are equivalent.

(1) v is a Weierstrass vertex

(2) $r(g(v)) \geq 1$

(3) $r(K_G - g(v)) \geq 0$

Proof.

(1) if and only if (2): Directly follows from the definition of Weierstrass vertex.
(2) if and only if (3): Let $D = g(v)$, $deg(D) = deg(g(v)) = g$. Then by Riemann - Roch theorem we have,
$1 - r(K_G - g(v)) \leq g - g + 1$ which implies $r(K_G - g(v)) \geq 0$. $\square$

CHAPTER 2: DHAR'S BURNING ALGORITHM AND Q-REDUCED DIVISORS

In this chapter we introduce the q-reduced divisors which is a special class of divisors, and then we explore the relationship between q-reduced divisors and calculating the rank when applying the Dhar's burning algorithm. Furthermore, we are discussing Some examples thoroughly to gain an understanding of the algorithm and how it is applied.

2.1 Dhar's Burning Algorithm

This algorithm was first introduced in 1989 by Deepak Dhar, who was a physicist, when he was studying Sandpile Automaton Models [8]. The algorithm was named after him. Over time, Dhar's burning algorithm has been modified and some improvements were introduced to the divisor theory on graphs.

So the current version is as follows;

The premise of the algorithm is a fire started from a vertex which is in debt, $D(v) < 0$ and spread eventually along its incident edges. If adjacent vertices are burned then they spread the fire. This process stops when the entire graph is burned or the vertex which is in debt, becomes out of debt. Even though the divisor is changing, the divisor degree is fixed throughout the whole process. When you look more closely, there are specific restrictions or conditions needed to be satisfied before applying the algorithm.

1. The divisor D corresponding to G must be effective away from the vertex which starts the fire (say fixed vertex).

2. A vertex v, other than the fixed vertex is burned when the number of chips at v, $D(v) <$ number of incident edges to v which carries fire at a specific move.

If the fire stops at a certain vertex $\setminus$ vertices, then all the unburned vertices can be fired simultaneously. If we take all the burned vertices as the set S and the set of unburned vertices as complement of S (S'), then we are only interested of the edges going from S to S'. This process can be continued until the whole graph is burned.

Example 2.1: Let G be the House graph with divisor $D = [q, 1, 1, 1, 1]$ where q is the fixed vertex.

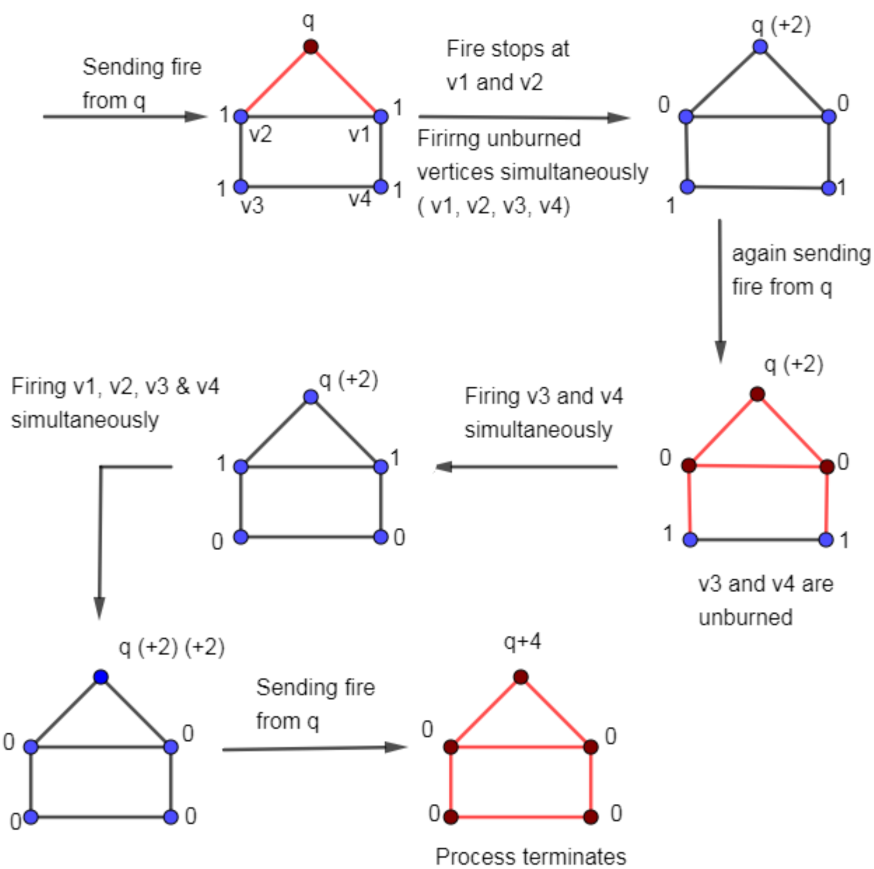

Figure 2.1: Dhar's burning algorithm applied to G with $D = [q, 1, 1, 1, 1]$.

2.2 q-Reduced Divisors

Reduced divisors play an important role in proving Riemann Roch Theorem for graphs. Matt Baker has used this special class of divisors to prove the Riemann Roch theorem for graphs. Such a proof can be found in [2, Section 3.2] .

Definition 2.2

A Divisor D on G is said to be $q-reduced$ if the following conditions are satisfied.

1. $D(v) \geq 0$ for all $v \neq q$, $v \in V(G)$. That is D is effective away from q.
2. For every non-empty set; $S \subset V(G) - \{q\}$, when all the vertices of S fired simultaneously, some vertex will go into debt.That is for some $v \in S$, $D(v) < 0$.

q - reduced Divisors can be obtained by Dhar's burning algorithm and the procedure is given below.

Step 1: Fix a vertex q of G for which the divisor D needs to be of q - reduced.

Step 2: D must be effective away from q. If it is not effective away from q initially, then convert D to D' which is effective away from q, by firing the q as of the need. Note that there are no limitations for negativity of $D(q)$, it can be a large negative number as well. At this point, G is ready to be burned.

Step 3: Start a fire from q and spread it along the incidence edges of q in G. If it is possible to burn the whole graph (every vertex in $V(G) - \{q\}$) at the end, $D'(\equiv D)$ is $q-reduced$.

Example 2.3: Consider the example 2.1 again, Fix $v_0 = q$ with $D = [q, 1, 1, 1, 1]$.
D is already effective away from q, therefore follow the procedure as described above. Then,
$D' = [q+4, 0, 0, 0]$ is the q-reduced divisor equivalent to D.

Proposition 2.4 [2]

For any given Divisor D of G, there exists a unique q - reduced Divisor D' with $D \equiv D'$ when q is fixed.

2.3 Calculating the rank of a given divisor

Usually calculating the rank of a divisor is a polynomial time problem and is difficult if one uses the definitions of rank directly. It can be calculated easily by Dhar's burning algorithm.

Steps to calculate the rank of a divisor D of G:

Step 1: If $deg(D) < 0$, then $r(D) = -1$.

Step 2: If $deg(D) \geq 2g-1$, then $r(D) = deg(D) - g$.

We can easily observe this by Riemann - Roch,
when $deg(D) \geq 2g-1$, $deg(K_G - D) < 0$ hence $r(K_G - D) = -1$. Substituting to the Riemann - Roch formula gives $r(D) - (-1) = deg(D) - g + 1$ implies $r(D) = deg(D) - g$.

Step 3: If $g \leq deg(D) \leq 2g-2$, then first compute rank of $(K_G - D)$ which is equivalent to a divisor of degree $0, ..., g-1$. Then using the Riemann - Roch formula we can find $r(D)$.

Since the degree of the canonical divisor is $2g-2$, then the degree of $K_G - D$
when $g \leq deg(D) \leq 2g-2$ is between $2g-2-g = g-2$ and $2g-2-(2g-2) = 0$.

Step 4: If $0 \leq deg(D) \leq g-1$, then compute the rank using Dhar's burning algorithm.

2.3.1 Applying Dhar's burning algorithm to compute the rank of a divisor of degree between 0 and g-1

In [14], Manjunath has given some other geometric computations to find the rank. However, in this thesis we are only using Dhar's burning algorithm and properties of the rank for the calculations.

Example 2.5: Applying Dhar's burning algorithm to calculate the rank of the following divisor $D = [2, 2, 1, -1, -1]$ of the House - X graph (G).

Calculating the genus of the House - X graph G: $g = |E(G)| - |V(G)| + 1$ implies $g = 8 - 5 + 1 = 4$. Degree of the divisor $D = 2 + 2 + 1 + (-1) + (-1) = 3$. Therefore, the degree of D equals to $g - 1$ and we can find the rank using Dhar's burning algorithm directly.

Step 1: Checking $r(D) \geq 0$.

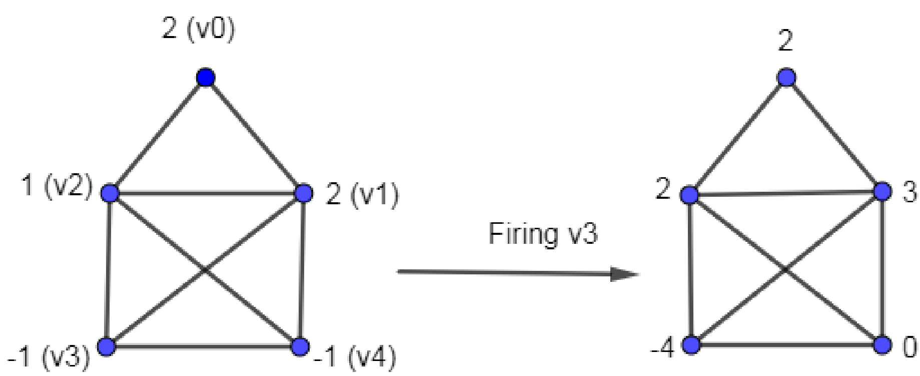

Figure 2.2: Firing v3 to get $D' = [2, 3, 2, -4, 0]$.

Now $D \equiv D' = [2, 3, 2, -4, 0]$ is effective away from v_3 and Dhar's burning algorithm is ready to be applied.

Remark 2.6: $v4$ also can be chosen for firing instead of $v3$.

Then after applying Dhar's burning algorithm to D', $(D' \equiv)D'' = [0,1,0,1,1]$ can be obtained. Therefore, since D'' is effective and equivalent to D, clearly $r(D) \geq 0$.

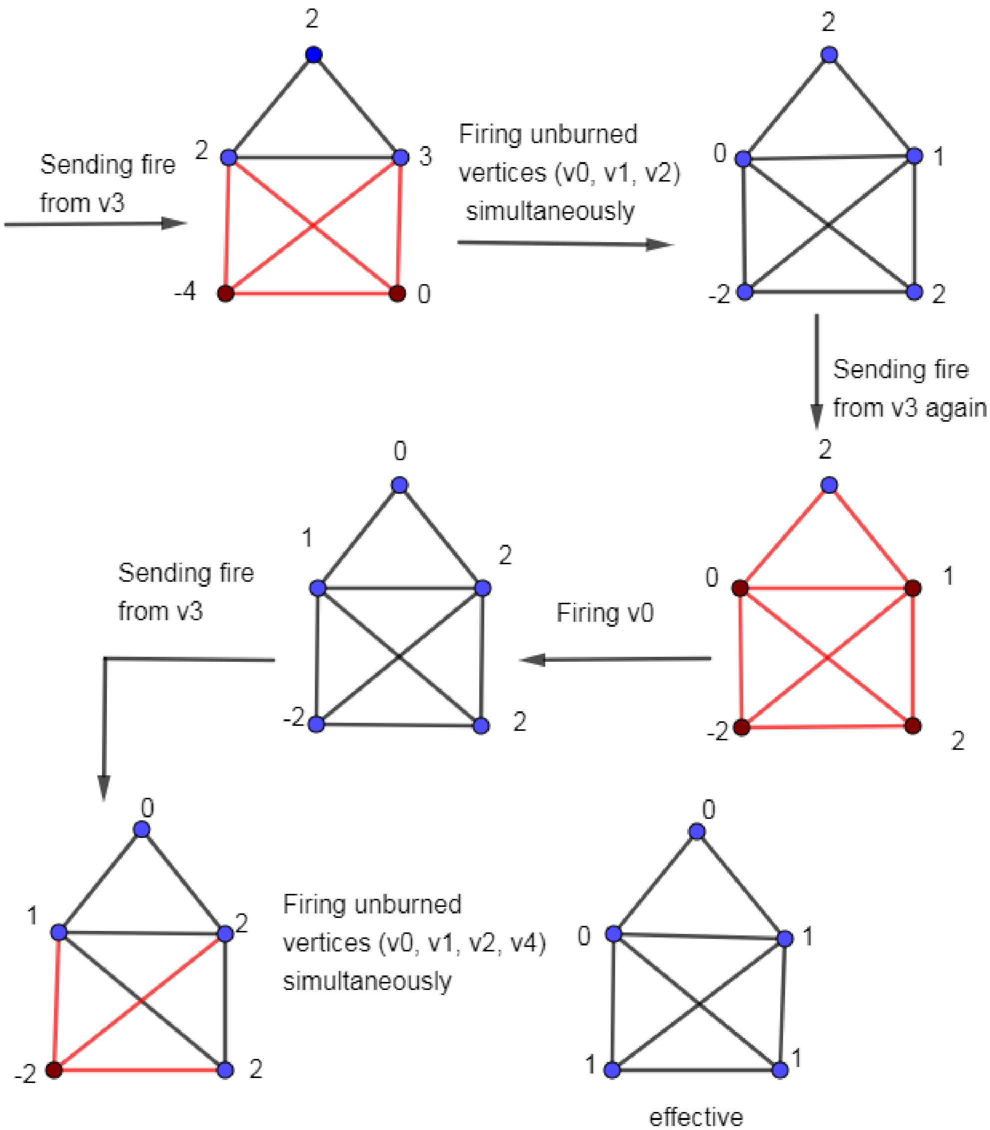

Figure 2.3: After applying Dhar's burning algorithm we obtain the effective divisor D''.

Next we need to check whether $r(D)$ is at least 1.

Step 2: Checking $r(D) \geq 1$.

We need to consider all the non-equivalent degree one effective divisors of G and according to this example we have 5 of those.

$[1,0,0,0,0]$

$[0,1,0,0,0]$

$[0,0,1,0,0]$

$[0,0,0,1,0]$

$[0,0,0,0,1]$

Then subtract each of these divisors from D'' and check whether the resulting divisor is effective or not. If it is not effective, by applying Dhar's burning algorithm we can deduce its equivalence to an effective divisor. Even after applying Dhar's burning algorithm, if we can't get an effective divisor (at least for one case), we can say that $r(D) < 1$. If we can get effective divisors for all the cases we move to the next level which is $r(D) \geq 2$, and this process continues until we get a contradiction.

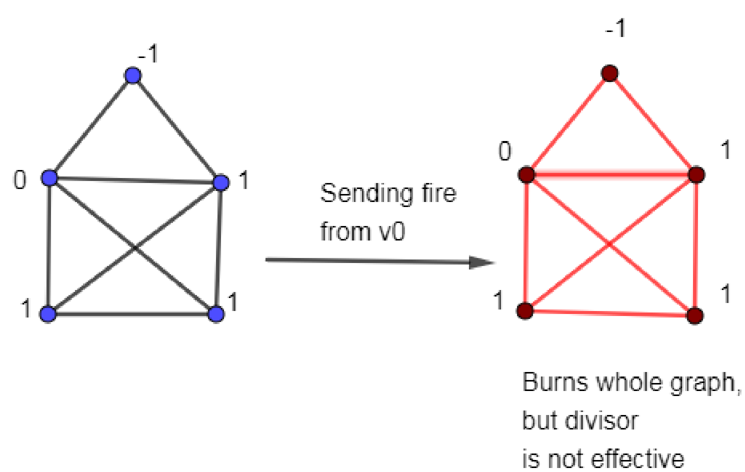

Figure 2.4: $D'' - [1,0,0,0,0] \not\equiv$ effective divisor.

Figure 2.4 shows that $D'' - [1,0,0,0,0]$ is not equivalent to an effective divisor. Hence $r(D) < 1$ implies $r(D) = 0$.

Remark 2.7: Once we obtain the q-reduced divisor for the chosen vertex, if it is not effective the process terminates, and if it is effective the process continues.

Remark 2.8: If you use the chip firing moves together with Dhar's burning algorithm to determine the rank of a given divisor, Dhar's burning algorithm is useful to determine the vertex set or the vertex that needs to be fired simultaneously to get an effective divisor, or to confirm that there is no possibility of firing or reverse firing in order to obtain an effective divisor. Otherwise, we can solely use Dhar's burning algorithm to calculate the rank, but it is inefficient.

Remark 2.9: In this thesis we have used Dhar's burning algorithm together with the chip firing moves to find the rank.

CHAPTER 3: GAP SEQUENCE

In this chapter we are introducing gap sequence, types of Weierstrass points according to their weights, and applying these concepts to the House - X graph and complete graphs of orders 4, 5 and 6. The House - X graph shows the properties parallel to Riemann surface case while complete graphs show a different pattern. Also in this chapter we determine the semigroups and the gap sequences for the House - X graph and for K_4, K_5, K_6 and predict the behavior for K_7 and so on. We are also introducing a conjecture regarding Weierstrass weights of complete graphs in this chapter.

For the complete graphs we only need to consider one vertex because of the symmetry, otherwise we need to consider all the vertices excluding symmetries if any. Then we are calculating $r(D) = r(n(p))$ for $n = 0, 1, 2, ...2g-2, 2g-1, 2g, ...$ for $D = n(p) = [n, 0, 0, ..., 0]$ assuming P is the first vertex of the vertex ordering of the graph. When $n \geq 2g-1$, $r(n(p)) = deg(D) - g = n - g$, therefore we are not interested in calculating the ranks for $n \geq 2g-1$. From this we can determine whether the vertex is Weierstrass or not by looking at $r(gP)$, when $n = g$. Then for the Weierstrass vertices, we are determining the gap sequence, weight and the corresponding semigroup.

All the ranks are calculated according to the properties of rank and applying Dhar's burning algorithm together with chip firing moves as described in section 2.3.

Definition 3.1 (*Weierstrass Gap*)

For a Weierstrass vertex $p \in V(G)$ the gap sequence G_p defined as all $n \in \mathbb{Z}^+$ such that
$r(np) = r((n-1)p)$. The cardinality of G_p is g.

Note that G_p is finite as we only examine finite graphs.

Definition 3.2 (*Weierstrass Semigroup*)

Weierstrass Semigroup H_p of a Weierstrass vertex $p \in V(G)$ defined as the set $\mathbb{N} - G_p$ with the additive operator.

Definition 3.3

Weierstrass weight of $p \in V(G)$ is,

$$w(p) = \sum_{n \in G_p} n - \frac{g(g+1)}{2} \tag{3.1}$$

A Weierstrass vertex is said to be a *normal Weierstrass vertex* if its gap sequence is $\{1, 2, \dots \text{g-1}, \text{g+1}\}$.

Proposition 3.4 [1, E-4 page 42]

The following are equivalent:

(1) p is a Weierstrass vertex,

(2) $w(p) \neq 0$,

(3) $r(g(p)) \neq 0$.

Proposition 3.5 [1, E-4 page 42]

Let p be a normal Weierstrass vertex if and only if $w(p) = 1$.

Proposition 3.6 [1, E-4 page 43]

If $g \geq 4$ and $w(p) = 2$, then either H_p = $\{g-1, g+2, g+3, g+4, ...\}$ or H_p = $\{g, g+1, g+3, ...\}$.

3.1 House - X Graph

There exist three types of vertices in the House - X graph and the symmetries marked as Q' and R'. The genus of this graph equals to 4. We need to calculate $r(nP), r(nQ)$ and $r(nR)$ up to $2g-2=6$ of n values. Suppose the general divisor is $D=[P,Q,Q',R,R']$.

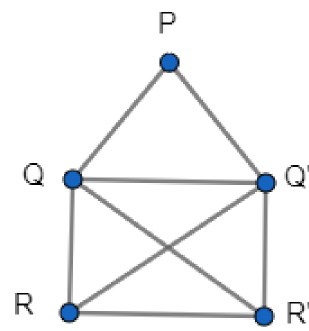

Figure 3.1: The House - X Graph with vertices marked.

3.1.1 Vertex P

1. $D=0P=[0,0,0,0,0]$: Clearly $r(0P)=0$.

2. $D=1P=[1,0,0,0,0]$: Clearly $r(1P)\geq 0$. To check $r(1P)\geq 1$,
   consider $[1,0,0,0,0]-[0,1,0,0,0]=[1,-1,0,0,0]$. Then sending fire from Q will burn the whole graph, but the divisor is not effective. Hence $r(1P)=0$.

3. $D=2P=[2,0,0,0,0]$: Checking $r(2P)\geq 1$;
   Consider $[2,0,0,0,0]-[0,0,0,1,0]=[2,0,0,-1,0]$. The divisor is not effective,
   hence $r(2P)=0$.

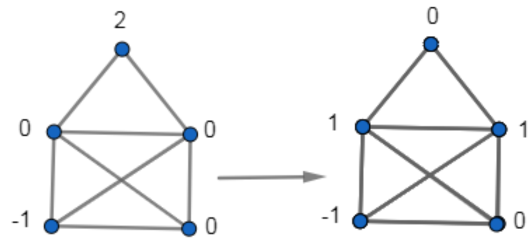

Figure 3.2: Sending fire from R will leave only P unburned and therefore we fire P. Again sending fire from R will burn the whole graph. But the divisor is not effective.

4. $D = 3P = [3,0,0,0,0]$: Checking $r(3P) \geq 1$; First we consider, $[3,0,0,0,0] - [0,0,0,1,0] = [3,0,0,-1,0]$. Then again following the same process as in 3 implies $r(3P) = 0$.

5. $D = 4P = [4,0,0,0,0]$: Since obviously $r(4P) \geq 0$, we need to check $r(4P) \geq 1$. For that we need to subtract all the effective divisors of degree 1 from D and check the effectiveness. Clearly $[4,0,0,0,0] - [1,0,0,0,0] = [3,0,0,0,0]$ is effective.
Next $[4,0,0,0,0] - [0,1,0,0,0] = [4,-1,0,0,0]$ is also effective after firing R once.We need to consider, $[4,0,0,0,0] - [0,0,0,1,0] = [4,0,0,-1,0]$ which is equivalent to [0, 0, 0, 1, 2] implies $r(4P) \geq 1$.

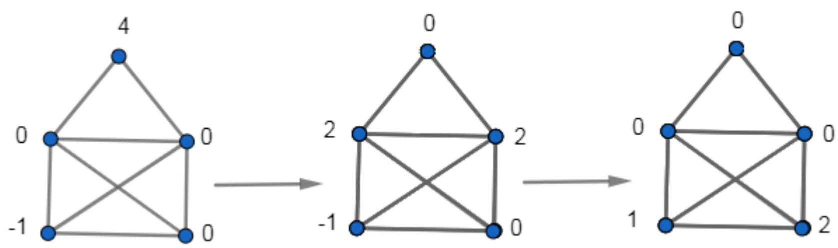

Figure 3.3: Sending fire from R will leave P unburned, therefore we fire P twice. Next, again sending fire from R will leave P, Q, Q' unburned. Finally, firing unburned vertices simultaneously lead to an effective divisor.

Since $r(4P) \geq 1$, we need to check whether $r(4P) \geq 2$; for that we consider,

$[4,0,0,0,0] - [0,1,0,1,0] = [4,-1,0,-1,0]$ which is not equivalent to an effective divisor. Hence $r(4P) = 1$.

6. $D = 5P = [5,0,0,0,0]$: Clearly $r(5P) \geq 1$, therefore checking $r(5P) \geq 2$. At this point we are using Riemann-Roch formula to calculate the rank.

   Consider, $K_G - 5P = [0,2,2,1,1] - [5,0,0,0,0] = [-5,2,2,1,1]$, which is not equivalent to an effective divisor. Hence $r(K_G - 5P) = -1$. Therefore, by Riemann - Roch we have,

   $r(5P) = deg(5P) - g + 1 + r(K_G - 5P) = 5 - 4 + 1 - 1 = 1$.

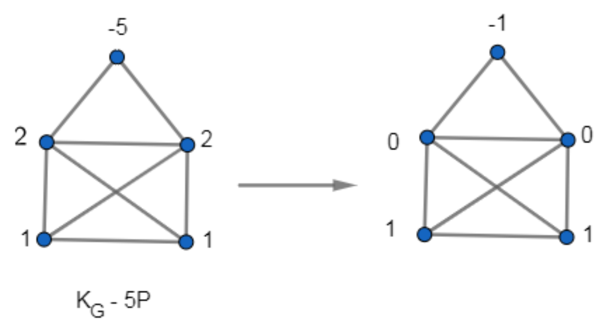

Figure 3.4: Reverse fire P twice and then sending fire from P will burn whole graph. But the divisor is not effective.

7. $D = 6P = [6,0,0,0,0]$: Similarly as above we use Riemann - Roch formula to calculate the rank of D. Clearly, $r(K_G - 6P) = r([-6,2,2,1,1]) = -1$,

   hence $r(6P) = deg(6P) - g + 1 + r(K_G - 6P) = 6 - 4 + 1 - 1 = 2$.

Remark: The ranks for all the divisors of the form $D = nP$ for $n \geq 7$ can be calculated using, $deg(D) - g$ as mentioned in section 2.3. But we are interested in finding the rank for first $2g - 2 = 6$, n values.

3.1.2 Vertex Q

1. $D = 0Q = [0,0,0,0,0]$: Clearly $r(0Q) = 0$.

2. $D = 1Q = [0,1,0,0,0]$: We have $r(1Q) \geq 0$. To check $r(1Q) \geq 1$, consider $[0,1,0,0,0] - [1,0,0,0,0] = [-1,1,0,0,0]$. Then sending fire from P will burn the whole graph, but divisor is not effective. Hence $r(1Q) = 0$.

3. $D = 2Q = [0,2,0,0,0]$: Checking $r(2Q) \geq 1$; Consider $[0,2,0,0,0] - [1,0,0,0,0] = [-1,2,0,0,0]$, which is not equivalent to an effective divisor. Thus $r(2Q) = 0$.

4. $D = 3Q = [0,3,0,0,0]$: We need to check whether $r(3Q) \geq 1$. For that consider $[0,3,0,0,0] - [1,0,0,0,0] = [-1,3,0,0,0]$. Then when you send fire from P, the whole graph burns as degree of Q vertex is 4 but its coefficient is 3. Therefore the divisor is not equivalent to an effective divisor. Hence $r(3Q) = 0$.

5. $D = 4Q = [0,4,0,0,0]$: Clearly $r(4Q) \geq 0$. Checking $r(4Q) \geq 1$;

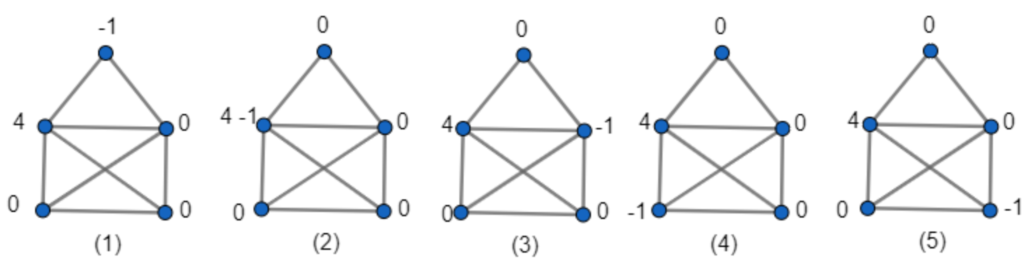

Figure 3.5: (1): $[0,4,0,0,0] - [1,0,0,0,0]$, (2): $[0,4,0,0,0] - [0,1,0,0,0]$, (3): $[0,4,0,0,0] - [0,0,1,0,0]$, (4): $[0,4,0,0,0] - [0,0,0,1,0]$, (5): $[0,4,0,0,0] - [0,0,0,0,1]$.

As shown in the figure, for (1), (3), (4) and (5), if we fire Q once, we will get an effective divisor and (2) is already effective. Therefore, $r(4Q) \geq 1$. Moreover, we need to check whether its rank is greater than or equals to 2. But $[0,4,0,0,0]-[0,1,0,1,0]=[0,3,0,-1,0]$ is not equivalent to an effective divisor, hence $r(4Q)=1$.

We are using Riemann - Roch formula to calculate the rank for the following two cases.

6. $D=5Q=[0,5,0,0,0]$: $K_G-5Q=[0,2,2,1,1]-[0,5,0,0,0]=[0,-3,2,1,1]\equiv[1,0,0,0,0]$ implies $r(K_G-5Q)=0$. Then $r(5Q)=deg(5Q)-g+r(K_G-5Q)+1=5-4+0+1=2$.

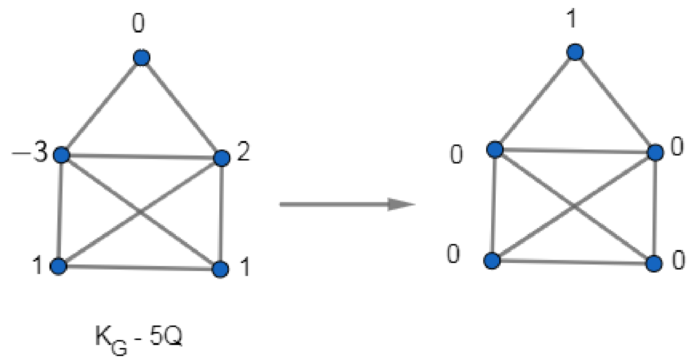

Figure 3.6: Sending fire from Q will only burn P, then reverse firing all the unburned vertices simultaneously will give an effective divisor $[1,0,0,0,0]$.

7. $D=6Q=[0,6,0,0,0]$: $K_G-6Q=[0,2,2,1,1]-[0,6,0,0,0]=[0,-4,2,1,1]\equiv[1,-1,0,0,0]$ implies $r(K_G-5Q)=-1$. Therefore by Riemann - Roch we have $r(6Q)=2$.

Because of symmetry, Q' has the same rank as Q.

3.1.3 Vertex R

1. $D = 0R = [0,0,0,0,0]$: Clearly $r(0R) = 0$.

2. $D = 1R = [0,0,0,1,0]$: As in the above two cases, we have $r(1R) = 0$ because degree of vertex R is equal to 3.

3. $D = 2R = [0,0,0,2,0]$: Similarly we can show that $r(2R) = 0$.

4. $D = 3R = [0,0,0,3,0]$: Since $r(3R) \geq 0$, we need to check whether $r(3R) \geq 1$. For that we list all the divisors of degree 1 and subtract from D as below and verify its equivalence to an effective divisor.

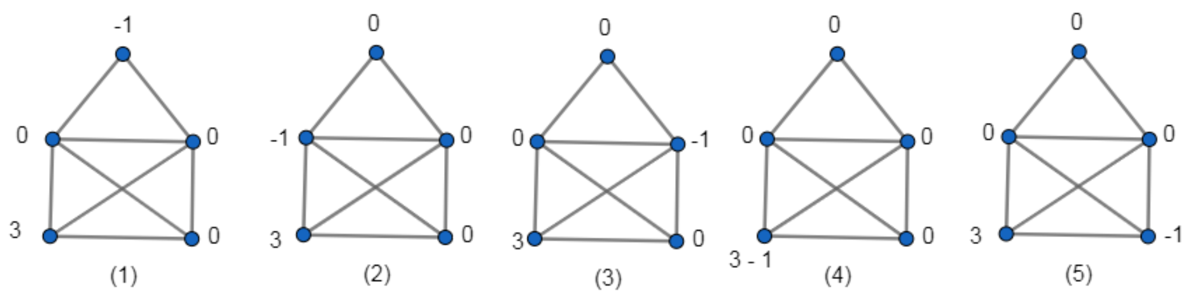

Figure 3.7: (1): $[0,0,0,3,0]-[1,0,0,0,0]$, (2): $[0,0,0,3,0]-[0,1,0,0,0]$, (3): $[0,0,0,3,0]-[0,0,1,0,0]$, (4): $[0,0,0,3,0]-[0,0,0,1,0]$, (5): $[0,0,0,3,0]-[0,0,0,0,1]$.

According to the figure, for cases $(2),(3)$ and (5), firing R once will give an effective divisor. Note that (4) is already effective. Then for case (1), firing R once and then reverse firing P once will lead to an effective divisor. But $r(3R)$ is not greater than or equals to 2, hence $r(3R) = 1$.

5. $D = 4R = [0,0,0,4,0]$: At this point we have $r(4R) \geq 1$, we need to determine whether $r(4R) \geq 2$. Consider $[0,0,0,4,0]-[1,0,1,0,0] = [-1,0,-1,4,0] \equiv [-1,1,0,0,1]$ which is not an equivalent divisor. Thus $r(4R) = 1$.

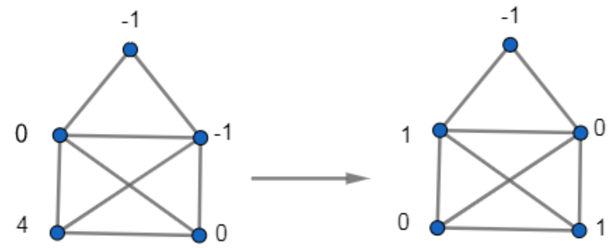

Figure 3.8: We first fire R once, then $[-1,0,-1,4,0] \equiv [-1,1,0,0,1]$. Next sending fire from P causes the burning of the whole graph. But the divisor is not effective.

6. $D = 5R = [0,0,0,5,0]$: $K_G - 5R = [0,2,2,1,1] - [0,0,0,5,0] = [0,2,2,-4,1] \equiv [0,1,1,-1,0]$ implies $r(K_G - 5R) = -1$. Therefore by Riemann - Roch we have,

   $r(5R) = deg(5R) - g + r(K_G - 5R) + 1 = 5 - 4 - 1 + 1 = 1$.

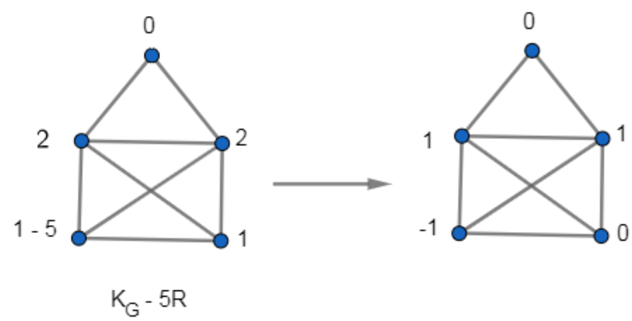

Figure 3.9: First reverse fire R once, then $[-1,0,-1,4,0] \equiv [0,1,1,-1,0]$. Next sending fire from R causes the burning of the whole graph. But the divisor is not effective. The process terminates.

7. $D = 6R = [0,0,0,6,0]$: Similarly we use Riemann - Roch to calculate the rank.

   Then $K_G - 6R = [0,2,2,1,1] - [0,0,0,6,0] = [0,2,2,-5,1] \equiv [0,1,1,-2,0]$ implies $r(K_G - 5R) = -1$. Then by Riemann - Roch formula we have $r(6R) = 2$.

Because of symmetry, R' also has the same rank as R.

All the calculations for vertices P, Q, Q', R and R' are summarized in the following table.

Table 3.1: Calculations for the House - X Graph ($g = 4$, $2g - 2 = 6$)

n	0	1	2	3	4	5	6	7	$8 \cdots$
$r(nP)$	0	0	0	0	1	1	2	3	$4 \cdots$
$r(nQ), r(nQ')$	0	0	0	0	1	2	2	3	$4 \cdots$
$r(nR), r(nR')$	0	0	0	1	1	1	2	3	$4 \cdots$

Following the definition of a Weierstrass vertex, we can see for all the vertices of House - X graph $r(4P), r(4Q), r(4Q'), r(4R), r(4R') \geq 1$, hence these vertices are Weierstrass. As described in definition 3.1, we now calculate the gap sequence for each vertex. Gap sequences for vertices $P = \{1,2,3,5\}$, $Q, Q' = \{1,2,3,6\}$ and $R, R' = \{1,2,4,5\}$.

Table 3.2: Gap Sequence, Semigroup, Weights for the House - X Graph

Vertex	Gap Sequence	Semigroup	Weight
P	$\{1,2,3,5\}$	$\{4,6,7,8,...\} =< 4,6,7,9 >$	$1+2+3+5-\frac{4*5}{2} = 1$
Q, Q'	$\{1,2,3,6\}$	$\{4,5,7,8,...\} =< 4,5,7 >$	$1+2+3+6-\frac{4*5}{2} = 2$
R, R'	$\{1,2,4,5\}$	$\{3,6,7,8,...\} =< 3,7,8 >$	$1+2+4+5-\frac{4*5}{2} = 2$

Therefore, we can conclude that only the vertex P is a normal Weierstrass vertex, but all the vertices are Weierstrass. Total weight of the House - X graph equals to $2*4+1 = 9$. Furthermore, vertices Q, Q', R and R' are examples for proposition 3.6.

3.2 Complete Graphs

In a complete graph of n vertices all the vertices are Weierstrass and behave in a similar manner when we are calculating the rank of a divisor preserving the symmetry. Thus for the calculations it is enough to consider only one vertex. In this work we are only presenting the calculations for K_4, K_5 and K_6. Furthermore, after closely observing these results, we predict the calculations for K_7 and so on. Normally in an algebraic curve, gaps occur before or including $n = 2g-2$ when we are calculating $r(nP)$, where P is a point on the curve. The house - X graph obeys that pattern, but complete graphs are slightly deviated maintaining their own structure.

Proposition 3.7

For the complete graphs on $n \geq 2$ vertices, $r(k(v)) = 0$ for $0 \leq k \leq n-2$, where v is any vertex of K_n.

Proof.

Let v and w be vertices of K_n and consider the divisor $D = k(v)$ for $0 \leq k \leq n-2$. Then the divisor $D - 1(w) = k(v) - 1(w)$ is not effective by Dhar's burning algorithm as $d_{K_n}(v) = n-1 > k$. So $r(D) < 1$, but, D is effective, hence $r(D) = 0$ $\square$

3.2.1 K_4

We are first considering the complete graph of 4 vertices. All the vertices are symmetric to each other. We define the general divisor on K_4 such that $D = [V_1, V_2, V_3, V_4]$. For the calculations, we are only considering V_1 because the results are the same for other vertices too. The genus of K_4 equals to 3, therefore, we are calculating $r(n(V_1))$ for $n = 0, 1, ...2g-2 = 4$.

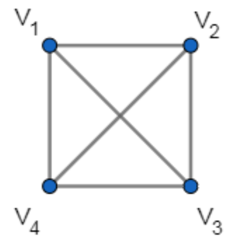

Figure 3.10: K_4 with vertices marked.

1. $D = 0V_1 = [0,0,0,0]$: Clearly $r(0V_1) = 0$.

2. $D = 1V_1 = [1,0,0,0]$: We have $r(1V_1) \geq 0$. Next, we need to check whether $r(1V_1) \geq 1$. For that we have to subtract all the effective divisors of degree 1 from D and check the effectiveness. Consider $[1,0,0,0] - [0,1,0,0] = [1,-1,0,0]$. Then, sending fire from V_2 will cause the burning of the whole graph but the divisor is not effective. Hence $r(1V_1) = 0$.

3. $D = 2V_1 = [2,0,0,0]$: To check whether $r(2V_1) \geq 1$, consider $[2,0,0,0] - [0,1,0,0] = [2,-1,0,0]$. Sending fire from V_2 will imply that $[2,-1,0,0]$ is not effective and, hence $r(2V_1) = 0$

4. $D = 3V_1 = [3,0,0,0]$: Obviously $r(3V_1) \geq 0$. Next, we need to check whether $r(3V_1) \geq 1$. Firing V_1 once in the following cases will produce effective divisors such that,
   $[3,0,0,0] - [1,0,0,0] = [2,0,0,0]$,
   $[3,0,0,0] - [0,1,0,0] = [3,-1,0,0] \equiv [0,0,1,1]$,
   $[3,0,0,0] - [0,0,1,0] = [3,0,-1,0] \equiv [0,1,0,1]$,
   $[3,0,0,0] - [0,0,0,1] = [3,0,0,-1] \equiv [0,1,1,0]$.
   Therefore, $r(3V_1) \geq 1$. Next, we need to check whether $r(3V_1) \geq 2$.

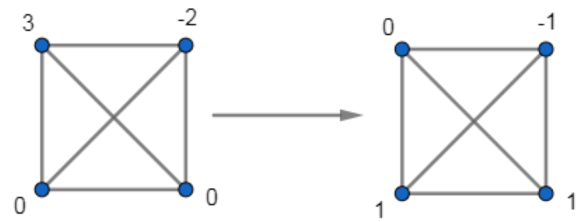

Figure 3.11: We first fire V_1 and then send fire from V_2. This causes the burning of the whole graph but the divisor, $[0,-1,1,1]$ is not effective. Thus we have $r(3V_1) = 1$.

For the next calculations, we are using the Riemann - Roch formula.

5. $D = 4V_1 = [4,0,0,0]$: Clearly $r(4V_1) \geq 1$. Next, consider $(K_G - 4V_1) = [-3,1,1,1]$, reverse firing V_1 produces $[0,0,0,0]$. Then, applying Riemann - Roch formula implies $r(4V_1) - 0 = 4 - 3 + 1 = 2$, that is $r(4V_1) = 2$.

All the calculations for vertices V_1, V_2, V_3 and V_4 are summarized in the following table. By symmetry all the results are the same. Therefore, we are taking a general vertex $v \in K_4$ to represent these vertices.

Table 3.3: Calculations for the K_4 Graph ($g = 3$, $2g - 2 = 4$)

n	0	1	2	3	4	5	$6\cdots$
$r(n(v))$	0	0	0	1	2	2	$3\cdots$

Table 3.4: Gap Sequence, Semigroup, Weights for the K_4 Graph

Vertex	Gap Sequence	Semigroup	Weight
v	$\{1,2,5\}$	$\{3,4,6,7,...\} =< 3,4 >$	$1+2+5-\frac{3*4}{2} = 2$

3.2.2 K_5

In K_5 all the vertices are symmetric to each other. We define the general divisor on K_5 such that $D = [V_1, V_2, V_3, V_4, V_5]$. For the calculations, we are only considering V_1 because the results are the same for the other vertices too. The genus of K_5 equals to 6, therefore, we are calculating $r(n(V_1))$ for $n = 0, 1, ...2g-2 = 10$.

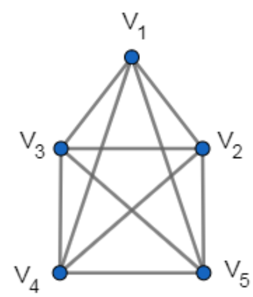

Figure 3.12: K_5 with vertices marked.

1. By proposition 3.7, we have $r(0V_1) = r(1V_1) = r(2V_1) = r(3V_1) = 0$.

2. $D = 4V_1 = [4,0,0,0,0]$: Clearly $r(4V_1) \geq 1$. We need to check whether $r(4V_1) \geq 2$.

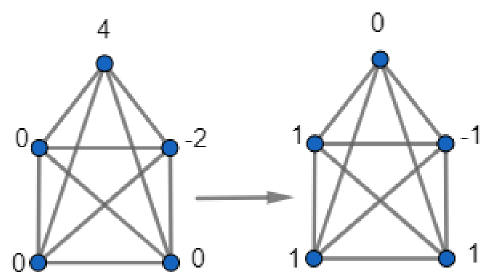

Figure 3.13: Consider the divisor $[4,-2,0,0,0]$. First, we fire V_1 and this produces $[0,-1,1,1,1]$, which is not effective. Hence $r(4V_1) = 1$

3. $D = 5V_1 = [5,0,0,0,0]$: We have $r(5V_1) \geq 1$. Next, consider all the effective divisors of degree 2. Clearly, $[5-1,-1,0,0,0]$, $[5-1,0,-1,0,0]$, $[5-1,0,0,-1,0]$ and $[5-1,0,0,0,-1]$ are the same as in the previous case. Therefore, these divisors are equivalent to the effective divisors.

$$\text{Fire } V_1 \text{ once} \begin{cases} [5,-1,-1,0,0] \equiv [1,0,0,1,1] \\ [5,-1,0,-1,0] \equiv [1,0,1,0,1] \\ [5,-1,0,0,-1] \equiv [1,0,1,1,0] \\ [5,0,-1,-1,0] \equiv [1,1,0,0,1] \\ [5,0,-1,0,-1] \equiv [1,1,0,1,0] \\ [5,0,0,-1,-1] \equiv [1,1,1,0,0] \end{cases}$$

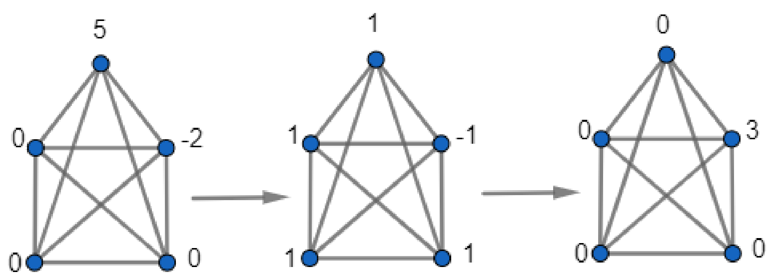

Figure 3.14: For $[5,-2,0,0,0]$, firing V_1 once and then reverse firing V_2 will produce the effective divisor $[0,3,0,0,0]$. Similarly $[5,0,-2,0,0]$, $[5,0,0,-2,0]$ and $[5,0,0,0,-2]$ are also equivalent to the effective divisors respectively.

Thus $r(5V_1) \geq 2$. However, $[5,0,0,0,0]-[0,2,1,0,0] = [5,-2,-1,0,0]$ is not equivalent to an effective divisor. Therefore, $r(5V_1) = 2$.

For the next calculations, we are using the Riemann - Roch formula.

4. $D = 6V_1 = [6,0,0,0,0]$: Consider $(K_G - 6V_1) = [2,2,2,2,2]-[6,0,0,0,0] = [-4,2,2,2,2]$ which is equivalent to $[0,1,1,1,1]$ when reverse firing V_1 once. Therefore, trivially

$r(K_G - 6V_1) \geq 1$. However, $[0,1,1,1,1] - [1,1,0,0,0] = [-1,0,1,1,1]$ is not equivalent to an effective divisor. Hence $r(K_G - 6V_1) = 1$. Then applying Riemann - Roch formula, we have $r(6V_1) - 1 = 6 - 6 + 1$ implies $r(6V_1) = 2$.

5. $D = 7V_1 = [7,0,0,0,0]$: Consider $(K_G - 7V_1) = [2,2,2,2,2] - [7,0,0,0,0] = [-5,2,2,2,2]$ which is equivalent to $[3,0,0,0,0]$ when reverse firing V_1 twice. This implies $r(K_G - 7V_1) = 0$. Therefore, by Riemann - Roch formula we have, $r(7V_1) - 0 = 7 - 6 + 1$ implies $r(7V_1) = 2$.

6. $D = 8V_1 = [8,0,0,0,0]$: Consider $(K_G - 8V_1) = [2,2,2,2,2] - [8,0,0,0,0] = [-6,2,2,2,2]$ which is equivalent to $[2,0,0,0,0]$ when reverse firing V_1 twice. Clearly, $r(K_G - 8V_1) = 0$. Therefore, by Riemann - Roch formula we have, $r(8V_1) - 0 = 8 - 6 + 1$ implies $r(8V_1) = 3$.

7. $D = 9V_1 = [9,0,0,0,0]$: Consider $(K_G - 9V_1) = [2,2,2,2,2] - [9,0,0,0,0] = [-7,2,2,2,2]$ which is equivalent to $[1,0,0,0,0]$ when reverse firing V_1 twice. Clearly, $r(K_G - 9V_1) = 0$. Therefore, by Riemann - Roch formula we have, $r(9V_1) - 0 = 9 - 6 + 1$ implies $r(9V_1) = 4$.

8. $D = 10V_1 = [10,0,0,0,0]$: Consider $(K_G - 10V_1) = [2,2,2,2,2] - [10,0,0,0,0] = [-8,2,2,2,2]$ which is equivalent to $[0,0,0,0,0]$ when reverse firing V_1 twice. Clearly, $r(K_G - 10V_1) = 0$. Therefore, by Riemann - Roch formula we have, $r(10V_1) - 0 = 10 - 6 + 1$ implies $r(10V_1) = 5$.

All the calculations for vertices V_1, V_2, V_3, V_4 and V_5 are summarized in the following table. By symmetry all the results are the same. Therefore, we are taking a general vertex $v \in K_5$ to represent these vertices.

Table 3.5: Calculations for the K_5 Graph ($g = 6$, $2g - 2 = 10$)

n	0	1	2	3	4	5	6	7	8	9	10	11	12	$13 \cdots$
$r(n(v))$	0	0	0	0	1	2	2	2	3	4	5	5	6	$7 \cdots$

Table 3.6: Gap Sequence, Semigroup, Weights for the K_5 Graph

Vertex	Gap Sequence	Semigroup	Weight
v	$\{1,2,3,6,7,11\}$	$\{4,5,8,9,10,12...\} =< 4,5 >$	$1+2+3+6+7+11-\frac{6*7}{2}=9$

3.2.3 K_6

Similarly in K_6 all the vertices are symmetric to each other. We define the general divisor on K_6 to be $D = [V_1,V_2,V_3,V_4,V_5,V_6]$. For the calculations we are only considering V_1 because the results are the same for the other vertices too. The genus of K_6 equals to 10. Therefore, we are calculating $r(n(V_1))$ for $n = 0,1,...2g-2 = 18$.

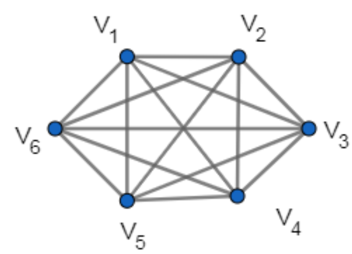

Figure 3.15: K_6 with vertices marked.

1. By proposition 3.7, we have $r(0V_1) = r(1V_1) = r(2V_1) = r(3V_1) = r(4V_1) = 0$.

2. $D = 5V_1 = [5,0,0,0,0,0]$: Trivially $r(5V_1) = 1$.

3. $D = 6V_1 = [6,0,0,0,0,0]$: Clearly $r(6V_1) \geq 1$. Next, we need to check the effectiveness of D - degree 2 effective divisors. Clearly, $[6-1,-1,0,0,0,0]$, $[6-1,0,-1,0,0,0]$, $[6-1,0,0,-1,0,0]$, $[6-1,0,0,0,-1,0]$ and $[6-1,0,0,0,0,-1]$ are the same as in the previous case. Therefore, these divisors are equivalent to the effective divisors.

$$\text{Fire } V_1 \text{ once} \begin{cases} [6,-1,-1,0,0,0] \equiv [1,0,0,1,1,1] \\ [6,-1,0,-1,0,0] \equiv [1,0,1,0,1,1] \\ [6,-1,0,0,-1,0] \equiv [1,0,1,1,0,1] \\ [6,-1,0,0,0,-1] \equiv [1,0,1,1,1,0] \\ [6,0,-1,-1,0,0] \equiv [1,1,0,0,1,1] \\ [6,0,-1,0,-1,0] \equiv [1,1,0,1,0,1] \\ [6,0,-1,0,0,-1] \equiv [1,1,0,1,1,0] \\ [6,0,0,-1,-1,0] \equiv [1,1,1,0,0,1] \\ [6,0,0,-1,0,-1] \equiv [1,1,1,0,1,0] \\ [6,0,0,0,-1,-1] \equiv [1,1,1,1,0,0] \end{cases}$$

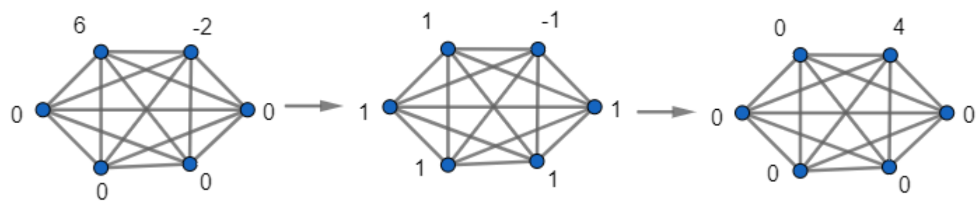

Figure 3.16: Next consider, $[6,-2,0,0,0,0]$ and fire V_1 once. This will produce the divisor $[1,-1,1,1,1,1]$. Then reverse firing V_2 produces $[0,4,0,0,0,0]$, which is effective. Similarly, $[6,0,-2,0,0,0]$, $[6,0,0,-2,0,0]$, $[6,0,0,0,-2,0]$ and $[6,0,0,0,0,-2]$ are also equivalent to the effective divisors.

Hence $r(6V_1) \geq 2$. However, $[6,0,0,0,0,0]-[0,2,1,0,0,0]=[6,-2,-1,0,0,0]$ is not equivalent to an effective divisor. Therefore, $r(6V_1)=2$.

4. $D = 7V_1 = [7,0,0,0,0,0]$: Clearly, $r(7V_1) \geq 2$. But, $[7,0,0,0,0,0]-[0,2,1,0,0,0]=[7,-2,-1,0,0,0]$ is not equivalent to an effective divisor. Hence $r(7V_1)=2$.

5. $D = 8V_1 = [8,0,0,0,0,0]$: We have $r(8V_1) \geq 2$. However, $[8,0,0,0,0,0] - [0,2,1,0,0,0] = [8,-2,-1,0,0,0]$ is not equivalent to an effective divisor. Therefore, $r(8V_1) = 2$.

6. $D = 9V_1 = [9,0,0,0,0,0]$: Clearly, $r(9V_1) \geq 2$. But,
   $[9,0,0,0,0,0] - [0,2,1,0,0,0] = [9,-2,-1,0,0,0]$ is not equivalent to an effective divisor. Thus $r(9V_1) = 2$.

   For the rest of calculations, we are using the Riemann - Roch formula.

7. $D = 10V_1 = [10,0,0,0,0,0]$: Consider $(K_G - 10v_1) = [3,3,3,3,3,3] - [10,0,0,0,0,0] = [-7,3,3,3,3,3]$. Reverse firing V_1 thrice will give $[8,0,0,0,0,0]$. By part 5, $r(K_G - 10v_1) = 2$. Then applying Riemann - Roch formula will give, $r(10V_1) - 2 = 10 - 10 + 1$ which implies, $r(10V_1) = 3$.

8. $D = 11V_1 = [11,0,0,0,0,0]$: Consider $(K_G - 11v_1) = [3,3,3,3,3,3] - [11,0,0,0,0,0] = [-8,3,3,3,3,3]$. Reverse firing V_1 thrice will give $[7,0,0,0,0,0]$. By part 4, $r(K_G - 11v_1) = 2$. Then applying Riemann - Roch formula will give, $r(11V_1) - 2 = 11 - 10 + 1$ which implies, $r(11V_1) = 4$.

9. $D = 12V_1 = [12,0,0,0,0,0]$: Consider $(K_G - 12v_1) = [3,3,3,3,3,3] - [12,0,0,0,0,0] = [-9,3,3,3,3,3]$. Reverse firing V_1 thrice will give $[6,0,0,0,0,0]$. By part 3, $r(K_G - 12v_1) = 2$. Then applying Riemann - Roch formula will give, $r(12V_1) - 2 = 12 - 10 + 1$ which implies, $r(12V_1) = 5$.

10. $D = 13V_1 = [13,0,0,0,0,0]$: Consider $(K_G - 13v_1) = [3,3,3,3,3,3] - [13,0,0,0,0,0] = [-10,3,3,3,3,3]$. Reverse firing V_1 thrice will give $[5,0,0,0,0,0]$. By part 2, $r(K_G - 13v_1) = 1$. Then applying Riemann - Roch formula will give, $r(13V_1) - 1 = 13 - 10 + 1$ which implies, $r(13V_1) = 5$.

11. $D = 14V_1 = [14,0,0,0,0,0]$: Consider $(K_G - 14v_1) = [3,3,3,3,3,3] - [14,0,0,0,0,0] =$

$[-11,3,3,3,3,3]$. Reverse firing V_1 thrice will give $[4,0,0,0,0,0]$. By part 1, $r(K_G - 14v_1) = 0$. Then applying Riemann - Roch formula will give, $r(14V_1) - 0 = 14 - 10 + 1$ which implies, $r(14V_1) = 5$.

12. $D = 15V_1 = [15,0,0,0,0,0]$: Consider $(K_G - 15v_1) = [3,3,3,3,3,3] - [15,0,0,0,0,0] = [-12,3,3,3,3,3]$. Reverse firing V_1 thrice will give $[3,0,0,0,0,0]$. By part 1, $r(K_G - 15v_1) = 0$. Then applying Riemann - Roch formula will give, $r(15V_1) - 0 = 15 - 10 + 1$ which implies, $r(15V_1) = 6$.

13. $D = 16V_1 = [16,0,0,0,0,0]$: Consider $(K_G - 16v_1) = [3,3,3,3,3,3] - [16,0,0,0,0,0] = [-13,3,3,3,3,3]$. Reverse firing V_1 thrice will give $[2,0,0,0,0,0]$. By part 1, $r(K_G - 16v_1) = 0$. Then applying Riemann - Roch formula will give, $r(16V_1) - 0 = 16 - 10 + 1$ which implies, $r(16V_1) = 7$.

14. $D = 17V_1 = [17,0,0,0,0,0]$: Consider $(K_G - 17v_1) = [3,3,3,3,3,3] - [17,0,0,0,0,0] = [-14,3,3,3,3,3]$. Reverse firing V_1 thrice will give $[1,0,0,0,0,0]$. By part 1, $r(K_G - 17v_1) = 0$. Then applying Riemann - Roch formula will give, $r(17V_1) - 0 = 17 - 10 + 1$ which implies, $r(17V_1) = 8$.

15. $D = 18V_1 = [18,0,0,0,0,0]$: Consider $(K_G - 18v_1) = [3,3,3,3,3,3] - [18,0,0,0,0,0] = [-15,3,3,3,3,3]$. Reverse firing V_1 thrice will give $[0,0,0,0,0,0]$. By part 1, $r(K_G - 18v_1) = 0$. Then applying Riemann - Roch formula will give, $r(18V_1) - 0 = 18 - 10 + 1$ which implies, $r(18V_1) = 9$.

    Next, we have $r(nV_1) = n - g = n - 10$ for all $n \geq 19$, from section 3.3, step 2.

All the calculations for vertices V_1, V_2, V_3, V_4, V_5 and V_6 are summarized in the following table. By symmetry all the results are the same. Therefore, we are taking a general vertex $v \in K_6$ to represent these vertices.

Table 3.7: Calculations for the K_6 Graph ($g = 10$, $2g-2 = 18$)

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
$r(n(v))$	0	0	0	0	0	1	2	2	2	2	3	4	5	5	5	6	7	8	9	9

n	20	21	$22\cdots$
$r(n(v))$	10	11	$12\cdots$

Table 3.8: Gap Sequence, Semigroup, Weights for the K_6 Graph

Gap Sequence	$\{1,2,3,4,7,8,9,13,14,19\}$
Semigroup	$\{5,6,10,11,12,15,16,...\} =< 5,6 >$
Weight	$(1+2+3+4+7+8+9+13+14+19) - \frac{10*11}{2} = 25$

3.2.4 Predictions and conjecture

After closely observing these calculations we can see a pattern in the $r(n(v))$ values of a complete graph. Moreover, we can observe that always the gap sequence of a complete graph contains $n = 2g-1$, as the last gap value deviates from the gap sequence structure of an algebraic curve. Furthermore, we can see gaps occurring when $r(n(v)) = 0,2(= g-1)$ for K_4, $r(n(v)) = 0,2,5(= g-1)$ for K_5 and $r(n(v)) = 0,2,5,9(= g-1)$ for K_6.

Therefore, we can predict the gap occurrence for K_7. Gaps must occur when $r(n(v)) = 0,2,5,9,14(= g-1)$, where the genus of K_7 is 15. Similarly, we can predict for K_n such that gaps must occur when $r(n(v)) = 0,2,5,9,14,...,g-1$ where g is the genus of K_n. Furthermore, we can predict how many n values are occurring at each of these $r(n(v))'s$. And also, from the proposition 3.7, we can determine the occurrence of zeros.

Next, without doing any direct calculations, we are going to predict the gap sequence, semigroup and weights for K_7.

The genus of K_7 equals to 15. Therefore, we are interested in predicting the $r(n(v))$ values for $n = 0, 1, 2, ..., 2g-2 = 28$.

Table 3.9: Calculations for the K_7 Graph ($g = 15$, $2g-2 = 28$)

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
$r(n(v))$	0	0	0	0	0	0	1	2	2	2	2	2	3	4	5	5	5	5	6	7

n	20	21	22	23	24	25	26	27	28	29	$30 \cdots$
$r(n(v))$	8	9	9	9	10	11	12	13	14	14	$15 \cdots$

Table 3.10: Gap Sequence, Semigroup, Weights for the K_7 Graph

Gap Sequence	$\{1,2,3,4,5,8,9,10,11,15,16,17,22,23,29\}$
Semigroup	$\{6,7,12,13,14,18,19,20,21,24,25,26,27,28,30...\} = < 6,7 >$
Weight	$(1+2+3+4+5+8+9+10+11+15+16+17+...+29) - \frac{15*16}{2} = 55$

Likewise, we can determine the weights for higher cases of complete graphs. Next, we are proposing following two lemmas and a proposition before presenting our main result.

Lemma 3.8

For any vertex v of K_n for $n \geq 4$, $r((n-1)v) = 1$.

Proof.

Clearly, $r((n-1)v) \geq 0$ as $D = (n-1)v$ is effective. Next, we need to check whether $r((n-1)v) \geq 1$. Since, $d_{K_n}(v) = n-1$, for any $w (\neq v) \in V(K_n)$, $D' = D - 1(w)$ is equivalent to an effective divisor if we fire v once. But, $D - 2(w)$ is not equivalent to an effective divisor by Dhar's burning algorithm. Therefore, $r((n-1)v) = 1$.

□

Lemma 3.9

For any vertex v of K_n for $n \geq 4$, $r(n(v)) = 2$.

Proof.

Clearly by lemma 3.8 we have $r(n(v)) \geq 1$. Next, we need to check whether $r(n(v)) \geq 2$. Let $D = n(v)$ and we need to subtract all the degree 2 effective divisors from D need to check the effectiveness of resulting divisor. We have 3 possibilities;

Case I: $n(v) - (1(v) + 1(w))$ for any $w(\neq v) \in V(K_n)$ is same as the previous case and hence, it is effective.

Case II: $n(v) - (1(w) + 1(x))$ is equivalent to an effective divisor when we fire v once. Where $w, x(\neq v) \in V(K_n)$

Case III: $n(v) - 2(v)$ is clearly effective and $n(v) - 2(w)$ for any $w(\neq v) \in V(K_n)$ is equivalent to an effective divisor when we first fire v and then reverse fire w.

Hence, $r(n(v)) \geq 2$. But for $n(v) - (1(v) + 2(w))$ is not equivalent to an effective divisor by the proof of lemma 3.8. Therefore, $r(n(v)) = 2$. $\square$

Proposition 3.10

Weierstrass Semigroup for complete graph on n vertices for $n \geq 4$ is 2 - generated and given by $< n-1, n >$.

Proof.

We are going to prove this by contradiction. Assume semigroup of K_n for$n \geq 4$ is 3 - generated. Then by the proposition 3.7, third generating element can't be less than or equal to $n-2$. Then it sould be between $n+1$ and $2n-2$ since, the last gap value occurs at $2n-1$. But, this violates cardinality of gap sequence. Hence, this is a contradiction. □

Next, considering these results and combining the weight sequence for complete graphs with an integer sequence [20], we present the following theorem.

Theorem 3.11

The Weierstrass weight of a vertex of K_n is given by the formula, $\frac{(n-1)(n-2)(n-3)(n+4)}{24}$.

Since for K_1, K_2 and K_3 no vertex is Weierstrass, from this formula we get the weight for a non-Weierstrass vertex is equals to 0.

CHAPTER 4: BOUND ON WEIERSTRASS WEIGHT

On a Riemann surface, the Weierstrass weight is exactly $g^3 - g$. But for graphs this is only an upper bound on the weight of $g^3 - g$. Here we introduce the conjecture that no graph with $g^3 - g$ vertices with genus g has exactly $g^3 - g$ normal Weierstrass vertices. For the algebraic curve case, when $g \geq 2$ Weierstrass points always exist, however, for graphs it is not the same. There are some graph families which do not contain Weierstrass vertices even though $g \geq 2$, and in some other cases such as complete graphs with $|V(G)| \geq 4$, all the vertices are Weierstrass. Refer to example 4.4 in [4].

In this study we prove this fact for the genus 2 case taking into consideration all the non-isomorphic graphs with 6 vertices and 7 edges, when $g = 2$, $g^3 - g = 6$. According to the Atlas of Graphs [18], there are 24 non-isomorphic graphs of genus 2 having 6 vertices and 7 edges. After eliminating disconnected graphs, 19 graphs are left for the calculations. Graph numbering was done according to [18], refer to page 10. We have used the properties of the rank, symmetry properties of the graph and Dhar's burning algorithm to determine whether a vertex is Weierstrass or not, according to the definition given in Chapter 2.

A vertex v has the Weierstrass property when $r(g(v)) \geq 1$ or $r(K_G - g(v)) \geq 0$ where, K_G is the canonical divisor. The vertices which satisfy the symmetry with a vertex (say x) are denoted by x', x'', etc. throughout this chapter.

Theorem 4.1

There is no finite, connected, undirected and unweighted multi graph without loop edges with 6 vertices and genus 2 with all vertices being normal Weierstrass vertices.

The proof of this theorem will be by direct examination of cases.

4.1 Calculations for the family of graphs of genus 2 with 6 vertices & 7 edges

4.1.1 Cycle of length 4 with handles and a chord (G111, G112, G113, G114, G115, G120, G123)

In this section we consider all the graphs with 6 vertices having a length 4 cycle with a chord and two handles or one subdivided handle. All of them share some kind of a common structure. The graphs in this category with their canonical divisors are shown below.

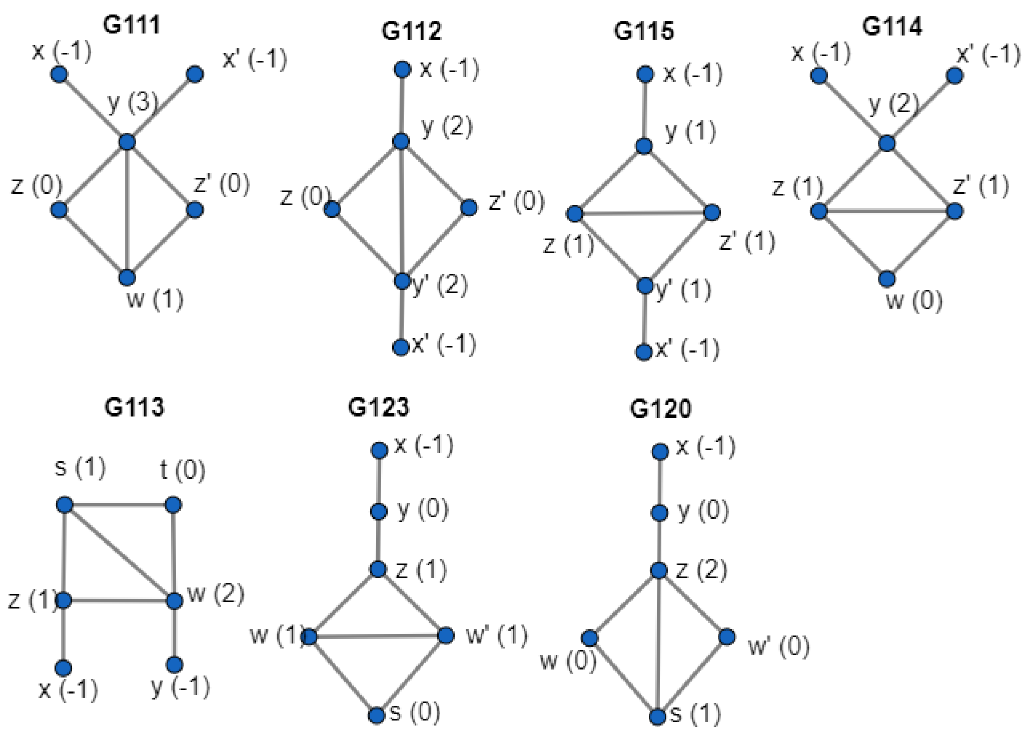

Figure 4.1: Cycle of length 4 with handles and a chord and their canonical divisor.

Corresponding to these 7 graphs there are 7 canonical divisors which can be transformed to become effective divisors by reverse firing moves as described in chip firing game (Baker's version). For the process of G123 and G120 it is not as straightforward. You must consider either G120 or G123 and first reverse fire *x*, next reverse fire *y* and finally reverse fire *x* again and do the same for the other graph. Next we need to check the Weierstrass property for all these graphs.

1. G111: $K_G = [x, x', y, z, z', w] \equiv [-1, -1, 3, 0, 0, 1] \equiv [0, 0, 1, 0, 0, 1]$.

   To verify the Weierstrass property of each vertex, we can easily use the second definition; $r(K_G - 2(v)) \geq 0$.

   $(K_G - 2(x)) \equiv [-2, 0, 1, 0, 0, 1]$, reverse firing x gives $[-1, 0, 0, 0, 0, 1]$. At this point sending fire from x will cause the burning of the entire graph but the divisor is not effective. Hence x and x' are not Weierstrass.

   $(K_G - 2(y)) \equiv [0, 0, -1, 0, 0, 1]$, then sending fire from y causes the whole graph to burn and the divisor is not effective. Therefore, y is not Weierstrass and because of the same structure w is also not Weierstrass.

   $(K_G - 2(z)) \equiv [0, 0, 1, -2, 0, 1]$, reverse firing z gives [0, 0, 0, 0, 0, 0] which is effective and hence $r(K_G - 2(z)) \geq 0$. Thus z and z' are Weierstrass.

2. G112: $K_G = [x, y, z, z', y', x'] \equiv [-1, 2, 0, 0, 2, -1] \equiv [0, 1, 0, 0, 1, 0]$. Because the format is same as G111, it implies only z and z' are the Weierstrass vertices of G112.

3. G120: $K_G = [x, y, z, w, w', s] \equiv [-1, 0, 2, 0, 0, 1] \equiv [0, 0, 1, 0, 0, 1]$. Similarly only w and w' are the Weierstrass vertices.

4. G113: $K_G = [x, y, z, w, t, s] \equiv [-1, -1, 1, 2, 0, 1] \equiv [0, 0, 0, 1, 0, 1]$. Using the definition $r(K_G - g(v)) \geq 0$, it's obvious that s and w are not Weierstrass. Consider $(K_G - 2(z)) \equiv [0, 0, -2, 1, 0, 1]$. Sending fire from z will burn only x then firing all the other vertices simultaneously produces $[0, 0, 0, 0, 0, 0]$ implying that z is Weierstrass. Similarly, t is Weierstrass. Next consider $(K_G - 2(x)) \equiv [-2, 0, 0, 1, 0, 1]$. First fire y, w, s, t simultaneously to produce $[-2, 0, 2, 0, 0, 0]$. Then reverse firing x will give the zero divisor, hence x is Weierstrass. For $(K_G - 2(y)) \equiv [0, -2, 0, 1, 0, 1]$ first reverse firing y and then sending fire from y will cause the burning of the whole graph but the divisor is not effective. Therefore y is not Weierstrass.

5. G114: Canonical divisor K_G is given by,

   $K_G = [x, x', y, z, z', w] \equiv [-1, -1, 2, 1, 1, 0] \equiv [0, 0, 0, 1, 1, 0]$. After observing the pattern from previous calculations it clearly implies that w is a Weierstrass vertex. And also y is Weierstrass by the same argument and that forces x and x' to be Weierstrass. But z and z' are not Weierstrass, because $(K_G - 2(z)) \equiv [0, 0, 0, -1, 1, 0]$ can't be transformed to an effective divisor and this is verified by Dhar's burning algorithm.

6. G115: $K_G = [x, y, z, z', y', x'] \equiv [-1, 1, 1, 1, 1, -1] \equiv [0, 0, 1, 1, 0, 0]$, almost same as G112 with zz' edge instead of yy' edge. Following the same pattern first y and y' are therefore Weierstrass and that implies x and x' are Weierstrass. Applying the same argument as in G114, z and z' are not Weierstrass.

7. G123: Except w and w', the rest are Weierstrass vertices which is the exact opposite of G120.

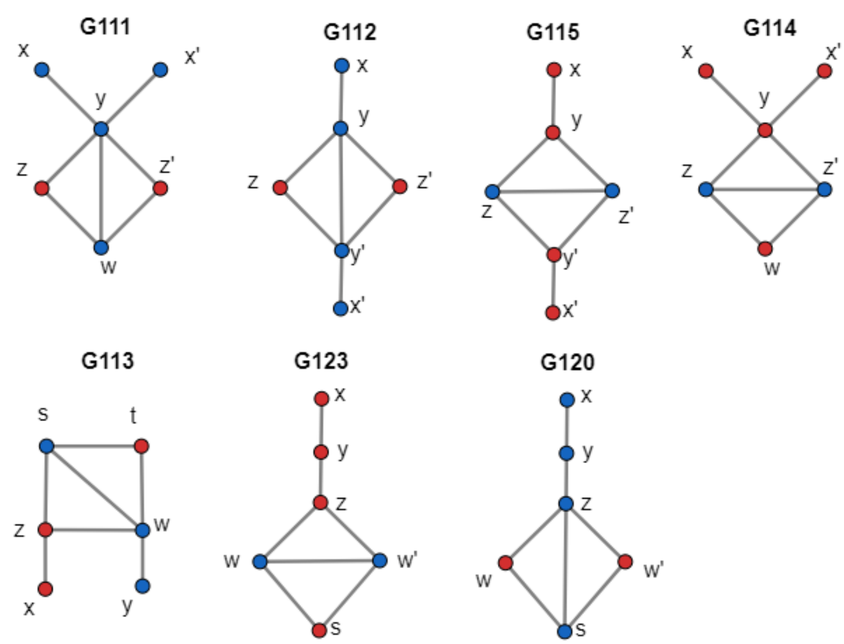

Figure 4.2: G111, G112, G113, G114, G115, G120, G123 with Weierstrass vertices marked in red.

4.1.2 Cycle of length 4 with a subdivided chord and a handle (G121, G125)

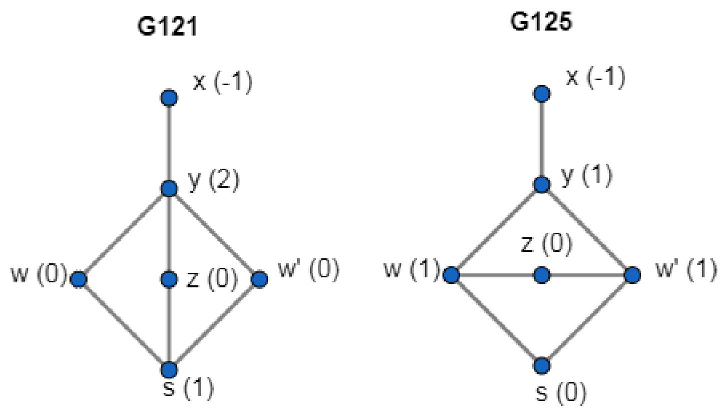

Figure 4.3: Cycle of length 4 with a subdivided chord and a handle and their canonical divisor.

1. G121: The canonical divisor of G121 $K_G = [x, y, w, z, w', s] \equiv [-1, 2, 0, 0, 0, 1]$ can be transformed to an effective divisor by reverse firing x one time; $K_G = [0, 1, 0, 0, 0, 1]$. Next we need to check the Weierstrass property of each vertex.

   Consider $(K_G - 2(x)) \equiv [-2, 1, 0, 0, 0, 1]$, when you send fire from x it will stop at y as divisor value is 1 at y. Then, fire all the other vertices except x simultaneously gives $(K_G - 2(x)) \equiv [-1, 0, 0, 0, 0, 1]$. Firing x again will cause the burning of the whole graph but the divisor is not effective. Hence x is not Weierstrass.

   Following the pattern in section 4.1.1 implies y and s are also not Weierstrass.

   Next consider $(K_G - 2(w)) \equiv [0, 1, -2, 0, 0, 1]$, firing x, y, z, s, w' simultaneously will give the zero divisor implying $r(K_G - 2(w)) \geq 0$. Therefore w and w' are Weierstrass. By a similar argument we can show that z is also Weierstrass.

2. G125: $K_G = [x, y, w, z, w', s] \equiv [-1, 1, 1, 0, 1, 0] \equiv [0, 0, 1, 0, 1, 0]$. Clearly w and w' are not Weierstrass. Consider $(K_G - 2(y)) \equiv [0, -2, 1, 0, 0, 1]$. Then sending fire from y will leave

w, w', z, s unburned. Firing these vertices simultaneously will give the zero divisor which is effective hence y is Weierstrass, and this implies x is also Weierstrass. Similarly z and s are also Weierstrass.

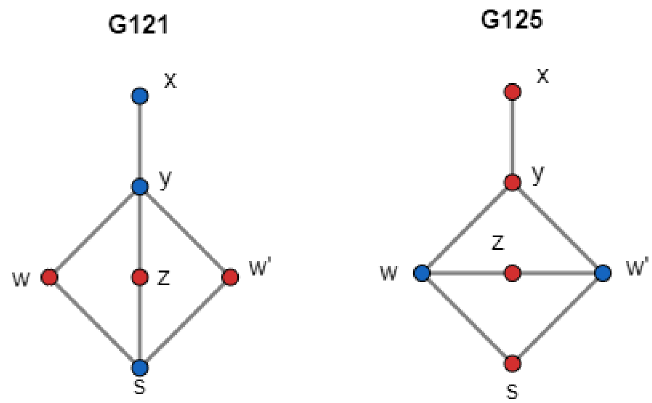

Figure 4.4: G121, G125 with Weierstrass vertices marked in red.

4.1.3 Cycle of length 5 with a handle and chord (G118, G122, G124)

In this subsection we consider all the non-isomorphic graphs having 6 vertices and 7 edges with cycle of length 5 and a chord and handle.

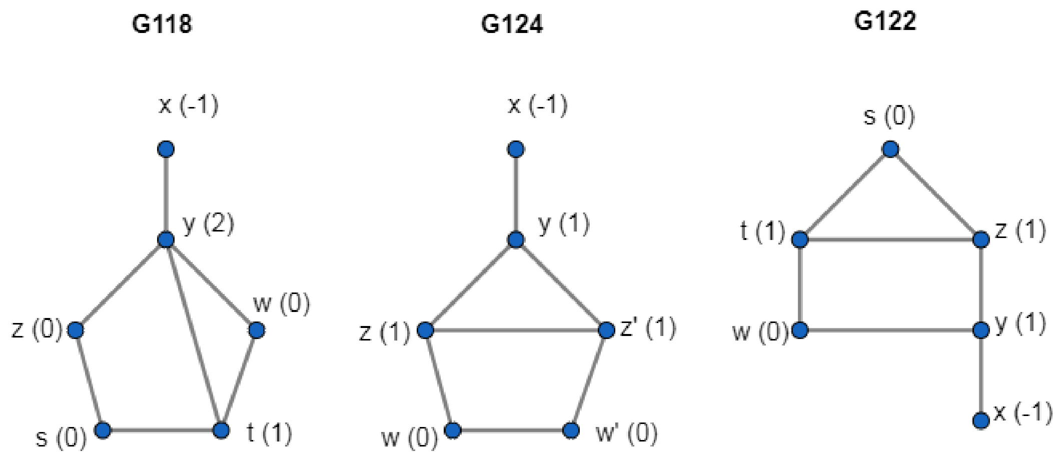

Figure 4.5: Cycle of length 5 with a handle and chord and their canonical divisor.

These canonical divisors are not effective, but easily can be converted to effective divisors by reverse firing x once.

1. G118: $K_G = [x,y,z,w,t,s] \equiv [-1,2,0,0,1,0] \equiv [0,1,0,0,1,0]$. First consider $(K_G - 2(x)) \equiv [-2,1,0,0,1,0]$, reverse fire x and then sending fire from x will cause the burning of the entire graph. But the divisor is not effective, hence x is not Weierstrass. Next, $(K_G - 2(y)) \equiv [0,-1,0,0,1,0]$ implies burning the whole graph when sending fire from y. Therefore y is also not Weierstrass. Next consider z and s, both are not Weierstrass which can be easily verified by Dhar's burning algorithm. Same as y, t is also not Weierstrass. We have $(K_G - 2(w)) \equiv [0,1,0,-2,1,0]$, reverse firing w is equivalent to the zero divisor hence w is Weierstrass.

2. G124: Follows the same pattern as G118. Therefore *y* is a Weierstrass vertex and this implies x is also Weierstrass. The rest is not.

3. G122: Similarly *s* is the only Weierstrass vertex of G122.

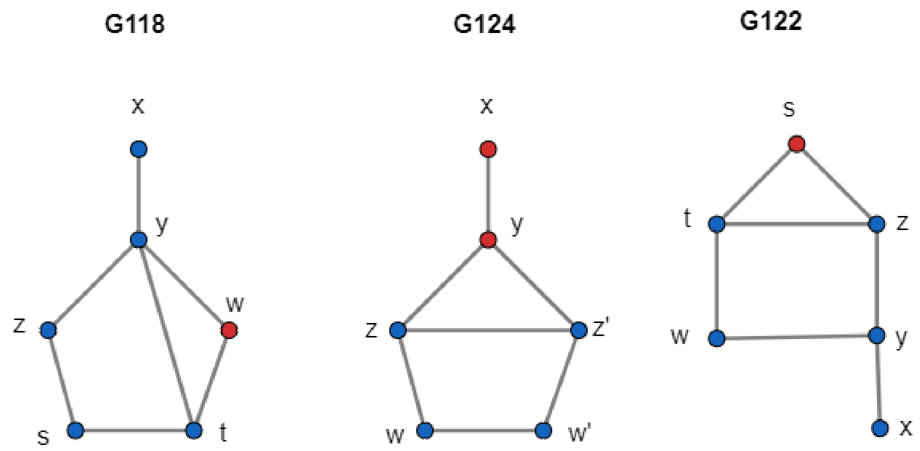

Figure 4.6: G118, G122, G124 with Weierstrass vertices marked in red.

4.1.4 Cycle of length 6 with a chord (G127, G128)

Here we have cycles of length 6. Since $|E(G)| = 7$, one edge becomes a chord. Canonical divisors are effective. Therefore we only need to check whether $r(K_G - 2(v)) \geq 0$ for all $v \in V(G)$.

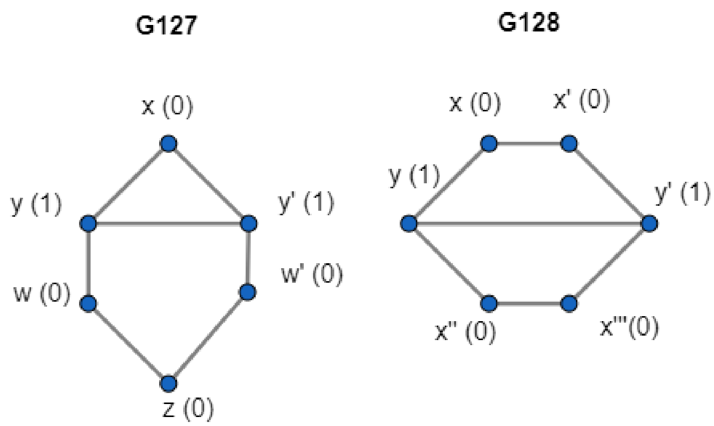

Figure 4.7: Cycle of length 6 with a chord and their canonical divisor.

1. G127: Here $K_G = [x, y, y', w, w', z] \equiv [0, 1, 1, 0, 0, 0]$. First consider $(K_G - 2(x)) = [-2, 1, 1, 0, 0, 0]$, reverse firing x will give the zero divisor. This implies $r(K_G - 2(x)) \geq 0$ and hence x is Weierstrass. Obviously y and y' are not Weierstrass because if you send fire from y or y' accordingly, the whole graph will burn, but the divisor is not effective.
   Next consider $(K_G - 2(w)) = [0, 1, 1, -2, 0, 0]$, sending fire from w will leave y, y', x unburned. Then firing y, y', x simultaneously gives $(K_G - 2(w)) = [0, 0, 0, -1, 1, 0]$ which is not equivalent to an effective divisor. Therefore, w and w' are not Weierstrass. By the same argument as for vertex x, we can show that z is also Weierstrass.

2. G128: From G127 it follows that G128 can't have any Weierstrass vertices.

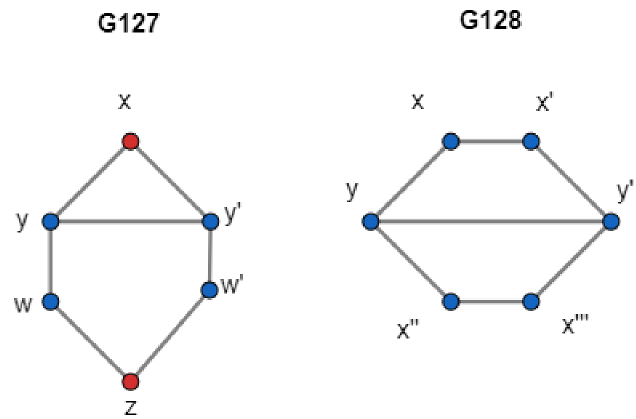

Figure 4.8: G127, G128 with Weierstrass vertices marked in red.

4.1.5 Bow-tie with a handle (G117, G119)

In this subsection we consider the following two graphs in a shape of a bow-tie with one vertex having degree 1.

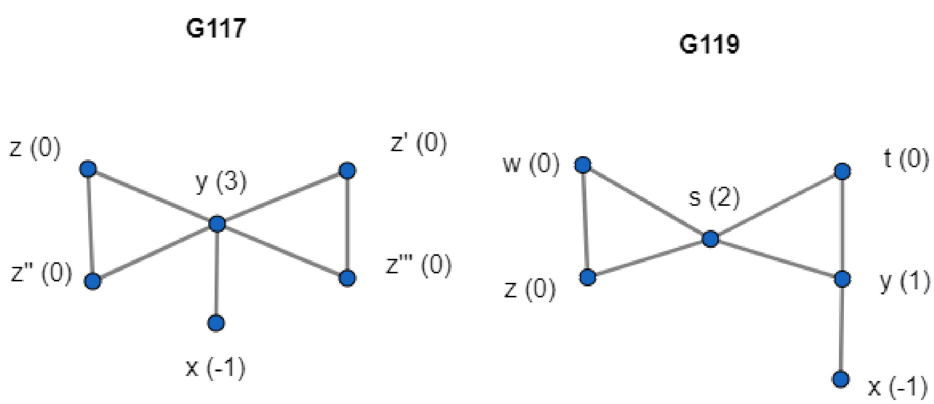

Figure 4.9: Bow-tie with a handle and their canonical divisor.

These canonical divisors are not effective, but reverse firing x will give effective divisors $[x,y,z,z',z'',z'''] \equiv [0,2,0,0,0,0]$ and $[x,y,z,w,s,t] \equiv [0,0,0,0,2,0]$, respectively.

1. G117: Since $(K_G - 2(y)) = [0,0,0,0,0,0]$, y is Weierstrass and hence x is also Weierstrass because we only need to reverse fire x twice.

   Next consider $(K_G - 2(z)) = [0,2,-2,0,0,0]$. Sending fire from z will leave x, y, z', z''' unburned. Then firing x, y, z', z''' simultaneously gives $(K_G - 2(z)) = [0,2,-2,0,0,0] \equiv [0,0,-1,0,1,0]$, and again sending fire from z will cause the burning of the whole graph and the divisor is not effective, hence z is Weierstrass. Because of symmetry z', z'', z''' are also not Weierstrass.

2. G119: From the same argument as in G117, it follows that only s is Weierstrass.

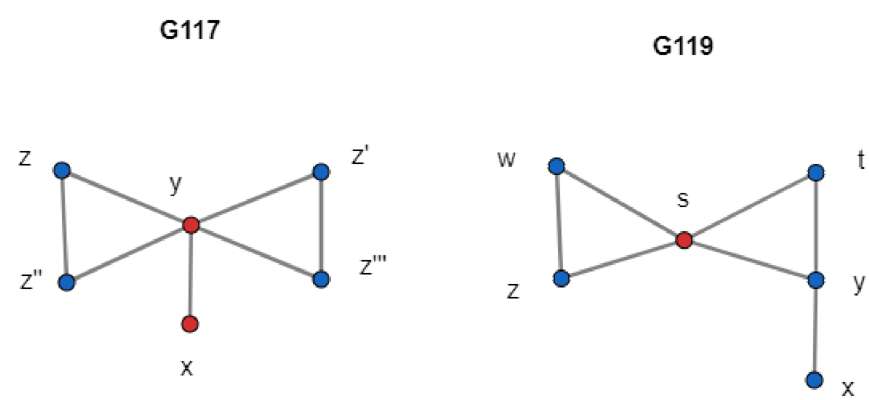

Figure 4.10: G117, G119 with Weierstrass vertices indicated in red.

4.1.6 Remaining Graphs (G126, G129, G130)

Here we have 3 graphs with minimum degree = 2. Therefore canonical divisors are effective. We only need to check the Weierstrass property of each vertex.

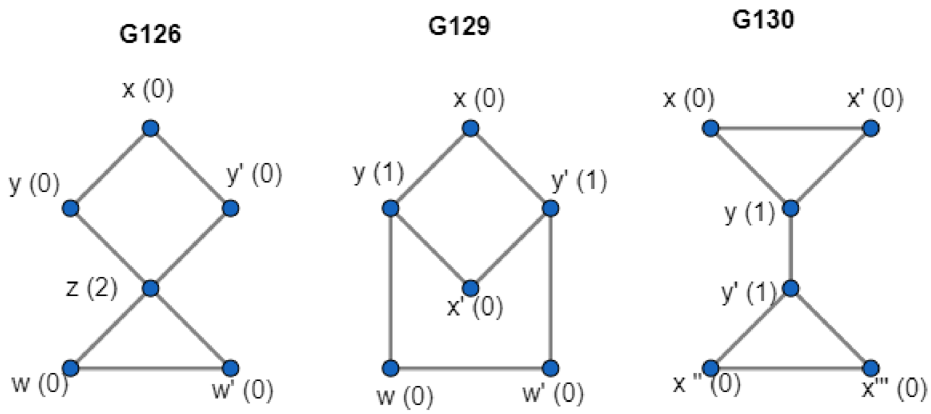

Figure 4.11: Remaining Graphs and their canonical divisors.

1. G126: $K_G = [x, y, y', z, w, w'] \equiv [0,0,0,2,0,0]$. Clearly z is Weierstrass as $(K_G - 2(z)) = [0,0,0,0,0,0]$.

   Consider $(K_G - 2(w)) = [0,0,0,2,-2,0]$, sending fire from w will leave z, y, y', x unburned. Firing these vertices simultaneously will lead to $[0,0,0,0,-1,1]$. Then again sending fire from w will burn the whole graph, but the divisor is not effective. Hence w is not Weierstrass. Because of the symmetry w' is also not Weierstrass. By a similar argument we can show that y and y' are also not Weierstrass.

   Take $(K_G - 2(x)) = [-2,0,0,2,0,0]$; sending fire from x will leave z, w, w' unburned. Firing z, w, w' simultaneously and reverse firing x will lead to an effective divisor. Therefore x is Weierstrass.

2. G129: Similarly x and x' are Weierstrass but y and y' are not.

Consider $(K_G - 2(w)) = [x, y, y', x, w, w'] \equiv [0, 1, 1, 0, -2, 0]$ sending fire from w will only burn w'. Then firing x, x', y, y' simultaneously will give $[0, 0, 0, 0, -1, 1]$ which can't be converted to an effective divisor. Thus w and also w' are not Weierstrass vertices.

3. G130: $K_G = [x, x', y, y', x'', x'''] \equiv [0, 0, 1, 1, 0, 0]$. Consider $(K_G - 2(x)) = [-2, 0, 1, 1, 0, 0]$. Sending fire from x will burn only x' and y. Then we need to fire the unburned vertices simultaneously which will lead to $[-2, 0, 2, 0, 0, 0]$. Now again firing x'', x''', y, y' simultaneously implies $[-1, 1, 0, 0, 0, 0]$. But this divisor is not equivalent to an effective divisor. Therefore x is not Weierstrass. x', x'', x''' are also not Weierstrass as they are symmetric with respect to x.

   Next consider $(K_G - 2(y)) = [0, 0, -1, 1, 0, 0] \equiv [0, 0, 0, 0, 0, 0]$, which can be easily verified using Dhar's burning algorithm. Therefore y and y' are Weierstrass.

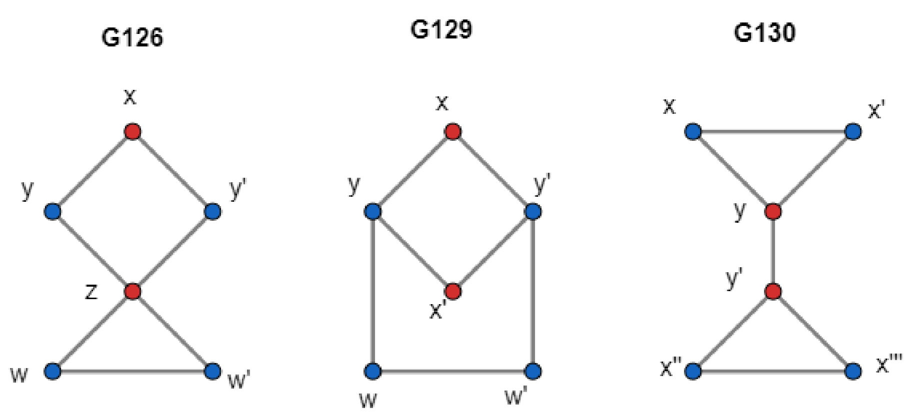

Figure 4.12: G126, G129, G130 with Weierstrass vertices marked in red.

4.2 Conjecture

Here we have considered graphs with genus 2 having exactly $g^3 - g = 2^3 - 2 = 6$ vertices. After carefully observing these results we can detect some interesting properties or behaviors of graphs. According to our calculations, the maximum number of Weierstrass vertices in this family of graphs equals to 4. Therefore if all the vertices are normal Weierstrass vertices that are having weight 1, maximum weight also must be equal to 4 which is strictly less than $2^3 - 2$.

Even though for algebraic curves when $g \geq 2$ there is always a finite number of Weierstrass veritices, but for graphs it is different. G128 works as a counter example. For curves there are at least $2g+2$ number of Weierstrass vertices. However, we have proved that for graphs this is not always true by considering all the non-isomorphic graphs of genus 2 and 6 vertices.
After considering all these observations we present the following conjecture.

Conjecture

There doesn't exist a finite, connected, undirected and unweighted multi graph $G = (V, E)$ without loop edges of genus g with $|V(G)| = g^3 - g$ all of whose vertices are normal Weierstrass vertices.

We have proved this conjecture for $g = 2$. Other cases remain open.

CHAPTER 5: CONCLUSIONS AND FUTURE WORK

In this thesis our main goal was to study the occurrence, density and the weights of Weierstrass vertices in certain families of graphs analogous to the algebraic curve case. We have studied complete graphs and genus 2 graphs having 6 vertices. In addition, we have calculated gap sequence for the House - X graph which is a direct application of proposition 3.6.

The family of complete graphs share many interesting properties. In [13, page 26], William D. Lindsay Jr. presents a corollary to say that all the verices of K_n for $n \geq 4$ are Weierstrass. Since there is a symmetry in complete graphs, we have taken a general vertex in K_n to represent all the vertices of it. Therefore, we can define a gap sequence and a Semigroup for K_n which is unique for itself for $n = 4, 5, 6, \ldots$. Additionally, we have provided a proposition together with a proof (proposition 3.7), to determine the occurrence of zeros in the gap sequence of K_n for $n \geq 3$. We have performed direct calculations for K_4, K_5 and K_6 with help of Dhar's burning algorithm and we have predicted the gap sequence, weights and Semigroup for K_7 and higher cases. Considering all these facts about complete graphs we have presented a theorem and two propositions with their proofs on the weight of a vertex in K_n and the Semigroup of K_n in the section 3.2.4.

Moreover, in the chapter 4 we have elaborated the results on family of genus 2 graphs having 6 vertices, performing direct calculations for total of 19 graphs. Therefore, we have proved that there is no finite, connected, undirected and unweighted multi graph without loop edges with 6 vertices and genus 2 with all vertices being normal Weierstrass vertices. Furthermore, we conjecture that there doesn't exist a finite, connected, undirected and unweighted multi graph $G = (V, E)$ without loop edges of genus g with $|V(G)| = g^3 - g$ all of whose vertices are normal Weierstrass vertices. This conjecture remains open for $g \geq 3$.

As an extension of this work, we can look on family of cycles with chords having for Weierstrass property. Moreover, we can study the two generated Semigroups obtained from the complete graphs for more interesting observations. Furthermore, we can study complete bipartite graphs. In this thesis we have presented a conjectures, therefore, we can try to come up with a reasonable proofs for this conjecture.

LIST OF REFERENCES

[1] E. Arbarello, M. Cornalba, P.A. Griffiths and J.Harris, *Geometry of Algebraic Curves*. Volume 1, Springer-Verlag New York, 1985.

[2] M. Baker and S. Norine, *Riemann-Roch and Abel-Jacobi theory on a finite graph*. Adv.Math., 215(2):766-788, 2007.

[3] M. Baker and S. Norine, *Harmonic morphisms and hyperelliptic Graphs*. Int. Math. Res. Not. IMRN, (15):2914-2955, 2009.

[4] M. Baker, *Specialization of linear systems from curves to graphs*. Algebra & Number theory, 2(6):613-653, 2008.

[5] M. Baker and F. Shokrieh, *Chip - Firing games, Potential theory on graphs and Spanning trees*. J. Combin. Theory Ser. A 120(1):164-182, 2013.

[6] A. Björner, L. Lovász and P. W. Shor, *Chip-firing games on graphs*. European J. Combin. 12(4):283-291, 1991.

[7] L. Caporaso, *Rank of divisors on graphs: an algebro-geometric analysis*. A celebration of algebraic geometry, 45-64, Clay Math. Proc., 18, Amer. Math. Soc., Providence, RI, 2013.

[8] D. Dhar, *Self-organized critical state of sandpile automaton models*. Phys. Rev. Lett. 64(14):1613-1616, 1990.

[9] W. Fulton, *Algebraic Curves: An Introduction to Algebraic Geometry*. 3rd edition, http://www.math.lsa.umich.edu/wfulton/CurveBook.pdf, January 28, 2008.

[10] E. Goles and M. Margenstern, *Universality of the chip-firing game*. Theoret. Comput. Sci. 172(1-2):121-134, 1997.

[11] P. A. Griffiths, *Introduction to Algebraic Curves*. Volume 76, Translations of Mathematical Monographs, Amer. Math. Soc.

[12] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*. Wiley Classics Library, Wiley, New York, 1994, reprint of the 1978 original.

[13] W. D. Lindsay Jr., *Weierstrass Points on Families of Graphs*. Master's Theses, 330, $http://digitalcommons.uconn.edu/gs_theses/$330, 2012.

[14] M. Manjunath, *The rank of a divisor on a finite graph: geometry and computation*. arXiv:1111.7251 [math.CO], December 1, 2011.

[15] C. Merino, *The chip-firing game*. Discrete Math. 302(1-3):188-210, 2005.

[16] R. Miranda, *Algebraic Curves and Riemann Surfaces*. Grad. Stud. Math., Vol. 5, Amer. Math. Soc., 1995.

[17] S. Payne, *Math 665: Tropical BrillNoether Theory Lecture Series*. Yale Math, http://users.math.yale.edu/sp547/Math665.html.

[18] R. C. Read and R. J. Wilson, *An Atlas of Graphs*. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1998. xii+454 pp. ISBN: 0-19-853289-X.

[19] E. Robeva and S. Payne, *Enumerating and computing divisor classes of given degree and rank*. http://math.mit.edu/$\sim$ erobeva/About.pdf.

[20] N. J. A. Sloane, *The On - Line Encyclopedia of Integer Sequences, 1964*. https://oeis.org/search?q=0.2%2c9%2c25&sort=references.

[21] J. Spencer, *Balancing Vectors in the max norm*. Combinatorica 6(1):55-65, 1986.

[22] *The burning algorithm, acyclic orientations, and maximal unwinnable divisors*. http://people.reed.edu/davidp/cameroon/handouts/dhar.pdf.