2016

Building Lax Integrable Variable-Coefficient Generalizations to Integrable PDEs and Exact Solutions to Nonlinear PDEs

Matthew Russo
University of Central Florida

BUILDING LAX INTEGRABLE VARIABLE-COEFFICIENT GENERALIZATIONS TO INTEGRABLE PDES AND EXACT SOLUTIONS TO NONLINEAR PDES

by

MATTHEW RUSSO
B.S. University of Central Florida, 2012
M.S. University of Central Florida, 2014

A dissertation submitted in partial fulfilment of the requirements
for the degree of Doctor of Philosophy
in the Department of Mathematics
in the College of Sciences
at the University of Central Florida
Orlando, Florida

Spring Term
2016

Major Professor: S. Roy Choudhury

ABSTRACT

This dissertation is composed of two parts. In Part I a technique based on extended Lax Pairs is first considered to derive variable-coefficient generalizations of various Lax-integrable NLPDE hierarchies recently introduced in the literature. It is demonstrated that the technique yields Lax- or S-integrable nonlinear partial differential equations (nlpdes) with both time- **and** space-dependent coefficients which are thus more general than almost all cases considered earlier via other methods such as the Painlevé Test, Bell Polynomials, and various similarity methods. However, this technique, although operationally effective, has the significant disadvantage that, for any integrable system with spatiotemporally varying coefficients, one must 'guess' a generalization of the structure of the known Lax Pair for the corresponding system with constant coefficients. Motivated by the somewhat arbitrary nature of the above procedure, we present a generalization to the well-known Estabrook-Wahlquist prolongation technique which provides a systematic procedure for the derivation of the Lax representation. In order to obtain a nontrivial Lax representation we must impose differential constraints on the variable coefficients present in the nlpde. The resulting constraints determine a class of equations which represent generalizations to a previously known integrable constant coefficient nlpde. We demonstate the effectiveness of this technique by deriving variable-coefficient generalizations to the nonlinear Schrodinger (NLS) equation, derivative NLS equation, PT-symmetric NLS, fifth-order KdV, and three equations in the MKdV hierarchy. In Part II of this dissertation, we introduce three types of singular manifold methods which have been successfully used in the literature to derive exact solutions to many nonlinear PDEs extending over a wide range of applications. The singular manifold methods considered are: truncated Painlevé analysis, Invariant Painlevé analysis, and a generalized Hirota expansion method. We then consider the KdV and KP-II equations as instructive examples before using each method to derive nontrivial solutions to a microstructure PDE and two generalized Pochhammer-Chree equations.

TABLE OF CONTENTS

LIST OF FIGURES

PART I: VARIABLE COEFFICIENT LAX-INTEGRABLE SYSTEMS

CHAPTER 1: INTRODUCTION

For a nonlinear partial differential equation (nlpde) the phase space is infinite dimensional and thus extension of integrability of a Hamiltonian system in the Liouville-Arnold sense becomes troublesome. As such, a universally accepted definition of integrability within the context of nlpdes does not exist in the literature [1]- [5]. For the purposes of this dissertation, we will consider a nlpde to be completely integrable if it can be expressed as the compatibility condition of a nontrivial Lax pair. Here we take a trivial Lax pair to be one for which dependence on the spectral parameter can be removed through a Gauge transformation. Indeed, if a nlpde is shown to possess a Lax pair then from it one may derive a variety of remarkable properties, including the existence of infinitely many conserved quantities. More specifically, consider a nlpde in $1+1$

$$F\left(u, \frac{\partial^{m+n} u}{\partial x^m \partial t^n}\right) = 0 \tag{1.1}$$

where $F : \mathscr{A} \to \mathscr{B}$ is a continuous function from a function space $\mathscr{A}$ to another function space $\mathscr{B}$. We say that (1.1) possesses a Lax representation if there exist matrices $\mathbf{U}, \mathbf{V} \in \mathscr{M}^{n\times n}(\mathbb{C})$, where $\mathscr{M}^{n\times n}(\mathbb{C})$ is the set of $n \times n$ matrices with entries in $\mathbb{C}$, such that compatibility of the set of equations

$$\Phi_x = \mathbf{U}\Phi, \qquad \Phi_t = \mathbf{V}\Phi \tag{1.2}$$

is achieved upon satisfaction of (1.1). We say that a Lax pair $\{\mathbf{U}, \mathbf{V}\}$ is nontrivial if it depends (nontrivially) on a spectral parameter. The matrices $\mathbf{U}$ and $\mathbf{V}$ are known as the space and time evolution matrices for the scattering problem (1.2), respectively. Compatability of (1.2) requires

the cross-derivative condition $\Phi_{xt} = \Phi_{tx}$ be satisfied. That is,

$$\begin{aligned} \dot{0} = \Phi_{xt}\Phi_{tx} &= \mathbf{U}_t\Phi + \mathbf{U}\Phi_t - \mathbf{V}_x\Phi - \mathbf{V}\Phi_x \\ &= \mathbf{U}_t\Phi + \mathbf{UV}\Phi - \mathbf{V}\Phi - \mathbf{VU}\Phi \\ &= (\mathbf{U}_t - \mathbf{V}_x + \mathbf{UV} - \mathbf{VU})\,\Phi, \end{aligned} \tag{1.3}$$

where $\dot{0}$ is equivalent to the zero matrix upon satisfaction of (1.1). Introducing the commutator operation $[\mathbf{A}, \mathbf{B}] = \mathbf{AB} - \mathbf{BA}$ the previous equation implies **U** and **V** satisfy

$$\mathbf{U}_t - \mathbf{V}_x + \mathbf{UV} - \mathbf{VU} = \dot{0}. \tag{1.4}$$

This equation is known as the *zero-curvature condition* and will form the basis of our investigation in Part I.

Variable-coefficient nonlinear partial differential equations have a long history dating from their derivations in a variety of physical contexts [6]- [18]. However, almost all studies, including those which derived exact solutions by a variety of techniques, as well as those which considered integrable sub-cases and various integrability properties by methods such as Painlevé analysis, Hirota's method, and Bell Polynomials, treat variable-coefficient nlpdes with coefficients which are functions of the time only. Due to their computational complexity and lack of an efficient method for deriving the conditions for Lax integrability, the question of integrability for equations with time and space dependent coefficients has largely been ignored.

In chapter 2 we introduce a method recently presented in the literature [19, 20] for deriving a Lax

pair for space and time dependent coefficient nlpdes. We then illstrate the method by deriving the Lax pair and integrability conditions for the nonlinear Schrodinger (NLS) equation, derivative NLS equation, PT-symmetric NLS, fifth-order KdV, and three equations in the MKdV hierarchy. This technique, although operationally effective, has the significant disadvantage that, for any integrable system with spatiotemporally varying coefficients, one must 'guess' a generalization of the structure of the known Lax Pair for the corresponding system with constant coefficients. This involves replacing constants in the Lax Pair for the constant coefficient integrable system, including powers of the spectral parameter, by functions. Provided that one has guessed correctly and generalized the constant coefficient system's Lax Pair sufficiently, and this is of course hard to be sure of 'a priori', one may then proceed to systematically deduce the Lax Pair for the corresponding variable-coefficient integrable system [21].

Motivated by the somewhat arbitrary nature of the above procedure, we embark in this dissertation on an attempt to systematize the derivation of Lax-integrable sytems with variable coefficients. Of the many techniques which have been employed for constant coefficient integrable systems, the Estabrook-Wahlquist (EW) prolongation technique [26]- [29] is among the most self-contained. The method directly proceeds to attempt construction of the Lax Pair or linear spectral problem, whose compatibility condition is the integrable system under discussion. While not at all guaranteed to work, any successful implementation of the technique means that Lax-integrability has already been verified during the procedure, and in addition the Lax Pair is algorithmically obtained. We note that failure of the technique does not necessarily imply non-integrability of the equation contained in the compatibility condition of the assumed Lax Pair for other definitions of integrability. It does however mean that the nlpde is not considered Lax-integrable. Due to the imprecise nature of the definition of integrability, a nlpde may be considered integrable in one sense but not

another. Indeed, consider the viscous Burgers' equation

$$u_t + uu_x = \nu u_{xx}, \tag{1.5}$$

where $\nu > 0$ is the (constant) viscosity. By making use of the Cole-Hopf transformation $u(x,t) = -2\nu\frac{\phi_x}{\phi}$ Burgers' equation becomes, after a little manipulation, the heat equation $v_t = \nu\phi_{xx}$. Since any linear partial differential equation is integrable we say that Burgers' equation is integrable in the sense that it can be linearized. Now consider the (focusing) NLS equation

$$i\psi_t = -\frac{1}{2}\psi_{xx} - k|\psi|^2\psi \tag{1.6}$$

where $k > 0$ is a dimensionless constant. There is no transformation which takes this equation to a linear equation, as was the case for Burgers' equation. However, this equation has been shown to be integrable in many other ways. For example, there exists a nontrivial Lax pair for the NLS and thus it is Lax-integrable. While transformations like the Cole-Hopf transformation are examples of *explicit* linearizations a Lax pair can be thought of as an *implicit* linearization of a nlpde.

In applications, the coefficients of a nlpde may include spatial dependence, in addition to the temporal variations that have been extensively considered using a variety of techniques. Both for this reason, as well as for their general mathematical interest, extending integrable hierarchies of nlpdes to include *both* spatial and temporal dependence of the coefficients is worthwhile. Hence, we attempt to apply the Estabrook-Wahlquist (EW) technique to generate a variety of such integrable systems with such spatiotemporally varying coefficients. However, this immediately requires that the technique be significantly generalized or broadened in several different ways which we outline in chapter 3. We then illustrate the effectiveness of this new and extended method by deriving the

Lax pair and integrability condition for the nonlinear Schrodinger (NLS) equation, derivative NLS equation, PT-symmetric NLS, fifth-order KdV, and the first equation in the MKdV hierarchy. As an instructive example of when the extended Estabrook-Wahlquist method correctly breaks down we then consider a generalization to the nonintegrable cubic-quintic nonlinear Schrödinger equation.

CHAPTER 2: KHAWAJA'S LAX PAIR METHOD

In this chapter we review Khawaja's Lax pair method for deriving the differential constraints necessary for compatibility of the Lax pair associated with a variable-coefficient nlpde. We then illustrate this method by deriving variable-coefficient generalizations to the nonlinear Schrodinger (NLS) equation, derivative NLS equation, PT-symmetric NLS, fifth-order KdV, and three equations in the MKdV hierarchy.

Outline of Khawaja's Method

In Khawaja's Lax pair method [19, 20], one seeks to represent a variable-coefficient generalization to a constant-coefficient nonlinear pde as the compatibility condition of a Lax pair. The method is based on the assumption that the Lax pair for the constant-coefficient nlpde is known. A Lax pair for an nlpde can be derived in various ways. Common methods include the Ablowitz-Kaup-Newell-Segur (AKNS) scheme, the Wadati-Konno-Ichikawa (WKI) scheme, and the Estabrook-Wahlquist method. Once a Lax pair is obtained, it may be decomposed into an expansion about powers of the unknown function and its derivatives where the coefficients are constant coefficient matrices. For example, we may write a Lax pair $\{\mathbf{U}, \mathbf{V}\}$ for the KdV equation

$$u_t + uu_x + 6u_{xxx} = 0 \tag{2.1}$$

as

$$\mathbf{U} = \begin{pmatrix} -ik & 1 \\ 0 & ik \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -\frac{1}{6} & 0 \end{pmatrix} u, \tag{2.2a}$$

$$\mathbf{V} = \begin{pmatrix} -4ik^3 & 4k^2 \\ 0 & 4ik^3 \end{pmatrix} + \begin{pmatrix} \frac{1}{3}ik & -\frac{1}{3} \\ -\frac{2}{3}k^2 & -\frac{1}{3}ik \end{pmatrix} u + \begin{pmatrix} 0 & 0 \\ \frac{1}{18} & 0 \end{pmatrix} u^2 + \begin{pmatrix} \frac{1}{6} & 0 \\ \frac{1}{3}ik & -\frac{1}{6} \end{pmatrix} u_x + \begin{pmatrix} 0 & 0 \\ \frac{1}{6} & 0 \end{pmatrix} u_{xx}, \tag{2.2b}$$

where here k is the spectral parameter associated with the linear eigenvalue problem (i.e. the Lax pair). In this method one replaces the elements in the constant-coefficient matrices present in the Lax pair expansion with undetermined functions of space and time. Compatibility of this new Lax pair is then enforced, allowing for the determination of the functions in the Lax Pair and derivation of the variable-coefficient constraints. Unfortunately, it is often the case that merely altering the existing form of the constant-coefficient Lax pair is not sufficient. Intuition and a little trial-and-error may be required to determine additional terms which may need to be added to the expansion in order to ensure nontrivial compatibility. In fact, in one of his papers [20] Khawaja alludes to the difficulty in this step of the procedure and further remarks that it took several attempts to find the correct form for the Lax pair he derived.

PT-Symmetric and Standard Nonlinear Schrödinger Equations

We begin with the derivation of the Lax pair and differential constraints for the standard NLS equation. To keep things somewhat general we will consider the system

$$iq_t(x,t) = -f(x,t)q_{xx}(x,t) - g(x,t)q^2(x,t)r(x,t) - v(x,t)q(x,t) - i\gamma(x,t)q(x,t) \quad (2.3)$$

$$ir_t(x,t) = f(x,t)r_{xx}(x,t) + g(x,t)r^2(x,t)q(x,t) + v(x,t)r(x,t) - i\gamma(x,t)r(x,t). \quad (2.4)$$

For the choice $r(x,t) = q^*(x,t)$, where * denotes the complex conjugate, this system is equivalent to a variable-coefficient NLS equation. However, with the choices $v(x,t) = \gamma(x,t) = 0$ and $r(x,t) = q^*(-x,t)$ this system is equivalent to the PT-symmetric NLS. Therefore, we may obtain the results for both the standard cubic NLS and the PT-symmetric NLS simultaneously by studying system (2.3). Following Khawaja's method we expand the **U** and **V** matrices in powers of q, r, and their partial derivatives. We therefore seek a Lax pair of the form

$$\mathbf{U} = \begin{bmatrix} f_1 + f_2 q & f_3 + f_4 q \\ f_5 + f_6 r & f_7 + f_8 r \end{bmatrix} \quad (2.5a)$$

$$\text{and} \quad \mathbf{V} = \begin{bmatrix} g_1 + g_2 q + g_3 q_x + g_4 qr & g_5 + g_6 q + g_7 q_x + g_8 qr \\ g_9 + g_{10} r + g_{11} r_x + g_{12} qr & g_{13} + g_{14} r + g_{15} r_x + g_{16} qr \end{bmatrix} \quad (2.5b)$$

where f_{1-8} and g_{1-16} are unknown functions of x and t. Compatibility of **U** and **V** requires

$$\mathbf{U}_t - \mathbf{V}_x + [\mathbf{U}, \mathbf{V}] = \dot{0} = \begin{bmatrix} 0 & p_1(x,t)F_1[q,r] \\ p_2(x,t)F_2[q,r] & 0 \end{bmatrix} \quad (2.6)$$

where $F_i[q,r]$ represents the i^{th} equation in (2.3), and $p_{1,2}$ are arbitrary real-valued functions. It should be clear that this off-diagonal compatibility condition requires that the coefficients of the q and the r on the off-diagonal of **U** be zero. Indeed upon plugging **U** and **V** into the compatibility

condition we immediately find that compatibility requires

$$f_2 = f_3 = f_5 = f_8 = g_2 = g_3 = g_5 = g_8 = g_9 = g_{12} = g_{14} = g_{15} = 0, f_4 = ip_1, f_6 = -ip_2,$$
$$g_7 = -fp_1, g_{11} = -fp_2, g_4 = -g_{16} = -ifp_1p_2.$$

The remaining constraints are given by

$$f_{1t} - g_{1x} = 0 \tag{2.7}$$
$$f_{7t} - g_{13x} = 0 \tag{2.8}$$
$$2fp_1p_2 + g = 0 \tag{2.9}$$
$$f_x p_1 - fp_1(f_1 - f_7) + fp_{1x} - g_6 = 0 \tag{2.10}$$
$$f_x p_2 + fp_2(f_1 - f_7) + fp_{2x} - g_{10} = 0 \tag{2.11}$$
$$g_6(f_1 - f_7) - ip_1(g_1 - g_{13} - iv + \gamma) - g_{6x} + ip_{1t} = 0 \tag{2.12}$$
$$g_{10}(f_1 - f_7) + ip_2(g_1 - g_{13} - iv - \gamma) + g_{10x} + ip_{2t} = 0 \tag{2.13}$$
$$(fp_1p_2)_x + g_{10}p_1 + g_6p_2 = 0. \tag{2.14}$$

We will now review the reduction of the system (2.7)-(2.14) to the differential constraints which may be found in [20]. Solving eq. (2.9) for fp_1p_2 and substituting the result into (2.14) we obtain

$$-\frac{1}{2}g_x + g_{10}p_1 + g_6p_2 = 0. \tag{2.15}$$

Upon multiplying equation (2.10) by p_2 and equation (2.11) by p_1 and adding, we obtain

$$(fp_1p_2)_x + p_1p_2f_x - g_6p_2 - g_{10}p_1 = 0. \tag{2.16}$$

Now utilizing equations (2.9), (2.15), and (2.16) we find

$$\frac{f_x}{f} = -2\frac{g_x}{g} \qquad \Rightarrow \qquad f(x,t) = \frac{c(t)}{g(x,t)^2}. \tag{2.17}$$

where $c(t)$ is arbitrary. Now multiplying equation (2.10) by p_2 and equation (2.11) by p_1 and subtracting, we get

$$f_1 - f_7 = -\frac{g_6}{fp_1} + \frac{1}{2}\left[\log\frac{p_1}{g}\right]_x + \frac{fp_1p_{2x}}{g}. \tag{2.18}$$

Multiplying equation (2.12) by p_2 and equation (2.13) by p_1 and adding, we get

$$(g_6p_2 + g_{10}p_1)(f_1 - f_7) - 2ip_1p_2\gamma - g_{6x}p_2 + g_{10x}p_1 + ip_{1t}p_2 + ip_{2t}p_1 = 0. \tag{2.19}$$

Substituting for f_1, g_{10}, p_2, and f using equations (2.18), (2.15), (2.9), and (2.17), respectively, we obtain

$$\frac{g^2}{c}\left(\frac{gg_6}{p_1}\right)_x - \frac{g_x^2}{g} + \frac{g_{xx}}{2} + i\left(\frac{g^3\gamma}{c} + \frac{\dot{c}g^3}{2c^2} - \frac{3g^2g_t}{2c}\right) = 0. \tag{2.20}$$

Solving for g_6 we find

$$g_6 = \frac{p_1}{g}\left[k_{1r} + ik_{1i} - \frac{cg_x}{2g^2} + \frac{i}{2}\int\left(3g_t - \frac{(2c\gamma + \dot{c})g}{c}\right)dx\right]. \tag{2.21}$$

where k_{1r} and k_{1i} are arbitrary real functions of t obtained through integration. On the other hand, multiplying equation (2.12) by p_2 and equation (2.13) by p_1 and subtracting, we get

$$(g_6p_2 - g_{10}p_1)(f_1 - f_7) - 2ip_1p_2(g_1 - g_{13} - iv) - g_{6x}p_2 - g_{10x}p_1 + ip_{1t}p_2 - ip_{2t}p_1 = 0. \tag{2.22}$$

Once again, substituting for f_1, g_{10}, p_2, and f using eqs. (2.18), (2.15), (2.9), and (2.17), respectively, we now obtain

$$i\frac{cg_x^2}{g^4} - i\frac{cg_{xx}}{g^3} + i\frac{g_6g_x}{gp_1} + i\frac{g_6^2g^2}{cp_1^2} - (g_1 - g_{13} - iv) - \frac{3g_t}{2g} + \frac{p_{1t}}{p_1} + \frac{\dot{c}}{2c} = 0. \tag{2.23}$$

Solving for $g_1 - g_{13}$ we get

$$g_1 - g_{13} = iv + i\frac{g^2g_6^2}{cp_1^2} + \frac{\dot{c}}{2c} + \frac{2gp_{1t} - 3p_1g_t + 2ig_6g_x}{2gp_1} - \frac{ic(gg_{xx} - 2g_x^2)}{2g^4}. \tag{2.24}$$

Now subtracting equation (2.8) from equation (2.7) and substituting for $f_1 - f_7$ and $g_1 - g_{13}$ using (2.18) and (2.24), respectively, we obtain

$$
\begin{aligned}
v_x &= \frac{2k_{1i}g_t}{c} - \frac{g}{c}(2k_{1i}\gamma + \dot{k}_{1i}) + \frac{g_t - g\gamma}{c}\int \left(3g_t - \frac{g}{c}(2c\gamma + \dot{c})\right) dx \\
&+ \frac{g}{2c^3}\int \left(c(g_t(2c\gamma + \dot{c}) - 3cg_{tt}) + g(c(2c\gamma_t + \ddot{c}) - \dot{c}^2)\right) dx - \frac{c}{2g}\left(\frac{1}{g}\right)_x \\
&+ i\left(\frac{g}{c}(2k_{1r}\gamma + \dot{k}_{1r}) - \frac{2}{c}k_{1r}g_t\right). \qquad (2.25)
\end{aligned}
$$

Since v is assumed to be real the imaginary part of (2.25) must vanish. That is, we require

$$
g(2k_{1r}\gamma + \dot{k}_{1r}) - 2k_{1r}g_t = 0 \qquad (2.26)
$$

from which we obtain

$$
\gamma(x,t) = \frac{g_t(x,t)}{g(x,t)} - \frac{1}{2}\frac{\dot{k}_{1r}(t)}{k_{1r}(t)}. \qquad (2.27)
$$

Now substituting $c(t) = f(x,t)g(x,t)^2$ into (2.25), combining integrals, and differentiating with respect to x we obtain the final condition

$$
\begin{aligned}
& fg^3(f_t(g_t - 2g\gamma) - f_{tt}g) + f_t^2g^4 + 2f^3g^3(gv_{xx} - g_xv_x) - 2f^2g^4(\gamma_t + 2\gamma^2) - 2f^2g^2g_t^2 \\
+ \;& f^2g^3(4g_t\gamma + g_{tt}) + f^4(36g_x^4 - 48gg_x^2g_{xx} + 10g^2g_xg_{xxx} + g^2(6g_{xx}^2 - gg_{xxxx})) = 0. \qquad (2.28)
\end{aligned}
$$

As the PT-symmetric NLS is a special case of the system (2.3) with $f = -a_1$, $g = -a_2$, $v = \gamma = 0$, and $r(x,t) = q^*(-x,t)$ we can exploit the Lax pair constraints derived above for standard NLS to

obtain the constraints for the PT-symmetric NLS. We therefore find, utilizing the same **U** and **U** that were given earlier in the section, that compatibility under the requirement that r and q satisfy the PT-symmetric NLS requires

$$f_2 = f_3 = f_5 = f_8 = g_2 = g_3 = g_5 = g_8 = g_9 = g_{12} = g_{14} = g_{15} = 0, f_4 = ip_1, f_6 = -ip_2,$$
$$g_7 = a_1 p_1, g_{11} = a_1 p_2, g_4 = -g_{16} = i a_1 p_1 p_2.$$

and the remaining constaints are given by

$$f_{1t} - g_{1x} = 0, \tag{2.29}$$
$$f_{7t} - g_{13x} = 0, \tag{2.30}$$
$$2a_1 p_1 p_2 + a_2 = 0, \tag{2.31}$$
$$-a_{1x} p_1 + a_1 p_1 (f_1 - f_7) - a_1 p_{1x} - g_6 = 0, \tag{2.32}$$
$$-a_{1x} p_2 - a_1 p_2 (f_1 - f_7) - a_1 p_{2x} - g_{10} = 0, \tag{2.33}$$
$$g_6 (f_1 - f_7) - i p_1 (g_1 - g_{13}) - g_{6x} + i p_{1t} = 0, \tag{2.34}$$
$$g_{10} (f_1 - f_7) + i p_2 (g_1 - g_{13}) + g_{10x} + i p_{2t} = 0, \tag{2.35}$$
$$-(a_1 p_1 p_2)_x + g_{10} p_1 + g_6 p_2 = 0. \tag{2.36}$$

From equations (2.17) and (2.27) we get

$$a_2(x,t) = f(t) g(x), \tag{2.37}$$

and

$$a_1(x,t) = \frac{c(t)}{a_2(x,t)^2} = \frac{c(t)}{f(t)^2 g(x)^2}. \tag{2.38}$$

Plugging these results into (2.28) we obtain

$$\begin{aligned} &c^4\left(36(g')^4 - 48g(g')^2g'' + 10g^2g'g''' + 6g^2(g'')^2 - g^3g''''\right) \\ &+ g^8\left(cf^3\dot{c}\dot{f} - 6c^2f^2\dot{f}^2 - cf^4\ddot{c} + 3c^2f^3\ddot{f} + f^4\dot{c}^2\right) = 0. \end{aligned} \tag{2.39}$$

Equation (2.39) may be further reduced to the system

$$36(g')^4 - 48g(g')^2g'' + 10g^2g'g''' + 6g^2(g'')^2 - g^3g'''' = 0, \tag{2.40}$$

$$cf^3\dot{c}\dot{f} - 6c^2f^2\dot{f}^2 - cf^4\ddot{c} + 3c^2f^3\ddot{f} + f^4\dot{c}^2 = 0. \tag{2.41}$$

This final set of equations represents the conditions on the variable coefficients in (2.3) required for Lax-integrability.

Derivative Nonlinear Schrodinger Equation

In this section we derive the Lax pair and differential constraints for the derivative nonlinear Schrödinger equation. We consider the equivalent system

$$iq_t(x,t) + a_1(x,t)q_{xx}(x,t) + ia_2(x,t)(q^2(x,t)r(x,t))_x = 0, \tag{2.42}$$

$$-ir_t(x,t) + a_1(x,t)r_{xx}(x,t) - ia_2(x,t)(r^2(x,t)q(x,t))_x = 0, \tag{2.43}$$

where $r(x,t) = q^*(x,t)$ and again * denotes the complex conjugate. The Lax pair **U** and **V** are expanded in powers of q and r and their partial derivatives as follows

$$\mathbf{U} = \begin{bmatrix} f_1 + f_2 q & f_3 + f_4 q \\ f_5 + f_6 r & f_7 + f_8 r \end{bmatrix}, \tag{2.44}$$

$$\mathbf{V} = \begin{bmatrix} g_1 + g_2 q + g_3 q_x + g_4 qr & g_5 + g_6 q + g_7 qr + g_8 q_x + g_9 q^2 r \\ g_{10} + g_{11} r + g_{12} qr + g_{13} r_x + g_{14} r^2 q & g_{15} + g_{16} r + g_{17} r_x + g_{18} qr \end{bmatrix}, \tag{2.45}$$

where f_{1-8} and g_{1-20} are unknown functions of x and t. Note that the compatibility condition

$$\mathbf{U}_t - \mathbf{V}_x + [\mathbf{U}, \mathbf{V}] = \dot{0} = \begin{bmatrix} 0 & p_1(x,t)F[q,r] \\ 0 & 0 \end{bmatrix} \tag{2.46}$$

we enforce, where $F[q,r]$ represents (2.42) and $p_1(x,t)$ is unknown, is chosen out of necessity. Upon considering a more standard compatibility condition as that considered by Khawaja,

$$U_t - V_x + [U, V] = \dot{0} = \begin{bmatrix} 0 & p_1(x,t)F_1[q,r] \\ p_2(x,t)F_2[q,r] & 0 \end{bmatrix} \tag{2.47}$$

we find that the only solution requires p_1 or p_2 be zero. We chose to let $p_2 = 0$ but it is important to note that the conditions would not change if we had set p_1 equal to zero instead. Given this modification to the zero-curvature condition compatibility of U and V immediately requires

$$f_2 = f_5 = f_6 = f_8 = g_2 = g_3 = g_4 = g_7 = g_{11} = g_{12} = g_{14} = g_{16} = g_{17} = g_{18} = 0,$$
$$g_8 = -p_1 a_1, f_4 = i p_1, g_9 = i p_1 a_2.$$

After substituting these into the compatibility conditions the remaining contraints are then given by

$$f_{1t} - g_{1x} = 0, \tag{2.48}$$
$$f_{7t} - g_{15x} = 0, \tag{2.49}$$
$$(p_1 a_2)_x + p_1 a_2 (f_7 - f_1) = 0, \tag{2.50}$$
$$(p_1 a_1)_x + p_1 a_1 (f_7 - f_1) - g_6 = 0, \tag{2.51}$$
$$-g_{5x} - g_5 (f_7 - f_1) = 0, \tag{2.52}$$
$$i p_{1t} - g_{6x} + i p_1 (g_{15} - g_1) - g_6 (f_7 - f_1) = 0. \tag{2.53}$$

Deriving a relation between a_1 and a_2

As it turns out, equations (2.48) and (2.50)-(2.53) may be solved exactly for f_1, f_7, g_6, g_5 and g_{15}, respectively. Upon solving equations (2.48) and (2.50)-(2.53) for f_1, f_7, g_6, g_5 and g_{15} we obtain

$$
\begin{aligned}
f_1 &= \int g_{1x} dt + F(x), && (2.54)\\
f_7 &= -\frac{(p_1 a_2)_x}{p_1 a_2} + f_1, && (2.55)\\
g_5 &= H(t) e^{\int (f_1 - f_7) dx}, && (2.56)\\
g_6 &= \frac{(p_1 a_1)_x a_2 - (p_1 a_2)_x a_1}{a_2}, && (2.57)\\
g_{15} &= -\frac{i}{p_1} \left(g_{6x} - i p_{1t} - g_6 (f_1 - f_7) \right) + g_1. && (2.58)
\end{aligned}
$$

Plugging these results into the remaining equation, (2.49), we obtain the constraint

$$
\begin{aligned}
&2p_1 a_2^2 a_{1x} p_{1xx} - 2p_{1x}^2 a_2^2 a_{1x} - 2p_1 a_2 a_{1x} p_{1x} a_{2x} + p_1^2 a_2^2 a_{1xxx} + 2p_{1x} a_2^2 p_1 a_{1xx} - p_1^2 a_2 a_{1x} a_{2xx}\\
&+2p_{1x}^2 a_2 a_1 a_{2x} + 2p_1 a_1 a_{2x}^2 p_{1x} - 2p_{1x} a_2 a_1 p_1 a_{2xx} + p_1^2 a_1 a_{2x} a_{2xx} - 2p_1 a_1 a_{2x} p_{1xx} a_2\\
&-p_1^2 a_2 a_1 a_{2xxx} + i p_1^2 (a_2 a_{2xt} - a_{2t} a_{2x}) = 0. \qquad (2.59)
\end{aligned}
$$

In order to have meaningful results we must require that the a_i are real-valued functions. Thus we can decouple the last constraint into the following equations

$$
\begin{aligned}
&2p_1 a_2^2 a_{1x} p_{1xx} - 2p_{1x}^2 a_2^2 a_{1x} - 2p_1 a_2 a_{1x} p_{1x} a_{2x} + p_1^2 a_2^2 a_{1xxx} + 2p_{1x} a_2^2 p_1 a_{1xx} - p_1^2 a_2 a_{1x} a_{2xx}\\
&+2p_{1x}^2 a_2 a_1 a_{2x} + 2p_1 a_1 a_{2x}^2 p_{1x} - 2p_{1x} a_2 a_1 p_1 a_{2xx} + p_1^2 a_1 a_{2x} a_{2xx} - 2p_1 a_1 a_{2x} p_{1xx} a_2\\
&-p_1^2 a_2 a_1 a_{2xxx} = 0, \qquad (2.60)\\
&a_2 a_{2xt} - a_{2t} a_{2x} = 0. \qquad (2.61)
\end{aligned}
$$

Taking $p_1 = a_2$ we obtain the final constraints

$$a_{2t}a_{2x} - a_{2xt}a_2 = 0, \tag{2.62}$$

$$a_2^3 a_{1xxx} - 3a_2^2 a_{2xx}a_{1x} - 4a_{2x}^3 a_1 + 5a_1 a_2 a_{2x}a_{2xx} + 4a_{2x}^2 a_2 a_{1x}$$
$$-a_2^2 a_1 a_{2xxx} - 2a_{2x}a_2^2 a_{1xx} = 0. \tag{2.63}$$

With the aid of MAPLE we find that the previous system is exactly solvable for a_1 and a_2 with solution given by

$$a_1(x,t) = F_4(t)F_2(x)(c_1 + c_2 x) - c_1 F_4(t)F_2(x)\int \frac{x\,dx}{F_2(x)} + c_1 x F_4(t)F_2(x)\int \frac{dx}{F_2(x)}, \tag{2.64}$$

$$a_2(x,t) = F_2(x)F_3(t). \tag{2.65}$$

This final set of expressions represents the forms for the variable coefficients in the variable-coefficient DNLS required for Lax-integrability.

Fifth-Order Korteweg-de-Vries Equation

In this section we derive the Lax pair and differential constraints for a generalized variable-coefficient fifth-order KdV equation (vcKdV) given by

$$u_t + a_1 u u_{xxx} + a_2 u_x u_{xx} + a_3 u^2 u_x + a_4 u u_x + a_5 u_{xxx} + a_6 u_{xxxxx} + a_7 u + a_8 u_x = 0. \quad (2.66)$$

Following Khawaja's method the Lax pair for the generalized vcKdV equation is expanded in powers of u and its derivatives as follows:

$$\mathbf{U} = \begin{bmatrix} f_1 + f_2 u & f_3 + f_4 u \\ f_5 + f_6 u & f_7 + f_8 u \end{bmatrix}, \quad (2.67)$$

$$\mathbf{V} = \begin{bmatrix} V_1 & V_2 \\ V_3 & V_4 \end{bmatrix}, \quad (2.68)$$

where $V_i = g_k + g_{k+1}u + g_{k+2}u^2 + g_{k+3}u^3 + g_{k+4}u_x + g_{k+5}u_x^2 + g_{k+6}u_{xx} + g_{k+7}uu_{xx} + g_{k+8}u_{xxx} + g_{k+9}uu_{xxx} + g_{k+10}u_{xxxx}$, $k = 11(i-1)+1$ and $f_{1-8}(x,t)$ and $g_{1-44}(x,t)$ are unknown functions. The compatibility condition

$$\mathbf{U}_t - \mathbf{V}_x + [\mathbf{U}, \mathbf{V}] = \dot{0} = \begin{bmatrix} 0 & p_1(x,t)F[u] \\ p_2(x,t)F[u] & 0 \end{bmatrix} \quad (2.69)$$

where $F[u]$ represents equation (2.66) and $p_{1-2}(x,t)$ are unknown functions requires immediately that some of the unknown functions be zero, indicative of a slightly incorrect initial guess. That is, we find that $g_{21} = g_{32} = f_2 = g_4 = g_{10} = g_{11} = f_8 = g_{37} = g_{43} = g_{44} = 0$. It is instructive to include this incorrect guess rather than remove them beforehand and include only the final, correct form. This indeed motivates the need for a new method, which we introduce in the next chapter, which would remove as much human error as possible while still remaining computationally tractable. We find that compatibility under the zero curvature condition requires that the remaining unknown functions satisfy a large coupled system of algebraic and partial differential equations

$$f_4 = p_1, g_{15} = -\frac{1}{3}p_1 a_3, g_{21} = -p_1 a_6, f_6 = p_2, g_{26} = -\frac{1}{3}p_2 a_6, g_{33} = -p_2 a_6,$$

$$2g_{41} - g_8 = 0, \qquad g_{39} - g_6 = 0,$$

$$p_2 g_{17} - p_1 g_{28} = 0, \qquad p_2 g_{19} - p_1 g_{30} = 0,$$

$$g_{19} + 2g_{17} = -p_1 a_2, \qquad g_{30} + 2g_{28} = -p_2 a_2,$$

$$g_{41} + 2g_{39} = 0, \qquad g_8 + 2g_6 = 0,$$

$$g_{39x} + f_3 g_{28} - f_5 g_{17} = 0, \qquad g_{6x} + f_5 g_{17} - f_3 g_{28} = 0,$$

$$2g_{25} + p_2(g_{38} - g_5) = -p_2 a_4, \qquad g_{19} + p_1(g_9 - g_{42}) = -p_1 a_1,$$

$$g_8 + p_2 g_{20} - p_1 g_{31} = 0, \qquad g_{41} + p_1 g_{31} - p_2 g_{20} = 0,$$

$$g_{30} + p_2(g_{42} - g_9) = -p_2 a_1, \qquad g_9 + a_6(p_2 f_3 - p_1 f_5) = 0,$$

$$g_{42} + a_6(p_1 f_5 - p_2 f_3) = 0, \qquad 2g_{14} + p_1(g_5 - g_{38}) = -p_1 a_4,$$

$$2g_3 + p_2 g_{16} - p_1 g_{27} = 0, \qquad 2g_{36} + p_1 g_{27} - p_2 g_{16} = 0,$$

$$g_2 + g_{5x} + f_5 g_{16} - f_3 g_{27} = 0, \qquad g_{35} + g_{38x} + f_3 g_{27} - f_5 g_{16} = 0,$$

$$g_7 + g_{9x} + f_5 g_{20} - f_3 g_{31} = 0, \qquad p_1\left(g_{25} + \frac{1}{3} f_5 a_3\right) - p_2\left(g_{14} + \frac{1}{3} f_3 a_3\right) = 0,$$

$$f_{1t} - g_{1x} + f_3 g_{23} - f_5 g_{12} = 0, \qquad f_{7t} - g_{34x} + f_5 g_{12} - f_3 g_{23} = 0,$$

$$g_5 + g_{7x} + f_5 g_{18} - f_3 g_{29} = 0, \qquad g_{38} + g_{40x} + f_3 g_{29} - f_5 g_{18} = 0,$$

$$g_{40} + g_{42x} + f_3 g_{31} - f_5 g_{20} = 0, \qquad (p_2 a_6)_x - g_{31} + p_2 a_6(f_1 - f_7) = 0,$$

$$g_{35x} + p_1 g_{23} + f_3 g_{24} - p_2 g_{12} - f_5 g_{13} = 0, \qquad g_{36x} + p_1 g_{24} + f_3 g_{25} - p_2 g_{13} - f_5 g_{14} = 0,$$

$$g_{2x} + p_2 g_{12} + f_5 g_{13} - p_1 g_{23} - f_3 g_{24} = 0, \qquad g_{3x} + p_2 g_{13} + f_5 g_{14} - p_1 g_{24} - f_3 g_{25} = 0,$$

$$g_{8x} + p_2 g_{18} + f_5 g_{19} - p_1 g_{29} - f_3 g_{30} = 0, \qquad g_{41x} + p_1 g_{29} + f_3 g_{30} - p_2 g_{18} - f_5 g_{19} = 0,$$

$$g_{28x} + g_{28}(f_1 - f_7) + f_5(g_{39} - g_6) = 0, \qquad g_{17x} + g_{17}(f_7 - f_1) + f_3(g_6 - g_{39}) = 0,$$

$$2g_{20} - (p_1a_6)_x + p_1a_6(f_1 - f_7) = 0, \qquad \frac{1}{3}(p_2a_3)_x + \frac{1}{3}p_2a_3(f_1 - f_7) + p_2(g_3 - g_{36}) = 0,$$

$$f_{5t} - g_{23x} + g_{23}(f_7 - f_1) + f_5(g_1 - g_{34}) = 0, \qquad g_{24} + g_{27x} + g_{27}(f_1 - f_7) + f_5(g_{38} - g_5) = -p_2a_8,$$

$$f_{3t} - g_{12x} + g_{12}(f_1 - f_7) + f_3(g_{34} - g_1) = 0, \qquad g_{29} + g_{31x} + g_{31}(f_1 - f_7) + f_5(g_{42} - g_9) = -p_2a_5,$$

$$\frac{1}{3}(p_1a_3)_x + \frac{1}{3}p_1a_3(f_7 - f_1) + p_1(g_{36} - g_3) = 0, \qquad g_{18} + g_{20x} + g_{20}(f_7 - f_1) + f_3(g_9 - g_{42}) = -p_1a_5,$$

$$g_{27} + g_{29x} + g_{29}(f_1 - f_7) + f_5(g_{40} - g_7) = 0, \qquad g_{16} + g_{18x} + g_{18}(f_7 - f_1) + f_3(g_7 - g_{40}) = 0,$$

$$g_{13} + g_{16x} + g_{16}(f_7 - f_1) + f_3(g_5 - g_{38}) = -p_1a_8,$$

$$g_{19x} + g_{19}(f_7 - f_1) + p_1(g_7 - g_{40}) + f_3(g_8 - g_{41}) = 0,$$

$$g_{30x} + g_{30}(f_1 - f_7) + p_2(g_{40} - g_7) + f_5(g_{41} - g_8) = 0,$$

$$g_{14x} + g_{14}(f_7 - f_1) + p_1(g_2 - g_{35}) + f_3(g_3 - g_{36}) = 0,$$

$$g_{25x} + g_{25}(f_1 - f_7) + p_2(g_{35} - g_2) + f_5(g_{36} - g_3) = 0,$$

$$p_{1t} - g_{13x} + g_{13}(f_1 - f_7) + p_1(g_{34} - g_1) + f_3(g_{35} - g_2) = p_1a_7,$$

$$p_{2t} - g_{24x} + g_{24}(f_7 - f_1) + p_2(g_1 - g_{34}) + f_5(g_2 - g_{35}) = p_2a_7.$$

Deriving a relation between the a_i

In this section we reduce the previous system down to equations which depend solely on the a_i's. We find that $g_k = 0$ for $k = 2, 3, 5 - 9, 35, 36, 38 - 42$ and

$f_7 = f_1, f_5 = f_3, g_{23} = g_{12}, g_{34} = g_1, g_{30} = -g_{19} = p_1a_1, g_{25} = g_{14} = -\frac{1}{2}p_1a_4, g_{31} = -g_{20} =$

$-(p_1 a_6)_x, g_{16} = -g_{27} = -g_{18x}, p_2 = p_1,$

$$
\begin{aligned}
g_{18} &= -g_{29} = -p_1 a_5 - (p_1 a_6)_{xx}, \\
g_{28} &= -g_{17} = -\frac{1}{2}(H_2(t) - H_1(t)), \\
g_{24} &= -g_{13} = -(p_1 a_5)_{xx} - (p_1 a_6)_{xxxx} - p_1 a_8, \\
p_1 &= \frac{H_1(t)}{a_1}, \\
a_{2-4} &= H_{2-4}(t) a_1,
\end{aligned}
$$

which leads to the PDE

$$\left(\frac{H_1}{a_1}\right)_t + \left(\frac{H_1 a_5}{a_1}\right)_{xxx} + \left(\frac{H_1 a_6}{a_1}\right)_{xxxxx} + \left(\frac{H_1 a_8}{a_1}\right)_x = \frac{H_1 a_7}{a_1}. \tag{2.72}$$

One clear solution to the equation above (for $H_1 \neq 0$) is

$$a_7 = \frac{a_1}{H_1}\left(\left(\frac{H_1}{a_1}\right)_t + \left(\frac{H_1 a_5}{a_1}\right)_{xxx} + \left(\frac{H_1 a_6}{a_1}\right)_{xxxxx} + \left(\frac{H_1 a_8}{a_1}\right)_x\right), \tag{2.73}$$

where a_1, a_5, a_6, a_8 and H_{1-4} are arbitrary functions in their respective variables.

Modified Korteweg-de-Vries Hierarchy

In this section we derive the Lax pairs and differential constraints for three equations in a variable-coefficient modified KdV hierarchy

$$v_t + a_1 v_{xxx} + a_2 v^2 v_x = 0, \tag{2.74}$$

$$v_t + b_1 v_{xxxxx} + b_2 v^2 v_{xxx} + b_3 v v_x v_{xx} + b_4 v_x^3 + b_5 v^4 v_x = 0, \tag{2.75}$$

and

$$\begin{aligned} &v_t + c_1 v_{xxxxxxx} + c_2 v^2 v_{xxxxx} + c_4 v v_{xx} v_{xxx} + c_5 v_x^2 v_{xxx} \\ &+ c_6 v_x v_{xx}^2 + c_7 v^4 v_{xxx} + c_8 v^3 v_x v_{xx} + c_9 v^2 v_x^3 + c_{10} v^6 v_x = 0. \end{aligned} \tag{2.76}$$

Once again following Khawaja's method the Lax pairs are expanded in powers of u and its derivatives as follows:

$$U = \begin{bmatrix} f_1 + f_2 v & f_3 + f_4 v \\ f_5 + f_6 v & f_7 + f_8 v \end{bmatrix}, \tag{2.77}$$

$$V^i = \begin{bmatrix} V_1^i & V_2^i \\ V_3^i & V_4^i \end{bmatrix}, \qquad (i = 1, 2, 3) \tag{2.78}$$

where

$$
\begin{aligned}
V_1^1 &= g_1 + g_2 v + g_3 v^2, \\
V_2^1 &= g_4 + g_5 v + g_6 v^2 + g_7 v^3 + g_8 v_x + g_9 v_{xx}, \\
V_3^1 &= g_{10} + g_{11} v + g_{12} v^2 + g_{13} v^3 + g_{14} v_x + g_{15} v_{xx}, \\
V_4^1 &= g_{16} + g_{17} v + g_{18} v^2,
\end{aligned}
$$

$$
\begin{aligned}
V_1^2 &= g_1 v_x^2 + g_2 v^4 + g_3 v v_{xx} + g_4 v^2 + g_5 + g_6 v_{xxxx} + g_7 v v_x^2 + g_8 v^2 v_{xx} + g_9 v^5 + g_{10} v_{xxx} \\
&\quad + g_{11} v^2 v_x + g_{12} v_{xx} + g_{13} v^3 + g_{14} v_x + g_{15} v, \\
V_2^2 &= g_{16} v_x^2 + g_{17} v^4 + g_{18} v v_{xx} + g_{19} v^2 + g_{20} + g_{21} v_{xxxx} + g_{22} v v_x^2 + g_{23} v^2 v_{xx} + g_{24} v^5 \\
&\quad + g_{25} v_{xxx} + g_{26} v^2 v_x + g_{27} v_{xx} + g_{28} v^3 + g_{29} v_x + g_{30} v, \\
V_3^2 &= g_{31} v_x^2 + g_{32} v^4 + g_{33} v v_{xx} + g_{34} v^2 + g_{35} + g_{36} v_{xxxx} + g_{37} v v_x^2 + g_{38} v^2 v_{xx} + g_{39} v^5 \\
&\quad + g_{40} v_{xxx} + g_{41} v^2 v_x + g_{42} v_{xx} + g_{43} v^3 + g_{44} v_x + g_{45} v, \\
V_4^2 &= g_{46} v_x^2 + g_{47} v^4 + g_{48} v v_{xx} + g_{49} v^2 + g_{50} + g_{51} v_{xxxx} + g_{52} v v_x^2 + g_{53} v^2 v_{xx} + g_{54} v^5 \\
&\quad + g_{55} v_{xxx} + g_{56} v^2 v_x + g_{57} v_{xx} + g_{58} v^3 + g_{59} v_x + g_{60} v,
\end{aligned}
$$

$$
\begin{aligned}
V_1^3 &= g_1 + g_2 v^2 + g_3 v v_{xx} + g_4 v_x v_{xxx} + g_5 v^2 v_x^2 + g_6 v^3 v_{xx} + g_7 v_{xx}^2 + g_8 v v_{xxxx} + g_9 v_x^2 + g_{10} v^4, \\
&\quad + g_{11} v^6 \\
V_2^3 &= g_{12} v^2 v_{xxx} + g_{13} v v_x^2 + g_{14} v^2 v_{xx} + g_{15} v^2 v_x + g_{16} v^4 v_x + g_{17} v_x^2 v_{xx} + g_{18} v v_{xx}^2 \\
&\quad + g_{19} v^2 v_{xxxx} + g_{20} v^3 v_x^2 + g_{21} v^4 v_{xx} + g_{22} v v_x v_{xx} + g_{23} v v_x v_{xxx} + g_{24} v + g_{25} v_x + g_{26} v_{xx} \\
&\quad + g_{27} v_{xxx} + g_{28} v_{xxxx} + g_{29} v_{xxxxx} + g_{30} v_{xxxxxx} + g_{31} v_x^3 + g_{32} v^3 + g_{33} v^5 + g_{34} v^7 + g_{35}, \\
V_3^3 &= g_{36} v^2 v_{xxx} + g_{37} v v_x^2 + g_{38} v^2 v_{xx} + g_{39} v^2 v_x + g_{40} v^4 v_x + g_{41} v_x^2 v_{xx} \\
&\quad + g_{42} v v_{xx}^2 + g_{43} v^2 v_{xxxx} + g_{44} v^3 v_x^2 + g_{45} v^4 v_{xx} + g_{46} v v_x v_{xx} + g_{47} v v_x v_{xxx} + g_{48} v + g_{49} v_x \\
&\quad + g_{50} v_{xx} + g_{51} v_{xxx} + g_{52} v_{xxxx} + g_{53} v_{xxxxx} + g_{54} v_{xxxxxx} + g_{55} v_x^3 + g_{56} v^3 + g_{57} v^5 + g_{58} v^7 + g_{59}, \\
V_4^3 &= g_{60} + g_{61} v^2 + g_{62} v v_{xx} + g_{63} v_x v_{xxx} + g_{64} v^2 v_x^2 + g_{65} v^3 v_{xx} + g_{66} v_{xx}^2 + g_{67} v v_{xxxx} + g_{68} v_x^2 + g_{69} v^4 \\
&\quad + g_{70} v^6.
\end{aligned}
$$

It is immediately clear from the latter two V matrices that finding the correct form of the time evolution matrix in the Lax pair via Khawaja's method can be very difficult. Through various insufficient guesses we arrived at the previous forms, for which we will now give the results. The compatibility condition gives

$$
U_t - V_x + [U, V] = \dot{0} = \begin{bmatrix} 0 & p_i(x,t) F_i[v] \\ q_i(x,t) F_i[v] & 0 \end{bmatrix} \tag{2.79}
$$

, where $F_i[v]$ represents the i^{th} equation in the MKDV hierarchy ($i = 1 - 3$).

Determining Equations for the First Equation

Requiring the compatibility condition yield a multiple of $F_1[v]$ in the off-diagonal as given above yields the following large coupled system of algebraic and partial differential equations

$f_4 = p_1, f_6 = q_1, g_7 = -\frac{1}{3}p_1a_2, g_9 = -p_1a_1, g_{13} = -\frac{1}{3}q_1a_2, g_{15} = -q_1a_1, f_2 = f_8 = g_{12} = g_6 = 0,$

$f_3q_1 - f_5p_1 = 0,$

$g_{3x} + q_1g_5 - p_1g_{11} = 0,$

$2g_{18} + p_1g_{14} - q_1g_8 = 0,$

$2g_3 + q_1g_8 - p_1g_{14} = 0,$

$f_3(g_{18} - g_3) + p_1(g_{17} - g_2) = 0,$

$f_{7t} - g_{16x} - f_3g_{10} + f_5g_4 = 0,$

$g_5 + g_{8x} + g_8(f_7 - f_1) = 0,$

$g_{14} - (q_1a_1)_x + q_1a_1(f_7 - f_1) = 0,$

$g_{2x} - f_3g_{11} + f_5g_5 - p_1g_{10} + q_1g_4 = 0,$

$\frac{1}{3}(q_1a_2)_x - \frac{1}{3}q_1a_2(f_7 - f_1) - q_1(g_{18} - g_3) = 0,$

$g_{18x} + p_1g_{11} - q_1g_5 = 0,$

$g_{17} + f_3g_{14} - f_5g_8 = 0,$

$g_2 + f_5g_8 - f_3g_{14} = 0,$

$f_5(g_{18} - g_3) + q_1(g_{17} - g_2) = 0,$

$f_{1t} - g_{1x} + f_3g_{10} - f_5g_4 = 0,$

$g_{11} + g_{14x} - g_{14}(f_7 - f_1) = 0,$

$g_8 - (p_1a_1)_x - p_1a_1(f_7 - f_1) = 0,$

$g_{17x} + f_3g_{11} - f_5g_5 + p_1g_{10} - q_1g_4 = 0,$

$\frac{1}{3}(p_1a_2)_x + \frac{1}{3}p_1a_2(f_7 - f_1) + p_1(g_{18} - g_3) = 0,$

$f_{3t} - g_{4x} - g_4(f_7 - f_1) - f_3(g_1 - g_{16}) = 0,$

$$f_{5t} - g_{10x} + g_{10}(f_7 - f_1) + f_5(g_1 - g_{16}) = 0,$$
$$p_{1t} - g_{5x} - g_5(f_7 - f_1) - p_1(g_1 - g_{16}) - f_3(g_2 - g_{17}) = 0,$$
$$p_{2t} - g_{11x} + g_{11}(f_7 - f_1) + q_1(g_1 - g_{16}) + f_5(g_2 - g_{17}) = 0.$$

Deriving a relation between the a_i

In this section we reduce the previous system down to equations which depend solely on the a_i's. In doing so we find that

$g_{16} = g_1 = f_7 = f_1 = g_{10} = g_4 = g_{17} = g_2 = f_5 = f_3 = 0, q_1 = -p_1 = -\frac{C(t)}{a_2}$,

$$g_{18} = -g_3 = -\frac{C(t)g_8}{a_2},$$
$$g_{11} = -g_5 = g_{8x},$$
$$g_{14} = -g_8 = -C(t)(\frac{a_1}{a_2})_x,$$

which leads to the condition

$$6a_1a_{2x}^3 - 6a_1a_2a_{2x}a_{2xx} + a_1a_2^2a_{2xxx} - \frac{K_t}{K}a_2^3 + a_2^2a_{2t} - a_2^3a_{1xxx}$$
$$+3a_{1xx}a_2^2a_{2x} - 6a_{1x}a_2a_{2x}^2 + 3a_{1x}a_2^2a_{2xx} = 0, \tag{2.81}$$

where $K(t)$ and $C(t)$ are arbitrary functions of t.

Determining Equations for the Second Equation

Requiring the compatability condition yield a multiple of $F_2[v]$ in the off-diagonal yields the following large coupled system of algebraic and partial differential equations

$q_1 = p_1, g_{51} = g_6 = g_{52} = g_7 = g_{53} = g_8 = g_{54} = g_9 = 0, g_{18} = -2g_{16}, g_{33} = -2g_{31}, g_{48} = -2g_{46}, g_{36} = g_{21} = -p_1b_1, g_{37} = g_{22} = -p_1b_4, g_{38} = g_{23} = -p_1b_2, g_{39} = g_{24} = -\frac{1}{5}p_1b_5,$

$$
\begin{aligned}
&b_3 = 2b_2 + 2b_4, && g_{1x} + f_3g_{16} - f_2g_{31} = 0,\\
&g_{55} + p_1b_1(f_3 - f_2) = 0, && g_{46x} + f_2g_{31} - f_3g_{16} = 0,\\
&2g_1 - g_{25} + g_{40} = 0, && g_{10} + p_1b_1(f_2 - f_3) = 0,\\
&g_{11} + 4g_{17} - g_{56} = 0, && g_{11} - 4g_{32} - g_{56} = 0,\\
&g_{14} - 2g_{34} - g_{59} = 0, && g_{14} + 2g_{19} - g_{59} = 0,\\
&g_{26} - 4g_{47} - g_{41} = 0, && g_{44} - 2g_4 - g_{29} = 0,\\
&g_{40} - 2g_{46} - g_{25} = 0, && g_{41} - 4g_2 - g_{26} = 0,\\
&g_{44} + 2g_{49} - g_{29} = 0, && g_{55} - 2g_{31} - g_{10} = 0,\\
&g_{55} + 2g_{16} - g_{10} = 0, && f_{4t} - g_{50x} + f_3g_{20} - f_2g_{35} = 0,\\
&f_{1t} - g_{5x} + f_2g_{35} - f_3g_{20} = 0, && g_{59} + g_{57x} + f_2g_{42} - f_3g_{27} = 0,\\
&g_{60} + g_{59x} + f_2g_{44} - f_3g_{29} = 0, && g_{14} + g_{12x} + f_3g_{27} - f_2g_{42} = 0,\\
&g_{15} + g_{14x} + f_3g_{29} - f_2g_{44} = 0, && 3g_{58} + g_{56x} + f_2g_{41} - f_3g_{26} = 0,\\
&g_{12} + g_{10x} + f_3g_{25} - f_2g_{40} = 0, && 3g_{13} + g_{11x} + f_3g_{26} - f_2g_{41} = 0,\\
&g_{32} - g_{17} + \frac{1}{5}p_1b_5(f_3 - f_2) = 0, && g_{57} + g_{55x} + f_2g_{40} - f_3g_{25} = 0,
\end{aligned}
$$

$$g_{27} - g_{42} + 2g_{46x} + 2f_2g_{31} - 2f_3g_{16} = 0, \qquad g_{50} + g_{45x} - g_5 - g_{45}(f_4 - f_1) - f_3(g_{15} - g_{60}) = 0,$$

$$g_{19} - g_{34} - g_{58x} + f_3g_{28} - f_2g_{43} = 0, \qquad g_{58} - g_{17x} - g_{13} - g_{17}(f_4 - f_1) - f_2(g_2 - g_{47}) = 0,$$

$$g_{42} + 2g_{1x} - g_{27} + 2f_3g_{16} - 2f_2g_{31} = 0, \qquad g_{58} + g_{32x} - g_{13} - g_{32}(f_4 - f_1) - f_3(g_{47} - g_2) = 0,$$

$$g_{45} + g_{49x} - g_{30} + f_2g_{34} - f_3g_{19} = 0, \qquad g_{34} - g_{13x} - g_{19} + f_2g_{43} - f_3g_{28} = 0,$$

$$g_{31x} + g_{31}(f_4 - f_1) + f_3(g_{46} - g_1) = 0, \qquad g_{25} - (p_1b_1)_x - p_1b_1(f_4 - f_1) = 0,$$

$$g_{40} - (p_1b_1)_x + p_1b_1(f_4 - f_1) = 0, \qquad g_{11} - 2g_{16} + 2g_{31} + p_1b_2(f_2 - f_3) = 0,$$

$$2g_{11} + g_{16} - g_{31} + p_1b_4(f_2 - f_3) = 0, \qquad g_{20} - g_{60x} - g_{35} + f_3g_{30} - f_2g_{45} = 0,$$

$$g_{28} - g_{47x} - g_{43} + f_3g_{27} - f_2g_{32} = 0, \qquad g_{31} + 2g_{56} - g_{16} - p_1b_4(f_2 - f_3) = 0,$$

$$2g_{31} - g_{56} - 2g_{16} + p_1b_2(f_2 - f_3) = 0, \qquad g_{35} - g_{15x} - g_{20} + f_2g_{45} - f_3g_{30} = 0,$$

$$g_{44} + g_{42x} - g_{42}(f_4 - f_1) + f_3(g_{57} - g_{12}) = 0, \qquad g_{42} + g_{40x} - g_{40}(f_4 - f_1) - f_3(g_{10} - g_{55}) = 0,$$

$$g_{27} + g_{25x} + g_{25}(f_4 - f_1) + f_2(g_{10} - g_{55}) = 0, \qquad g_{30} + g_{29x} + g_{29}(f_4 - f_1) + f_2(g_{14} - g_{59}) = 0,$$

$$f_{2t} - g_{20x} - g_{20}(f_4 - f_1) - f_2(g_5 - g_{50}) = 0, \qquad f_{3t} - g_{35x} + g_{35}(f_4 - f_1) + f_3(g_5 - g_{50}) = 0,$$

$$3g_{43} + g_{41x} - g_{41}(f_4 - f_1) - f_3(g_{11} - g_{56}) = 0, \qquad g_{45} + g_{44x} - g_{44}(f_4 - f_1) - f_3(g_{14} - g_{59}) = 0,$$

$$3g_{28} + g_{26x} + g_{26}(f_4 - f_1) + f_2(g_{11} - g_{56}) = 0, \qquad g_{29} + g_{27x} + g_{27}(f_4 - f_1) + f_2(g_{12} - g_{57}) = 0,$$

$$g_{26} + 2g_{46} - 2g_1 - (p_1b_2)_x - p_1b_2(f_4 - f_1) = 0, \qquad g_{47} - g_2 + \frac{1}{5}(p_1b_5)_x + \frac{1}{5}p_1b_5(f_4 - f_1) = 0,$$

$$g_{49} - g_{28x} - g_4 - g_{28}(f_4 - f_1) - f_2(g_{13} - g_{58}) = 0, \; g_{43} - g_{28} - g_{2x} + f_2g_{32} - f_3g_{17} = 0,$$

$$g_{50} - g_{30x} - g_5 - g_{30}(f_4 - f_1) - f_2(g_{15} - g_{60}) = 0, \; g_{16x} + g_{16}(f_4 - f_1) + f_2(g_1 - g_{46}) = 0,$$

$$g_{60} + g_{34x} - g_{15} - g_{34}(f_4 - f_1) - f_3(g_4 - g_{49}) = 0, \; g_{46} - 2g_{26} - g_1 + (p_1b_4)_x + p_1b_4(f_4 - f_1) = 0,$$

$$g_{46} + 2g_{41} - g_1 - (p_1b_4)_x + p_1b_4(f_4 - f_1) = 0, \qquad 2g_{46} - g_{41} - 2g_1 + (p_1b_2)_x - p_1b_2(f_4 - f_1) = 0,$$

$$g_2 - g_{47} + \frac{1}{5}(p_1b_5)_x - \frac{1}{5}p_1b_5(f_4 - f_1) = 0, \qquad g_{60} - g_{19x} - g_{15} - g_{19}(f_4 - f_1) - f_2(g_{49} - g_4) = 0,$$

$$g_{49} + g_{43x} - g_4 - g_{43}(f_4 - f_1) - f_3(g_{13} - g_{58}) = 0, g_{45} - g_{4x} - g_{30} + f_2g_{34} - f_3g_{19} = 0,$$

$$g_{57} + 2g_{16x} - g_{12} + 2g_{16}(f_4 - f_1) - 2f_2(g_{46} - g_1) = 0,$$
$$g_{12} + 2g_{31x} - g_{57} - 2g_{31}(f_4 - f_1) - 2f_3(g_1 - g_{46}) = 0.$$

Deriving a relation between the b_i

In this section we reduce the previous system down to equations which depend solely on the b_i's. In doing so we find that $g_k = 0$ for $k = 1, 2, 4, 5, 10 - 17, 29, 31, 32, 44, 46, 47, 49, 50, 55 - 58$, $f_4 = f_1 = 0, f_3 = f_2, p_2 = \frac{H(t)}{b_5}, g_{34} = -g_{19}, g_{42} = g_{27}, g_{59} = 2g_{19}, g_{40} = g_{25}, g_{41} = g_{26}, g_{60} = g_{19x}$,

$$g_{45} = g_{30} + 2g_{19}f_2, \qquad g_{19} = \frac{1}{2}f_2(g_{43} - g_{28}),$$
$$g_{35} = g_{20} - 2f_2^2 g_{19}, \qquad g_{25} = H(t)\left(\frac{b_1}{b_5}\right)_x,$$
$$g_{26} = H(t)\left(\frac{b_2}{b_5}\right)_x, \qquad g_{27} = -H(t)\left(\frac{b_2}{b_5}\right)_{xx},$$
$$g_{43} = -\frac{1}{3}H(t)\left(\frac{b_2}{b_5}\right)_{xx}, g_{30} = -\left(\frac{b_1}{b_5}\right)_{xxxx},$$

and

$$g_{28} = \frac{4}{3}H(t)\left(\frac{b_2}{b_5}\right)_x - \frac{1}{3}H(t)\left(\frac{b_2}{b_5}\right)_{xx}.$$

This leads to the following conditions on the b_i,

$$b_3 = 2b_2 + 2b_4, \tag{2.85}$$

$$b_5 = C(t)(2b_2 - b_4), \tag{2.86}$$

$$\begin{aligned} &12b_{2x}b_{2xx}b_4^2 + 12b_2^2b_{2xx}b_{4x} - 3b_{2xx}b_{4x}b_4^2 + 4b_2^2b_{2xxx}b_4 - 4b_2b_{2xxx}b_4^2 - 24b_2b_{2x}^2b_{4x} + 24b_2b_{2x}b_{4x}^2 \\ &+12b_2^2b_{2x}b_{4xx} - 12b_2^2b_{4x}b_{4xx} + 4b_2^2b_4b_{4xxx} - b_2b_4^2b_{4xxx} - 24b_{2x}^2b_4b_{4x} + 6b_{2x}b_4b_{4x}^2 - 3b_{2x}b_4^2b_{4xx} \\ &+b_{2xxx}b_4^3 - 6b_2b_{4x}^3 - 4b_2^3b_{4xxx} + 24a_{2x}^3b_4 - 24b_2b_{2x}b_{2xx}b_4 + 6b_2b_4b_{4x}b_{4xx} = 0, \end{aligned} \tag{2.87}$$

and

$$\begin{aligned} &-9600b_1b_{2x}^4b_{4x} + 9600b_1b_{2x}^3b_{4x}^2 - 4800b_1b_{2x}^2b_{4x}^3 + 1200b_1b_{2x}b_{4x}^4 - 3840b_{1x}b_2b_{2x}^4 - 240b_{1x}b_2b_{4x}^4 + \\ &1920b_{1x}b_{2x}^4b_4 + 120b_{1x}b_4b_{4x}^4 + 3840b_1b_{2x}^5 - 120b_1b_{4x}^5 + 7680b_{1x}b_2b_{2x}^3b_{4x} - 5760b_{1x}b_2b_{2x}^2b_{4x}^2 + \\ &1920b_{1x}b_2b_{2x}b_{4x}^3 - 3840b_{1x}b_{2x}^3b_4b_{4x} + 2880b_{1x}b_{2x}^2b_4b_{4x}^2 - 960b_{1x}b_{2x}b_4b_{4x}^3 - 32b_{1xxxxx}b_2^5 + \\ &b_{1xxxxx}b_4^5 + 10b_{1xxxx}b_{2x}b_4^4 - 80b_{1xxxx}b_2^4b_{4x} - 5b_{1xxxx}b_4^4b_{4x} + 80b_{1xxxxx}b_2^4b_4 - 80b_{1xxxxx}b_2^3b_4^2 + \\ &40b_{1xxxxx}b_2^2b_4^3 - 10b_{1xxxxx}b_2b_4^4 + 32b_1b_2^4b_{2xxxxx} + 2b_1b_{2xxxxx}b_4^4 - 16b_1b_2^4b_{4xxxxx} - b_1b_4^4b_{4xxxxx} - \\ &960b_{1x}b_2^3b_{2xx}^2 - 240b_{1x}b_2^3b_{4xx}^2 + 30b_{1x}b_4^3b_{4xx}^2 + 160b_{1x}b_2^4b_{2xxxx} - 80b_{1x}b_2^4b_{4xxxx} + 10b_{1x}b_4^4b_{2xxxx} - \\ &5b_{1x}b_4^4b_{4xxxx} + 1920b_{1xx}b_2^2b_{2x}^3 - 240b_{1xx}b_2^2b_{4x}^3 + 480b_{1xx}b_{2x}^3b_4^2 - 60b_{1xx}b_4^2b_{4x}^3 + 320b_{1xx}b_2^4b_{2xxx} - \\ &160b_{1xx}b_2^4b_{4xxx} + 20b_{1xx}b_{2xxx}b_4^4 - 10b_{1xx}b_4^4b_{4xxx} - 640b_{1xxx}b_2^3b_{2x}^2 - 160b_{1xxx}b_2^3b_{4x}^2 + \\ &80b_{1xxx}b_{2x}^2b_4^3 + 20b_{1xxx}b_4^3b_{4x}^2 + 320b_{1xxx}b_2^4b_{2xx} - 160b_{1xxx}b_2^4b_{4xx} + 20b_{1xxx}b_{2xx}b_4^4 - \\ &10b_{1xxx}b_4^4b_{4xx} + 960b_1b_2^2b_{2xx}b_{2xxx}b_4 - 480b_1b_2b_{2xx}b_{4xxx}b_4^2 - 480b_1b_2^2b_{4xx}b_{2xxx}b_4 + 240b_1b_2b_{2xx}b_4^2 + \end{aligned}$$

$$240b_1b_2^2b_4b_{4xx}b_{4xxx} - 120b_1b_2b_4^2b_{4xx}b_{4xxx} + 480b_1b_2^2b_{2x}b_{2xxxx}b_4 - 240b_1b_2b_{2x}b_{2xxxx}b_4^2 -$$
$$240b_1b_2^2b_{2x}b_4b_{4xxxx} + 120b_1b_2b_{2x}b_4^2b_{4xxxx} - 240b_1b_2^2b_{2xxxx}b_4b_{4x} + 120b_1b_2b_{2xxxx}b_4^2b_{4x} +$$
$$120b_1b_2^2b_4b_{4x}b_{4xxxx} - 60b_1b_2b_4^2b_{4x}b_{4xxxx} - 5760b_{1x}b_2^2b_{2x}b_{2xx}b_{4x} + 2880b_{1x}b_2^2b_{2x}b_{4x}b_{4xx} -$$
$$5760b_{1x}b_2b_{2x}^2b_{2xx}b_4 + 2880b_{1x}b_2b_{2x}^2b_4b_{4xx} - 1440b_{1x}b_2b_{2xx}b_4b_{4x}^2 + 720b_{1x}b_2b_4b_{4x}^2b_{4xx} -$$
$$1440b_{1x}b_{2x}b_{2xx}b_4^2b_{4x} + 720b_{1x}b_2b_4b_{4x}^2b_{4xx} - 1440b_{1x}b_{2x}b_{2xx}b_4^2b_{4x} + 720b_{1x}b_{2x}b_4^2b_{4x}b_{4xx} +$$
$$1920b_{1x}b_2^2b_{2x}b_{2xxx}b_4 - 960b_{1x}b_2^2b_{2x}b_4b_{4xxx} - 960b_{1x}b_2^2b_{2xxx}b_4b_{4x} + 480b_{1x}b_2^2b_4b_{4x}b_{4xxx} -$$
$$960b_{1x}b_2b_{2x}b_{2xxx}b_4^2 + 480b_{1x}b_2b_{2x}b_4^2b_{4xxx} + 480b_{1x}b_2b_{2xxx}b_4^2b_{4x} - 240b_{1x}b_2b_4^2b_{4x}b_{4xxx} +$$
$$2880b_{1xx}b_2^2b_{2x}b_{2xx}b_4 - 1440b_{1xx}b_2^2b_{2x}b_4b_{4xx} - 1440b_{1xx}b_2^2b_{2xx}b_4b_{4x} + 720b_{1xx}b_2^2b_4b_{4x}b_{4xx} -$$
$$1440b_{1xx}b_2b_{2x}b_{2xx}b_4^2 + 720b_{1xx}b_2b_{2x}b_4^2b_{4xx} + 720b_{1xx}b_2b_{2xx}b_4^2b_{4x} - 360b_{1xx}b_2b_4^2b_{4x}b_{4xx} +$$
$$11520b_1b_2b_{2x}^2b_{2xx}b_{4x} - 5760b_1b_{2x}^2b_{2xx}b_4b_{4x} - 5760b_1b_2b_{2x}^2b_{4x}b_{4xx} + 2880b_1b_{2x}^2b_4b_{4x}b_{4xx} -$$
$$5760b_1b_2b_{2x}b_{2xx}b_{4x}^2 + 2880b_1b_{2x}b_{2xx}b_4b_{4x}^2 + 2880b_1b_2b_{2x}b_{4x}^2b_{4xx} - 1440b_1b_{2x}b_4b_{4x}^2b_{4xx} -$$
$$2880b_1b_2b_{2x}b_{2xx}^2b_4 - 2880b_1b_2^2b_{2x}b_{2xx}b_{4xx} - 720b_1b_{2x}b_{2xx}b_4^2b_{4xx} - 720b_1b_2b_{2x}b_4b_{4xx}^2 +$$
$$1440b_1b_2b_{2xx}^2b_4b_{4x} + 1440b_1b_2^2b_{2xx}b_{4x}b_{4xx} + 360b_1b_{2xx}b_4^2b_{4x}b_{4xx} + 360b_1b_2b_4b_{4x}b_{4xx}^2 -$$
$$1920b_1b_2b_{2x}^3b_{2xxx}b_4 + 960b_1b_2b_{2x}^3b_4b_{4xxx} - 1920b_1b_2^2b_{2x}b_{2xxx}b_{4x} - 480b_1b_{2x}b_{2xxx}b_4^2b_{4x} +$$
$$960b_1b_2^2b_{2x}b_{4x}b_{4xxx} + 240b_1b_{2x}b_4^2b_{4x}b_{4xxx} - 480b_1b_2b_{2xxx}b_4b_{4x}^2 + 240b_1b_2b_4b_{4x}^2b_{4xxx} -$$
$$1440b_{1x}b_2^2b_{2xx}b_4b_{4xx} + 720b_{1x}b_2b_{2xx}b_4^2b_{4xx} + 2880b_{1xx}b_2b_{2x}^2b_4b_{4x} - 1440b_{1xx}b_2b_{2x}b_4b_{4x}^2 -$$
$$960b_{1xxx}b_2^2b_{2x}b_4b_{4x} + 480b_{1xxx}b_2b_{2x}b_4^2b_{4x} + 5760b_{1x}b_2b_{2x}b_{2xx}b_4b_{4x} - 2880b_{1x}b_2b_{2x}b_4b_{4x}b_{4xx} -$$
$$1440b_1b_2b_{2xx}b_4b_{4x}b_{4xx} + 1920b_1b_2b_{2x}b_{2xxx}b_4b_{4x} - 960b_1b_2b_{2x}b_4b_{4x}b_{4xxx} - 640b_1b_2^3b_{2xx}b_{2xxx} +$$
$$80b_1b_{2xx}b_{2xxx}b_4^3 + 320b_1b_2^3b_{2xx}b_{4xxx} - 40b_1b_{2xx}b_4^3b_{4xxx} + 320b_1b_2^3b_{2xxx}b_{4xx} - 40b_1b_{2xxx}b_4^3b_{4xx} -$$
$$160b_1b_2^3b_{4xx}b_{4xxx} + 20b_1b_4^3b_{4xx}b_{4xxx} - 320b_1b_2^3b_{2x}b_{2xxxx} + 40b_1b_{2x}b_{2xxxx}b_4^3 + 160b_1b_2^3b_{2x}b_{4xxxx} -$$
$$20b_1b_{2x}b_4^3b_{4xxxx} + 160b_1b_2^3b_{2xxxx}b_{4x} - 20b_1b_{2xxxx}b_4^3b_{4x} - 80b_1b_2^3b_{4x}b_{4xxxx} + 10b_1b_4^3b_{4x}b_{4xxxx} +$$
$$5760b_{1x}b_2^2b_{2x}^2b_{2xx} - 2880b_{1x}b_2^2b_{2x}^2b_{4xx} + 1440b_{1x}b_2^2b_{2xx}b_{4x}^2 - 720b_{1x}b_2^2b_{4x}^2b_{4xx} + 1440b_{1x}b_{2x}^2b_{2xx}b_4^2 -$$
$$720b_{1x}b_{2x}^2b_4^2b_{4xx} + 360b_{1x}b_{2xx}b_4^2b_{4x}^2 - 180b_{1x}b_4^2b_{4x}^2b_{4xx} - 1280b_{1x}b_2^3b_{2x}b_{2xxx} + 640b_{1x}b_2^3b_{2x}b_{4xxx} +$$
$$640b_{1x}b_2^3b_{2xxx}b_{4x} - 320b_{1x}b_2^3b_{4x}b_{4xxx} + 160b_{1x}b_{2x}b_{2xxx}b_4^3 - 80b_{1x}b_{2x}b_4^3b_{4xxx} - 80b_{1x}b_{2xxx}b_4^3b_{4x} +$$
$$40b_{1x}b_4^3b_{4x}b_{4xxx} - 1920b_{1xx}b_2^3b_{2x}b_{2xx} + 960b_{1xx}b_2^3b_{2x}b_{4xx} + 960b_{1xx}b_2^3b_{2xx}b_{4x} - 480b_{1xx}b_2^3b_{4x}b_{4xx} +$$
$$240b_{1xx}b_{2x}b_{2xx}b_4^3 - 120b_{1xx}b_{2x}b_4^3b_{4xx} - 120b_{1xx}b_{2xx}b_4^3b_{4x} + 60b_{1xx}b_4^3b_{4x}b_{4xx} - 320b_{1xxxx}b_2^3b_{2x}b_4 +$$

$$240b_{1xxxx}b_2^2b_{2x}b_4^2 - 80b_{1xxxx}b_2b_{2x}b_4^3 + 160b_{1xxxx}b_2^3b_4b_{4x} - 120b_{1xxxx}b_2^2b_4^2b_{4x} + 40b_{1xxxx}b_2b_4^3b_{4x} -$$
$$7680b_1b_2b_{2x}^3b_{2xx} + 3840b_1b_{2x}^3b_{2xx}b_4 + 3840b_1b_2b_{2x}^3b_{4xx} - 1920b_1b_{2x}^3b_4b_{4xx} + 960b_1b_2b_{2xx}b_{4x}^3 -$$
$$480b_1b_{2xx}b_4b_{4x}^3 - 480b_1b_2b_{4x}^3b_{4xx} + 240b_1b_4b_{4x}^3b_{4xx} + 2880b_1b_2^2b_{2x}b_{2xx}^2 + 720b_1b_{2x}b_{2xx}^2b_4^2 +$$
$$720b_1b_2^2b_{2x}b_{4xx}^2 + 180b_1b_{2x}b_4^2b_{4xx}^2 - 1440b_1b_2^2b_{2xx}^2b_{4x} - 360b_1b_{2xx}^2b_4^2b_{4x} - 360b_1b_2^2b_{4x}b_{4xx}^2 -$$
$$90b_1b_4^2b_{4x}b_{4xx}^2 + 1920b_1b_2^2b_{2x}^2b_{2xxx} + 480b_1b_{2x}^2b_{2xxx}b_4^2 - 960b_1b_2^2b_{2x}^2b_{4xxx} - 240b_1b_{2x}^2b_4^2b_{4xxx} +$$
$$480b_1b_2^2b_{2xxx}b_{4x}^2 + 120b_1b_{2xxx}b_4^2b_{4x}^2 - 240b_1b_2^2b_{4x}^2b_{4xxx} - 60b_1b_4^2b_{4x}^2b_{4xxx} - 64b_1b_2^3b_{2xxxxx}b_4 +$$
$$48b_1b_2^2b_{2xxxxx}b_4^2 - 16b_1b_2b_{2xxxxx}b_4^3 + 32b_1b_2^3b_4b_{4xxxxx} - 24b_1b_2^2b_4^2b_{4xxxxx} + 8b_1b_2b_4^3b_{4xxxxx} +$$
$$960b_{1x}b_2^3b_{2xx}b_{4xx} + 1440b_{1x}b_2^2b_{2xx}^2b_4 + 360b_{1x}b_2^2b_4b_{4xx}^2 - 720b_{1x}b_2b_{2xx}^2b_4^2 - 180b_{1x}b_2b_4^2b_{4xx}^2 -$$
$$120b_{1x}b_{2xx}b_4^3b_{4xx} - 320b_{1x}b_2^3b_{2xxxx}b_4 + 160b_{1x}b_2^3b_4b_{4xxxx} + 240b_{1x}b_2^2b_4^2b_{2xxxx} - 120b_{1x}b_2^2b_4^2b_{4xxxx} -$$
$$80b_{1x}b_2b_{2xxxx}b_4^3 + 40b_{1x}b_2b_4^3b_{4xxxx} - 2880b_{1xx}b_2^2b_{2x}^2b_{4x} + 1440b_{1xx}b_2^2b_{2x}b_{4x}^2 - 1920b_{1xx}b_2b_{2x}^3b_4 +$$
$$240b_{1xx}b_2b_4b_{4x}^3 - 720b_{1xx}b_{2x}^2b_4^2b_{4x} + 360b_{1xx}b_{2x}b_4^2b_{4x}^2 - 640b_{1xx}b_2^3b_{2xxx}b_4 + 320b_{1xx}b_2^3b_4b_{4xxx} +$$
$$480b_{1xx}b_2^2b_{2xxx}b_4^2 - 240b_{1xx}b_2^2b_4^2b_{4xxx} - 160b_{1xx}b_2b_{2xxx}b_4^3 + 80b_{1xx}b_2b_4^3b_{4xxx} + 640b_{1xxx}b_2^3b_{2x}b_{4x} +$$
$$960b_{1xxx}b_2^2b_{2x}^2b_4 + 240b_{1xxx}b_2^2b_4b_{4x}^2 - 480b_{1xxx}b_2b_{2x}^2b_4^2 - 120b_{1xxx}b_2b_4^2b_{4x}^2 - 80b_{1xxx}b_{2x}b_4^3b_{4x} -$$
$$640b_{1xxx}b_2^3b_{2xx}b_4 + 320b_{1xxx}b_2^3b_4b_{4xx} + 480b_{1xxx}b_2^2b_{2xx}b_4^2 - 240b_{1xxx}b_2^2b_4^2b_{4xx} - 160b_{1xxx}b_2b_{2xx}b_4^3 +$$
$$80b_{1xxx}b_2b_4^3b_{4xx} + 160b_{1xxxx}b_2^4b_{2x} + 120b_{1x}b_{2xx}^2b_4^3 + 2880b_1b_2b_{2x}b_{2xx}b_4b_{4xx} = 0,$$

where $H(t)$ and $C(t)$ are arbitrary functions of t.

Determining Equations for the Third Equation

Requiring the compatability condition yield a multiple of $F_3[v]$ in the off-diagonal yields the following large coupled system of algebraic and partial differential equations

$f_2 = f_8 = 0, g_{34} = -\frac{1}{7}p_1a_{10}, g_{20} = -\frac{1}{3}p_3a_9, f_4 = p_3, g_{30} = -p_3a_1, g_{21} = -p_3a_7, g_{19} = -p_3a_2, g_{54} = q_3a_1, g_{58} = \frac{1}{7}q_3a_{10}, g_{44} = \frac{1}{3}q_3a_9, g_{45} = q_3a_7, g_{43} = q_3a_2, f_6 = -q_3, g_{63} = -g_{67} = -g_8 = g_4 = G_4(t), g_{66} = g_7 = G_7(t), g_{68} = -\frac{1}{2}g_3 = g_9 = G_9(t), f_5 = f_3, f_7 = f_1, q_3 = -p_3,$

$$f_3(g_{i+24} - g_i) = 0, \ \ (i = 12 - 18, 22, 23, 25 - 29, 31 - 33)$$

$$f_3(g_{i+59} - g_i) = 0, \ \ (i = 2, 3, 5, 6, 8, 10, 11)$$

$$\pm\frac{1}{7}(p_3 c_{10})_x + p_3(g_{70} - g_{11}) = 0,$$

$$g_i + g_{i+1,x} = 0, \ \ (i = 24 - 29, 48 - 52)$$

$$\frac{2}{3}p_3c_9 + 4p_3a_7 = p_3c_8, \qquad g_{47} - p_3(2c_2 - c_3) = 0, \qquad g_{18} + 2g_{17} = -p_3c_6,$$

$$g_{23} - p_3(2c_2 - c_3) = 0, \qquad G_7 = -\frac{1}{2}G_4, \qquad g_{42} + 2g_{41} = -p_3c_6,$$

$$G_{66} = -\frac{1}{2}G_{63}, \qquad G_{68} = -\frac{1}{2}G_{62}, \qquad g_3 - p_3(g_{51} - g_{27}) = 0,$$

$$g_{23} + g_{17} = -p_3c_5, \qquad g_{23} + 2g_{18} = -p_3c_4, \qquad g_2 - \frac{1}{2}p_3(g_{49} - g_{25}) = 0,$$

$$g_{47} + g_{41} = -p_3c_5, \qquad g_{47} + 2g_{42} = -p_3c_4, \qquad g_5 - \frac{1}{2}p_3(g_{55} - g_{31}) = 0,$$

$$g_6 - p_3(g_{36} - g_{16}) = 0, \qquad g_8 - p_3(g_{53} - g_{29}) = 0, \qquad g_{10} - \frac{1}{4}p_3(g_{39} - g_{15}) = 0,$$

$$g_{11} - \frac{1}{6}p_3(g_{40} - g_{16}) = 0, \qquad g_{61} + \frac{1}{2}p_3(g_{49} - g_{25}) = 0, \qquad g_{62} + p_3(g_{51} - g_{27}) = 0,$$

$$g_{65} + p_3(g_{36} - g_{12}) = 0, \qquad g_{64} + \frac{1}{2}p_3(g_{55} - g_{31}) = 0, \qquad g_{67} + p_3(g_{53} - g_{29}) = 0,$$

$$g_{69} + \frac{1}{4}p_3(g_{39} - g_{15}) = 0, \qquad g_{70} + \frac{1}{6}p_3(g_{40} - g_{16}) = 0, \qquad g_{2x} - p_3(g_{48} - g_{24}) = 0,$$

$$g_{5x} - p_3(g_{37} - g_{13}) = 0, \qquad g_{3x} - p_3(g_{50} - g_{26}) = 0, \qquad g_{6x} - p_3(g_{38} - g_{14}) = 0,$$

$$g_{8x} - p_3(g_{52} - g_{28}) = 0, \qquad g_{10x} - p_3(g_{56} - g_{32}) = 0, \qquad g_{11x} - p_3(g_{57} - g_{33}) = 0,$$

$$g_{62x} + p_3(g_{50} - g_{26}) = 0, \qquad g_{61x} + p_3(g_{48} - g_{24}) = 0, \qquad g_{64x} + p_3(g_{37} - g_{13}) = 0,$$

$$g_{65x} + p_3(g_{38} - g_{14}) = 0, \qquad g_{67x} + p_3(g_{52} - g_{28}) = 0, \qquad g_{69x} + p_3(g_{56} - g_{32}) = 0,$$

$$g_{12} - (p_3c_2)_x - p_3(g_{67} - g_8) = 0, \qquad 2g_{12} + g_{23x} + g_{22} = 0, \qquad g_{70x} + p_3(g_{57} - g_{33}) = 0,$$

$g_{13} + g_{31x} = 0,$ $g_{14} + g_{12x} = 0,$ $g_{15} + g_{14x} + p_3(g_3 - g_{62}) = 0,$

$g_{16} - (p_3c_7)_x - p_3(g_{65} - g_6) = 0,$ $2g_{15} + g_{13x} = 0,$ $4g_{16} - \frac{1}{3}(p_3c_9)_x - p_3(g_{64} - g_5) = 0,$

$4g_{40} - \frac{1}{3}(p_3c_9)_x + p_3(g_{64} - g_5) = 0,$ $g_{22} + g_{18x} = 0,$ $3g_{32} + g_{15x} = 0,$

$g_{37} + g_{55x} = 0,$ $g_{38} + g_{36x} = 0,$ $g_{39} + g_{38x} - p_3(g_3 - g_{62}) = 0,$

$g_{40} - (p_3c_7)_x + p_3(g_{65} - g_6) = 0,$ $g_{46} + g_{42x} = 0,$ $g_{46} + g_{47x} + 2g_{36} = 0,$

$3g_{56} + g_{39x} = 0,$ $5g_{57} + g_{40x} = 0,$ $p_3(g_{59} - g_{35}) + f_3(g_{48} - g_{24}) = 0,$

$g_{16} + g_{16x} + 5g_{33} = 0,$ $2g_{13} + g_{22x} + 2g_{14} = 0,$ $g_{22} + g_{17x} + 3g_{31} = 0,$

$2g_{37} + g_{46x} + 2g_{38} = 0,$ $g_{46} + g_{41x} + 3g_{55} = 0,$ $2g_5 + 3g_6 - p_3(g_{46} - g_{22}) = 0,$

$f_{1t} - g_{1x} - f_3(g_{35} + g_{59}) = 0,$ $2g_{64} + 3g_{65} + p_3(g_{46} - g_{22}) = 0,$ $f_{1t} - g_{60x} + f_3(g_{35} - g_{59}) = 0,$

$f_{3t} - g_{35x} + f_3(g_{60} - g_1) = 0,$ $f_{3t} - g_{59x} - f_3(g_{60} - g_1) = 0,$ $g_{56x} - p_3(g_2 - g_{61}) = 0,$

$g_{32x} + p_3(g_2 - g_{61}) = 0,$ $g_{33x} + p_3(g_{10} - g_{69}) = 0,$ $g_{57x} - p_3(g_{10} - g_{69}) = 0,$

$g_{29} - (p_3c_1)_x = 0,$ $g_{36} - (p_3c_2)_x + p_3(g_{67} - g_8) = 0,$ $p_{1t} - g_{24x} - p_3(g_1 - g_{60}) = 0,$

$g_{53} - (p_3c_1)_x = 0,$ $2g_{39} + g_{37x} = 0,$ $p_{1t} - g_{48x} + p_3(g_1 - g_{60}) = 0.$

Deriving a relation between the c_i

In this section we reduce the previous system down to equations which depend solely on the c_i's. In doing so we find that

$g_{64} = g_5 = g_{65} = g_6 = g_{61} = g_2 = g_{10} = g_{69} = g_{70} = g_{11} = G_4(t) = G_9(t) = 0, g_{59} = g_{35}, g_{60} = g_1, g_{42} = g_{18}, g_{40} = g_{16}, g_{52} = g_{28}, g_{53} = g_{29}, g_{55} = g_{31}, g_{36} = g_{12}, g_{39} = g_{15}, g_{56} = g_{32}, g_{41} = g_{17}, g_{57} = g_{33}, g_{37} = g_{13}, g_{47} = g_{23}, p_3 = \frac{H(t)}{c_{10}}, g_{29} = (p_3c_1)_x, g_{12} = (p_3c_2)_x, g_{15} = -\frac{1}{2}g_{13x}, g_{32} =$

$-\frac{1}{3}g_{15x}, g_{28} = -(p_3c_1)_{xx}, g_{33} = G_{33}(t), g_i = -g_{ix}, \ \ (i = 24-27, 48-51)$

$$
\begin{aligned}
g_{23} &= -p_3c_5 - g_{17}, & g_{13} &= \frac{1}{3}(p_3(2c_3 - c_5 - 6c_2))_{xx},\\
g_{18} &= -p_3c_6 - 2g_{17}, & g_{22} &= g_{17x} - 2g_{12} + (p_3c_5)_x,\\
g_{16} &= (p_3c_7)_x, & g_{46} &= g_{17x} - 2g_{12} + (p_3c_5)_x,\\
g_{14} &= g_{12x} - \frac{1}{2}g_{17xx} - g_{13} - \frac{1}{2}(p_3c_5)_{xx}, & g_{31} &= \frac{1}{3}(2g_{12} - 2g_{17x} - (p_3c_5)_x),\\
g_{38} &= -g_{13} + g_{12x} - \frac{1}{2}g_{17xx} - \frac{1}{2}(p_3c_5)_{xx}, & g_{17} &= p_3(c_3 - 2c_2 - c_5),
\end{aligned}
$$

which leads to the following conditions on the c_i,

$$c_4 = -10c_2 + 5c_3 + 2c_6 - 4c_5, \tag{2.91}$$

$$c_8 = 4c_7 + \frac{2}{3}c_9, \tag{2.92}$$

$$c_{10x}c_6 - c_6c_{10x} - 2c_5c_{10x} + 2c_{5x}c_{10} + c_{10x}c_3 - c_{3x}c_{10} = 0, \tag{2.93}$$

$$-12c_{10x}c_7 + 12c_{7x}c_{10} + c_{10x}c_9 - c_{9x}c_{10} = 0, \tag{2.94}$$

$$5Gc_{10}^3 - 2Hc_{10x}c_{7x}c_{10} + Hc_{10}^2c_{7xx} + 2Hc_{10x}^2c_7 - Hc_7c_{10}c_{10xx} = 0, \tag{2.95}$$

$$14c_3c_{10x}^2 - 7c_3c_{10}c_{10xx} - 60c_2c_{10x}^2 + 30c_2c_{10}c_{10xx} + 60c_{2x}c_{10}c_{10x} - 30c_{2xx}c_{10}^2 + 4c_{5x}c_{10}c_{10x}$$
$$-14c_{3x}c_{10}c_{10x} + 7c_{3xx}c_{10}^2 - 4c_5c_{10x}^2 + 2c_5c_{10}c_{10xx} - 2c_{5xx}c_{10}^2 = 0, \tag{2.96}$$

$$18c_3c_{10x}^2 - 18c_3c_{10}c_{10x}c_{10xx} + 3a_3c_{10}^3c_{10xxx} - 18c_{3x}c_{10}c_{10x}^2 + 9c_{3x}c_{10}^2c_{10xx} - 60c_2c_{10x}^3$$
$$+60c_2c_{10}c_{10x}c_{10xx} - 10c_2c_{10}^2c_{10xxx} + 60c_{2x}c_{10}c_{10x}^2 - 30c_{2x}c_{10}^2c_{10xx} - 30c_{2xx}c_{10}^2c_{10x}$$
$$+10c_{2xxx}c_{10}^3 + 6c_{5x}c_{10}c_{10x}^2 - 3c_{5x}c_{10}^2c_{10xx} - 3c_{5xx}c_{10}^2c_{10x} + 9c_{3xx}c_{10}^2c_{10x} - 3c_{3xxx}c_{10}^3$$
$$-6c_5c_{10x}^3 + c_5c_{10}c_{10x}c_{10xx} - c_5c_{10}^2c_{10xxx} + c_{5xxx}c_{10}^3 = 0, \tag{2.97}$$

$$c_{1xxxxxxx}c_{10}^7 + 5040c_{1x}c_{10}c_{10x}^6 - 7c_{1xxxxxx}c_{10}^6c_{10x} - c_1c_{10}^6c_{10xxxxxxx} - 5040c_1c_{10x}^7$$
$$-7c_{1x}c_{10}^6c_{10xxxxxx} - 2520c_{1xx}c_{10}^2c_{10x}^5 + \frac{H_t}{H}c_{10}^7 + 210c_{1xxx}c_{10}^5c_{10xx}^2 - c_{10}^6c_{10t}$$
$$-35c_{1xxx}c_{10}^6c_{10xxxx} - 210c_{1xxxx}c_{10}^4c_{10x}^3 - 21c_{1xxxxx}c_{10}^6c_{10xx} + 140c_{1x}c_{10}^5c_{10xxx}^2$$
$$-630c_1c_{10}^4c_{10x}c_{10xx}c_{10xxxx} - 630c_{1x}c_{10}^4c_{10xx}^3 + 5040c_1c_{10}^3c_{10x}^2c_{10xx}c_{10xxx}$$
$$-630c_1c_{10}^4c_{10xx}^2c_{10xxx} - 126c_1c_{10}^4c_{10x}^2c_{10xxxxx} + 70c_1c_{10}^5c_{10xxx}c_{10xxxx}$$
$$+14c_1c_{10}^5c_{10x}c_{10xxxxxx} - 12600c_1c_{10}^2c_{10x}^3c_{10xx}^2 + 2520c_1c_{10}^3c_{10x}c_{10xx}^3$$
$$-12600c_{1x}c_{10}^2c_{10x}^4c_{10xx} + 7560c_{1x}c_{10}^3c_{10x}^2c_{10xx}^2 + 3360c_{1x}c_{10}^3c_{10x}^3c_{10xxx}$$
$$+210c_{1x}c_{10}^5c_{10xx}c_{10xxxx} + 84c_{1x}c_{10}^5c_{10x}c_{10xxxxx} + 5040c_{1xx}c_{10}^3c_{10x}^3c_{10xx}$$
$$-1260c_{1xx}c_{10}^4c_{10x}^2c_{10xxx} + 420c_{1xx}c_{10}^5c_{10xx}c_{10xxx} + 210c_{1xx}c_{10}^5c_{10x}c_{10xxxx}$$
$$-2520c_{1x}c_{10}^4c_{10x}c_{10xx}c_{10xxx} + 840c_1c_{10}^3c_{10x}^3c_{10xxxx} - 1260c_{1xxx}c_{10}^4c_{10x}^2c_{10xx}$$
$$+42c_1c_{10}^5c_{10xx}c_{10xxxxx} - 420c_1c_{10}^4c_{10x}c_{10xxx}^2 - 630c_{1x}c_{10}^4c_{10x}^2c_{10xxxx}$$
$$-1890c_{1xx}c_{10}^4c_{10x}c_{10xx}^2 + 840c_{1xxx}c_{10}^3c_{10x}^4 - 4200c_1c_{10}^2c_{10x}^4c_{10xxx}$$
$$+42c_{1xxxxx}c_{10}^5c_{10x}^2 + 280c_{1xxx}c_{10}^5c_{10x}c_{10xxx} + 210c_{1xxxx}c_{10}^5c_{10x}c_{10xx}$$
$$-21c_{1xx}c_{10}^6c_{10xxxxx} - 35c_{1xxxx}c_{10}^6c_{10xxx} + 15210c_1c_{10}c_{10x}^5c_{10xx} = 0, \tag{2.98}$$

and

$$
\begin{aligned}
&-240c_3c_{10x}^5+720c_2c_{10x}^5+120c_5c_{10x}^5+40c_{3xxx}c_{10}^3c_{10x}^2-20c_{3xxx}c_{10}^4c_{10xx}\\
&-180c_{2x}c_{10}^3c_{10xx}^2+30c_{2x}c_{10}^4c_{10xxxx}+360c_{2xx}c_{10}^2c_{10x}^3+60c_{2xx}c_{10}^4c_{10xxx}\\
&+60c_{2xxx}c_{10}^4c_{10xx}-30c_{5x}c_{10}^3c_{10xx}^2+5c_{5x}c_{10}^4c_{10xxxx}+60c_{5xx}c_{10}^2c_{10x}^3\\
&+10c_{5xxx}c_{10}^4c_{10xx}+c_5c_{10}^4c_{10xxxxx}-120c_{3xx}c_{10}^2c_{10x}^3-20c_{3xx}c_{10}^4c_{10xxx}\\
&-10c_{3x}c_{10}^4c_{10xxxx}+30c_{2xxxx}c_{10}^4c_{10x}+5c_{5xxxx}c_{10}^4c_{10x}-10c_{3xxxx}c_{10}^4c_{10x}\\
&-c_{5xxxxx}c_{10}^5-720c_{2x}c_{10}c_{10x}^4-120c_{5x}c_{10}c_{10x}^4+240c_{3x}c_{10}c_{10x}^4\\
&-120c_3c_{10}^2c_{10x}^2c_{10xxx}+40c_3c_{10}^3c_{10xx}c_{10xxx}+20c_3c_{10}^3c_{10x}c_{10xxxx}\\
&+120c_{3xx}c_{10}^3c_{10x}c_{10xx}-1440c_2c_{10}c_{10x}^3c_{10xx}+540c_2c_{10}^2c_{10x}c_{10xx}^2\\
&-60c_2c_{10}^3c_{10x}c_{10xxxx}+1080c_{2x}c_{10}^2c_{10x}^2c_{10xx}-240c_{2x}c_{10}^3c_{10x}c_{10xxx}\\
&-40c_{5x}c_{10}^3c_{10x}c_{10xxx}-60c_{5xx}c_{10}^3c_{10x}c_{10xx}-240c_5c_{10}c_{10x}^3c_{10xx}\\
&-20c_{5xxx}c_{10}^3c_{10x}^2+60c_{3x}c_{10}^3c_{10xx}^2+2c_{3xxxxx}c_{10}^5-180c_3c_{10}^2c_{10x}c_{10xx}^2\\
&+180c_{5x}c_{10}^2c_{10x}^2c_{10xx}-120c_2c_{10}^3c_{10xx}c_{10xxx}+80c_{3x}c_{10}^3c_{10x}c_{10xxx}\\
&+10c_{5xx}c_{10}^4c_{10xxx}-2c_3c_{10}^4c_{10xxxxx}-6c_{2xxxxx}c_{10}^5-360c_{3x}c_{10}^2c_{10x}^2c_{10xx}\\
&-20c_5c_{10}^3c_{10xx}c_{10xxx}-10c_5c_{10}^3c_{10x}c_{10xxxx}+60c_5c_{10}^2c_{10x}^2c_{10xxx}+6c_2c_{10}^4c_{10xxxxx}\\
&+480c_3c_{10}c_{10x}^3c_{10xx}+360c_2c_{10}^2c_{10x}^2c_{10xxx}-360c_{2xx}c_{10}^3c_{10x}c_{10xx}\\
&+90c_5c_{10}^2c_{10x}c_{10xx}^2-120c_{2xxx}c_{10}^3c_{10x}^2=0,
\end{aligned}
\tag{2.99}
$$

where $G(t)$ and $H(t)$ are arbitrary functions of t.

CHAPTER 3: THE EXTENDED ESTABROOK-WAHLQUIST METHOD

In this chapter we present the extended Estabrook-Wahlquist method for deriving the differential constraints necessary for compatibility of the Lax pair associated with a variable-coefficient nlpde. We then illustrate the effectiveness of this method by deriving variable-coefficient generalizations to the nonlinear Schrödinger (NLS) equation, derivative NLS equation, PT-symmetric NLS, fifth-order KdV, and the first equation in the MKdV hierarchy. As a final example, we consider a variable-coefficient extension to the nonintegrable cubic-quintic NLS and show that the extended Estabrook-Wahlquist method correctly breaks down upon attempting to satisfy the consistency conditions.

Outline of the Extended Estabrook-Wahlquist Method

In the standard Estabrook-Wahlquist method one begins with a constant-coefficient nlpde and assumes an implicit dependence on the unknown function(s) and its (their) partial derivatives for the spatial and time evolution matrices $(\mathbb{F}, \mathbb{G})$ involved in the linear scattering problem

$$\psi_x = \mathbb{F}\psi, \quad \psi_t = \mathbb{G}\psi$$

The evolution matrices $\mathbb{F}$ and $\mathbb{G}$ are connected via a zero-curvature condition (independence of path in spatial and time evolution) derived by requiring $\psi_{xt} = \psi_{tx}$. That is, it requires

$$\mathbb{F}_t - \mathbb{G}_x + [\mathbb{F}, \mathbb{G}] = 0 \tag{3.1}$$

upon satisfaction of the nlpde.

For simplicity in outlining the details of the method we will consider a system with only one unknown function, namely $u(x,t)$. Generalization of the procedure to follow to systems with $n \geq 2$ unknown functions is straightforward. Let

$$\mathbb{F} = \mathbb{F}(u, u_x, u_t, \ldots, u_{mx,nt}), \qquad \mathbb{G} = \mathbb{G}(u, u_x, u_t, \ldots, u_{kx,jt})$$

represent the space and time evolution matrices of a Lax pair, respectively, where $u_{px,qt} = \frac{\partial^{p+q} u}{\partial x^p \partial t^q}$. Plugging these into (3.1) we obtain the equivalent condition

$$\sum_{m,n} \mathbb{F}_{u_{mx,nt}} u_{mx,(n+1)t} - \sum_{j,k} \mathbb{G}_{u_{jx,kt}} u_{(j+1)x,kt} + [\mathbb{F}, \mathbb{G}] = 0$$

As we are now letting $\mathbb{F}$ and $\mathbb{G}$ have explicit dependence on x and t and for notational clarity, it will be more convenient to consider the following version of the zero-curvature condition

$$\mathrm{D}_t \mathbb{F} - \mathrm{D}_x \mathbb{G} + [\mathbb{F}, \mathbb{G}] = 0 \tag{3.2}$$

where D_t and D_x are the total derivative operators on time and space, respectively. Recall the definition of the total derivative

$$\mathrm{D}_x f(x, t, u_1(x,t), u_2(x,t), \ldots, u_n(x,t)) = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial u_1}\frac{\partial u_1}{\partial x} + \frac{\partial f}{\partial u_2}\frac{\partial u_2}{\partial x} + \cdots + \frac{\partial f}{\partial u_n}\frac{\partial u_n}{\partial x}$$

Thus we can write the compatibility condition as

$$\mathbb{F}_t + \sum_{m,n} \mathbb{F}_{u_{mx,nt}} u_{mx,(n+1)t} - \mathbb{G}_x - \sum_{j,k} \mathbb{G}_{u_{jx,kt}} u_{(j+1)x,kt} + [\mathbb{F}, \mathbb{G}] = 0$$

It is important to note that the subscripted x and t denotes the partial derivative with respect to only the x and t elements, respectively. That is, although u and it's derivatives depend on x and t this will not invoke use of the chain rule as they are treated as independent variables. This will become more clear in the examples of the next section.

From here there is often a systematic approach [26]- [29] to determining the explicit dependence of $\mathbb{F}$ and $\mathbb{G}$ on u and its derivatives which is outlined in [28] and will be utilized in the examples to follow.

Typically one takes $\mathbb{F}$ to depend on all terms in the nlpde for which there is a partial time derivative present. Similarly one may take $\mathbb{G}$ to depend on all terms for which there is a partial space derivative present. For example, given the Camassa-Holm equation,

$$u_t + 2ku_x - u_{xxt} + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0,$$

one would consider $\mathbb{F} = \mathbb{F}(u, u_{xx})$ and $\mathbb{G} = \mathbb{G}(u, u_x, u_{xx})$. Imposing compatibility allows one to determine the explicit form of $\mathbb{F}$ and $\mathbb{G}$ in a very algorithmic way. Additionally the compatibility condition induces a set of constraints on the coefficient matrices in $\mathbb{F}$ and $\mathbb{G}$. These coefficient matrices subject to the constraints generate a finite dimensional matrix Lie algebra.

In the extended Estabrook-Wahlquist method we once again take $\mathbb{F}$ and $\mathbb{G}$ to be functions of u and its partial derivatives but now additionally allow dependence on x and t. Although the details

change, the general procedure will remain essentially the same. This typically entails equating the coefficient of the highest partial derivative in space of the unknown function(s) to zero and working our way down until we have eliminated all partial derivatives of the unknown function(s). This is simply because the spatial derivative of the time evolution matrix $\mathbb{G}$ will introduce a term which contains a spatial derivative of the unknown function of degree one greater than that which $\mathbb{G}$ depends on. For example, in the case of the Camassa-Holm equation with $\mathbb{G}$ as it was given above one would have

$$D_x\mathbb{G} = \mathbb{G}_x + \mathbb{G}_u u_x + \mathbb{G}_{u_x} u_{xx} + \mathbb{G}_{u_{xx}} u_{xxx}.$$

The final term in the above expression involves u_{xxx} (i.e. a spatial derivative of u one degree higher than $\mathbb{G}$ depends on.) This result, coupled with the terms resulting from the elimination of the u_t using the evolution equation, yield a u_{xxx} term whose coefficient must necessarily vanish as $\mathbb{F}$ and $\mathbb{G}$ do not depend on u_{xxx}. Before considering some examples we make the following observations.

This extended method usually results in a large partial differential equation (in the standard Estabrook-Wahlquist method, this is an algebraic equation) which can be solved by equating the coefficients of the different powers of the unknown function(s) to zero. This final step induces a set of constraints on the coefficient matrices in $\mathbb{F}$ and $\mathbb{G}$. *Another big difference which we will see in the examples comes in the final and, arguably, the hardest step. In the standard Estabrook-Wahlquist method the final step involves finding explicit forms for the set of coefficient matrices such that they satisfy the contraints derived in the procedure and depend on a spectral parameter. Note these constraints are nothing more than a system of algebraic matrix equations. In the extended Estabrook-Wahlquist method these constraints will be in the form of matrix partial differential equations which can be used to derive an integrability condition on the coefficients in the nlpde.*

Note that compatibility of the time and space evolution matrices will yield a set of constraints which contain the constant coefficient constraints as a subset. In fact, taking the variable coefficients to be the appropriate constants will yield exactly the Estabrook-Wahlquist results for the constant coefficient version of the nlpde. That is, the constraints given by the Estabrook-Wahlquist method for a constant coefficient nlpde are always a proper subset of the constraints given by a variable-coefficient version of the nlpde. This can easily be shown. Letting $\mathbb{F}$ and $\mathbb{G}$ not depend explicitly on x and t and taking the coefficients in the NLPDE to be constant the zero-curvature condition as it is written above becomes

$$\sum_{m,n} \mathbb{F}_{u_{mx,nt}} u_{mx,(n+1)t} - \sum_{j,k} \mathbb{G}_{u_{jx,kt}} u_{(j+1)x,kt} + [\mathbb{F}, \mathbb{G}] = 0,$$

which is exactly the standard Estabrook-Wahlquist method.

The conditions derived via mandating (3.2) be satisfied upon solutions of the variable-coefficient nlpde may be used to determine conditions on the coefficient matrices and variable coefficients (present in the NLPDE). Successful closure of these conditions is equivalent to the system being S-integrable. A major advantage to using the Estabrook-Wahlquist method that carries forward with the extension is the fact that it requires little guesswork and yields quite general results.

PT-Symmetric and Standard Nonlinear Schrödinger Equation

We begin with the derivation of the Lax pair and differential constraints for the variable-coefficient standard NLS equation. Following with the procedure outlined above we choose

$$\mathbb{F} = \mathbb{F}(x, t, q, r), \quad \text{and} \quad \mathbb{G} = \mathbb{G}(x, t, r, q, r_x, q_x).$$

Compatibility requires these matrices satisfy the zero-curvature conditions given by (3.2). Plugging $\mathbb{F}$ and $\mathbb{G}$ into (3.2) we have

$$\mathbb{F}_t + \mathbb{F}_r r_t + \mathbb{F}_q q_t - \mathbb{G}_x - \mathbb{G}_r r_x - \mathbb{G}_q q_x - \mathbb{G}_{r_x} r_{xx} - \mathbb{G}_{q_x} q_{xx} + [\mathbb{F}, \mathbb{G}] = 0. \tag{3.3}$$

Now requiring this be satisfied upon solutions of (2.3) we follow the standard technique of eliminating r_t and u_t via (2.3) from which we obtain

$$\begin{aligned} &\mathbb{F}_t - i\mathbb{F}_r(f r_{xx} + g r^2 q + (\upsilon - i\gamma) r) + i\mathbb{F}_q(f q_{xx} + g q^2 r + (\upsilon + i\gamma) q) \\ &-\mathbb{G}_x - \mathbb{G}_r r_x - \mathbb{G}_q q_x - \mathbb{G}_{r_x} r_{xx} - \mathbb{G}_{q_x} q_{xx} + [\mathbb{F}, \mathbb{G}] = 0. \end{aligned} \tag{3.4}$$

Since $\mathbb{F}$ and $\mathbb{G}$ do not depend on q_{xx} or r_{xx} we collect the coefficients of q_{xx} and r_{xx} and equate them to zero. This requires

$$-if\mathbb{F}_r - \mathbb{G}_{r_x} = 0, \text{ and } if\mathbb{F}_q - \mathbb{G}_{q_x} = 0. \tag{3.5}$$

Solving this linear system yields

$$\mathbb{G} = if(\mathbb{F}_q q_x - \mathbb{F}_r r_x) + \mathbb{K}^0(x, t, q, r). \tag{3.6}$$

Plugging this expression for $\mathbb{G}$ into equation (3.4) gives us the updated requirement

$$\mathbb{F}_t - i\mathbb{F}_r(gr^2q + (\upsilon - i\gamma)r) + i\mathbb{F}_q(gq^2r + (\upsilon + i\gamma)q) - if_x(\mathbb{F}_q q_x - \mathbb{F}_r r_x)$$
$$-\mathbb{K}^0_q q_x - \mathbb{K}^0_r r_x - if(\mathbb{F}_{qx} q_x - \mathbb{F}_{rx} r_x) - if(\mathbb{F}_{qq} q_x^2 - \mathbb{F}_{rr} r_x^2) - \mathbb{K}^0_x + if q_x \left[\mathbb{F}, \mathbb{F}_q\right]$$
$$-if r_x \left[\mathbb{F}, \mathbb{F}_r\right] + \left[\mathbb{F}, \mathbb{K}^0\right] = 0. \tag{3.7}$$

Now since $\mathbb{F}$ and $\mathbb{K}^0$ do not depend on q_x and r_x we collect the coefficients of the q_x^2 and r_x^2 and equate them to zero. This now requires

$$if\mathbb{F}_{rr} = 0 = -if\mathbb{F}_{qq}.$$

From this, it follows via simple integration that $\mathbb{F}$ depends on q and r explicitly as follows,

$$\mathbb{F} = \mathbb{X}_1(x,t) + \mathbb{X}_2(x,t)q + \mathbb{X}_3(x,t)r + \mathbb{X}_4(x,t)rq,$$

where the $\mathbb{X}_i$ in this expression are arbitrary matrices whose elements are functions of x and t but do not depend on q, r, or their partial derivatives. Plugging this expression for $\mathbb{F}$ into equation (3.7) we obtain

$$\mathbb{X}_{1,t}+\mathbb{X}_{2,t}q+\mathbb{X}_{3,t}r+\mathbb{X}_{4,t}rq-i(\mathbb{X}_3+\mathbb{X}_4q)(gr^2q+(\upsilon-i\gamma)r)+i(\mathbb{X}_2+\mathbb{X}_4r)(gq^2r+(\upsilon+i\gamma)q)$$
$$-if_x((\mathbb{X}_2+\mathbb{X}_4r)q_x-(\mathbb{X}_3+\mathbb{X}_4q)r_x)-\mathbb{K}^0_qq_x-\mathbb{K}^0_rr_x-if((\mathbb{X}_{2,x}+\mathbb{X}_{4,x}r)q_x-(\mathbb{X}_{3,x}+\mathbb{X}_{4,x}q)r_x)$$
$$-\mathbb{K}^0_x+if[\mathbb{X}_1,\mathbb{X}_2]q_x+if[\mathbb{X}_3,\mathbb{X}_2]rq_x+if[\mathbb{X}_4,\mathbb{X}_2]rqq_x+if[\mathbb{X}_1,\mathbb{X}_4]rq_x+if[\mathbb{X}_2,\mathbb{X}_4]rqq_x$$
$$+if[\mathbb{X}_3,\mathbb{X}_4]r^2q_x-if[\mathbb{X}_1,\mathbb{X}_3]r_x-if[\mathbb{X}_2,\mathbb{X}_3]qr_x-if[\mathbb{X}_4,\mathbb{X}_3]qrr_x-if[\mathbb{X}_1,\mathbb{X}_4]qr_x$$
$$-if[\mathbb{X}_2,\mathbb{X}_4]q^2r_x-if[\mathbb{X}_3,\mathbb{X}_4]qrr_x+[\mathbb{X}_1,\mathbb{K}^0]+[\mathbb{X}_2,\mathbb{K}^0]q+[\mathbb{X}_3,\mathbb{K}^0]r+[\mathbb{X}_4,\mathbb{K}^0]qr=0. \quad (3.8)$$

Noting the antisymmetry of the commutator, that is $[A,B]=-[B,A]$ (which is required if the $\mathbb{X}_i$ are elements of a Lie algebra), we can further simplify the previous expression to obtain

$$\mathbb{X}_{1,t}+\mathbb{X}_{2,t}q+\mathbb{X}_{3,t}r+\mathbb{X}_{4,t}rq-i(\mathbb{X}_3+\mathbb{X}_4q)(gr^2q+(\upsilon-i\gamma)r)+i(\mathbb{X}_2+\mathbb{X}_4r)(gq^2r+(\upsilon+i\gamma)q)$$
$$-if_x((\mathbb{X}_2+\mathbb{X}_4r)q_x-(\mathbb{X}_3+\mathbb{X}_4q)r_x)-\mathbb{K}^0_qq_x-\mathbb{K}^0_rr_x-if((\mathbb{X}_{2,x}+\mathbb{X}_{4,x}r)q_x-(\mathbb{X}_{3,x}+\mathbb{X}_{4,x}q)r_x)$$
$$-\mathbb{K}^0_x+if[\mathbb{X}_1,\mathbb{X}_2]q_x+if[\mathbb{X}_3,\mathbb{X}_2]rq_x+if[\mathbb{X}_1,\mathbb{X}_4]rq_x+if[\mathbb{X}_3,\mathbb{X}_4]r^2q_x-if[\mathbb{X}_1,\mathbb{X}_3]r_x$$
$$-if[\mathbb{X}_2,\mathbb{X}_3]qr_x-if[\mathbb{X}_1,\mathbb{X}_4]qr_x-if[\mathbb{X}_2,\mathbb{X}_4]q^2r_x=0. \quad (3.9)$$

As before, since the $\mathbb{X}_i$ and $\mathbb{K}^0$ do not depend on r_x or q_x we equate the coefficients of the q_x and r_x terms to zero. We therefore require the following equations are satisfied,

$$-if_x(\mathbb{X}_2+\mathbb{X}_4 r)-\mathbb{K}_q^0-if(\mathbb{X}_{2,x}+\mathbb{X}_{4,x}r)+if[\mathbb{X}_1,\mathbb{X}_2]+ifr[\mathbb{X}_1,\mathbb{X}_4]$$
$$-ifr[\mathbb{X}_2,\mathbb{X}_3]+ifr^2[\mathbb{X}_3,\mathbb{X}_4]=0, \tag{3.10}$$
$$if_x(\mathbb{X}_3+\mathbb{X}_4 q)-\mathbb{K}_r^0+if(\mathbb{X}_{3,x}+\mathbb{X}_{4,x}q)-if[\mathbb{X}_1,\mathbb{X}_3]-ifq[\mathbb{X}_1,\mathbb{X}_4]$$
$$-ifq[\mathbb{X}_2,\mathbb{X}_3]-ifq^2[\mathbb{X}_2,\mathbb{X}_4]=0. \tag{3.11}$$

Upon trying to integrate this system one finds that the system is in its current state inconsistent unless a consistency condition is satisfied. Recall that given a system of PDEs

$$\Psi_q=\xi(q,r) \text{ and } \Psi_r=\eta(q,r),$$

if we are to recover Ψ we must satisfy a consistency condition. That is, we must have $\xi_r=\Psi_{qr}=\Psi_{rq}=\eta_q$. In terms of equations (3.10) and (3.11) we have

$$\begin{aligned}\xi(q,r) &= -if_x(\mathbb{X}_2+\mathbb{X}_4 r)-if(\mathbb{X}_{2,x}+\mathbb{X}_{4,x}r)+if[\mathbb{X}_1,\mathbb{X}_2]+ifr[\mathbb{X}_1,\mathbb{X}_4]\\ &\quad -ifr[\mathbb{X}_2,\mathbb{X}_3]+ifr^2[\mathbb{X}_3,\mathbb{X}_4]=0 \qquad (3.12)\\ \eta(q,r) &= if_x(\mathbb{X}_3+\mathbb{X}_4 q)+if(\mathbb{X}_{3,x}+\mathbb{X}_{4,x}q)-if[\mathbb{X}_1,\mathbb{X}_3]-ifq[\mathbb{X}_1,\mathbb{X}_4]\\ &\quad -ifq[\mathbb{X}_2,\mathbb{X}_3]-ifq^2[\mathbb{X}_2,\mathbb{X}_4]=0 \qquad (3.13)\end{aligned}$$

Thus the consistency condition ($\xi_r=\eta_q$) requires that

$$-if_x\mathbb{X}_4 - if\mathbb{X}_{4,x} + if[\mathbb{X}_1,\mathbb{X}_4] - if[\mathbb{X}_2,\mathbb{X}_3] + 2if[\mathbb{X}_3,\mathbb{X}_4]r = if_x\mathbb{X}_4 + if\mathbb{X}_{4,x} - if[\mathbb{X}_1,\mathbb{X}_4]$$
$$-if[\mathbb{X}_2,\mathbb{X}_3] - 2if[\mathbb{X}_2,\mathbb{X}_4]q$$

hold. But this means we must have

$$2if_x\mathbb{X}_4 + 2if\mathbb{X}_{4,x} - 2if[\mathbb{X}_1,\mathbb{X}_4] - 2if[\mathbb{X}_3,\mathbb{X}_4](r+q) = 0. \tag{3.14}$$

One easy choice to make the system consistent, and for the purpose of demonstrating how this method can reproduce results previously obtained in the literature, is to set $\mathbb{X}_4 = 0$. Thus the system becomes

$$\mathbb{K}^0_q = -if_x\mathbb{X}_2 - if\mathbb{X}_{2,x} + if[\mathbb{X}_1,\mathbb{X}_2] - ifr[\mathbb{X}_2,\mathbb{X}_3], \tag{3.15}$$
$$\mathbb{K}^0_r = if_x\mathbb{X}_3 + if\mathbb{X}_{3,x} - if[\mathbb{X}_1,\mathbb{X}_3] - ifq[\mathbb{X}_2,\mathbb{X}_3]. \tag{3.16}$$

Integrating the first equation with respect to q we obtain

$$\mathbb{K}^0 = -if_x\mathbb{X}_2 q - if\mathbb{X}_{2,x}q + if[\mathbb{X}_1,\mathbb{X}_2]q - if[\mathbb{X}_2,\mathbb{X}_3]rq + \mathbb{K}^*(x,t,r).$$

Now differentiating this and requiring that it equal our previous expression for K^0_r we find that $\mathbb{K}^*$ must satisfy

$$\mathbb{K}_r^* = if_x\mathbb{X}_3 + if\mathbb{X}_{3,x} - if[\mathbb{X}_1, \mathbb{X}_3].$$

Integrating this expression with respect to r we easily find that

$$\mathbb{K}^* = if_x\mathbb{X}_3 r + if\mathbb{X}_{3,x} r - if[\mathbb{X}_1, \mathbb{X}_3]r + \mathbb{X}_0(x,t),$$

where $\mathbb{X}_0$ is an arbitrary matrix whose elements are functions of x and t and does not depend on q, r, or their partial derivatives. Finally, plugging this expression for $\mathbb{K}^*$ into our previous expression for $\mathbb{K}^0$ we have

$$\mathbb{K}^0 = if_x(\mathbb{X}_3 r - \mathbb{X}_2 q) + if(\mathbb{X}_{3,x} r - \mathbb{X}_{2,x} q) + if[\mathbb{X}_1, \mathbb{X}_2]q - if[\mathbb{X}_1, \mathbb{X}_3]r - if[\mathbb{X}_2, \mathbb{X}_3]qr + \mathbb{X}_0. \quad (3.17)$$

Plugging this into equation (3.9) we obtain

$$\mathbb{X}_{1,t}+\mathbb{X}_{2,t}q+\mathbb{X}_{3,t}r-i\mathbb{X}_3(gr^2q+(\upsilon-i\gamma)r)+i\mathbb{X}_2(gq^2r+(\upsilon+i\gamma)q)-if_{xx}(\mathbb{X}_3r-\mathbb{X}_2q)$$
$$-2if_x(\mathbb{X}_{3,x}r-\mathbb{X}_{2,x}q)-if(\mathbb{X}_{3,xx}r-\mathbb{X}_{2,xx}q)-i(f[\mathbb{X}_1,\mathbb{X}_2])_xq+i(f[\mathbb{X}_1,\mathbb{X}_3])_xr$$
$$-\mathbb{X}_{0,x}+if_x([\mathbb{X}_1,\mathbb{X}_3]r-[\mathbb{X}_1,\mathbb{X}_2]q)+if([\mathbb{X}_1,\mathbb{X}_{3,x}]r-[\mathbb{X}_1,\mathbb{X}_{2,x}]q)+if[\mathbb{X}_1,[\mathbb{X}_1,\mathbb{X}_2]]q$$
$$-if[\mathbb{X}_1,[\mathbb{X}_1,\mathbb{X}_3]]r-if[\mathbb{X}_1,[\mathbb{X}_2,\mathbb{X}_3]]qr+[\mathbb{X}_1,\mathbb{X}_0]+if_x[\mathbb{X}_2,\mathbb{X}_3]qr+i(f[\mathbb{X}_2,\mathbb{X}_3])_xqr$$
$$+if[\mathbb{X}_2,[\mathbb{X}_1,\mathbb{X}_2]]q^2-if[\mathbb{X}_2,[\mathbb{X}_1,\mathbb{X}_3]]qr-if[\mathbb{X}_2,[\mathbb{X}_2,\mathbb{X}_3]]q^2r+[\mathbb{X}_2,\mathbb{X}_0]q-if_x[\mathbb{X}_3,\mathbb{X}_2]rq$$
$$+if([\mathbb{X}_3,\mathbb{X}_{3,x}]r^2-[\mathbb{X}_3,\mathbb{X}_{2,x}]rq)+if[\mathbb{X}_3,[\mathbb{X}_1,\mathbb{X}_2]]rq-if[\mathbb{X}_3,[\mathbb{X}_1,\mathbb{X}_3]]r^2-if[\mathbb{X}_3,[\mathbb{X}_2,\mathbb{X}_3]]qr^2$$
$$+if([\mathbb{X}_2,\mathbb{X}_{3,x}]qr-[\mathbb{X}_2,\mathbb{X}_{2,x}]q^2)+[\mathbb{X}_3,\mathbb{X}_0]r=0. \tag{3.18}$$

Since the $\mathbb{X}_i$ are independent of r and q we equate the coefficients of the different powers of r and q to zero and thus obtain the following constraints at each order in r and q:

$$
\begin{aligned}
O(1) \quad &: \quad \mathbb{X}_{1,t} - \mathbb{X}_{0,x} + [\mathbb{X}_1, \mathbb{X}_0] = 0, && (3.19)\\
O(q) \quad &: \quad \mathbb{X}_{2,t} + i\mathbb{X}_2(\upsilon + i\gamma) + i(f\mathbb{X}_2)_{xx} - i(f[\mathbb{X}_1, \mathbb{X}_2])_x - if_x[\mathbb{X}_1, \mathbb{X}_2] - if[\mathbb{X}_1, \mathbb{X}_{2,x}] \\
&\qquad +if[\mathbb{X}_1, [\mathbb{X}_1, \mathbb{X}_2]] + [\mathbb{X}_2, \mathbb{X}_0] = 0, && (3.20)\\
O(r) \quad &: \quad \mathbb{X}_{3,t} - i\mathbb{X}_3(\upsilon - i\gamma) - i(f\mathbb{X}_3)_{xx} + i(f[\mathbb{X}_1, \mathbb{X}_3])_x + if_x[\mathbb{X}_1, \mathbb{X}_3] + if[\mathbb{X}_1, \mathbb{X}_{3,x}] \\
&\qquad -if[\mathbb{X}_1, [\mathbb{X}_1, \mathbb{X}_3]] + [\mathbb{X}_3, \mathbb{X}_0] = 0, && (3.21)\\
O(qr) \quad &: \quad 2i(f[\mathbb{X}_2, \mathbb{X}_3])_x - if[\mathbb{X}_1, [\mathbb{X}_2, \mathbb{X}_3]] + if_x[\mathbb{X}_2, \mathbb{X}_3] - if[\mathbb{X}_2, [\mathbb{X}_1, \mathbb{X}_3]] \\
&\qquad +if[\mathbb{X}_3, [\mathbb{X}_1, \mathbb{X}_2]] = 0, && (3.22)\\
O(q^2) \quad &: \quad if[\mathbb{X}_2, \mathbb{X}_{2,x}] - if[\mathbb{X}_2, [\mathbb{X}_1, \mathbb{X}_2]] = 0, && (3.23)\\
O(r^2) \quad &: \quad if[\mathbb{X}_3, \mathbb{X}_{3,x}] - if[\mathbb{X}_3, [\mathbb{X}_1, \mathbb{X}_3]] = 0, && (3.24)\\
O(q^2 r) \quad &: \quad ig\mathbb{X}_2 - if[\mathbb{X}_2, [\mathbb{X}_2, \mathbb{X}_3]] = 0, && (3.25)\\
O(r^2 q) \quad &: \quad ig\mathbb{X}_3 + if[\mathbb{X}_3, [\mathbb{X}_2, \mathbb{X}_3]] = 0. && (3.26)
\end{aligned}
$$

These equations collectively determine the conditions for the Lax-integrability of the system (2.3). Note that in general, as with the standard Estabrook-Wahlquist method, the solution to the above system is not unique. Moreover, as we will show, one finds that resolution of the system of equations derived in the process of requiring compatibility under the zero-curvature equation (such as the system above) cannot be done in general. Rather it will require additional compatibility conditions be satisfied between the coefficients in system (2.3). This result is not unique to this system but in fact typical of most systems (if not all). Provided we can find representations for the $\mathbb{X}_i$, and in doing so derive the necessary compatibility conditions between the coefficient functions in (2.3), we will obtain our Lax pair $\mathbb{F}$ and $\mathbb{G}$. We will now show how to reproduce the results given

in Khawaja's paper. Let us consider Khawaja's choices for the $\mathbb{X}_i$ matrices. That is, we take

$$\mathbb{X}_0 = \begin{bmatrix} g_1 & 0 \\ 0 & g_{13} \end{bmatrix}, \ \mathbb{X}_1 = \begin{bmatrix} f_1 & 0 \\ 0 & f_7 \end{bmatrix}, \ \mathbb{X}_2 = \begin{bmatrix} 0 & ip_1 \\ 0 & 0 \end{bmatrix}, \ \mathbb{X}_3 = \begin{bmatrix} 0 & 0 \\ -ip_2 & 0 \end{bmatrix}. \tag{3.27}$$

Plugging these into our integrability conditions yields

$$\begin{aligned}
O(1) \ &: \ f_{1t} - g_{1x} = 0, && (3.28)\\
O(1) \ &: \ f_{7t} - g_{13x} = 0, && (3.29)\\
O(q) \ &: \ ip_{1t} - ip_1(g_1 - g_{13} - i\upsilon + \gamma) - (fp_1)_{xx} + 2(f_1 - f_7)(p_1 f)_x \\
&\quad -fp_1(f_1 - f_7)^2 + fp_1(f_1 - f_7)_x = 0, && (3.30)\\
O(r) \ &: \ ip_{2t} + ip_2(g_1 - g_{13} - i\upsilon - \gamma) + (fp_2)_{xx} + 2(f_1 - f_7)(fp_2)_x \\
&\quad +(f_1 - f_7)^2 fp_2 + fp_2(f_1 - f_7)_x = 0, && (3.31)\\
O(qr) \ &: \ f_x p_1 p_2 + 2(fp_1 p_2)_x = 0, && (3.32)\\
O(q^2 r) \text{ and } O(r^2 q) \ &: \ g + 2fp_1 p_2 = 0. && (3.33)
\end{aligned}$$

Note that equations (3.23) and (3.24) were identically satisfied and thus omitted here. In Khawaja's paper we see equations (3.28), (3.29), (3.32) and (3.33) given exactly as they are above. To see that the other conditions are equivalent we note that in his paper he had the additional determining equations

$$(fp_1)_x - fp_1(f_1 - f_7) - g_6 = 0, \tag{3.34}$$

$$(fp_2)_x + fp_2(f_1 - f_7) - g_{10} = 0, \tag{3.35}$$

$$g_6(f_1 - f_7) - ip_1(g_1 - g_{13} - i\upsilon + \gamma) - g_{6x} + ip_{1t} = 0, \tag{3.36}$$

$$g_{10}(f_1 - f_7) + ip_2(g_1 - g_{13} - i\upsilon - \gamma) + g_{10x} + ip_{2t} = 0. \tag{3.37}$$

We begin by solving equations (3.34) and (3.35) for g_6 and g_{10}, respectively. Now plugging g_6 into equation (3.36) and g_{10} into equation (3.37) we obtain the system

$$\begin{aligned} &2(fp_1)_x(f_1 - f_7) - fp_1(f_1 - f_7)^2 - ip_1(g_1 - g_{13} - i\upsilon + \gamma) - (fp_1)_{xx} \\ &+fp_1(f_1 - f_7)_x + ip_{1t} = 0, \end{aligned} \tag{3.38}$$

$$\begin{aligned} &(fp_2)_x(f_1 - f_7) + fp_2(f_1 - f_7)^2 + ip_2(g_1 - g_{13} - i\upsilon - \gamma) + (fp_2)_{xx} \\ &+fp_2(f_1 - f_7)_x + ip_{2t} = 0, \end{aligned} \tag{3.39}$$

which are exactly equations (3.30) and (3.31). At this point the derivation of the final conditions on the coefficient functions is exactly as it was given in the previous chapter and thus will be omitted here. The Lax pair for this system is given by

$$\mathbb{F} = \mathbb{X}_1 + \mathbb{X}_2 q + \mathbb{X}_3 r \tag{3.40}$$

$$\mathbb{G} = if(\mathbb{X}_2 q_x - \mathbb{X}_3 r_x) + if_x(\mathbb{X}_3 r - \mathbb{X}_2 q) + if(\mathbb{X}_{3,x} r - \mathbb{X}_{2,x} q) + if[\mathbb{X}_1, \mathbb{X}_2]q - if[\mathbb{X}_1, \mathbb{X}_3]r$$
$$-if[\mathbb{X}_2, \mathbb{X}_3]qr + \mathbb{X}_0 \tag{3.41}$$

As the PT-symmetric NLS is a special case of the system considered above we may obtain the results through the necessary reductions. Letting $\gamma = v = 0$, $f = -a_1$, $g = -a_2$ and taking $r(x,t) = q^*(-x,t)$ where * denotes the complex conjugate we obtain

$$O(1) : \mathbb{X}_{1,t} - \mathbb{X}_{0,x} + [\mathbb{X}_1, \mathbb{X}_0] = 0, \tag{3.42}$$

$$O(q) : \mathbb{X}_{2,t} - i(a_1 \mathbb{X}_2)_{xx} + i(a_1[\mathbb{X}_1, \mathbb{X}_2])_x + ia_{1x}[\mathbb{X}_1, \mathbb{X}_2] + ia_1[\mathbb{X}_1, \mathbb{X}_{2,x}]$$
$$+if[\mathbb{X}_1, [\mathbb{X}_1, \mathbb{X}_2]] + [\mathbb{X}_2, \mathbb{X}_0] = 0, \tag{3.43}$$

$$O(r) : \mathbb{X}_{3,t} + i(a_1 \mathbb{X}_3)_{xx} - i(a_1[\mathbb{X}_1, \mathbb{X}_3])_x - ia_{1x}[\mathbb{X}_1, \mathbb{X}_3] - ia_1[\mathbb{X}_1, \mathbb{X}_{3,x}]$$
$$ia_1[\mathbb{X}_1, [\mathbb{X}_1, \mathbb{X}_3]] + [\mathbb{X}_3, \mathbb{X}_0] = 0, \tag{3.44}$$

$$O(qr) : -2i(a_1[\mathbb{X}_2, \mathbb{X}_3])_x + ia_1[\mathbb{X}_1, [\mathbb{X}_2, \mathbb{X}_3]] - ia_{1x}[\mathbb{X}_2, \mathbb{X}_3] + ia_1[\mathbb{X}_2, [\mathbb{X}_1, \mathbb{X}_3]]$$
$$-ia_1[\mathbb{X}_3, [\mathbb{X}_1, \mathbb{X}_2]] = 0, \tag{3.45}$$

$$O(q^2) : -ia_1[\mathbb{X}_2, \mathbb{X}_{2,x}] + ia_1[\mathbb{X}_2, [\mathbb{X}_1, \mathbb{X}_2]] = 0, \tag{3.46}$$

$$O(r^2) : -ia_1[\mathbb{X}_3, \mathbb{X}_{3,x}] + ia_1[\mathbb{X}_3, [\mathbb{X}_1, \mathbb{X}_3]] = 0, \tag{3.47}$$

$$O(q^2 r) : -ia_2 \mathbb{X}_2 + ia_1[\mathbb{X}_2, [\mathbb{X}_2, \mathbb{X}_3]] = 0, \tag{3.48}$$

$$O(r^2 q) : -ia_2 \mathbb{X}_3 - ia_1[\mathbb{X}_3, [\mathbb{X}_2, \mathbb{X}_3]] = 0. \tag{3.49}$$

Utilizing the same set of generators

$$\mathbb{X}_0 = \begin{bmatrix} g_1 & 0 \\ 0 & g_{13} \end{bmatrix}, \ \mathbb{X}_1 = \begin{bmatrix} f_1 & 0 \\ 0 & f_7 \end{bmatrix}, \ \mathbb{X}_2 = \begin{bmatrix} 0 & ip_1 \\ 0 & 0 \end{bmatrix}, \ \mathbb{X}_3 = \begin{bmatrix} 0 & 0 \\ -ip_2 & 0 \end{bmatrix}, \tag{3.50}$$

we have the following set of conditions

$$\begin{aligned}
O(1) \ &: \ f_{1t} - g_{1x} = 0, && (3.51)\\
O(1) \ &: \ f_{7t} - g_{13x} = 0, && (3.52)\\
O(q) \ &: \ ip_{1t} - ip_1(g_1 - g_{13}) + (a_1p_1)_{xx} - 2(f_1 - f_7)(p_1a_1)_x \\
&\quad + a_1p_1(f_1 - f_7)^2 - a_1p_1(f_1 - f_7)_x = 0, && (3.53)\\
O(r) \ &: \ ip_{2t} + ip_2(g_1 - g_{13}) - (a_1p_2)_{xx} - 2(f_1 - f_7)(a_1p_2)_x \\
&\quad - (f_1 - f_7)^2a_1p_2 - a_1p_2(f_1 - f_7)_x = 0, && (3.54)\\
O(qr) \ &: \ -a_{1x}p_1p_2 - 2(a_1p_1p_2)_x = 0, && (3.55)\\
O(q^2r) \text{ and } O(r^2q) \ &: \ a_2 + 2a_1p_1p_2 = 0. && (3.56)
\end{aligned}$$

As with the standard NLS, the final conditions on the coefficient functions can be found in the previous chapter. The Lax pair for this system is then given by

$$\begin{aligned}
\mathbb{F} &= \mathbb{X}_1 + \mathbb{X}_2q + \mathbb{X}_3r, && (3.57)\\
\mathbb{G} &= -ia_1(\mathbb{X}_2q_x - \mathbb{X}_3r_x) - ia_{1x}(\mathbb{X}_3r - \mathbb{X}_2q) - ia_1(\mathbb{X}_{3,x}r - \mathbb{X}_{2,x}q) - ia_1[\mathbb{X}_1, \mathbb{X}_2]q + ia_1[\mathbb{X}_1, \mathbb{X}_3]r \\
&\quad + ia_1[\mathbb{X}_2, \mathbb{X}_3]qr + \mathbb{X}_0. && (3.58)
\end{aligned}$$

Derivative Nonlinear Schrodinger Equation

In this section we derive the Lax pair and differential constraints for the variable-coefficient derivative nonlinear Schrödinger equation. The details of this example will be similar to that of the standard NLS and PT-symmetric NLS. Following the extended Estabrook-Wahlquist procedure as outlined at the beginning of the chapter we let

$$\mathbb{F} = \mathbb{F}(x, t, r, q), \quad \text{and} \quad \mathbb{G} = \mathbb{G}(x, t, r, q, r_x, q_x).$$

Plugging $\mathbb{F}$ and $\mathbb{G}$ as given above into equation (3.2) we obtain

$$\mathbb{F}_t + \mathbb{F}_q q_t + \mathbb{F}_r r_t - \mathbb{G}_x - \mathbb{G}_q q_x - \mathbb{G}_{q_x} q_{xx} - \mathbb{G}_r r_x - \mathbb{G}_{r_x} r_{xx} + [\mathbb{F}, \mathbb{G}] = 0. \tag{3.59}$$

Now using substituting for q_t and r_t using equations (2.42) and (2.43), respectively, we obtain the following equation,

$$\begin{aligned} &\mathbb{F}_t + (ia_1\mathbb{F}_q - \mathbb{G}_{q_x})q_{xx} - (ia_1\mathbb{F}_r + \mathbb{G}_{r_x})r_{xx} - \mathbb{F}_q(2a_2 rqq_x + a_2 q^2 r_x) \\ &-\mathbb{F}_r(2a_2 qrr_x + a_2 r^2 q_x) - \mathbb{G}_x - \mathbb{G}_q q_x - \mathbb{G}_r r_x + [\mathbb{F}, \mathbb{G}] = 0. \end{aligned} \tag{3.60}$$

Since $\mathbb{F}$ and $\mathbb{G}$ do not depend on q_{xx} or r_{xx} we can set the coefficients of the q_{xx} and r_{xx} terms to

zero. This requires $\mathbb{F}$ and $\mathbb{G}$ satisfy the equations

$$ia_1\mathbb{F}_q - \mathbb{G}_{q_x} = 0 \text{ and } ia_1\mathbb{F}_r + \mathbb{G}_{r_x} = 0. \tag{3.61}$$

Solving this in the same manner as in the NLS example we obtain

$$\mathbb{G} = ia_1(\mathbb{F}_q q_x - \mathbb{F}_r r_x) + \mathbb{K}^0(x,t,r,q). \tag{3.62}$$

Plugging this expression for $\mathbb{G}$ into equation (3.60) we obtain

$$\begin{aligned}&\mathbb{F}_t - \mathbb{F}_q(2a_2rqq_x + a_2q^2r_x) - \mathbb{F}_r(2a_2qrr_x + a_2r^2q_x) - i(a_1\mathbb{F}_q)_x q_x + i(a_1\mathbb{F}_r)_x r_x - \mathbb{K}^0_x \\ &+ia_1\mathbb{F}_{rr}r_x^2 - \mathbb{K}^0_r r_x - ia_1\mathbb{F}_{qq}q_x^2 - \mathbb{K}^0_q q_x + ia_1[\mathbb{F},\mathbb{F}_q]q_x - ia_1[\mathbb{F},\mathbb{F}_r]r_x + [\mathbb{F},\mathbb{K}^0] = 0.\end{aligned} \tag{3.63}$$

Now since $\mathbb{F}$ and $\mathbb{K}^0$ do not depend on q_x or r_x we can set the coefficients of the different powers of r_x and q_x to zero. Setting the coefficients of the q_x^2 and r_x^2 terms to zero we obtain the conditions

$$-ia_1\mathbb{F}_{qq} = ia_1\mathbb{F}_{rr} = 0. \tag{3.64}$$

From these conditions it follows that $\mathbb{F}$ must be of the form $\mathbb{F} = \mathbb{X}_1(x,t) + \mathbb{X}_2(x,t)r + \mathbb{X}_3(x,t)q + \mathbb{X}_4(x,t)qr$ where the $\mathbb{X}_i$ are matrices whose elements are functions of x and t. As with the previous examples these matrices are independent of q, r, and their partial derivatives. Now setting the coefficients of the q_x and r_x terms to zero we obtain the following conditions,

$$-a_2\mathbb{X}_3q^2 - a_2\mathbb{X}_4q^2r + i(a_1\mathbb{X}_2)_x + i(a_1\mathbb{X}_4)_xq - \mathbb{K}_r^0 - ia_1[\mathbb{X}_1,\mathbb{X}_2] - ia_1[\mathbb{X}_1,\mathbb{X}_4]q$$
$$-ia_1[\mathbb{X}_3,\mathbb{X}_2]q - ia_1[\mathbb{X}_3,\mathbb{X}_4]q^2 - 2a_2\mathbb{X}_2rq - 2a_2\mathbb{X}_4q^2r = 0, \tag{3.65}$$
$$-a_2\mathbb{X}_2r^2 - a_2\mathbb{X}_4r^2q - i(a_1\mathbb{X}_3)_x - i(a_1\mathbb{X}_4)_xr - \mathbb{K}_q^0 + ia_1[\mathbb{X}_1,\mathbb{X}_3] + ia_1[\mathbb{X}_1,\mathbb{X}_4]r$$
$$+ia_1[\mathbb{X}_2,\mathbb{X}_3]r + ia_1[\mathbb{X}_2,\mathbb{X}_4]r^2 - 2a_2\mathbb{X}_3rq - 2a_2\mathbb{X}_4r^2q = 0. \tag{3.66}$$

In much the same way as for the NLS we denote the left-hand side of (3.65) as $\xi(r,q)$ and the left-hand side of (3.66) as $\eta(r,q)$. For recovery of $\mathbb{K}^0$ we require that $\xi_q = \eta_r$. Computing ξ_q and η_r we find

$$\begin{aligned}\xi_q &= -2a_2\mathbb{X}_3q - 2a_2\mathbb{X}_4qr + i(a_1\mathbb{X}_4)_x - ia_1[\mathbb{X}_1,\mathbb{X}_4] - ia_1[\mathbb{X}_3,\mathbb{X}_2] - 2ia_1[\mathbb{X}_3,\mathbb{X}_4]q \\ &\quad -2a_2\mathbb{X}_2r - 4a_2\mathbb{X}_4qr, \qquad (3.67)\\ \eta_r &= -2a_2\mathbb{X}_2r - 2a_2\mathbb{X}_4rq - i(a_1\mathbb{X}_4)_x + ia_1[\mathbb{X}_1,\mathbb{X}_4] + ia_1[\mathbb{X}_2,\mathbb{X}_3] + 2ia_1[\mathbb{X}_2,\mathbb{X}_4]r \\ &\quad -2a_2\mathbb{X}_3q - 4a_2\mathbb{X}_4rq. \qquad (3.68)\end{aligned}$$

As with the NLS, consistency requires ξ_q and η_r be equal. This is equivalent to the condition

$$2i(a_1\mathbb{X}_4)_x - 2ia_1[\mathbb{X}_1,\mathbb{X}_4] - 2ia_1[\mathbb{X}_3,\mathbb{X}_4]q - 2ia_1[\mathbb{X}_2,\mathbb{X}_4]r = 0. \tag{3.69}$$

Since the $\mathbb{X}_i$ do not depend on q or r this previous condition requires

$$2i(a_1\mathbb{X}_4)_x - 2ia_1[\mathbb{X}_1, \mathbb{X}_4] = 0, \tag{3.70}$$

$$-2ia_1[\mathbb{X}_3, \mathbb{X}_4] = 0, \tag{3.71}$$

$$-2ia_1[\mathbb{X}_2, \mathbb{X}_4] = 0. \tag{3.72}$$

Once again in following with the standard NLS we will take $\mathbb{X}_4 = 0$ in order to simplify computations. Therefore we have

$$\mathbb{K}^0_q = -a_2\mathbb{X}_2 r^2 - i(a_1\mathbb{X}_3)_x + ia_1[\mathbb{X}_1, \mathbb{X}_3] + ia_1[\mathbb{X}_2, \mathbb{X}_3]r - 2a_2\mathbb{X}_3 rq, \tag{3.73}$$

$$\mathbb{K}^0_r = -a_2\mathbb{X}_3 q^2 + i(a_1\mathbb{X}_2)_x - ia_1[\mathbb{X}_1, \mathbb{X}_2] - ia_1[\mathbb{X}_3, \mathbb{X}_2]q - 2a_2\mathbb{X}_2 rq. \tag{3.74}$$

Integrating the first equation with respect to q yields

$$\mathbb{K}^0 = -a_2\mathbb{X}_2 r^2 q - i(a_1\mathbb{X}_3)_x q + ia_1[\mathbb{X}_1, \mathbb{X}_3]q + ia_1[\mathbb{X}_2, \mathbb{X}_3]rq - a_2\mathbb{X}_3 q^2 r + \mathbb{K}^*(x, t, r).$$

Differentiating this equation with respect to r and requiring that it equal our previous expression for $\mathbb{K}^0_r$ we find that $\mathbb{K}^*$ must satisfy

$$\mathbb{K}^*_r = i(a_1\mathbb{X}_2)_x - ia_1[\mathbb{X}_1, \mathbb{X}_2].$$

From this it follows

$$\mathbb{K}^* = i(a_1\mathbb{X}_2)_x r - ia_1[\mathbb{X}_1, \mathbb{X}_2]r + \mathbb{X}_0(x,t),$$

where $\mathbb{X}_0$ a matrix whose elements are functions of x and t and which does not depend on q, r, or their partial derivatives. Plugging this expression for $\mathbb{K}^*$ into our previous expression for $\mathbb{K}^0$ we obtain the following final expression for $\mathbb{K}^0$,

$$\begin{aligned}\mathbb{K}^0 &= i(a_1\mathbb{X}_2)_x r - i(a_1\mathbb{X}_3)_x q - ia_1[\mathbb{X}_1, \mathbb{X}_2]r + ia_1[\mathbb{X}_1, \mathbb{X}_3]q + ia_1[\mathbb{X}_2, \mathbb{X}_3]rq - a_2\mathbb{X}_2 r^2 q \\ &\quad -a_2\mathbb{X}_3 q^2 r + \mathbb{X}_0(x,t). \end{aligned} \tag{3.75}$$

Now plugging this and our expression for $\mathbb{F}$ into equation (3.63) we get

$$\begin{aligned}&\mathbb{X}_{1,t} + \mathbb{X}_{2,t}r + \mathbb{X}_{3,t}q - i(a_1\mathbb{X}_2)_{xx}r + i(a_1\mathbb{X}_3)_{xx}q + i(a_1[\mathbb{X}_1, \mathbb{X}_2])_x r - i(a_1[\mathbb{X}_1, \mathbb{X}_3])_x q \\ &-i(a_1[\mathbb{X}_2, \mathbb{X}_3])_x rq + (a_2\mathbb{X}_2)_x r^2 q + (a_2\mathbb{X}_3)_x q^2 r - \mathbb{X}_{0,x} + i[\mathbb{X}_1, (a_1\mathbb{X}_2)_x]r - [\mathbb{X}_1, (a_1\mathbb{X}_3)_x]q \\ &-ia_1[\mathbb{X}_1, [\mathbb{X}_1, \mathbb{X}_2]]r + ia_1[\mathbb{X}_1, [\mathbb{X}_1, \mathbb{X}_3]]q + ia_1[\mathbb{X}_1, [\mathbb{X}_2, \mathbb{X}_3]]rq - a_2[\mathbb{X}_1, \mathbb{X}_2]r^2 q - a_2[\mathbb{X}_1, \mathbb{X}_3]q^2 r \\ &+[\mathbb{X}_1, \mathbb{X}_0] + i[\mathbb{X}_2, (a_1\mathbb{X}_2)_x]r^2 - i[\mathbb{X}_2, (a_1\mathbb{X}_3)_x]rq - ia_1[\mathbb{X}_2, [\mathbb{X}_1, \mathbb{X}_2]]r^2 + ia_1[\mathbb{X}_2, [\mathbb{X}_1, \mathbb{X}_3]]rq \\ &+ia_1[\mathbb{X}_2, [\mathbb{X}_2, \mathbb{X}_3]]r^2 q + [\mathbb{X}_2, \mathbb{X}_0]r + i[\mathbb{X}_3, (a_1\mathbb{X}_2)_x]rq - i[\mathbb{X}_3, (a_1\mathbb{X}_3)_x]q^2 - ia_1[\mathbb{X}_3, [\mathbb{X}_1, \mathbb{X}_2]]rq \\ &+ia_1[\mathbb{X}_3, [\mathbb{X}_1, \mathbb{X}_3]]q^2 + ia_1[\mathbb{X}_3, [\mathbb{X}_2, \mathbb{X}_3]]q^2 r + [\mathbb{X}_3, \mathbb{X}_0]q = 0. \end{aligned} \tag{3.76}$$

Since the $\mathbb{X}_i$ are independent of r and q we equate the coefficients of the different powers of r and q to zero thereby obtaining the following final constraints:

$$O(1) \;:\; \mathbb{X}_{1,t} - \mathbb{X}_{0,x} + [\mathbb{X}_1, \mathbb{X}_0] = 0, \tag{3.77}$$

$$O(q) \;:\; \mathbb{X}_{3,t} + i(a_1\mathbb{X}_3)_{xx} - i(a_1[\mathbb{X}_1, \mathbb{X}_3])_x - i[\mathbb{X}_1, (a_1\mathbb{X}_3)_x] + ia_1[\mathbb{X}_1, [\mathbb{X}_1, \mathbb{X}_3]] + [\mathbb{X}_3, \mathbb{X}_0] = 0, \tag{3.78}$$

$$O(r) \;:\; \mathbb{X}_{2,t} - i(a_1\mathbb{X}_2)_{xx} + i(a_1[\mathbb{X}_1, \mathbb{X}_2])_x + i[\mathbb{X}_1, (a_1\mathbb{X}_2)_x] - ia_1[\mathbb{X}_1, [\mathbb{X}_1, \mathbb{X}_2]] + [\mathbb{X}_2, \mathbb{X}_0] = 0, \tag{3.79}$$

$$O(rq) \;:\; -(a_1[\mathbb{X}_2, \mathbb{X}_3])_x + a_1[\mathbb{X}_1, [\mathbb{X}_2, \mathbb{X}_3]] - [\mathbb{X}_2, (a_1\mathbb{X}_3)_x] + a_1[\mathbb{X}_2, [\mathbb{X}_1, \mathbb{X}_3]] + [\mathbb{X}_3, (a_1\mathbb{X}_2)_x] - a_1[\mathbb{X}_3, [\mathbb{X}_1, \mathbb{X}_2]] = 0, \tag{3.80}$$

$$O(q^2) \;:\; -[\mathbb{X}_3, (a_1\mathbb{X}_3)_x] + a_1[\mathbb{X}_3, [\mathbb{X}_1, \mathbb{X}_3]] = 0, \tag{3.81}$$

$$O(r^2) \;:\; [\mathbb{X}_2, (a_1\mathbb{X}_2)_x] - a_1[\mathbb{X}_2, [\mathbb{X}_1, \mathbb{X}_2]] = 0, \tag{3.82}$$

$$O(r^2q) \;:\; (a_2\mathbb{X}_2)_x - a_2[\mathbb{X}_1, \mathbb{X}_2] + ia_1[\mathbb{X}_2, [\mathbb{X}_2, \mathbb{X}_3]] = 0, \tag{3.83}$$

$$O(q^2r) \;:\; (a_2\mathbb{X}_3)_x - a_2[\mathbb{X}_1, \mathbb{X}_3] + ia_1[\mathbb{X}_3, [\mathbb{X}_2, \mathbb{X}_3]] = 0. \tag{3.84}$$

We take the following forms for the generators

$$\mathbb{X}_0 = \begin{bmatrix} g_1 & g_2 \\ g_3 & g_4 \end{bmatrix}, \;\; \mathbb{X}_1 = \begin{bmatrix} f_1 & 0 \\ 0 & f_2 \end{bmatrix}, \;\; \mathbb{X}_2 = \begin{bmatrix} 0 & f_3 \\ 0 & 0 \end{bmatrix}, \;\; \mathbb{X}_3 = \begin{bmatrix} 0 & 0 \\ f_4 & 0 \end{bmatrix}, \tag{3.85}$$

where the f_i and g_j are yet to be determined functions of x and t. Note that with this choice the (3.81) and (3.82) equations are immediately satisfed. From equations (3.78) and (3.79) we obtain the following conditions,

$$g_3 f_4 = g_2 f_3 = 0, \tag{3.86}$$

$$f_{4t} + i(a_1 f_4)_{xx} - i(a_1 f_4(f_1 - f_2))_x + (f_2 - f_1)(a_1 f_4)_x + i a_1 f_4 (f_1 - f_2)^2$$

$$+ f_4(g_4 - g_1) = 0, \tag{3.87}$$

$$f_{3t} - i(a_1 f_3)_{xx} - i(a_1 f_3(f_1 - f_2))_x + (f_2 - f_1)(a_1 f_3)_x - i a_1 f_3 (f_1 - f_2)^2$$

$$- f_3(g_4 - g_1) = 0. \tag{3.88}$$

To keep $\mathbb{X}_2$ and $\mathbb{X}_3$ nonzero (and thus obtain nontrivial results) we force $g_2 = g_3 = 0$. The condition given by equation (3.80) reduces to the single equation

$$(a_1 f_3 f_4)_x + f_3(a_1 f_4)_x + f_4(a_1 f_3)_x = 0. \tag{3.89}$$

The final two conditions now yield the system

$$(a_2 f_3)_x - a_2 f_3(f_2 - f_1) - 2i a_1 f_3^2 f_4 = 0, \tag{3.90}$$

$$(a_2 f_4)_x + a_2 f_4(f_2 - f_1) + 2i a_1 f_4^2 f_3 = 0. \tag{3.91}$$

At this point the resolution of the system given by equations (3.77) and (3.87) - (3.91) such that the a_i are real-valued functions requires either $f_3 = 0$ or $f_4 = 0$. Without loss of generality we choose $f_3 = 0$ from which we obtain the new system of equations

$$f_{1t} - g_{1x} = 0, \tag{3.92}$$

$$f_{2t} - g_{4x} = 0, \tag{3.93}$$

$$f_{4t} + i(a_1 f_4)_{xx} - i(a_1 f_4(f_1 - f_2))_x + (f_2 - f_1)(a_1 f_4)_x + i a_1 f_4 (f_1 - f_2)^2$$

$$+ f_4(g_4 - g_1) = 0, \tag{3.94}$$

$$(a_2 f_4)_x + a_2 f_4 (f_2 - f_1) = 0. \tag{3.95}$$

Solving equations (3.92), (3.94) and (3.95) for f_1, g_4, and f_2, respectively, we obtain

$$f_1 = \int g_{1x} dt + F_1(x), \tag{3.96}$$

$$f_2 = -\frac{(a_2 f_4)_x}{a_2 f_4} + \int g_{1x} dt + F_1(x), \tag{3.97}$$

$$g_4 = \frac{-i a_2^2 f_4 a_{1xx} + i a_1 a_2 f_4 a_{2xx} - 2i a_1 a_{2x}^2 f_4 + 2i a_2 a_{2x} a_{1x} f_4 - f_{4t} a_2^2}{a_2^2 f_4} + g_1, \tag{3.98}$$

where $F_1(x)$ is an arbitrary function of x. Plugging these expressions into equation (3.93) yields the integrability condition

$$a_2^3 a_{1xxx} - i a_{2t} a_{2x} a_2 + i a_{2xt} a_2^2 - 3 a_2^2 a_{2xx} a_{1x} - 4 a_{2x}^3 a_1 + 5 a_1 a_2 a_{2x} a_{2xx} + 4 a_{2x}^2 a_2 a_{1x}$$

$$- a_2^2 a_1 a_{2xxx} - 2 a_{2x} a_2^2 a_{1xx} = 0. \tag{3.99}$$

Since we require that the a_i be real this equation splits into the conditions

$$a_{2t}a_{2x} - a_{2xt}a_2 = 0, \tag{3.100}$$

$$a_2^3 a_{1xxx} - 3a_2^2 a_{2xx}a_{1x} - 4a_{2x}^3 a_1 + 5a_1 a_2 a_{2x}a_{2xx} + 4a_{2x}^2 a_2 a_{1x}$$

$$-a_2^2 a_1 a_{2xxx} - 2a_{2x}a_2^2 a_{1xx} = 0. \tag{3.101}$$

With the aid of MAPLE we find that the previous system is exactly solvable with solution given by

$$a_1(x,t) = F_4(t)F_2(x)\left(c_1 + c_2 x - c_1 \int \frac{x\,dx}{F_2(x)} + c_1 x \int \frac{dx}{F_2(x)}\right), \tag{3.102}$$

$$a_2(x,t) = F_2(x)F_3(t), \tag{3.103}$$

where F_2, F_3, and F_4 are arbitrary functions in their respective variables and c_1 and c_2 are arbitrary constants. The Lax pair for this system is then found to be

$$\mathbb{F} = \mathbb{X}_1 + \mathbb{X}_3 q, \tag{3.104}$$

$$\mathbb{G} = ia_1\mathbb{X}_3 q_x - i(a_1\mathbb{X}_3)_x q + ia_1[\mathbb{X}_1, \mathbb{X}_3]q - a_2\mathbb{X}_3 q^2 r + \mathbb{X}_0. \tag{3.105}$$

Fifth-Order Korteweg-de-Vries Equation

In this section we derive the Lax pair and differential constraints for the generalized fifth-order variable-coefficient KdV. Following the procedure outlined earlier in the chapter we let

$$\mathbb{F} = \mathbb{F}(x, t, u) \text{ and } \mathbb{G} = \mathbb{G}(x, t, u, u_x, u_{xx}, u_{xxx}, u_{xxxx}).$$

Plugging $\mathbb{F}$ and $\mathbb{G}$ into equation (3.2) we obtain

$$\mathbb{F}_t + \mathbb{F}_u u_t - \mathbb{G}_x - \mathbb{G}_u u_x - \mathbb{G}_{u_x} u_{xx} - \mathbb{G}_{u_{xx}} u_{xxx} - \mathbb{G}_{u_{xxx}} u_{xxxx} - \mathbb{G}_{u_{xxxx}} u_{xxxxx} + [\mathbb{F}, \mathbb{G}] = 0. \quad (3.106)$$

Next, substituting for u_t in the previous expression using equation (2.66) we obtain the equation

$$\begin{aligned} &\mathbb{F}_t - \mathbb{F}_u \left(a_1 u u_{xxx} + a_2 u_x u_{xx} + a_3 u^2 u_x + a_4 u u_x + a_5 u_{xxx} + a_6 u_{xxxxx} + a_7 u + a_8 u_x \right) - \mathbb{G}_x \\ &- \mathbb{G}_u u_x - \mathbb{G}_{u_x} u_{xx} - \mathbb{G}_{u_{xx}} u_{xxx} - \mathbb{G}_{u_{xxx}} u_{xxxx} - \mathbb{G}_{u_{xxxx}} u_{xxxxx} + [\mathbb{F}, \mathbb{G}] = 0. \end{aligned} \quad (3.107)$$

Since $\mathbb{F}$ and $\mathbb{G}$ do not depend on u_{xxxxx} we can equate the coefficient of the u_{xxxxx} term to zero. This requires that $\mathbb{F}$ and $\mathbb{G}$ satisfy

$$\mathbb{G}_{u_{xxxx}} + a_6 \mathbb{F}_u = 0 \Rightarrow \mathbb{G} = -a_6 \mathbb{F}_u u_{xxxx} + \mathbb{K}^0(x, t, u, u_x, u_{xx}, u_{xxx}).$$

Updating equation (3.107) using the above expression for $\mathbb{G}$ we obtain the equation

$$\mathbb{F}_t - \mathbb{F}_u \left(a_1 u u_{xxx} + a_2 u_x u_{xx} + a_3 u^2 u_x + a_4 u u_x + a_5 u_{xxx} + a_7 u + a_8 u_x\right) + a_{6x}\mathbb{F}_u u_{xxxx}$$
$$+a_6 \mathbb{F}_{xu} u_{xxxx} - \mathbb{K}^0_x - \mathbb{K}^0_u u_x - \mathbb{K}^0_{u_x} u_{xx} - \mathbb{K}^0_{u_{xx}} u_{xxx} - \mathbb{K}^0_{u_{xxx}} u_{xxxx} + a_6 \mathbb{F}_{uu} u_x u_{xxxx}$$
$$-[\mathbb{F}, \mathbb{F}_u] a_6 u_{xxxx} + [\mathbb{F}, \mathbb{K}^0] = 0. \tag{3.108}$$

Since $\mathbb{F}$ and $\mathbb{K}^0$ do not depend on u_{xxxx} we can equate the coefficient of the u_{xxxx} term to zero. This requires that $\mathbb{K}^0$ satisfies

$$a_{6x}\mathbb{F}_u + a_6 \mathbb{F}_{xu} + a_6 \mathbb{F}_{uu} u_x - \mathbb{K}^0_{u_{xxx}} - [\mathbb{F}, \mathbb{F}_u] a_6 = 0. \tag{3.109}$$

Integrating with respect to u_{xxx} and solving for $\mathbb{K}^0$ we obtain

$$\mathbb{K}^0 = a_{6x}\mathbb{F}_u u_{xxx} + a_6 \mathbb{F}_{xu} u_{xxx} + a_6 \mathbb{F}_{uu} u_x u_{xxx} - [\mathbb{F}, \mathbb{F}_u] a_6 u_{xxx} + \mathbb{K}^1(x, t, u, u_x, u_{xx}). \tag{3.110}$$

Updating equation (3.108) by plugging in our expression for $\mathbb{K}^0$ we obtain the equation

$$\mathbb{F}_t - \mathbb{F}_u\left(a_1uu_{xxx} + a_2u_xu_{xx} + a_3u^2u_x + a_4uu_x + a_5u_{xxx} + a_7u + a_8u_x\right) - a_{6xx}\mathbb{F}_uu_{xxx}$$
$$-2a_{6x}\mathbb{F}_{xu}u_{xxx} - a_6\mathbb{F}_{xxu}u_{xxx} - a_{6x}\mathbb{F}_{uu}u_xu_{xxx} - a_6\mathbb{F}_{xuu}u_xu_{xxx} + [\mathbb{F}_x, \mathbb{F}_u]a_6u_{xxx}$$
$$+[\mathbb{F}, \mathbb{F}_{xu}]a_6u_{xxx} + [\mathbb{F}, \mathbb{F}_u]a_{6x}u_{xxx} - \mathbb{K}^1_x - a_{6x}\mathbb{F}_{uu}u_xu_{xxx} - a_6\mathbb{F}_{xuu}u_xu_{xxx} - \mathbb{K}^1_uu_x$$
$$-a_6\mathbb{F}_{uuu}u_x^2u_{xxx} + [\mathbb{F}, \mathbb{F}_{uu}]a_6u_xu_{xxx} - a_6\mathbb{F}_{uu}u_{xx}u_{xxx} - \mathbb{K}^1_{u_x}u_{xx} - \mathbb{K}^1_{u_{xx}}u_{xxx} + [\mathbb{F}, \mathbb{K}^1]$$
$$+a_{6x}[\mathbb{F}, \mathbb{F}_u]u_{xxx} + a_6[\mathbb{F}, \mathbb{F}_{xu}]u_{xxx} + a_6[\mathbb{F}, \mathbb{F}_{uu}]u_xu_{xxx} - [\mathbb{F}, [\mathbb{F}, \mathbb{F}_u]]a_6u_{xxx} = 0. \qquad (3.111)$$

Since $\mathbb{F}$ and $\mathbb{K}^1$ do not depend on u_{xxx} we can equate the coefficient of the u_{xxx} term to zero. This requires that $\mathbb{K}^1$ satisfies the equation

$$-\mathbb{F}_u(a_1u + a_5) - a_{6xx}\mathbb{F}_u - 2a_{6x}\mathbb{F}_{xu} - a_6\mathbb{F}_{xxu} - a_{6x}\mathbb{F}_{uu}u_x - a_6\mathbb{F}_{xuu}u_x$$
$$+[\mathbb{F}_x, \mathbb{F}_u]a_6 + [\mathbb{F}, \mathbb{F}_{xu}]a_6 + [\mathbb{F}, \mathbb{F}_u]a_{6x} - a_{6x}\mathbb{F}_{uu}u_x - a_6\mathbb{F}_{xuu}u_x - a_6\mathbb{F}_{uuu}u_x^2$$
$$+[\mathbb{F}, \mathbb{F}_{uu}]a_6u_x - a_6\mathbb{F}_{uu}u_{xx} - \mathbb{K}^1_{u_{xx}} + a_{6x}[\mathbb{F}, \mathbb{F}_u] + a_6[\mathbb{F}, \mathbb{F}_{xu}] + a_6[\mathbb{F}, \mathbb{F}_{uu}]u_x$$
$$-[\mathbb{F}, [\mathbb{F}, \mathbb{F}_u]]a_6 = 0. \qquad (3.112)$$

Integrating with respect to u_{xx} and solving for $\mathbb{K}^1$ and collecting like terms we have

$$\begin{aligned}\mathbb{K}^1 \;=\; & -\mathbb{F}_u(a_1u + a_5)u_{xx} - (a_6\mathbb{F}_u)_{xx}u_{xx} - 2(a_6\mathbb{F}_{uu})_xu_xu_{xx} + 2(a_6[\mathbb{F}, \mathbb{F}_u])_xu_{xx} \\ & -a_6\mathbb{F}_{uuu}u_x^2u_{xx} + 2a_6[\mathbb{F}, \mathbb{F}_{uu}]u_xu_{xx} - \frac{1}{2}a_6\mathbb{F}_{uu}u_{xx}^2 - a_6[\mathbb{F}_x, \mathbb{F}_u]u_{xx} \\ & -a_6[\mathbb{F}, [\mathbb{F}, \mathbb{F}_u]]u_{xx} + \mathbb{K}^2(x, t, u, u_x). \qquad (3.113)\end{aligned}$$

Plugging the expression for $\mathbb{K}^1$ into equation (3.111) and simplifying a little bit we obtain the large equation

$$\begin{aligned}
&\mathbb{F}_t - \mathbb{F}_u(a_2u_xu_{xx} + a_3u^2u_x + a_4uu_x + a_7u + a_8u_x) + (a_1\mathbb{F}_u)_xuu_{xx} + (a_5\mathbb{F}_u)_xu_{xx} \\
&+(a_6\mathbb{F}_u)_{xxx}u_{xx} + 2(a_6\mathbb{F}_{uu})_{xx}u_xu_{xx} - (a_6[\mathbb{F},\mathbb{F}_u])_{xx}u_{xx} + (a_6\mathbb{F}_{uuu})_xu_x^2u_{xx} \\
&+\frac{1}{2}(a_6\mathbb{F}_{uu})_xu_{xx}^2 - ([\mathbb{F},(a_6\mathbb{F}_u)_x])_xu_{xx} + (a_6[\mathbb{F},[\mathbb{F},\mathbb{F}_u]])_xu_{xx} - \mathbb{K}_x^2 + \mathbb{F}_{uu}a_1uu_xu_{xx} \\
&+\mathbb{F}_ua_1u_xu_{xx} + \mathbb{F}_{uu}a_5u_xu_{xx} + (a_6\mathbb{F}_{uu})_{xx}u_xu_{xx} + 2(a_6\mathbb{F}_{uuu})_xu_x^2u_{xx} + a_6\mathbb{F}_{uuuu}u_x^3u_{xx} \\
&-a_6[\mathbb{F}_u,\mathbb{F}_{uu}]u_x^2u_{xx} - a_6[\mathbb{F},\mathbb{F}_{uuu}]u_x^2u_{xx} + \frac{5}{2}a_6\mathbb{F}_{uuu}u_{xx}^2u_x - [\mathbb{F}_u,(a_6\mathbb{F}_u)_x]u_xu_{xx} \\
&-2[\mathbb{F},(a_6\mathbb{F}_{uu})_x]u_xu_{xx} - a_6[\mathbb{F}_u,\mathbb{F}_{uu}]u_x^2u_{xx} - a_6[\mathbb{F},\mathbb{F}_{uuu}]u_x^2u_{xx} + a_6[\mathbb{F}_u,[\mathbb{F},\mathbb{F}_u]]u_xu_{xx} \\
&+a_6[\mathbb{F},[\mathbb{F},\mathbb{F}_{uu}]]u_xu_{xx} - \mathbb{K}_u^2u_x + 2(a_6\mathbb{F}_{uu})_xu_{xx}^2 - \frac{3}{2}a_6[\mathbb{F},\mathbb{F}_{uu}]u_{xx}^2 - \mathbb{K}_{u_x}^2u_{xx} \\
&-a_5[\mathbb{F},\mathbb{F}_u]u_{xx} - [\mathbb{F},(a_6\mathbb{F}_u)_{xx}]u_{xx} - [\mathbb{F},(a_6\mathbb{F}_{uu})_x]u_xu_{xx} + [\mathbb{F},(a_6[\mathbb{F},\mathbb{F}_u])_x]u_{xx} \\
&-a_6[\mathbb{F},\mathbb{F}_{uuu}]u_x^2u_{xx} + 2a_6[\mathbb{F},[\mathbb{F},\mathbb{F}_{uu}]]u_xu_{xx} + [\mathbb{F},[\mathbb{F},(a_6\mathbb{F}_u)_x]]u_{xx} - a_1[\mathbb{F},\mathbb{F}_u]uu_{xx} \\
&-a_6[\mathbb{F},[\mathbb{F},[\mathbb{F},\mathbb{F}_u]]]u_{xx} - 3(a_6[\mathbb{F},\mathbb{F}_{uu}])_xu_xu_{xx} + [\mathbb{F},\mathbb{K}^2] = 0. \qquad (3.114)
\end{aligned}$$

Since $\mathbb{K}^2$ and $\mathbb{F}$ do not depend on u_{xx} we can start by setting the coefficients of the u_{xx}^2 and the u_{xx} terms to zero. Note the difference here to that of the previous steps. Here we have multiple powers of u_{xx} present in equation (3.114). Setting the $O(u_{xx}^2)$ term to zero we obtain the condition

$$\frac{3}{2}(a_6\mathbb{F}_{uu})_x + \frac{5}{2}a_6\mathbb{F}_{uuu}u_x - \frac{3}{2}a_6[\mathbb{F},\mathbb{F}_{uu}] = 0. \qquad (3.115)$$

Since $\mathbb{F}$ does not depend on u_x we must additionally require that the coefficient of the u_x term in this previous expression is zero. This is equivalent to $\mathbb{F}$ satisfying

$$\mathbb{F}_{uuu} = 0 \Rightarrow \mathbb{F} = \mathbb{X}_1(x,t) + \mathbb{X}_2(x,t)u + \frac{1}{2}\mathbb{X}_3(x,t)u^2,$$

where the $\mathbb{X}_i$ are matrices whose elements are functions of x and t and do not depend on u or its partial derivatives. Plugging this expression for $\mathbb{F}$ into equation (3.115) we obtain the condition

$$3(a_6\mathbb{X}_3)_x - 3a_6([\mathbb{X}_1, \mathbb{X}_3] + [\mathbb{X}_2, \mathbb{X}_3]u) = 0. \tag{3.116}$$

Since the $\mathbb{X}_i$ do not depend on u we can set the coefficient of the u term to zero. That is, we require that $\mathbb{X}_2$ and $\mathbb{X}_3$ commute. We find now that equation (3.116) reduces to

$$(a_6\mathbb{X}_3)_x - a_6[\mathbb{X}_1, \mathbb{X}_3] = 0. \tag{3.117}$$

For ease of computation and in order to immediately satisfy equation (3.117) we set $\mathbb{X}_3 = 0$. Plugging our expression for $\mathbb{F}$ into equation (3.114) we obtain the large equation

$$\mathbb{X}_{1,t}+\mathbb{X}_{2,t}u-\mathbb{X}_2(a_2u_xu_{xx}+a_3u^2u_x+a_4uu_x+a_7u+a_8u_x)+(a_1\mathbb{X}_2)_xuu_{xx}+(a_5\mathbb{X}_2)_xu_{xx}$$
$$+(a_6\mathbb{X}_2)_{xxx}u_{xx}-(a_6[\mathbb{X}_1,\mathbb{X}_2])_{xx}u_{xx}-\mathbb{K}_x^2+\mathbb{X}_2a_1u_xu_{xx}-[\mathbb{X}_2,(a_6\mathbb{X}_2)_x]u_xu_{xx}$$
$$-([\mathbb{X}_1,(a_6\mathbb{X}_2)_x])_xu_{xx}-([\mathbb{X}_2,(a_6\mathbb{X}_2)_x])_xuu_{xx}+(a_6[\mathbb{X}_1,[\mathbb{X}_1,\mathbb{X}_2]])_xu_{xx}+(a_6[\mathbb{X}_2,[\mathbb{X}_1,\mathbb{X}_2]])_xuu_{xx}$$
$$-\mathbb{K}_u^2u_x-\mathbb{K}_{u_x}^2u_{xx}-a_1[\mathbb{X}_1,\mathbb{X}_2]uu_{xx}+a_6[\mathbb{X}_2,[\mathbb{X}_1,\mathbb{X}_2]]u_xu_{xx}+[\mathbb{X}_2,(a_6[\mathbb{X}_1,\mathbb{X}_2])_x]uu_{xx}$$
$$-a_5[\mathbb{X}_1,\mathbb{X}_2]u_{xx}-[\mathbb{X}_1,(a_6\mathbb{X}_2)_{xx}]u_{xx}-[\mathbb{X}_2,(a_6\mathbb{X}_2)_{xx}]uu_{xx}+[\mathbb{X}_1,(a_6[\mathbb{X}_1,\mathbb{X}_2])_x]u_{xx}$$
$$+[\mathbb{X}_1,[\mathbb{X}_1,(a_6\mathbb{X}_2)_x]]u_{xx}+[\mathbb{X}_1,[\mathbb{X}_2,(a_6\mathbb{X}_2)_x]]uu_{xx}+[\mathbb{X}_2,[\mathbb{X}_1,(a_6\mathbb{X}_2)_x]]uu_{xx}+[\mathbb{X}_1,\mathbb{K}^2]$$
$$-a_6[\mathbb{X}_1,[\mathbb{X}_1,[\mathbb{X}_1,\mathbb{X}_2]]]u_{xx}-a_6[\mathbb{X}_1,[\mathbb{X}_2,[\mathbb{X}_1,\mathbb{X}_2]]]uu_{xx}-a_6[\mathbb{X}_2,[\mathbb{X}_1,[\mathbb{X}_1,\mathbb{X}_2]]]uu_{xx}$$
$$+[\mathbb{X}_2,[\mathbb{X}_2,(a_6\mathbb{X}_2)_x]]u^2u_{xx}-a_6[\mathbb{X}_2,[\mathbb{X}_2,[\mathbb{X}_1,\mathbb{X}_2]]]u^2u_{xx}+[\mathbb{X}_2,\mathbb{K}^2]u=0. \tag{3.118}$$

Again using the fact that the $\mathbb{X}_i$ and $\mathbb{K}^2$ do not depend on u_{xx} we can set the coefficient of the u_{xx} term in equation (3.118) to zero. This requires

$$(a_6\mathbb{X}_2)_{xxx}-(a_6[\mathbb{X}_1,\mathbb{X}_2])_{xx}+a_1\mathbb{X}_2u_x-[\mathbb{X}_2,(a_6\mathbb{X}_2)_x]u_x-a_2\mathbb{X}_2u_x$$
$$-([\mathbb{X}_1,(a_6\mathbb{X}_2)_x])_x-([\mathbb{X}_2,(a_6\mathbb{X}_2)_x])_xu+(a_6[\mathbb{X}_1,[\mathbb{X}_1,\mathbb{X}_2]])_x+(a_6[\mathbb{X}_2,[\mathbb{X}_1,\mathbb{X}_2]])_xu$$
$$-\mathbb{K}_{u_x}^2-a_1[\mathbb{X}_1,\mathbb{X}_2]u+a_6[\mathbb{X}_2,[\mathbb{X}_1,\mathbb{X}_2]]u_x+[\mathbb{X}_2,(a_6[\mathbb{X}_1,\mathbb{X}_2])_x]u+(a_5\mathbb{X}_2)_x$$
$$-a_5[\mathbb{X}_1,\mathbb{X}_2]-[\mathbb{X}_1,(a_6\mathbb{X}_2)_{xx}]-[\mathbb{X}_2,(a_6\mathbb{X}_2)_{xx}]u+[\mathbb{X}_1,(a_6[\mathbb{X}_1,\mathbb{X}_2])_x]$$
$$+[\mathbb{X}_1,[\mathbb{X}_1,(a_6\mathbb{X}_2)_x]]+[\mathbb{X}_1,[\mathbb{X}_2,(a_6\mathbb{X}_2)_x]]u+[\mathbb{X}_2,[\mathbb{X}_1,(a_6\mathbb{X}_2)_x]]u+(a_1\mathbb{X}_2)_xu$$
$$-a_6[\mathbb{X}_1,[\mathbb{X}_1,[\mathbb{X}_1,\mathbb{X}_2]]]-a_6[\mathbb{X}_1,[\mathbb{X}_2,[\mathbb{X}_1,\mathbb{X}_2]]]u-a_6[\mathbb{X}_2,[\mathbb{X}_1,[\mathbb{X}_1,\mathbb{X}_2]]]u$$
$$+[\mathbb{X}_2,[\mathbb{X}_2,(a_6\mathbb{X}_2)_x]]u^2-a_6[\mathbb{X}_2,[\mathbb{X}_2,[\mathbb{X}_1,\mathbb{X}_2]]]u^2=0. \tag{3.119}$$

Integrating equation (3.119) with respect to u_x and solving for $\mathbb{K}^2$ we find

$$\begin{aligned}
\mathbb{K}^2 \;=\;& (a_6\mathbb{X}_2)_{xxx}u_x + \frac{1}{2}a_1\mathbb{X}_2u_x^2 - \frac{1}{2}[\mathbb{X}_2,(a_6\mathbb{X}_2)_x]u_x^2 - \frac{1}{2}a_2\mathbb{X}_2u_x^2 + (a_6[\mathbb{X}_2,[\mathbb{X}_1,\mathbb{X}_2]])_xuu_x \\
&-(a_6[\mathbb{X}_1,\mathbb{X}_2])_{xx}u_x - ([\mathbb{X}_1,(a_6\mathbb{X}_2)_x])_xu_x - ([\mathbb{X}_2,(a_6\mathbb{X}_2)_x])_xuu_x + (a_6[\mathbb{X}_1,[\mathbb{X}_1,\mathbb{X}_2]])_xu_x \\
&-a_1[\mathbb{X}_1,\mathbb{X}_2]uu_x + \frac{1}{2}a_6[\mathbb{X}_2,[\mathbb{X}_1,\mathbb{X}_2]]u_x^2 + [\mathbb{X}_2,(a_6[\mathbb{X}_1,\mathbb{X}_2])_x]uu_x + (a_1\mathbb{X}_2)_xuu_x \\
&-a_5[\mathbb{X}_1,\mathbb{X}_2]u_x - [\mathbb{X}_1,(a_6\mathbb{X}_2)_{xx}]u_x - [\mathbb{X}_2,(a_6\mathbb{X}_2)_{xx}]uu_x + [\mathbb{X}_1,(a_6[\mathbb{X}_1,\mathbb{X}_2])_x]u_x \\
&+[\mathbb{X}_1,[\mathbb{X}_1,(a_6\mathbb{X}_2)_x]]u_x + [\mathbb{X}_1,[\mathbb{X}_2,(a_6\mathbb{X}_2)_x]]uu_x + [\mathbb{X}_2,[\mathbb{X}_1,(a_6\mathbb{X}_2)_x]]uu_x + (a_5\mathbb{X}_2)_xu_x \\
&-a_6[\mathbb{X}_1,[\mathbb{X}_1,[\mathbb{X}_1,\mathbb{X}_2]]]u_x - a_6[\mathbb{X}_1,[\mathbb{X}_2,[\mathbb{X}_1,\mathbb{X}_2]]]uu_x - a_6[\mathbb{X}_2,[\mathbb{X}_1,[\mathbb{X}_1,\mathbb{X}_2]]]uu_x \\
&+[\mathbb{X}_2,[\mathbb{X}_2,(a_6\mathbb{X}_2)_x]]u^2u_x - a_6[\mathbb{X}_2,[\mathbb{X}_2,[\mathbb{X}_1,\mathbb{X}_2]]]u^2u_x + \mathbb{K}^3(x,t,u). \qquad (3.120)
\end{aligned}$$

It is helpful at this stage to define the following new matrices

$$\mathbb{X}_4 = [\mathbb{X}_1,\mathbb{X}_2],\;\; \mathbb{X}_5 = [\mathbb{X}_1,\mathbb{X}_4],\;\; \mathbb{X}_6 = [\mathbb{X}_2,\mathbb{X}_4] \qquad (3.121)$$

$$\mathbb{X}_7 = [\mathbb{X}_1,\mathbb{X}_5],\;\; \mathbb{X}_8 = [\mathbb{X}_2,\mathbb{X}_5],\;\; \mathbb{X}_9 = [\mathbb{X}_1,\mathbb{X}_6],\;\; \mathbb{X}_{10} = [\mathbb{X}_2,\mathbb{X}_6] \qquad (3.122)$$

for clarity and in order to reduce the size of the equations to follow. Plugging the expression for $\mathbb{K}^2$ into equation (3.118) we obtain the updated equation

$$\mathbb{X}_{1,t}+\mathbb{X}_{2,t}u-\mathbb{X}_2(a_3u^2u_x+a_4uu_x+a_7u+a_8u_x)-(a_6\mathbb{X}_6)_xu_x^2+a_1\mathbb{X}_4u_x^2$$
$$-(a_6\mathbb{X}_2)_{xxxx}u_x-\frac{1}{2}(a_1\mathbb{X}_2)_xu_x^2+\frac{1}{2}([\mathbb{X}_2,(a_6\mathbb{X}_2)_x])_xu_x^2+\frac{1}{2}(a_2\mathbb{X}_2)_xu_x^2$$
$$+(a_6\mathbb{X}_4)_{xxx}u_x+([\mathbb{X}_1,(a_6\mathbb{X}_2)_x])_{xx}u_x+([\mathbb{X}_2,(a_6\mathbb{X}_2)_x])_{xx}uu_x-(a_6\mathbb{X}_5)_{xx}u_x$$
$$+(a_1\mathbb{X}_4)_xuu_x-\frac{1}{2}(a_6\mathbb{X}_6)_xu_x^2-([\mathbb{X}_2,(a_6\mathbb{X}_4)_x])_xuu_x-(a_5\mathbb{X}_2)_{xx}u_x-(a_1\mathbb{X}_2)_{xx}uu_x$$
$$+(a_5\mathbb{X}_4)_xu_x+([\mathbb{X}_1,(a_6\mathbb{X}_2)_{xx}])_xu_x+([\mathbb{X}_2,(a_6\mathbb{X}_2)_{xx}])_xuu_x-([\mathbb{X}_1,(a_6\mathbb{X}_4)_x])_xu_x$$
$$-([\mathbb{X}_1,[\mathbb{X}_1,(a_6\mathbb{X}_2)_x]])_xu_x-([\mathbb{X}_1,[\mathbb{X}_2,(a_6\mathbb{X}_2)_x]])_xuu_x-([\mathbb{X}_2,[\mathbb{X}_1,(a_6\mathbb{X}_2)_x]])_xuu_x$$
$$-([\mathbb{X}_2,[\mathbb{X}_2,(a_6\mathbb{X}_2)_x]])_xu^2u_x+(a_6\mathbb{X}_{10})_xu^2u_x-\mathbb{K}_x^3+([\mathbb{X}_2,(a_6\mathbb{X}_2)_x])_xu_x^2+a_6\mathbb{X}_9u_x^2$$
$$-[\mathbb{X}_2,(a_6\mathbb{X}_4)_x]u_x^2+[\mathbb{X}_2,(a_6\mathbb{X}_2)_{xx}]u_x^2+a_6\mathbb{X}_8u_x^2+(a_6\mathbb{X}_9)_xuu_x+(a_6\mathbb{X}_8)_xuu_x$$
$$-[\mathbb{X}_1,[\mathbb{X}_2,(a_6\mathbb{X}_2)_x]]u_x^2-[\mathbb{X}_2,[\mathbb{X}_1,(a_6\mathbb{X}_2)_x]]u_x^2-(a_1\mathbb{X}_2)_xu_x^2-(a_6\mathbb{X}_6)_{xx}uu_x$$
$$-2[\mathbb{X}_2,[\mathbb{X}_2,(a_6\mathbb{X}_2)_x]]uu_x^2+2a_6\mathbb{X}_{10}uu_x^2-\mathbb{K}_u^3u_x+[\mathbb{X}_1,(a_6\mathbb{X}_5)_x]u_x-\frac{1}{2}a_2\mathbb{X}_4u_x^2$$
$$+[\mathbb{X}_1,(a_6\mathbb{X}_2)_{xxx}]u_x+\frac{1}{2}a_1\mathbb{X}_4u_x^2-\frac{1}{2}[\mathbb{X}_1,[\mathbb{X}_2,(a_6\mathbb{X}_2)_x]]u_x^2+[\mathbb{X}_1,(a_6\mathbb{X}_6)_x]uu_x$$
$$-[\mathbb{X}_1,(a_6\mathbb{X}_4)_{xx}]u_x-[\mathbb{X}_1,([\mathbb{X}_1,(a_6\mathbb{X}_2)_x])_x]u_x-[\mathbb{X}_1,([\mathbb{X}_2,(a_6\mathbb{X}_2)_x])_x]uu_x$$
$$-a_1\mathbb{X}_5uu_x+\frac{1}{2}a_6\mathbb{X}_9u_x^2+[\mathbb{X}_1,[\mathbb{X}_2,(a_6\mathbb{X}_4)_x]]uu_x+[\mathbb{X}_1,(a_5\mathbb{X}_2)_x]u_x+(a_6\mathbb{X}_7)_xu_x$$
$$-a_5\mathbb{X}_5u_x-[\mathbb{X}_1,[\mathbb{X}_1,(a_6\mathbb{X}_2)_{xx}]]u_x-[\mathbb{X}_1,[\mathbb{X}_2,(a_6\mathbb{X}_2)_{xx}]]uu_x+[\mathbb{X}_1,[\mathbb{X}_1,(a_6\mathbb{X}_4)_x]]u_x$$
$$+[\mathbb{X}_1,[\mathbb{X}_1,[\mathbb{X}_1,(a_6\mathbb{X}_2)_x]]]u_x+[\mathbb{X}_1,[\mathbb{X}_1,[\mathbb{X}_2,(a_6\mathbb{X}_2)_x]]]uu_x+[\mathbb{X}_1,[\mathbb{X}_2,[\mathbb{X}_1,(a_6\mathbb{X}_2)_x]]]uu_x$$
$$-a_6[\mathbb{X}_1,\mathbb{X}_7]u_x-a_6[\mathbb{X}_1,\mathbb{X}_9]uu_x-a_6[\mathbb{X}_1,\mathbb{X}_8]uu_x+[\mathbb{X}_2,[\mathbb{X}_1,(a_6\mathbb{X}_4)_x]]uu_x$$
$$+[\mathbb{X}_1,[\mathbb{X}_2,[\mathbb{X}_2,(a_6\mathbb{X}_2)_x]]]u^2u_x-a_6[\mathbb{X}_1,\mathbb{X}_{10}]u^2u_x+[\mathbb{X}_1,\mathbb{K}^3]+[\mathbb{X}_1,(a_1\mathbb{X}_2)_x]uu_x$$
$$+[\mathbb{X}_2,(a_6\mathbb{X}_2)_{xxx}]uu_x-\frac{1}{2}[\mathbb{X}_2,[\mathbb{X}_2,(a_6\mathbb{X}_2)_x]]uu_x^2+[\mathbb{X}_2,(a_6\mathbb{X}_6)_x]u^2u_x+[\mathbb{X}_2,(a_6\mathbb{X}_5)_x]uu_x$$
$$-[\mathbb{X}_2,(a_6\mathbb{X}_4)_{xx}]uu_x-[\mathbb{X}_2,([\mathbb{X}_1,(a_6\mathbb{X}_2)_x])_x]uu_x-[\mathbb{X}_2,([\mathbb{X}_2,(a_6\mathbb{X}_2)_x])_x]u^2u_x$$
$$-a_1\mathbb{X}_6u^2u_x+\frac{1}{2}a_6\mathbb{X}_{10}uu_x^2+[\mathbb{X}_2,[\mathbb{X}_2,(a_6\mathbb{X}_4)_x]]u^2u_x+[\mathbb{X}_2,(a_5\mathbb{X}_2)_x]uu_x$$
$$-a_5\mathbb{X}_6uu_x-[\mathbb{X}_2,[\mathbb{X}_1,(a_6\mathbb{X}_2)_{xx}]]uu_x-[\mathbb{X}_2,[\mathbb{X}_2,(a_6\mathbb{X}_2)_{xx}]]u^2u_x-a_6[\mathbb{X}_2,\mathbb{X}_8]u^2u_x$$

$$+[\mathbb{X}_2,[\mathbb{X}_1,[\mathbb{X}_1,(a_6\mathbb{X}_2)_x]]]uu_x+[\mathbb{X}_2,[\mathbb{X}_1,[\mathbb{X}_2,(a_6\mathbb{X}_2)_x]]]u^2u_x+[\mathbb{X}_2,(a_1\mathbb{X}_2)_x]u^2u_x$$
$$+[\mathbb{X}_2,[\mathbb{X}_2,[\mathbb{X}_1,(a_6\mathbb{X}_2)_x]]]u^2u_x-a_6[\mathbb{X}_2,\mathbb{X}_7]uu_x-a_6[\mathbb{X}_2,\mathbb{X}_9]u^2u_x-a_6[\mathbb{X}_2,\mathbb{X}_{10}]u^3u_x$$
$$+[\mathbb{X}_2,[\mathbb{X}_2,[\mathbb{X}_2,(a_6\mathbb{X}_2)_x]]]u^3u_x+[\mathbb{X}_2,\mathbb{K}^3]u=0. \tag{3.123}$$

Now since $\mathbb{K}^3$ and the $\mathbb{X}_i$ do not depend on u_x we can set the coefficient of the u_x^2 term to zero in equation (3.123). This is equivalent to the requirement that

$$3([\mathbb{X}_2,(a_6\mathbb{X}_2)_x])_x-3(a_1\mathbb{X}_2)_x+(a_2\mathbb{X}_2)_x-3(a_6\mathbb{X}_6)_x+3a_1\mathbb{X}_4+3a_6\mathbb{X}_9$$
$$-3[\mathbb{X}_1,[\mathbb{X}_2,(a_6\mathbb{X}_2)_x]]-2[\mathbb{X}_2,[\mathbb{X}_1,(a_6\mathbb{X}_2)_x]]-2[\mathbb{X}_2,(a_6\mathbb{X}_4)_x]+2[\mathbb{X}_2,(a_6\mathbb{X}_2)_{xx}]$$
$$-a_2\mathbb{X}_4+2a_6\mathbb{X}_8+5a_6\mathbb{X}_{10}-5[\mathbb{X}_2,[\mathbb{X}_2,(a_6\mathbb{X}_2)_x]]=0. \tag{3.124}$$

Further since we know that the $\mathbb{X}_i$ do not depend on u we can decouple this condition into the system

$$3([\mathbb{X}_2,(a_6\mathbb{X}_2)_x])_x-3(a_1\mathbb{X}_2)_x+(a_2\mathbb{X}_2)_x-3(a_6\mathbb{X}_6)_x+3a_1\mathbb{X}_4+3a_6\mathbb{X}_9$$
$$-3[\mathbb{X}_1,[\mathbb{X}_2,(a_6\mathbb{X}_2)_x]]-2[\mathbb{X}_2,[\mathbb{X}_1,(a_6\mathbb{X}_2)_x]]-2[\mathbb{X}_2,(a_6\mathbb{X}_4)_x]+2[\mathbb{X}_2,(a_6\mathbb{X}_2)_{xx}]$$
$$-a_2\mathbb{X}_4+2a_6\mathbb{X}_8=0, \tag{3.125}$$
$$a_6\mathbb{X}_{10}-[\mathbb{X}_2,[\mathbb{X}_2,(a_6\mathbb{X}_2)_x]]=0. \tag{3.126}$$

Taking these conditions into account and once again noting the fact that $\mathbb{K}^3$ and the $\mathbb{X}_i$ do not depend on u_x we can simplify and equate the coefficient of the u_x in equation (3.123) to zero. This results in the condition

$$
\begin{aligned}
&-\mathbb{X}_2(a_3u^2+a_4u+a_8)-(a_6\mathbb{X}_2)_{xxxx}-\mathbb{K}^3_u+[\mathbb{X}_1,(a_6\mathbb{X}_5)_x]\\
&+(a_6\mathbb{X}_4)_{xxx}+([\mathbb{X}_1,(a_6\mathbb{X}_2)_x])_{xx}+([\mathbb{X}_2,(a_6\mathbb{X}_2)_x])_{xx}u-(a_6\mathbb{X}_5)_{xx}\\
&+(a_1\mathbb{X}_4)_xu-([\mathbb{X}_2,(a_6\mathbb{X}_4)_x])_xu-(a_5\mathbb{X}_2)_{xx}-(a_1\mathbb{X}_2)_{xx}u-a_1\mathbb{X}_6u^2\\
&+(a_5\mathbb{X}_4)_x+([\mathbb{X}_1,(a_6\mathbb{X}_2)_{xx}])_x+([\mathbb{X}_2,(a_6\mathbb{X}_2)_{xx}])_xu-([\mathbb{X}_1,(a_6\mathbb{X}_4)_x])_x\\
&-([\mathbb{X}_1,[\mathbb{X}_1,(a_6\mathbb{X}_2)_x]])_x-([\mathbb{X}_1,[\mathbb{X}_2,(a_6\mathbb{X}_2)_x]])_xu-([\mathbb{X}_2,[\mathbb{X}_1,(a_6\mathbb{X}_2)_x]])_xu\\
&+(a_6\mathbb{X}_9)_xu+(a_6\mathbb{X}_8)_xu-(a_6\mathbb{X}_6)_{xx}u+[\mathbb{X}_1,(a_6\mathbb{X}_2)_{xxx}]+[\mathbb{X}_1,(a_6\mathbb{X}_6)_x]u\\
&-[\mathbb{X}_1,(a_6\mathbb{X}_4)_{xx}]-[\mathbb{X}_1,([\mathbb{X}_1,(a_6\mathbb{X}_2)_x])_x]-[\mathbb{X}_1,([\mathbb{X}_2,(a_6\mathbb{X}_2)_x])_x]u\\
&-a_1\mathbb{X}_5u+[\mathbb{X}_1,[\mathbb{X}_2,(a_6\mathbb{X}_4)_x]]u+[\mathbb{X}_1,(a_5\mathbb{X}_2)_x]+(a_6\mathbb{X}_7)_x+[\mathbb{X}_2,[\mathbb{X}_2,(a_6\mathbb{X}_4)_x]]u^2\\
&-a_5\mathbb{X}_5-[\mathbb{X}_1,[\mathbb{X}_1,(a_6\mathbb{X}_2)_{xx}]]-[\mathbb{X}_1,[\mathbb{X}_2,(a_6\mathbb{X}_2)_{xx}]]u+[\mathbb{X}_1,[\mathbb{X}_1,(a_6\mathbb{X}_4)_x]]\\
&+[\mathbb{X}_1,[\mathbb{X}_1,[\mathbb{X}_1,(a_6\mathbb{X}_2)_x]]]+[\mathbb{X}_1,[\mathbb{X}_1,[\mathbb{X}_2,(a_6\mathbb{X}_2)_x]]]u+[\mathbb{X}_1,[\mathbb{X}_2,[\mathbb{X}_1,(a_6\mathbb{X}_2)_x]]]u\\
&-a_6[\mathbb{X}_1,\mathbb{X}_7]-a_6[\mathbb{X}_1,\mathbb{X}_9]u-a_6[\mathbb{X}_1,\mathbb{X}_8]u+[\mathbb{X}_2,[\mathbb{X}_1,(a_6\mathbb{X}_4)_x]]u\\
&+[\mathbb{X}_2,(a_6\mathbb{X}_2)_{xxx}]u+[\mathbb{X}_2,(a_6\mathbb{X}_6)_x]u^2+[\mathbb{X}_2,(a_6\mathbb{X}_5)_x]u+[\mathbb{X}_1,(a_1\mathbb{X}_2)_x]u\\
&-[\mathbb{X}_2,(a_6\mathbb{X}_4)_{xx}]u-[\mathbb{X}_2,([\mathbb{X}_1,(a_6\mathbb{X}_2)_x])_x]u-[\mathbb{X}_2,([\mathbb{X}_2,(a_6\mathbb{X}_2)_x])_x]u^2\\
&-a_5\mathbb{X}_6u-[\mathbb{X}_2,[\mathbb{X}_1,(a_6\mathbb{X}_2)_{xx}]]u-[\mathbb{X}_2,[\mathbb{X}_2,(a_6\mathbb{X}_2)_{xx}]]u^2+[\mathbb{X}_2,(a_5\mathbb{X}_2)_x]u\\
&+[\mathbb{X}_2,[\mathbb{X}_1,[\mathbb{X}_1,(a_6\mathbb{X}_2)_x]]]u+[\mathbb{X}_2,[\mathbb{X}_1,[\mathbb{X}_2,(a_6\mathbb{X}_2)_x]]]u^2+[\mathbb{X}_2,(a_1\mathbb{X}_2)_x]u^2\\
&+[\mathbb{X}_2,[\mathbb{X}_2,[\mathbb{X}_1,(a_6\mathbb{X}_2)_x]]]u^2-a_6[\mathbb{X}_2,\mathbb{X}_7]u-a_6[\mathbb{X}_2,\mathbb{X}_9]u^2-a_6[\mathbb{X}_2,\mathbb{X}_8]u^2=0. \quad (3.127)
\end{aligned}
$$

Integrating with respect to u and solving for $\mathbb{K}^3$ in equation (3.127) we find

$$
\begin{aligned}
\mathbb{K}^3 \;=\;& -\frac{1}{3}a_3\mathbb{X}_2u^3-\frac{1}{2}a_4\mathbb{X}_2u^2-a_8\mathbb{X}_2u-(a_6\mathbb{X}_2)_{xxxx}u+[\mathbb{X}_1,(a_6\mathbb{X}_5)_x]u-\frac{1}{2}a_6[\mathbb{X}_2,\mathbb{X}_7]u^2\\
&+(a_6\mathbb{X}_4)_{xxx}u+([\mathbb{X}_1,(a_6\mathbb{X}_2)_x])_{xx}u+\frac{1}{2}([\mathbb{X}_2,(a_6\mathbb{X}_2)_x])_{xx}u^2-(a_6\mathbb{X}_5)_{xx}u\\
&+\frac{1}{2}(a_1\mathbb{X}_4)_xu^2-\frac{1}{2}([\mathbb{X}_2,(a_6\mathbb{X}_4)_x])_xu^2-(a_5\mathbb{X}_2)_{xx}u-\frac{1}{2}(a_1\mathbb{X}_2)_{xx}u^2-\frac{1}{3}a_1\mathbb{X}_6u^3\\
&+(a_5\mathbb{X}_4)_xu+([\mathbb{X}_1,(a_6\mathbb{X}_2)_{xx}])_xu+\frac{1}{2}([\mathbb{X}_2,(a_6\mathbb{X}_2)_{xx}])_xu^2-([\mathbb{X}_1,(a_6\mathbb{X}_4)_x])_xu\\
&-([\mathbb{X}_1,[\mathbb{X}_1,(a_6\mathbb{X}_2)_x]])_xu-\frac{1}{2}([\mathbb{X}_1,[\mathbb{X}_2,(a_6\mathbb{X}_2)_x]])_xu^2-\frac{1}{2}([\mathbb{X}_2,[\mathbb{X}_1,(a_6\mathbb{X}_2)_x]])_xu^2\\
&+\frac{1}{2}(a_6\mathbb{X}_9)_xu^2+\frac{1}{2}(a_6\mathbb{X}_8)_xu^2-\frac{1}{2}(a_6\mathbb{X}_6)_{xx}u^2+[\mathbb{X}_1,(a_6\mathbb{X}_2)_{xxx}]u+\frac{1}{2}[\mathbb{X}_1,(a_6\mathbb{X}_6)_x]u^2\\
&-[\mathbb{X}_1,(a_6\mathbb{X}_4)_{xx}]u-[\mathbb{X}_1,([\mathbb{X}_1,(a_6\mathbb{X}_2)_x])_x]u-\frac{1}{2}[\mathbb{X}_1,([\mathbb{X}_2,(a_6\mathbb{X}_2)_x])_x]u^2-\frac{1}{2}a_1\mathbb{X}_5u^2\\
&+\frac{1}{2}[\mathbb{X}_1,[\mathbb{X}_2,(a_6\mathbb{X}_4)_x]]u^2+[\mathbb{X}_1,(a_5\mathbb{X}_2)_x]u+(a_6\mathbb{X}_7)_xu+\frac{1}{3}[\mathbb{X}_2,[\mathbb{X}_2,(a_6\mathbb{X}_4)_x]]u^3\\
&-a_5\mathbb{X}_5u-[\mathbb{X}_1,[\mathbb{X}_1,(a_6\mathbb{X}_2)_{xx}]]u-\frac{1}{2}[\mathbb{X}_1,[\mathbb{X}_2,(a_6\mathbb{X}_2)_{xx}]]u^2+[\mathbb{X}_1,[\mathbb{X}_1,(a_6\mathbb{X}_4)_x]]u\\
&+[\mathbb{X}_1,[\mathbb{X}_1,[\mathbb{X}_1,(a_6\mathbb{X}_2)_x]]]u+\frac{1}{2}[\mathbb{X}_1,[\mathbb{X}_1,[\mathbb{X}_2,(a_6\mathbb{X}_2)_x]]]u^2+\frac{1}{2}[\mathbb{X}_1,[\mathbb{X}_2,[\mathbb{X}_1,(a_6\mathbb{X}_2)_x]]]u^2\\
&-a_6[\mathbb{X}_1,\mathbb{X}_7]u-\frac{1}{2}a_6[\mathbb{X}_1,\mathbb{X}_9]u^2-\frac{1}{2}a_6[\mathbb{X}_1,\mathbb{X}_8]u^2+\frac{1}{2}[\mathbb{X}_2,[\mathbb{X}_1,(a_6\mathbb{X}_4)_x]]u^2\\
&+\frac{1}{2}[\mathbb{X}_2,(a_6\mathbb{X}_2)_{xxx}]u^2+\frac{1}{3}[\mathbb{X}_2,(a_6\mathbb{X}_6)_x]u^3+\frac{1}{2}[\mathbb{X}_2,(a_6\mathbb{X}_5)_x]u^2+\frac{1}{2}[\mathbb{X}_1,(a_1\mathbb{X}_2)_x]u^2\\
&-\frac{1}{2}[\mathbb{X}_2,(a_6\mathbb{X}_4)_{xx}]u^2-\frac{1}{2}[\mathbb{X}_2,([\mathbb{X}_1,(a_6\mathbb{X}_2)_x])_x]u^2-\frac{1}{3}[\mathbb{X}_2,([\mathbb{X}_2,(a_6\mathbb{X}_2)_x])_x]u^3\\
&-\frac{1}{2}a_5\mathbb{X}_6u^2-\frac{1}{2}[\mathbb{X}_2,[\mathbb{X}_1,(a_6\mathbb{X}_2)_{xx}]]u^2-\frac{1}{3}[\mathbb{X}_2,[\mathbb{X}_2,(a_6\mathbb{X}_2)_{xx}]]u^3+\frac{1}{2}[\mathbb{X}_2,(a_5\mathbb{X}_2)_x]u^2\\
&+\frac{1}{2}[\mathbb{X}_2,[\mathbb{X}_1,[\mathbb{X}_1,(a_6\mathbb{X}_2)_x]]]u^2+\frac{1}{3}[\mathbb{X}_2,[\mathbb{X}_1,[\mathbb{X}_2,(a_6\mathbb{X}_2)_x]]]u^3+\frac{1}{3}[\mathbb{X}_2,(a_1\mathbb{X}_2)_x]u^3\\
&+\frac{1}{3}[\mathbb{X}_2,[\mathbb{X}_2,[\mathbb{X}_1,(a_6\mathbb{X}_2)_x]]]u^3-\frac{1}{3}a_6[\mathbb{X}_2,\mathbb{X}_9]u^3-\frac{1}{3}a_6[\mathbb{X}_2,\mathbb{X}_8]u^3+\mathbb{X}_0(x,t). \qquad (3.128)
\end{aligned}
$$

where $\mathbb{X}_0$ is a matrix whose elements are functions of x and t and does not depend on u or its partial derivatives. Next, we update equation (3.123) by plugging in the expression for $\mathbb{K}^3$. Upon doing this we will have a rather large expression which is nothing more than an algebraic equation in u. In this final step, as the $\mathbb{X}_i$ do not depend on u, we set the coefficients of the different powers

of u in this last, lengthy expression to zero. At the different orders of u we find the following conditions

$$O(1) \quad : \quad \mathbb{X}_{1,t} - \mathbb{X}_{0,x} + [\mathbb{X}_1, \mathbb{X}_0] = 0, \tag{3.129}$$

$$\begin{aligned} O(u) \quad : \quad & [\mathbb{X}_2, \mathbb{X}_0] - a_6[\mathbb{X}_1, [\mathbb{X}_1, \mathbb{X}_7]] + [\mathbb{X}_1, [\mathbb{X}_1, [\mathbb{X}_1, (a_6\mathbb{X}_4)_x]]] + [\mathbb{X}_1, [\mathbb{X}_1, [\mathbb{X}_1, [\mathbb{X}_1, (a_6\mathbb{X}_2)_x]]]] \\ & + [\mathbb{X}_1, [\mathbb{X}_1, (a_5\mathbb{X}_2)_x]] - [\mathbb{X}_1, [\mathbb{X}_1, [\mathbb{X}_1, (a_6\mathbb{X}_2)_{xx}]]] + [\mathbb{X}_1, (a_6\mathbb{X}_7)_x] - [\mathbb{X}_1, [\mathbb{X}_1, (a_6\mathbb{X}_4)_{xx}]] \\ & - a_5\mathbb{X}_7 - [\mathbb{X}_1, [\mathbb{X}_1, ([\mathbb{X}_1, (a_6\mathbb{X}_2)_x])_x]] + [\mathbb{X}_1, [\mathbb{X}_1, (a_6\mathbb{X}_2)_{xxx}]] - [\mathbb{X}_1, ([\mathbb{X}_1, (a_6\mathbb{X}_4)_x])_x] \\ & - [\mathbb{X}_1, ([\mathbb{X}_1, [\mathbb{X}_1, (a_6\mathbb{X}_2)_x]])_x] + [\mathbb{X}_1, (a_5\mathbb{X}_4)_x] + [\mathbb{X}_1, ([\mathbb{X}_1, (a_6\mathbb{X}_2)_{xx}])_x] - [\mathbb{X}_1, (a_5\mathbb{X}_2)_{xx}] \\ & - [\mathbb{X}_1, (a_6\mathbb{X}_5)_{xx}] + [\mathbb{X}_1, (a_6\mathbb{X}_4)_{xxx}] + [\mathbb{X}_1, ([\mathbb{X}_1, (a_6\mathbb{X}_2)_x])_{xx}] - a_8\mathbb{X}_4 - [\mathbb{X}_1, (a_6\mathbb{X}_2)_{xxxx}] \\ & + [\mathbb{X}_1, [\mathbb{X}_1, (a_6\mathbb{X}_5)_x]] + (a_6[\mathbb{X}_1, \mathbb{X}_7])_x - ([\mathbb{X}_1, [\mathbb{X}_1, [\mathbb{X}_1, (a_6\mathbb{X}_2)_x]]])_x - ([\mathbb{X}_1, [\mathbb{X}_1, (a_6\mathbb{X}_4)_x]])_x \\ & + (a_8\mathbb{X}_2)_x + ([\mathbb{X}_1, [\mathbb{X}_1, (a_6\mathbb{X}_2)_{xx}]])_x - ([\mathbb{X}_1, (a_5\mathbb{X}_2)_x])_x - (a_6\mathbb{X}_7)_{xx} + ([\mathbb{X}_1, (a_6\mathbb{X}_4)_{xx}])_x \\ & + ([\mathbb{X}_1, ([\mathbb{X}_1, (a_6\mathbb{X}_2)_x])_x])_x - ([\mathbb{X}_1, (a_6\mathbb{X}_2)_{xxx}])_x + ([\mathbb{X}_1, [\mathbb{X}_1, (a_6\mathbb{X}_2)_x]])_{xx} + (a_5\mathbb{X}_5)_x \\ & + \mathbb{X}_{2,t} - ([\mathbb{X}_1, (a_6\mathbb{X}_2)_{xx}])_{xx} - (a_5\mathbb{X}_4)_{xx} + (a_5\mathbb{X}_2)_{xxx} + (a_6\mathbb{X}_5)_{xxx} - ([\mathbb{X}_1, (a_6\mathbb{X}_2)_x])_{xxx} \\ & - ([\mathbb{X}_1, (a_6\mathbb{X}_5)_x])_x + (a_6\mathbb{X}_2)_{xxxxx} - (a_6\mathbb{X}_4)_{xxxx} - a_7\mathbb{X}_2 + ([\mathbb{X}_1, (a_6\mathbb{X}_4)_x])_{xx} = 0, \end{aligned} \tag{3.130}$$

$$
\begin{aligned}
O(u^2) \quad : \quad & -2a_5\mathbb{X}_8 - 2a_6[\mathbb{X}_2,[\mathbb{X}_1,\mathbb{X}_7]] + 2[\mathbb{X}_2,[\mathbb{X}_1,[\mathbb{X}_1,[\mathbb{X}_1,(a_6\mathbb{X}_2)_x]]]] + 2[\mathbb{X}_2,[\mathbb{X}_1,[\mathbb{X}_1,(a_6\mathbb{X}_4)_x]]] \\
& -2[\mathbb{X}_2,[\mathbb{X}_1,[\mathbb{X}_1,(a_6\mathbb{X}_2)_{xx}]]] + 2[\mathbb{X}_2,[\mathbb{X}_1,(a_5\mathbb{X}_2)_x]] - 2[\mathbb{X}_2,[\mathbb{X}_1,([\mathbb{X}_1,(a_6\mathbb{X}_2)_x])_x]] - a_4\mathbb{X}_4 \\
& +2[\mathbb{X}_2,(a_6\mathbb{X}_7)_x] - 2[\mathbb{X}_2,[\mathbb{X}_1,(a_6\mathbb{X}_4)_{xx}]] + 2[\mathbb{X}_2,[\mathbb{X}_1,(a_6\mathbb{X}_2)_{xxx}]] - 2[\mathbb{X}_2,([\mathbb{X}_1,(a_6\mathbb{X}_4)_x])_x] \\
& -2[\mathbb{X}_2,([\mathbb{X}_1,[\mathbb{X}_1,(a_6\mathbb{X}_2)_x]])_x] + 2[\mathbb{X}_2,(a_5\mathbb{X}_4)_x] + 2[\mathbb{X}_2,([\mathbb{X}_1,(a_6\mathbb{X}_2)_{xx}])_x] - 2[\mathbb{X}_2,(a_5\mathbb{X}_2)_{xx}] \\
& +2[\mathbb{X}_2,(a_6\mathbb{X}_4)_{xxx}] + 2[\mathbb{X}_2,([\mathbb{X}_1,(a_6\mathbb{X}_2)_x])_{xx}] - 2[\mathbb{X}_2,(a_6\mathbb{X}_2)_{xxxx}] + 2[\mathbb{X}_2,[\mathbb{X}_1,(a_6\mathbb{X}_5)_x]] \\
& -2[\mathbb{X}_2,(a_6\mathbb{X}_5)_{xx}] - a_5\mathbb{X}_9 + [\mathbb{X}_1,[\mathbb{X}_2,(a_5\mathbb{X}_2)_x]] + [\mathbb{X}_1,[\mathbb{X}_2,[\mathbb{X}_1,[\mathbb{X}_1,(a_6\mathbb{X}_2)_x]]]] \\
& -[\mathbb{X}_1,[\mathbb{X}_2,[\mathbb{X}_1,(a_6\mathbb{X}_2)_{xx}]]] + [\mathbb{X}_1,[\mathbb{X}_1,(a_1\mathbb{X}_2)_x]] - [\mathbb{X}_1,[\mathbb{X}_2,([\mathbb{X}_1,(a_6\mathbb{X}_2)_x])_x]] \\
& +[\mathbb{X}_1,[\mathbb{X}_1,[\mathbb{X}_1,[\mathbb{X}_2,(a_6\mathbb{X}_2)_x]]]] - [\mathbb{X}_1,[\mathbb{X}_2,(a_6\mathbb{X}_4)_{xx}]] + [\mathbb{X}_1,[\mathbb{X}_2,[\mathbb{X}_1,(a_6\mathbb{X}_4)_x]]] \\
& -a_6[\mathbb{X}_1,[\mathbb{X}_1,\mathbb{X}_9]] + [\mathbb{X}_1,[\mathbb{X}_2,(a_6\mathbb{X}_2)_{xxx}]] + [\mathbb{X}_1,[\mathbb{X}_1,[\mathbb{X}_2,[\mathbb{X}_1,(a_6\mathbb{X}_2)_x]]]] \\
& +[\mathbb{X}_1,[\mathbb{X}_2,(a_6\mathbb{X}_5)_x]] - [\mathbb{X}_1,[\mathbb{X}_1,[\mathbb{X}_2,(a_6\mathbb{X}_2)_{xx}]]] - [\mathbb{X}_1,[\mathbb{X}_1,([\mathbb{X}_2,(a_6\mathbb{X}_2)_x])_x]] \\
& -a_6[\mathbb{X}_1,[\mathbb{X}_1,\mathbb{X}_8]] + [\mathbb{X}_1,[\mathbb{X}_1,[\mathbb{X}_2,(a_6\mathbb{X}_4)_x]]] - a_1\mathbb{X}_7 + [\mathbb{X}_1,[\mathbb{X}_1,(a_6\mathbb{X}_6)_x]] \\
& -[\mathbb{X}_1,([\mathbb{X}_2,[\mathbb{X}_1,(a_6\mathbb{X}_2)_x]])_x] + [\mathbb{X}_1,(a_6\mathbb{X}_9)_x] + [\mathbb{X}_1,(a_6\mathbb{X}_8)_x] - [\mathbb{X}_1,(a_6\mathbb{X}_6)_{xx}] \\
& -[\mathbb{X}_1,([\mathbb{X}_1,[\mathbb{X}_2,(a_6\mathbb{X}_2)_x]])_x] + [\mathbb{X}_1,([\mathbb{X}_2,(a_6\mathbb{X}_2)_{xx}])_x] - [\mathbb{X}_1,([\mathbb{X}_2,(a_6\mathbb{X}_4)_x])_x] \\
& -[\mathbb{X}_1,(a_1\mathbb{X}_2)_{xx}] + [\mathbb{X}_1,(a_1\mathbb{X}_4)_x] + [\mathbb{X}_1,([\mathbb{X}_2,(a_6\mathbb{X}_2)_x])_{xx}] - a_6[\mathbb{X}_1,[\mathbb{X}_2,\mathbb{X}_7]] \\
& -([\mathbb{X}_2,(a_5\mathbb{X}_2)_x])_x - ([\mathbb{X}_2,[\mathbb{X}_1,[\mathbb{X}_1,(a_6\mathbb{X}_2)_x]]])_x + ([\mathbb{X}_2,[\mathbb{X}_1,(a_6\mathbb{X}_2)_{xx}]])_x \\
& +(a_5\mathbb{X}_6)_x + ([\mathbb{X}_2,([\mathbb{X}_1,(a_6\mathbb{X}_2)_x])_x])_x - ([\mathbb{X}_2,(a_6\mathbb{X}_5)_x])_x - ([\mathbb{X}_1,(a_1\mathbb{X}_2)_x])_x \\
& +([\mathbb{X}_2,(a_6\mathbb{X}_4)_{xx}])_x + (a_6[\mathbb{X}_1,\mathbb{X}_9])_x + (a_6[\mathbb{X}_1,\mathbb{X}_8])_x - ([\mathbb{X}_2,[\mathbb{X}_1,(a_6\mathbb{X}_4)_x]])_x \\
& -([\mathbb{X}_2,(a_6\mathbb{X}_2)_{xxx}])_x - ([\mathbb{X}_1,[\mathbb{X}_1,[\mathbb{X}_2,(a_6\mathbb{X}_2)_x]]])_x - ([\mathbb{X}_1,[\mathbb{X}_2,[\mathbb{X}_1,(a_6\mathbb{X}_2)_x]]])_x \\
& +([\mathbb{X}_1,[\mathbb{X}_2,(a_6\mathbb{X}_2)_{xx}]])_x - ([\mathbb{X}_1,[\mathbb{X}_2,(a_6\mathbb{X}_4)_x]])_x + ([\mathbb{X}_1,([\mathbb{X}_2,(a_6\mathbb{X}_2)_x])_x])_x \\
& +(a_1\mathbb{X}_5)_x - ([\mathbb{X}_1,(a_6\mathbb{X}_6)_x])_x - (a_6\mathbb{X}_9)_{xx} - (a_6\mathbb{X}_8)_{xx} + ([\mathbb{X}_1,[\mathbb{X}_2,(a_6\mathbb{X}_2)_x]])_{xx} \\
& +(a_6\mathbb{X}_6)_{xxx} + ([\mathbb{X}_2,[\mathbb{X}_1,(a_6\mathbb{X}_2)_x]])_{xx} - ([\mathbb{X}_2,(a_6\mathbb{X}_2)_{xx}])_{xx} + ([\mathbb{X}_2,(a_6\mathbb{X}_4)_x])_{xx} \\
& +(a_1\mathbb{X}_2)_{xxx} - (a_1\mathbb{X}_4)_{xx} - ([\mathbb{X}_2,(a_6\mathbb{X}_2)_x])_{xxx} + (a_6[\mathbb{X}_2,\mathbb{X}_7])_x + (a_4\mathbb{X}_2)_x = 0, \qquad (3.131)
\end{aligned}
$$

$$
\begin{aligned}
O(u^3) \;:\; & -2([\mathbb{X}_2,(a_6\mathbb{X}_6)_x])_x+3[\mathbb{X}_2,[\mathbb{X}_2,[\mathbb{X}_1,[\mathbb{X}_1,(a_6\mathbb{X}_2)_x]]]]-3[\mathbb{X}_2,[\mathbb{X}_2,[\mathbb{X}_1,(a_6\mathbb{X}_2)_{xx}]]]\\
& +2(a_3\mathbb{X}_2)_x-3[\mathbb{X}_2,[\mathbb{X}_2,(a_6\mathbb{X}_4)_{xx}]]+3[\mathbb{X}_2,[\mathbb{X}_2,(a_6\mathbb{X}_5)_x]]+3[\mathbb{X}_2,[\mathbb{X}_1,(a_1\mathbb{X}_2)_x]]\\
& -3[\mathbb{X}_2,[\mathbb{X}_2,([\mathbb{X}_1,(a_6\mathbb{X}_2)_x])_x]]-3a_6[\mathbb{X}_2,[\mathbb{X}_1,\mathbb{X}_9]]+3[\mathbb{X}_2,[\mathbb{X}_1,[\mathbb{X}_2,[\mathbb{X}_1,(a_6\mathbb{X}_2)_x]]]]\\
& -3a_6[\mathbb{X}_2,[\mathbb{X}_1,\mathbb{X}_8]]+3[\mathbb{X}_2,[\mathbb{X}_2,[\mathbb{X}_1,(a_6\mathbb{X}_4)_x]]]+3[\mathbb{X}_2,[\mathbb{X}_1,[\mathbb{X}_1,[\mathbb{X}_2,(a_6\mathbb{X}_2)_x]]]]\\
& +3[\mathbb{X}_2,[\mathbb{X}_2,(a_6\mathbb{X}_2)_{xxx}]]-3[\mathbb{X}_2,[\mathbb{X}_1,[\mathbb{X}_2,(a_6\mathbb{X}_2)_{xx}]]]-3[\mathbb{X}_2,[\mathbb{X}_1,([\mathbb{X}_2,(a_6\mathbb{X}_2)_x])_x]]\\
& -3a_1\mathbb{X}_8+3[\mathbb{X}_2,[\mathbb{X}_1,[\mathbb{X}_2,(a_6\mathbb{X}_4)_x]]]+3[\mathbb{X}_2,[\mathbb{X}_1,(a_6\mathbb{X}_6)_x]]-3[\mathbb{X}_2,(a_6\mathbb{X}_6)_{xx}]\\
& -3[\mathbb{X}_2,([\mathbb{X}_2,[\mathbb{X}_1,(a_6\mathbb{X}_2)_x]])_x]+3[\mathbb{X}_2,(a_6\mathbb{X}_9)_x]-3[\mathbb{X}_2,([\mathbb{X}_1,[\mathbb{X}_2,(a_6\mathbb{X}_2)_x]])_x]\\
& +3[\mathbb{X}_2,([\mathbb{X}_2,(a_6\mathbb{X}_2)_{xx}])_x]-3[\mathbb{X}_2,(a_1\mathbb{X}_2)_{xx}]+3[\mathbb{X}_2,([\mathbb{X}_2,(a_6\mathbb{X}_2)_x])_{xx}]+3[\mathbb{X}_2,(a_1\mathbb{X}_4)_x]\\
& +3[\mathbb{X}_2,(a_6\mathbb{X}_8)_x]-3[\mathbb{X}_2,([\mathbb{X}_2,(a_6\mathbb{X}_4)_x])_x]-3a_6[\mathbb{X}_2,[\mathbb{X}_2,\mathbb{X}_7]]-2a_6[\mathbb{X}_1,[\mathbb{X}_2,\mathbb{X}_9]]\\
& +2[\mathbb{X}_1,[\mathbb{X}_2,(a_1\mathbb{X}_2)_x]]+2[\mathbb{X}_1,[\mathbb{X}_2,[\mathbb{X}_2,[\mathbb{X}_1,(a_6\mathbb{X}_2)_x]]]]-2[\mathbb{X}_1,[\mathbb{X}_2,([\mathbb{X}_2,(a_6\mathbb{X}_2)_x])_x]]\\
& +2[\mathbb{X}_1,[\mathbb{X}_2,[\mathbb{X}_1,[\mathbb{X}_2,(a_6\mathbb{X}_2)_x]]]]-2[\mathbb{X}_1,[\mathbb{X}_2,[\mathbb{X}_2,(a_6\mathbb{X}_2)_{xx}]]]-2a_6[\mathbb{X}_1,[\mathbb{X}_2,\mathbb{X}_8]]\\
& +2[\mathbb{X}_1,[\mathbb{X}_2,(a_6\mathbb{X}_6)_x]]+2[\mathbb{X}_1,[\mathbb{X}_2,[\mathbb{X}_2,(a_6\mathbb{X}_4)_x]]]-2a_1\mathbb{X}_9-2([\mathbb{X}_2,(a_1\mathbb{X}_2)_x])_x\\
& -2a_3\mathbb{X}_4+2(a_6[\mathbb{X}_2,\mathbb{X}_9])_x+2(a_6[\mathbb{X}_2,\mathbb{X}_8])_x-2([\mathbb{X}_2,[\mathbb{X}_1,[\mathbb{X}_2,(a_6\mathbb{X}_2)_x]]])_x\\
& -2([\mathbb{X}_2,[\mathbb{X}_2,[\mathbb{X}_1,(a_6\mathbb{X}_2)_x]]])_x+2([\mathbb{X}_2,[\mathbb{X}_2,(a_6\mathbb{X}_2)_{xx}]])_x+2([\mathbb{X}_2,([\mathbb{X}_2,(a_6\mathbb{X}_2)_x])_x])_x\\
& -2([\mathbb{X}_2,[\mathbb{X}_2,(a_6\mathbb{X}_4)_x]])_x+2(a_1\mathbb{X}_6)_x=0, \qquad (3.132)
\end{aligned}
$$

$$
\begin{aligned}
O(u^4) \;:\; & [\mathbb{X}_2,[\mathbb{X}_2,[\mathbb{X}_2,[\mathbb{X}_1,(a_6\mathbb{X}_2)_x]]]]-a_6[\mathbb{X}_2,[\mathbb{X}_2,\mathbb{X}_8]]+[\mathbb{X}_2,[\mathbb{X}_2,[\mathbb{X}_1,[\mathbb{X}_2,(a_6\mathbb{X}_2)_x]]]]\\
& -a_6[\mathbb{X}_2,[\mathbb{X}_2,\mathbb{X}_9]]-[\mathbb{X}_2,[\mathbb{X}_2,[\mathbb{X}_2,(a_6\mathbb{X}_2)_{xx}]]]-[\mathbb{X}_2,[\mathbb{X}_2,([\mathbb{X}_2,(a_6\mathbb{X}_2)_x])_x]]\\
& +[\mathbb{X}_2,[\mathbb{X}_2,(a_6\mathbb{X}_6)_x]]+[\mathbb{X}_2,[\mathbb{X}_2,[\mathbb{X}_2,(a_6\mathbb{X}_4)_x]]]=0. \qquad (3.133)
\end{aligned}
$$

Note that if we decouple equation (3.125) into the following three conditions

$$([\mathbb{X}_2,(a_6\mathbb{X}_2)_x])_x-(a_6\mathbb{X}_6)_x+a_6\mathbb{X}_9-[\mathbb{X}_1,[\mathbb{X}_2,(a_6\mathbb{X}_2)_x]]=0, \tag{3.134}$$

$$[\mathbb{X}_2,[\mathbb{X}_1,(a_6\mathbb{X}_2)_x]]+[\mathbb{X}_2,(a_6\mathbb{X}_4)_x]-[\mathbb{X}_2,(a_6\mathbb{X}_2)_{xx}]-a_6\mathbb{X}_8=0, \tag{3.135}$$

$$((a_2-3a_1)\mathbb{X}_2)_x-(a_2-3a_1)\mathbb{X}_4=0, \tag{3.136}$$

then the $O(u^4)$ equation is identically satisfied. To reduce the complexity of the $O(u^3)$ equation we can decouple it into the following two equations

$$\begin{aligned}&[\mathbb{X}_2,[\mathbb{X}_1,[\mathbb{X}_1,(a_6\mathbb{X}_2)_x]]]-[\mathbb{X}_2,[\mathbb{X}_1,(a_6\mathbb{X}_2)_{xx}]]-[\mathbb{X}_2,(a_6\mathbb{X}_4)_{xx}]+[\mathbb{X}_2,(a_6\mathbb{X}_5)_x]\\&+[\mathbb{X}_1,(a_1\mathbb{X}_2)_x]-[\mathbb{X}_2,([\mathbb{X}_1,(a_6\mathbb{X}_2)_x])_x]+[\mathbb{X}_2,[\mathbb{X}_1,(a_6\mathbb{X}_4)_x]]+[\mathbb{X}_2,(a_6\mathbb{X}_2)_{xxx}]\\&-a_1\mathbb{X}_5-(a_1\mathbb{X}_2)_{xx}+(a_1\mathbb{X}_4)_x-a_6[\mathbb{X}_2,\mathbb{X}_7]=0,\end{aligned} \tag{3.137}$$

$$(a_3\mathbb{X}_2)_x+[\mathbb{X}_1,[\mathbb{X}_2,(a_1\mathbb{X}_2)_x]]-a_1\mathbb{X}_9-([\mathbb{X}_2,(a_1\mathbb{X}_2)_x])_x-a_3\mathbb{X}_4+(a_1\mathbb{X}_6)_x=0 \tag{3.138}$$

From this last condition, we can use equations (3.134)-(3.138) to reduce the $O(u^2)$ condition to the following equation

$$
\begin{aligned}
&-a_5\mathbb{X}_8 - a_6[\mathbb{X}_2,[\mathbb{X}_1,\mathbb{X}_7]] + [\mathbb{X}_2,[\mathbb{X}_1,[\mathbb{X}_1,[\mathbb{X}_1,(a_6\mathbb{X}_2)_x]]]] + [\mathbb{X}_2,[\mathbb{X}_1,[\mathbb{X}_1,(a_6\mathbb{X}_4)_x]]] \\
&-[\mathbb{X}_2,[\mathbb{X}_1,[\mathbb{X}_1,(a_6\mathbb{X}_2)_{xx}]]] + [\mathbb{X}_2,[\mathbb{X}_1,(a_5\mathbb{X}_2)_x]] - [\mathbb{X}_2,[\mathbb{X}_1,([\mathbb{X}_1,(a_6\mathbb{X}_2)_x])_x]] \\
&+[\mathbb{X}_2,(a_6\mathbb{X}_7)_x] - [\mathbb{X}_2,[\mathbb{X}_1,(a_6\mathbb{X}_4)_{xx}]] + [\mathbb{X}_2,[\mathbb{X}_1,(a_6\mathbb{X}_2)_{xxx}]] - [\mathbb{X}_2,([\mathbb{X}_1,(a_6\mathbb{X}_4)_x])_x] \\
&-[\mathbb{X}_2,([\mathbb{X}_1,[\mathbb{X}_1,(a_6\mathbb{X}_2)_x]])_x] + [\mathbb{X}_2,(a_5\mathbb{X}_4)_x] + [\mathbb{X}_2,([\mathbb{X}_1,(a_6\mathbb{X}_2)_{xx}])_x] - [\mathbb{X}_2,(a_5\mathbb{X}_2)_{xx}] \\
&+[\mathbb{X}_2,(a_6\mathbb{X}_4)_{xxx}] + [\mathbb{X}_2,([\mathbb{X}_1,(a_6\mathbb{X}_2)_x])_{xx}] - [\mathbb{X}_2,(a_6\mathbb{X}_2)_{xxxx}] + [\mathbb{X}_2,[\mathbb{X}_1,(a_6\mathbb{X}_5)_x]] \\
&-[\mathbb{X}_2,(a_6\mathbb{X}_5)_{xx}] - \frac{1}{2}a_5\mathbb{X}_9 + \frac{1}{2}[\mathbb{X}_1,[\mathbb{X}_2,(a_5\mathbb{X}_2)_x]] - \frac{1}{2}([\mathbb{X}_2,(a_5\mathbb{X}_2)_x])_x + \frac{1}{2}(a_5\mathbb{X}_6)_x \\
&-\frac{1}{2}a_4\mathbb{X}_4 + \frac{1}{2}(a_4\mathbb{X}_2)_x = 0.
\end{aligned} \tag{3.139}
$$

Decoupling this equation allows for the simplification of the $O(u)$ equation. Thus we write the previous condition as the following system of equations

$$
\begin{aligned}
&-[\mathbb{X}_1,(a_6\mathbb{X}_4)_x] - [\mathbb{X}_1,[\mathbb{X}_1,(a_6\mathbb{X}_2)_x]] + a_5\mathbb{X}_4 + [\mathbb{X}_1,(a_6\mathbb{X}_2)_{xx}] - (a_5\mathbb{X}_2)_x \\
&+(a_6\mathbb{X}_4)_{xx} + ([\mathbb{X}_1,(a_6\mathbb{X}_2)_x])_x - (a_6\mathbb{X}_2)_{xxx} - (a_6\mathbb{X}_5)_x = 0,
\end{aligned} \tag{3.140}
$$

$$
-a_5\mathbb{X}_9 + [\mathbb{X}_1,[\mathbb{X}_2,(a_5\mathbb{X}_2)_x]] - ([\mathbb{X}_2,(a_5\mathbb{X}_2)_x])_x + (a_5\mathbb{X}_6)_x - \frac{1}{2}a_4\mathbb{X}_4 + \frac{1}{2}(a_4\mathbb{X}_2)_x = 0. \tag{3.141}
$$

Using the previous system the $O(u)$ equation is reduced to the following, simpler equation

$$
\mathbb{X}_{2,t} + [\mathbb{X}_2,\mathbb{X}_0] - a_8\mathbb{X}_4 + (a_8\mathbb{X}_2)_x - a_7\mathbb{X}_2 = 0. \tag{3.142}
$$

We therefore find that the final, reduced constraints are given by equations (3.126), (3.134)-(3.138)

and (3.140)-(3.142). In order to satisfy these constraints we begin with the following forms for our generators,

$$\mathbb{X}_0 = \begin{bmatrix} g_1(x,t) & g_2(x,t) \\ g_3(x,t) & g_4(x,t) \end{bmatrix}, \quad \mathbb{X}_1 = \begin{bmatrix} 0 & f_1(x,t) \\ f_2(x,t) & 0 \end{bmatrix}, \quad \mathbb{X}_2 = \begin{bmatrix} 0 & f_3(x,t) \\ f_4(x,t) & 0 \end{bmatrix}.$$

To get more general results we will assume $a_2 \neq 3a_1$. Note that had we instead opted for the forms

$$\mathbb{X}_0 = \begin{bmatrix} g_1(x,t) & g_{12}(x,t) \\ g_{23}(x,t) & g_{34}(x,t) \end{bmatrix}, \quad \mathbb{X}_1 = \begin{bmatrix} f_1(x,t) & f_3(x,t) \\ f_5(x,t) & f_7(x,t) \end{bmatrix}, \quad \mathbb{X}_2 = \begin{bmatrix} f_2(x,t) & f_4(x,t) \\ f_6(x,t) & f_8(x,t) \end{bmatrix},$$

we would obtain an equivalent system to that obtained in [21]. The additional unknown functions which appear in Khawaja's method [21] can be introduced with the proper substitutions via their functional dependence on the twelve unknown functions given above.

Taking the naive approach of beginning with the smaller conditions first we begin with equation (3.136) which, utilizing the given forms for $\mathbb{X}_0, \mathbb{X}_1$, and $\mathbb{X}_2$, becomes

$$(a_2 - 3a_1)(f_1 f_4 - f_2 f_3) = 0, \tag{3.143}$$

$$((a_2 - 3a_1) f_j)_x = 0, \quad j = 3, 4. \tag{3.144}$$

Solving this system for f_2, f_3 and f_4 in this previous system yields

$$f_3(x,t) = \frac{F_3(t)}{a_2(x,t) - 3a_1(x,t)}, \tag{3.145}$$

$$f_4(x,t) = \frac{F_4(t)}{a_2(x,t) - 3a_1(x,t)}, \tag{3.146}$$

$$f_2(x,t) = \frac{f_1(x,t)F_4(t)}{F_3(t)}, \tag{3.147}$$

$$\tag{3.148}$$

where $F_{3,4}(t)$ are arbitrary functions of t. With these choices we've elected to satisfy $\mathbb{X}_4 = 0$ rather than $a_2 = 3a_1$. This will greatly reduce the complexity of the remaining conditions while allowing for the possibility of a less trivial relation between a_1 and a_2. Looking next at equation (3.142) with $\mathbb{X}_4 = 0$ we obtain the system

$$\left(\frac{F_j}{a_2 - 3a_1}\right)_t - \frac{F_j a_7}{a_2 - 3a_1} + \left(\frac{F_j a_8}{a_2 - 3a_1}\right)_x + \frac{1}{2}\left(\frac{F_j a_4}{a_2 - 3a_1}\right)_x$$
$$+(-1)^j \frac{F_j(g_4 - g_1)}{a_2 - 3a_1} = 0 \quad j = 3, 4, \tag{3.149}$$

$$\frac{F_3 g_3}{a_2 - 3a_1} - \frac{F_4 g_2}{a_2 - 3a_1} = 0. \tag{3.150}$$

Solving the second equation for g_3 yields

$$g_3 = \frac{F_4(t)g_2(x,t)}{F_3(t)}$$

Considering next the $O(1)$ equation, we obtain the following system of equations

$$g_{1x} = g_{4x} = 0, \tag{3.151}$$

$$f_{1t} - g_{2x} + f_1(g_4 - g_1) = 0, \tag{3.152}$$

$$F_3(F_4 f_1)_t - f_1 F_4 F_{3t} - g_{2x} F_4 F_3 + F_3 F_4 f_1 (g_1 - g_4) = 0. \tag{3.153}$$

It follows that we must have $g_1(x,t) = G_1(t)$ and $g_4(x,t) = G_4(t)$ where G_1 and G_4 are arbitrary functions of t. Since equations (3.152) and (3.153) do not depend on the a_i they will not affect the conditions on the a_i required for Lax-integrability of (2.66). On the other hand the remaining conditions have been reduced to conditions involving soley the a_i and the previously introduced arbitrary functions of t. The remaining conditions are given by

$$\left(\frac{a_5}{a_2 - 3a_1}\right)_x + \left(\frac{a_6}{a_2 - 3a_1}\right)_{xxx} = 0, \tag{3.154}$$

$$\left(\frac{a_3}{a_2 - 3a_1}\right)_x = 0, \tag{3.155}$$

$$\left(\frac{a_1}{a_2 - 3a_1}\right)_{xx} = 0, \tag{3.156}$$

$$\left(\frac{a_4}{a_2 - 3a_1}\right)_x = 0. \tag{3.157}$$

One can easily solve the system of equations given by equations (3.149), (3.154)-(3.157) yielding

$$
\begin{aligned}
F_4 &= c_1 F_3 e^{2\int (G_4 - G_1)dt} && (3.158)\\
g_2 &= \int (f_{1t} + f_1(G_4 - G_1))dx + F_{10} && (3.159)\\
a_2 &= -\frac{(3F_1 - 1 - 3F_2 x)a_1}{F_2 x - F_1} && (3.160)\\
a_3 &= \frac{F_5 a_1}{F_2 x - F_1} && (3.161)\\
a_4 &= \frac{F_6 a_1}{F_2 x - F_1} && (3.162)\\
a_6 &= \frac{(F_7 + F_8 x + F_9 x^2)a_1}{F_2 x - F_1} - \int^x \int^y \frac{a_5(z,t)\, dz\, dy}{a_2(z,t) - 3a_1(z,t)} && (3.163)\\
a_7 &= \frac{a_2 - 3a_1}{F_3}\left(\frac{F_3}{a_2 - 3a_1}\right)_t + (a_2 - 3a_1)\left(\frac{a_8}{a_2 - 3a_1}\right)_x + G_4 - G_1 && (3.164)
\end{aligned}
$$

where F_{5-10} are arbitrary functions of t. Note that a_1, a_5 and a_8 have no restrictions beyond the appropriate differentiability and integrability conditions. The Lax pair for the generalized variable-coefficient KdV equation with the previous integrability conditions on the variable coefficients is therefore given by

$$
\begin{aligned}
\mathbb{F} &= \mathbb{X}_1 + \mathbb{X}_2 u, && (3.165)\\
\mathbb{G} &= -a_6\mathbb{X}_2 u_{xxxx} + (a_6\mathbb{X}_2)_x u_{xxx} - \mathbb{X}_2(a_1 u + a_5)u_{xx} - (a_6\mathbb{X}_2)_{xx}u_{xx} - a_8\mathbb{X}_2 u \\
&\quad + \frac{1}{2}a_1\mathbb{X}_2 u_x^2 - \frac{1}{2}a_2\mathbb{X}_2 u_x^2 + (a_1\mathbb{X}_2)_x u u_x - \frac{1}{3}a_3\mathbb{X}_2 u^3 - \frac{1}{2}a_4\mathbb{X}_2 u^2 + \mathbb{X}_0. && (3.166)
\end{aligned}
$$

Modified Korteweg-de-Vries Equation

In this section we derive the Lax pair and differential constraints for the first equation in the variable-coefficient mKdV hierarchy. Following the procedure outlined at the beginning of the chapter we let

$$\mathbb{F} = \mathbb{F}(x, t, u) \text{ and } \mathbb{G} = \mathbb{G}(x, t, u, u_x, u_{xx}).$$

Plugging $\mathbb{F}$ and $\mathbb{G}$ into equation (3.2) we obtain

$$\mathbb{F}_t + \mathbb{F}_v v_t - \mathbb{G}_x - \mathbb{G}_v v_x - \mathbb{G}_{v_x} v_{xx} - \mathbb{G}_{v_{xx}} v_{xxx} + [\mathbb{F}, \mathbb{G}] = 0. \tag{3.167}$$

Using equation (2.74) to substitute for v_t in the equation given above we have

$$\mathbb{F}_t - \mathbb{G}_x - (\mathbb{G}_v + b_2 \mathbb{F}_v v^2) v_x - \mathbb{G}_{v_x} v_{xx} - (\mathbb{G}_{v_{xx}} + b_1 \mathbb{F}_v) v_{xxx} + [\mathbb{F}, \mathbb{G}] = 0. \tag{3.168}$$

Since $\mathbb{F}$ and $\mathbb{G}$ do not depend on v_{xxx} we must set the coefficient of the v_{xxx} term to zero from which we find that $\mathbb{F}$ and $\mathbb{G}$ must satisfy

$$\mathbb{G}_{v_{xx}} + b_1 \mathbb{F}_v = 0 \Rightarrow \mathbb{G} = -b_1 \mathbb{F}_v v_{xx} + \mathbb{K}^0(x, t, v, v_x).$$

Substituting this expression for $\mathbb{G}$ into equation (3.168) we obtain the updated equation

$$\mathbb{F}_t + (b_1\mathbb{F}_v)_x v_{xx} - \mathbb{K}^0_x + b_1\mathbb{F}_{vv}v_x v_{xx} - \mathbb{K}^0_v v_x - b_2\mathbb{F}_v v^2 v_x - \mathbb{K}^0_{v_x} v_{xx} - b_1[\mathbb{F},\mathbb{F}_v]v_{xx} + [\mathbb{F},\mathbb{K}^0] = 0. \tag{3.169}$$

Since $\mathbb{F}$ and $\mathbb{K}^0$ do not depend on v_{xx} we can equate the coefficient of the v_{xx} term to zero which is equivalent to the requirement

$$(b_1\mathbb{F}_v)_x + b_1\mathbb{F}_{vv}v_x - \mathbb{K}^0_{v_x} - b_1[\mathbb{F},\mathbb{F}_v] = 0. \tag{3.170}$$

Integrating with respect to v_x and solving for $\mathbb{K}^0$ we have

$$\mathbb{K}^0 = (b_1\mathbb{F}_v)_x v_x + \frac{1}{2}b_1\mathbb{F}_{vv}v_x^2 - b_1[\mathbb{F},\mathbb{F}_v]v_x + \mathbb{K}^1(x,t,v).$$

Substituting this expression for $\mathbb{K}^0$ into equation (3.169) we have the following updated equation

$$\begin{aligned}&\mathbb{F}_t - (b_1\mathbb{F}_v)_{xx}v_x - \frac{1}{2}(b_1\mathbb{F}_{vv})_x v_x^2 + (b_1[\mathbb{F},\mathbb{F}_v])_x v_x - \mathbb{K}^1_x - (b_1\mathbb{F}_{vv})_x v_x^2 - \frac{1}{2}b_1\mathbb{F}_{vvv}v_x^3\\ &-\mathbb{K}^1_v v_x + b_1[\mathbb{F},\mathbb{F}_{vv}]v_x^2 - b_2\mathbb{F}_v v^2 v_x + [\mathbb{F},(b_1\mathbb{F}_v)_x]v_x + \frac{1}{2}b_1[\mathbb{F},\mathbb{F}_{vv}]v_x^2 - b_1[\mathbb{F},[\mathbb{F},\mathbb{F}_v]]v_x\\ &+[\mathbb{F},\mathbb{K}^1] = 0.\end{aligned} \tag{3.171}$$

Since $\mathbb{F}$ and $\mathbb{K}^1$ do not depend on v_x we can equate the coefficients of the v_x, v_x^2 and v_x^3 terms to zero from which we obtain the following system,

$$O(v_x^3) \quad : \quad \mathbb{F}_{vvv} = 0, \tag{3.172}$$

$$O(v_x^2) \quad : \quad \frac{1}{2}(b_1\mathbb{F}_{vv})_x + (b_1\mathbb{F}_{vv})_x - b_1[\mathbb{F}, \mathbb{F}_{vv}] - \frac{1}{2}b_1[\mathbb{F}, \mathbb{F}_{vv}] = 0, \tag{3.173}$$

$$O(v_x) \quad : \quad (b_1\mathbb{F}_v)_{xx} - (b_1[\mathbb{F}, \mathbb{F}_v])_x + \mathbb{K}_v^1 + b_2\mathbb{F}_v v^2 - [\mathbb{F}, (b_1\mathbb{F}_v)_x] + b_1[\mathbb{F}, [\mathbb{F}, \mathbb{F}_v]] = 0 \tag{3.174}$$

Since the MKdV equation does not contain a vv_t term and for ease of computation we take $\mathbb{F}_{vv} = 0$ from which we have $\mathbb{F} = \mathbb{X}_1(x,t) + \mathbb{X}_2(x,t)v$ where $\mathbb{X}_1(x,t)$ and $\mathbb{X}_2(x,t)$ are matrices whose elements are functions of x and t and do not depend on v or its partial derivatives. With this requirement on $\mathbb{F}$ the $O(v_x^3)$ and $O(v_x^2)$ equations are immediately satisfied. Integrating the $O(v_x^3)$ equations with respect to v and solving for $\mathbb{K}^1$ we find

$$\begin{aligned} \mathbb{K}^1 \quad = \quad & -(b_1\mathbb{X}_2)_{xx}v + (b_1[\mathbb{X}_1, \mathbb{X}_2])_x v + [\mathbb{X}_1, (b_1\mathbb{X}_2)_x]v + \frac{1}{2}[\mathbb{X}_2, (b_1\mathbb{X}_2)_x]v^2 - b_1[\mathbb{X}_1, [\mathbb{X}_1, \mathbb{X}_2]]v \\ & -\frac{1}{3}b_2\mathbb{X}_2 v^3 - \frac{1}{2}b_1[\mathbb{X}_2, [\mathbb{X}_1, \mathbb{X}_2]]v^2 + \mathbb{X}_0(x,t), \end{aligned} \tag{3.175}$$

where $\mathbb{X}_0$ is a matrix whose elements are functions of x and t and does not depend on v or its partial derivatives. Substituting the expression for $\mathbb{K}^1$ into equation (3.171) we obtain

$$\mathbb{X}_{1,t} + (b_1\mathbb{X}_2)_{xxx}v - (b_1[\mathbb{X}_1,\mathbb{X}_2])_{xx}v + \frac{1}{3}(b_2\mathbb{X}_2)_x v^3 - ([\mathbb{X}_1,(b_1\mathbb{X}_2)_x])_x v - \frac{1}{2}([\mathbb{X}_2,(b_1\mathbb{X}_2)_x])_x v^2$$
$$+(b_1[\mathbb{X}_1,[\mathbb{X}_1,\mathbb{X}_2]])_x v + \frac{1}{2}(b_1[\mathbb{X}_2,[\mathbb{X}_1,\mathbb{X}_2]])_x v^2 - \mathbb{X}_{0,x} - [\mathbb{X}_1,(b_1\mathbb{X}_2)_{xx}]v + [\mathbb{X}_1,(b_1[\mathbb{X}_1,\mathbb{X}_2])_x]v$$
$$+\mathbb{X}_{2,t}v - \frac{1}{3}b_2[\mathbb{X}_1,\mathbb{X}_2]v^3 + [\mathbb{X}_1,[\mathbb{X}_1,(b_1\mathbb{X}_2)_x]]v + \frac{1}{2}[\mathbb{X}_1,[\mathbb{X}_2,(b_1\mathbb{X}_2)_x]]v^2 - b_1[\mathbb{X}_1,[\mathbb{X}_1,[\mathbb{X}_1,\mathbb{X}_2]]]v$$
$$-\frac{1}{2}b_1[\mathbb{X}_1,[\mathbb{X}_2,[\mathbb{X}_1,\mathbb{X}_2]]]v^2 - [\mathbb{X}_2,(b_1\mathbb{X}_2)_{xx}]v^2 + [\mathbb{X}_2,(b_1[\mathbb{X}_1,\mathbb{X}_2])_x]v^2 + [\mathbb{X}_2,[\mathbb{X}_1,(b_1\mathbb{X}_2)_x]]v^2$$
$$+[\mathbb{X}_1,\mathbb{X}_0] + \frac{1}{2}[\mathbb{X}_2,[\mathbb{X}_2,(b_1\mathbb{X}_2)_x]]v^3 - b_1[\mathbb{X}_2,[\mathbb{X}_1,[\mathbb{X}_1,\mathbb{X}_2]]]v^2 - \frac{1}{2}b_1[\mathbb{X}_2,[\mathbb{X}_2,[\mathbb{X}_1,\mathbb{X}_2]]]v^3$$
$$+[\mathbb{X}_2,\mathbb{X}_0]v = 0 \tag{3.176}$$

Since the $\mathbb{X}_i$ do not depend on v we can equate the coefficients of the different powers of v to zero. In doing so we obtain the following system of equations

$$O(1) \;:\; \mathbb{X}_{1,t} - \mathbb{X}_{0,x} + [\mathbb{X}_1,\mathbb{X}_0] = 0, \tag{3.177}$$

$$O(v) \;:\; \mathbb{X}_{2,t} - ([\mathbb{X}_1,(b_1\mathbb{X}_2)_x])_x + (b_1[\mathbb{X}_1,[\mathbb{X}_1,\mathbb{X}_2]])_x - [\mathbb{X}_1,(b_1\mathbb{X}_2)_{xx}] + [\mathbb{X}_1,(b_1[\mathbb{X}_1,\mathbb{X}_2])_x]$$
$$-(b_1[\mathbb{X}_1,\mathbb{X}_2])_{xx} + (b_1\mathbb{X}_2)_{xxx} + [\mathbb{X}_1,[\mathbb{X}_1,(b_1\mathbb{X}_2)_x]] - b_1[\mathbb{X}_1,[\mathbb{X}_1,[\mathbb{X}_1,\mathbb{X}_2]]]$$
$$+[\mathbb{X}_2,\mathbb{X}_0] = 0, \tag{3.178}$$

$$O(v^2) \;:\; -\frac{1}{2}([\mathbb{X}_2,(b_1\mathbb{X}_2)_x])_x + \frac{1}{2}(b_1[\mathbb{X}_2,[\mathbb{X}_1,\mathbb{X}_2]])_x + \frac{1}{2}[\mathbb{X}_1,[\mathbb{X}_2,(b_1\mathbb{X}_2)_x]] - [\mathbb{X}_2,(b_1\mathbb{X}_2)_{xx}]$$
$$-\frac{1}{2}b_1[\mathbb{X}_1,[\mathbb{X}_2,[\mathbb{X}_1,\mathbb{X}_2]]] + [\mathbb{X}_2,(b_1[\mathbb{X}_1,\mathbb{X}_2])_x] - b_1[\mathbb{X}_2,[\mathbb{X}_1,[\mathbb{X}_1,\mathbb{X}_2]]]$$
$$+[\mathbb{X}_2,[\mathbb{X}_1,(b_1\mathbb{X}_2)_x]] = 0, \tag{3.179}$$

$$O(v^3) \;:\; \frac{1}{3}(b_2\mathbb{X}_2)_x - \frac{1}{3}b_2[\mathbb{X}_1,\mathbb{X}_2] + \frac{1}{2}[\mathbb{X}_2,[\mathbb{X}_2,(b_1\mathbb{X}_2)_x]] - \frac{1}{2}b_1[\mathbb{X}_2,[\mathbb{X}_2,[\mathbb{X}_1,\mathbb{X}_2]]] = 0. \tag{3.180}$$

Note that if we decouple the $O(v^3)$ equation into the following system of equations

$$b_1[\mathbb{X}_1, \mathbb{X}_2] - (b_1\mathbb{X}_2)_x = 0, \tag{3.181}$$

$$b_2[\mathbb{X}_1, \mathbb{X}_2] - (b_2\mathbb{X}_2)_x = 0, \tag{3.182}$$

we find that the $O(v^2)$ equation is immediately satisfied and the $O(v)$ equation reduces to

$$[\mathbb{X}_2, \mathbb{X}_0] + \mathbb{X}_{2,t} = 0. \tag{3.183}$$

Again at this point we should note that should we opt for the forms

$$\mathbb{X}_0 = \begin{bmatrix} g_1(x,t) & g_4(x,t) \\ g_{10}(x,t) & g_{16}(x,t) \end{bmatrix}, \quad \mathbb{X}_1 = \begin{bmatrix} f_1(x,t) & f_3(x,t) \\ f_5(x,t) & f_7(x,t) \end{bmatrix}, \quad \mathbb{X}_2 = \begin{bmatrix} f_2(x,t) & f_4(x,t) \\ f_6(x,t) & f_8(x,t) \end{bmatrix}$$

we would obtain an equivalent system of equations to that obtained in [21] for the mKdV. The additional unknown functions which appear in Khawaja's method [21] can be introduced (as in the case of the fifth-order KdV equation) with the proper substitutions via their functional dependence on the twelve unknown functions given above. Therefore, utilizing the same generators as in the generalized KdV equation of the previous section we obtain the system of equations

$$f_1 f_4 - f_2 f_3 = 0, \tag{3.184}$$

$$(b_1 f_j)_x = (b_2 f_j)_x = 0 \quad j = 3, 4, \tag{3.185}$$

$$f_3 g_3 - f_4 g_2 = 0, \tag{3.186}$$

$$f_{jt} + (-1)^j f_j (g_1 - g_4) = 0 \quad j = 3, 4, \tag{3.187}$$

$$g_{jx} + (-1)^j (f_1 g_3 - f_2 g_2) = 0 \quad j = 1, 4, \tag{3.188}$$

$$f_{jt} - g_{(j+1)x} + (-1)^j f_j (g_4 - g_1) = 0 \quad j = 2, 3. \tag{3.189}$$

Solving this system with the aid of MAPLE yields

$$f_j(x,t) = \frac{F_j(t)}{b_1(x,t)} \quad j = 3, 4, \tag{3.190}$$

$$g_3(x,t) = \frac{F_4(t) g_2(x,t)}{F_3(t)}, \tag{3.191}$$

$$f_2(x,t) = \frac{F_4(t) f_1(x,t)}{F_3(t)}, \tag{3.192}$$

$$g_1(x,t) = G_1(t), \tag{3.193}$$

$$g_4(x,t) = G_4(t), \tag{3.194}$$

such that F_3, F_4, G_1, G_4, and the b_i are subject to the constraints

$$\left(\frac{F_j}{b_1}\right)_t + (-1)^j(G_1 - G_4) = 0 \quad j = 3, 4, \tag{3.195}$$

$$\left(\frac{b_2 F_j}{b_1}\right)_x = 0 \quad j = 3, 4, \tag{3.196}$$

$$f_{jt} - g_{(j+1)x} + (-1)^j f_j(G_4 - G_1) = 0 \quad j = 2, 3. \tag{3.197}$$

Solving (3.195) and (3.196) for F_4, b_1, b_2 and G_4 we obtain

$$F_4(t) = \frac{c_1 F_2(t)^2}{F_3(t)}, \tag{3.198}$$

$$b_1(x,t) = F_1(x)F_2(t), \tag{3.199}$$

$$b_2(x,t) = F_1(x)F_5(t), \tag{3.200}$$

$$G_4(t) = \frac{F_3(t)F_2'(t) - F_2(t)F_3'(t) + G_1(t)F_2(t)F_3(t)}{F_2(t)F_3(t)}, \tag{3.201}$$

$$g_2(x,t) = \int \frac{[f_1(x,t)]_t F_3(t)F_2(t) - f_1(x,t)F_3'(t)F_2(t) + f_1(x,t)F_3(t)F_2'(t)}{F_3(t)F_2(t)} dx + F_6(t), \tag{3.202}$$

where F_1 and F_2 are arbitrary functions in their respective variables and c_1 is an arbitrary constant.

The Lax pair for the variable-coefficient MKdV equation with the previous integrability conditions is then given by

$$\mathbb{F} = \mathbb{X}_1 + \mathbb{X}_2 v \tag{3.203}$$

$$\mathbb{G} = -b_1 \mathbb{X}_2 v_{xx} - \frac{1}{3} b_2 \mathbb{X}_2 v^3 + \mathbb{X}_0 \tag{3.204}$$

Cubic-Quintic Nonlinear Schrödinger Equation

In this section we consider an extended version of a standard non-integrable nlpde, namely the cubic-quintic nonlinear Schrödinger equation (CQNLS). Since the constant-coefficient systems can be obtained as reductions of the extended systems in which the coefficient functions are taken to be the appropriate constants, one would expect this method to breakdown in the case of the CQNLS. As we will show in this section, the extended Estabrook-Wahlquist method indeed breaks down for the CQNLS by requiring that the quintic term be removed or that the Lax pair be trivial (i.e. $\mathbb{F}$ and $\mathbb{G}$ are both the zero matrix). Consider the following variable-coefficient generalization to the nonintegrable CQNLS,

$$i\psi_t + f\psi_{xx} + h\psi + g_1|\psi|^2\psi + g_2|\psi|^4\psi = 0, \tag{3.205}$$

where f, h, g_1, and g_2 are real functions of x and t. It is imperative that the condition $g_2 \neq 0$ hold. Otherwise equation (3.205) reduces to the well-known cubic-NLS for which the results are given in the PT-symmetric and standard nonlinear Schrödinger section. As with the cubic NLS, it will be notationally cleaner to work with the following equivalent system,

$$iq_t + fq_{xx} + hq + g_1q^2r + g_2q^3r^2 = 0, \tag{3.206a}$$

$$-ir_t + fr_{xx} + hr + g_1r^2q + g_2r^3q^2 = 0. \tag{3.206b}$$

Following the procedure outlined earlier in the chapter we make an initial assumption only on the implicit dependence of the Lax pair on the unknown function and its derivatives augmented by al-

lowing dependence on x and t as well. That is, we take $\mathbb{F} = \mathbb{F}(q, r, x, t)$ and $\mathbb{G} = \mathbb{G}(q, r, q_x, r_x, x, t)$. Plugging these into the zero curvature condition yields

$$\mathbb{F}_q q_t + \mathbb{F}_r r_t + \mathbb{F}_t - \mathbb{G}_q q_x - \mathbb{G}_r r_x - \mathbb{G}_{q_x} q_{xx} - \mathbb{G}_{r_x} r_{xx} - \mathbb{G}_x + [\mathbb{F}, \mathbb{G}] = 0. \tag{3.207}$$

Utilizing equations (3.206) to eliminate the q_t and r_t terms in the equation above we obtain

$$\begin{aligned} &if\mathbb{F}_q q_{xx} + ih\mathbb{F}_q q + ig_1\mathbb{F}_q q^2 r + i\mathbb{F}_q q^3 r^2 - if\mathbb{F}_r r_{xx} - ih\mathbb{F}_r r - ig_1\mathbb{F}_r r^2 q - ig_2\mathbb{F}_r r^3 q^2 \\ &+\mathbb{F}_t - \mathbb{G}_q q_x - \mathbb{G}_r r_x - \mathbb{G}_{q_x} q_{xx} - \mathbb{G}_{r_x} r_{xx} - \mathbb{G}_x + [\mathbb{F}, \mathbb{G}] = 0. \end{aligned} \tag{3.208}$$

Since $\mathbb{F}$ and $\mathbb{G}$ do not depend on q_{xx} or r_{xx} the coefficients of the q_{xx} and r_{xx} must be zero. This is equivalent to the conditions

$$if\mathbb{F}_q - \mathbb{G}_{q_x} = 0, \tag{3.209a}$$

$$if\mathbb{F}_r + \mathbb{G}_{r_x} = 0. \tag{3.209b}$$

Solving this system for $\mathbb{G}$ in a method analogous to that discussed in the case of the NLS we find $\mathbb{G} = if\mathbb{F}_q q_x - if\mathbb{F}_r r_x + \mathbb{K}^0(q, r, x, t)$. Plugging this expression for $\mathbb{G}$ into equation (3.208) we obtain the updated zero-curvature condition

$$ih\mathbb{F}_q q + ig_1\mathbb{F}_q q^2 r + ig_2\mathbb{F}_q q^3 r^2 - ih\mathbb{F}_r r - ig_1\mathbb{F}_r r^2 q - ig_2\mathbb{F}_r r^3 q^2 + \mathbb{F}_t - i(f\mathbb{F}_q)_x q_x$$
$$-if\mathbb{F}_{qq} q_x^2 + if\mathbb{F}_{rr} r_x^2 + i(f\mathbb{F}_r)_x r_x + if[\mathbb{F}, \mathbb{F}_q] q_x - if[\mathbb{F}, \mathbb{F}_r] r_x - \mathbb{K}_q^0 q_x - \mathbb{K}_r^0 r_x$$
$$-\mathbb{K}_x^0 + [\mathbb{F}, \mathbb{K}^0] = 0. \tag{3.210}$$

Since $\mathbb{F}$ and $\mathbb{K}^0$ do not depend on q_x or r_x the coefficients of the different powers of q_x and r_x must be zero. This is equivalent to the system

$$\mathbb{F}_{qq} = \mathbb{F}_{rr} = 0, \tag{3.211a}$$
$$\mathbb{K}_q^0 + i(f\mathbb{F}_q)_x - if[\mathbb{F}, \mathbb{F}_q] = 0, \tag{3.211b}$$
$$\mathbb{K}_r^0 - i(f\mathbb{F}_q)_x + if[\mathbb{F}, \mathbb{F}_r] = 0. \tag{3.211c}$$

From the former two conditions it is clear that $\mathbb{F}$ must be of the form $\mathbb{F} = \mathbb{X}_1 + \mathbb{X}_2 q + \mathbb{X}_3 r + \mathbb{X}_4 qr$ where the $\mathbb{X}_i$ are matrices whose elements are functions of x and t and do not depend on q, r, or their partial derivatives. The inclusion of the $\mathbb{X}_4 qr$ term will not lead to any specific terms present in equation (3.206) and thus we will take $\mathbb{X}_4$ to be zero in order to satisfy the consistency conditions

$$(f\mathbb{X}_4)_x - f[\mathbb{X}_1, \mathbb{X}_4] = 0, \qquad [\mathbb{X}_2, \mathbb{X}_4] = [\mathbb{X}_3, \mathbb{X}_4] = 0, \tag{3.212}$$

which arise in the process of recovering $\mathbb{K}^0$ from equations (3.211b) and (3.211c). Using the derived explicit form for $\mathbb{F}$ we find that $\mathbb{K}^0$ takes the form

$$\mathbb{K}^0 = -i(f\mathbb{X}_2)_x q + if[\mathbb{X}_1, \mathbb{X}_2]q + if[\mathbb{X}_3, \mathbb{X}_2]rq + i(f\mathbb{X}_2)_x r - if[\mathbb{X}_1, \mathbb{X}_3]r \quad (3.213)$$

$$+\mathbb{X}_0, \quad (3.214)$$

where $\mathbb{X}_0$ is a matrix whose elements are functions of x and t and does not depend on q, r, or their partial derivatives. Using $\mathbb{X}_4 = 0$ and the expressions for $\mathbb{F}$ and $\mathbb{K}^0$ to update the zero-curvature condition (3.210) we obtain a rather long expression which is nothing more than a polynomial in q and r. In order for the zero-curvature condition to be satisfied upon solutions to equations (3.206) we must require that the coefficients of the different powers of q and r be zero in much the same fashion as we have done previously. Decoupling this large expression results in the following system,

$$O(1) \quad : \quad \mathbb{X}_{1,t} - \mathbb{X}_{0,x} + [\mathbb{X}_1, \mathbb{X}_0] = 0, \tag{3.215}$$

$$O(q) \quad : \quad \mathbb{X}_{2,t} + ih\mathbb{X}_2 + i(f\mathbb{X}_2)_{xx} - i(f[\mathbb{X}_1, \mathbb{X}_2])_x - i[\mathbb{X}_1, (f\mathbb{X}_2)_x] + if[\mathbb{X}_1, [\mathbb{X}_1, \mathbb{X}_2]] + [\mathbb{X}_2, \mathbb{X}_0] = 0, \tag{3.216}$$

$$O(r) \quad : \quad \mathbb{X}_{3,t} - ih\mathbb{X}_3 - i(f\mathbb{X}_2)_{xx} + i(f[\mathbb{X}_1, \mathbb{X}_3])_x + i[\mathbb{X}_1, (f\mathbb{X}_2)_x] - if[\mathbb{X}_1, [\mathbb{X}_1, \mathbb{X}_2]] + [\mathbb{X}_3, \mathbb{X}_0] = 0, \tag{3.217}$$

$$O(rq) \quad : \quad -i(f[\mathbb{X}_3, \mathbb{X}_2])_x + if[\mathbb{X}_1, [\mathbb{X}_3, \mathbb{X}_2]] + if[\mathbb{X}_2, \mathbb{X}_{2,x}] - if[\mathbb{X}_2, [\mathbb{X}_1, \mathbb{X}_3]] - i[\mathbb{X}_3, (f\mathbb{X}_2)_x] + if[\mathbb{X}_3, [\mathbb{X}_1, \mathbb{X}_2]] = 0, \tag{3.218}$$

$$O(r^2) \quad : \quad [\mathbb{X}_3, (f\mathbb{X}_2)_x] - f[\mathbb{X}_3, [\mathbb{X}_1, \mathbb{X}_3]] = 0, \tag{3.219}$$

$$O(q^2) \quad : \quad [\mathbb{X}_2, \mathbb{X}_{2,x}] - [\mathbb{X}_2, [\mathbb{X}_1, \mathbb{X}_2]] = 0, \tag{3.220}$$

$$O(r^2 q) \quad : \quad f[\mathbb{X}_3, [\mathbb{X}_3, \mathbb{X}_2]] - g_1\mathbb{X}_3 = 0, \tag{3.221}$$

$$O(q^2 r) \quad : \quad f[\mathbb{X}_2, [\mathbb{X}_3, \mathbb{X}_2]] + g_1\mathbb{X}_2 = 0, \tag{3.222}$$

$$O(q^3 r^2) \quad : \quad \mathbb{X}_2 g_2 = 0, \tag{3.223}$$

$$O(r^3 q^2) \quad : \quad \mathbb{X}_3 g_2 = 0. \tag{3.224}$$

At this point from the final two conditions it is clear that we must either have $\mathbb{X}_2 = \mathbb{X}_3 = 0$ or $g_2 = 0$. As we required $g_2 \neq 0$ this means we must take $\mathbb{X}_2 = \mathbb{X}_3 = 0$. But this completely removes the q and r dependence of both $\mathbb{F}$ and $\mathbb{G}$ rendering the Lax pair trivial. Therefore no nontrivial Lax pair exists to equation (3.205).

PART II: SOLUTIONS OF CONSTANT COEFFICIENT INTEGRABLE SYSTEMS

CHAPTER 4: EXACT SOLUTIONS OF NONLINEAR PDES

Introduction

In this chapter we employ three types of singular manifold methods, namely the truncated Painlevé, invariant truncated Painlevé, and generalized Hirota expansions to derive exact traveling and non-traveling wave solutions to various PDEs which occur in mathematical physics. Truncated Painlevé, invariant truncated Painlevé, and the generalized Hirota expansion methods have been extensively used over the past 30 years to derive solutions to a wide variety of nlpdes. In this chapter we will give a brief introduction to each method and subsequently demonstrate each method on a classic example for which the results are well known. Among the nlpdes considered will be the KdV equation, Kadomtsev-Petviashvili II (KP-II) equation, a microstructure PDE which arises within the context of one-dimensional wave propagation in microstructured solids [34] - [38]

$$v_{tt} - bv_{xx} - \frac{\mu}{2}(v^2)_{xx} - \delta(\beta v_{tt} - \gamma v_{xx})_{xx} = 0 \tag{4.1}$$

where b, μ, δ, β, and γ are dimensionless parameters and v denotes the macro-deformation, and two versions

$$(u - u_{xx})_{tt} - (a_1 u + a_2 u^2 + a_3 u^3)_{xx} = 0 \tag{4.2}$$

and

$$(u - u_{xx})_{tt} - (a_1 u + a_3 u^3 + a_5 u^5)_{xx} = 0 \tag{4.3}$$

of a Pochhammer-Chree equation which describe the propagation of longitudinal deformation waves [39] - [44] in elastic rods. While the results for the KdV and KP equations are known, the solutions derived by the aforementioned methods for equations (4.1), (4.2), and (4.3) are new.

Singular Manifold Methods

In this section we will give a brief description of the truncated Painlevé, invariant Painlevé, and generalized Hirota expansion methods which will be subsequently utilized to derive exact solutions. Each of these methods stem from a broader class of solution methods known as singular manifold methods and thus will begin quite similarly. Given a nonlinear PDE we seek a Laurent series solution centered about a movable singular manifold $\phi(\mathbf{x}, t) = 0$. That is, for a nonlinear PDE

$$N\left(u, \frac{\partial^{i+l} u}{\partial t^i \partial x_1^{l_1} \cdots \partial x_n^{l_n}}\right) = 0, \qquad l_1 + \cdots l_n = l \tag{4.4}$$

in $(n+1)$, we seek solution of the form

$$u(\mathbf{x}, t) = \phi(\mathbf{x}, t)^{-\alpha} \sum_{n=0}^{\alpha} u_n(\mathbf{x}, t)\phi(\mathbf{x}, t)^n \tag{4.5}$$

where the u_n are functions to be determined and α is the singularity degree. This truncated expansion of course only makes sense for $\alpha \in \mathbb{N}$ thus implying that the function $u(\mathbf{x}, t)$ is single-valued about the movable, singularity manifolds. This condition was given by Weiss, Tabor, and Carnevale [45] as an extension of the well-known Painlevé property from ODEs to PDEs. However, this may prove to be too strict a condition while endeavoring to obtain exact solutions. To

circumvent this issue and thus extend the applicability of the truncated singular manifold method for the cases when $\alpha \notin \mathbb{N}$ one may define a substitution $v(\mathbf{x}, t) = F(u(\mathbf{x}, t))$ for some suitably well-defined (invertible) function $F(\cdot)$ such that for the transformed equation

$$\tilde{N}\left(u, \frac{\partial^{i+l} u}{\partial t^i \partial x_1^{l_1} \cdots \partial x_n^{l_n}}\right) = 0, \qquad l_1 + \cdots l_n = l \tag{4.6}$$

we may seek a solution of the form

$$v(\mathbf{x}, t) = \phi(\mathbf{x}, t)^{-\beta} \sum_{n=0}^{\beta} v_n(\mathbf{x}, t) \phi(\mathbf{x}, t)^n \tag{4.7}$$

where the v_n are functions to be determined and $\beta \in \mathbb{N}$ is the singularity degree. The solution to the original nonlinear PDE is then given by $u(\mathbf{x}, t) = F^{-1}(v(\mathbf{x}, t))$.

Plugging the truncated series expansion for $u(\mathbf{x}, t)$ into the nonlinear PDE and reconciling the powers of the unknown function ϕ will yield a recurrence relation from which we will determine the u_n and α.

Truncated Painlevé Analysis Method

In this section we will briefly outline the truncated Painlevé analysis method for nonlinear PDEs as it was introduced by Weiss, Tabor, and Carnevale [45] in 1983. Following their procedure and continuing from where we left off in the previous section, we may reduce the order and complexity of the recurrence relations with the introduction of the additional functions

$$C_0(x_0,\ldots,x_n,t) = \frac{\phi_t}{\phi_{x_0}} \tag{4.8}$$

$$C_1(x_0,\ldots,x_n,t) = \frac{\phi_{x_1}}{\phi_{x_0}} \tag{4.9}$$

$$\vdots \tag{4.10}$$

$$C_n(x_0,\ldots,x_n,t) = \frac{\phi_{x_n}}{\phi_{x_0}} \tag{4.11}$$

$$V(x_0,\ldots,x_n,t) = \frac{\phi_{x_0x_0}}{\phi_{x_0}} \tag{4.12}$$

It is clear that after making use of these additonal functions we will have eliminated all partial derivatives of ϕ excepting ϕ_x. For simplicity it is common to allow the $C_i(\mathbf{x},t)$ and $V(\mathbf{x},t)$ to be constants, thereby reducing a system of PDEs (more than likely nonlinear) in $\{C_i(\mathbf{x},t), V(\mathbf{x},t)\}$ to an algebraic system in $\{C_i, V\}$ for $(i = 0,\ldots,n)$. It is straightforward to see that this simplification is equivalent assuming a traveling-wave form for ϕ.

Once we have determined the unknown coefficient functions u_n in the truncated series solution as well as the unknown $C_i(\mathbf{x},t)$ and $V(\mathbf{x},t)$ we will have the required information to recover the singularity manifold ϕ. Upon determining ϕ we may plug it into the truncated expansion, thus obtaining an exact solution to the original PDE.

Example: KP-II Equation

Before presenting the main results for the microstructure PDE and Pochhammer-Chree equations we will demonstrate the effectiveness of the truncated Painlevé analysis method on a classic example in $(2+1)$, the KP-II equation. The KP-II equation [51]- [54], a two-dimensional generalization of the well-known KdV equation describing weakly transverse water waves in a long-wave regime

with small surface tension is given by

$$(u_t + uu_x + \epsilon^2 u_{xxx})_x + \lambda u_{yy} = 0. \tag{4.13}$$

Plugging $\phi^{-\alpha}$ into (4.13) and balancing the $(uu_x)_x$ and u_{xxxx} terms we find that the degree of the singularity, α, is equal to 2. Therefore we seek a solution of the form

$$u(x,y,t) = \frac{u_0(x,y,t)}{\phi(x,y,t)^2} + \frac{u_1(x,y,t)}{\phi(x,y,t)} + u_2(x,y,t) \tag{4.14}$$

for equation (4.13). Plugging the truncated expansion (4.14) into (4.13) and resolving the powers of ϕ yields the recurrence relation

$$\begin{aligned}
&u_{n-4,xt} + (n-5)u_{n-3,x}\phi_t + (n-5)u_{n-3}\phi_{xt} + (n-4)(n-5)u_{n-2}\phi_x\phi_t \\
&+ (n-5)u_{n-3,t}\phi_x + \epsilon^2(u_{n-4,xxxx} + 4(n-5)u_{n-3,xxx}\phi_x + 6(n-5)u_{n-3,xx}\phi_{xx} \\
&+ 6(n-4)(n-5)u_{n-2,xx}\phi_x^2 + 4(n-5)u_{n-3,x}\phi_{xxx} + 12(n-4)(n-5)u_{n-2,x}\phi_x\phi_{xx} \\
&+ 4(n-3)(n-4)(n-5)u_{n-1,x}\phi_x^3 + (n-5)u_{n-3}\phi_{xxxx} + 4(n-4)(n-5)u_{n-2}\phi_x\phi_{xxx} \\
&+ 3(n-4)(n-5)u_{n-2}\phi_{xx}^2 + 6(n-3)(n-4)(n-5)u_{n-1}\phi_x^2\phi_{xx} \\
&+ (n-2)(n-3)(n-4)(n-5)u_n\phi_x^4) + \lambda(u_{n-4,yy} + 2(n-5)u_{n-3,y}\phi_y + (n-5)u_{n-3}\phi_{yy} \\
&+ (n-4)(n-5)u_{n-2}\phi_y^2) + \sum_{k=0}^{n}(u_{k,x}u_{n-k-2,x} + (n-k-3)u_{k,x}u_{n-k-1}\phi_x + (k-2)u_k u_{n-k-1,x}\phi_x \\
&+ (k-2)(n-k-2)u_k u_{n-k}\phi_x^2 + u_k(u_{n-k-2,xx} + 2(n-k-3)u_{n-k-1,x}\phi_x \\
&+ (n-k-3)u_{n-k-1}\phi_{xx} + (n-k-2)(n-k-3)u_{n-k}\phi_x^2)) = 0
\end{aligned} \tag{4.15}$$

where $u_n = 0$ if $n < 0$ and $u_{n,x} \equiv \frac{\partial}{\partial x}(u_n)$. Upon satisfaction of the recurrence relation at the lowest powers of ϕ we find

$$n = 0 \quad : \quad u_0 = -12\epsilon^2\phi_x^2 \tag{4.16}$$

$$n = 1 \quad : \quad u_1 = 12\epsilon^2\phi_{xx} \tag{4.17}$$

$$n = 2 \quad : \quad u_2 = \phi_x^{-2}(3\epsilon^2\phi_{xx}^2 - \phi_x\phi_t - 4\epsilon^2\phi_x\phi_{xxx} - \lambda\phi_y^2) \tag{4.18}$$

which we may write in the cleaner form

$$u(x, y, t) = -2[\log(\phi)]_{xx} + u_2(x, y, t) \tag{4.19}$$

For simplicity we allow $C(x, y, t)$ (our $C_0(x, y, t)$ function), $V(x, y, t)$ and $Q(x, y, t)$ (our $C_1(x, y, t)$ function) to be the constants C, V, and Q, respectively. Upon making this simplification we easily reconcile the remainder of the coefficients of ϕ thus obtaining the solution

$$u_0(x, y, t) = -12\epsilon^2\phi_x^2 \tag{4.20}$$

$$u_1(x, y, t) = 12\epsilon^2 V\phi_x \tag{4.21}$$

$$u_2(x, y, t) = 3\epsilon^2 V^2 - C - 4\epsilon^2 V^2 - \lambda Q^2 \tag{4.22}$$

$$\phi(x, y, t) = c_1 + c_2 e^{V(x+Qy+Ct)} \tag{4.23}$$

where c_1 and c_2 are arbitrary constants. Our solution can be written in the more compact form

$$u(x,y,t) = -12\epsilon^2 \left[\log(\phi(x,y,t))\right]_{xx} + 3\epsilon^2 V^2 - C - 4\epsilon^2 V^2 - \lambda Q^2 \tag{4.24}$$

From this form of the solution it follows that a nontrivial (in this context non-constant) solution requires $c_1 \neq 0$.

We will now plot the solution derived above for two differents choices for the set of parameters (see figure 4.1). For the first choice we take $(\epsilon, \lambda, V, Q, C, c_1, c_2) = (1, 1, -1, 1, i, 1, 1)$. The solution (4.24) at $t = 1$ then becomes

$$u(x,y,1) = -\frac{12e^{-x-y-i}}{(1+e^{-x-y-i})^2} - 2 - i. \tag{4.25}$$

It is clear that for this set of parameters the solution is complex-valued. Therefore to better visualize the solution we plot the real and imaginary parts separately.

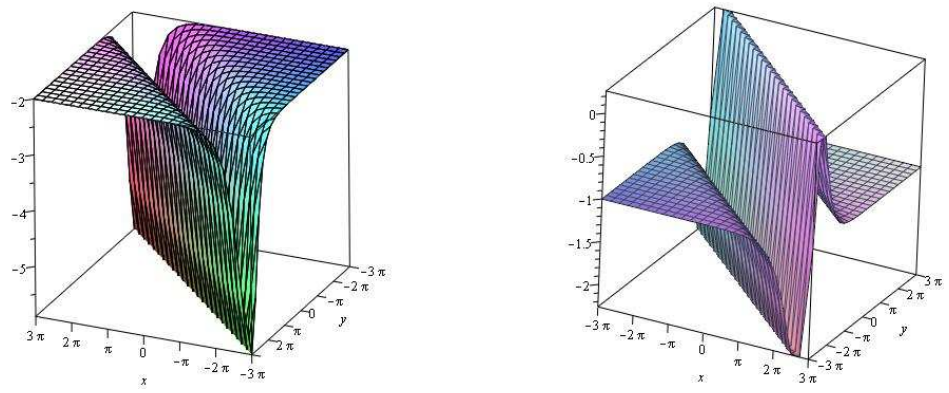

Figure 4.1: Plots of the real (left) and imaginary (right) parts of the solution to the KP-II equation at $t = 1$ on the rectangle $[-3\pi, 3\pi] \times [-3\pi, 3\pi]$ with the choice of parameters $(\epsilon, \lambda, V, Q, C, c_1, c_2) = (1, 1, -1, 1, i, 1, 1)$.

As a second choice for the set of parameters we will take $(\epsilon, \lambda, V, Q, C, c_1, c_2) = (1, 1, -1, -1, -2, 1, 1)$ for which the solution (4.24) at $t = 1$ becomes

$$u(x, y, 1) = -\frac{12e^{-x+y+2}}{(1 + e^{-x+y+2})^2} = -6\text{sech}^2\left((1/2)(-x + y + 2)\right). \tag{4.26}$$

The plot for this solution is given in figure 4.2.

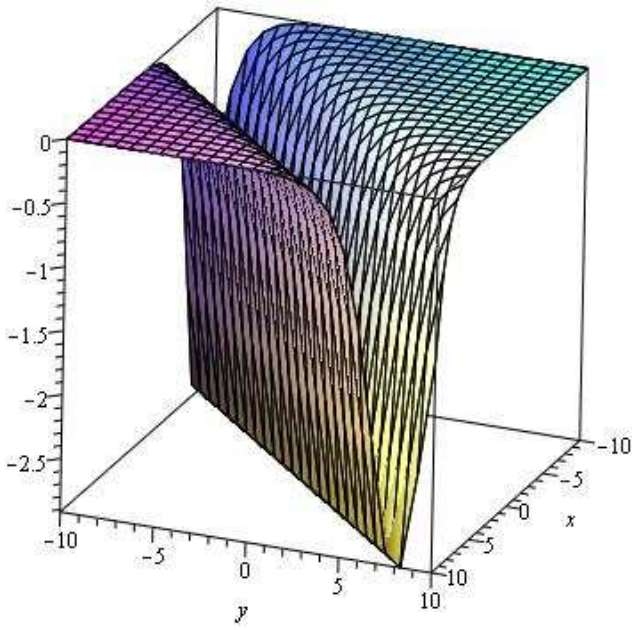

Figure 4.2: Plot of the solution to the KP-II equation at $t = 1$ on the rectangle $[-10, 10] \times [-10, 10]$ with the choice of parameters $(\epsilon, \lambda, V, Q, C, c_1, c_2) = (1, 1, -1, -1, -2, 1, 1)$.

Microstructure PDE

In this section we use the truncated Painlevé analysis method to construct an exact solution to the microstructure PDE given earlier in the section by equation (4.1). Plugging $\phi^{-\alpha}$ into (4.1) and balancing the $(v^2)_{xx}$ and v_{xxxx} terms we find that the degree of the singularity, α, is equal to 2. In fact, this amounted to balancing the same terms that we did for the KP-II equation so we could have borrowed the results from the leading order analysis in the previous section. Since $\alpha = 2$, we seek a solution of the form

$$u(x,t) = \frac{u_0(x,t)}{\phi(x,t)^2} + \frac{u_1(x,t)}{\phi(x,t)} + u_2(x,t) \tag{4.27}$$

for equation (4.1). Plugging the truncated expansion (4.27) into (4.1) and resolving the powers of ϕ yields the large recurrence relation

$$
\begin{aligned}
&u_{m-4,tt} - bu_{m-4,xx} + (m-5)(2u_{m-3,t}\phi_t - 2bu_{m-3,x}\phi_x + u_{m-3}(\phi_{tt} - b\phi_{xx}) \\
&+(m-4)u_{m-2}(\phi_t^2 - b\phi_x^2)) - \mu\sum_{k=0}^{m}\{u_{k,x}u_{m-k-2,x} + u_k u_{m-k-2,xx} \\
&+(m-k-3)(u_{k,x}u_{m-k-1}\phi_x + 2u_k u_{m-k-1,x}\phi_x + (m-k-2)u_k u_{m-k}\phi_x^2 \\
&+u_k u_{m-k-1}\phi_{xx}) + (k-2)(u_k u_{m-k-1,x}\phi_x + (m-k-2)u_k u_{m-k}\phi_x^2)\} - \delta\beta(u_{m-4,xxtt} \\
&+(m-5)(2u_{m-3,xxt}\phi_t + u_{m-3,xx}\phi_{tt} + 2u_{m-3,xtt}\phi_x + 4u_{m-3,xt}\phi_{xt} + 2u_{m-3,x}\phi_{xtt} \\
&+u_{m-3,tt}\phi_{xx} + 2u_{m-3,t}\phi_{xxt} + u_{m-3}\phi_{xxtt} + (m-4)(u_{m-2,xx}\phi_t^2 + u_{m-2,tt}\phi_x^2 \\
&+4u_{m-2,xt}\phi_x\phi_t + 4u_{m-2,x}\phi_t\phi_{xt} + 2u_{m-2,x}\phi_x\phi_{tt} + 4u_{m-2,t}\phi_x\phi_{xt} + 2u_{m-2}\phi_{xt}^2 \\
&+2u_{m-2}\phi_x\phi_{xtt} + 2u_{m-2,t}\phi_t\phi_{xx} + u_{m-2}\phi_t\phi_{xxt} + u_{m-2}\phi_{xx}\phi_{tt} + u_{m-2}\phi_{xxt}\phi_t \\
&+(m-3)(2u_{m-1,x}\phi_x\phi_t^2 + 2u_{m-1,t}\phi_x^2\phi_t + 4u_{m-1}\phi_x\phi_t\phi_{xt} + u_{m-1}\phi_x^2\phi_{tt} + u_{m-1}\phi_{xx}\phi_t^2 \\
&+(m-2)u_m\phi_x^2\phi_t^2)))) + \delta\gamma(u_{m-4,xxxx} + (m-5)(u_{m-3}\phi_{xxxx} + 4u_{m-3,x}\phi_{xxx} \\
&+6u_{m-3,xx}\phi_{xx} + 4u_{m-3,xxx}\phi_x + (m-4)(u_{m-2}(4\phi_x\phi_{xxx} + 3\phi_{xx}^2) + 12u_{m-2,x}\phi_x\phi_{xx} \\
&+6u_{m-2,xx}\phi_x^2 + (m-3)(4u_{m-1,x}\phi_x^3 + 6u_{m-1}\phi_x^2\phi_{xx} + (m-2)u_m\phi_x^4)))) = 0
\end{aligned}
\tag{4.28}
$$

where once again $u_n = 0$ if $n < 0$ and $u_{n,x} \equiv \frac{\partial}{\partial x}(u_n)$. Upon satisfaction of the recurrence relation at the lowest powers of ϕ we find

$$m=0 \quad : \quad u_0 = -\frac{12\delta}{\mu}(\gamma\phi_t^2 - \beta\phi_x^2) \tag{4.29}$$

$$m=1 \quad : \quad u_1 = \frac{12\delta}{5\mu(\gamma\phi_x^2 - \beta\phi_t^2)}\left(5\gamma^2\phi_x^2\phi_{xx} + 5\beta^2\phi_t^2\phi_{tt} - \beta\gamma(\phi_t^2\phi_{xx} + \phi_x^2\phi_{tt}) - 8\beta\gamma\phi_t\phi_x\phi_{xt}\right) \tag{4.30}$$

$$\begin{aligned} m=2 \quad : \quad u_2 = {} & \frac{1}{25\phi_x^3\mu(\beta\phi_t^2 - \gamma\phi_x^2)^3}(25\beta^3\phi_x\phi_t^8 - 25\gamma^3\phi_x^7\phi_t^2 + 25\gamma^3 b\phi_x^9 - 25\beta^3 b\phi_x^3\phi_t^9 - 75\beta^2\gamma\phi_x^3\phi_t^6 \\ & + 75\beta\gamma^2\phi_x^5\phi_t^4 + 171\delta\gamma^4\phi_x^7\phi_{xx}^2 - 120\delta\gamma^4\phi_x^8\phi_{xxx} + 75\beta^2\gamma b\phi_x^5\phi_t^4 - 75\beta\gamma^2 b\phi_x^2\phi_t^2 \\ & - 120\delta\beta^3\gamma\phi_x^2\phi_{xx}\phi_{xt}\phi_t^5 + 292\delta\beta^2\gamma^2\phi_x^4\phi_{xx}\phi_{xt}\phi_t^3 - 856\delta\beta\gamma^3\phi_x^6\phi_{xx}\phi_{xt}\phi_t^3 - 128\delta\beta^3\gamma\phi_x^4\phi_{xt}\phi_t^3\phi_{tt} \\ & - 10\delta\beta^3\gamma\phi_x^3\phi_{xx}\phi_t^4\phi_{tt} + 340\delta\beta^2\gamma^2\phi_x^5\phi_{xx}\phi_t^2\phi_{tt} - 356\delta\beta^2\gamma^2\phi_x^6\phi_{xt}\phi_t\phi_{tt} + 160\delta\beta^3\gamma\phi_x^5\phi_t^3\phi_{ttt} \\ & - 60\delta\beta^2\gamma^2\phi_x^7\phi_t\phi_{ttt} - 476\delta\beta^3\gamma\phi_x^3\phi_{xt}^2\phi_t^4 + 584\delta\beta^2\gamma^2\phi_x^5\phi_{xt}^2\phi_t^2 + 215\delta\beta^2\gamma^2\phi_x^3\phi_{xx}^2\phi_t^4 \\ & - 110\delta\beta\gamma^3\phi_x^5\phi_{xx}^2\phi_t^2 - 100\delta\beta^3\gamma\phi_x\phi_{xx}^2\phi_t^6 - 100\delta\beta^3\gamma\phi_x^2\phi_{xxx}\phi_t^6 - 60\delta\beta^3\gamma\phi_x^3\phi_{xxt}\phi_t^5 \\ & + 120\delta\beta^2\gamma^2\phi_x^4\phi_{xxx}\phi_t^4 - 360\delta\beta^2\gamma^2\phi_x^5\phi_{xxt}\phi_t^3 + 100\delta\beta\gamma^3\phi_x^6\phi_{xxx}\phi_t^2 + 300\delta\beta\gamma^3\phi_x^7\phi_{xxt}\phi_t \\ & - 220\delta\beta^4\phi_x^2\phi_{xt}\phi_t^5\phi_{tt} + 22\delta\beta\gamma^3\phi_x^7\phi_{xx}\phi_{tt} + 480\delta\beta^3\gamma\phi_x^4\phi_{xtt}\phi_t^4 - 420\delta\beta^2\gamma^2\phi_x^6\phi_{xtt}\phi_t^2 \\ & + 150\delta\beta^3\gamma\phi_x^5\phi_t^2\phi_{tt}^2 - 100\delta\beta^4\phi_x^3\phi_t^5\phi_{ttt} + 336\delta\beta^4\phi_x\phi_{xt}^2\phi_t^6 + 120\delta\beta^4\phi_x\phi_{xxt}\phi_t^7 + 260\delta\beta\gamma^3\phi_x^7\phi_{xt}^2 \\ & - 20\delta\beta^4\phi_{xx}\phi_{xt}\phi_t^7 - 140\delta\beta^4\phi_x^2\phi_{xtt}\phi_t^6 + 80\delta\beta\gamma^3\phi_x^8\phi_{xtt} + 75\delta\beta^4\phi_x^3\phi_t^4\phi_{tt}^2 - 49\delta\beta^2\gamma^2\phi_x^7\phi_{tt}^2) \end{aligned} \tag{4.31}$$

Once again we will find that without the assumption that $C(x,t)$ and $V(x,t)$ be constants the calculations will become unnecessarily complicated. Under this reduction we find that elimination of the remaining coefficients of ϕ requires that $C = 1$ while V can remain arbitrary. Using $C(x,t) = 1$ and $V(x,t) = V$ (constant) the previous equations reduce to

$$u_0(x,t) = \frac{12\delta(\gamma-\beta)\phi_x^2}{\mu} \tag{4.32}$$

$$u_1(x,t) = -\frac{12\delta(\gamma-\beta)V\phi_x}{\mu} \tag{4.33}$$

$$u_2(x,t) = \frac{V^2\delta(\gamma-\beta)+1-b}{\mu} \tag{4.34}$$

and we obtain the solution for the singularity manifold

$$\phi(x,t) = c_1 + c_2 e^{V(x+t)} \tag{4.35}$$

where c_1 and c_2 are arbitrary constants. As with the KP-II equation we may write the the equation in a more convenient form

$$u(x,t) = \frac{12\delta(\beta-\gamma)}{\mu}\left(\log(\phi(x,t))\right)_{xx} + \frac{V^2\delta(\gamma-\beta)+1-b}{\mu} \tag{4.36}$$

From this form of the solution we make the observation that a nontrivial (once again taken to mean non-constant) solution will require that $c_1 \neq 0$ but also $\delta \neq 0$ and $\beta \neq \gamma$.

We will now plot the solution derived above for the set of parameters $(c_1, c_2, V, \gamma, \mu, \delta, \beta, b) = (1, 1, -1, 1, 1, 1, -1, 1)$. The solution (4.36) then becomes

$$u(x,t) = 2\frac{(e^{-2x-2t} - 10e^{-x-t} + 1)}{(1+e^{-x-t})^2}. \tag{4.37}$$

The plot for this solution is given in figure 4.3.

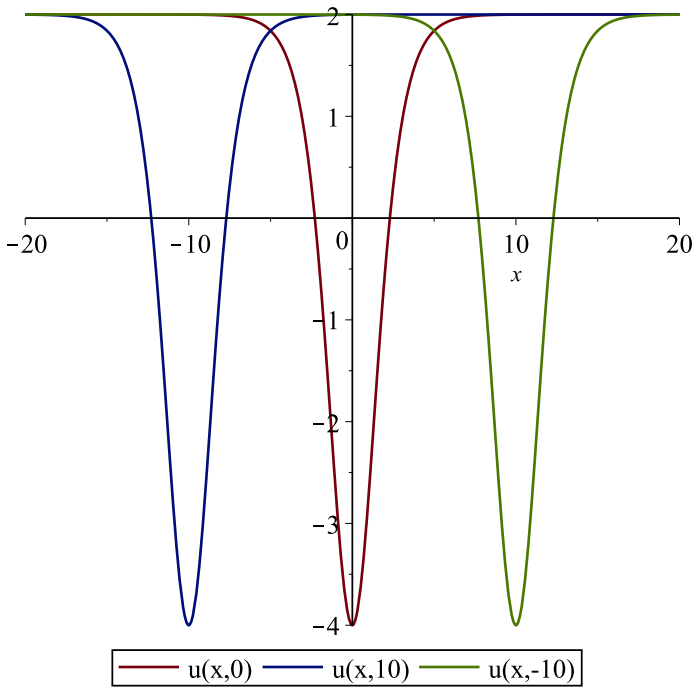

Figure 4.3: Plot of the solution (4.37) for $t = -10, 0, 10$ on the interval $[-20, 20]$ with the choice of parameters $(c_1, c_2, V, \gamma, \mu, \delta, \beta, b) = (1, 1, -1, 1, 1, 1, -1, 1)$.

Generalized Pochhammer-Chree Equations

In this section we use the truncated Painlevé analysis method to construct an exact solution to the generalized Pochhammer-Chree equations given earlier in the section by equations (4.2) and (4.3). Plugging $\phi^{-\alpha}$ into (4.2) and balancing the $(u^3)_{xx}$ and u_{xxtt} terms we find that the degree of the singularity, α, is equal to 1. Following the procedure outlined earlier in the chapter using $\alpha = 1$ and thus we seek a solution of the form

$$u(x,t) = \frac{u_0(x,t)}{\phi(x,t)} + u_1(x,t). \tag{4.38}$$

for equation (4.2). Now consider the following generalized higher-order Pochhammer-Chree equa-

tion

$$(u - u_{xx})_{tt} - \left(\sum_{i=0}^{n} a_{2i+1} u^{2i+1} \right)_{xx} = 0, \tag{4.39}$$

where $a_{2n+1} \neq 0$. Plugging $\phi^{-\alpha}$ into (4.39) and balancing the $(u^{2n+1})_{xx}$ and u_{xxtt} terms leads to $\alpha = \frac{1}{n}$. For $n \geq 2$ (e.g. polynomial nonlinearity of degree 5 or higher), α will be noninteger. We may circumvent this by substituting $u(x,t) = [v(x,t)]^{\frac{1}{n}}$. for which α becomes 1 for all $n \geq 1$. Plugging $u(x,t) = [v(x,t)]^{\frac{1}{n}}$ into equation (4.39) and multiplying through by $n^4[v(x,t)]^{4-\frac{1}{n}}$ we obtain the following complicated new NLPDE

$$\begin{aligned} 0 \;=\; & -(1-n)(1-2n)(1-3n)v_t^2 v_x^2 - n(1-n)(1-2n)v(v_x^2 v_{tt} + 4v_x v_t v_{xt} + v_t^2 v_{xx}) \\ & + n^2(1-n)v^2(v_t^2 - 2v_x v_{xtt} - v_{xx}v_{tt} - 2v_t v_{xxt} - 2v_{xt}^2) + n^3 v^3(v_{tt} - v_{xxtt}) \\ & - v^2 n^2 \sum_{i=0}^{n} a_{2i+1}(2i+1)v^{\frac{2i}{n}} \left\{ (2i+1-n)v_x^2 + nvv_{xx} \right\}. \end{aligned} \tag{4.40}$$

Note that if $n = 1$ (corresponding to equation (4.2)) then most of the terms in equation (4.40) vanish yielding fewer (and smaller) determining equations for the u_i, C, and V. For $n \neq 1$, however, these terms do not vanish. We then find that the resulting system of equations for the u_i, C, and V are overdetermined and in fact inconsistent. For this reason we will proceed with only the derivation of the solution to equation (4.2) here. Plugging (4.38) into (4.2) with $\alpha = 1$ gives us the following recurrence relation for the u_m's

$$
\begin{aligned}
0 \;=\;& -2(m-2)(m-3)(m-4)\phi_x\phi_t^2 u_{m-1,x} - (m-3)(m-4)(2\phi_t\phi_{xxt} + \phi_{tt}\phi_{xx} + 2\phi_x\phi_{xtt})u_{m-2} \\
& -2(m-3)(m-4)(\phi_t\phi_{xx} + 2\phi_x\phi_{xt})u_{m-2,t} - 4(m-3)(m-4)\phi_t\phi_x u_{m-2,xt} \\
& -2(m-3)(m-4)(\phi_x\phi_{tt} + 2\phi_t\phi_{xt})u_{m-2,x} + u_{m-4,tt} - (m-1)(m-2)(m-3)(m-4)\phi_t^2\phi_x^2 u_m \\
& -u_{m-4,xxtt} - 2(m-2)(m-3)(m-4)\phi_t\phi_x^2 u_{m-1,t} - 2(m-3)(m-4)\phi_{xt}^2 u_{m-2} \\
& -(m-3)(m-4)\phi_t^2 u_{m-2,xx} - (m-2)(m-3)(m-4)\phi_t^2\phi_{xx}u_{m-1} + (m-3)(m-4)\phi_t^2 u_{m-2} \\
& -(m-2)(m-3)(m-4)\phi_x^2\phi_{tt}u_{m-1} - (m-3)(m-4)\phi_x^2 u_{m-2,tt} - a_1((m-3)(m-4)\phi_x^2 u_{m-2} \\
& +2(m-4)\phi_x u_{m-3,x} + (m-4)\phi_{xx}u_{m-3} + u_{m-4,xx}) - 4(m-2)(m-3)(m-4)\phi_x\phi_t\phi_{xt}u_{m-1} \\
& +2(m-4)\phi_t u_{m-3,t} - 2(m-4)\phi_t u_{m-3,xxt} - 2(m-4)\phi_x u_{m-3,xtt} - (m-4)\phi_{xx}u_{m-3,tt} \\
& -2(m-4)\phi_{xtt}u_{m-3,x} - (m-4)\phi_{xxtt}u_{m-3} - 2(m-4)\phi_{xxt}u_{m-3,t} - (m-4)\phi_{tt}u_{m-3,xx} \\
& -4(m-4)\phi_{xt}u_{m-3,xt} + (m-4)\phi_{tt}u_{m-3} - 2a_2\sum_{j=0}^{m}[(m-j-2)(m-j-3)\phi_x^2 u_j u_{m-j-1} \\
& +2(m-j-3)\phi_x u_j u_{m-j-2,x} + (m-j-3)\phi_{xx}u_j u_{m-j-2} + u_j u_{m-j-3,xx}] \\
& -3a_3\sum_{j=0}^{m}\sum_{k=0}^{m-j}[(m-k-j-1)(m-k-j-2)\phi_x^2 u_j u_k u_{m-k-j} + u_j u_k u_{m-k-j-2,xx} \\
& +(m-k-j-2)\phi_{xx}u_j u_k u_{m-k-j-1} + 2(m-k-j-2)\phi_x u_j u_k u_{m-k-j-1,x}] \\
& -6a_3\sum_{j=0}^{m}\sum_{k=0}^{m-j}[(m-k-j-1)(k-1)\phi_x^2 u_j u_k u_{m-k-j} + (m-k-j-2)\phi_x u_j u_{k,x} u_{m-k-j-1} \\
& +u_j u_{k,x} u_{m-k-j-2,x} + (k-1)\phi_x u_j u_k u_{m-j-k-1,x}] - 2a_2\sum_{j=0}^{m}[(j-1)(m-j-2)\phi_x^2 u_j u_{m-j-1} \\
& +(j-1)\phi_x u_j u_{m-j-2,x} + (m-j-3)\phi_x u_{j,x} u_{m-j-2} + u_{j,x} u_{m-j-3,x}]
\end{aligned}
$$

where $u_m = 0$ if $m < 0$ and $u_{m,x} \equiv \frac{\partial}{\partial x}(u_m)$.

Iterating through m values we obtain the coefficients of the different powers of ϕ. Solving the first two for u_0 and u_1 we have

$$m = 0 \quad : \quad u_0 = -\frac{2\phi_t}{\sqrt{-2a_3}} \tag{4.41}$$

$$m = 1 \quad : \quad u_1 = -\frac{1}{3}\frac{a_2\phi_t\sqrt{-2a_3} - 3a_3\phi_{tt}}{a_3\phi_t\sqrt{-2a_3}} \tag{4.42}$$

$$\tag{4.43}$$

Substituting in C and V and letting both be constants the solution to the remaining system of equations is easily found to be

$$u_0(x,t) = \frac{2C\phi_x}{\sqrt{-2a_3}} \tag{4.44}$$

$$u_1(x,t) = \frac{\sqrt{2}(3CVa_3 + a_2\sqrt{-2a_3})}{6(-a_3)^{3/2}} \tag{4.45}$$

$$C = -\text{sgn}\left(3a_1a_3 - a_2^2\right) \tag{4.46}$$

$$V = \frac{\sqrt{6a_3(3a_1a_3 - a_2^2 - 3a_3)}}{3a_3} \tag{4.47}$$

$$\phi(x,t) = c_1 + c_2e^{V(x+Ct)} \tag{4.48}$$

where $\text{sgn}(x) = \begin{cases} 1\,, & x > 0 \\ -1\,, & x < 0 \end{cases}$. This again lends itself to a rather nice expression for $u(x,t)$ given by

$$u(x,t) = \frac{2C}{\sqrt{-2a_3}}\left(\log(\phi(x,t))\right)_x + \frac{\sqrt{2}(3CVa_3 + a_2\sqrt{-2a_3})}{6(-a_3)^{3/2}} \tag{4.49}$$

for equation (4.2). We will now plot the solution derived above for the set of parameters $(c_1, c_2, a_1, a_2, a_3) =$

$(1, 1, 1, 1, -1)$. The solution (4.49) then becomes

$$u(x,t) = -\frac{2}{\sqrt{3}} \frac{e^{-\sqrt{\frac{2}{3}}(x+t)}}{1 + e^{-\sqrt{\frac{2}{3}}(x+t)}} + \frac{1+\sqrt{3}}{3}. \tag{4.50}$$

The plot for this solution is given in figure 4.4.

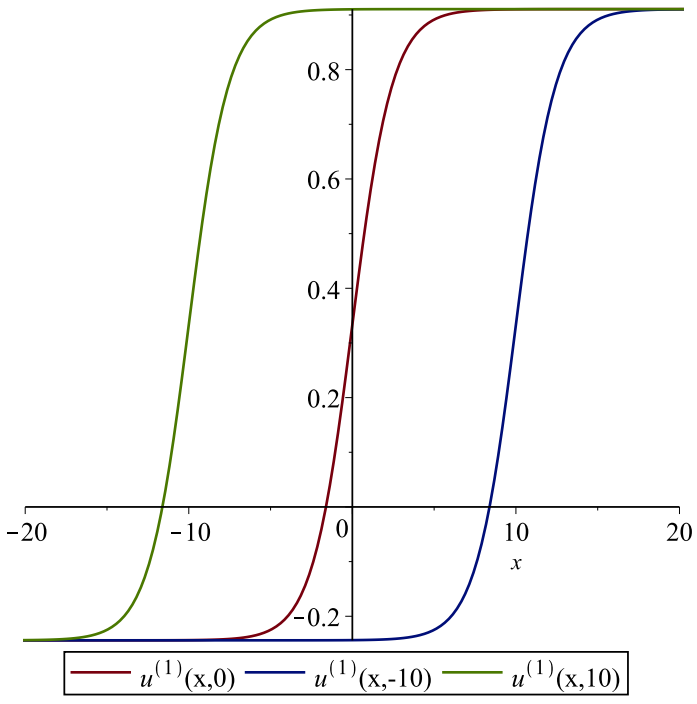

Figure 4.4: Plot of the solution (4.50) for $t = -10, 0, 10$ on the interval $[-20, 20]$ with the choice of parameters $(c_1, c_2, a_1, a_2, a_3) = (1, 1, 1, 1, -1)$.

Invariant Painlevé Analysis Method

In 1989 Conte [46] showed that the Painlevé analysis of PDEs was in fact invariant under an arbitrary homographic transformation of the singularity manifold ϕ. That is, the analysis was invariant under any transformation of ϕ of the form

$$\phi \mapsto \frac{a\phi + b}{c\phi + d} \qquad \text{s.t.} \quad ad - bc = 1. \tag{4.51}$$

He found the "best" choice of new expansion function to be the function

$$\chi \equiv \frac{\psi}{\psi_{x_1}} = \left(\frac{\phi_{x_1}}{\phi - \phi_0} - \frac{\phi_{x_1 x_1}}{2\phi_{x_1}} \right)^{-1} \tag{4.52}$$

$$\psi = (\phi - \phi_0)\phi_{x_1}^{-1/2} \tag{4.53}$$

From this definition it can then be shown that the new expansion variable χ satisfies the following Ricatti equations

$$\begin{aligned}
\chi_{x_1} &= 1 + \frac{1}{2} S\chi^2 \\
\chi_t &= -C_1 + C_{1,x_1}\chi - \frac{1}{2}(C_1 S + C_{1,x_1 x_1})\chi^2 \\
\chi_{x_2} &= -C_2 + C_{2,x_1}\chi - \frac{1}{2}(C_2 S + C_{2,x_1 x_1})\chi^2 \\
&\vdots \\
\chi_{x_n} &= -C_n + C_{n,x_1}\chi - \frac{1}{2}(C_n S + C_{n,x_1 x_1})\chi^2
\end{aligned} \tag{4.54}$$

and thus ψ satisfies the following linear equations

$$\begin{aligned}
\psi_{x_1x_1} &= -\frac{1}{2}S\psi \\
\psi_t &= \frac{1}{2}C_{1,x_1}\psi - C_1\psi_{x_1} \\
\psi_{x_2} &= \frac{1}{2}C_{1,x_1}\psi - C_1\psi_{x_1} \\
&\vdots \\
\psi_{x_n} &= \frac{1}{2}C_{n,x_1}\psi - C_n\psi_{x_1}
\end{aligned} \tag{4.55}$$

where S (the Schwarzian derivative) and the C_i are defined by

$$S = \frac{\phi_{x_1x_1x_1}}{\phi_{x_1}} - \frac{3}{2}\left(\frac{\phi_{x_1x_1}}{\phi_{x_1}}\right)^2 \tag{4.56}$$

$$C_1 = -\frac{\phi_t}{\phi_{x_1}} \tag{4.57}$$

$$C_i = -\frac{\phi_{x_i}}{\phi_{x_1}} \ (2 \geq i \leq n) \tag{4.58}$$

and are also invariant under the group of homographic transformations. It is important to note that the systems (4.54) and (4.55) are equivalent to each other. The C_i and S are linked by the cross derivative conditions

$$\phi_{x_1x_1x_1t} = \phi_{tx_1x_1x_1} \tag{4.59}$$

$$\phi_{x_1x_1x_1x_2} = \phi_{x_2x_1x_1x_1} \tag{4.60}$$

$$\vdots \tag{4.61}$$

$$\phi_{x_1x_1x_1x_n} = \phi_{x_nx_1x_1x_1} \tag{4.62}$$

which are equivalent to the conditions

$$S_t + C_{1,x_1x_1x_1} + 2C_{1,x_1}S + C_1S_{x_1} = 0 \tag{4.63}$$

$$S_{x_2} + C_{2,x_1x_1x_1} + 2C_{2,x_1}S + C_2S_{x_1} = 0 \tag{4.64}$$

$$\vdots \tag{4.65}$$

$$S_{x_n} + C_{n,x_1x_1x_1} + 2C_{n,x_1}S + C_nS_{x_1} = 0 \tag{4.66}$$

Upon determining the unknown coefficients in our truncated expansion and resolving these conditions, thereby determining our expansion function χ we will have obtained an exact solution to the original PDE.

Example: KdV Equation

Before presenting the main results for the microstructure PDE and Pochhammer-Chree equations we would like to demonstrate the effectiveness of the truncated invariant Painlevé analysis method on the KdV equation. The KdV equation enjoys a wide variety of applications accross multiple

disciplines. The KdV equation [55] - [57], which describes the propagation of weakly dispersive and weakly nonlinear water waves, is often given by

$$u_t + u_{xxx} - 6uu_x = 0 \tag{4.67}$$

Plugging $\chi^{-\alpha}$ into (4.67) and balancing the uu_x and u_{xxx} terms we find that the degree of the singularity, α, is equal to 2. Therefore we seek a solution of the form

$$u(x,t) = \frac{u_0(x,t)}{\chi(x,t)^2} + \frac{u_1(x,t)}{\chi(x,t)} + u_2(x,t) \tag{4.68}$$

for equation (4.67). Plugging the truncated expansion (4.68) into (4.67), eliminating the partial derivatives of χ, and collecting in powers of χ yields the system of equations

$$O(\chi^{-5}) \quad : \quad -24u_0 + 12u_0^2 = 0 \tag{4.69}$$

$$O(\chi^{-4}) \quad : \quad -6u_0(u_{0,x} - u_1) + 12u_0u_1 + 18u_{0,x} - 6u_1 = 0 \tag{4.70}$$

$$O(\chi^{-3}) \quad : \quad 2u_0C - 20u_0S - 6u_0(u_{1,x} - u_0S) - 6u_1(u_{0,x} - u_1) + 12u_2u_0 - 6u_{0,xx} + 6u_{1,x} = 0 \tag{4.71}$$

$$\begin{aligned} O(\chi^{-2}) \quad : \quad & u_{0,t} - u_{1,xx} - 4u_1S + 21u_{0,x}S - 6u_0\left(u_{2,x} - \frac{1}{2}u_1S\right) - 6u_1(u_{1,x} - u_0S) \\ & -6u_2(u_{0,x} - u_1) + 5u_0S_x - 2u_0C_x + u_1C + u_{0,xxx} = 0 \end{aligned} \tag{4.72}$$

$$\begin{aligned} O(\chi^{-1}) \quad : \quad & u_{1,t} + 3u_{1,x}S - 3u_{0,x}S_x - 6u_1\left(u_{2,x} - \frac{1}{2}u_1S\right) - 6u_2(u_{1,x} - u_0S) \\ & +u_0(CS + C_{xx}) + u_1S_x - 3u_{0,xx}S - 2u_0\left(\frac{1}{2}S_{xx} + 2S^2\right) - u_1C_x + u_{1,xxx} = 0 \end{aligned} \tag{4.73}$$

$$\begin{aligned} O(\chi^{0}) \quad : \quad & \frac{3}{2}u_{0,x}S^2 - u_1\left(\frac{1}{2}S_{xx} + 2S^2\right) + \frac{3}{2}u_1S^2 + \frac{3}{2}u_0SS_x + u_{2,t} - \frac{3}{2}u_{1,x}S_x - \frac{3}{2}u_{1,xx}S \\ & -6u_2\left(u_{2,x} - \frac{1}{2}u_1S\right) + u_{2,xxx} + \frac{1}{2}u_1(CS + C_{xx}) = 0 \end{aligned} \tag{4.74}$$

We now have a system of equations for the unknown functions u_0, u_1, u_2, C and S. It is often useful to impose certain conditions (such as C, S constant) to reduce computational complexity, however a major drawback as one may deduce is that our solutions will become more trivial. Instead, leaving S and C as functions of x and t and solving this system with the aid of MAPLE yields the following results

$$
\begin{aligned}
u_0(x,t) &= 2, && (4.75)\\
u_1(x,t) &= 0, && (4.76)\\
u_2(x,t) &= \frac{2}{3}S(x,t) - \frac{1}{6}C(x,t), && (4.77)\\
C(x,t) &= \frac{x}{3t+C_1}, && (4.78)\\
S(x,t) &= \frac{x}{3t+C_1}. && (4.79)
\end{aligned}
$$

Using our equations for ψ and χ we are then able to find the following rather complicated solutions

$$
\begin{aligned}
\psi &= \frac{(3t+C_1)^{1/3}\left(C_3 x\ {}_0F_1(,\frac{4}{3};-\frac{x^3}{54t+18C_1}) + C_2(3t+C_1)^{1/3}\ {}_0F_1(,\frac{2}{3};-\frac{x^3}{54t+18C_1})\right)}{\sqrt{9t+3C_1}}, && (4.80)\\
\chi &= \frac{C_3 x\ {}_0F_1(,\frac{4}{3};-\frac{x^3}{54t+18C_1}) + C_2(3t+C_1)^{1/3}\ {}_0F_1(,\frac{2}{3};-\frac{x^3}{54t+18C_1})}{C_3\ {}_0F_1(,\frac{4}{3};-\frac{x^3}{54t+18C_1}) - \frac{9}{4}\frac{C_3x^3\ {}_0F_1(,\frac{7}{3};-\frac{x^3}{54t+18C_1})}{54t+18C_1} - \frac{9}{2}\frac{C_2(3t+C_1)^{1/3}x^2\ {}_0F_1(,\frac{5}{3};-\frac{x^3}{54t+18C_1})}{54t+18C_1}}, && (4.81)
\end{aligned}
$$

from which our equation for $u(x,t)$ gives us the following solution

$$
\begin{aligned}
u(x,t) &= 2\left(\frac{C_3\ {}_0F_1(,\frac{4}{3};-\frac{x^3}{54t+18C_1}) - \frac{9}{4}\frac{C_3x^3\ {}_0F_1(,\frac{7}{3};-\frac{x^3}{54t+18C_1})}{54t+18C_1} - \frac{9}{2}\frac{C_2(3t+C_1)^{1/3}x^2\ {}_0F_1(,\frac{5}{3};-\frac{x^3}{54t+18C_1})}{54t+18C_1}}{C_3 x\ {}_0F_1(,\frac{4}{3};-\frac{x^3}{54t+18C_1}) + C_2(3t+C_1)^{1/3}\ {}_0F_1(,\frac{2}{3};-\frac{x^3}{54t+18C_1})}\right)^2\\
&\quad + \frac{x}{6t+2C_1}, && (4.82)
\end{aligned}
$$

where ${}_0F_1(,b;z)$ is the hypergeometric function defined by

$$ {}_0F_1(,b;z) = \sum_{k=0}^{\infty} \frac{\Gamma(b) z^k}{\Gamma(b+k)k!}, \tag{4.83} $$

and $\Gamma(a)$ is the gamma function.

We will now plot the solution derived above for the set of parameters $(C_1, C_2, C_3) = (1, 1, 0)$ for which the solution (4.82) then becomes

$$ u(x,t) = -\frac{x}{3t+1} \left[\frac{I_{2/3}\left(\frac{\sqrt{2}}{3}\sqrt{-\frac{x^3}{3t+1}}\right)}{I_{-1/3}\left(\frac{\sqrt{2}}{3}\sqrt{-\frac{x^3}{3t+1}}\right)} \right]^2 . \tag{4.84} $$

The plot for this solution is given in figure 4.5.

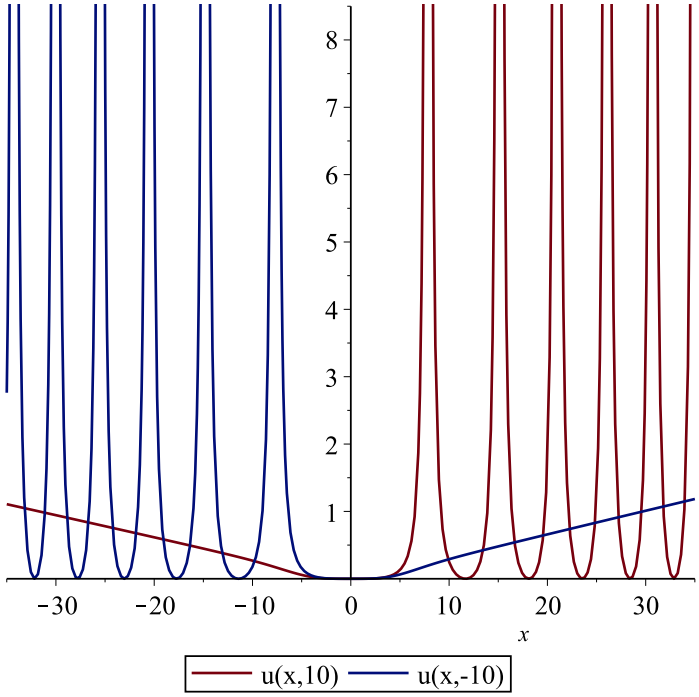

Figure 4.5: Plot of the solution (4.84) for $t = -2$ and 2 on the interval $[-35, 35]$ with the choice of parameters $(C_1, C_2, C_3) = (1, 1, 0)$.

We remark that $u(x, t)$ is not of traveling wave form. The inclusion of both $u(x, -2)$ and $u(x, 2)$ in the same plot was to show near symmetry of the solution (4.84) in t for $|t| \geq 1$.

MicroStructure PDE

In this section we use the truncated invariant Painlevé analysis method to construct an exact solution to the microstructure PDE given earlier in the chapter by equation (4.1). Plugging $\chi^{-\alpha}$ into (4.1) and balancing the $(v^2)_{xx}$ and v_{xxxx} terms we find that the degree of the singularity, α, is equal

to 2. Since $\alpha = 2$, we seek a solution of the form

$$v(x,t) = \frac{v_0(x,t)}{\chi(x,t)^2} + \frac{v_1(x,t)}{\chi(x,t)} + v_2(x,t) \tag{4.85}$$

for equation (4.1). Plugging the truncated expansion (4.85) into (4.1) and and eliminating all derivatives of χ yields the Painleve-Bäcklund equations order by order in χ. Due to the complexity of the Painlevé-Bäcklund equations we shall make a further assumption on $C(x,t)$ that it be constant. With this assumption on the function $C(x,t)$ the first three Painleve-Bäcklund equations become

$$12\beta\delta C^2 v_0 - 12\gamma\delta v_0 + \mu v_0^2 = 0, \tag{4.86}$$

$$-\frac{\mu}{2}\left(8v_0(v_1 - v_{0,x}) + 2v_0(2v_1 - 4v_{0,x}) + 12v_0v_1\right) - \delta[\beta(24C^2v_1 - 48C^2v_{0,x}$$
$$+48Cv_{0,t}) - \gamma(24v_1 - 96v_{0,x})] = 0, \tag{4.87}$$

$$6C^2v_0 - 6bv_0 - \frac{\mu}{2}\left(-8v_0v_{1,x} + 2(v_{0,x} - v_1)^2 + 2v_0(v_{0,xx} - 2v_{1,x}) + 2v_1(2v_1 - 4v_{0,x})\right.$$
$$\left.+12v_0v_2\right) - \delta\left(\beta(24v_0C^2S - 12C^2v_{1,x} + 6C^2v_{0,xx} + 12Cv_{1,t} - 24Cv_{0,xt} + 6v_{0,tt})\right.$$
$$\left.-\gamma(24v_0S - 24v_{1,x} + 36v_{0,xx})\right) - 8\mu v_0^2 S - 96\delta v_0 S\left(\beta C^2 - \gamma\right) = 0, \tag{4.88}$$

where it is important to keep in mind that at this point we have not made any additional assumptions on the function $S(x,t)$. Solving the three equations above for v_0, v_1, and v_2 with the aid of MAPLE we obtain

$$v_0 = -\frac{12\delta(C^2\beta-\gamma)}{\mu}, \tag{4.89}$$

$$v_1 = 0, \tag{4.90}$$

$$v_2 = -\frac{4C^2\beta\delta S-4\gamma\delta S-C^2+b}{\mu}. \tag{4.91}$$

The next step is to use the remaining Painleé-Bäcklund equations to determine the function $S(x,t)$ and, if necessary, the constant C. Using the previous expressions for v_0, v_1, and v_2 with C constant the remaining Painlevé-Bäcklund equations become

$$24\mu^{-1}\left(C^2\beta-\gamma\right)\delta^2\left[\left(18C^2\beta+\gamma\right)S_x+19C\beta S_t\right]=0, \tag{4.92}$$

$$24\mu^{-1}\delta^2\left(\gamma-C^2\beta\right)\left[\left(3C^2\beta+\gamma\right)S_{xx}+4C\beta S_{xt}\right]=0, \tag{4.93}$$

$$12\mu^{-1}\delta\left(C^2\beta-\gamma\right)\left[\left(8C^2\delta\beta+\frac{\delta\gamma}{2}\right)(S^2)_x+\frac{17C\delta\beta}{2}(S^2)_t+\delta\left(C\beta S_t+\gamma S_x\right)_{xx}\right.$$
$$\left.-C^2S_x-CS_t\right]=0, \tag{4.94}$$

$$2\mu^{-1}\delta\left(\gamma-C^2\beta\right)\left[\delta\left(14C^2\beta+4\gamma\right)SS_{xx}+18C\delta\beta SS_{xt}+\delta\left(8C^2\beta+\gamma\right)S_x^2\right.$$
$$\left.-18\delta\beta S_t^2-9C\delta\beta S_xS_t+2S_{tt}-2C^2S_{xx}+2\delta\gamma S_{xxxx}-2\delta\beta S_{xxtt}\right]=0. \tag{4.95}$$

It follows that a rather simple solution would be to take $C=\pm\sqrt{\frac{\gamma}{\beta}}$. Looking back at the forms of u_0 and u_2 we see that this choice would make $u_0=0$ and $u_2=$ constant thereby giving only a constant solution. For less trivial results we will require that $C\neq\pm\sqrt{\frac{\gamma}{\beta}}$. Another simple, yet sufficient, solution to the previous system is to take S to be a constant. Using $C(x,t)=C$, $S(x,t)=S$ (where C and S are constants) and the determining equations for $\psi(x,t)$ we find that

$\psi(x,t)$ takes the form

$$\psi(x,t) = c_1 \sin\left(\sqrt{\frac{S}{2}}(x - Ct)\right) + c_2 \cos\left(\sqrt{\frac{S}{2}}(x - Ct)\right), \tag{4.96}$$

where c_1 and c_2 are arbitrary constants. Substituting this expression for ψ into the expression for χ in terms of ψ (given earlier in the section) we arrive at the following traveling wave solution to equation (4.1),

$$\begin{aligned} v(x,t) &= \frac{6S\delta(\gamma - C^2\beta)\left[c_1 \cos\left(\sqrt{\frac{S}{2}}(x - Ct)\right) - c_2 \sin\left(\sqrt{\frac{S}{2}}(x - Ct)\right)\right]^2}{\mu\left[c_1 \sin\left(\sqrt{\frac{S}{2}}(x - Ct)\right) + c_2 \cos\left(\sqrt{\frac{S}{2}}(x - Ct)\right)\right]^2} \\ &\quad -\mu^{-1}(4S\delta(C^2\beta - \gamma) - C^2 + b). \end{aligned} \tag{4.97}$$

From this form of the solution we make the observation that a nontrivial (once again taken to mean non-constant) solution will require that $c_1 \neq 0$ and $c_2 \neq 0$. We make the additional remark that the solution will have two qualitatively different forms depending on the sign of S. On one hand, for $S > 0$ and $t = t_0$ fixed the solution will be periodic with infinitely many singularities a $x = Ct_0 - \sqrt{\frac{2}{S}} \tan^{-1}\left(-\frac{c_2}{c_1}\right)$. On the other hand, for $S < 0$ the solution will involve hyperbolic sines and hyperbolic cosines and thus will be continuous on $\mathbb{R}^2$ for most parameter set choices. Note that to ensure the solution is real-valued we will take c_2 to be imaginary. We will now plot the solution derived above for two parameter sets such that $S > 0$ in the first set and $S < 0$ in the second set. For the choice $(c_1, c_2, S, C, \gamma, \mu, \delta, \beta, b) = (1, 1, 1, -1, 1, 1, 1, -1, 1)$ ($S > 0$) the

solution (4.97) simplifies to

$$v(x,t) = -\frac{4\sin\left(\sqrt{2}(x+t)\right)+5}{\sin\left(\sqrt{2}(x+t)\right)-1}, \tag{4.98}$$

whereas for the choice $(c_1, c_2, S, C, \gamma, \mu, \delta, \beta, b) = (1, 2i, -1, 1, 1, 1, 1, -1, 1)$ ($S < 0$) the solution (4.97) simplifies to

$$v(x,t) = \frac{4\left[5\cosh^2\left(\frac{t-x}{\sqrt{2}}\right)+4\cosh\left(\frac{t-x}{\sqrt{2}}\right)\sinh\left(\frac{t-x}{\sqrt{2}}\right)-40\right]}{5\cosh^2\left(\frac{t-x}{\sqrt{2}}\right)+4\cosh\left(\frac{t-x}{\sqrt{2}}\right)\sinh\left(\frac{t-x}{\sqrt{2}}\right)-1}. \tag{4.99}$$

The plots of these solutions are given in figure 4.6.

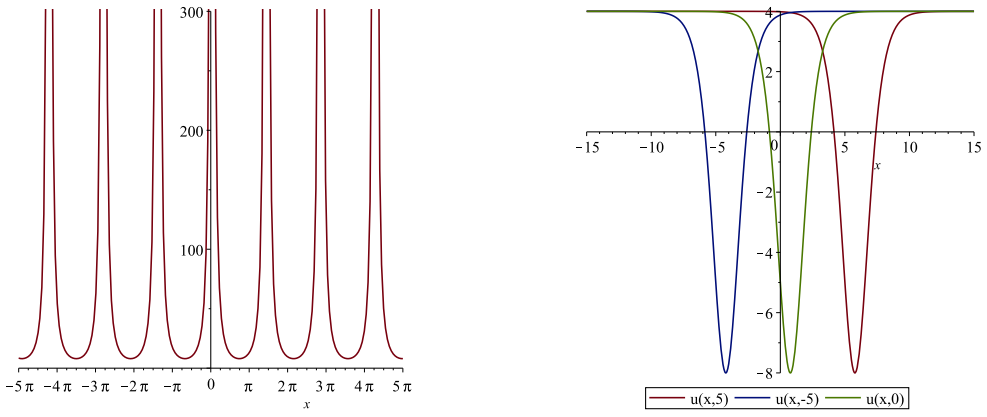

Figure 4.6: (Left): Plot of the solution (4.97) for the choice of parameters $(c_1, c_2, S, C, \gamma, \mu, \delta, \beta, b) = (1, 1, 1, -1, 1, 1, 1, -1, 1)$ at $t = 1$ in the rectangle $[-5\pi, 5\pi]$. (Right): Plot of the solution (4.97) for the choice of parameters $(c_1, c_2, S, C, \gamma, \mu, \delta, \beta, b) = (1, 2i, -1, 1, 1, 1, 1, -1, 1)$ at $t = -5, 0, 5$ on the interval $[-15, 15]$.

Generalized Pochhammer-Chree Equations

In this section we use the truncated invariant Painlevé analysis method to construct an exact solution to the generalized Pochhammer-Chree equations given earlier in the chapter by equations (4.2) and (4.3). Plugging $\chi^{-\alpha}$ into (4.2) and balancing the $(u^3)_{xx}$ and u_{xxtt} terms we find that the degree of the singularity, $\alpha^{(1)}$, is equal to 1. When balancing the $(u^5)_{xx}$ and u_{xxtt} terms in equation (4.3), however, we find that $\alpha^{(2)}$ is equal to $1/2$. Due to the non-integer value of $\alpha^{(2)}$ we need to find an appropriate substitution to make $\alpha^{(2)}$ a nonnegative integer. The substitution we use to accomplish this is $u^{(2)}(x,t) = \left(v^{(2)}(x,t)\right)^{1/2}$ where it is important to keep in mind that the superscripts in the parenthesis are not powers nor derivatives but rather indicate which equation the term corresponds to. That is, any term with $^{(1)}$ corresponds to equation (4.2) and any term with $^{(2)}$ corresponds to equation (4.3). Running through the leading order analysis again on the new system now yields $\alpha^{(2)} = 1$. Therefore we seek solutions of the form

$$u^{(1)}(x,t) = \frac{u_0^{(1)}(x,t)}{\chi^{(1)}(x,t)} + u_1^{(1)}(x,t) \tag{4.100}$$

and

$$v^{(2)}(x,t) = \frac{v_0^{(2)}(x,t)}{\chi^{(2)}(x,t)} + v_1^{(2)}(x,t) \tag{4.101}$$

for equations (4.2) and (4.3), respectively. Plugging these truncated expansions into their respective equations and eliminating all derivatives of χ yields the Painlevé-Bäcklund equations order by order in χ for equations (4.2) and (4.3). Due to the complexity of these systems of equations in both cases we shall once again require that $C(x,t)$ be a constant. For equation (4.2) the first two Painlevé-Bäcklund equations are then

$$-12u_0^{(1)}\left(2C^2+a_3\left(u_0^{(1)}\right)^2\right)=0, \tag{4.102}$$

$$2Cu_{0,t}^{(1)}+3a_3\left(u_0^{(1)}\right)^2u_1^{(1)}-3a_3\left(u_0^{(1)}\right)^2u_{0,x}^{(1)}+a_2\left(u_0^{(1)}\right)^2-2C^2u_{0,x}^{(1)}=0, \tag{4.103}$$

and for equation (4.3) the first two Painlevé-Bäcklund equations are

$$-\frac{35}{4}\left(v_0^{(1)}\right)^4\left(a_5\left(v_0^{(1)}\right)^2+\frac{3}{4}C^2\right)=0, \tag{4.104}$$

$$-\frac{5}{8}\left(v_0^{(1)}\right)^3\left(64a_5\left(v_0^{(1)}\right)^2v_1^{(1)}-20a_5\left(v_0^{(1)}\right)^2v_{0,x}^{(1)}+36C^2v_1^{(1)}-3C^2v_{0,x}^{(1)}\right.$$

$$\left.+6a_3\left(v_0^{(1)}\right)^2+3Cv_{0,t}^{(1)}\right)=0. \tag{4.105}$$

Solving the first two equations for both cases yields the following results

$$u_0^{(1)} = -C\sqrt{\frac{2}{-a_3}} \tag{4.106}$$

$$u_1^{(1)} = -\frac{a_2}{3a_3} \tag{4.107}$$

$$v_0^{(2)} = \frac{\sqrt{3}C}{2\sqrt{-a_5}} \tag{4.108}$$

$$v_1^{(2)} = -\frac{3a_3}{8a_5} \tag{4.109}$$

At this point taking $S^{(i)}(x,t)$ to be a constant for $i=1,2$ reduces the remaining Painlevé-Bäcklund

equations to a system of algebraic equations in the $C^{(i)}$ and $S^{(i)}$. Solving these systems for the $S^{(i)}(x,t)$ and $C^{(i)}(x,t)$ we find

$$C^{(1)} = C, \tag{4.110}$$

$$S^{(1)} = \frac{1}{3}\frac{a_2^3 + 3C^2a_3 - 3a_1a_3}{C^2a_3}, \tag{4.111}$$

and

$$C^{(2)} = \frac{1}{4}\sqrt{\frac{2(9a_3^2 - 32a_1a_5)}{a_5(S^{(2)} - 4)}}, \tag{4.112}$$

$$S^{(2)} = \frac{6a_3^2}{16a_1a_5 - 3a_3^2}. \tag{4.113}$$

Note that for equation (4.3) the C term must take a specific form (dependent on the a_i) whereas for equation (4.2) we can take the C term to be arbitrary. In keeping with the notation of this section we will use $\psi^{(1)}(x,t)$ and $\psi^{(2)}(x,t)$ for equations (4.2) and (4.3), respectively. From our values for the $S^{(i),}s$ and $C^{(i),}s$ and the determining equations for the $\psi^{(i),}s$ given earlier in the section we find that $\psi^{(1)}$ and $\psi^{(2)}$ take the following forms

$$\psi^{(1)}(x,t) = c_1\cos\left(\lambda(Ct - x)\right) + c_2\sin\left(\lambda(Ct - x)\right), \tag{4.114}$$

where $\lambda = \sqrt{\frac{3C^2a_3 - 3a_1a_3 + a_2^2}{6Ca_3}}$, and

$$\psi^{(2)}(x,t) = c_1 \cos\left(\frac{\sqrt{3}a_3\left(4x\sqrt{a_5} + t\sigma\right)}{4\sqrt{a_5}\sigma}\right) + c_2 \sin\left(\frac{\sqrt{3}a_3\left(4x\sqrt{a_5} + t\sigma\right)}{4\sqrt{a_5}\sigma}\right), \tag{4.115}$$

where $\sigma = \sqrt{16a_1a_5 - 3a_3^2}$. Therefore we have the following traveling wave solutions,

$$u^{(1)}(x,t) = \frac{C\sqrt{-\frac{2}{a_3}}\lambda(c_1 \sin(\lambda(\frac{x}{C} - t)) - c_2 \cos(\lambda(\frac{x}{C} - t))))}{c_1 \cos(\lambda(\frac{x}{C} - t))) + c_2 \sin(\lambda(\frac{x}{C} - t))} - \frac{a_2}{3a_3} \tag{4.116}$$

and, noting that $u^{(2)}(x,t) = \sqrt{v^{(2)}(x,t)}$,

$$u^{(2)}(x,t) = \left(\frac{3\left(32a_1a_5 - 9a_3^2\right)a_3(-c_1 \sin(y(x,t)) + c_2 \cos(y(x,t)))}{4\sqrt{-a_5}(c_1 \cos(y(x,t)) + c_2 \sin(y(x,t)))} - \frac{3a_3}{8a_5}\right)^{1/2}, \tag{4.117}$$

where $y(x,t) = \frac{\sqrt{3}a_3(4x\sqrt{a_5} + t\sigma)}{4\sqrt{a_5}\sigma}$. These solutions have the potential to become complex-valued but may be taken to be real provided we make suitable choices for the arbitrary constants. For example, a rather simple requirement for $u^{(1)}$ is $a_3 < 0$ and $\lambda, C \in \mathbb{R}$. The condition $\lambda \in \mathbb{R}$ is equivalent to requiring that a_1 satisfy $a_1 < \frac{a_2^3}{3a_3} + C^2$. Due to the nature of $u^{(2)}$ being that of a rational expression involving trigonometric functions inside a radical one cannot guarantee the solutions are real for all x and t as we did previously without eliminating the x and t dependence. However, given adequate choices it is possible to ensure the solutions are real for some spatial and time intervals.

We will now plot the solutions derived above for parameter sets which lead to qualitatively different plots. We find that the choice of parameter sets may lead to solutions with zero, one, or infinitely

many singularities. The case of zero or one singularity corresponds to parameters sets such that $\lambda \in i\mathbb{R}$ (the set of imaginary numbers). That is, if $\lambda \in i\mathbb{R}$ then $\frac{c_1}{c_2} \leq 1$ corresponds to one singularity whereas $\frac{c_1}{c_2} > 1$ corresponds to no singularities. On the other hand, if $\lambda \in \mathbb{R}$ then there will be infinitely many singularities for all choices of $c_1, c_2 \in \mathbb{R}$. For the parameter set $(c_1, c_2, C, a_1, a_2, a_3) = (1, 1, 2, 1, 1, -1)$ the solution (4.116) becomes

$$u^{(1)}(x,t) = -\frac{1}{3}\frac{(2\sqrt{6}-1)\sin\left(3^{-1/2}(2t-x)\right) + (2\sqrt{6}+1)\cos\left(3^{-1/2}(2t-x)\right)}{\sin\left(3^{-1/2}(2t-x)\right) - \cos\left(3^{-1/2}(2t-x)\right)}, \qquad (4.118)$$

whereas for the choice $(c_1, c_2, C, a_1, a_2, a_3) = (3, -2i, 1, 2, 1, -1)$ the solution becomes

$$u^{(1)}(x,t) = -\frac{1}{3}\frac{(6\sqrt{3}-2)\sinh\left(\sqrt{\frac{2}{3}}(t-x)\right) + (4\sqrt{3}-3)\cosh\left(\sqrt{\frac{2}{3}}(t-x)\right)}{3\cosh\left(\sqrt{\frac{2}{3}}(t-x)\right) + 2\sinh\left(\sqrt{\frac{2}{3}}(t-x)\right)}. \qquad (4.119)$$

The plots of these solutions are given in figure 4.7.

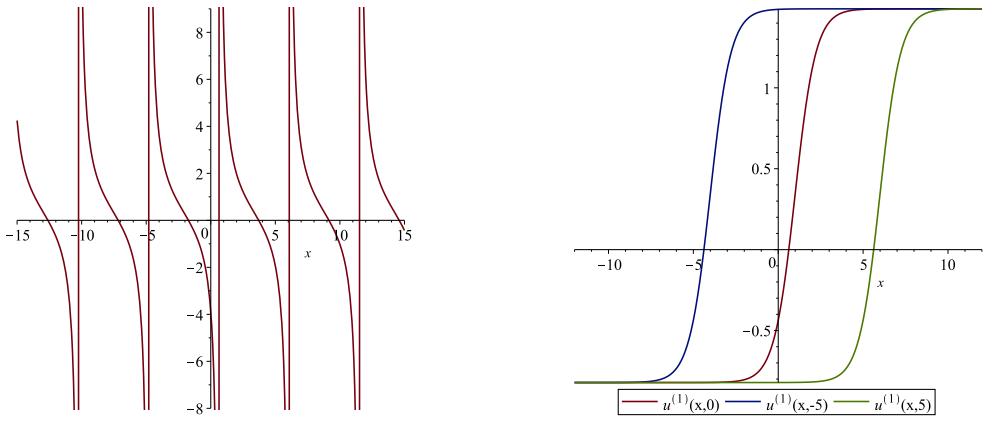

Figure 4.7: (Left): Plot of the solution (4.116) for the choice of parameters $(c_1, c_2, C, a_1, a_2, a_3) = (1, 1, 2, 1, 1, -1)$ at $t = 1$ in the interval $[-15, 15]$. (Right): Plot of the solution (4.116) for the choice of parameters $(c_1, c_2, C, a_1, a_2, a_3) = (3, -2i, 1, 2, 1, -1)$ at $t = -5, 0, 5$ in the interval $[-12, 12]$.

As mentioned above, for some choices of parameter set $(c_1, c_2, a_1, a_3, a_5)$ the term inside the square root in the solution (4.117) may oscillate between positive and negative values and thus will only be real-valued in certain regions of the $xt-$plane. For this reason we will only consider cases where the solution is real-valued for all $(x, t) \in \mathbb{R}^2$. In particular, we consider the parameter set $(c_1, c_2, a_1, a_3, a_5) = (2i, 1, 1, 2, -1)$ for which the solution (4.117) becomes

$$u^{(2)}(x,t) = \frac{3}{2}\left(\frac{\sinh\left(\frac{\sqrt{21}(t\sqrt{7}+2x)}{14}\right) + \cosh\left(\frac{\sqrt{21}(t\sqrt{7}+2x)}{14}\right)}{\sinh\left(\frac{\sqrt{21}(t\sqrt{7}+2x)}{14}\right) + 2\cosh\left(\frac{\sqrt{21}(t\sqrt{7}+2x)}{14}\right)}\right)^{1/2} \tag{4.120}$$

The plot of this solution for $t = -5, 0, 5$ is given in figure 4.8.

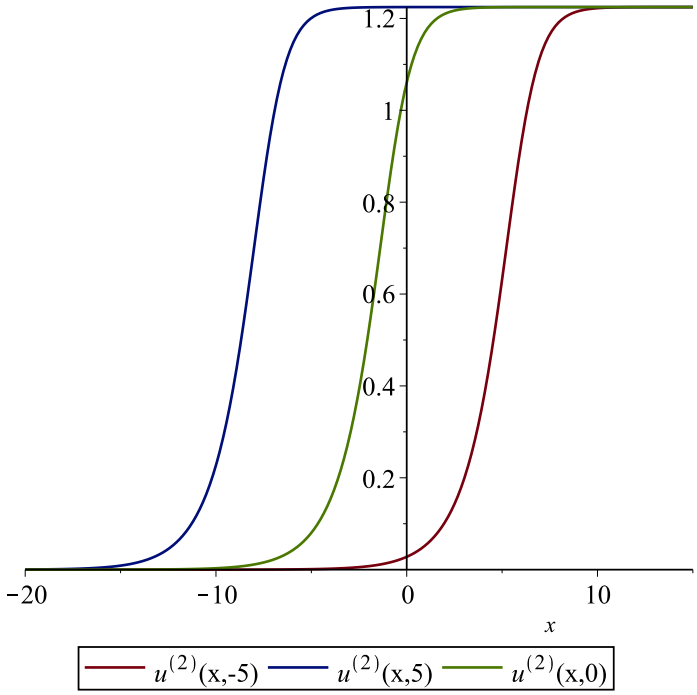

Figure 4.8: Plot of the solution (4.117) for $t = -5, 0, 5$ on the interval $[-20, 15]$ with the choice of parameters $(c_1, c_2, a_1, a_3, a_5) = (2i, 1, 1, 2, -1)$.

Generalized Hirota Expansion Method

In this section we will give a brief description of the generalized Hirota expansion method [50]. As previously mentioned, after plugging the truncated series solution into the PDE and reconciling the powers of ϕ we will obtain a recurrence relation from which we will determine the u_n and α. In general we find that, after arranging the equations according to increasing order in ϕ, the first equation will determine u_0, the second u_1, etc. We continue this process until we have found $u_0, \ldots, u_{\alpha-1}$ and keep the remaining u_α unknown. The final term can be expanded in a power

series about $x_1 = 0$

$$u_\alpha(x_1, \ldots, x_n, t) = \sum_{i=0}^{\infty} u_{\alpha,i}(x_2, \ldots, x_n, t) x_1^i \tag{4.121}$$

Since α is finite and in general we will have an PDE with finite order of nonlinearity we will have a finite number of remaining conditions to satisfy. Therefore plugging (4.121) into the remaining equations will yield a heavily underdetermined system. Thus there will exist some $N \in \mathbb{N}$ such that $u_{\alpha,i}(x_2, \ldots, x_n, t) = 0$ for all $i \geq N$. That is, $u_\alpha(x_1, \ldots, x_n, t)$ can be represented by a finite series. For ease of computation we force $\phi(x_1, \ldots, x_n, t)$ to be of the form $\phi(x_1, \ldots, x_n, t) = 1 + \exp\{\Gamma(t) + \sum_{l=1}^{n} x_l \Omega_l(t)\}$. This functional form is somewhat reminiscent of the standard ansatz one considers whilst searching for a one-soliton solution via Hirota's bilinear method. To further reduce complexity we let $\Gamma(t) = k_1 + k_2 t$ and $\Omega_l(t) = k_{l+2}$ where $k_l \in \mathbb{C}$ $(l = 1, \ldots, n+2)$. Plugging the new expansion with known u_n, $n = 0, 1, \ldots, \alpha - 1$ into the remaining equations gives rise to a new set of equations for each equation in the previous set of equations. From these we will determine the unknown $u_{\alpha,i}(t)$ and k_l. If, in theory, we can solve for these terms we will have found the last term in our truncated series expansion, as well as the form of the singularity manifold ϕ and therefore will have a solution to the original NLPDE.

Example: KP-II Equation

Before presenting the main results for the microstructure PDE and Pochhammer-Chree equations we will demonstrate the effectiveness of the generalized Hirota expansion method on a classic example in $(2+1)$, the KP-II equation. As the procedure begins exactly as the standard truncated Painlevé method we find that the initial steps in this procedure yield the same results. That is, for the KP-II equation given earlier by equation (4.13) we find that the degree of the singularity α,

found by balancing the $(uu_x)_x$ and u_{xxxx} terms, is equal to 2. Therefore, we will once again seek a solution of the form

$$u(x,y,t) = \frac{u_0(x,y,t)}{\phi(x,y,t)^2} + \frac{u_1(x,y,t)}{\phi(x,y,t)} + u_2(x,y,t). \tag{4.122}$$

Plugging the truncated expansion (4.122) into (4.13) yields the previously found recurrence relation

$$\begin{aligned}
&u_{n-4,xt} + (n-5)u_{n-3,x}\phi_t + (n-5)u_{n-3}\phi_{xt} + (n-4)(n-5)u_{n-2}\phi_x\phi_t \\
&+ (n-5)u_{n-3,t}\phi_x + \epsilon^2(u_{n-4,xxxx} + 4(n-5)u_{n-3,xxx}\phi_x + 6(n-5)u_{n-3,xx}\phi_{xx} \\
&+ 6(n-4)(n-5)u_{n-2,xx}\phi_x^2 + 4(n-5)u_{n-3,x}\phi_{xxx} + 12(n-4)(n-5)u_{n-2,x}\phi_x\phi_{xx} \\
&+ 4(n-3)(n-4)(n-5)u_{n-1,x}\phi_x^3 + (n-5)u_{n-3}\phi_{xxxx} + 4(n-4)(n-5)u_{n-2}\phi_x\phi_{xxx} \\
&+ 3(n-4)(n-5)u_{n-2}\phi_{xx}^2 + (n-3)(n-4)(n-5)u_{n-1}(6\phi_x^2\phi_{xx} + (n-2)u_n\phi_x^4)) \\
&+\lambda(u_{n-4,yy} + 2(n-5)u_{n-3,y}\phi_y + (n-5)u_{n-3}\phi_{yy} + (n-4)(n-5)u_{n-2}\phi_y^2) \\
&+ \sum_{k=0}^{n}(u_{k,x}u_{n-k-2,x} + (n-k-3)u_{k,x}u_{n-k-1}\phi_x + (k-2)u_k u_{n-k-1,x}\phi_x \\
&+ (k-2)(n-k-2)u_k u_{n-k}\phi_x^2 + u_k(u_{n-k-2,xx} + 2(n-k-3)u_{n-k-1,x}\phi_x \\
&+ (n-k-3)u_{n-k-1}\phi_{xx} + (n-k-2)(n-k-3)u_{n-k}\phi_x^2)) = 0
\end{aligned} \tag{4.123}$$

where $u_n = 0$ if $n < 0$ and $u_{n,x} \equiv \frac{\partial}{\partial x}(u_n)$. The difference here is that we will solve for only the functions u_0 and u_1, leaving the determination of u_2 to the end. Upon satisfaction of the recurrence

relation at the lowest powers of ϕ we find that

$$u_0(x,y,t) = -12\epsilon^2\phi_x^2 \quad \text{and} \quad u_1(x,y,t) = 12\epsilon^2\phi_{xx} \tag{4.124}$$

Following the procedure as outlined earlier in the section with the assumed forms for $\Gamma(t)$ and $\Omega(t)$ we find that u_2 takes the relatively simple form

$$u_2(x,y,t) = -\frac{k_3k_4 + \epsilon^2k_3^4 + \lambda k_2^2}{k_3^2} \tag{4.125}$$

where k_{1-4} are arbitrary constants. With this the remaining equations at each order of ϕ are identically satisfied and thus we find the final solution given by

$$\begin{aligned} u(x,y,t) &= \frac{(\epsilon^2k_3^4 + k_3k_4 + \lambda k_2^2)e^{2k_1+2k_2y+2k_3x+2k_4t} + 2(\lambda k_2^2 + k_3k_4 - 5\epsilon^2k_3^4)e^{k_1+k_2y+k_3x+k_4t}}{k_3^2(1+e^{k_1+k_2y+k_3x+k_4t})^2} \\ &\quad + \frac{\epsilon^2k_3^4 + k_3k_4 + \lambda k_2^2}{k_3^2(1+e^{k_1+k_2y+k_3x+k_4t})^2} \end{aligned} \tag{4.126}$$

We will now plot (see figure 4.9) the solution given by (4.126) for two different choices for the set of parameters just as we did in truncated Painlevé analsis section. For the first choice we take $(k_1,k_2,k_3,k_4,\epsilon,\lambda) = (1,-1,-1,i,1,1)$. The solution (4.126) at $t=1$ then becomes

$$u(x,y,1) = \frac{(2-i)\left(1+e^{2+2i-2y-2x}\right) - (8+2i)e^{1+i-y-x}}{\left(1+e^{1+i-y-x}\right)^2}. \tag{4.127}$$

It is clear that for this set of parameters the solution is complex-valued. Therefore, to better visu-

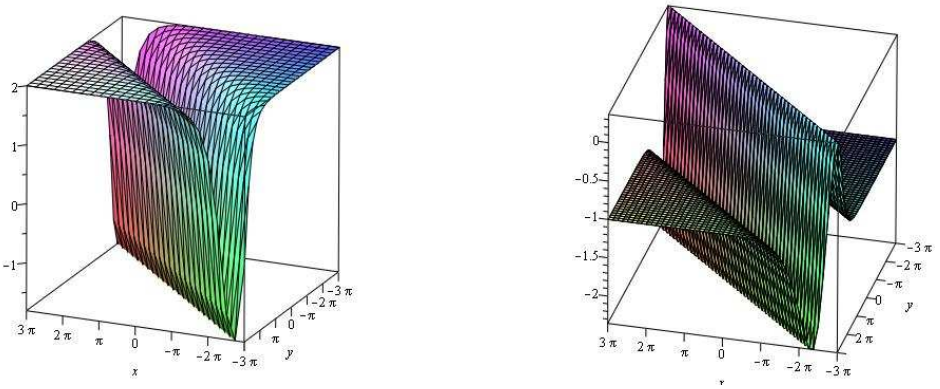

Figure 4.9: Plots of the real (left) and imaginary (right) parts of the solution to the KP-II equation at $t = 1$ on the rectangle $[-3\pi, 3\pi] \times [-3\pi, 3\pi]$ with the choice of parameters $(k_1, k_2, k_3, k_4, \epsilon, \lambda) = (1, -1, -1, i, 1, 1)$.

alize the solution we plot the real and imaginary parts separately.

As a second choice for the set of parameters we will take $(k_1, k_2, k_3, k_4, \epsilon, \lambda) = (1, -1, -1, -1, 1, 1)$ for which the solution (4.24) at $t = 1$ becomes

$$u(x, y, 1) = \frac{3\left(1 - e^{-x-y}\right)^2}{\left(1 + e^{-x-y}\right)^2} = 3\tanh^2\left((1/2)(-x - y)\right). \tag{4.128}$$

The plot for this solution is given in figure 4.10.

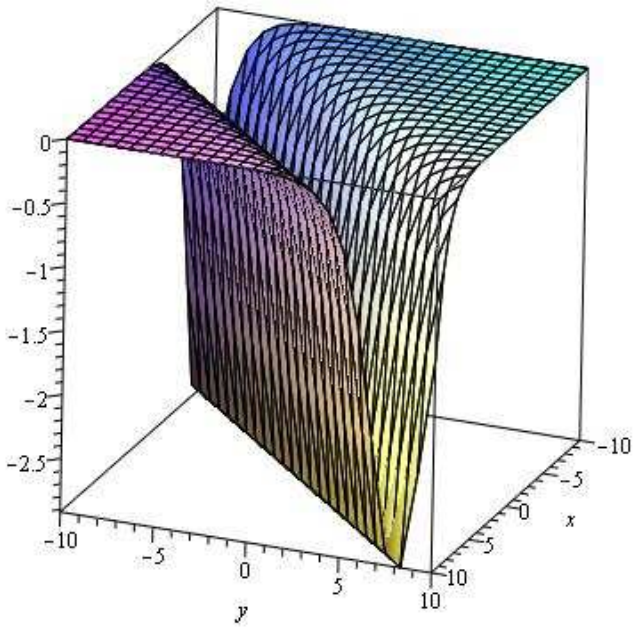

Figure 4.10: Plot of the solution to the KP-II equation at $t = 1$ on the rectangle $[-10, 10] \times [-10, 10]$ with the choice of parameters $(k_1, k_2, k_3, k_4, \epsilon, \lambda) = (1, -1, -1, -1, 1, 1)$.

Microstructure PDE

In this section we use the generalized Hirota expansion method to construct an exact solution to the microstructure PDE given earlier in the chapter by equation (4.1). Plugging $\phi^{-\alpha}$ into equation (4.1) and balancing the $(v^2)_{xx}$ and v_{xxxx} terms we find that the degree of the singularity, α, is equal to 2. Since $\alpha = 2$, we seek a solution of the form

$$v(x,t) = \frac{v_0(x,t)}{\phi(x,t)^2} + \frac{v_1(x,t)}{\phi(x,t)} + v_2(x,t) \tag{4.129}$$

for equation (4.1). Plugging the truncated expansion (4.129) into equation (4.1) yields the recurrence relation given previously in the chapter in the Painlevé analyis section,

$$
\begin{aligned}
&v_{m-4,tt} - bv_{m-4,xx} + (m-5)(2v_{m-3,t}\phi_t - 2bv_{m-3,x}\phi_x + v_{m-3}(\phi_{tt} - b\phi_{xx}) \\
&+(m-4)v_{m-2}(\phi_t^2 - b\phi_x^2)) - \mu\sum_{k=0}^{m}\{v_{k,x}v_{m-k-2,x} + v_k v_{m-k-2,xx} \\
&+(m-k-3)(v_{k,x}v_{m-k-1}\phi_x + 2v_k v_{m-k-1,x}\phi_x + (m-k-2)v_k u_{m-k}\phi_x^2 \\
&+v_k v_{m-k-1}\phi_{xx}) + (k-2)(v_k v_{m-k-1,x}\phi_x + (m-k-2)v_k v_{m-k}\phi_x^2)\big\} - \delta\beta(v_{m-4,xxtt} \\
&+(m-5)(2v_{m-3,xxt}\phi_t + v_{m-3,xx}\phi_{tt} + 2v_{m-3,xtt}\phi_x + 4v_{m-3,xt}\phi_{xt} + 2v_{m-3,x}\phi_{xtt} \\
&+v_{m-3,tt}\phi_{xx} + 2v_{m-3,t}\phi_{xxt} + v_{m-3}\phi_{xxtt} + (m-4)(v_{m-2,xx}\phi_t^2 + v_{m-2,tt}\phi_x^2 \\
&+4v_{m-2,xt}\phi_x\phi_t + 4v_{m-2,x}\phi_t\phi_{xt} + 2v_{m-2,x}\phi_x\phi_{tt} + 4v_{m-2,t}\phi_x\phi_{xt} + 2v_{m-2}\phi_{xt}^2 \\
&+2v_{m-2}\phi_x\phi_{xtt} + 2v_{m-2,t}\phi_t\phi_{xx} + v_{m-2}\phi_t\phi_{xxt} + v_{m-2}\phi_{xx}\phi_{tt} + v_{m-2}\phi_{xxt}\phi_t \\
&+(m-3)(2v_{m-1,x}\phi_x\phi_t^2 + 2v_{m-1,t}\phi_x^2\phi_t + 4v_{m-1}\phi_x\phi_t\phi_{xt} + v_{m-1}\phi_x^2\phi_{tt} + v_{m-1}\phi_{xx}\phi_t^2 \\
&+(m-2)v_m\phi_x^2\phi_t^2)))) + \delta\gamma(v_{m-4,xxxx} + (m-5)(v_{m-3}\phi_{xxxx} + 4v_{m-3,x}\phi_{xxx} \\
&+6v_{m-3,xx}\phi_{xx} + 4v_{m-3,xxx}\phi_x + (m-4)(v_{m-2}(4\phi_x\phi_{xxx} + 3\phi_{xx}^2) + 12v_{m-2,x}\phi_x\phi_{xx} \\
&+6v_{m-2,xx}\phi_x^2 + (m-3)(4v_{m-1,x}\phi_x^3 + 6v_{m-1}\phi_x^2\phi_{xx} + (m-2)v_m\phi_x^4)))) = 0
\end{aligned}
\tag{4.130}
$$

where once again $v_n = 0$ if $n < 0$ and $v_{n,x} \equiv \frac{\partial}{\partial x}(v_n)$. Solving the first two equations ($m = 0$ and $m = 1$) for v_0 and v_1 we again find

$$
v_0 = -\frac{12\delta}{\mu}(\gamma\phi_t^2 - \beta\phi_x^2) \tag{4.131}
$$

$$
v_1 = \frac{12\delta\,(5\gamma^2\phi_x^2\phi_{xx} + 5\beta^2\phi_t^2\phi_{tt} - \beta\gamma(\phi_t^2\phi_{xx} + \phi_x^2\phi_{tt}) - 8\beta\gamma\phi_t\phi_x\phi_{xt})}{5\mu(\gamma\phi_x^2 - \beta\phi_t^2)} \tag{4.132}
$$

Note that we do not solve for v_2 at this time as we did previously when following the Painlevé

analysis method. Rather, following the procedure for the generalized Hirota expansion method as outlined earlier in the section and expanding the v_2 in x with the assumed forms for $\Gamma(t)$ and $\Omega(t)$ we find that v_2 takes the relatively simple form

$$v_2(x,t) = \frac{\mu}{k_3^2}\left(\delta k_3^2(k_3^2\gamma - \beta k_2^2) - bk_3^2 + k_2^2\right) \tag{4.133}$$

where k_{1-3} are arbitrary constants. With this the remaining equations at each order of ϕ are identically satisfied and thus we find the final solution given by

$$\begin{aligned} v(x,t) = & - \frac{12\delta(\beta k_2^2 - \gamma k_3^2)\text{exp}\{k_1 + k_2 t + k_3 x\}}{\mu(1 + \text{exp}\{k_1 + k_2 t + k_3 x\})^2} \\ & - \frac{12\delta(10\beta\gamma k_2^2 k_3^2 - 5\gamma^2 k_3^4 - 5\beta^2 k_2^4)\text{exp}\{k_1 + k_2 t + k_3 x\}}{\mu(\beta k_2^2 - \gamma k_3^2)(1 + \text{exp}\{k_1 + k_2 t + k_3 x\})} \\ & + \frac{\delta\gamma k_3^4 - \beta k_3^2 - \delta\beta k_2^2 k_3^2 + k_2^2}{\mu k_3^2} \end{aligned} \tag{4.134}$$

We will now plot the solution derived for the parameter set $(k_1, k_2, k_3, \gamma, \mu, \delta, \beta, b) = (1, -1, -1, 1, 1, 1, -1, 1)$. For this choice of parameter set the solution (4.134) simplifies to

$$v(x,t) = \frac{2\left(e^{-2t-2x+2} - 10e^{-t-x+1} + 1\right)}{\left(1 + e^{-t-x+1}\right)^2}. \tag{4.135}$$

The plot of this solution for $t = -5, 0, 5$ is given in figure 4.11.

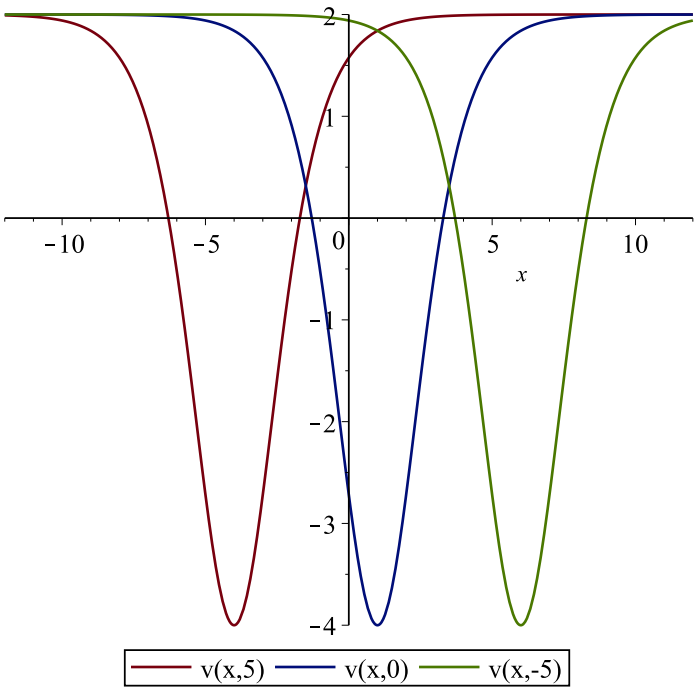

Figure 4.11: Plot of the solution (4.134) to the microstructure PDE at $t = -5, 0, 5$ on the interval $[-12, 12]$ with the choice of parameters $(k_1, k_2, k_3, \gamma, \mu, \delta, \beta, b) = (1, -1, -1, 1, 1, 1, -1, 1)$.

Generalized Pochhammer-Chree Equations

In this section we use the generalized Hirota expansion method to construct an exact solution to the generalized Pochhammer-Chree equations given earlier in the chapter by equations (4.2) and (4.3). Plugging $\phi^{-\alpha}$ into equation (4.2) and balancing the $(u^3)_{xx}$ and u_{xxtt} terms we find that the degree of the singularity, α, is equal to 1. Following the procedure and using the proper substitution ($u(x,t) = \sqrt{v(x,t)}$) for equation (4.3) as outlined in the previous sections we find that $\alpha^{(i)} = 1$ $(i = 1, 2)$. From this point on we will adopt the same notation used in the previous section for differentiating between results for equations (4.2) and (4.3). That is, we will use the superscript $^{(1)}$ for terms corresponding to equation (4.2) and the superscript $^{(2)}$ for terms corresponding to

equation (4.3). From the leading order analysis we seek solutions of the form

$$u(x,t) = \frac{u_0(x,t)}{\phi^{(1)}(x,t)} + u_1(x,t) \tag{4.136}$$

$$v(x,t) = \frac{v_0(x,t)}{\phi^{(2)}(x,t)} + v_1(x,t) \tag{4.137}$$

Plugging (4.136) and (4.137) into equation (4.2) we obtain the recurrence relation found earlier in the chapter in the Painlevé analysis section for equation (4.2),

$$-2(m-2)(m-3)(m-4)\phi_x\phi_t^2u_{m-1,x}-(m-3)(m-4)(2\phi_t\phi_{xxt}+\phi_{tt}\phi_{xx}+2\phi_x\phi_{xtt})u_{m-2}$$

$$-2(m-3)(m-4)(\phi_t\phi_{xx}+2\phi_x\phi_{xt})u_{m-2,t}-4(m-3)(m-4)\phi_t\phi_xu_{m-2,xt}$$

$$-2(m-3)(m-4)(\phi_x\phi_{tt}+2\phi_t\phi_{xt})u_{m-2,x}+u_{m-4,tt}-(m-1)(m-2)(m-3)(m-4)\phi_t^2\phi_x^2u_m$$

$$-u_{m-4,xxtt}-2(m-2)(m-3)(m-4)\phi_t\phi_x^2u_{m-1,t}-2(m-3)(m-4)\phi_{xt}^2u_{m-2}$$

$$-(m-3)(m-4)\phi_t^2u_{m-2,xx}-(m-2)(m-3)(m-4)\phi_t^2\phi_{xx}u_{m-1}+(m-3)(m-4)\phi_t^2u_{m-2}$$

$$-(m-2)(m-3)(m-4)\phi_x^2\phi_{tt}u_{m-1}-(m-3)(m-4)\phi_x^2u_{m-2,tt}-a_1((m-3)(m-4)\phi_x^2u_{m-2}$$

$$+2(m-4)\phi_xu_{m-3,x}+(m-4)\phi_{xx}u_{m-3}+u_{m-4,xx})-4(m-2)(m-3)(m-4)\phi_x\phi_t\phi_{xt}u_{m-1}$$

$$+2(m-4)\phi_tu_{m-3,t}-2(m-4)\phi_tu_{m-3,xxt}-2(m-4)\phi_xu_{m-3,xtt}-(m-4)\phi_{xx}u_{m-3,tt}$$

$$-2(m-4)\phi_{xtt}u_{m-3,x}-(m-4)\phi_{xxtt}u_{m-3}-2(m-4)\phi_{xxt}u_{m-3,t}-(m-4)\phi_{tt}u_{m-3,xx}$$

$$-4(m-4)\phi_{xt}u_{m-3,xt}+(m-4)\phi_{tt}u_{m-3}-2a_2\sum_{j=0}^{m}[(m-j-2)(m-j-3)\phi_x^2u_ju_{m-j-1}$$

$$+2(m-j-3)\phi_xu_ju_{m-j-2,x}+(m-j-3)\phi_{xx}u_ju_{m-j-2}+u_ju_{m-j-3,xx}]$$

$$-3a_3\sum_{j=0}^{m}\sum_{k=0}^{m-j}[(m-k-j-1)(m-k-j-2)\phi_x^2u_ju_ku_{m-k-j}+u_ju_ku_{m-k-j-2,xx}$$

$$+(m-k-j-2)\phi_{xx}u_ju_ku_{m-k-j-1}+2(m-k-j-2)\phi_xu_ju_ku_{m-k-j-1,x}]$$

$$-6a_3\sum_{j=0}^{m}\sum_{k=0}^{m-j}[(m-k-j-1)(k-1)\phi_x^2u_ju_ku_{m-k-j}+(m-k-j-2)\phi_xu_ju_{k,x}u_{m-k-j-1}$$

$$+u_ju_{k,x}u_{m-k-j-2,x}+(k-1)\phi_xu_ju_ku_{m-j-k-1,x}]-2a_2\sum_{j=0}^{m}[(j-1)(m-j-2)\phi_x^2u_ju_{m-j-1}$$

$$+(j-1)\phi_xu_ju_{m-j-2,x}+(m-j-3)\phi_xu_{j,x}u_{m-j-2}+u_{j,x}u_{m-j-3,x}]=0,$$

where $u_m = 0$ if $m < 0$ and $u_{m,x} \equiv \frac{\partial}{\partial x}(u_m)$. As was the case in the Painlevé section we must transform (4.3) according to $u(x,t) = \sqrt{v(x,t)}$. The resulting equation will then follow from equation (4.40) with $n = 2$. As one can see from the recurrence relation for equation (4.2) the

recurrence relation for (4.40) would be unecessarily messy and therefore will be omitted. Iterating through m values in the recurrence relation for equation (4.2) we obtain the coefficients of the different powers of ϕ. Solving the equations for the coefficient of the lowest order of ϕ in each case for u_0 and v_0 we find

$$u_0(x,t) = -\frac{\sqrt{2}\phi_t}{\sqrt{-a_3}} \quad \text{and} \quad v_0(x,t) = -\frac{\sqrt{3}\phi_t}{2\sqrt{-a_5}}. \tag{4.138}$$

Here again we do not solve for u_2 or v_2 at this time as we did previously when following the Painlevé analysis method. Rather, following the procedure for the generalized Hirota expansion method as outlined earlier in the section and expanding the u_2 and v_2 in x with the assumed forms for the $\Gamma^{(i)}(t)$ and $\Omega^{(i)}(t)$ we find that u_2 and v_2 take the relatively simple forms

$$u_1(x,t) = -\frac{\sqrt{2}(-2k_2^{(1)}a_3 + a_2\sqrt{-2a_3})}{6(-a_3)^{3/2}} \quad \text{and} \quad v_1(x,t) = \frac{\sqrt{6}(9a_3^2 + 2a_2\sqrt{6a_3a_5})}{36a_3\sqrt{a_3a_5}} \tag{4.139}$$

where $k_2^{(1)}$ is an arbitrary constant. Unlike the results for the KP equation and microstructure PDE we cannot assume here that the $k_j^{(i)}$ $(j = 1, 2, 3)$ in the $\phi^{(i)}$ are arbitrary. Rather we find that only $k_1^{(1)}$, $k_2^{(1)}$, and $k_1^{(2)}$ will be allowed to be arbitrary. Solving for the remaining $k_j^{(i)}$ in terms of the coefficients in their respective PDEs ((4.2) or (4.3)) and the previous $k_m^{(i)}$ the $\phi^{(i)}$ are readily found to be

$$\phi^{(1)}(x,t) = 1 + \exp\left\{k_1 + k_2 t - \frac{k_2\sqrt{6}\sqrt{a_3(-3k_2^2a_3 + 6a_1a_3 - 2a_2^2)}}{-3k_2^2a_3 + 6a_1a_3 - 2a_2^2}x\right\} \tag{4.140}$$

and

$$\phi^{(2)}(x,t) = 1 + \exp\left\{ k_1 - \frac{3a_3}{2\sqrt{-3a_5}}t - \frac{6\sqrt{2}a_3\sqrt{a_5(32a_1a_5 - 6a_3^2)}}{\sqrt{-3a_5}(32a_1a_5 - 6a_3^2)}x \right\}. \tag{4.141}$$

With this the remaining equations at each order of $\phi^{(i)}$ are identically satisfied and thus we find the final solutions are given by

$$u^{(1)}(x,t) = -\frac{2k_2\exp\{y^{(1)}(x,t)\}}{\sqrt{-2a_3}(1+\exp\{y^{(1)}(x,t)\})} - \frac{\sqrt{2}(-2k_2a_3 + a_2\sqrt{-2a_3})}{6(-a_3)^{3/2}} \tag{4.142}$$

and, noting again that $u(x,t) = \sqrt{v(x,t)}$,

$$u^{(2)}(x,t) = \left(\frac{3k_2\exp\{y^{(2)}(x,t)\}}{2\sqrt{-3a_5}(1+\exp\{y^{(2)}(x,t)\})} + \frac{\sqrt{6}(9a_3^2 + 2a_1\sqrt{6a_3a_5})}{36a_3\sqrt{a_3a_5}} \right)^{1/2}, \tag{4.143}$$

where

$$y^{(1)}(x,t) = k_1 + k_2t - \frac{k_2\sqrt{6}\sqrt{a_3(-3k_2^2a_3 + 6a_1a_3 - 2a_2^2)}}{-3k_2^2a_3 + 6a_1a_3 - 2a_2^2}x \tag{4.144}$$

and

$$y^{(2)}(x,t) = k_1 - \frac{3a_3}{2\sqrt{-3a_5}}t - \frac{6\sqrt{2}a_3\sqrt{a_5(32a_1a_5 - 6a_3^2)}}{\sqrt{-3a_5}(32a_1a_5 - 6a_3^2)}x. \tag{4.145}$$

Again, we see that these solutions may be complex-valued but can be taken to be real with suitable choices of the arbitrary constants involved. For example, for $u^{(1)}$ we may take $a_3 < 0$ and $a_1 <$

$\frac{a_2^2}{3a_3} + \frac{k_2^2}{2}$. For $u^{(2)}$ we may take $a_5 < 0$, $a_3 < 0$, and $-\frac{3\sqrt{6}a_3^2}{4\sqrt{a_3 a_5}} < a_1 < \frac{3a_3^2}{16a_5}$.

We will now plot the solutions derived above for parameter sets which lead to real-valued solutions.

For the parameter set $(k_1, k_2, a_1, a_2, a_3) = (0, 1, 1, 1, -1)$ the solution (4.142) becomes

$$u^{(1)}(x,t) = \frac{\sqrt{2}}{6} \frac{3 + \sqrt{2} + (2\sqrt{2} - 6)e^{t+\sqrt{6/5}x}}{2e^{t+\sqrt{6/5}x} + 1}. \tag{4.146}$$

The plot of this solution for $t = -5, 0, 5$ is given in figure 4.12.

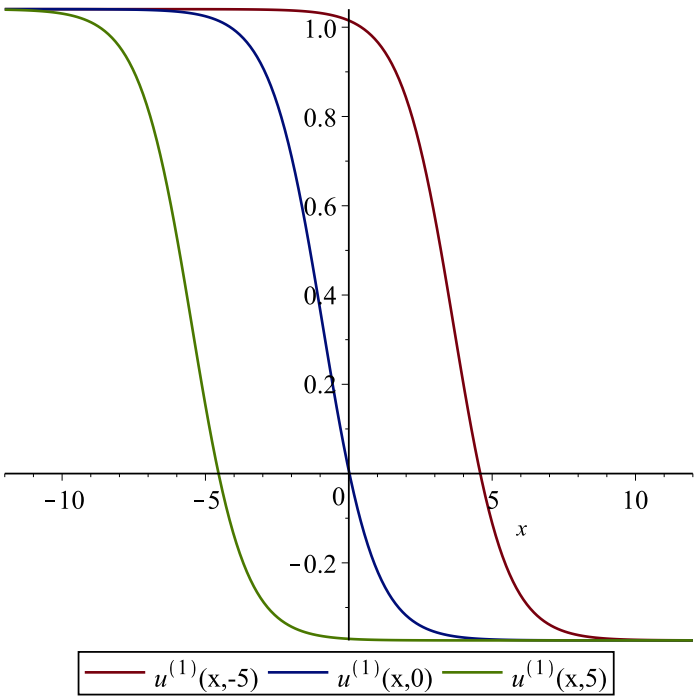

Figure 4.12: Plot of the solution (4.142) for $t = -5, 0, 5$ on the interval $[-12, 12]$ with the choice of parameters $(k_1, k_2, a_1, a_2, a_3) = (0, 1, 1, 1, -1)$.

As with the previous section, for some choices of parameter set (k_1, a_1, a_3, a_5) the term inside the square root in the solution (4.143) may oscillate between positive and negative values and thus will only be real-valued in certain regions of the $xt-$plane. For this reason we will only consider cases where the solution is real-valued for all $(x, t) \in \mathbb{R}^2$. In particular, we consider the parameter set $(k_1, a_1, a_3, a_5) = (0, 1, 1, -1)$ for which the solution (4.143) becomes

$$u^{(2)}(x,t) = \frac{\sqrt{3}}{2}\left(1 + e^{-\frac{\sqrt{3}}{2}t + \frac{2\sqrt{3}}{\sqrt{19}}x}\right)^{-1/2} \tag{4.147}$$

The plot of this solution for $t = -5, 0, 5$ is given in figure 4.13.

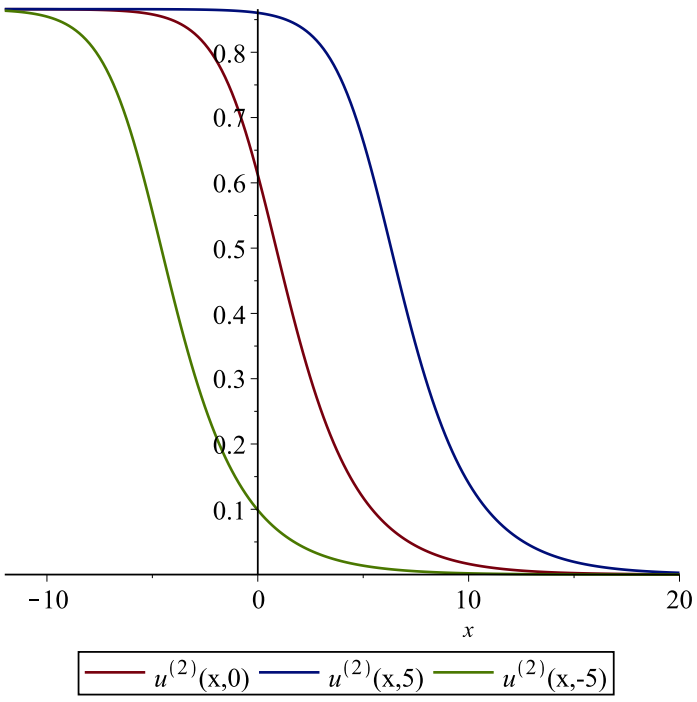

Figure 4.13: Plot of the solution (4.143) for $t = -5, 0, 5$ on the interval $[-12, 20]$ with the choice of parameters $(k_1, a_1, a_3, a_5) = (0, 1, 1, -1)$.

CHAPTER 5: CONCLUSION

In the first part of this dissertation we presented two methods for deriving the Lax pair for variable-coefficient nlpdes, namely Khawaja's Lax pair method and the extended Estabrook-Wahlquist method. In doing so we determined the necessary conditions on the variable coefficients for the variable-coefficient nonlinear PDEs to be Lax-integrable. The latter technique is introduced here in the context of space and time dependent coefficients for the first time. As the extended Estabrook-Wahlquist method requires many fewer assumptions and is algorithmic it proves to be a vast improvement over Khawaja's Lax pair method.

As stated in the introduction, an accepted definition for integrability of a nonlinear PDE does not currently exist in the literature. However, many nonlinear PDEs which have been classified as integrable share a remarkable number of properties. Perhaps the most important of these properties is the existence of a nontrivial Lax pair. We have derived variable-coefficient extensions to several well-known integrable nonlinear PDEs from the requirement that they possess nontrivial Lax representations as well as proving the nonexistence of a nontrivial Lax pair to an extension to a known nonintegrable nonlinear PDE.

In the second part of this dissertation we gave a brief introduction to three distinct types of singular manifold methods: truncated Painlevé analysis, truncated invariant Painlevé analysis, and a generalized Hirota expansion method. These methods were then demonstrated on the well-known integrable KdV and KP-II equations. Plots of the derived solutions were given for various choices of the arbitrary constants and system parameters involved. Following these examples we employed each method of solution to derive nontrivial solutions to a microstructure PDE and two generalized Pochhammer-Chree equations. We found that the truncated Painlevé analysis method failed to produce a solution for the second of the two Pochhammer-Chree equations (equation (4.3))

but was successful in all other cases considered. On the other hand, the invariant Painlevé analysis and generalized Hirota expansion methods successfully produced solutions in all cases considered. Plots of the derived solutions were given for various choices of the arbitrary constants and system parameters involved.

Future work will be centered around the results presented in part 1 of this dissertation. Due to the algorithmic nature of the extended Estabrook-Wahlquist method the natural question of whether the whole procedure can be programmed arises. Although the results in part 1 were derived partially by hand, the entire process (or at least up to derivation of the determining equations for the $\mathbb{X}_i$) can be programmed in a computer algebra system such as MAPLE or MATHEMATICA.

As a first possible direction for future research we would like to look into developing a program (for MAPLE or MATHEMATICA) which, given a variable-coefficient nlpde, would carry out the extended Estabrook-Wahlquist procedure.

Many PDEs which are integrable by this definition have been shown to possess a variety of other interesting properties, e.g. the existence of infinitely many conserved quantities, a biHamiltonian representation, solvability by the Inverse Scattering Transform, etc..

It would be of interest to study these new extended systems to determine if they share the same properties common to integrable systems as their constant coefficient predecessors.

As these generalized systems contain as limiting subcases the constant coefficient equations from which they stem their study may lead to interesting results such as generalized biHamiltonian structures which in turn could determine hierarchies of variable-coefficient nonlinear PDEs.

In [**?**]- [**?**] we considered variable-coefficient extensions to the KdV, mKdV, NLS, DNLS, and PT-symmetric NLS equations. We have derived extensions to only a few of the many known PDEs in the field of integrable equations.

As a third direction we may utilize the extended Estabrook-Wahlquist method to derive integrable extensions to many other known systems thus broadening the field.

LIST OF REFERENCES

[1] M. J. Ablowitz and H. Segur, Solitons and the IST (SIAM, Philadelphia, 1981).

[2] P. G. Drazin and R. S. Johnson, Solitons: an introduction (Cambridge U. Press, Cambridge, 1989.

[3] M. J. Ablowitz and Z. Musslimani, Integrable Nonlocal Nonlinear Schrodinger Equation, Physical Review Letters 110, (2013), 064105 (4 pages).

[4] M.J. Ablowitz, D.J. Kaup, A.C. Newell, H. Segur, Stud. Appl. Math., 53 (1974) 249-315.

[5] V.E. Zakharov and A.B. Shabat, Functional Analysis and Its Applications, 8 (3): (1974) 226-235

[6] R. Grimshaw and S. Pudjaprasetya, Stud. Appl. Math., 112 (2004) 271.

[7] G Ei, R. Grimshaw, and A. Kamchatov, J. Fluid Mech., 585 (2007) 213.

[8] G. Das Sharma and M. Sarma, Phys. Plasmas 7 (2000) 3964.

[9] E. G. Fan, Phys. Lett A294 (2002), 26.

[10] R. Grimshaw, Proc. Roy. Soc., A368 (1979) 359.

[11] N. Joshi, Phys. Lett., A125 (1987), 456.

[12] Y. C. Zhang, J. Phys. A, 39 (2006) 14353.

[13] E. G. Fan, Phys. Lett A375 (2011), 493.

[14] X. Hu and Y. Chen, J. Nonlin. Math. Phys., 19 (2012) 1250002

[15] S. F. Tian and H. Q. Zhang, Stud. Appl. Math., doi 10.1111/sapm.12026, 13 pages (2013).

[16] S. K. Suslov, arXiv: 1012.3661v3 [math-ph].

[17] Z. Y Sun, Y. T. Gao, Y. Liu and X. Yu, Phys.Rev E84 (2011) 026606.

[18] J. He and Y. Li, Stud. Appl. Math., 126 (2010), 1.

[19] U. Al Khawaja, J. Phys. A: Math. Gen. 39 (2006) 9679 9691.

[20] U. Al Khawaja, Journal of Mathematical Physics, 51(2010), 053506.

[21] M. Russo and S. Roy Choudhury, J. of Phys. Conf. Series 2014: 482(1): 012038. doi 10.1088/1742-6596/482/1/012038.

[22] M. Russo and S.R. Choudhury, *Integrable Spatiotemporally Varying KdV and MKdV: Generalized Lax Pairs and an Extended Estabrook Method*, accepted for publication in Quarterly of Applied Mathematics

[23] S.R. Choudhury and M. Russo, *The Extended Estabrook-Wahlquist method*, accepted for publication in Physica D: Nonlinear Phenomena

[24] M. Russo and S.R. Choudhury, *Integrable spatiotemporally varying NLS, PT-symmetric NLS, and DNLS equations: Generalized Lax pairs and Lie algebras*, Submitted

[25] M. Russo and S.R. Choudhury, *Integrable generalized KdV, mKdV, and nonlocal PT-symmetric NLS equations with spatiotemporally varying coefficients*, Submitted

[26] F. B. Estabrook and H. D. Wahlquist, J. Math. Phys., 16 (1975) 1; ibid: 17 (1976) 1293.

[27] R. K. Dodd and A. P. Fordy, Proc. Roy. Soc. London A385 (1983) 389.

[28] R. K. Dodd and A. P. Fordy, J. Phys. A17 (1984) 3249.

[29] D. J. Kaup, Physica D1 (1980) 391.

[30] X. Yu, Y.-T. Gao, Z.-Y. Sun, and Y. Liu, N-soliton solutions, Baklund transformation and Lax pair for a generalized variable-coefficient fifth-order Korteweg Vries equation, Phys. Scr. 81 (2010), 045402

[31] X. Yong, H. Wang, and Y. Zhang, Symmetry, Intregrability and Exact Solutions of a Variable-Coefficient Korteweg-de-Vries (vcKDV) Equation, International Journal of Nonlinear Science 14 (2012)

[32] B. Fuchssteiner, Some tricks from the symmetry-toolbox for nonlinear equations: generalizations of the Camassa-Holm equation. *Physica D* **95** (1996) 229-243.

[33] A. Fokas, On a Class of physically important integrable equations. *Physica D* **87** (1995) 145-150.

[34] A. Salupere, K. Tamm, J. Engelbrecht and P. Peterson, Interaction of solitary deformation waves in microstructured solids, Proc. 5th. Intl. Conf. on Nonlin. Wave Equations, Athens, Georgia, April 2007.

[35] J. Engelbrecht and F. Pastrone, Waves in microstructured solids with nonlinearities in microscale, Proc. Estonian Acad. Sci. Phy. Math., 52 (2003), 12-20.

[36] J. Engelbrecht, A. Berezovski, F. Pastroneand M. Braun, Waves in microstructured materials and dispersion, Phil. Mag., 85 (2005), 4127.

[37] J. Janno and J. Engelbrecht, Solitary waves in nonlinear microstructured materials, J. Phys. A, 38 (2005), 5159.

[38] J. Janno and J. Engelbrecht, And inverse solitary wave problem related to microstructured materials, Inv. Problems, 21 (2005), 2019.

[39] I. L. Bogolubsky, Some examples of inelsatic soliton interaction. Comp. Phys., 31 (1977) 149-155.

[40] R. J. LeVeque, P. A. Clarkson and R. Saxton, Solitary wave interactions in elastic rods, Stud. Appl. Math., 75 (1986) 95-122.

[41] R. Saxton, Existence of solutions for a finite nonlinearly hyperelastic rod, J. Math. Anal. Appl., 105 (1985) 59 - 75.

[42] Z. Weiguo and W. X. Ma, Explicit solitary wave solutions to generalized Pochhammer-Chree equations, Appl. Math. and Mech., 20 (1999) 666-674.

[43] L. Pochhammer, Biegung des KreiscylindersFortpflanzungs-Geschwindigkeit kleiner Schwingungen in einem Kreiscylinder, Journal für die Reine und Angewandte Mathematik, vol. 81, p. 326, 1876.

[44] C. Chree, Longitudinal vibrations of a circular bar, Quarterly Journal of Mathematics, vol. 21, pp. 287298, 1886.

[45] J. Weiss, M. Tabor and G. Carnevale, The Painlevé property for partial differential equations, *J. Math. Phys.* **24** (1983), 522.

[46] R. Conte, Invariant painlevé analysis of partial differential equations, *Phyics Letters A*, **140** (1989) 383-390.

[47] M. Russo, R. Van Gorder, S.R. Choudhury, Communications in Nonlinear Science and Numerical Simulation **18** (2013), no. 7, 1623-1634.

[48] M. Russo and S.R. Choudhury, *Exact solutions of Pochammer-Chree and Boussinesq Equations by Invariant Painlevé Analysis and Generalized Hirota Techniques,* accepted for publication in Discontinuity, Nonlinearity, and Complexity

[49] M. Russo and S.R. Choudhury, *Analytic Solutions of a Microstructure PDE and the KdV and Kadomtsev-Petviashvili Equations by Invariant Painlevé Analysis and Generalized Hirota Techniques,* Submitted

[50] Y. T. Gao and B. Tian, Some two-dimensional and non-traveling-wave observable effects in shallow water waves, Phys. Lett., A301 (2002), 74.

[51] B.B. Kadomtsev and V.I. Petviashvili, On the stability of solitary waves in weakly dispersive media, Sov. Phys. Dokl. 15, 539-541 (1970).

[52] M. Hadac, S. Herr, and H. Koch, Well-posedness and scattering for the KP-II equation in a critical space, arXiv:0708.2011v3 (2009).

[53] J. Hammack, N. Scheffner, and H. Segur, Two-dimensional periodic waves in shallow water, J. Fluid Mech. 209, 567-589 (1989).

[54] G. Biondini and S. Chakravarty, Soliton solutions of the Kadomtsev-Petviashvili II equation, J. Math. Phys. 47, 033514:1-26 (2006).

[55] J. Boussinesq, Essai sur la theorie des eaux courantes, Memoires presentes par divers savants, l'Acad. des Sci. Inst. Nat. France, XXIII, pp. 1680 (1877).

[56] D.J. Korteweg, G. de Vries, On the Change of Form of Long Waves Advancing in a Rectangular Canal, and on a New Type of Long Stationary Waves, Philosophical Magazine 39 (240): 422443 (1895).

[57] N.J. Zabusky, M.D. Kruskal, Interaction of 'Solitons' in a Collisionless Plasma and the Recurrence of Initial States, Phys. Rev. Lett. 15 (6): 240243 (1965)