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Quasi-Gorenstein Modules

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by

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ABSTRACT

This thesis will study the various roles that quasi-Gorenstein modules and their properties play in the study of homological dimensions and linkage of modules. To that effect we begin by studying these modules in their own right. An R-module M of grade g will be quasi-Gorenstein if $\mathrm{Ext}^i_R(M, R) = 0$ for $i \neq g$ and there is an isomorphism $M \cong \mathrm{Ext}^g_R(M, R)$. Such modules have many nice properties which we will explore throughout this thesis. We will show they help extend a characterization of diagonalizable matrices over principal ideal domains to more general rings. We will use their properties to help lay a foundation for a study of homological dimensions, helping to generalize the concept of Gorenstein dimension to modules of larger grade and present a connection to these new dimensions with certain generalized Serre conditions.

We then give a categorical construction to the concept of linkage. The main motivation of such a construction is to generalize ideal and module linkage into one unified theory. By using the defintion of linkage presented by Nagel [53], we can use categorical language to define linkage between categories. One of the focuses of this thesis is to show that the history of linkage has been wrought with a misunderstanding of which classes of objects to study. We give very compelling evidence to suggest that linkage is a tool to gain information about the even linkage classes of objects. Further, scattered among the literature is a wide array of results pertaining to module linkage, homological dimensions, duality, and adjoint functor pairs and for which we show that these fall under the umbrella of this unified theory. This leads to an intimate relationship between associated homological dimensions and the linkage of objects in a category. We will give many applications of the theory to modules allowing one to cover vast grounds from Gorenstein dimensions to Auslander and Bass classes to local cohomology and local homology. Each of these gives useful insight into certain classes of modules by applying this categorical approach to linkage.

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This is dedicated to my family: Malia, Sherlock & Watson

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CHAPTER 1: INTRODUCTION

This dissertation is concerned with exploring the homological properties of objects using the tools developed in the theory of homological dimensions, ideal linkage, and duality. We capture the properties of duality and use them to discover the intimate connection between homological dimensions and linkage.

Let R be a Noetherian ring and R-Mod be the category of R-modules. We are concerned with the homological properties of a class of R-modules called the quasi-Gorenstein modules. An R-module M is quasi-Gorenstein if for some $g \geq 0$ there is some isomorphism $\alpha : M \to \mathrm{Ext}_R^g(M, R)$ and $\mathrm{Ext}_R^i(M, R) = 0$ for $i \neq g$. Finitely generated free R-modules are quasi-Gorenstein where $g = 0$ and $R/\bar{x}$ where $\bar{x}$ is a regular sequence of length g in R is a quasi-Gorenstein R-module with $\mathrm{Ext}_R^g(R/\bar{x}, R) \cong R/\bar{x}$. Clearly these are very specialized modules. They are formally defined by Nagel [53], but have appeared in various forms [8, 9, 12, 29, 43]. These modules play a role in the study of the structure of modules and matrices, the study of linkage using modules, and the study of homological dimensions associated to certain functors in R-Mod.

To see their connection to matrices, let m be a square $n \times n$ matrix with entries in a ring R. If m is of full rank there is a short exact sequence

$$0 \to R^n \xrightarrow{m} R^n \to Q \to 0.$$

Then we get another short exact sequence

$$0 \to R^n \xrightarrow{m^T} R^n \to \mathrm{Ext}_R^1(Q, R) \to 0.$$

which is obtained from the first sequence by applying the functor $\text{Hom}_R(-, R)$. Assuming m is symmetric, $m = m^T$ and we have that $Q \cong \text{Ext}_R^1(Q, R)$ and $\text{Hom}_R(Q, R) = 0 = \text{Ext}_R^i(Q, R)$ for $i > 1$. Thus Q is quasi-Gorenstein and this construction gives entire families of quasi-Gorenstein modules that are not free or $R/\bar{x}$. This situation arises in other areas of mathematics such as combinatorics, matrix theory, and graph theory.

Given a graph G, one can associate to it two matrices D and A, the degree matrix and adjacency matrix, respectively. One can then form the matrix $L = D - A$. The laplacian matrix associated to G is the matrix $L(1, 1)$ where the first row and column of L have been deleted. This is a square full rank symmetric matrix, and so it fits into the short exact sequence

$$0 \to \mathbb{Z}^n \xrightarrow{L(1,1)} \mathbb{Z}^n \to Q \to 0,$$

where from the above remarks Q is quasi-Gorenstein where $Q \cong \text{Ext}_{\mathbb{Z}}^1(Q, \mathbb{Z})$. Q is called the Sandpile group or critical group of the graph G, see [7, 45–47].

It is clear that quasi-Gorenstein modules are special modules and in this thesis we explore them as well as their role in other aspects of commutative and homological algebra. The layout of the thesis is as follows.

Chapter 2 will provide introductory and background material and preliminary and motivational results.

In Chapter 3, we define quasi-Gorenstein modules which have finite projective dimension and

explore their properties and homological characteristics. These are described in the following way:

Definition 1.0.1. *Let R be a Noetherian commutative ring and Q an R-module of finite projective dimension. We say that Q is quasi-Gorenstein with projective dimension q if the following holds:*

(i) $pd_R(Q) = grade_R(Q)$

(ii) $Q \cong Ext_R^q(Q, R)$

In general, these can be defined without finite projective dimension and we will have a much more general definition attached to the category pair with linkage definition in Chapter 5. Our goal in this chapter is to present an extension the structure theorem of finitely generated modules over a principal ideal domain. The classical structure theorem can be stated as follows:

Theorem 1.0.2. *Let R be a principal ideal domain. If M is a finitely generated R-module then $pd_R(M) \leq 1$ and the following holds:;*

(i) $M \cong \oplus_{i=1}^n R/(\lambda_i)$, $\lambda_i \in R$

(ii) M *is presented by a full rank $n \times n$ matrix m which is diagonalizable.*

(iii) M *is presented by a diagonal matrix $\Lambda = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$.*

The focus of the chapter is to present properties and results concerning quasi-Gorenstein modules which will lead to an extension of this theorem.

In Section 3.2 we begin our focus on quasi-Gorenstein R-modules of projective dimension one. This gives rise to the characterization of such modules using matrices over R.

Corollary 1.0.3. *Suppose that M is an R-module of projective dimension one presented by a full rank $n \times n$ matrix m. Then m is equivalent to m^T if and only if M is a quasi-Gorenstein R-module.*

In Section 3.3 we present a filtration associated to modules which utilizes their associated primes. Given an R-modules M one has a lattice of associated primes under inclusion associated to M. We can then consider an increasing chain of submodules of M,

$$0 = M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq M_2 \subsetneq \cdots \subsetneq M,$$

where each consecutive quotient $M_{i+1}/M_i \cong R/I_{i+1}$ has I_{i+1} an associated prime of M. We call such a filtration a cyclic-filtration of M. These are used to help decompose the symmetric matrix associated to a quasi-Gorenstein module of projective dimension one.

In Section 3.4 we use these definitions and properties of quasi-Gorenstein modules and cyclic-filtrations to prove the following theorem:

Theorem 1.0.4. *Let R be a commutative factorial domain and M an R-module of projective dimension one presented by a full rank $n \times n$ matrix m. The following are equivalent:*

(i) *M is quasi-Gorenstein and M has a minimal cyclic-filtration consisting of quasi-Gorenstein submodules.*

(ii) *m is equivalent to m^T and M has a minimal cyclic filtration consisting of quasi-Gorenstein submodules.*

(iii) *$M \cong \bigoplus_{i=1}^{n} R/(\lambda_i)$, $\lambda_i \in R$.*

(iv) *The matrix m is diagonalizable.*

This gives an extension to the structure theorem for finitely generated modules over a principal ideal domain. This is a useful characterization of diagonalizable matrices over more general rings. In fact, a useful application of these ideas is to the study of laplacian matrices of edge-weighted graphs and graph operations. By giving an indeterminate weight to an edge of a graph, the laplacian has elements in $\mathbb{Z}[q]$ where q is the indeterminate. This polynomial ring is a commutative factorial domain and that allows us to use this theorem to understand when the laplacian is diagonalizable and what this means for families of graphs.

In Chapter 4, we begin our study of homological dimensions associated to the functors $\text{Ext}^i_R(-, R)$ over a commutative ring R. The definitions and ideas mirror those of C-Gorenstein dimensions, see [15, 18]. The ideas and intuition built up from quasi-Gorenstein modules helps understand the homological characteristics of these new modules which can be defined in the following way:

Definition 1.0.5. *Suppose that C is a semidualizing R-module. Let M be an R-module where* $grade_R(M) = g$. *We say that M has G^g_C-dimension zero if the following holds:*

(i) *M is finitely generated*

(ii) *$Ext^{g+i}_R(M, C) = 0 = Ext^{g+i}_R(Ext^g_R(M, C), C)$ for $i \neq 0$.*

(iii) *The biduality map $\delta^g_C(M) : M \to Ext^g_R(Ext^g_R(M, C), C)$ is an isomorphism.*

This biduality map can be thought of as an extension of the natural evaluation map $\delta_M : M \to \text{Hom}_R(\text{Hom}_R(M, C), C)$ given by $\delta_M(m)(\varphi) = \varphi(m)$. Clearly, quasi-Gorenstein R-modules of grade g have G^g_C-dimension zero. One can use G^g_C-dimension zero modules to construct resolutions of modules with grade at least g. This gives rise to a collection of homological dimension associated to each grade, which are called the intermediate C-Gorenstein dimensions.

In Section 4.1, we provide results determining what these dimensions are and how they are connected together. We prove the following result which gives an exact value for the dimension if it is finite.

Theorem 1.0.6. *Let C be a semidualizing R-module. For a finitely generated R-module M with $grade_R(M) \geq g$, the following are equivalent:*

(i) $G_C^g\text{-}dim_R(M) \leq n$

(ii) $G_C^g\text{-}dim_R(M) < \infty$ and $n \geq \alpha_R(M) - g$

(iii) *In any G_C^g-resolution,*

$$\cdots \to M_i \to M_{i-1} \to \cdots \to M_0 \to M \to 0$$

the kernel $K_n = \ker(M_{n-1} \to M_{n-2})$ has G_C^g-dimension zero.

This also shows how these dimensions are connected together.

Theorem 1.0.7. *Let C be a semidualizing R-module and M a finitely generated R-module. Then $G_C^{grade_R(M)}\text{-}dim_R(M) < \infty$ if and only if $G_C^i\text{-}dim_R(M) < \infty$ for any $i \leq grade_R(M)$.*

This leads to an Auslander-Bridger formula for these dimensions

Corollary 1.0.8. *Let C be a semidualizing module for a local ring $(R, \mathfrak{m}, k)$ with $G_C^j\text{-}dim_R(M) < \infty$ for an R-module M. Then*

$$G_C^j\text{-}dim_R(M) = depth(R) - depth_R(M) + j$$

In Section 4.2, we use a more general definition of a condition related to the Serre conditions for rings to give results concerning these new dimensions. This condition can be stated in the following way:

Definition 1.0.9. *Let C be a semidualizing R-module and M and R-module. We say that M satisfies $\tilde{S}_n^g$ if*

$$depth_{R_p}(M_p) + g \geq \min\{n, depth(R_p)\} \quad \forall p \in Spec(R)$$

This leads to a generalization of [18, Proposition 2.4] which gives a characterization of modules satisfying this general Serre-like condition.

Theorem 1.0.10. *Let C be a semidualizing R-module, M an R-module with $grade_R(M) = g$, $n \geq g$, and M have locally finite G_C^g-dimension. Then the following are equivalent:*

(i) $Ext_R^{g+i}(D_C^g M, C) = 0$ *for* $1 \leq i \leq n - g$

(ii) M *is an* $(n-g)^{th}\mathcal{G}_C^g$*-syzygy*

(iii) M *satisfies* $\tilde{S}_n^g$

(iv) $grade_{C_p}(Ext_{R_p}^{grade_{C_p}(M_p)+i}(M_p, C_P)) \geq i + n$ *for* $i \geq 1$ *and* $p \in Spec(R)$ *where* $depth(R_p) \leq i + n - 1$.

In Chapter 5, we begin our study of linkage by defining linkage from a categorical perspective. Linkage, under the lense of algebra, was originally studied by Peskine and Szpiro, see [57]. They were concerned with connections between invariants of curves and the study of the intersection of curves. The linkage they define is hereafter referred to as ideal linkage. There is an extremely large literature on ideal linkage for which [32, 39, 40, 52, 57] are just a few resources. We are more concerned with the generalization of ideal linkage to module linkage and beyond. Exactly how to

generalize ideal linkage to module linkage is not an easy question and one that has been studied thoroughly, see [18, 44, 48, 49, 52, 53, 60]. We are concerned with preserving the homological characteristics and properties of the associated modules to these ideals. To that end, we choose to emulate the generalization of ideal linkage to module linkage presented by Nagel, see [53]. He uses quasi-Gorenstein modules to define a class of modules through which linkage is achieved. Given an R-module M with $\text{grade}_R(M) = g$, we can approximate M by a quasi-Gorenstein R-module Q of grade g with a short exact sequence

$$0 \to K_M \to Q \to M \to 0$$

Then by applying $\text{Hom}_R(-, R)$ to this we get the exact sequence

$$0 \to \text{Ext}_R^g(M, R) \to \text{Ext}_R^g(Q, R) \to \text{Ext}_R^g(K_M, R) \to \text{Ext}_R^{g+1}(M, R) \to 0$$

and using the isomorphism $Q \to \text{Ext}_R^g(Q, R)$ we can obtain an R-module we denote with $\mathcal{L}_Q(M)$ where

$$0 \to \text{Ext}_R^g(M, R) \to Q \to \mathcal{L}_Q(M) \to 0$$

is a short exact sequence. We then have the following

Definition 1.0.11. *We say that two R-modules M and N are directly linked by the quasi-Gorenstein module Q if $\mathcal{L}_Q(M) \cong N$ and $\mathcal{L}_Q(N) \cong M$.*

We will explore this exact situation in more detail in Chapter 6. The most important property utilized by the quasi-Gorenstein R-modules is their self-dual property. In our definition of a category pair with linkage we use this property to help formulate an adequate situation in which to perform linkage.

Definition 1.0.12. *Let $\mathcal{X}$ and $\mathcal{Y}$ be homological categories where $S : \mathcal{X} \to \mathcal{Y}$ and $T : \mathcal{Y} \to \mathcal{X}$ are*

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additive contravariant left exact functors. We say that S and T are linkage functors, and (S, T) form a linkage functor pair for $\mathcal{X}$ and $\mathcal{Y}$ if there exists a category $\mathcal{B}$ with a pair of full and faithful functors $F_{\mathcal{X}} : \mathcal{X} \to \mathcal{X}$ and $F_{\mathcal{Y}} : \mathcal{B} \to \mathcal{Y}$ such that the following holds:

(i) *There exists functors $X_{\mathcal{B}} : \mathcal{X}|_{F_{\mathcal{X}}(\mathcal{B})} \to \mathcal{Y}|_{F_{\mathcal{Y}}(\mathcal{B})}$ and $Y_{\mathcal{B}} : \mathcal{Y}|_{F_{\mathcal{Y}}(\mathcal{B})} \to \mathcal{X}|_{F_{\mathcal{X}}(\mathcal{B})}$ such that $X_{\mathcal{B}} \circ F_{\mathcal{X}} = F_{\mathcal{Y}}$ and $Y_{\mathcal{B}} \circ F_{\mathcal{Y}} = F_{\mathcal{X}}$. Moreover, $X_{\mathcal{B}}$ and $Y_{\mathcal{B}}$ are such that $S|_{F_{\mathcal{B}}(\mathcal{B})} = X_{\mathcal{B}}$ and $T|_{F_{\mathcal{Y}}(\mathcal{B})} = Y_{\mathcal{B}}$.*

(ii) *For each $B \in \mathcal{B}$, $D^i S(F_{\mathcal{X}}(B)) = 0$ and $D^i T(F_{\mathcal{Y}}(B)) = 0$ for $i > 0$ where $D^i S(-)$ and $D^i T(-)$ are the derived functors of S and T, respectively.*

In this case we say that $\mathcal{B}$ is a Fossum category, both $F_{\mathcal{X}}(\mathcal{B})$ and $F_{\mathcal{Y}}(\mathcal{B})$ are linking classes of $\mathcal{X}$ and $\mathcal{Y}$, and $((\mathcal{X}, \mathcal{Y}), \mathcal{B}, (S, T))$ is a category pair with linkage.

Before we define a generalization of the linkage defined previously, we want to point out the use of two categories and two functors here. In order to construct an adequate theory of linkage, one needs two functors in which to apply one after the other to different sequences. It turns out that the pairs of functors which give the desired theory (and we give substantial proof of this) are two contravariant left exact functors, two contravariant right exact functors, or one covariant right exact functor and one covariant left exact functor. The reason for this is the way in which we compare a module with what is hopefully its direct link. After application of the functor, one uses the original object and the self-dual (we will call them Fossum) object to construct the link. Depending upon the functor this puts the link on either the opposite side or the same side of the short exact sequence as the original object. Due to this, the three pairs mentioned above are the ones which form an adequate theory, and which emulates the already proven theory of ideal and module linkage. So there are similar definitions in which S and T are both contravariant right exact and where both are covariant and one is left exact and one is right exact. These three situations are explored in

sections 5.1 and 5.2.

Furthermore, the functors $X_\mathcal{B}$ and $Y_\mathcal{B}$ capture the duality used in Nagel's definition. For instance, in the situation of both categories being R-Mod and we use the contravariant left exact functors $S = T = \mathrm{Ext}^g_R(-, R)$ we have that $X_\mathcal{B}$ and $Y_\mathcal{B}$ are the isomorphisms connecting a quasi-Gorenstein R-module Q with $\mathrm{Ext}^g_R(Q, R)$.

We define linkage in almost exactly the same manner as before, given an object X which can be approximated by a Fossum object B we have a short exact sequence

$$0 \to K_X \to F_\mathcal{X}(B) \to X \to 0$$

in $\mathcal{X}$, and after applying the functor S we get

$$0 \to S(X) \to S(F_\mathcal{X}(B)) \to S(K_X) \to R^1 S(X) \to 0$$

in $\mathcal{Y}$. From this exact sequence we obtain the short exact sequence

$$0 \to S(X) \to F_\mathcal{Y}(B) \to \mathcal{L}^S_B(X) \to 0$$

as $S \circ F_\mathcal{X} = X_\mathcal{B} \circ F_\mathcal{X} = F_\mathcal{Y}$. Then we have a similar definition for linkage.

Definition 1.0.13. *Let* $X \in \mathcal{X}$ *and* $Y \in \mathcal{Y}$ *be objects. We say that* X *is directly linked to* Y *by* $B \in \mathcal{B}$ *if* $\mathcal{L}^S_B(X) \cong Y$ *and* $\mathcal{L}^T_B(Y) \cong X$.

One can then discuss objects which are linked through any number of these direct links, and further break this down into an even number of direct links and an odd number of direct links. These

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are the even and odd linkage classes of an object in either $\mathcal{X}$ or $\mathcal{Y}$. Note, by using two different categories $\mathcal{X}$ and $\mathcal{Y}$ we given strong evidence to suggest that properties which should be shared throughout linked objects should only be shared by those in the same category. In ideal and module linkage, one large area of study are properties which are shared across an entire linkage class. One of the more popular properties to consider are those ideals which are in the same linkage class as a complete intersection ideal. This is similar to asking about objects which are in the same linkage class as an object in $\mathcal{B}$. However, the question should instead be what properties are shared by those in the even linkage class of such an object as those are the ones which fall in the same category. Therefore, many of the results we prove show that objects in the same even linkage class share many nice properties not shared by the entire linkage class.

Next, we prove which homological properties are shared by linked objects. An interesting object to consider is one which is invariant under the composition $S \circ T$ or $T \circ S$. We define these as the perfect objects in $\mathcal{X}$ or $\mathcal{Y}$, see Definition 5.1.7. These are quite similar to the G_C^g-Gorenstein dimension zero modules discuss in Chapter 4. We prove this property is invariant under linkage giving the following result.

Corollary 1.0.14. *Let $((\mathcal{X}, \mathcal{Y}), \mathcal{B}, (S, T))$ be a category pair with linkage. Suppose that X is S-perfect. Then every object in the even linkage class of X is S-perfect and every object in the odd linkage class of X is T-perfect.*

What is left out is the natural biduality map used to define G_C^g-Gorenstein dimension zero. If such a mapping exists using the composition $S \circ T$ or $T \circ S$ for any object we say the category pair with linkage is perfect. More specifically, if there is some natural transformation $\delta_{\mathcal{X}}^S(-)$ between the identity functor and $T \circ S$ or vice versa for $\mathcal{Y}$, we say that the category pair with linkage is S-perfect. This is exactly when S and T form an adjoint functor pair. In such categories one can

prove many generalizations of results from many sources such as $[18, 36, 49, 61, 62]$. Specifically, one can see that the four term exact sequences associated each natural biduality map is a specific instance of the following:

Proposition 1.0.15. *Let $((\mathcal{X}, \mathcal{Y}), \mathcal{B}, (S, T))$ be a category pair with linkage which is also both S-perfect and T-perfect. Then*

(i) *For any $X \in \mathcal{X}$ we have an exact complex*

$$0 \longrightarrow R^1 T(D_S(X)) \longrightarrow X \xrightarrow{\delta_{\mathcal{X}}^S(X)} T(S(X)) \longrightarrow R^2 T(D_S(X)) \longrightarrow 0$$

(ii) *For any $Y \in \mathcal{Y}$ we have an exact complex*

$$0 \longrightarrow R^1 S(D_T(Y)) \longrightarrow Y \xrightarrow{\delta_{\mathcal{Y}}^T(Y)} S(T(Y)) \longrightarrow R^2 S(D_T(Y)) \longrightarrow 0$$

The objects $D_S(X)$ and $D_T(Y)$ are called the S and T-duals of X and Y, respectively. Furthermore, we give characterizations of perfect objects and conditions for objects to have nonempty linkage classes. The culmination of this is the following:

Theorem 1.0.16. *Let $((\mathcal{X}, \mathcal{Y}), \mathcal{B}, (S, T))$ be perfect and suppose that $X \in \mathcal{X}$ has no summands in $\mathcal{B}$. If X is directly linked to Y by B and X' is directly linked to Y by B' then $R^i T(D_S(X)) \cong R^i T(D_S(X'))$ and $R^i S(X) \cong R^i S(X')$ for $i > 0$.*

In Section 5.3, we use the perfect objects in these linkage categories to define homological dimensions associated to the functors S and T, called S and T-dimensions. As the perfect objects are generalizations of G_C^g-Gorenstein dimension zero modules, among many other types of modules or objects, the results concerning these dimensions will mirror those surrounding G_C^g-dimension. We first give the exact value for these dimensions, when it is finite.

Theorem 1.0.17. *For a category pair with linkage $((\mathcal{X}, \mathcal{Y}), \mathcal{B}, (S, T))$ we have the following:*

(i) *For $X \in \mathcal{X}$ with $S\text{-}dim_{\mathcal{X}}(X) < \infty$, $S\text{-}dim_{\mathcal{X}}(X) = \sup\{i : D^i S(X) \neq 0\}$.*

(ii) *For $Y \in \mathcal{Y}$ with $T\text{-}dim_{\mathcal{Y}}(Y) < \infty$, $T\text{-}dim_{\mathcal{Y}}(Y) = \sup\{i : D^i T(Y) \neq 0\}$.*

With these homological dimensions we are able to discuss their connection with the linkage classes of objects in $\mathcal{X}$ and $\mathcal{Y}$. We are able to show, under additional conditions on S and T, that the dimension is preserved in linkage.

Corollary 1.0.18. *Let $((\mathcal{X}, \mathcal{Y}), \mathcal{B}, (S, T))$ be perfect where S and T are also perfect. Objects which are evenly linked have the same S or T-dimension.*

Chapter 6 is used to present applications of the results found in Chapter 5. We give the example of module linkage defined by Nagel, except using semidualizing R-modules. This is the situation where $\mathcal{X} = \mathcal{Y} = R\text{-Mod}$ and $S = T = \text{Ext}_R^g(-, C)$ where C is a semidualizing R-module. Therefore the associated homological dimension is the G_C^g-dimension. However, we first answer a question about when the linkage class of an R-module is nonempty. This is a topic of discussion by Martsinkovsky and Strooker, see [49]. They use the term horizontal linkage to describe when a module has a nonempty linkage class using only free modules from R. The following definition gives the situation.

Definition 1.0.19. *Let M be an R-module with $grade_R(M) = g$. We say that M is horizontally linked by Q if $M \cong \mathcal{L}_Q^2(M) := \mathcal{L}_Q(\mathcal{L}_Q(M))$.*

It is then clear that M is horizontally linked by *some* Q if and only if the linkage class of M is nonempty. Given an R-module with no quasi-Gorenstein summands we can construct a short exact sequence

$$0 \to \text{Ext}_R^{g+1}(D_C^g M, C) \to M \to \mathcal{L}_Q^2(M) \to 0$$

from the associated four term exact sequence. This leads to the following result.

Theorem 1.0.20. *A finitely generated R-module M is horizontally linked by some Q if and only if $Ext_R^{g+1}(D_C^g M, C) = 0$.*

This gives a characterization of when the linkage class of any R-module (for this type of linkage) is nonempty. It is not unexpected that this is purely a homological property since it follows from categorically defined concepts.

In Section 6.1.2, we extend the results from Section 4.2 to include results connecting together G_C^g-dimension, the generalized Serre-like conditions, and linkage. We prove the following result:

Theorem 1.0.21. *Let R be a Cohen-Macaulay ring, C a semidualizing R-module, and M an R-module with $\mathrm{grade}_R(M) = g$ with no quasi-Gorenstein summands. Suppose that $n \geq g$, M is in the Auslander class with respect to C, and the G_C^g-dimension of M is locally finite. Then the following are equivalent:*

   (i) *M satisfies $\tilde{S}_n^g$*

   (ii) *M is horizontally linked by some C-quasi-Gorenstein R-module Q and $Ext_R^{g+i}(\mathcal{L}_Q(M), C) = 0$ for $0 < i < n - g$.*

In Section 6.2, we give another application of the results from Chapter 5. Here we specialize to the situation where $\mathcal{X} = \mathcal{Y} = R$-Mod and $S = C \otimes_R -$ and $T = \mathrm{Hom}_R(C, -)$. Here $C \otimes_R -$ is a covariant right exact functor and $\mathrm{Hom}_R(C, -)$ is a covariant left exact functor. There is then a category pair with linkage associated to these functors. It is more difficult to say what R-modules are in the Fossum category $\mathcal{B}$ associated to these functors, but it isn't difficult to see what the perfect objects are. The Auslander class and Bass class with respect to C are defined in the following way:

Definition 1.0.22. *Let M be an R-module and C a semidualizing R-module. We say that*

(*i*) *M is in the Auslander class with respect to C, Aus(C) if*

    (*a*) *$Tor_i^R(C, M) = 0$ for $i > 0$,*

    (*b*) *$Ext_R^I(C, C \otimes_R M) = 0$ for $i > 0$,*

    (*c*) *the natural evaluation map $M \to Hom_R(C, C \otimes_R M)$ is an isomorphism.*

(*ii*) *M is in the Bass class with respect to C, Bass(C) if*

    (*a*) *$Ext_R^i(C, M) = 0$ for $i > 0$,*

    (*b*) *$Tor_i^R(C, Hom_R(C, M)) = 0$ for $i > 0$,*

    (*c*) *the natural evaluation map $C \otimes_R (Hom_R(C, M)) \to M$ is an isomorphism.*

It is straightforward to see that these are exactly the conditions needed to be perfect objects and in fact the natural evaluation maps show that the linkage category is perfect. Therefore the perfect objects are the modules in the Auslander and Bass classes. These classes have been the object of study from many different avenues [5, 24, 25, 36, 61, 62] and were originally defined by Foxby, see [25]. Using results in Chapter 5, this leads to a new characterization of modules in these classes.

Theorem 1.0.23. *Let M be an R-module. Then*

(*i*) *$M \in Aus(C) \Leftrightarrow Tor_i^R(C, M) = 0 = Ext_R^i(C, D_{C \otimes_R -} M)$ for $i > 0$.*

(*ii*) *$M \in Bass(C) \Leftrightarrow Ext_R^i(C, M) = 0 = Tor_i^R(C, D^{Hom_R(C, -)} M)$ for $i > 0$.*

Further, the linkage defined by these functors moves objects from one class to another. It is easy to see that injective modules are in the Bass class and flat modules are in the Auslander class.

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Further, modules which are used to construct resolutions used for Gorenstein Injective and Flat dimension are also in these classes, so called C-flat and C-injective modules. Consider the associated homological dimensions associated to these functors, called the Aus(C)-dimension and Bass(C)-dimension, we get the following result.

Theorem 1.0.24. *For any R-module M we have the following inequalities*

$$Aus(C)\text{-}dim_R(M) \leq Gfd_R(M) \leq fd_R(M)$$
$$Bass(C)\text{-}dim_R(M) \leq Gid_R(M) \leq id_R(M)$$

Furthermore, these dimensions are preserved through linkage.

Corollary 1.0.25. *For any R-module, the Auslander dimension and the Bass dimension with respect to a semidualizing R-module C are preserved through even linkage using the functors $C \otimes_R -$ and $Hom_R(C, -)$.*

Lastly, in Section 6.3 we define linkage using the local cohomology and local homology functors. Here we move between two different categories, local cohomology takes Noetherian R-modules to Artinian R-modules and local homology takes Artinian R-modules to Noetherian R-modules. The ith local cohomology functor $H^i_{\mathfrak{m}}(-)$ is a covariant left exact functor and the ith local homology functor $H^{\mathfrak{m}}_i(-)$ is a covariant right exact functor. These form an adjoint pair of functors and so we get results concerning linkage and the homological dimensions defined by local cohomology and local homology. The culmination of these ideas gives a characterization of Cohen-Macaulay rings.

Corollary 1.0.26. *Let $(R, \mathfrak{m}, k)$ be a complete Noetherian local ring which is the homomorphic image of a Gorenstein ring. Let $d = depth_R(R)$. Then the following are equivalent:*

(i) *R is Cohen-Macaulay*

(ii) *R is evenly linked to a Cohen-Macaulay R-module*

(iii) *R is oddly linked to a co-Cohen-Macaulay R-module*

(iv) $\Gamma_{\mathfrak{m}}$-*dim*$_R(R) = d$, *i.e. there is an exact sequence*

$$0 \to R \to C_0 \to C_1 \to C_2 \to \cdots \to C_d \to 0$$

where C_i is Cohen-Macaulay of depth 0 for $0 \leq i \leq d$

(v) $H_{\mathfrak{m}}^d(R)$ *is a co-Cohen-Macaulay module*

(vi) $\Lambda_{\mathfrak{m}}$-*dim*$_R(H_{\mathfrak{m}}^d(R)) = d$, *i.e. there is an exact sequence*

$$0 \to N_d \to \cdots \to N_2 \to N_1 \to N_0 \to H_{\mathfrak{m}}^d(R) \to 0$$

where N_i is co-Cohen-Macaulay of width 0 for $0 \leq i \leq d$.

CHAPTER 2: PRELIMINARIES

This chapter will serve as a location for those results which we rely on in the subsequent chapters. Our references for most of the material concerning basic commutative algebra are [19, 21, 41, 42]. For basic category theory we reference [2, 63]. For an introduction to the theory of homological dimensions we will reference [15, 19, 63]. Lastly, for ideal linkage and module linkage we reference [39, 49, 52, 53, 57]. Those results and definitions which do not originate from the aforementioned references will be cited when presented. Further, we will recall results and definitions which motivate ideas that will appear in the rest of this thesis. The reader is advised to use this chapter as referential material.

2.1   Rings and Modules

We will be working with Noetherian commutative rings (unless otherwise stated) R with identity and all modules will assumed to be finitely generated. The category of R-modules will be denoted by R-Mod. We let the set of prime ideals of a ring R be denoted by $\mathrm{Spec}(R)$. If R is local we will denote the maximal ideal by $\mathfrak{m}$ and let k be the residue class field $R/\mathfrak{m}$. A local ring $(R, \mathfrak{m}, k)$ is called a regular local ring if size of a minimal set of generators for $\mathfrak{m}$ is equal to the Krull dimension of R. A ring R is regular if the localization R_P is a regular local ring for every prime ideal P of R.

Given an R-module M we let the annihilator of M be the ideal $\mathrm{Ann}_R(M) = \{r \in R : rM = 0\}$. A prime ideal P of R is an associated prime of M if P is the annihilator of an element of M. The collection of all associated primes of M is written $\mathrm{Ass}_R(M)$.

A complex of R-modules is a diagram

$$\mathcal{M} : \cdots \to M_{i+1} \overset{\delta_{i+1}}{\to} M_i \overset{\delta_i}{\to} M_{i-1} \to \cdots \to M_1 \to M_0 \to M_{-1} \to \cdots$$

in R-Mod where $\operatorname{im} \delta_{i+1} \subseteq \ker \delta_i$. We use the notation $\sup \mathcal{M} = \sup\{i : M_i \neq 0\}$ and $\inf \mathcal{M} = \inf\{i : M_i \neq 0\}$. The complex is exact at M_i if $\operatorname{im} \delta_{i+1} = \ker \delta_i$. A complex

$$0 \to A \overset{\alpha}{\to} B \overset{\beta}{\to} C \to 0$$

is called a short exact sequence if it is exact at A, B, and C. That is, for modules, α is a monomorphism (injection) and β is an epimorphism (surjection). Such a short exact sequence is called split if any of the following equivalent conditions are met:

(i) There is a morphism $\gamma : B \to A$ such that $\gamma\alpha = 1_A$.

(ii) There is a morphism $\gamma : C \to B$ such that $\beta\gamma = 1_C$.

(iii) $B \cong A \oplus C$.

Note that these conditions are not equivalent in general, but are for the category R-Mod. Short exact sequences are a useful homological tool to help compare module theoretic and categorical properties of the modules or objects in the sequence.

Proposition 2.1.1. *If* $0 \to M' \to M \to M'' \to 0$ *is a short exact sequence of R-modules, then*
$$Ass_R(M') \subseteq Ass_R(M) \subseteq (Ass_R(M')) \cup (Ass_R(M'')).$$

For a proof see [19, Lemma 3.6 b].

2.2 Categories and Long Exact Sequences

Given a category $\mathcal{C}$, we call the category $\mathcal{C}^{\mathrm{op}}$ the opposite category of $\mathcal{C}$ where the arrows are reversed. A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be covariant and a functor $G : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ is said to be contravariant. We will suppress the opposite notation and just use the words covariant and contravariant.

It is assumed the reader is familiar with the regular notation and terminology, see [2, 63], for categories including, zero objects, products, coproducts, equalizers, coequalizers, pullbacks, pushforwards, limits, colimits, monomorphisms, epimorphisms, and isomorphisms.

In this thesis we would like to work in a category in which we can use the Snake Lemma, but one which is more general than an Abelian category. A homological category, see [10], is one such category. A homological category $\mathcal{H}$ is a category which satisfies the following conditions:

(i) $\mathcal{H}$ is pointed, i.e. it has a zero object.

(ii) $\mathcal{H}$ is regular, i.e. $\mathcal{H}$ admits all finite limits, the kernel pair of any morphism $f : A \to B$ admits a coequalizer Q where $A \times_B A \rightrightarrows A \to Q$, and the pullback of any regular epimorphism along any morphism is a regular epimorphism.

(iii) $\mathcal{H}$ is protomodular, i.e. $\mathcal{H}$ is regular and given a regular epimorphism p and a commutative diagram

$$
\begin{array}{ccccc}
A & \xrightarrow{u} & B & \xrightarrow{v} & C \\
\downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle\gamma} \\
X & \xrightarrow{x} & Y & \xrightarrow{y} & Z
\end{array}
$$

if the outer rectangle is a pullback and the left square is a pullback, then the right square is a

pullback.

For the rest of this thesis, all categories will be assumed to be homological categories, and in most instances semi-abelian (homological and has finite coproducts) or abelian categories (homological and exact). In fact, many of the notation and terminology we pull from abelian categories, but the results are stated for homological categories. Examples of homological categories are any abelian category which includes modules categories and abelian groups; semi-abelian categories which includes categories of groups, rings, rings without unit, associative and Lie algebras; as well as topological groups and the dual of the category of pointed objects in a topos. For more specific examples see [10, Section 4.6]. This allows us to move out of the realm of module categories and consider many different situations.

A functor is called left exact if after application of the functor to a short exact sequence, the sequence stays exact on the left. Similarly for right exact functors. To each left exact functor F there are associated to it right derived functors $R^i F(-)$ for each $i \geq 0$, see [2], where $R^0 F(-) = F(-)$ such that given a short exact sequence

$$0 \to A \to B \to C \to 0$$

there is an exact complex

$$0 \to F(A) \to F(B) \to F(C) \to R^1 F(A) \to R^1 F(B) \to R^1 F(C) \to R^2 F(A) \to \cdots$$

supposing F is covariant, and reversing the order of A, B, and C for F contravariant. Similarly to each right exact functor G there are associated to it left derived functors $L_i G(-)$ for each $i \geq 0$,

see [2], where $L_0G(-) = G(-)$ such that given a short exact sequence

$$0 \to A \to B \to C \to 0$$

there is an exact complex

$$\cdots \to L_2G(C) \to L_1G(A) \to L_1G(B) \to L_1G(C) \to G(A) \to G(B) \to G(C) \to 0$$

supposing G is covariant, and reversing the order of A, B, and C for G contravariant.

These exact complexes are called long exact sequences in homology for the associated short exact sequences. There are many avenues to their discovery. One such way to approach the theory is to use comparison lemmas which take multiple short exact sequences and compare them with mappings to construct new complexes. Perhaps the most useful is the Snake Lemma which we state now.

Lemma 2.2.1. *Let $\mathcal{C}$ be a homological category and consider the following commutative diagram*

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\
0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0
\end{array}
$$

in $\mathcal{C}$. Then there exists an exact sequence

$$0 \to \ker(\alpha) \to \ker(\beta) \to \ker(\gamma) \to coker(\alpha) \to coker(\beta) \to coker(\gamma) \to 0$$

in $\mathcal{C}$.

2.2.1   Derived Functors; Tor and Ext

There are three functors between categories of R-modules we specify now, $M \otimes_R -$, $\text{Hom}_R(M, -)$, and $\text{Hom}_R(-, M)$, for a fixed R-module M.

The functor $M \otimes_R -$ is right exact covariant and the left derived functors of it are the Tor modules associated to M and are denoted $\text{Tor}_i^R(M, -)$. Note that $M \otimes_R N \cong N \otimes_R M$ for any two R-modules M and N. From this we say that Tor is balanced as $\text{Tor}_i^R(M, N) \cong \text{Tor}_i^R(N, M)$ for all i.

The functor $\text{Hom}_R(M, -)$ is left exact and covariant and the right derived functors are the Ext modules associated to M and are denoted $\text{Ext}_R^i(M, -)$. Similarly, the functor $\text{Hom}_R(-, M)$ is left exact and contravariant and the right derived functors are also the Ext modules associated to M yet they have the form $\text{Ext}_R^i(-, M)$.

2.2.2   Basic Homological Dimensions

There are classes of modules which preserve exactness of short exact sequences for certain functors. These classes give rise to certain homological dimensions which help motivate the ideas and definitions in [15].

Definition 2.2.2. *We say that an R-module M is*

(i) **free** *if it is a direct sum of copies of R.*

(ii) **projective** *if $\text{Ext}_R^1(M, N) = 0$ for any R-module N.*

(iii) **injective** if $Ext^1_R(N, M) = 0$ *for any R-module N.*

(iv) **flat** if $Tor^R_1(M, N) = 0$ *for any R-module N.*

We will let $\mathcal{P}, \mathcal{I}$, and $\mathcal{F}$ denote the classes of projective, injective, and flat R-modules, respectively. The vanishing of each of these derived functors is equivalent to each of these types of modules preserving exactness of any short exact sequence when a specific functor is applied.

The following results can be found in [19, Appendix A3.3-5].

Proposition 2.2.3. *Let P be an R-module. The following are equivalent:*

(i) $P \in \mathcal{P}$

(ii) *P is a direct summand of a free module.*

(iii) *Every epimorphism $\alpha : M \to P$ splits, i.e. there exists a morphism $\beta : P \to M$ such that $\alpha\beta = 1_P$.*

(iv) *For every epimorphism of modules $\alpha : M \to N$, the induced morphism $Hom_R(P, M) \to Hom_R(P, N)$ is an epimorphism.*

(v) *For some epimorphism $\alpha : F \to P$ where F is free, the induced morphism $Hom_R(P, F) \to Hom_R(P, P)$ is an epimorphism.*

Note that every free module is projective. We say that a E is an essential extension of an R-module M if every nonzero submodule of E intersects M nontrivially.

Proposition 2.2.4. *Suppose that M is an R-module. Then there is a unique essential extension $E(M)$ of M that is an injective R-module, i.e. there is a monomorphism $i : M \to E(M)$ where $E(M)$ is injective and is minimal in the sense that if $M \subset E'$ is essential then $E' = E(M)$.*

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Also, by [19, Corollary 6.6] the following is true.

Proposition 2.2.5. *Every free R-module is a flat R-module.*

Every R-module can be approximated by an R-module from each class, $\mathcal{P}$, $\mathcal{I}$, and $\mathcal{F}$ as well as by free modules. Given an R-module M we can approximate it by a projective module P_0 with an epimorphism $P_0 \to M$. Then consider the short exact sequence

$$0 \to K_0 \to P_0 \to M \to 0$$

where K_0 is the kernel of the map $P_0 \to M$. We can then approximate K_0 with a projective module and repeat the process. This leads to an exact sequence

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

called a **projective resolution** of M. This is a way to approximate M using projective modules; it is a way to see how close M is to being projective. In much that same way we can approximate M by free modules and construct **free resolutions** of M.

Similarly, we can construct a **injective resolution** of M

$$0 \to M \to I_0 \to I_1 \to \cdots \to I_{n-1} \to I_n \to \cdots$$

where $I_i \in \mathcal{I}$ for $i \geq 0$, and we can construct a **flat resolution** of M

$$\cdots \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$$

where $F_i \in \mathcal{F}$ for $i \geq 0$.

So to each R-module M there are collections of free, projective, injective, and flat resolutions associated to it. The length of a short resolution of each type is an invariant of the module M.

Definition 2.2.6. *Let M be an R-module M. Then the length of a shortest*

    (i) *free resolution is the **free dimension** of M, free-$dim_R(M)$.*

    (ii) *projective resolution is the **projective dimension** of M, $pd_R(M)$.*

    (iii) *injective resolution is the **injective dimension** of M, $id_R(M)$.*

    (iv) *flat resolution is the **flat dimension** of M, $fd_R(M)$.*

If there is no such shortest resolution or none are of finite length, then we say the respective dimension is infinite.

Lastly, a local ring $(R, \mathfrak{m}, k)$ is called Gorenstein if $\mathrm{id}_R(R) < \infty$ where R is considered as an R-module over itself.

2.3   Grade and Depth

In this section we will be working with a Noetherian commutative ring R.

Definition 2.3.1. *Given an R-module M, an element $r \in R$ is called a non-zero-divisor if $rm = 0$ implies $m = 0$ for $m \in M$. Then a sequence $r_1, r_2, \ldots, r_n$ is called an **M-regular sequence** if r_i is a non-zero-divisor on $M/(r_1, r_2, \ldots, r_{i-1})M$ for $i = 0, 1, \ldots, n$.*

A regular sequence over a module mirrors a linearly independent set in a vector space. We can also define a regular sequence in an ideal of R for a module M, i.e. the above sequence $r_1, r_2, \ldots, r_n$ lies in an ideal of R.

Definition 2.3.2. *Let I be an ideal in R and M and R-module such that $IM \neq M$. The **depth of I on M** is the length of a maximal M-regular sequence in I, written depth(I, M).*

If $(R, \mathfrak{m}, k)$ is a local ring, then we say the depth of M is depth$(\mathfrak{m}, M)$ and is denoted by depth$_R(M)$.

Definition 2.3.3. *Let $(R, \mathfrak{m}, k)$ be a local ring. We say that R is **Cohen-Macaulay** if depth$_R(R) = \dim_R(R)$ where $\dim_R(R)$ is the Krull dimension of R. In general a ring R is Cohen-Macaulay if the localization R_P is Cohen-Macaulay for every prime ideal P of R.*

Theorem 2.3.4. *Let $(R, \mathfrak{m}, k)$ be a local ring and M an R-module. If $id_R(M) < \infty$ then $id_R(M) = depth_R(R)$.*

For a proof see [6, Lemma 3.3]

Proposition 2.3.5. *Let M and N be R-modules. Then depth$(Ann(M), N) = \inf\{i : Ext_R^i(M, N) \neq 0\}$.*

For a proof see [19, Proposition 18.4].

With this proposition the following definition arises as it is a useful invariant of R-modules.

Definition 2.3.6. *Let M be an R-module. We say that the **grade of M** is defined as*

$$grade_R(M) = depth(Ann(M), R) = \inf\{i : Ext_R^i(M, R) \neq 0\}$$

The grade of an R-module M is also referred to as the codimension of M. With these definitions we can present a fundamental result connecting together these invariants with projective dimension.

Theorem 2.3.7 (Auslander-Buchsbaum Formula). *Let $(R, \mathfrak{m}, k)$ be a local ring and M an R-module. If $pd_R(M) < \infty$ then*

$$depth_R(M) + pd_R(M) = depth_R(R)$$

For a proof see [19, Theorem 19.9]

The Auslander-Buchsbaum formula is useful in calculating the depth or projective dimension of R-modules or even the ring R. It becomes extremely useful when attempting to characterize regular local rings.

Theorem 2.3.8. *Let $(R, \mathfrak{m}, k)$ be a local ring. The following are equivalent:*

(i) *R is a regular local ring*

(ii) *$pd_R(k) < \infty$*

(iii) *$pd_R(M) < \infty$ for any R-module M.*

For a proof see [51, Theorem 19.2].

This theorem is often referred to as the regularity theorem. It is one of the more impressive results showing that properties of a ring can be gained by approximations of its modules with a subclass of modules.

2.4   Gorenstein Dimension

In this section we will suppose that R is a Noetherian commutative ring. One of the concerns with the regularity theorem in the previous section is that the quality of being a projective module is more than just homological. Therefore one would hope that properties of a ring can be characterized by purely homological properties of its modules. To that end this section presents results leading up to the characterization of Gorenstein rings using Gorenstein dimension.

In a commutative ring R, given a module M there are a few natural maps concerning the functors $\mathrm{Hom}_R(M, -)$, $\mathrm{Hom}_R(-, M)$, and $M \otimes_R -$. The first we will discuss is called the biduality map, it is given by

$$\delta_M : M \to \mathrm{Hom}_R(\mathrm{Hom}_R(M, R), R)$$

where $\delta(m)(\varphi) = \varphi(m)$ for $\varphi \in \mathrm{Hom}_R(M, R)$ and $m \in M$. An R-module is called reflexive if this map is an isomorphism.

Definition 2.4.1. *Let M be an R-module. We say that M has **Gorenstein dimension zero** if the following conditions hold:*

  (i) *M is finitely generated*

  (ii) *$Ext^i_R(M, R) = 0 = Ext^i_R(Hom_R(M, R), R)$ for $i > 0$*

  (iii) *The natural biduality map δ_M is an isomorphism.*

It is clear that both free and projective modules satisfy these conditions. Therefore we can construct resolutions of R-modules using Gorenstein dimension zero modules, called G-resolutions.

Definition 2.4.2. *Let M be an R-module. The length of a shortest resolution of M constructed using Gorenstein dimension zero modules is the **Gorenstein dimension of** M, denoted by G-dim$_R(M)$.*

If there is no finite length resolution then the dimension is infinite.

When the Gorenstein dimension of an R-module is finite, we can say exactly what it is.

Proposition 2.4.3. *Let M be an R-module and $n \in \mathbb{N}$. The following are equivalent:*

(i) *G-dim$_R(M) \leq n$.*

(ii) *G-dim$_R(M) < \infty$ and Ext$_R^i(M, R) = 0$ for $i > n$.*

(iii) *For any G-resolution of M,*

$$\cdots \to G_i \to G_{i-1} \to \cdots \to G_1 \to G_0 \to M \to 0,$$

the kernel $K_n = \ker(G_{n-1} \to G_{n-2})$ has Gorenstein dimension zero.

For a proof see [15, Theorem 1.2.7].

Therefore if G-dim$_R(M) < \infty$ we have that G-dim$_R(M) = \sup\{i : \text{Ext}_R^i(M, R) \neq 0\}$. Gorenstein dimension shares many of the properties that projective dimension does. In fact, G-dim$_R(M) \leq$ pd$_R(M)$ for any R-module M with equality if pd$_R(M) < \infty$. Further, there is also a formula involving Gorenstein dimension and the depth of the module discovered by Auslander and Bridger, see [3].

Theorem 2.4.4 (Auslander-Bridger Formula). *Let $(R, \mathfrak{m}, k)$ be a local ring and M an R-module. If M has finite Gorenstein dimension, then*

$$G\text{-}dim_R(M) + depth_R(M) = d$$

For a proof see [3, Theorem 4.13] or [15, Theorem 1.4.8].

This result has been extended to coherent rings by Hummel and Marly, see [38, Theorem 4.4.]. This formula leads to a characterization of Gorenstein rings.

Theorem 2.4.5. *Let $(R, \mathfrak{m}, k)$ be a local ring. The following are equivalent:*

(i) *R is Gorenstein.*

(ii) *G-$dim_R(k) < \infty$.*

(iii) *G-$dim_R(M) < \infty$ for all finitely generated R-modules M.*

For a proof see [15, Theorem 1.4.9].

As stated earlier, there are other natural maps besides the biduality map worth considering in this context. These will be explored in Chapter 5.

2.5   Ideal Linkage

In this section R is a Noetherian commutative ring. An ideal $\mathfrak{c}$ in R is called a complete intersection if a minimal generating set for $\mathfrak{c}$ is a regular sequence in R. A ring S is called a complete intersection ring if $S \cong R/I$ where I is a complete intersection ideal.

Given ideals I and J of a ring R we can define the colon ideal of I and J, or ideal quotient, as $(I : J) = \{r \in R : rJ \subseteq I\}$. One can write annihilators of ideals in terms of colon ideals where

$\mathrm{Ann}(I) = (0 : I)$.

Definition 2.5.1. *Let I and J be ideals in R. We say that I is linked to J if there is a complete intersection $\mathfrak{c} \subseteq I \cap J$ such that $(\mathfrak{c} : I) = J$ and $(\mathfrak{c} : J) = I$.*

Ideal linkage was first defined by Peskine and Szpiro, see [57], and has blossomed into a tremendous theory concerning algebraic structures from modules to schemes. When discussing linkage we will be dealing with modules, and so we give an equivalent module theoretic definition.

Definition 2.5.2. *We say that the two quotients R/I and R/J or R are directly linked if there is a regular sequence $r_1, r_2, \ldots, r_n$ in $I \cap J$ such that*

$$Hom_R(R/I, R/(r_1, r_2, \ldots, r_n)) = J/(r_1, r_2, \ldots, r_n)$$

and

$$Hom_R(R/J, R/(r_1, r_2, \ldots, r_n)) = I/(r_1, r_2, \ldots, r_n).$$

This definition connects the concept of ideal linkage with the homological properties of the functor $Hom_R(-, R/\mathfrak{c})$ where $\mathfrak{c}$ is a complete intersection. In general two quotients R/I and R/J are linked if there is a sequence of direct links between the two quotients. Two ideals or quotients are evenly linked if there are an even number of links between them, and similarly for oddly linked ideals or quotients.

Ideal linkage can be generalized to schemes and this generalization helps connect the geometric properties with algebraic ones. Many results discuss the properties shared by ideals in the same linkage class. Linkage of schemes is defined by linkage of their respective ideal representations.

Proposition 2.5.3. *Suppose the two schemes V_1 and V_2 are linked by X. Then V_1 is locally Cohen-Macaulay if and only if V_2 is.*

For a proof see [52, Corollary 5.2.12]

Proposition 2.5.4. *Suppose the two schemes V_1 and V_2 are evenly linked and V_1 and V_2 are of dimension n. Then $Ext_R^{n-i+1}(R/I_{V_1}, R) \cong Ext_R^{n-i+1}(R/I_{V_2}, R)$ for all $i = 1, \ldots, n$.*

We won't be pushing further into the geometric side of linkage, but it is worth noting that many of the historical motivations for linkage arise from a desire to understand the geometric connections between curves. Perhaps the most sought after properties are those preserved by being linked to a complete intersection ideal. Such ideals are called licci ideals, as they are in the linkage class of a complete intersection. Licci ideals share many nice properties as outlined in [39, 52, 57].

2.6   Module Linkage

In this section we let R be a Noetherian commutative semiperfect ring. Module linkage arose as a way to generalize ideal linkage. There are many ways to define module linkage that generalize certain types of ideal linkage and we have chosen the definition which most closely resembles module theoretic ideal linkage definition.

First consider an R-module M. It has a projective presentation

$$P_1 \to P_0 \to M \to 0$$

where $P_0, P_1 \in \mathcal{P}$. Define the module $D_R(M) = \text{coker}(\text{Hom}_R(P_0, R) \to \text{Hom}_R(P_1, R))$ which lies in the exact sequence

$$0 \to \text{Hom}_R(M, R) \to \text{Hom}_R(P_0, R) \to \text{Hom}_R(P_1, R) \to D_R(M) \to 0.$$

It is clear here that $D_R(M)$ is only unique up to projective equivalence, but using a minimal projective presentation $D_R(M)$ is uniquely defined. Then considering

$$\text{Hom}_R(P_0, R) \to \text{Hom}_R(P_1, R) \to D_R(M) \to 0$$

as a projective presentation of $D_R(M)$, we define $\Omega D_R(M) = \text{ker}(\text{Hom}_R(P_1, R) \to D_R(M))$.

Definition 2.6.1. *Let M and N be R-modules. We say that M and N are **directly linked** if $\Omega D_R(M) \cong N$ and $\Omega D_R(N) \cong M$.*

Breaking down this definition one has the following equivalent defintion. Given a module M, take a projective presentation

$$P_1 \to P_0 \to M \to 0$$

and let $K = \text{ker}(P_0 \to M)$ so that

$$0 \to K \to P_0 \to M \to 0$$

is short exact. Then define $\lambda_R M = \text{coker}(\text{Hom}_R(M, R) \to \text{Hom}_R(P_0, R))$. So it follows that two modules M and N are directly linked if $\lambda_R M \cong N$ and $\lambda_R N \cong M$. Just as with ideals, we say two modules are linked, or in the same linkage class, if there is a sequence of direct links from one module to the other. We further distinguish between evenly linked modules and oddly linked modules.

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Many homological properties are shared between modules in the same linkage class.

Proposition 2.6.2. *Let M and N be R-modules. If M has Gorenstein dimension zero and is directly linked to N, then N has Gorenstein dimension zero.*

This is a consequence of [49, Theorem 1].

In fact, modules in the same linkage class share homological dimensions.

Theorem 2.6.3. *If R is Gorenstein, then Gorenstein dimension is preserved under module linkage.*

For a proof see [49, Corollary 14].

Further, modules in the same even linkage class also share interesting properties.

Theorem 2.6.4. *If R is Gorenstein, then projective dimension is preserved in even linkage classes for modules.*

For a proof see [49, Corollary 18].

In fact, Gorenstein dimension is preserved in even linkage classes even when R is not a Gorenstein ring, see Chapter 6. Note that in Chapter 6 a more general notion of module linkage due to Nagel, see [53], is explored as a special case of the theory in Chapter 5.

CHAPTER 3: QUASI-GORENSTEIN MODULES

In this chapter we will assume our rings are Noetherian and commutative and all modules are finitely generated. Our goal is to explore the soon to be defined quasi-Gorenstein modules. We present some properties the modules in this colleciton have, and prove an extension of the classical structure theorem for modules over a principal ideal domain attributed to Jacobson as well as Frobenius and Stickelberger. The results in this chapter give the content of our paper [12].

3.1   Definition and Properties

In this section we will give definitions and properties that will be useful in proving our results in the rest of the chapter.

Definition 3.1.1. *A finitely generated R-module M with finite projective dimension is* ***quasi-Gorenstein*** *if the following hold:*

   (i) $pd(M) = grade(M)$

  (ii) $Ext_R^{pd(M)}(M, R) \cong M$

There are many motivations for this definition. The paper of Fossum, [23], focuses on the interplay between $\text{Ext}_R^i(-, R)$ and collection of R-modules giving many useful results concerning sequences of functors and $\text{Ext}_R^i(-, R)$. The name and defintion for quasi-Gorenstein modules appears in [53] to facilitate a generalization of ideal linkage to module linkage. These two papers [23,53] provided the inspiration for this chapter.

Nagel gave these modules this name since R/I is quasi-Gorenstein if and only if I is a Gorenstein ideal when R/I is finitely generated. Futher, Grassi in [29] defines *Koszul Modules* which are quasi-Gorenstein modules with certain types of free resolutions. Furthermore, a *Gorenstein algebra* is a quasi-Gorenstein module and there is a collection of work about such algebras [8,9,22,43].

A module M is said to be perfect if grade$(M) = \text{pd}(M)$. Perfect modules are Cohen-Macaulay and thus so are quasi-Gorenstein modules, as they are clearly perfect. Quasi-Gorenstein modules have many nice properties (aside from those inherited by being Cohen-Macaulay). We present a few of them here for completeness as well as their usefulness in proofs later in this chapter.

Proposition 3.1.2 ((see [14] Exercise 1.4.26)). *Suppose that M is an R-module of projective dimension m which has a projective resolution, $\mathcal{J}$. Then M is quasi-Gorenstein if and only if $\mathcal{J}$ and $\mathcal{J}^* = Hom_R(\mathcal{J}, R)$ are homotopy equivalent up to shift, i.e. $\mathcal{J}^*$ is a projective resolution of M.*

Proof : Note that the ith homology of $\mathcal{J}^*$ is exactly $\text{Ext}^i_R(M, R)$. Therefore $\mathcal{J}^*$ is exact except at the right where the homology is $\text{Ext}^m_R(M, R) \cong M$ if and only if M is quasi-Gorenstein of projective dimension m.

$\square$

The previous result gives us another method of determining when a module is quasi-Gorenstein. We only need to know how a free or projective resolution of the module behaves with its dual. Finitely generated free R-modules are examples of quasi-Gorenstein modules of projective dimensions zero. An example of a module of projective dimension greater than 1 that is quasi-Gorenstein is a complete intersection over regular local ring. This follows as a free resolution of the complete intersection is the Koszul complex on the regular sequence that generates the ideal, and it is well

known that this complex is self-dual.

Proposition 3.1.3. *Suppose that M and $\{M_i\}_{i=1}^{\ell}$ are quasi-Gorenstein R-modules of projective dimension m and N is a quasi-Gorenstein R-module of projective dimension n. Then*

(i) $\bigoplus_{i=1}^{\ell} M_i$ *is quasi-Gorenstein of projective dimension m.*

(ii) *M is quasi-Gorenstein of projective dimension m if and only if $M_{\mathfrak{m}}$ is quasi-Gorenstein of projective dimension m for all maximal ideals $\mathfrak{m}$ of R.*

(iii) *M is quasi-Gorenstein of projective dimension m if and only if the completion $\hat{M}$ is a quasi-Gorenstein $\hat{R}$-module of projective dimension m.*

(iv) *$M \otimes_R N$ is a quasi-Gorenstein R-module of projective dimension $m + n - 1$*

Proof :

(i) This is clear by properties of Ext as

$$\operatorname{Ext}_R^j \left(\bigoplus_{i=1}^{\ell} M_i, R \right) \cong \prod_{i=1}^{\ell} \operatorname{Ext}_R^j(M_i, R) \cong \begin{cases} 0 & j \neq m \\ \bigoplus_{i=1}^{\ell} M_i & j = m \end{cases}$$

(ii) $(\Rightarrow)$ Let $\mathcal{J}$ be a projective (free) resolution of M. Then

$$\begin{aligned} \operatorname{Ext}_R^i(M, R)_{\mathfrak{m}} = (H^i(\operatorname{Hom}_R(\mathcal{J}, R)))_{\mathfrak{m}} &= H^i((\operatorname{Hom}_R(\mathcal{J}, R))_{\mathfrak{m}}) \\ &= H^i(\operatorname{Hom}_{R_{\mathfrak{m}}}(\mathcal{J}_{\mathfrak{m}}, R_{\mathfrak{m}})) \\ &= \operatorname{Ext}_{R_{\mathfrak{m}}}^i(M_{\mathfrak{m}}, R_{\mathfrak{m}}) \end{aligned}$$

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holds as localization is exact and we use $\mathcal{J}_\mathfrak{m}$ to denote the exact sequence obtained by localizing each module and morphism in $\mathcal{J}$. Therefore $\mathrm{Ext}^i_{R_\mathfrak{m}}(M_\mathfrak{m}, R_\mathfrak{m}) \cong M_\mathfrak{m}$ if $i = m$ and 0 otherwise. This shows that $M_\mathfrak{m}$ is a quasi-Gorenstein $R_\mathfrak{m}$-module of projective dimension m for any maximal ideal $\mathfrak{m}$ of R.

($\Longleftarrow$) Suppose $M_\mathfrak{m}$ is a quasi-Gorenstein $R_\mathfrak{m}$-module of projective dimension m for every maximal ideal $\mathfrak{m}$ of R. Since $M = \oplus_\mathfrak{m} M_\mathfrak{m}$ we have

$$
\mathrm{Ext}^i_R(M, R) = \oplus_\mathfrak{m} (\mathrm{Ext}^i_R(M, R))_\mathfrak{m} = \oplus_\mathfrak{m} \mathrm{Ext}^i_{R_\mathfrak{m}}(M_\mathfrak{m}, R_\mathfrak{m})
$$

$$
\cong
\begin{cases}
\oplus_\mathfrak{m} M_\mathfrak{m} & i = m, \\
0 & i \neq m
\end{cases}
$$

$$
=
\begin{cases}
M & i = m, \\
0 & i \neq m.
\end{cases}
$$

(iii) Let $\mathcal{J}$ be a projective (free) resolution of M. Since $\hat{M} = M \otimes_R \hat{R}$ we have

$$
\mathrm{Ext}^i_{\hat{R}}(\hat{M}, \hat{R}) \cong \mathrm{Ext}^i_{\hat{R}}(\hat{R} \otimes_R M, \hat{R}) = H^i\left(\mathrm{Hom}_{\hat{R}}(\hat{R} \otimes_R \mathcal{J}, \hat{R})\right)
$$

$$
= H^i\left(\mathrm{Hom}_{\hat{R}}(\hat{R} \otimes_R \mathcal{J}, \hat{R} \otimes_R R)\right)
$$

$$
\cong H^i\left(\hat{R} \otimes_R \mathrm{Hom}_R(\mathcal{J}, R)\right)
$$

$$
\cong \hat{R} \otimes_R H^i(\mathrm{Hom}_R(\mathcal{J}, R))
$$

$$
= \hat{R} \otimes_R \mathrm{Ext}^i_R(M, R)
$$

$$
= (\mathrm{Ext}^i_R(M, R))^\wedge
$$

showing that M and $\hat{M}$ are simultaneously quasi-Gorenstein over R and $\hat{R}$, respectively.

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(iv) Let $\mathcal{J}_M$ and $\mathcal{J}_N$ be projective (free) resolutions of M and N, respectively. It then follows by Proposition 3.1.2 that $\text{Hom}_R(\mathcal{J}_M, R)$ and $\text{Hom}_R(\mathcal{J}_N, R)$ are projective resolutions of M and N, respectively. Further $\mathcal{J}_M \otimes_R \mathcal{J}_N$ is a projective resolution of $M \otimes_R N$ and as

$$\text{Hom}_R(\mathcal{J}_M \otimes_R \mathcal{J}_N, R) \cong \text{Hom}_R(\mathcal{J}_M, R) \otimes_R \text{Hom}_R(\mathcal{J}_N, R)$$

we see that $\mathcal{J}_M \otimes_R \mathcal{J}_N$ and $\text{Hom}_R(\mathcal{J}_M \otimes_R \mathcal{J}_N, R)$ are homotopy equivalent. Therefore by Proposition 3.1.2 we have that $M \otimes_R N$ is quasi-Gorenstein. It is of projective dimension $m + n - 1$ as the length of $\mathcal{J}_M \otimes \mathcal{J}_N$ is $m + n - 1$.

$\square$

Proposition 3.1.4. *Suppose that M and N are quasi-Gorenstein R-modules with $pd_R(M) = m$ and $pd_R(N) = n$. Then $Tor_i^R(M, N) \cong Ext_R^{m-i}(M, N)$. Moreover, $Ext_R^{m-i}(M, N) \cong Ext_R^{n-i}(M, N)$ for all $i =, 1, \ldots, \min\{m, n\}$.*

Proof : As $pd_R(N) = n$, N has finite Tor dimension as an R-module, see [63, Section 10.8]. We have

$$\mathbf{R}\text{Hom}_R(M, R) \otimes_R^{\mathbf{L}} N \cong \mathbf{R}\text{Hom}_R(M, R \otimes_R^{\mathbf{L}} N)$$

in $\mathcal{D}(R)$ the derived category over R-Mod as N has finite Tor dimension. What is not represented in the above isomorphisms is the shift in the complexes representing each module. The difference in shift is exactly the projective dimension of M, which when taking homology gives

$$H^i(\mathbf{R}\text{Hom}_R(M, R) \otimes_R^{\mathbf{L}} N) \cong H_{m-i}(\mathbf{R}\text{Hom}_R(M, R \otimes_R^{\mathbf{L}} N)).$$

Further, in $\mathcal{D}(R)$ we have that $\mathbf{R}\text{Hom}_R(M, R) \cong \oplus_j Ext_R^j(M, R)$ and $R \otimes_R^{\mathbf{L}} N \cong \oplus_j Tor_j^R(R, N)$.

Therefore

$$\operatorname{Tor}_i^R(M, N) \cong \operatorname{Tor}_i^R(\oplus_j \operatorname{Ext}_R^j(M, R), N) \cong H^i(\mathbf{R}\operatorname{Hom}_R(M, R) \otimes_R^{\mathbf{L}} N)$$

$$\cong H_{m-i}(\mathbf{R}\operatorname{Hom}_R(M, R \otimes_R^{\mathbf{L}} N))$$

$$\cong \operatorname{Ext}_R^{m-i}(M, \oplus_j \operatorname{Tor}_j^R(R, N))$$

$$\cong \operatorname{Ext}_R^{m-i}(M, N).$$

Moreover, as Tor is balanced we have

$$\operatorname{Ext}_R^{m-i}(M, N) \cong \operatorname{Tor}_i^R(M, N) \cong \operatorname{Tor}_i^R(N, M) \cong \operatorname{Ext}_R^{n-i}(N, M).$$

$\square$

Corollary 3.1.5. *If M and N are quasi-Gorenstein R-modules of projective dimension r, then* $\operatorname{Ext}_R^r(M, N) \cong \operatorname{Ext}_R^r(N, M)$.

Proof : Follows immediately from Proposition 3.1.4.

$\square$

These results show that the collection of quasi-Gorenstein modules of a fixed projective dimension form a monoid under direct sum with an associated product using tensor product. Therefore one can form a ring of isomorphism classes of quasi-Gorenstein R-modules, not unlike a ring associated to a Grothendieck group.

3.2 Projective Dimension One

Throughout the rest of this chapter R will be a Noetherian factorial domain. We change gears and begin exploring additional module theoretic properties by comparing quasi-Gorenstein modules of projective dimension one. Notice that an R-module M of projective dimension one has a free resolution

$$0 \to F_1 \xrightarrow{m} F_0 \to M \to 0$$

where m is a matrix with entries in R. In fact, we can write the resolution as

$$0 \to R^s \xrightarrow{m} R^t \to M \to 0$$

where m is a $t \times s$ matrix. It is clear that there is a connection between M being a quasi-Gorenstein R-module and the matrix m. Note that if we apply the functor $\text{Hom}_R(-, R)$ to the resolution we get an exact sequence

$$0 \longrightarrow \text{Hom}_R(M, R) \longrightarrow \text{Hom}_R(R^t, R) \xrightarrow{m^T} \text{Hom}_R(R^s, R) \longrightarrow \text{Ext}^1_R(M, R) \longrightarrow 0.$$

which reduces to the sequence

$$0 \longrightarrow \text{Hom}_R(M, R) \longrightarrow R^t \xrightarrow{m^T} R^s \longrightarrow \text{Ext}^1_R(M, R) \longrightarrow 0$$

where m^T is the transpose of m.

Definition 3.2.1. *Let m and m' be two $n \times n$ matrices over R. We say that m and m' are equivalent if there are isomorphisms $\Phi : R^n \to R^n$ and $\Psi : R^n \to R^n$ such that $\Phi m = m^T \Psi$.*

In addition to this definition, we say that a matrix is diagonalizable if it is equivalent to a diagonal

matrix. If we are considering modules that are candidates to be quasi-Gorenstein, then the matrix presenting it must be a square matrix. We then get the following result.

Corollary 3.2.2. *Suppose that M is an R-module of projective dimension one presented by a full rank $n \times n$ matrix m. Then m is equivalent to m^T if and only if M is a quasi-Gorenstein R-module.*

Proof : Clear by Proposition 3.1.2.

$\square$

3.3   Minimal Cyclic-Filtrations

We begin this section with a result connecting together quasi-Gorenstein modules with their associated primes.

Proposition 3.3.1. *Suppose M is a quasi-Gorenstein R-module of projective dimension one and M' a submodule of M. Then $Ext^1_R(M/M', R)$ is isomorphic to a submodule of M. Moreover, if M/M' is quasi-Gorenstein of projective dimension 1 then M/M' is isomorphic to a submodule of M and $Ass(M/M') \subset Ass(M)$.*

Proof : Note that by [23] Proposition $3(a)$ we have that

$$\mathrm{grade}(M') \geq \mathrm{grade}(M) = 1.$$

The result then follows by dualizing the short exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M/M' \longrightarrow 0$$

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to get

$$0 \longrightarrow \operatorname{Ext}_R^1(M/M', R) \longrightarrow \operatorname{Ext}_R^1(M, R) \longrightarrow \operatorname{Ext}_R^1(M', R) \longrightarrow \operatorname{Ext}_R^2(M/M', R) \longrightarrow 0$$

which shows that $\operatorname{Ext}_R^1(M/M', R)$ is isomorphic to a submodule of $M \cong \operatorname{Ext}_R^1(M, R)$. Further, if M/M' is quasi-Gorenstein of projective dimension 1 then the sequence above shows that M/M' is isomorphic to a submodule of M.

□

This shows that the associated primes of a module and whether or not it is a quasi-Gorenstein module are intimately related. It says that if the quotient of a module is quasi-Gorenstein then the associated primes of the quotient are among those of the quasi-Gorenstein module. One way to guarantee this is to restrict ourselves to such modules. We can continue this process of looking at quotients with the submodule and obtain a filtration of M. Many types of filtrations exist in the literature and we define a type of filtration closely related to those of clean and pretty clean filtrations explored by Herzog and Popescu in [35]. We will denote by $L(M)$ the lattice of ideals in R containing ideals of the form $\operatorname{Ann}(x)$ for $x \in M$ under inclusion, that is the lattice of associated primes of M.

Definition 3.3.2. *Let M be a finitely generated R-module and*

$$\mathcal{M} : 0 = M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq M_3 \subsetneq \ldots \subsetneq M$$

*be an increasing chain of submodules of M. We say that $\mathcal{M}$ is a **cyclic-filtration** of M if*

$$M_{i+1}/M_i \cong R/I_{i+1} \qquad \text{with } I_{i+1} \in L(M) \quad \text{for } i = 0, 1, \ldots$$

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Since R is Noetherian, the length of a cyclic-filtration is finite. We will say that a cyclic-filtration is a **minimal cyclic-filtration** if each module M_i is a minimal submodule of M_{i+1} such that $M_{i+1}/M_i \cong R/I_{i+1}$ is maximal for some $I_{i+1} \in L(M_{i+1})$. In other words, the filtration is minimal at M_{i+1} if the annihilators of the quotients are as small as possible in $\mathcal{L}(M_{i+1})$. In order to see the difference between these consider the module $\mathbb{Z}/4\mathbb{Z}$ over $\mathbb{Z}$. We have the obvious filtration

$$0 \subsetneq \mathbb{Z}/4\mathbb{Z}$$

and the one by using the associated prime of $\mathbb{Z}/4\mathbb{Z}$ which is

$$0 \subsetneq \mathbb{Z}/2\mathbb{Z} \subsetneq \mathbb{Z}/4\mathbb{Z}.$$

These are both cyclic-filtrations of $\mathbb{Z}/4\mathbb{Z}$, but only the first is a *minimal* cyclic-filtration of $\mathbb{Z}/4\mathbb{Z}$. Notice that the sequence

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/4\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

is not split. We will see in the next section that having the filtration be minimal and consisting of quasi-Gorenstein submodules is enough to guarantee such a sequence splits.

3.4 An Extension of a Theorem of Frobenius and Stickelberger

This section is a culmination of the chapters material where we present results directly used in proving the following theorem, which is a useful application of quasi-Gorenstein modules.

Theorem 3.4.1. *Let R be a Noetherian commutative factorial domain and M an R-module of projective dimension one presented by a full rank $n \times n$ matrix m. The following are equivalent:*

(i) *M is quasi-Gorenstein and M has a minimal cyclic-filtration consisting of quasi-Gorenstein submodules.*

(ii) *m is equivalent to m^T and M has a minimal cyclic filtration consisting of quasi-Gorenstein submodules.*

(iii) *$M \cong \bigoplus\limits_{i=1}^{n} R/(\lambda_i)$, $\lambda_i \in R$.*

(iv) *The matrix m is diagonalizable.*

This theorem is an extension of the structure theorem for finitely generated modules over a principal ideal domain, see [26] for the original result and see [2, Section 6.5] for a modern proof, which is a futher generalization of the fundamental theorem of finitely generated abelian groups, see [41]. In essence this theorem is a characterization of the class of modules which are in some sense are free over R. To prove Theorem 3.4.1 the following lemma is key.

Lemma 3.4.2. *Let R be a Noetherian factorial domain. Suppose M is a quasi-Gorenstein R-module of projective dimension one and let $\mathcal{M} : 0 \subsetneq M' \subsetneq M$ be a minimal cyclic-filtration of M where M' is a perfect R-module. Then $M \cong M' \oplus M/M'$.*

Proof : Consider the short exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M/M' \longrightarrow 0$$

Where $M/M' \cong R/I$ for some height one ideal I of R. Note that I is principal as R is a factorial domain, and so R/I is a quasi-Gorenstein R-module. Rewrite the above sequence as

$$(i) : 0 \longrightarrow M' \xrightarrow{\eta} M \xrightarrow{\lambda} R/I \longrightarrow 0$$

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and suppose that this sequence is not split exact. There exists $m \in M$ such that $\lambda(m)$ generates R/I. Now $\text{Ann}(m) \subseteq I$, but as the sequence is not split $0 \neq I \cdot m \subset \eta(M')$, and so $\text{Ann}(m) \subsetneq I$. Then as $R/\text{Ann}(m)$ is a submodule of M we have the short exact sequence

$$(ii) : 0 \longrightarrow R/\text{Ann}(m) \xrightarrow{\gamma} M \xrightarrow{\delta} K \longrightarrow 0.$$

If we take the dual of (i) we get

$$0 \longrightarrow R/I \xrightarrow{\lambda^*} M \xrightarrow{\eta^*\varphi} \text{Ext}^1_R(M', R) \longrightarrow 0$$

where φ is the isomorphism between M and $\text{Ext}^1_R(M, R)$. We get a commutative diagram

$$
(*) \qquad
\begin{array}{ccccccccc}
0 & \longrightarrow & R/I & \xrightarrow{\lambda^*} & \text{Ext}^1_R(M, R) & \xrightarrow{\eta^*} & \text{Ext}^1_R(M', R) & \longrightarrow & 0 \\
 & & \downarrow{\alpha} & & \downarrow{\varphi} & & \downarrow{\beta} & & \\
0 & \longrightarrow & R/\text{Ann}(m) & \xrightarrow{\gamma} & M & \xrightarrow{\delta} & K & \longrightarrow & 0
\end{array}
$$

where α and β are induced by universal properties as $\delta\varphi\lambda^* = 0$. Note that α is an injection and β

is a surjection by the Snake Lemma. Taking a dual of $(*)$ we get a commutative diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & & & & & I/\mathrm{Ann}(m) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Ext}^1_R(K,R) & \xrightarrow{\delta^*} & \mathrm{Ext}^1_R(M,R) & \xrightarrow{\gamma^*} & R/\mathrm{Ann}(m) & \longrightarrow \mathrm{Ext}^2_R(K,R) \longrightarrow 0\\
 & & \downarrow{\beta^*} & & \downarrow{\varphi} & & \vdots\,{\varepsilon} & & \\
0 & \longrightarrow & M' & \xrightarrow{\eta} & M & \xrightarrow{\lambda} & R/I & \longrightarrow 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \mathrm{coker}(\beta^*) & & 0 & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}
$$

Where the left square is commutative as it is the dual of the right square in $(*)$. This induces the mapping ε which is a surjection. We claim that $\mathrm{Ext}^2_R(K,R) = 0$. $\mathrm{Ext}^2_R(K,R)$ is the cokernel of the mapping γ^*. The image of γ^* in $R/\mathrm{Ann}(m)$ is isomorphic to R/I as it is the same as $\alpha(R/I)$ in $(*)$ and the image of M through λ is R/I. Therefore $\mathrm{Ext}^2_R(K,R) = I/\mathrm{Ann}(m)$, As R is a factorial domain, if $\mathrm{Ext}^2_R(K,R)$ is non zero, it only has associated primes of height at least two. This is a contradiction as $I/\mathrm{Ann}(m) \cong R/(\mathrm{Ann}(m) :_R I)$ and $(\mathrm{Ann}(m) :_R I) \subset \mathrm{Ann}(m)$ is principal because both I and $\mathrm{Ann}(m)$ are principal. So since R is factorial, $R/(\mathrm{Ann}(m) :_R I)$ has only associated primes of height one. Therefore we must have that $\mathrm{Ext}^2_R(K,R) = 0$. However, $\mathrm{Ext}^2_R(K,R) = I/\mathrm{Ann}(M)$ and so $\mathrm{Ann}(M) = I$ a contradiction to the original assumption that (i) is not split. Therefore (i) must be split.

$\square$

Proposition 3.4.3. *Suppose M is a quasi-Gorenstein R-module of projective dimension one and*

$$\mathcal{M} : 0 = M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots \subsetneq M_{n-1} \subsetneq M_n = M$$

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is a minimal cyclic-filtration of M where M_i is a quasi-Gorenstein submodule of M for $i =$
$1, \ldots, n-1$. Then $M \cong \bigoplus_{i=0}^{n-1} M_{i+1}/M_i$.

Proof : We prove this result by induction on n. The case $n = 1$ is trivial as $M_1 = M = M_1/M_0 \cong$
R/I for some $I \in \mathcal{L}(M)$. The case $n = 2$ is Lemma 3.4.2. So suppose that $n > 2$. We claim that

$$\mathcal{M}' : 0 = M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots \subsetneq M_{n-2} \subsetneq M_{n-1}$$

is a minimal cyclic-filtration of M_{n-1}. Indeed it is a cyclic filtration as $M_{i+1}/M_i \cong R/I_{i+1}$ with
$I_{i+1} \in \mathcal{L}(M_{i+1}) \subset \mathcal{L}(M_{n-1})$ for $i = 0, \ldots, n-2$. It is minimal as each module M_i is a minimal
submodule of M_{i+1} with $M_{i+1}/M_i \cong R/I_{i+1}$ with $I_{i+1} \in \mathcal{L}(M_{i+1})$, for $i = 1, \ldots, n-2$. So by
induction $M_{n-1} \cong \bigoplus_{i=0}^{n-2} M_{i+1}/M_i \cong \bigoplus_{i=0}^{n-2} R/I_{i+1}$. Now M_{n-1} is a minimal submodule of M
with $M/M_{n-1} \cong R/I$ for $I \in \mathcal{L}(M)$. We have the following sequence

$$0 \longrightarrow \bigoplus_{i=0}^{n-1} R/I_{i+1} \stackrel{\alpha}{\longrightarrow} M \stackrel{\beta}{\longrightarrow} R/I \longrightarrow 0$$

Using the same argument as that of Lemma 3.4.2 we see that is split and $M \cong \bigoplus_{i=0}^{n-1} M_{i+1}/M_i$.

$\square$

Now we can prove Theorem 3.4.1.

Proof : *(of Theorem 3.4.1)*

$(i) \Leftrightarrow (ii)$ is Corollary 3.2.2.

$(iii) \Leftrightarrow (iv)$ is trivial.

$(i) \Rightarrow (iv)$ is Proposition 3.4.3.

So we are left to prove $(iv) \Rightarrow (i)$. We know that if m is diagonalizable then M is a quasi-Gorenstein module as m is equivalent to m^T. Next as $(iii) \Leftrightarrow (iv)$ we can take a decomposition of $M \cong \bigoplus_{i=1}^{m} R/(\lambda_i)$ for $\lambda_i \in R$. Consider $\mathcal{L}(M)$ and choose (λ_j) such that (λ_j) is minimal among all (λ_i) for $i = 1, \ldots, m$. Note that there may be more than one choice of such ideals. Let $M_{m-1} = \bigoplus_{i=1, i\neq j}^{m} R/(\lambda_i)$. Then $M_{m-1} \subsetneq M$ is a piece of a minimal cyclic-filtration of M. It is clear that $M/M_{m-1} = R/(\lambda_j)$ and M_{m-1} is minimal with this property by the choice of (λ_j). We repeat this process for M_{m-1} in $\mathcal{L}(M_{m-1})$ to obtain a minimal submodule M_{m-2} with $M_{m-1}/M_{m-2} = R/(\lambda_k)$ for some $k \in \{1, 2, \ldots, m\} \smallsetminus \{j\}$. Continuing in this fashion we obtain $M_1, M_2, \ldots, M_{m-1}$ and a cyclic-filtration of M

$$\mathcal{M} : 0 = M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots \subsetneq M_{m-2} \subsetneq M_{m-1} \subsetneq M$$

which is a minimal cyclic-filtration of M by the choice of each quotient M_{i+1}/M_i for $i = 0, \ldots, m-1$. Note that each M_i is a quasi-Gorenstein module by Proposition 3.1.2 (a) as it is a finite direct sum of quasi-Gorenstein modules $R/(\lambda_i)$. Thus $\mathcal{M}$ is a minimal cyclic-filtration of M consisting of quasi-Gorenstein submodules of M. This proves $(iv) \Rightarrow (i)$ and the Theorem is shown.

$\square$

CHAPTER 4: C-GORENSTEIN DIMENSIONS AND SERRE-LIKE CONDITIONS

Throughout this chapter R will be a Noetherian ring and all R-modules are assumed to be finitely generated. The goal of this chapter is to build up a theory of intermediate Gorenstein dimensions for modules which will allow us to generalize results pertaining to generalized Serre conditions and module linkage.

4.1 Intermediate C-Gorenstein Dimensions

This section lays the groundwork for a generalization of G-dimension to modules of nonzero grade. The theory is presented in full generality by moving to a semidualizing module over the ring.

Recall the definition of a semidualizing R-module, first studied in [25] and [28].

Definition 4.1.1. *An R-module C is called **semidualizing** if the homothety map $R \to Hom_R(C, C)$ is an isomorphism and $Ext_R^i(C, C) = 0$ for $i > 0$.*

Obvious examples of semidualizing modules are the ring R and the canonical module ω_R of a Cohen-Macaulay ring R. There has been a large body of research into semidualizing modules over rings and their connections with homolgical dimensions, [58], [59], [25], and [28].

For a semidualizing R-module C, we can define the C-grade of M as

$$\mathrm{grade}_C(M) = \mathrm{depth}_R(\mathrm{Ann}_R(M), C) = \inf\{i \mid \mathrm{Ext}^i_R(M, C) \neq 0\}.$$

Note that $\mathrm{grade}_R(M) = \mathrm{grade}_C(M)$ for any semidualizing R-module C, and so we will continue to use $\mathrm{grade}_R(M)$. In a similar fashion, we define $\alpha_R(M) = \sup\{i : \mathrm{Ext}^i_R(M, R) \neq 0\}$ and so we can also define $\alpha_C(M) = \sup\{i \mid \mathrm{Ext}^i_R(M, C) \neq 0\}$. Moreover, $\alpha_C(M) = \alpha_R(M)$ for any semidualizing R-module C and so we will use $\alpha_R(M)$ where appropriate. There is a similar natural map $\delta^C_M : M \to \mathrm{Hom}_R(\mathrm{Hom}_R(M, C), C)$ to the bi-duality map for any semidualizing R-module C.

Due to this the Gorenstein dimension of a module has been extended to the C-Gorenstein dimension of a module [24]. In order to do this, one only needs to replace R with C in the appropriate places in the definition. However, one of the disadvantages of C-Gorenstein dimension (or even Gorenstein dimension) is that the modules that comprise the resolutions sit in grade zero whereas the module being examined may not. It is useful to be able to use modules in the same grade as it helps sift out unwanted information and makes proof techniques simpler. We can illustrate this with an example.

Example 4.1.2. *Consider the polynomial ring $R = k[x, y]$ where k is an infinite field and let $C = R$ be the semidualizing module and k be the R-module under consideration. We can take a G_C-resolution of k as*

$$0 \longrightarrow R \xrightarrow{\begin{pmatrix} x \\ y \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} -y & x \end{pmatrix}} R \longrightarrow k \longrightarrow 0$$

which is the Koszul complex of k. If we also consider the R-module $R/(x)$, we see that these

modules sit in different grades, k in grade 2 and $R/(x)$ in grade 1. If we wanted to consider the R-module $Hom_R(k, R/(x))$ (this arises when discussing depth and local cohomology) we can apply $Hom_R(-, R/(x))$ to the Koszul complex of k. However, it may be simpler instead to consider the following way of representing k:

$$0 \longrightarrow R/(x) \xrightarrow{\cdot y} R/(x) \longrightarrow k \longrightarrow 0.$$

Notice that since $R/(x)$ is of grade 1, both modules presenting k are of grade 1. Then it is much simpler to understand $Hom_R(k, R/(x))$ by using this short exact sequence. In fact, we have

$$0 \longrightarrow Hom_R(k, R/(x)) \longrightarrow Hom_R(R/(x), R/(x)) \xrightarrow{\cdot y} Hom_R(R/(x), R/(x))$$

which shows that $Hom_R(k, R/(x))$ is the kernel of the right map. This shows that there is an advantage to using representations of modules in higher grades.

Notice that $R/(x)$ as an R-module satisfies

(i) $grade_R(R/(x)) = 1$

(ii) $Hom_R(R/(x), R) = 0$, $Hom_R(Ext^1_R(R/(x), R), R) = 0$, and $Ext^2_R(R/(x), R) = 0$

(iii) $R/(x) \cong Ext^1_R(Ext^1_R(R/(x), R), R)$.

In other words, these conditions are similar to those of G_C-dimension zero but for a higher index in Ext.

In order to define an analogous C-Gorenstein dimension for higher grade, we will need a "biduality" map in higher grade. It follows from results in [3] or [23] that for an R-module M with

$\text{grade}_R(M) \geq g$ there is a natural map

$$\delta_C^g(M) : M \longrightarrow \text{Ext}_R^g(\text{Ext}_R^g(M, C), C)$$

we denote by $\delta_C^g(M)$.

This motivates the following definition.

Definition 4.1.3. *Suppose that C is a semidualizing R-module. We say that an R-module M with $\text{grade}_R(M) = g$ has* **G_C^g-dimension zero**, *or $G_C^g\text{-dim}_R(M) = 0$, if the following conditions hold:*

(i) *$Ext_R^{g+i}(M, C) = 0$ for $i > 0$*

(ii) *$Ext_R^{g+i}(Ext_R^g(M, C), C) = 0$ for $i \neq 0$*

(iii) *The map $\delta_C^g(M) : M \to Ext_R^g(Ext_R^g(M, C), C)$ is an isomorphism*

We will call the class of all G_C^g-dimension zero modules $\mathcal{G}_C^g$. Notice that if $M \in \mathcal{G}_C^j$ then so is $\text{Ext}_R^j(M, C)$. Then just as with C-Gorenstein dimension we can resolve any module by modules in $\mathcal{G}_C^g$ and define

Definition 4.1.4. *Suppose that C is a semidualizing R-module and $\text{grade}_R(M) \geq j$ for an R-module M. Given an exact sequence*

$$\mathcal{C} : 0 \longrightarrow M_n \longrightarrow M_{n-1} \longrightarrow \cdots \longrightarrow M_1 \longrightarrow M_0 \longrightarrow M \longrightarrow 0$$

where $M_i \in \mathcal{G}_C^j$ for $i = 0, \ldots, n$, we will say that $\mathcal{C}$ is a G_C^j-resolution of M.

If M has a G_C^j-resolution of length $n + 1$, such as the one above, then we say that the **jth intermediate C−Gorenstein dimension** of M is less than or equal to n, or $G_C^j\text{-dim}_R(M) \leq n$.

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Note that this agrees exactly with $G_C\text{-dim}_R(M)$ when $j = 0$. A question one may have after this definition is if every module has such a resolution. The assumption that $\text{grade}_R(M) \geq j$ was made to provide an affirmative answer to this inquiry. If $\text{grade}_R(M) \geq j$, then there exists a C-regular sequence $\bar{x} = (x_1, \ldots, x_j)$ in $\text{Ann}_R(M)$. Then there is a natural surjective map from $(R/(\bar{x}) \otimes_R C)^\alpha \cong (C/(\bar{x})C)^\alpha \to M \to 0$ for some $\alpha \geq 0$. Clearly $(C/(\bar{x})C)^\alpha \in \mathcal{G}_C^j$. Since $\text{grade}_R(\ker(C/(\bar{x})C \to M) \geq \text{grade}_R(M) \geq j$ we continue the process with $\ker(C/(\bar{x})C \to M)$. Continuing in this fashion we can iteratively construct a G_C^j-resolution of any R-module M with $\text{grade}_R(M) \geq j$.

There are analogous results for G_C^j-dimension for those on G-dimension. We list a few which are necessary for proofs later on, but many of these results are special cases of results in Chapters 5 and 6. The next result is analogous to [15, Lemma 1.1.10].

Proposition 4.1.5. *Suppose that $0 \to M' \to M \to M'' \to 0$ is short exact with $\text{grade}_R(M') = \text{grade}_R(M) = \text{grade}_R(M'') = g$. Then*

(a) *If $M'' \in \mathcal{G}_C^g$, then $M \in \mathcal{G}_C^g \Leftrightarrow M' \in \mathcal{G}_C^g$*

(b) *If $M \in \mathcal{G}_C^g$, then $\text{Ext}_R^{g+i}(M', C) \cong \text{Ext}_R^{g+i+1}(M'', C)$ for $i > 0$*

(c) *If the sequence splits, then $M \in \mathcal{G}_C^g \Leftrightarrow M', M'' \in \mathcal{G}_C^g$*

Proof : Both (a) and (b) are clear by applying $\text{Hom}_R(-, C)$ and considering the long exact sequence, and (c) follows by the naturality of $\delta_C^g(-)$ and the commutativity of Ext and direct sum. $\square$

The next two lemmas will make the proof of the following results simpler.

Lemma 4.1.6. *Suppose that*

$$0 \longrightarrow M_n \longrightarrow M_{n-1} \longrightarrow \cdots \longrightarrow M_1 \longrightarrow M_0 \longrightarrow M \longrightarrow 0$$

is an exact complex of R-module with $M_i \in \mathcal{G}_C^g$ for $i = 0, \ldots, n-1$. Let $K_i = \ker(M_{i-1} \to M_{i-2})$ for $i = 2, \ldots, n$ with $K_n = M_n$ and $K_0 = M_0$. Then

$$Ext_R^{grade_R(M_{i-1})+j}(K_i, C) \cong Ext_R^{grade_R(M_{i-1})+i+j}(M, R) \text{ for } j > 0 \text{ and } i = 0, \ldots, n.$$

Proof : Clear by breaking the complex into short exact sequences and applying $\text{Hom}_R(-, C)$ or using Proposition 4.1.5 (b).

$\square$

Lemma 4.1.7. *Suppose that*

$$0 \longrightarrow M_n \longrightarrow M_{n-1} \longrightarrow \cdots \longrightarrow M_1 \longrightarrow M_0 \longrightarrow 0$$

is exact with $grade_R(M_i) = \alpha_R(M_i) = j$ for all $i \geq 0$. Then there is an exact sequence

$$0 \to Ext_R^j(M_0, C) \to Ext_R^j(M_1, C) \to \cdots \to Ext_R^j(M_{n-1}, C) \to Ext_R^j(M_n, C) \to 0$$

Proof : Clear by breaking the complex into short exact sequences.

$\square$

In fact, one can show that the ith homology of the complex after applying $\text{Hom}_R(-, C)$ to a resolution of M is exactly $\text{Ext}_R^{j+i}(M, C)$. The next result is analogous to [15, Lemma 1.2.6] and is useful in the proof of the next theorem.

Proposition 4.1.8. *Suppose that* $G_C^j\text{-}dim_R(M) < \infty$ *and* $grade_R(M) = j = \alpha_R(M)$. *Then* $M \in \mathcal{G}_C^j$.

Proof : Suppose $G_C^j\text{-}dim_R(M) = n$. We prove by induction on n. We are done if $n = 0$, so suppose $n = 1$ and consider a shortest G_C^j-resolution of M

$$0 \longrightarrow M_1 \longrightarrow M_0 \longrightarrow M \longrightarrow 0$$

By Lemma 4.1.7 we get the following short exact sequence

$$0 \longrightarrow \mathrm{Ext}_R^j(M, C) \longrightarrow \mathrm{Ext}_R^j(M_0, C) \longrightarrow \mathrm{Ext}_R^j(M_1, C) \longrightarrow 0.$$

By Proposition 4.1.5 (a) we see that $\mathrm{Ext}_R^j(M, C) \in \mathcal{G}_C^j$. Therefore $\mathrm{Ext}_R^j(\mathrm{Ext}_R^j(M, C), C) \in \mathcal{G}_C^j$. If we apply $\mathrm{Hom}_R(-, C)$ to the above short exact sequence we get

$$0 \longrightarrow \mathrm{Ext}_R^j(\mathrm{Ext}_R^j(M_1, C), C) \longrightarrow \mathrm{Ext}_R^j(\mathrm{Ext}_R^j(M_0, C), C) \longrightarrow \mathrm{Ext}_R^j(\mathrm{Ext}_R^j(M, C), C) \longrightarrow 0$$

and it is then clear using $\delta_C^j(-)$ that $M \cong \mathrm{Ext}_R^j(\mathrm{Ext}_R^j(M, C), C)$ and thus $M \in \mathcal{G}_C^j$.

Now suppose that $G_C^j\text{-}dim_R(M) = n$ with $n > 1$. Take a shortest G_C^j-resolution of M

$$0 \longrightarrow M_n \longrightarrow \cdots \longrightarrow M_0 \longrightarrow M \longrightarrow 0$$

and let $K = \ker(M_0 \to M)$. Then $G_C^j\text{-}dim_R(K) \leq n - 1 < \infty$ and by applying $\mathrm{Hom}_R(-, C)$ to the short exact sequence

$$0 \longrightarrow K \longrightarrow M_0 \longrightarrow M \longrightarrow 0$$

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we see that $\text{Ext}_R^{j+i}(K, C) = 0$ for $i > 0$. Therefore $\alpha_R(K) = j = \text{grade}_R(K)$ and so by the induction hypothesis $K \in \mathcal{G}_C^j$. Then $G_C^j\text{-dim}_R(M) \leq 1$ and again by the induction hypothesis $M \in \mathcal{G}_C^j$.

$\square$

Using these results we get the following theorem about G_C^g-dimension.

Theorem 4.1.9. *Let C be a semidualizing R-module. For a finitely generated R-module M with $\text{grade}_R(M) \geq j$, the following are equivalent:*

(i) *G_C^j-dim$_R(M) \leq n$*

(ii) *G_C^j-dim$_R(M) < \infty$ and $n \geq \alpha_R(M) - j$*

(iii) *In any G_C^j-resolution,*

$$\cdots \longrightarrow M_i \longrightarrow M_{i-1} \longrightarrow \cdots \longrightarrow M_0 \longrightarrow M \longrightarrow 0$$

the kernel $K_n = \ker(M_{n-1} \to M_{n-2}) \in \mathcal{G}_C^j$.

Moreover, if G_C^j-dim$_R(M) < \infty$, then G_C^j-dim$_R(M) = \alpha_R(M) - j$.

Proof : Clearly, the equivalence of (i) and (ii) will imply the last statement.

$(i) \Rightarrow (ii)$ Suppose that G_C^j-dim$_R(M) \leq n$ and consider a G_C^j-resolution of M

$$0 \longrightarrow M_n \longrightarrow M_{n-1} \longrightarrow \cdots \longrightarrow M_1 \longrightarrow M_0 \longrightarrow M \longrightarrow 0$$

By Lemma 4.1.6 we have $0 = \operatorname{Ext}_R^{\operatorname{grade}_R(M_n)+i}(M_n, C) \cong \operatorname{Ext}_R^{\operatorname{grade}_R(M_n)+n+i}(M, C)$ for $i > 0$ since $M_n \in \mathcal{G}_C^j$. Then

$$\operatorname{grade}_R(M_n) + n \geq \alpha_R(M) \Rightarrow n \geq \alpha_R(M) - \operatorname{grade}_R(M_n)$$

$(ii) \Rightarrow (i)$ Suppose that $G_C^j\text{-dim}_R(M) = p$. If $p \leq n$ we are done, and so we may assume that $p > n$. Consider a G_C^j-resolution of M

$$0 \longrightarrow M_p \longrightarrow \cdots \longrightarrow M_n \longrightarrow M_{n-1} \longrightarrow \cdots \longrightarrow M_1 \longrightarrow M_0 \longrightarrow M \longrightarrow 0$$

and let $K_n = \ker(M_{n-1} \to M_{n-2})$. Then by Lemma 4.1.6 again we have

$$\operatorname{Ext}_R^{\operatorname{grade}_R(M_{n-2})+i}(K_n, C) \cong \operatorname{Ext}_R^{\operatorname{grade}_R(M_{n-1})+n+i}(M, C).$$

Since $n \geq \alpha_R(M) - j$ and $j = \operatorname{grade}_R(M_{n-1})$, we have $\operatorname{grade}_R(M_{n-1}) + n + i \geq \alpha_R(M) + i$. Thus $\operatorname{Ext}_R^{\operatorname{grade}_R(M_{n-1})+i}(K_n, R) = 0$ for $i > 0$. Therefore $\operatorname{grade}_R(K_n) = j = \alpha_R(K_n)$. So by Proposition 4.1.8 we see that $K_n \in \mathcal{G}_C^j$. Therefore $G_C^j\text{-dim}_R(M) \leq n$.

$(i) \Rightarrow (iii)$ Suppose $G_C^j\text{-dim}_R(M) \leq n$. Consider a G_C^j-resolution of M of length n

$$0 \longrightarrow M_n \longrightarrow M_{n-1} \longrightarrow \cdots \longrightarrow M_0 \longrightarrow M \longrightarrow 0$$

and let $K_n = \ker(M_{n-1} \to M_{n-2})$. Consider a specific G_C^j-resolution constructed by using projective $R/(\bar{x})$-modules, where $\bar{x}$ is a regular C-sequence in $\operatorname{Ann}_R(M)$ of length j,

$$0 \longrightarrow S_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

It is then sufficient to prove that $S_n \in \mathcal{G}_C^j$ if and only if $K_n \in \mathcal{G}_C^j$. Since P_i is projective (as $R/(\bar{x})$-modules) we get mappings from one complex to the other. Then the result follows by considering the mapping cone and using Proposition 4.1.5 (a) and (c).

$(iii) \Rightarrow (i)$ Clear.

$\square$

Corollary 4.1.10. *If C is a semidualizing R-module and M is a finitely generated R-module, then*

$$G_C^i\text{-}dim_R(M) \leq G_C^j\text{-}dim_R(M)$$

for any $i \geq j$. Moreover, if $G_C^j\text{-}dim_R(M) < \infty$ then $G_C^i\text{-}dim_R(M) = G_C^j\text{-}dim_R(M) + (j - i)$ for all $i \geq j$.

So we get this chain of inequalities

$$\mathrm{G}_C^{\mathrm{grade}_R(M)}\text{-}\dim_R(M) \leq \mathrm{G}_C^{\mathrm{grade}_R(M)-1}\text{-}\dim_R(M) \leq \cdots \leq \mathrm{G}_C^1\text{-}\dim_R(M) \leq \mathrm{G}_C^0\text{-}\dim_R(M)$$

In fact we get the following

Theorem 4.1.11. *Let C be a semidualizing R-module and M a finitely generated R-module. Then $G_C^{grade_R(M)}\text{-}dim_R(M) < \infty$ if and only if $G_C^i\text{-}dim_R(M) < \infty$ for any $i \leq grade_R(M)$.*

Proof : Clearly if $\mathrm{G}_C^i\text{-}\dim_R(M) < \infty$, then $\mathrm{G}_C^{\mathrm{grade}_R(M)}\text{-}\dim_R(M) < \infty$ by Corollary 4.1.10. So suppose that $\mathrm{G}_C^{\mathrm{grade}_R(M)}\text{-}\dim_R(M) < \infty$. Then for a suitable $\mathrm{G}_C^{\mathrm{grade}_R(M)}$-resolution of M (constructed in $R/(\bar{x})$ for $\bar{x}$ a regular C-sequence in $\mathrm{Ann}_R(M)$) we can convert it into a G_C-resolution

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of M by considering it as a complex over $R/(\bar{x})$.. As finiteness of G_C-dimension is preserved between R and $R/(\bar{x})$ when $\bar{x}$ is a regular sequence [15, Proposition 1.5.3] we are done by Corollary 4.1.10.

$\square$

Corollary 4.1.12. *Let C be a semidualizing R-module and M a finitely generated R-module. If G_C-dim$_R(M) < \infty$, then*

$$G_C\text{-}dim_R(M) = G_C^j\text{-}dim_R(M) + j$$

Note that from now on, any assumption about the finiteness of G_C-dimension is the same as assuming that any or all of the G_C^i-dimensions are finite. So in the rest of the results we will use notation that fits the theme of the result. The next result is an Auslander-Bridger formula for these dimensions.

Corollary 4.1.13 (Auslander-Bridger Formula for $\mathcal{G}_C^j$). *Let C be a semidualizing module for a local ring $(R, \mathfrak{m}, k)$ with G_C^j-dim$_R(M) < \infty$ for an R-module M. Then*

$$G_C^j\text{-}dim_R(M) = depth(R) - depth_R(M) - j$$

4.1.1 C-Duals and C-Gorenstein Dimension

Given a semidualizing R-module C and a finitely generated R-module M, one can take a projective presentation

$$P_1 \to P_0 \to M \to 0$$

and by applying $\text{Hom}_R(-, C)$ on obtains the exact sequence

$$0 \longrightarrow \text{Hom}_R(M, C) \longrightarrow \text{Hom}_R(P_0, C) \longrightarrow \text{Hom}_R(P_1, C) \longrightarrow D_C M \longrightarrow 0$$

where $D_C(M) = \mathrm{coker}(\mathrm{Hom}_R(P_0, C) \to \mathrm{Hom}_R(P_1, C))$ is the C-dual of M. In [18], this is called the C transpose of M with notation $\mathrm{Tr}_C M$. Using the notation $(-)^\nabla = \mathrm{Hom}_R(-, C)$, one can write the following exact sequences which arise from the above sequence

$$0 \longrightarrow \mathrm{Ext}^1_R(D_C M, C) \longrightarrow M \longrightarrow M^{\nabla\nabla} \longrightarrow \mathrm{Ext}^2_R(D_C M, C) \longrightarrow 0$$

$$0 \longrightarrow \mathrm{Ext}^1_R(M, C) \longrightarrow D_C M \longrightarrow (D_C M)^{\nabla\nabla} \longrightarrow \mathrm{Ext}^2_R(M, C) \longrightarrow 0$$

Both of which are stated in [18].

Now suppose that $\mathrm{grade}_R(M) \geq j$. In almost exactly the same manner we can take a G^j_C-presentation of M, i.e. a presentation

$$M_1 \to M_0 \to M \to 0$$

where $M_1, M_0 \in \mathcal{G}^j_C$ and after applying $\mathrm{Ext}^j_R(-, C)$ we get the exact sequence

$$0 \longrightarrow \mathrm{Ext}^j_R(M, C) \longrightarrow \mathrm{Ext}^j_R(M_0, C) \longrightarrow \mathrm{Ext}^j_R(M_1, C) \longrightarrow D^j_C M \longrightarrow 0$$

which leads to the following result.

Lemma 4.1.14. *Let C be a semidualizing R-module and M an R-module with $\mathrm{grade}_R(M) \geq j$. Then there is an exact sequence*

$$0 \longrightarrow Ext^{j+1}_R(D^g_C M, C) \longrightarrow M \xrightarrow{\delta^j_C(M)} Ext^j_R(Ext^j_R(M, C), C) \longrightarrow Ext^{j+2}_R(D^g_C M, C) \longrightarrow 0$$

Proof : Consider the presentation of M (as above)

$$(\Pi) : \qquad M_1 \xrightarrow{\ \alpha\ } M_0 \xrightarrow{\ \beta\ } M \longrightarrow 0$$

Applying $\mathrm{Ext}^j_R(-, C)$ to this and splitting the resulting complex into short exact sequences we get

$$(\Pi_0) : \qquad 0 \longrightarrow \mathrm{Ext}^j_R(M, C) \xrightarrow{\ \beta^*\ } \mathrm{Ext}^j_R(M_0, C) \xrightarrow{\ \gamma_0\ } Q \longrightarrow 0$$

$$(\Pi_1) : \qquad 0 \longrightarrow Q \xrightarrow{\ \gamma_1\ } \mathrm{Ext}^j_R(M_1, C) \longrightarrow D^g_C M \longrightarrow 0$$

where $\gamma_1 \circ \gamma_0 = \alpha^*$. Applying $\mathrm{Ext}^j_R(-, C)$ to both of these gives the exact sequences

$$(\Pi_0^*) : 0 \longrightarrow \mathrm{Ext}^j_R(Q, C) \xrightarrow{\ \gamma_0^*\ } \mathrm{Ext}^j_R(\mathrm{Ext}^j_R(M_0, C), C) \xrightarrow{\ \beta^{**}\ } \mathrm{Ext}^j_R(\mathrm{Ext}^j_R(M, C), C) \longrightarrow \mathrm{Ext}^{j+1}_R(Q, C) \longrightarrow 0$$

$$(\Pi_1^*) : 0 \to \mathrm{Ext}^j_R(D^g_C M, C) \to \mathrm{Ext}^j_R(\mathrm{Ext}^j_R(M_1, C), C) \xrightarrow{\ \gamma_1^*\ } \mathrm{Ext}^j_R(Q, C) \longrightarrow \mathrm{Ext}^{j+1}_R(D^g_C M, C) \to 0.$$

Then if we consider the following commutative diagram

$$
\begin{array}{ccccccc}
M_1 & \xrightarrow{\ \ \alpha\ \ } & M_0 & \xrightarrow{\ \ \beta\ \ } & M & \longrightarrow & 0 \\
\big\downarrow{\scriptstyle \gamma_1^* \circ \delta^j_C(M_1)} & & \big\downarrow{\scriptstyle \delta^j_C(M_0)} & & \big\downarrow{\scriptstyle \delta^j_C(M)} & & \\
0 \longrightarrow \mathrm{Ext}^j_R(Q, C) & \xrightarrow{\ \gamma_0^*\ } & \mathrm{Ext}^j_R(\mathrm{Ext}^j_R(M_0, C), C) & \xrightarrow{\ \beta^{**}\ } & \mathrm{Ext}^j_R(\mathrm{Ext}^j_R(M, C), C), & &
\end{array}
$$

the $\mathrm{coker}(\gamma_1^* \circ \delta^j_C(M_1)) = \mathrm{coker}(\gamma_1^*) = \mathrm{Ext}^{j+1}_R(D^g_C M, C)$ according to (Π_1^*). Therefore by the Snake Lemma, $\ker(\delta^j_C(M)) \cong \mathrm{coker}(\gamma_1^* \circ \delta^j_C(M_1)) \cong \mathrm{Ext}^{j+1}_R(D^j_C M, C)$.

Further, $\mathrm{coker}(\delta^g_C(M)) \cong \mathrm{coker}(\beta^{**}) \cong \mathrm{Ext}^{j+1}_R(Q, C) \cong \mathrm{Ext}^{j+2}_R(D^g_C M, C)$, where the last isomorphism comes from applying $\mathrm{Ext}^j_R(-, C)$ to (Π_1).

Lastly, we see from applying $\text{Ext}_R^j(-, C)$ to (Π_0) and (Π_1) that

$$\text{Ext}_R^{j+i}(\text{Ext}_R^j(M, C), C) \cong \text{Ext}_R^{j+1+i}(Q, C)$$

and

$$\text{Ext}_R^{j+i}(Q, C) \cong \text{Ext}_R^{j+i+1}(D_C^j M, C)$$

for $i > 0$, respectively. This says that $\text{Ext}_R^{j+i}(\text{Ext}_R^j(M, C), C) \cong \text{Ext}_R^{j+i+2}(D_C^g M, C)$ for $i > 0$, as desired.

$\square$

In the same way one can obtain a sequence of functors

$$0 \longrightarrow \text{Ext}_R^{j+1}(D_C^g M, -) \longrightarrow M \otimes_R - \longrightarrow \text{Ext}_R^j(\text{Ext}_R^j(M, C), -) \longrightarrow \text{Ext}_R^{j+2}(D_C^g M, -) \longrightarrow 0$$

for any dualizing R-module C.

Corollary 4.1.15. *Let C be a semidualizing R-module and M an R-module with $\text{grade}_R(M) = g$. Then $M \in \mathcal{G}_C^g$ if and only if $\text{Ext}_R^{g+i}(M, C) = \text{Ext}_R^{g+i}(D_C^g M, C) = 0$ for $i > 0$. Consequently, $M \in \mathcal{G}_C^g$ if and only if $D_C^g M \in \mathcal{G}_C^g$.*

Proof : From the previous proof we see that $\text{Ext}_R^{g+i+2}(D_C^g M, C) \cong \text{Ext}_R^{g+i}(\text{Ext}_R^g(M, C), C)$ for $i > 0$. This along with Lemma 4.1.14 gives the result.

$\square$

Corollary 4.1.16. *Suppose that $0 \to M' \to M \to M'' \to 0$ is a short exact sequence with*

$$\min\{\text{grade}_R(M'), \text{grade}_R(M), \text{grade}_R(M'')\} \geq g.$$

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Then there is a long exact sequence

$$0 \to Ext_R^g(M'', C) \to Ext_R^g(M, C) \to Ext_R^g(M', C) \to D_C^g M'' \to D_C^g M \to D_C^g M' \to 0$$

4.2 Generalized Serre-like Conditions

The goal of this section is to provide results concerning Serre-like conditions using these newly developed dimensions. Later in this thesis we will connect these Serre-like conditions with module linkage.

Recall that an R-module M satisfies (S_n) if $\text{depth}_{R_p}(M_p) \geq \min\{n, \dim(R_p)\}$ for $p \in \text{Spec}(R)$. Then one says that an R-module M satisfies $\tilde{S}_n$ if $\text{depth}_{R_p}(N_p) \geq \min\{n, \text{depth}(R_p)\}$ for $p \in \text{Spec}(R)$. It is natural for us to generalize $\tilde{S}_n$ since we will want to use our Auslander-Bridger type formula which uses the depth of R and not the dimension. So, we define the following

Definition 4.2.1 (Generalized Serre Condition)**.** *Let C be a semidualizing R-module and M an R-module. We say that M satisfies $\tilde{S}_n^g$ if*

$$depth_{R_p}(M_p) + g \geq \min\{n, depth(R_p)\} \quad \forall p \in Spec(R)$$

Note, $\tilde{S}_n^g$ is always satisfied when $n \leq g$. There is a similar generalized condition, (S_n^j) for rings which has been shown to preserve many of the same results as for (S_n), see [37]. We also make the following definition

Definition 4.2.2. *Let C be a semidualizing R-module and M an R-module with $grade_R(M) \geq g$. Then we say that M is $\mathbf{C_n^g}$-torsionless if $Ext_R^{g+i}(D_C^g M, C) = 0$ for $1 \leq i \leq n$.*

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This definition originates from [3] as a way to gauge how far M is from having dimension zero.

Lemma 4.2.3. *Let* $0 \to M' \to M \to M'' \to 0$ *be a short exact sequence with*

$$\min\{grade_R(M'), grade_R(M), grade_R(M'')\} \geq g$$

and $Q = coker(Ext_R^g(M, C) \to Ext_R^g(M', C))$. *If* M' *is* C_{k+1}^g-*torsionless,* M *is* C_k^g-*torsionless, and* $grade_R(Q) \geq k + 1$ *then* M'' *is* C_k^g-*torsionless.*

Proof : From Corollary 4.1.16 we have the exact sequence

$$0 \longrightarrow Q \longrightarrow D_C^g M'' \longrightarrow D_C^g M \longrightarrow D_C^g M' \longrightarrow 0.$$

Breaking this into short exacts sequences and looking at the corresponding long exact sequences in Ext gives the desired result.

$\square$

With these definitions and results we will prove Theorem 4.2.5 and Corollary 4.2.6, generalizing [18, Proposition 2.4] and [18, Proposition 2.7], respectively. Recall that in a resolution

$$\cdots \to X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 \to M \to 0$$

of M, the ith syzygy of M is $\ker(X_{i-1} \to X_{i-2})$ where $X_{-1} = M$. The next proposition is given to help give intuition into how these definitions fit together. We show that a module being C_{n-g}^g-torsionless is stronger than satisfying $\tilde{S}_n^g$.

Proposition 4.2.4. *Let* C *be a semidualizing* R-*module and* M *an* R-*module with* $grade_R(M) = g$ *and* $n \geq g$, *then for the following conditions:*

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(i) M is C^g_{n-g}-torsionless

(ii) M is an $(n-g)^{th}$ $\mathcal{G}^g_C$-syzygy

(iii) M satisfies $\tilde{S}^g_n$

we have $(i) \Rightarrow (ii) \Rightarrow (iii)$.

Proof : We exclude $n = g$ as that is vacuously satisfied.

$(i) \Rightarrow (ii)$ Consider a $\mathcal{G}^g_C$-resolution of $\mathrm{Ext}^g_R(M, C)$

$$\cdots \longrightarrow M_{n-g-1} \longrightarrow \cdots \longrightarrow M_0 \longrightarrow \mathrm{Ext}^g_R(M, C) \longrightarrow 0$$

If we apply $\mathrm{Ext}^g_R(-, C)$ we get a complex

$$(*) : 0 \longrightarrow \mathrm{Ext}^g_R(\mathrm{Ext}^g_R(M, C), C) \longrightarrow \mathrm{Ext}^g_R(M_0, C) \longrightarrow \cdots \longrightarrow \mathrm{Ext}^g_R(M_{n-g-1}, C)$$

When $n - g = 1$ we have that $M \subset \mathrm{Ext}^g_R(\mathrm{Ext}^g_R(M, C), C)$ by Lemma 4.1.14 and are done using $(*)$. If $n - g = 2$, then $M \cong \mathrm{Ext}^g_R(\mathrm{Ext}^g_R(M, C), C)$ by Lemma 4.1.14 and it is clear that M is a 2nd $\mathcal{G}^g_C$-syzygy. So suppose that $n - g > 2$. Then as $\mathrm{Ext}^{g+i}_R(D^g_C M, C) \cong \mathrm{Ext}^{g+i-2}_R(\mathrm{Ext}^g_R(M, C), C)$ for $2 < i \leq n - g$ we see that $(*)$ is exact.

$(ii) \Rightarrow (iii)$ If we take an exact complex

$$0 \longrightarrow M \longrightarrow M_1 \longrightarrow \cdots \longrightarrow M_{n-g}$$

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then clearly $\text{depth}_{R_p}(M_p) \geq \min\{\text{depth}_{R_p}((M_{n-g})_p), n-g\}$. Using Corollary 4.1.13 we have

$$\text{depth}_{R_p}(M_{n-g}) = \text{depth}(R_p) - \text{grade}_{C_p}(M_p)$$

and so

$$\text{depth}_{R_p}(M_p) \geq \min\{\text{depth}(R_p) - \text{grade}_{C_p}(M_p), n-g\} \geq \min\{\text{depth}(R_p), n\} - \text{grade}_{C_p}(M_p)$$

$\square$

We can extend this to an equivalence of statements assuming that the G_C^g-dimension of M is locally finite.

Theorem 4.2.5. *Let C be a semidualizing R-module, M an R-module with $\text{grade}_R(M) = g$, $n \geq g$, and M have locally finite G_C^g-dimension. Then the following are equivalent:*

(i) *M is C_{n-g}^g-torsionless*

(ii) *M is an $(n-g)^{th}$ $\mathcal{G}_C^g$-syzygy*

(iii) *M satisfies $\tilde{S}_n^g$*

(iv) *$\text{grade}_{C_p}(\text{Ext}_{R_p}^{\text{grade}_{C_p}(M_p)+i}(M_p, C_p)) \geq i+n$ for $i \geq 1$ and $p \in \text{Spec}(R)$ where $\text{depth}(R_p) \leq i+n-1$.*

Proof : We have already seen that $(i) \Rightarrow (ii) \Rightarrow (iii)$. We will now show that $(iii) \Rightarrow (iv)$ and $(iv) \Rightarrow (i)$.

$(iii) \Rightarrow (iv)$ Fix $i \geq 1$ and a prime $p \in \text{Spec}(R)$ with $\text{depth}(R_p) < i + n$. Let $\text{grade}_{C_p}(M_p) = g_p$. We need to show that $\text{Ext}_{R_p}^{g_p+i}(M_p, C_p) = 0$, and so we will show that $p \notin \text{Supp}(\text{Ext}_{R_p}^{g_p+i}(M_p, C_p))$. Using Corollary 4.1.13 we have

$$G_{C_p}^{g_p}\text{-dim}_{R_p}(M_p) = \text{depth}(R_p) - \text{depth}_{R_p}(M_p) - g_p$$
$$\leq \text{depth}(R_p) - \min\{n, \text{depth}(R_p)\}$$
$$= \max\{0, \text{depth}(R_p) - n\}$$

and since $\text{depth}(R_p) < i + n$ we have $G_{C_p}^{g_p}\text{-dim}_{R_p}(M_p) < i + n - n = i$. This says that $\alpha_{C_p}(M_p) < i + g_p$ and so $\text{Ext}_{R_p}^{g_p+i}(M_p, C_p) = 0$.

$(iv) \Rightarrow (i)$ It is enough to show this in a local ring (R, p) for $p \in \text{Spec}(R)$. So we may assume, $G_C^g\text{-dim}_R(M) = \alpha_R(M) - g < \infty$. By Corollary 4.1.15 the result holds if $G_C^g\text{-dim}_R(M) = 0$. So suppose that $G_C^g\text{-dim}_R(M) = p > 0$ and we proceed by induction on p. Let

$$0 \rightarrow K \rightarrow N \rightarrow M \rightarrow 0$$

be a short exact sequence with $N \in \mathcal{G}_C^g$ and so $G_C^g\text{-dim}_R(K) = p - 1$. Then we have

$$\text{grade}_R(\text{Ext}_R^{g+i}(K, C)) = \text{grade}_R(\text{Ext}_R^{g+i+1}(M, C)) \geq i + n + 1$$

for $i \geq 1$. So by induction $\text{Ext}_R^{g+i}(D_C^g K, C) = 0$ for $1 \leq i \leq n - g + 1$. That is K is C_{n-g+1}^g-torsionless. The result will now follow by the following Lemma.

$\square$

Corollary 4.2.6. *Let C be a semidualizing R-module, M an R-module with $\text{grade}_R(M) = g$, $n \geq g$, and $\text{id}_R(C) <_{loc} \infty$. Then the following are equivalent:*

(i) M is C_{n-g}^g-torsionless

(ii) $M \otimes_R C$ is an $(n-g)^{th}$ $\mathcal{G}_C^g$-syzygy

(iii) $M \otimes_R C$ satisfies $\tilde{S}_n^g$

(iv) $grade_{C_p}(Ext_{R_p}^{grade_{R_p}(M_p)+i}(M_p, R_p)) \geq i + n$ for $i \geq 1$ and $p \in X^{i+n-1}(R)$

Proof : Let $N = M \otimes_R C$. Since C is semidualizing

$$\mathrm{Ext}_R^g(\mathrm{Ext}_R^g(N, C), C) = \mathrm{Ext}_R^g(\mathrm{Ext}_R^g(M \otimes_R C, C), C) \cong \mathrm{Ext}_R^g(\mathrm{Ext}_R^g(M, R), C).$$

Then using the exact sequence from Lemma 4.1.14 and the remark following the lemma we have the following commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ext}_R^{g+1}(D_R^g M, C) & \longrightarrow & N & \longrightarrow & \mathrm{Ext}_R^g(\mathrm{Ext}_R^g(M, R), C) & \longrightarrow & \mathrm{Ext}_R^{g+2}(D_R^g M, C) & \longrightarrow & 0 \\
& & \downarrow & & \| & & \downarrow{\scriptstyle\cong} & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Ext}_R^{g+1}(D_C^g N, C) & \longrightarrow & N & \longrightarrow & \mathrm{Ext}_R^g(\mathrm{Ext}_R^g(N, C), C) & \longrightarrow & \mathrm{Ext}_R^{g+2}(D_C^g N, C) & \longrightarrow & 0
\end{array}
$$

So it follows naturally that $\mathrm{Ext}_R^{g+i}(D_R^g M, C) \cong \mathrm{Ext}_R^{g+i}(D_C^g N, C)$ for $i > 0$ using the above diagram and $\mathrm{Ext}_R^g(M, R) \cong \mathrm{Ext}_R^g(N, C)$. Then the result follows by replacing M with N in Theorem 4.2.5.

$\square$

To end this chapter, we will prove a result analogous to [18, Theorem 2.12]. We first need to recall the Auslander class of a semidualizing module. Such modules were defined by Foxby [25] and further developed by Avramov and Foxby in [5].

Definition 4.2.7. *Let C be a semidualizing R-module. The **Auslander class with respect to C**, $\mathcal{A}_C$, consists of all R-modules M satisfying:*

(i) *The map $M \rightarrow Hom_R(C, M \otimes_R C)$ is an isomorphism*

(ii) *$Tor_i^R(M, C) = 0 = Ext_R^i(C, M \otimes_R C)$ for all $i > 0$*

Then using [18, Lemma 2.11] we get

Proposition 4.2.8. *Let C be a semidualizing R-module, $M \in \mathcal{A}_C$, $grade_R(M) = g$, $n \geq g$, and M have localy finite G_C^g-dimension. Then the following are equivalent:*

(i) *M is R_{n-g}^g-torsionless*

(ii) *M is C_{n-g}^g-torsionless*

(iii) *$M \otimes_R C$ satisfies $\tilde{S}_n^g$*

(iv) *M satisfies $\tilde{S}_n^g$*

Proof : The equivalence of (iii) and (iv) follows from [18, Lemma 2.11] and that $grade_R(M) = grade_R(M \otimes_R C)$ since

$$Ext_R^i(M \otimes_R C, C) \cong Ext_R^i(M, Hom_R(C, C)) \cong Ext_R^i(M, R)$$

and $grade_R(M) = grade_C(M)$. Note that (i) and (iv) are equivalent from Theorem 4.2.5 by replacing C with R. Further note that (ii) and (iii) are equivalent from Corollary 4.2.6.

$\square$

CHAPTER 5: CATEGORIES WITH LINKAGE

This chapter marks the start of a shift to the study of linkage and certain associated homological dimensions over general categories. We will assume that our categories are homological categories. This chapter will serve to define linkage in this generality and we provide results generalizing the results of Chapter 4 in the section C-duals and C-Gorenstein dimensions along with a study of the even linkage classes of objects in these categories with linkage.

It should be stated here to importance of moving to this generality. One of the many defining characteristics of linkage is the ability to compare a multitude of properties with ideals in the same linkage class. One of the many aims of the theory, and perhaps some would say the most important, is the study of ideals in the linkage class of a complete intersection, so called licci ideals. There is an extraordinary amount of literature on the subject, but what has yet to have been made clear is the misguidedness of concerning ones self with the entire linkage class. We will make extremely clear here why it is more important to consider the even linkage class of an object when studying linkage and how this class should be considered when looking for homological properties of these objects.

One of the many ways one studies category theory is through the lens of functors. This will be our approach as the only difference between these types of categories will be the functors used in their definitions. The prototypical example one can think of is module linkage in the category R-Mod. There is a single functor used in defining linkage, $\mathrm{Hom}_R(-, R)$, but in fact there are two used that are identical which will become apparent when we define linkage. $\mathrm{Hom}_R(-, R)$ is contravariant and left exact, and we will begin our study of linkage with contravariant functors in a hope to

emulate the example in R-Mod.

5.1 Contravariant Linkage Functors

We begin by defining a category pair with linkage.

Definition 5.1.1. *Suppose $\mathcal{X}$ and $\mathcal{Y}$ are homological categories where $S : \mathcal{X} \to \mathcal{Y}$ and $T : \mathcal{Y} \to \mathcal{X}$ are additive contravariant functors. Also, suppose that S and T are either both right exact or left exact (RE or LE). We say S and T are **linkage functors**, and (S, T) form a **linkage functor pair** for $\mathcal{X}$ and $\mathcal{Y}$ if there exists a category $\mathcal{B}$ with a pair of full and faithful functors $F_{\mathcal{X}} : \mathcal{B} \to \mathcal{X}$ and $F_{\mathcal{Y}} : \mathcal{B} \to \mathcal{Y}$ such that the following holds:*

   (i) For $B \in \mathcal{B}$ there exist functors $X_{\mathcal{B}} : \mathcal{X}|_{F_{\mathcal{X}}(\mathcal{B})} \to \mathcal{Y}|_{F_{\mathcal{Y}}(\mathcal{B})}$ and $Y_{\mathcal{B}} : \mathcal{Y}|_{F_{\mathcal{Y}}(\mathcal{B})} \to \mathcal{X}|_{F_{\mathcal{X}}(\mathcal{B})}$ such that $X_{\mathcal{B}} \circ F_{\mathcal{X}} = F_{\mathcal{Y}}$ and $Y_{\mathcal{B}} \circ F_{\mathcal{Y}} = F_{\mathcal{X}}$. Moreover, $X_{\mathcal{B}}$ and $Y_{\mathcal{B}}$ are such that $S|_{F_{\mathcal{X}}(\mathcal{B})} = X_{\mathcal{B}}$ and $T|_{F_{\mathcal{Y}}(\mathcal{B})} = Y_{\mathcal{B}}$.

   (ii) For each $B \in \mathcal{B}$, $D^i S(F_{\mathcal{X}}(B)) = 0$ and $D^i T(F_{\mathcal{Y}}(B)) = 0$ for $i > 0$ where $D^i S(-)$ and $D^i T(-)$ are the derived functors of S and T, respectively.

*In this case we say that $\mathcal{B}$ is a **Fossum** category, both $F_{\mathcal{X}}(\mathcal{B})$ and $F_{\mathcal{Y}}(\mathcal{B})$ are **linking classes** in $\mathcal{X}$ and $\mathcal{Y}$, and $((\mathcal{X}, \mathcal{Y}), \mathcal{B}, (S, T))$ is a **category pair with linkage**.*

Note that $\mathcal{B}$ is exactly the collection of objects in $\mathcal{X}$ and $\mathcal{Y}$ in which (i) and (ii) hold. This means that given two contravariant LE or RE functors we can determine the collection $\mathcal{B}$ and obtain a category pair with linkage. In the rest of the paper we will suppress the notation for the category $\mathcal{B}$. An object B in $\mathcal{X}$ or $\mathcal{Y}$ will denote the representative of B in $\mathcal{X}$ and $\mathcal{Y}$, which are equivalent, and B will denote both $F_{\mathcal{X}}(\mathcal{B})$ and $F_{\mathcal{Y}}(\mathcal{B})$. We give the category $\mathcal{B}$ the name Fossum due to his influence

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into the study of certain types of modules and the functors associated to them. $\mathcal{B}$ represents the collection of quasi-Gorenstein modules of a fixed grade in the example of linkage for R-Mod.

We take a moment here to continue our discussion about the entire linkage class versus the even linkage class. In this generality it is easy to see why we pay more attention to the even linkage classes as opposed to the entire linkage class. Linkage switches from one category to another and so properties that are captured in a single category will be captured through even linkage, not necessarily through direct linkage.

Let $((\mathcal{X}, \mathcal{Y}), \mathcal{B}, (S, T))$ be a category pair with linkage. We say that X is **presented by** $\mathcal{B}$ or $X \in \mathrm{Pres}(\mathcal{B})$ if there exists a short exact sequence

$$0 \longrightarrow K \longrightarrow B \longrightarrow X \longrightarrow 0.$$

in the appropriate category, where $B \in \mathcal{B}$, and we say that X is **copresented by** $\mathcal{B}$ or $X \in \mathrm{Copres}(\mathcal{B})$ if there exists a short exact sequence

$$0 \longrightarrow X \longrightarrow B \longrightarrow Q \longrightarrow 0$$

in the appropriate category, where $B \in \mathcal{B}$.

Definition 5.1.2. *Let $((\mathcal{X}, \mathcal{Y}), \mathcal{B}, (S, T))$ be a category pair with linkage. Suppose*

    (i) $X \in \mathcal{X}$ and $X \in Pres(\mathcal{B})$ and both S and T are left exact. Then there is a short exact sequence

$$0 \longrightarrow K \xrightarrow{\ \beta\ } B \xrightarrow{\ \alpha\ } A \longrightarrow 0$$

in $\mathcal{X}$ which induces an exact sequence

$$0 \longrightarrow S(A) \xrightarrow{Y_{\mathcal{B}} \circ S(\alpha)} B \xrightarrow{S(\beta) \circ X_{\mathcal{B}}} S(K).$$

From this exact sequence we define an object $\mathcal{L}_B^S(X) = coker(Y_{\mathcal{B}} \circ S(\alpha))$. Similarly, we define $\mathcal{L}_B^T(Y) = coker(X_{\mathcal{B}} \circ T(\alpha))$ for $Y \in \mathcal{Y}$ where $Y \in Pres(\mathcal{B})$.

(ii) *$X \in \mathcal{X}$ and $X \in Copres(\mathcal{B})$ and both S and T are right exact. Then there is a short exact sequence*

$$0 \longrightarrow X \xrightarrow{\alpha} B \xrightarrow{\beta} Q \longrightarrow 0$$

in $\mathcal{X}$ which induces an exact sequence

$$S(Q) \xrightarrow{Y_{\mathcal{B}} \circ S(\beta)} B \xrightarrow{S(\alpha) \circ X_{\mathcal{B}}} S(A) \longrightarrow 0.$$

in $\mathcal{Y}$. From this exact sequence we define an object $\mathcal{L}_S^B(X) = \ker(S(\alpha) \circ X_{\mathcal{B}})$. Similarly, we define $\mathcal{L}_T^B(Y) = \ker(T(\alpha) \circ Y_{\mathcal{B}})$ for $Y \in \mathcal{Y}$ where $Y \in Copres(\mathcal{B})$.

Definition 5.1.3. *Let $((\mathcal{X}, \mathcal{Y}), \mathcal{B}, (S,T))$ be a category pair with linkage.*

(i) *Suppose that S and T are left exact. Given objects $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$ in $Pres(\mathcal{B})$ we say X is **directly S-linked** to Y (Y is directly T-linked to X) by $B \in \mathcal{B}$ if $\mathcal{L}_B^S(X) \cong Y$ and $\mathcal{L}_B^T(Y) \cong X$ and we use the notation $X \overset{B}{\underset{S}{\rightsquigarrow}} Y$ ($Y \overset{B}{\underset{T}{\rightsquigarrow}} X$). Moreover, we say that Z_1 is **linked** to Z_2 if there is a sequence of direct links from Z_1 to Z_2.*

(ii) *Suppose that S and T are right exact. Given objects $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$ in $Copres(\mathcal{B})$ we say that X is **directly S-linked** to Y (Y is directly T-linked to X) by $B \in \mathcal{B}$ if $\mathcal{L}_S^B(X) \cong Y$ and $\mathcal{L}_T^B(Y) \cong X$, and we use the notation $X \overset{S}{\underset{B}{\rightsquigarrow}} Y$ ($Y \overset{T}{\underset{B}{\rightsquigarrow}} X$). Moreover, we say that Z_1 is linked to Z_2 if there is a sequence of direct links from Z_1 to Z_2.*

Two objects which are linked will be in the same *linkage class*. Objects which are linked in an even number of steps will be in the same *even linkage class*. Similarly, objects which are linked in an odd number of steps will be in the same *odd linkage class*. We will use the notation $[Z]$ for the linkage class of an object Z.

Note that when discussing direct linkage there is no difference a priori between being directly S-linked and directly T-linked. So in most of the results concerning direct linkage we will assume directly S-linked.

Proposition 5.1.4. *In a category pair with linkage* $((\mathcal{X}, \mathcal{Y}), \mathcal{B}, (S, T))$ *every object in* $\mathcal{B}$ *is in the same linkage class.*

Proof : Let $B_1, B_2 \in \mathcal{B}$, then $B_1 \oplus B_2 \in \mathcal{B}$ as S and T are additive. Then by the short exact sequences

$$0 \to B_1 \to B_1 \oplus B_2 \to B_2 \to 0$$

we see that B_2 is directly linked to B_1.

$\square$

In fact, this shows that every pair of objects in $\mathcal{B}$ are directly linked (both $S-$ and $T-$ linked) lending more credence to our decision to suppress the notation involving $\mathcal{B}$. The following lemma is instrumental in finding the homological connections between an object and its linkage class.

Lemma 5.1.5. *Let* $((\mathcal{X}, \mathcal{Y}), \mathcal{B}, (S, T))$ *be a category pair with linkage where* S *and* T *are left exact and suppose* $X \overset{B}{\underset{S}{\rightsquigarrow}} Y$. *Then the following holds:*

(i) *The two sequences*

$$0 \to X \to T(S(X)) \to R^1 T(Y) \to 0$$

$$0 \to Y \to S(T(Y)) \to R^1 S(X) \to 0$$

are exact.

(ii) $R^i T(S(X)) \cong R^{i+1} T(Y) \, for \, i > 0$

(iii) $R^i S(T(Y)) \cong R^{i+1} S(X) \, for \, i > 0$

Proof : As $X \overset{B}{\underset{S}{\rightsquigarrow}} Y$ it is assumed that $X \in \mathrm{Pres}(\mathcal{B})$, and the sequence

$$0 \to K \to B \to X \to 0.$$

exists. By applying S we get the short exact sequence

$$* : 0 \to S(X) \to B \to Y \to 0.$$

Similarly $Y \in \mathrm{Pres}(\mathcal{B})$, and we have the short exact sequence

$$** : 0 \to T(Y) \to B \to X \to 0.$$

So after applying T to $*$ and S to $**$ we obtain the following

$$0 \longrightarrow T(Y) \longrightarrow T(B) \longrightarrow T(S(X)) \longrightarrow R^1 T(Y) \longrightarrow 0$$

$$0 \longrightarrow R^i T(S(X)) \longrightarrow R^{i+1} T(Y) \longrightarrow 0, \qquad i > 0$$

$$0 \longrightarrow S(X) \longrightarrow S(B) \longrightarrow S(T(Y)) \longrightarrow R^1 S(X) \longrightarrow 0$$

$$0 \longrightarrow R^i S(T(Y)) \longrightarrow R^{i+1} S(X) \longrightarrow 0, \qquad i > 0$$

which shows (ii) and (iii). Once again, using that X and Y are directly linked, we break the two long sequences into two short exact sequences to show (i).

$\square$

Using the same technique we get a similar result when S and T are right exact.

Lemma 5.1.6. *Let $((\mathcal{X}, \mathcal{Y}), \mathcal{B}, (S, T))$ be a category pair with linkage where S and T are right exact and suppose $X \overset{S}{\underset{B}{\leadsto}} Y$. Then the following holds:*

(i) *The two sequences*

$$0 \to L_1 T(Y) \to T(S(X)) \to X \to 0$$

$$0 \to L_1 S(X) \to S(T(Y)) \to Y \to 0$$

are exact.

(ii) $L_i T(S(X)) \cong L_{i+1} T(Y)$ *for $i > 0$*

(iii) $L_i S(T(Y)) \cong L_{i+1} S(X)$ *for $i > 0$*

With this lemma in mind, it is easy to see that objects in the same category which are linked (so evenly linked) share many homological properties. We define a collection of objects which capture when these derived functors vanish.

Definition 5.1.7. *Let $((\mathcal{X}, \mathcal{Y}), \mathcal{B}, (S, T))$ be a category pair with linkage. An object X will be called **S-perfect** (**T-perfect**) if $T(S(X)) \cong X$ $(S(T(X)) \cong X)$ and $D^i S(X) = 0 = D^i T(S(X))$ for $i > 0$ $(D^i T(X) = 0 = D^i S(T(X))$ for $i > 0)$. We will use the notation $Per(S)$ $(Per(T))$ for the collection of S-perfect $(T$-perfect$)$ objects.*

The following lemma will show that this property separates linkage classes from those which consist of perfect objects and those that do not.

Lemma 5.1.8. *Let* $((\mathcal{X}, \mathcal{Y}), \mathcal{B}, (S, T))$ *be a category pair with linkage. Suppose* X *is* S*-perfect. Then*

    (i) $S(X) \in Per(T)$

   (ii) *if* X *is directly* S*-linked to* Y *by* B, Y *is* T*-perfect.*

Proof :

    (i) Given that $X \in \text{Per}(S)$, we know that $D^i T(S(X)) = 0$ for $i > 0$. Further as $T(S(X)) \cong X$ we have that $S(T(S(X)) \cong S(X)$ and $D^i S(T(S(X)) \cong D^i S(X) = 0$ for $i > 0$. Thus $S(X)$ is T-perfect.

   (ii) Suppose that S and T are left exact. By Lemma 5.1.5 (i),

$$0 \to Y \to S(T(Y)) \to R^1 S(X) \to 0$$

shows that $Y \cong S(T(Y))$. Also by Lemma 5.1.5 (i),

$$0 \to X \to T(S(X)) \to R^1 T(Y) \to 0$$

shows that $R^1 T(Y) = 0$ as $X \cong T(S(X))$. Further, by Lemma 5.1.5 (ii) and (iii) we get that $R^i S(T(Y)) \cong R^{i+1} S(X) = 0$ for $i > 0$ and $R^i T(Y) \cong R^{i+1} T(S(X)) = 0$ for $i > 1$. So Y is then T-perfect.

Similarly, if S and T are right exact, we can use Lemma 5.1.6 to show that Y is T-perfect.

$\square$

Clearly, the same holds true for Y T-perfect with S and T switched. So we immediately get the following consequence.

Corollary 5.1.9. *Let $((\mathcal{X}, \mathcal{Y}), \mathcal{B}, (S, T))$ be a category pair with linkage. Suppose that X is S-perfect. Then every object in the even linkage class of X is S-perfect and every object in the odd linkage class of X is T-perfect.*

When defining perfect objects in these categories, it is similar to how one defines Gorenstein dimension zero modules except without some natural bi-duality map. We make this idea formal which allows us to connect together this class of objects with linkage and in the next chapter associated a homological dimension to these perfect categories.

Definition 5.1.10. *Suppose that $((\mathcal{X}, \mathcal{Y}), \mathcal{B}, (S, T))$ is a category pair with linkage.*

(i) *(LE) Suppose that S and T are left exact. We say that $((\mathcal{X}, \mathcal{Y}), \mathcal{B}, (S, T))$ is **S-perfect** (**T-perfect**) if $\mathcal{X} = Pres(Per(S))$ ($\mathcal{Y} = Pres(Per(T))$) and given a morphism $X \xrightarrow{\alpha} X'$ in $\mathcal{X}$ ($Y \xrightarrow{\beta} Y'$ in $\mathcal{Y}$), there exists morphisms $\delta_{\mathcal{X}}^{S}(X) : X \to T(S(X))$ and $\delta_{\mathcal{X}}^{S}(X') : X' \to T(S(X'))$ ($\delta_{\mathcal{Y}}^{T}(Y) : Y \to S(T(Y))$ and $\delta_{\mathcal{Y}}^{T}(Y') : Y' \to S(T(Y'))$) such that the following square commutes:*

$$
\begin{array}{ccc}
X & \xrightarrow{\ \alpha\ } & X' \\
{\scriptstyle \delta_{\mathcal{X}}^{S}(X)}\downarrow & & \downarrow{\scriptstyle \delta_{\mathcal{X}}^{S}(X')} \\
T(S(X)) & \xrightarrow{T(S(\alpha))} & T(S(X'))
\end{array}
\qquad
\left(
\begin{array}{ccc}
Y & \xrightarrow{\ \beta\ } & Y' \\
{\scriptstyle \delta_{\mathcal{Y}}^{T}(Y)}\downarrow & & \downarrow{\scriptstyle \delta_{\mathcal{Y}}^{T}(Y')} \\
S(T(Y)) & \xrightarrow{S(T(\beta))} & S(T(Y'))
\end{array}
\right)
$$

That is, $\delta_{\mathcal{X}}^{S}(-)$ ($\delta_{\mathcal{Y}}^{T}(-)$) is a natural transformation between the identity functor and $T(S(-))$ ($S(T(-))$).

(ii) *(RE) Suppose that S and T are right exact. We say that $((\mathcal{X}, \mathcal{Y}), \mathcal{B}, (S, T))$ is **S-perfect** (**T-perfect**) if $\mathcal{X} = Copres(Per(S))$ ($\mathcal{Y} = Copres(Per(T))$) and given a morphism $X \xrightarrow{\alpha} X'$*

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in $\mathcal{X}$ ($Y \xrightarrow{\beta} Y'$ in $\mathcal{Y}$), there exists morphisms $\delta_S^{\mathcal{X}}(X) : T(S(X)) \rightarrow X$ and $\delta_S^{\mathcal{X}}(X') :$
$T(S(X')) \rightarrow X'$ ($\delta_T^{\mathcal{Y}}(Y) : S(T(Y)) \rightarrow Y$ and $\delta_T^{\mathcal{Y}}(Y') : S(T(Y')) \rightarrow Y'$) such that the
following square commutes:

$$
\begin{array}{ccc}
T(S(X)) & \xrightarrow{T(S(\alpha))} & T(S(X')) \\
\downarrow{\scriptstyle\delta_S^{\mathcal{X}}(X)} & & \downarrow{\scriptstyle\delta_S^{\mathcal{X}}(X')} \\
X & \xrightarrow{\quad\alpha\quad} & X'
\end{array}
\qquad
\left(
\begin{array}{ccc}
S(T(Y)) & \xrightarrow{S(T(\beta))} & S(T(Y')) \\
\downarrow{\scriptstyle\delta_T^{\mathcal{Y}}(Y)} & & \downarrow{\scriptstyle\delta_T^{\mathcal{Y}}(Y')} \\
Y & \xrightarrow{\quad\beta\quad} & Y'
\end{array}
\right)
$$

That is, $\delta_S^{\mathcal{X}}(-)$ ($\delta_T^{\mathcal{Y}}(-)$) is a natural transformation between the identity functor and $T(S(-))$
($S(T(-))$).

We will say that $((\mathcal{X}, \mathcal{Y}), \mathcal{B}, (S, T))$ is **perfect** if it is both S-perfect and T-perfect.

To simplify the notation we will use a subscript $(S, T)_L$ to signify that S and T are left exact and $(S, T)_R$ to signify that S and T are right exact. Suppose that $((\mathcal{X}, \mathcal{Y}), \mathcal{B}, (S, T))_L)$ is perfect. As $\mathcal{X} = \mathrm{Pres}(\mathrm{Per}(S))$, given an object X we can take an S-perfect presentation of X

$$P_1 \rightarrow P_0 \rightarrow X \rightarrow 0$$

where $P_0, P_1 \in \mathrm{Per}(S)$. If we apply S to this we get

$$0 \rightarrow S(X) \rightarrow S(P_0) \rightarrow S(P_1) \rightarrow D_S(X) \rightarrow 0$$

where $D_S(X) = \mathrm{coker}(S(P_0) \rightarrow S(P_1))$ is defined to be the **S-dual** of X. Similarly, we can define $D_T(Y) = \mathrm{coker}(T(P_0) \rightarrow T(P_1))$ the **T-dual** of Y by taking a T-perfect presentation of Y. Also, for $((\mathcal{X}, \mathcal{Y}), \mathcal{B}, (S, T))_R)$ perfect we can consider an S-perfect co-presentation of X, or T-perfect co-presentation of Y

$$0 \rightarrow X \rightarrow P_0 \rightarrow P_1$$

or

$$0 \to Y \to Q_0 \to Q_1$$

and define $D^S(X) = \ker(S(P_1) \to S(P_0))$ or $D^T(Y) = \ker(T(Q_1) \to T(Q_0))$, the S and T duals of X and Y, respectively.

The S and T duals will help us determine a way to measure how close two objects are to being in the same linkage class. The next proposition will give us a way to do this.

Proposition 5.1.11. *Let $((\mathcal{X}, \mathcal{Y}), \mathcal{B}, (S, T))_L)$ be a category pair with linkage. Then*

(i) *If $((\mathcal{X}, \mathcal{Y}), \mathcal{B}, (S, T))_L)$ is S-perfect, then for any $X \in \mathcal{X}$ we have an exact complex*

$$0 \longrightarrow R^1 T(D_S(X)) \longrightarrow X \xrightarrow{\delta^S_{\mathcal{X}}(X)} T(S(X)) \longrightarrow R^2 T(D_S(X)) \longrightarrow 0$$

(ii) *If $((\mathcal{X}, \mathcal{Y}), \mathcal{B}, (S, T))_L)$ is T-perfect, then for any $Y \in \mathcal{Y}$ we have an exact complex*

$$0 \longrightarrow R^1 S(D_T(Y)) \longrightarrow Y \xrightarrow{\delta^T_{\mathcal{Y}}(Y)} S(T(Y)) \longrightarrow R^2 S(D_T(Y)) \longrightarrow 0$$

Proof :

(i) Given an S-perfect presentation of X

$$P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} X \longrightarrow 0$$

we get the exact sequence

$$0 \longrightarrow S(X) \xrightarrow{S(p_0)} S(P_0) \xrightarrow{S(p_1)} S(P_1) \longrightarrow D_S(X) \longrightarrow 0.$$

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Let $K = \ker(S(P_1) \to D_S(X))$. Then we have two short exact sequences

$$0 \longrightarrow S(X) \xrightarrow{S(p_0)} S(P_0) \xrightarrow{\varphi_0} K \longrightarrow 0$$

$$0 \longrightarrow K \xrightarrow{\varphi_1} S(P_1) \longrightarrow D_S(X) \longrightarrow 0.$$

Applying T to these yields the following exact sequences

$$0 \longrightarrow T(K) \xrightarrow{T(\varphi_0)} T(S(P_0)) \xrightarrow{T(S(p_0))} T(S(X)) \longrightarrow R^1 T(K) \longrightarrow 0$$

$$0 \longrightarrow T(D_S(X)) \longrightarrow T(S(P_1)) \xrightarrow{T(\varphi_1)} T(K) \longrightarrow R^1 T(D_S(X)) \longrightarrow 0$$

$$0 \longrightarrow R^1 T(K) \longrightarrow R^2 T(D_S(X)) \longrightarrow 0.$$

From these sequences there arises a commutative diagram

$$
\begin{array}{ccccccc}
P_1 & \xrightarrow{\ p_1\ } & P_0 & \xrightarrow{\ p_0\ } & X & \longrightarrow & 0 \\
\downarrow{\scriptstyle T(\varphi_1)\circ\delta^S_{\mathcal{X}}(P_1)} & & \downarrow{\scriptstyle \delta^S_{\mathcal{X}}(P_0)} & & \downarrow{\scriptstyle \delta^S_{\mathcal{X}}(X)} & & \\
0 \longrightarrow T(K) & \xrightarrow{T(\varphi_0)} & T(S(P_0)) & \xrightarrow{T(S(p_0))} & T(S(X)) & &
\end{array}
$$

Note that $\delta^S_{\mathcal{X}}(P_0)$ is an isomorphism. Therefore, by the Snake Lemma

$$\ker(\delta^S_{\mathcal{X}}(X)) = \mathrm{coker}(T(\varphi_1) \circ \delta^S_{\mathcal{X}}(P_1)) = \mathrm{coker}(T(\varphi_1)) = R^1 T(D_S(X)).$$

Further, the cokernel of $\delta^S_{\mathcal{X}}(X)$ is the same as the cokernel of $T(S(p_0))$ as $\delta^S_{\mathcal{X}}(P_0)$ is an iso-morphism and p_0 is an epimorphism. Therefore $\mathrm{coker}(\delta^S_{\mathcal{X}}(X)) = R^1 T(K) = R^2 T(D_S(X))$.

(ii) Similar to (i) with S and T switched with an object Y of $\mathcal{Y}$.

$\square$

Corollary 5.1.12. *Let* $((\mathcal{X}, \mathcal{Y}), \mathcal{B}, (S,T))_L)$ *be perfect. Then* W *is* S-*perfect* $(T$-*perfect) if and only if* $D^i S(W) = 0 = D^i T(D_S(W))$ *for* $i > 0$ $(D^i T(W) = 0 = D^i S(D_T(W))$ *for* $i > 0$).

There is an analogous result for right exact linkage functors which we now state.

Proposition 5.1.13. *Let* $((\mathcal{X}, \mathcal{Y}), \mathcal{B}, (S,T))_R)$ *be a category pair with linkage. Then*

(i) *If* $((\mathcal{X}, \mathcal{Y}), \mathcal{B}, (S,T)_R)$ *is* S-*perfect, then for any* $X \in \mathcal{X}$ *we have an exact complex*

$$0 \longrightarrow L_2 T(D^S(X)) \longrightarrow T(S(X)) \xrightarrow{\delta_S^{\mathcal{X}}(X)} X \longrightarrow L_1 T(D^S(X)) \longrightarrow 0$$

(ii) *If* $((\mathcal{X}, \mathcal{Y}), \mathcal{B}, (S,T))_R)$ *is* T-*perfect, then for any* $Y \in \mathcal{Y}$ *we have an exact complex*

$$0 \longrightarrow L_2 S(D^T(Y)) \longrightarrow S(T(Y)) \xrightarrow{\delta_T^{\mathcal{Y}}(Y)} Y \longrightarrow L_1 S(D^T(Y)) \longrightarrow 0$$

Proof :

(i) Given an S-perfect co-presentation of X

$$0 \longrightarrow X \xrightarrow{i_0} I_0 \xrightarrow{i_1} I_1$$

we get the exact sequence

$$0 \longrightarrow D^S(X) \longrightarrow S(I_1) \xrightarrow{S(i_1)} S(I_0) \xrightarrow{S(i_0)} S(X) \longrightarrow 0.$$

Let $Q = \mathrm{coker}(D^S(X) \to S(I_1))$. Then we have two short exact sequences

$$0 \longrightarrow D^S(X) \longrightarrow S(I_1) \xrightarrow{\psi_1} Q \longrightarrow 0$$

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$$0 \longrightarrow Q \xrightarrow{\psi_0} S(I_0) \xrightarrow{S(i_0)} S(X) \longrightarrow 0,$$

Applying T to these yields the following exact sequences

$$0 \longrightarrow L_1 T(D^S(X)) \longrightarrow T(Q) \xrightarrow{T(\psi_1)} T(S(I_1)) \longrightarrow T(D^S(X)) \longrightarrow 0$$

$$0 \longrightarrow L_1 T(Q) \longrightarrow T(S(X)) \xrightarrow{T(S(i_0))} T(S(I_0)) \xrightarrow{T(\psi_0)} T(Q) \longrightarrow 0$$

$$0 \longrightarrow L_2 T(D^S(A)) \longrightarrow L_1 T(Q) \longrightarrow 0.$$

From these sequences there arise a commutative diagram

$$
\begin{array}{ccccccc}
T(S(X)) & \xrightarrow{T(S(i_0))} & T(S(I_0)) & \xrightarrow{T(\psi_0)} & T(Q) & \longrightarrow & 0 \\
\downarrow{\scriptstyle \delta_S^{\mathcal{X}}(X)} & & \downarrow{\scriptstyle \delta_S^{\mathcal{X}}(I_0)} & & \downarrow{\scriptstyle \delta_S^{\mathcal{X}}(I_1)\circ T(\psi_1)} & & \\
0 \longrightarrow X & \xrightarrow{i_0} & I_0 & \xrightarrow{i_1} & I_1 & &
\end{array}
$$

Note that $\delta_S^{\mathcal{X}}(I_0)$ is an isomorphism. Therefore, by the Snake Lemma

$$\text{coker}(\delta_S^{\mathcal{X}}(X)) = \ker(\delta_S^{\mathcal{X}}(I_1) \circ T(\psi_1)) = \ker(T(\psi_1)) = L_1 T(D^S(X)).$$

Further, the kernel of $\delta_S^{\mathcal{X}}(X)$ is the same as the kernel of $T(S(i_0))$ as i_0 is a monomorphism and $\delta_S^{\mathcal{X}}(X)$ is an isomorphism. Therefore $\ker(\delta_S^{\mathcal{X}}(X)) = L_1 T(Q) = L_2 T(D^S(X))$.

(ii) Similar to (i) with S and T switched.

$\square$

Corollary 5.1.14. *Let* $((\mathcal{X}, \mathcal{Y}), \mathcal{B}, (S,T))_R)$ *be perfect. Then* W *is* S-*perfect* (T-*perfect*) *if and only if* $D^i S(W) = 0 = D^i T(D^S(W))$ *for* $i > 0$ ($D^i T(W) = 0 = D^i S(D^T(W))$ *for* $i > 0$).

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Notice the parallel between these results and those in Chapter 4. These results are why we choose to be in a homological category where one can have long exact sequences in homology and the Snake Lemma.

Let $((\mathcal{X}, \mathcal{Y}), \mathcal{B}, (S, T))_L)$ be perfect. We will say that W is stable if W has no summands in $\mathcal{B}$, i.e. if $W \in \mathcal{X}$ then there are no summands of the form $F_{\mathcal{X}}(\mathcal{B})$ and vice versa for $W \in \mathcal{Y}$. If $X \in \mathcal{X}$ is stable and also in $\mathrm{Pres}(\mathcal{B})$ then there is a short exact sequence

$$0 \longrightarrow K \longrightarrow B \xrightarrow{\alpha} X \longrightarrow 0$$

which leads to

$$0 \longrightarrow S(X) \xrightarrow{Y_{\mathcal{B}} \circ S(\alpha)} B \longrightarrow \mathcal{L}_B^S(X) \longrightarrow 0$$

and

$$0 \longrightarrow T(\mathcal{L}_B^S(X)) \longrightarrow B \xrightarrow{T(S(\alpha)) \circ T(Y_{\mathcal{B}})} \mathcal{L}_B^T(\mathcal{L}_B^S(X)) \longrightarrow 0$$

So there is an epimorphism $T(S(B)) \xrightarrow{T(S(\alpha))} \mathcal{L}_B^T(\mathcal{L}_B^S(X))$ which is "minimal" as X is stable. Similarly there is an epimorphism $S(T(B)) \xrightarrow{S(T(\alpha))} \mathcal{L}_B^S(\mathcal{L}_B^T(Y))$ for $Y \in \mathcal{Y}$.

Proposition 5.1.15. *Let $((\mathcal{X}, \mathcal{Y}), \mathcal{B}, (S, T))_L)$ be perfect. Then*

(i) for $X \in \mathcal{X}$ where $X \in Pres(\mathcal{B})$ we have the short exact sequence

$$0 \longrightarrow R^1 T(D_S(X)) \longrightarrow X \xrightarrow{\delta_{\mathcal{X}}^S(X)} \mathcal{L}_B^T(\mathcal{L}_B^S(X)) \longrightarrow 0$$

(ii) for $Y \in \mathcal{Y}$ where $Y \in Pres(\mathcal{B})$ we have the short exact sequence

$$0 \longrightarrow R^1 F(D_G(A)) \longrightarrow A \xrightarrow{\delta_{\mathcal{A}}^G(A)} \mathcal{L}_B^F(\mathcal{L}_B^G(A)) \longrightarrow 0$$

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Proof :

(i) Consider the commutative square

$$
\begin{array}{ccc}
B & \xrightarrow{\;\;\alpha\;\;} & X \\
\big\downarrow{\scriptstyle \delta_{\mathcal{X}}^{S}(B)} & & \big\downarrow{\scriptstyle \delta_{\mathcal{X}}^{S}(X)} \\
T(S(B)) & \xrightarrow{T(S(\alpha))} & T(S(X))
\end{array}
$$

where $\delta_{\mathcal{X}}^{S}(B)$ is an isomorphism and α is an epimorphism. Since $0 \to \mathcal{L}_{B}^{T}(\mathcal{L}_{B}^{S}(X)) \to$ $T(S(X))$ and $T(S(B)) \xrightarrow{T(S(\alpha))} \mathcal{L}_{B}^{T}(\mathcal{L}_{B}^{S}(X)) \to 0$ we have that $X \xrightarrow{\delta_{\mathcal{X}}^{S}(X)} \mathcal{L}_{B}^{T}(\mathcal{L}_{B}^{S}(X)) \to 0$. That is to say that the image of $T(S(\alpha))$ is the same as the image of $\delta_{\mathcal{X}}^{S}(X)$ for the maps in the square. Therefore we get the short exact sequence

$$
0 \longrightarrow R^{1}T(D_{S}(X)) \longrightarrow X \xrightarrow{\delta_{\mathcal{X}}^{S}(X)} \mathcal{L}_{B}^{T}(\mathcal{L}_{B}^{S}(X)) \longrightarrow 0
$$

using Proposition 5.1.11.

(ii) Same as (i) except with S and T switched.

$\square$

It is clear that there is an analogous result for right exact functors using Proposition 5.1.13. It is also clear by the previous result that if $X \overset{B}{\underset{S}{\rightsquigarrow}} \mathcal{L}_{B}^{S}(X)$ then $X \cong \mathcal{L}_{B}^{T}(\mathcal{L}_{B}^{S}(X))$ and so $R^{1}T(D_{S}(X)) = 0$, and similarly for $R^{1}S(D_{T}(Y))$. So the vanishing of $R^{1}T(D_{S}(X))$ (or $R^{1}S(D_{T}(Y))$) is a check to see if the S-linkage class (or T-linkage class) of X is nonempty for X stable. However, if X is not stable then we have the following (compare with [53, Lemma 3.11]).

Proposition 5.1.16. *Let* $((\mathcal{X}, \mathcal{Y}), \mathcal{B}, (S, T))_{L})$ *be perfect and suppose* $X \cong X' \oplus B$ *is not stable, i.e.* $B \in \mathcal{B}$. *Then if* $[X'] \neq \emptyset$, $X' \oplus B$ *is evenly linked to* X.

Proof : We can assume that $X \in \mathcal{X}$ as the same proof will hold in $\mathcal{Y}$ with S and T switched. As $[X'] \neq \emptyset$ we have that $X' \overset{B'}{\underset{S}{\rightsquigarrow}} \mathcal{L}_{B'}^S(X')$ for some $B' \in \mathcal{B}$ and so it follows that

$$X = X' \oplus B \overset{B \oplus B'}{\underset{S}{\rightsquigarrow}} \mathcal{L}_{B'}^S(X') \overset{B'}{\underset{T}{\rightsquigarrow}} X'.$$

$\square$

Therefore, if we have an object $X \cong X' \oplus \mathcal{B}$ which is not stable, then it is evenly linked to a stable module as long as there is an object that X' is linked to. We get a similar result for right exact functors which we state for completeness.

Corollary 5.1.17. *Let $((\mathcal{X}, \mathcal{Y}), \mathcal{B}, (S, T))_R)$ be perfect and suppose $X \cong X' \oplus B$ is not stable, i.e. $B \in \mathcal{B}$. Then if $[X'] \neq \emptyset$, $X' \oplus B$ is evenly linked to X.*

Using these stable representatives in the even linkage classes of an object we can connect the homological properties of an object with any in its even linkage class using its S or T dual.

Theorem 5.1.18. *Let $((\mathcal{X}, \mathcal{Y}), \mathcal{B}, (S, T))_L)$ be perfect and suppose that $X \in \mathcal{X}$ is stable. If $X \overset{B}{\underset{S}{\rightsquigarrow}} Y$ and $X' \overset{B'}{\underset{S}{\rightsquigarrow}} Y$ then $R^i T(D_S(X)) \cong R^i T(D_S(X'))$ and $R^i S(X) \cong R^i S(X')$ for $i > 0$.*

Proof : By Proposition 5.1.11 (i) and Proposition 5.1.15 we have the two short exact sequences

$$0 \longrightarrow D^1 T(D_S(X)) \longrightarrow X \longrightarrow \mathcal{L}_B^T(\mathcal{L}_B^S(X)) \longrightarrow 0$$

$$0 \longrightarrow \mathcal{L}_B^T(\mathcal{L}_B^S(X)) \longrightarrow T(S(X)) \longrightarrow D^2 T(D_S(X)) \longrightarrow 0.$$

Since $X \overset{B}{\underset{S}{\rightsquigarrow}} Y$ we have that $D^1 T(D_S(X)) = 0$ as $X \cong \mathcal{L}_B^T(Y) = \mathcal{L}_B^T(\mathcal{L}_B^S(X))$. Thus the second sequence becomes

$$0 \longrightarrow X \longrightarrow T(S(X)) \longrightarrow D^2 T(D_S(X)) \longrightarrow 0.$$

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However, by Lemma 5.1.5 (i) we have the short exact sequence

$$0 \longrightarrow X \longrightarrow T(S(X)) \longrightarrow R^1 T(Y) \longrightarrow 0$$

and so we must have $R^1 T(Y) \cong R^2 T(D_S(X))$ since both morphisms $X \to T(S(X))$ are the same. Therefore as $X' \overset{B'}{\underset{S}{\rightsquigarrow}} Y$ we also have $R^1 T(Y) \cong R^2 T(D_S(X'))$. So $R^2 T(D_S(X)) \cong R^2 T(D_S(X'))$. Next, by Lemma 5.1.5 (ii) we have that $R^i T(S(X)) \cong R^{i+1} T(Y) \cong R^i T(S(X'))$ for $i > 0$. By the proof of Proposition 5.1.11 (i) we have that $R^{i+2} T(D_S(X)) \cong R^i T(S(X))$ for $i > 0$. Putting these together give $R^{i+2} T(D_S(X)) \cong R^{i+2} T(D_S(X'))$ for $i > 0$ and thus we have that $R^i T(D_S(X)) \cong R^i T(D_S(X))$ for $i > 0$.

To show that $R^i S(X) \cong R^i S(X')$ for $i > 0$. By reversing the roles of X and Y in the above argument we have $R^1 S(X) \cong R^2 S(D_T(Y))$ and thus $R^1 S(X) \cong R^1 S(X')$. Lastly, by Lemma 5.1.5 (iii) we have $R^{i+1} S(X) \cong R^i S(T(Y)) \cong R^{i+1} S(X')$ for $i > 0$. So $R^i S(X) \cong R^i S(X')$ for $i > 0$.

$\square$

The same holds true for $Y \in \mathcal{Y}$, but it isn't necessary to state this result as it only concerns even linkage classes. The analogous result for right exact linkage functors is as follows.

Corollary 5.1.19. *Let* $((\mathcal{X}, \mathcal{Y}), \mathcal{B}, (S, T))_R$ *be perfect and suppose that* $X \in \mathcal{X}$ *is stable. If* $X \overset{S}{\underset{B}{\rightsquigarrow}} Y$ *and* $X' \overset{S}{\underset{B'}{\rightsquigarrow}} Y$ *then* $L_i T(D^S(X)) \cong L_i T(D^S(X'))$ *and* $L_i S(X) \cong L_i S(X')$ *for* $i > 0$.

Proof : Analogous to Theorem 5.1.18 using Lemma 5.1.6 and 5.1.13.

$\square$

From now on we will use the notation $[Z]_e$ and $[Z]_o$ for the even and odd linkage class of an object Z.

Corollary 5.1.20. *Let $((\mathcal{X}, \mathcal{Y}), \mathcal{B}, (S, T))_L)$ be perfect and suppose that $X \in \mathcal{X}$ is stable. If $|[X]_e| > 0$, then there is an object X_e for which*

$$0 \longrightarrow X' \longrightarrow T(S(X')) \longrightarrow X_e \longrightarrow 0$$

is exact for any $X' \in [X]_e$. There is a corresponding statement for each class $[Y]_e$, $[X]_o$, and $[Y]_o$ for $Y \in \mathcal{Y}$.

Corollary 5.1.21. *Let $((\mathcal{X}, \mathcal{Y}), \mathcal{B}, (S, T))_R)$ be perfect and suppose that X is stable. If $|[X]_e| > 0$, then there is an object X^e for which*

$$0 \longrightarrow X^e \longrightarrow T(S(X')) \longrightarrow X' \longrightarrow 0$$

is exact for any $X' \in [X]_e$. There is a corresponding statement for each class $[Y]_e$, $[X]_o$, $[Y]_o$ for $Y \in \mathcal{Y}$.

Therefore we have come across a way to compare the even linkage class of different objects. These special objects X_e and X^e exist for any object whose even linkage class is empty. In fact, these exist for any object whose entire linkage class is empty (as they are then evenly linked to themselves). So given two objects X and X' in $\mathcal{X}$ whose linkage classes are nonempty (similar for Y and Y' in $\mathcal{Y}$) we can compare X_e and X'_e. If $X_e \neq X'_e$ then X and X' are not evenly linked.

5.2 Covariant Linkage Functors

We now present the analogous definitions and results when our functors are covariant. One has to be more careful in this situation in order to define linkage in a way that makes sense. In this case we must have one functor be left exact and the other right exact so that we end up with a theory that produces the results we desire. Many of the results are extremely similar in presentation and proof, so we may leave some proofs out unless they illuminate some other proof technique or are altogether different.

Definition 5.2.1. *Suppose $\mathcal{X}$ and $\mathcal{Y}$ are homological categories where $S : \mathcal{X} \to \mathcal{Y}$ and $T : \mathcal{Y} \to \mathcal{X}$ are additive covariant functors. Suppose that S is right exact (RE) and T is left exact (LE). We say S and T are **linkage functors**, and (S, T) form a **linkage functor pair** for $\mathcal{X}$ and $\mathcal{Y}$ if there exists a category $\mathcal{B}$ with a pair of full and faithful functors $F_{\mathcal{X}} : \mathcal{B} \to \mathcal{X}$ and $F_{\mathcal{Y}} : \mathcal{B} \to \mathcal{Y}$ such that the following holds:*

(i) *For $B \in \mathcal{B}$ there exists functors $X_{\mathcal{X}} : \mathcal{X}|_{F_{\mathcal{X}}(\mathcal{B})} \to \mathcal{Y}|_{F_{\mathcal{Y}}(\mathcal{B})}$ and $Y_{\mathcal{B}} : \mathcal{Y}|_{F_{\mathcal{Y}}(\mathcal{B})} \to \mathcal{X}|_{F_{\mathcal{X}}(\mathcal{B})}$ such that $X_{\mathcal{B}} \circ F_{\mathcal{X}} = F_{\mathcal{Y}}$ and $Y_{\mathcal{B}} \circ F_{\mathcal{Y}} = F_{\mathcal{X}}$. Moreover, $X_{\mathcal{B}}$ and $Y_{\mathcal{B}}$ are such that $S|_{F_{\mathcal{X}}(\mathcal{B})} = X_{\mathcal{B}}$ and $T|_{F_{\mathcal{Y}}(\mathcal{B})} = Y_{\mathcal{B}}$.*

(ii) *For each $B \in \mathcal{B}$, $L_i S(F_{\mathcal{X}}(B)) = 0$ and $R^i T(F_{\mathcal{Y}}(B)) = 0$ for $i > 0$.*

*In this case we say that $\mathcal{B}$ is a **Fossum** category, both $F_{\mathcal{X}}(\mathcal{B})$ and $F_{\mathcal{Y}}(\mathcal{B})$ are **linking classes** in $\mathcal{X}$ and $\mathcal{Y}$, and $((\mathcal{X}, \mathcal{Y}), \mathcal{B}, (S, T))$ is a **category pair with linkage**.*

Just as in the contravariant case, the linkage is determined by the functors S and T. Further still, since we have already set which functors are left and right exact we have less cases to deal with in the covariant situation.

Let $((\mathcal{X}, \mathcal{Y}), \mathcal{B}, (S, T))$ be a category pair with linkage. We can define $\mathcal{L}_S^B(X)$ and $\mathcal{L}_T^G(Y)$ in a similar manner to the contravariant case. Suppose $X \in \mathrm{Pres}(\mathcal{B})$, i.e. there is a short exact sequence

$$0 \longrightarrow K \longrightarrow B \xrightarrow{\alpha} X \longrightarrow 0$$

in $\mathcal{X}$ where $B \in \mathcal{B}$. Using S we can define $\mathcal{L}_S^B(X) = \ker(S(\alpha) \circ X_{\mathcal{B}})$. If $Y \in \mathrm{Copres}(\mathcal{B})$ then we have a short exact sequence

$$0 \longrightarrow Y \xrightarrow{\alpha} B \longrightarrow Q \longrightarrow 0$$

in $\mathcal{Y}$ where $B \in \mathcal{B}$. Using T we can define $\mathcal{L}_B^T(Y) = \mathrm{coker}(Y_{\mathcal{B}} \circ T(\alpha))$.

Definition 5.2.2. *Let $((\mathcal{X}, \mathcal{Y}), \mathcal{B}, (S, T))$ be a category pair with linkage. Given two objects $X \in$ Pres$(\mathcal{B})$ and $Y \in$ Copres$(\mathcal{B})$ we say that X is **directly S-linked** to Y (Y is directly T-linked to X) by $B \in \mathcal{B}$ if $\mathcal{L}_S^B(X) \cong Y$ and $\mathcal{L}_B^T(Y) \cong X$ and we use the notation $X \overset{S}{\underset{B}{\rightsquigarrow}} Y$ ($Y \overset{B}{\underset{T}{\rightsquigarrow}} X$). Moreover, we say that X is linked to W if there is a chain of links starting at X which ends in W.*

Just as in the contravariant case we can consider the linkage class, even linkage class, and odd linkage class of an object.

Proposition 5.2.3. *Every object in $\mathcal{B}$ is in the same linkage class.*

Just as before, every pair of objects in $\mathcal{B}$ are directly linked as any object in $\mathcal{B}$ is in both $\mathrm{Pres}(\mathcal{B})$ and $\mathrm{Copres}(\mathcal{B})$ by the appropriate sequence as was used in the proof of Proposition 5.1.4. Next is the crucial lemma is connecting the homological properties of linked objects.

Lemma 5.2.4. *Let $((\mathcal{X}, \mathcal{Y}), \mathcal{B}, (S, T))$ be a category pair with linkage and suppose $X \overset{S}{\underset{B}{\rightsquigarrow}} Y$. Then the following holds:*

(i) *The two sequences*

$$0 \longrightarrow X \longrightarrow T(S(X)) \longrightarrow R^1 T(Y) \longrightarrow 0$$

$$0 \longrightarrow L_1 S(X) \longrightarrow S(T(Y)) \longrightarrow Y \longrightarrow 0$$

are exact.

(ii) $R^i T(S(X)) \cong R^{i+1} T(Y)$ *for* $i > 0$

(ii) $L_i S(T(Y)) \cong L_{i+1} S(X)$ *for* $i > 0$.

Proof : As $X \overset{S}{\underset{B}{\rightsquigarrow}} Y$ it is assumed that $X \in \mathrm{Pres}(\mathcal{B})$ and a sequence

$$0 \to K \to B \to X \to 0$$

exists. Applying S we get the short exact sequence

$$0 \to Y \to B \to S(X) \to 0.$$

as $\mathcal{L}_S^B(X) = Y$. This gives us the first sequence in (i) and (ii) by applying T to this short exact sequence. Similarly as $Y \overset{B}{\underset{T}{\rightsquigarrow}} X$ we have $Y \in \mathrm{Copres}(\mathcal{B})$ and have the short exact sequence

$$0 \to Y \to B \to Q \to 0.$$

Applying T we get the short exact sequence

$$0 \to T(Y) \to B \to X \to 0$$

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as $\mathcal{L}_B^T(Y) = X$. This gives us the second sequence in (i) and (iii) after applying S to this short exact sequence.

$\square$

In a similar fashion to the contravariant case we define S-perfect and T-perfect objects in exactly the same manner. We will continue to use $\text{Per}(S)$ and $\text{Per}(T)$ to denote the S-perfect and T-perfect objects in $\mathcal{X}$ and $\mathcal{Y}$, respectively.

Lemma 5.2.5. *Let $((\mathcal{X}, \mathcal{Y}), \mathcal{B}, (S, T))$ be a category pair with linkage. Suppose X is S-perfect. Then*

(i) $S(X) \in Per(T)$

(ii) *If X is directly S-linked to Y by B, Y is T-perfect.*

Proof : Similar to the proof of Lemma 5.1.8 using Lemma 5.2.4.

$\square$

Also the same holds true for Y T-perfect with S and T switched, we have the following

Corollary 5.2.6. *Let $((\mathcal{X}, \mathcal{Y}), \mathcal{B}, (S, T))$ be a category pair with linkage. Suppose that X is S-perfect. Then every object in $[X]_e$ is S-perfect and every object in $[X]_o$ is T-perfect.*

Definition 5.2.7. *Let $((\mathcal{X}, \mathcal{Y}), \mathcal{B}, (S, T))$ be a category pair with linkage. We say that $((\mathcal{X}, \mathcal{Y}), \mathcal{B}, (S, T))$ is **S-perfect (T-perfect)** if $\mathcal{X} = Pres(Per(S))$ ($\mathcal{Y} = Copres(Per(T))$) and given a morphism $X \xrightarrow{\alpha} X'$ in $\mathcal{X}$ ($Y \xrightarrow{\beta} Y'$ in $\mathcal{Y}$), there exists morphisms $\delta_{\mathcal{X}}^S(X) : X \to T(S(X))$ and $\delta_{\mathcal{X}}^S(X') : X' \to T(S(X'))$ ($\delta_T^{\mathcal{Y}}(Y) : S(T(Y)) \to Y$ and $\delta_T^{\mathcal{Y}}(Y') : S(T(Y')) \to Y'$) such that the*

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following square commutes:

$$
\begin{array}{ccc}
X & \xrightarrow{\ \alpha\ } & X' \\
\downarrow{\scriptstyle \delta_{\mathcal{X}}^S(X)} & & \downarrow{\scriptstyle \delta_{\mathcal{X}}^S(X')} \\
T(S(X)) & \xrightarrow{T(S(\alpha))} & T(S(X'))
\end{array}
\qquad
\left(
\begin{array}{ccc}
S(T(Y)) & \xrightarrow{S(Y(\beta))} & S(T(Y')) \\
\downarrow{\scriptstyle \delta_T^{\mathcal{Y}}(Y)} & & \downarrow{\scriptstyle \delta_T^{\mathcal{Y}}(Y')} \\
Y & \xrightarrow{\ \beta\ } & Y'
\end{array}
\right)
$$

That is, $\delta_{\mathcal{X}}^S(-)$ ($\delta_T^{\mathcal{Y}}(-)$) is a natural transformation between the identity functor and $T(S(-))$ ($S(T(-))$).

Once again, we say that $((\mathcal{X}, \mathcal{Y}), \mathcal{B}, (S, T))$ is **perfect** if it is both S-perfect and T-perfect. Now suppose that $((\mathcal{X}, \mathcal{Y}), \mathcal{B}, (S, T))$ is perfect. As $\mathcal{X} = \mathrm{Pres}(\mathrm{Per}(S))$, given any object X we can take an S-perfect presentation

$$P_1 \to P_0 \to X \to 0$$

where $P_0, P_1 \in \mathrm{Per}(S)$. If we apply S to this we get

$$0 \to D_S(X) \to S(P_1) \to S(P_0) \to X \to 0$$

where $D_S(X) = \ker(S(P_1) \to S(P_0))$ is the S-dual of X. Similarly, $\mathcal{Y} = \mathrm{Copres}(\mathrm{Per}(T))$, given any object Y we can take a T-perfect copresentation

$$0 \to Y \to I_0 \to I_1$$

where $I_0, I_1 \in \mathrm{Per}(T)$. If we apply T to this we get

$$0 \to T(Y) \to T(I_0) \to T(I_1) \to D^T(Y) \to 0$$

where $D^T(Y) = \mathrm{coker}(T(I_0) \to T(I_1))$ is the T-dual of Y.

Proposition 5.2.8. *Let* $((\mathcal{X}, \mathcal{Y}), \mathcal{B}, (S, T))$ *be a category pair with linkage. Then*

(i) *If* $((\mathcal{X}, \mathcal{Y}), \mathcal{B}, (S, T))$ *is* S*-perfect, then for any* $X \in \mathcal{X}$ *we have an exact complex*

$$0 \longrightarrow R^1 T(D_S(X)) \longrightarrow X \xrightarrow{\delta^S_{\mathcal{X}}(X)} T(S(X)) \longrightarrow R^2 T(D_S(X)) \longrightarrow 0$$

(ii) *If* $((\mathcal{X}, \mathcal{Y}), \mathcal{B}, (S, T))$ *is* T*-perfect, then for any* $Y \in \mathcal{Y}$ *we have an exact complex*

$$0 \longrightarrow L_2 S(D^T(Y)) \longrightarrow S(T(Y)) \xrightarrow{\delta^{\mathcal{Y}}_T(Y)} Y \longrightarrow L_1 S(D^T(Y)) \longrightarrow 0$$

Proof :

(i) Similar proof to Proposition 5.1.11 (i).

(ii) Similar proof to Proposition 5.1.13 (ii).

$\square$

Corollary 5.2.9. *Let* $((\mathcal{X}, \mathcal{Y}), \mathcal{B}, (S, T))$ *be perfect. Then* W *is* S*-perfect (*T*-perfect) if and only if* $L_i S(W) = 0 = R^i T(D_S(W))$ *for* $i > 0$ *(*$R^i T(W) = 0 = L_i S(D^T(W))$ *for* $i > 0$*).*

Once again, we say that W is **stable** if it has no summands in $\mathcal{B}$ and we get a result concerning stable objects as follows.

Proposition 5.2.10. *Let* $((\mathcal{X}, \mathcal{Y}), \mathcal{B}, (S, T))$ *be perfect.*

(i) *If* $X \in Pres(\mathcal{B})$ *is stable, then there is a short exact sequence*

$$0 \longrightarrow R^1 T(D_S(X)) \longrightarrow X \xrightarrow{\delta^S_{\mathcal{X}}(X)} \mathcal{L}^T_B(\mathcal{L}^B_S(X)) \longrightarrow 0$$

(ii) *If $Y \in Copres(\mathcal{B})$ is stable, then there is a short exact sequence*

$$0 \longrightarrow \mathcal{L}_S^B(\mathcal{L}_B^T(Y)) \xrightarrow{\delta_T^{\mathcal{Y}}(Y)} Y \longrightarrow L_1 S(D^T(Y)) \longrightarrow 0$$

Proof : Similar to Proposition 5.1.15

□

Now in the same way as before, each nonempty even linkage class has a stable representative.

Proposition 5.2.11. *Let $((\mathcal{X}, \mathcal{Y}), \mathcal{B}, (S, T))$ be perfect with $X \cong X' \oplus B$ not stable, i.e. $B \in \mathcal{B}$. Then if $[X'] \neq \emptyset$, $X' \oplus B$ is evenly linked to X'.*

Proof : Similar to the proof of Proposition 5.1.16

□

We also get a theorem connecting the homological properties of the even linkage class of a stable object in $\mathcal{X}$ or $\mathcal{Y}$.

Theorem 5.2.12. *Let $((\mathcal{X}, \mathcal{Y}), \mathcal{B}, (S, T))$ be perfect and suppose that X is stable. If $X \overset{B}{\underset{S}{\rightsquigarrow}} Y$ and $X' \overset{B'}{\underset{S}{\rightsquigarrow}} Y$ then $R^i T(D_S(X)) \cong R^i T(D_S(X'))$ and $L_i S(X) \cong L_i S(X')$ for $i > 0$.*

Proof : Similar to the proof of Theorem 5.1.18 using Lemma 5.2.4 and Propositions 5.2.8 and 5.2.10.

□

Corollary 5.2.13. *Let $((\mathcal{X}, \mathcal{Y}), \mathcal{B}, (S, T))$ be perfect and suppose that Y is stable. If $Y \overset{T}{\underset{B}{\rightsquigarrow}} X$ and $Y' \overset{T}{\underset{B'}{\rightsquigarrow}} X$ then $L^i S(D^T(Y)) \cong L^i S(D^T(Y'))$ and $R^i T(Y) \cong R^i T(Y')$ for $i > 0$.*

Proof : Analogous to Theorem 5.2.12.

$\square$

Corollary 5.2.14. *Let* $((\mathcal{X}, \mathcal{Y}), \mathcal{B}, (S, T))$ *be perfect and suppose that* X *is stable.*

(i) *If* $|[X]^S_e| > 0$, *then there is an object* X^F_e *for which*

$$0 \longrightarrow X' \longrightarrow T(S(X')) \longrightarrow X^F_e \longrightarrow 0$$

is exact for any $X' \in [X]^S_e$.

(ii) *If* $|[Y]^T_e| > 0$, *then there is an object* Y^T_e *for which*

$$0 \longrightarrow Y^T_e \longrightarrow S(T(Y')) \longrightarrow Y' \longrightarrow 0$$

is exact for any $Y' \in [Y]^T_e$.

<div align="center">

5.3   Associated Homological Dimensions

</div>

In this section we will begin the study of the homological dimensions associated to perfect categories with linkage. As each object in the categories can be presented or copresented by perfect objects we can associate to each functor a dimension. This will mirror the homological aspects of Gorenstein dimension and other homological dimensions in the literature including Gorenstein injective, projective, and flat dimension (see [15], [20], and [16]) as well as Noetherian dimension and width (see [56], [27], and [17]).

From now on we will use the following notation to differentiate between types of functor pairs:

(i) $(S, T)_L$ for left exact contravariant functors

(ii) $(S,T)_R$ for right exact contravariant functors

(iii) (S,T) when S is right exact covariant and T is left exact covariant.

We will present the results in the order of categories stated above.

5.3.1   Left Exact Contravariant Functors

Definition 5.3.1. *Let* $((\mathcal{X},\mathcal{Y}),\mathcal{B},(S,T))_L)$ *be perfect. As* $\mathcal{X} = Pres(Per(S))$ *we can iteratively construct an exact complex*

$$\cdots \to P_i \to P_{i-1} \to \cdots \to P_1 \to P_0 \to X \to 0$$

where $P_j \in Per(S)$ *for* $j \geq 0$ *for any object* X*. We call such a complex an **S-resolution** of* X *and let* $\mathcal{S}(X)$ *be the collection of all* S*-resolutions of* X*. Then we define the **S-dimension** of* X *as*

$$S - dim_{\mathcal{X}}(X) = \inf\{\sup \mathcal{S} : \mathcal{S} \in \mathcal{S}(X)\},$$

i.e. the length of a shortest S*-resolution of* X*.*

Similarly, as $\mathcal{Y} = Pres(Per(T))$ *we can construct* T*-resolutions of* Y *and let* $\mathcal{T}(Y)$ *be the collection of such resolutions. We also define the **T-dimension** of* Y *as*

$$T - dim_{\mathcal{Y}}(Y) = \inf\{\sup \mathcal{T} : \mathcal{T} \in \mathcal{T}(Y)\},$$

i.e. the length of a shortest T*-resolution of* Y*.*

Proposition 5.3.2. *Let* $((\mathcal{X},\mathcal{Y}),\mathcal{B},(S,T))_L)$ *be perfect.*

(*i*) *Suppose that* $0 \to X' \to X \to X'' \to 0$ *in* $\mathcal{X}$ *is exact. Then*

if $X'' \in Per(S)$, then $X \in Per(S) \Leftrightarrow X' \in Per(S)$.

if $X \in Per(S)$, then $R^i S(X') \cong R^{i+1} S(X'')$ for $i > 0$.

if $X \cong X' \oplus X''$, then $X \in Per(S) \Leftrightarrow X', X'' \in Per(S)$.

(*ii*) *Suppose that* $0 \to Y' \to Y \to Y'' \to 0$ *in* $\mathcal{Y}$ *is exact. Then*

if $Y'' \in Per(T)$, then $Y \in Per(T) \Leftrightarrow Y' \in Per(T)$.

if $Y \in Per(T)$, then $R^i T(Y') \cong R^{i+1} T(Y'')$ for $i > 0$.

if $Y \cong Y' \oplus Y''$, then $Y \in Per(T) \Leftrightarrow Y', Y'' \in Per(T)$.

Proof : Clear by the additivity of S and T and the long exact sequence in $R^i S(-)$ and $R^i T(-)$.

□

The following result connects together the homological properties of an object X with the homology of a resolution of X after applying S or T.

Proposition 5.3.3. *Let* $((\mathcal{X}, \mathcal{Y}), \mathcal{B}, (S, T))_L)$ *be perfect. Suppose that*

$$\mathcal{P} : \cdots \longrightarrow P_i \longrightarrow P_{i-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0$$

is an S-resolution (T-resolution) of X. Then $H^i(S(\mathcal{P})) = R^i S(X)$ $(H^i(T(\mathcal{P})) = R^i T(X))$ *for* $i > 0$.

Proof : We will show this is true for an S-resolution. The proof for a T-resolution is exactly the same with S and T interchanged.

We have the following diagram

$$
\begin{array}{ccccccc}
0 & & 0 & & 0 & & 0 \\
\searrow & & \nearrow & & \searrow & & \nearrow \\
& K_{n-2} & & & & K_0 & \\
\nearrow & & \searrow & & \nearrow & & \searrow \\
\cdots \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \to X \to 0 \\
\searrow & & \nearrow & & \searrow & & \nearrow \\
& K_{n-1} & & & & K_1 & \\
\nearrow & & \searrow & & \nearrow & & \searrow \\
0 & & 0 & & 0 & & 0
\end{array}
$$

which gives the following exact sequences

$$0 \longrightarrow S(X) \longrightarrow S(P_0) \longrightarrow S(K_0) \longrightarrow R^1 S(A) \longrightarrow 0$$

$$0 \longrightarrow S(K_{i-1}) \longrightarrow S(P_i) \longrightarrow S(K_i) \longrightarrow R^1 S(K_{i-1}) \longrightarrow 0, \qquad i > 0$$

and shows that $H^1(S(\mathcal{P})) = R^1 S(X)$, and $H^i(S(\mathcal{P})) = R^1 S(K_{i-2})$ for $i > 1$. Now as

$R^1 S(K_{i-2}) \cong R^i S(X)$ by Proposition 5.3.2 (ii) we are done.

$\square$

We will now endeavor to pinpoint what the S-dimension (or T-dimension) of an object X is when

it is finite. First we will show the following result which will lead to the characterization. Notice

the similarities with Proposition 4.1.8.

Lemma 5.3.4. *Let $((\mathcal{X}, \mathcal{Y}), \mathcal{B}, (S, T))_L)$ be perfect and X an object of $\mathcal{X}$ (Y an object of $\mathcal{Y}$).*

If $S{-}dim_{\mathcal{X}}(X) < \infty$ ($T{-}dim_{\mathcal{Y}}(Y) < \infty$) and $R^i S(X) = 0$ ($R^i T(Y) = 0$) for $i > 0$, then

$X \in Per(S)$ ($Y \in Per(T)$).

Proof : Once again we will show this holds true for S-dimension. The proof for T-dimension will

be exactly the same except with S and T interchanged.

We prove by induction on $S-\dim_{\mathcal{X}}(X)$. If $S-\dim_{\mathcal{X}}(X) = 1$, then an S-resolution

$$0 \to P_1 \to P_0 \to X \to 0$$

gives

$$0 \to S(X) \to S(P_0) \to S(P_1) \to 0.$$

as $R^1 S(X) = 0$. Therefore $S(X) \in \mathrm{Per}(T)$ by Proposition 5.3.2 (i) as $S(P_0), S(P_1) \in \mathrm{Per}(T)$ by Lemma 5.1.8 (i). So $T(S(X)) \in \mathrm{Per}(S)$ by Lemma 5.1.8 (i). We get that $X \cong T(S(X))$ by using the sequence

$$0 \to T(S(P_1)) \to T(S(P_0)) \to T(S(X)) \to 0$$

and the Snake Lemma. Thus $X \in \mathrm{Per}(S)$.

Now suppose that $S-\dim_{\mathcal{X}}(X) = p > 1$ and take a shortest S-resolution of X

$$0 \to P_p \to \cdots \to P_1 \to P_0 \to X \to 0.$$

Let $K = \ker(P_0 \to X)$. Then $S-\dim_{\mathcal{X}}(K) \leq p - 1$ and $R^i S(K) \cong R^{i+1} S(X) = 0$ for $i > 0$. So by the induction hypothesis $K \in \mathrm{Per}(S)$. Then by the short exact sequence

$$0 \to K \to P_0 \to X \to 0$$

we see that $S-\dim_{\mathcal{X}}(X) \leq 1$. So again by the induction hypothesis $X \in \mathrm{Per}(S)$.

$\square$

We end this section by showing exactly what each dimension is when it is finite. Not unexpectedly it turns out to be the largest index of a non vanishing derived functor much like Gorenstein

dimension or G_C^g-dimension.

Theorem 5.3.5. *Let $((\mathcal{X}, \mathcal{Y}), \mathcal{B}, (S, T))_L)$ be perfect.*

(i) *Let $X \in \mathcal{X}$. Then*

$$S - dim_{\mathcal{X}}(X) \leq n \Leftrightarrow S - dim_{\mathcal{X}}(X) < \infty \text{ and } n \geq \sup\{i : R^i S(X) \neq 0\}.$$

That is, if $S - dim_{\mathcal{X}}(X) < \infty$ then $S - dim_{\mathcal{X}}(X) = \sup\{i : R^i S(X) \neq 0\}$.

(ii) *Let $Y \in \mathcal{Y}$. Then*

$$T - dim_{\mathcal{Y}}(Y) \leq n \Leftrightarrow T - dim_{\mathcal{Y}}(Y) < \infty \text{ and } n \geq \sup\{i : R^i T(Y) \neq 0\}.$$

That is, if $T - dim_{\mathcal{Y}}(Y) < \infty$ then $T - dim_{\mathcal{Y}}(Y) = \sup\{i : R^i T(Y) \neq 0\}$.

Proof :

(i) Suppose $S - \dim_{\mathcal{X}}(X) \leq n$. Take a shortest S-resolution of X

$$0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to X \to 0.$$

Then $R^i S(P_n) \cong R^{i+n} S(X) = 0$ for $i > 0$ as $P_n \in \mathrm{Per}(S)$. So $n \geq \sup\{i : R^i S(X) \neq 0\}$.

Suppose now that $S - \dim_{\mathcal{X}}(X) < \infty$ and $n \geq \sup\{i : R^i S(X) \neq 0\}$. If $S - \dim_{\mathcal{X}}(X) = p \leq n$, we are done, so suppose $p > n$ and take a shortest S-resolution

$$0 \to P_p \to \cdots \to P_n \to P_{n-1} \to \cdots \to P_0 \to X \to 0.$$

Let $K_n = \ker(P_{n-1} \to P_{n-2})$. Then $R^i S(K_n) \cong R^{i+n} S(X) = 0$ for $i > 0$ as $n \geq \sup\{i :$
$R^i S(X) \neq 0\}$. By Lemma 5.3.4 $K_n \in \text{Per}(S)$ which will make

$$0 \to K_n \to P_{n-1} \to \cdots \to P_0 \to X \to 0$$

a shorter right S-resolution. Thus p cannot be greater than n and so $S-\dim_{\mathcal{X}}(X) \leq n$.

(ii) The same proof as (i) except with S and T interchanged.

$\square$

5.3.2 Right Exact Contravariant Functors

Now we will build up the same theory for right exact contravariant functors. We can define a resolution and dimension in an analogous fashion.

Definition 5.3.6. *Let $((\mathcal{X}, \mathcal{Y}), \mathcal{B}, (S, T))_R)$ be perfect. As $\mathcal{X} = Copres(Per(S))$*
($\mathcal{Y} = Copres(Per(T))$) we can construct an exact complex

$$0 \to X \to I_0 \to I_1 \to \cdots \to I_{i-1} \to I_i \to \cdots$$

where $I_j \in Per(S)$ $(Per(T))$ for $j \geq 0$ for any object X. Such a complex is called an S-coresolution
(T-coresolution) of X and we let $\mathcal{S}(X)$ $(\mathcal{T}(X))$ be the collection of such coresolutions. So we
*define the **S-dimension (T-dimension)** of X as*

$$S - dim_{\mathcal{X}}(X) = \sup\{\inf \mathcal{S} : \mathcal{S} \in \mathcal{S}(X)\} \quad (T - dim_{\mathcal{Y}}(Y) = \sup\{\inf \mathcal{T} : \mathcal{T} \in \mathcal{T}(Y)\}),$$

i.e. the length of a shortest S-coresolution (T-coresolution) of X.

We then get analogous results which we state without proof here.

Proposition 5.3.7. *Let $((\mathcal{X}, \mathcal{Y}), \mathcal{B}, (S, T))_R)$ be perfect.*

(i) *Suppose that $0 \to X' \to X \to X'' \to 0$ in $\mathcal{X}$ is exact. Then*

   if $X' \in Per(S)$, then $X \in Per(S) \Leftrightarrow X'' \in Per(S)$.

   if $X \in Per(S)$, then $L_i S(X'') \cong L_{i+1} S(X')$ for $i > 0$.

   if $X \cong X' \oplus X''$, then $X \in Per(S) \Leftrightarrow X', X'' \in Per(S)$.

(ii) *Suppose that $0 \to Y' \to Y \to Y'' \to 0$ in $\mathcal{Y}$ is exact. Then*

   if $Y' \in Per(T)$, then $Y \in Per(T) \Leftrightarrow Y'' \in Per(T)$.

   if $Y \in Per(T)$, then $L_i T(Y'') \cong L_{i+1} T(Y')$ for $i > 0$.

   if $Y \cong Y' \oplus Y''$, then $Y \in Per(T) \Leftrightarrow Y', Y'' \in Per(T)$.

Proposition 5.3.8. *Let $((\mathcal{X}, \mathcal{Y}), \mathcal{B}, (S, T))_R)$ be perfect. Suppose that*

$$\mathcal{I} : 0 \to X \to I_0 \to I_1 \to \cdots \to I_{i-1} \to I_i \to \cdots$$

is an S-coresolution (T-coresolution) of X. Then $H_i(S(\mathcal{I})) = L_i S(X)$ ($H_i(T(\mathcal{I})) = L_i T(X)$) for $i > 0$.

Lemma 5.3.9. *Let $((\mathcal{X}, \mathcal{Y}), \mathcal{B}, (S, T))_R)$ be perfect and X an object of $\mathcal{X}$ (Y an object of $\mathcal{Y}$). If $S{-}dim_\mathcal{X}(X) < \infty$ ($T{-}dim_\mathcal{Y}(Y) < \infty$ and $L_i S(X) = 0$ ($L_i T(Y) = 0$) for $i > 0$, then $X \in Per(S)$ ($Y \in Per(T)$).*

Theorem 5.3.10. *Let $((\mathcal{X}, \mathcal{Y}), \mathcal{B}, (S, T)_R)$ be perfect.*

(i) *Suppose that $X \in \mathcal{X}$. Then*

$$S\text{-}dim_\mathcal{X}(X) \leq n \Leftrightarrow S\text{-}dim_\mathcal{X}(X) < \infty \text{ and } n \geq \sup\{i : L_i S(X) \neq 0\}$$

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(ii) *Suppose that $Y \in \mathcal{Y}$. Then*

$$T\text{-}dim_{\mathcal{Y}}(Y) \leq n \Leftrightarrow T\text{-}dim_{\mathcal{Y}}(Y) < \infty \text{ and } n \geq \sup\{i : L_i T(Y) \neq 0\}$$

5.3.3   Covariant Functors

Lastly, for covariant functors we must switch between a resolution and a coresolution when applying each functor. Since we have already built up the theory for resolutions and coresolution separately this does not pose a problem here.

Definition 5.3.11. *Let $((\mathcal{X}, \mathcal{Y}), \mathcal{B}, (S, T))$ be perfect. As $\mathcal{X} = Pres(Per(S))$ we can construct an exact complex*

$$\cdots \to P_i \to P_{i-1} \to \cdots \to P_1 \to P_0 \to X \to 0$$

*where $P_j \in Per(S)$ for $j \geq 0$ for any object $X \in \mathcal{X}$. We call such a complex an S-resolution of X and let $\mathcal{S}(X)$ be the collection of all S-resolutions of X. Then we define the **S-dimension** of X as*

$$S\text{-}dim_{\mathcal{X}}(X) = \inf\{\sup \mathcal{S} : \mathcal{S} \in \mathcal{S}(X)\},$$

i.e. the length of a shortest S-resolution of X.

Similarly, as $\mathcal{Y} = Copres(Per(T))$ we can construct T-coresolutions of $Y \in \mathcal{Y}$

$$0 \to Y \to I_0 \to I_1 \to \cdots \to I_{i-1} \to I_i \to \cdots$$

and let $\mathcal{T}(Y)$ *be the collection of such coresolutions. We also define the **T-dimension** of* Y *as*

$$T\text{-}dim_{\mathcal{Y}}(Y) = \sup\{\inf \mathcal{T} : \mathcal{T} \in \mathcal{T}(Y)\},$$

i.e the length of a shortest T*-coresolution of* Y.

Proposition 5.3.12. *Let* $((\mathcal{X},\mathcal{Y}),\mathcal{B},(S,T))$ *be perfect*

(i) *Suppose that* $0 \to X' \to X \to X'' \to 0$ *in* $\mathcal{X}$ *is exact. Then*

(a) *if* $X' \in Per(S)$, *then* $X \in Per(S) \Leftrightarrow X'' \in Per(S)$.

(b) *if* $X \in Per(S)$, *then* $L^i S(X'') \cong L_{i+1}S(X')$ *for* $i > 0$.

(c) *if* $X \cong X' \oplus X''$, *then* $X \in Per(S) \Leftrightarrow X', X'' \in Per(S)$.

(ii) *Suppose that* $0 \to Y' \to Y \to Y'' \to 0$ *in* $\mathcal{Y}$ *is exact. Then*

(a) *if* $Y'' \in Per(T)$, *then* $Y \in Per(T) \Leftrightarrow Y' \in Per(T)$.

(b) *if* $Y \in Per(T)$, *then* $R^i T(Y') \cong R^{i+1}T(Y'')$ *for* $i > 0$.

(c) *if* $Y \cong Y' \oplus Y''$, *then* $Y \in Per(T) \Leftrightarrow Y', Y'' \in Per(T)$.

Proposition 5.3.13. *Let* $((\mathcal{X},\mathcal{Y}),\mathcal{B},(S,T))$ *be perfect where* $X \in \mathcal{X}$ *and* $Y \in \mathcal{Y}$. *If* $\mathcal{P} \in \mathcal{S}(X)$ *then* $H_{-i}(S(\mathcal{P})) = L_i S(X)$, *and if* $\mathcal{I} \in \mathcal{T}(Y)$ *then* $H^i(T(\mathcal{I})) = R^i T(Y)$.

Lemma 5.3.14. *Let* $((\mathcal{X},\mathcal{Y}),\mathcal{B},(S,T))$ *be perfect and* X *and object of* $\mathcal{X}$ (Y *an object of* $\mathcal{Y}$). *If* $S\text{-}dim_{\mathcal{X}}(X) < \infty$ ($T\text{-}dim_{\mathcal{Y}}(Y) < \infty$) *and* $L_i S(X) = 0$ ($R^i T(Y) = 0$) *for* $i > 0$, *then* $X \in Per(S)$ ($Y \in Per(T)$).

Theorem 5.3.15. *Let* $((\mathcal{X},\mathcal{Y}),\mathcal{B},(S,T))$ *be perfect.*

(i) *Let* $X \in \mathcal{X}$. *Then*

$$S - dim_{\mathcal{X}}(X) \leq n \Leftrightarrow S - dim_{\mathcal{X}}(X) < \infty \text{ and } n \geq \sup\{i : L_i S(X) \neq 0\}.$$

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That is, if $S-dim_{\mathcal{X}}(X) < \infty$ then $S-dim_{\mathcal{X}}(X) = \sup\{i : L_iS(X) \neq 0\}$.

(ii) *Let $Y \in \mathcal{Y}$. Then*

$$T - dim_{\mathcal{Y}}(Y) \leq n \Leftrightarrow T - dim_{\mathcal{Y}}(Y) < \infty \text{ and } n \geq \sup\{i : R^iT(Y) \neq 0\}.$$

That is, if $T-dim_{\mathcal{Y}}(Y) < \infty$ then $T-dim_{\mathcal{Y}}(Y) = \sup\{i : R^iT(Y) \neq 0\}$.

5.3.4   Perfect Functors

We now explore the connection between the associated homological dimensions for S and T and linkage classes of objects in a perfect category pair with linkage. First we make a definition specializing the type of functor we would like to consider.

Definition 5.3.16. *Suppose that $((\mathcal{X}, \mathcal{Y}), \mathcal{B}, -)$ is perfect where $-$ is one of (S, T), $(S, T)_L$, or $(S, T)_R$.*

(i) *Suppose we are in the case $(S, T)_L$, then we say that S (T) is perfect if we can complete a diagram*

$$
\begin{array}{ccccccccc}
& & 0 & & & & 0 & & \\
& & \uparrow & & & & \uparrow & & \\
0 & \longrightarrow & W' & \longrightarrow & W & \longrightarrow & W'' & \longrightarrow & 0 \\
& & \uparrow & & & & \uparrow & & \\
& & M' & & & & M'' & &
\end{array}
$$

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in $\mathcal{X}$ ($\mathcal{Y}$) with $M', M'' \in Per(S)$ (Per(T)) to a diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & W' & \longrightarrow & W & \longrightarrow & W'' & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & K' & \longrightarrow & K & \longrightarrow & K'' & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

where $M \in Per(S)$ (Per(T)).

(ii) *Suppose we are in the case $(S,T)_R$, then we say that S (T) is perfect if we can complete a diagram*

$$
\begin{array}{ccccccccc}
 & & M' & & & & M'' & & \\
 & & \uparrow & & & & \uparrow & & \\
0 & \longrightarrow & W' & \longrightarrow & W & \longrightarrow & W'' & \longrightarrow & 0 \\
 & & \uparrow & & & & \uparrow & & \\
 & & 0 & & & & 0 & &
\end{array}
$$

in $\mathcal{X}$ ($\mathcal{Y}$) with $M', M'' \in Per(S)$ $(Per(T))$ to a diagram

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & 0 & & \\
& & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & K' & \longrightarrow & K & \longrightarrow & K'' & \longrightarrow & 0 \\
& & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\
& & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & W' & \longrightarrow & W & \longrightarrow & W'' & \longrightarrow & 0 \\
& & \uparrow & & \uparrow & & \uparrow & & \\
& & 0 & & 0 & & 0 & &
\end{array}
$$

where $M \in Per(S)$ $(Per(T))$.

(iii) *Suppose we are in the case (S,T), then we say that S is perfect if it satisfies the conditions in (i) and T is perfect if it satisfies the conditions in (ii).*

We make this definition in order to be able to compare S and T dimension along a short exact sequence. This will allow us to compare S and T dimension among the links of an object.

Proposition 5.3.17. *Suppose that $((\mathcal{X}, \mathcal{Y}), \mathcal{B}, (S,T))_L)$ is perfect and both S and T are perfect.*

(i) *If $0 \to X' \to X \to X'' \to 0$ is short exact in $\mathcal{X}$, then the following holds:*

(a) *$S\text{-}dim_{\mathcal{X}}(X'') \leq n \Rightarrow S\text{-}dim_{\mathcal{X}}(X') \leq n \Leftrightarrow S\text{-}dim_{\mathcal{X}}(X) \leq n$*

(b) *If $S\text{-}dim_{\mathcal{X}}(X') > S\text{-}dim_{\mathcal{X}}(X'')$ or $S\text{-}dim_{\mathcal{X}}(X) > S\text{-}dim_{\mathcal{X}}(X'')$, then $S\text{-}dim_{\mathcal{X}}(X') = S\text{-}dim_{\mathcal{X}}(X)$*

(c) *If $S\text{-}dim_{\mathcal{X}}(X'') > 0$ and $X \in Per(S)$, then $S\text{-}dim_{\mathcal{X}}(X') = S\text{-}dim_{\mathcal{X}}(X'') - 1$.*

Moreover, if any two have finite S-dimension then so does the other.

(*ii*) *If* $0 \to Y' \to Y \to Y'' \to 0$ *is short exact in* $\mathcal{Y}$, *then the following holds:*

(*a*) T-*dim*$_\mathcal{Y}(Y'') \leq n \Rightarrow T$-*dim*$_\mathcal{Y}(Y') \leq n \Leftrightarrow T$-*dim*$_\mathcal{Y}(Y) \leq n$

(*b*) *If* T-*dim*$_\mathcal{Y}(Y') > T$-*dim*$_\mathcal{Y}(Y'')$ *or* T-*dim*$_\mathcal{Y}(Y) > T$-*dim*$_\mathcal{Y}(Y'')$, *then* T-*dim*$_\mathcal{Y}(Y') = T$-*dim*$_\mathcal{Y}(Y)$

(*c*) *If* T-*dim*$_\mathcal{Y}(Y'') > 0$ *and* $Y \in Per(T)$, *then* T-*dim*$_\mathcal{Y}(Y') = T$-*dim*$_\mathcal{Y}(Y'') - 1$.

Moreover, if any two have finite T-*dimension then so does the other.*

Proof :

(*i*)   (*a*) Clearly true if S-dim$_\mathcal{X}(X'') \leq 0$ by Proposition 5.3.2 (*i*). So suppose that S-dim$_\mathcal{X}(X'') \leq n$. Using S-resolutions of X'' and X' we can get a commutative diagram

as S is perfect. Then as $K_n'' \in \mathrm{Per}(S)$ we have that $K_n' \in \mathrm{Per}(S) \Leftrightarrow K_n \in \mathrm{Per}(S)$. The inequalities then follow.

(b) Clear by the inequalities in part (a).

(c) Follows by Proposition 5.3.2 (ii).

It then clearly follows that if any two have finite dimension then so does the third.

(ii) (a), (b), and (c) are proved in the same way as in (i) with S and T switched.

$\square$

Proposition 5.3.18. *Suppose that* $((\mathcal{X}, \mathcal{Y}), \mathcal{B}, (S, T))_R)$ *is perfect and both S and T are perfect.*

(i) *If* $0 \to X' \to X \to X'' \to 0$ *is short exact in $\mathcal{X}$, then the following holds:*

(a) *S-dim$_{\mathcal{X}}(X') \le n \Rightarrow S$-dim$_{\mathcal{X}}(X'') \le n \Leftrightarrow S$-dim$_{\mathcal{X}}(X) \le n$*

(b) *If S-dim$_{\mathcal{X}}(X'') > S$-dim$_{\mathcal{X}}(X')$ or S-dim$_{\mathcal{X}}(X) > S$-dim$_{\mathcal{X}}(X')$, then S-dim$_{\mathcal{X}}(X'') = S$-dim$_{\mathcal{X}}(X)$*

(c) *If S-dim$_{\mathcal{X}}(X') > 0$ and $X \in Per(S)$, then S-dim$_{\mathcal{X}}(X'') = S$-dim$_{\mathcal{X}}(X') - 1$.*

Moreover, if any two have finite S-dimension then so does the other.

(ii) *If* $0 \to Y' \to Y \to Y'' \to 0$ *is short exact in $\mathcal{Y}$, then the following holds:*

(a) *T-dim$_{\mathcal{Y}}(Y') \le n \Rightarrow T$-dim$_{\mathcal{Y}}(Y'') \le n \Leftrightarrow T$-dim$_{\mathcal{Y}}(Y) \le n$*

(b) *If T-dim$_{\mathcal{Y}}(Y'') > T$-dim$_{\mathcal{Y}}(Y')$ or T-dim$_{\mathcal{Y}}(Y) > T$-dim$_{\mathcal{Y}}(Y')$, then T-dim$_{\mathcal{Y}}(Y'') = T$-dim$_{\mathcal{Y}}(Y)$*

(c) If T-$dim_\mathcal{Y}(Y') > 0$ and $Y \in Per(T)$, then T-$dim_\mathcal{Y}(Y'') = T$-$dim_\mathcal{Y}(Y') - 1$.

Moreover, if any two have finite T-dimension then so does the other.

Proof : Exactly the same as the proof of Proposition 5.3.17 except using 5.3.7 and with the arrows reversed.

$\square$

Corollary 5.3.19. *Suppose that $((\mathcal{X}, \mathcal{Y}), \mathcal{B}, (S, T)))$ is perfect and both S and T are perfect.*

(i) *If $0 \to X' \to X \to X'' \to 0$ is short exact in $\mathcal{X}$, then the following holds:*

(a) S-$dim_\mathcal{X}(X'') \leq n \Rightarrow S$-$dim_\mathcal{X}(X') \leq n \Leftrightarrow S$-$dim_\mathcal{X}(X) \leq n$

(b) If S-$dim_\mathcal{X}(X') > S$-$dim_\mathcal{X}(X'')$ or S-$dim_\mathcal{X}(X) > S$-$dim_\mathcal{X}(X'')$, then S-$dim_\mathcal{X}(X') = S$-$dim_\mathcal{X}(X)$

(c) If S-$dim_\mathcal{X}(X'') > 0$ and $X \in Per(S)$, then S-$dim_\mathcal{X}(X') = S$-$dim_\mathcal{X}(X'') - 1$.

Moreover, if any two have finite S-dimension then so does the other.

(ii) *If $0 \to Y' \to Y \to Y'' \to 0$ is short exact in $\mathcal{Y}$, then the following holds:*

(a) T-$dim_\mathcal{Y}(Y') \leq n \Rightarrow T$-$dim_\mathcal{Y}(Y'') \leq n \Leftrightarrow T$-$dim_\mathcal{Y}(Y) \leq n$

(b) If T-$dim_\mathcal{Y}(Y'') > T$-$dim_\mathcal{Y}(Y')$ or T-$dim_\mathcal{Y}(Y) > T$-$dim_\mathcal{Y}(Y')$, then T-$dim_\mathcal{Y}(Y'') = T$-$dim_\mathcal{Y}(Y)$

(c) If T-$dim_\mathcal{Y}(Y') > 0$ and $Y \in Per(T)$, then T-$dim_\mathcal{Y}(Y'') = T$-$dim_\mathcal{Y}(Y') - 1$.

Moreover, if any two have finite T-dimension then so does the other.

We will call the collection of objects with finite S-dimension $\mathcal{S}(\mathcal{X})$ and the objects with finite T-dimension $\mathcal{T}(\mathcal{Y})$. Then if S and T are perfect $\mathcal{S}(\mathcal{X})$ and $\mathcal{T}(\mathcal{Y})$ form thick subcategories of $\mathcal{X}$ and $\mathcal{Y}$, respectively.

If we are in a perfect category pair with linkage, then from the previous results we can show that S and T dimension are preserved through even linkage when S and T are perfect.

Proposition 5.3.20. *Suppose that X is directly linked to Y by B and X' is directly linked to Y by B' where $S\text{-}dim_{\mathcal{X}}(X) = n$. Then $S\text{-}dim_{\mathcal{X}}(X') = n$.*

Proof : Suppose we are in the case $(S,T)_L$. Then by Proposition 5.3.17 and the sequences

$$0 \longrightarrow T(Y) \longrightarrow B \longrightarrow X \longrightarrow 0$$

$$0 \longrightarrow T(Y) \longrightarrow B' \longrightarrow X' \longrightarrow 0$$

we have that $S\text{-}\dim_{\mathcal{X}}(T(Y)) = n - 1$ from the first sequence and then $S\text{-}\dim_{\mathcal{X}}(X') = n$ by the second.

Suppose we are in the case $(S,T)_R$. Then by Proposition 5.3.18 and the sequences

$$0 \longrightarrow X \longrightarrow B \longrightarrow T(Y) \longrightarrow 0$$

$$0 \longrightarrow X' \longrightarrow B' \longrightarrow T(Y) \longrightarrow 0$$

we have that $S\text{-}\dim_{\mathcal{X}}(T(Y)) = n - 1$ from the first sequence and then $S\text{-}\dim_{\mathcal{X}}(X') = n$ by the second.

It is then clear by these two arguments that if we are in the case (S, T) we are done.

$\square$

Corollary 5.3.21. *Let $((\mathcal{X}, \mathcal{Y}), \mathcal{B}, -)$ be perfect where $-$ is one of (S, T), $(S, T)_L$, or $(S, T)_R$.*
Objects which are evenly linked have the same S or T-dimension.

This result is what generalizes the results which discuss the preservation of projective dimension, Gorenstein dimension, and C-Gorenstein dimension through linkage. Those are all special cases of this result and we will show other dimensions are preserved through linkage in the next chapter.

CHAPTER 6: APPLICATIONS OF LINKAGE

In this chapter we have combined together a collection of applications of the linkage theory presented in the previous chapters. We give the classical examples of ideals and module linkage in the category of R-modules over a ring R, but we also present linkage using different functors in R-Mod which will extend the theory of certain homological dimension including the C-Gorenstein dimension defined earlier and Gorenstein injective and flat dimensions.

6.1 R-Mod and C-quasi-Gorenstein R-modules

Our first example will be extending the classical ideal and module linkage theory using our intermediate C-Gorenstein dimensions. This will help us extend some results aboue module linkage that are found in [49] and [18]. Throughtout this section R will be a semiperfect ring. Semiperfect rings are products of commutative local Noetherian rings, and so we do not lose too much generality (with regards to the rest of the thesis) in this setting. One of the reasons for choosing such a setting is that in a semiperfect ring every finitely generated module has a projective cover, (see [21] Ch. 18). In fact, we will only need this hypothesis for Proposition **??** and the results following it.

We first will present how module linkage is defined according to Nagel. Given a semidualizing R-module C, an R-module Q is said to be **C-quasi-Gorenstein** if $Q \in \mathcal{G}_C^{\mathrm{grade}_R(Q)}$ and there is some isomorphism $\alpha : Q \to \mathrm{Ext}_R^{\mathrm{grade}_R(Q)}(Q, C)$. C-Quasi-Gorenstein modules, Q, have nice properties outlined in Chapter 3 and in [54], [12], and [53]. Given a C-quasi-Gorenstein module Q of grade g we will denote by $\mathrm{Epi}(Q)$ the set of all R-module homomorphisms $\varphi : Q \to M$ where $\mathrm{im}\,\varphi$ has

the same grade as Q. Given such a homomorphism φ we have a short exact sequence

$$0 \longrightarrow \ker \varphi \longrightarrow Q \longrightarrow \operatorname{im} \varphi \longrightarrow 0$$

which induces a long exact sequence

$$0 \longrightarrow \operatorname{Ext}_R^g(\operatorname{im} \varphi, C) \longrightarrow \operatorname{Ext}_R^g(Q, C) \xrightarrow{\psi} \operatorname{Ext}_R^g(\ker \varphi, C) \longrightarrow \operatorname{Ext}_R^{g+1}(\operatorname{im} \varphi, C) \longrightarrow \cdots$$

If $\alpha : C \to \operatorname{Ext}_R^g(Q, C)$ is an isomorphism, then we can construct a short exact sequence from the long sequence above as

$$0 \longrightarrow \operatorname{Ext}_R^g(\operatorname{im} \varphi, C) \longrightarrow Q \xrightarrow{L_Q(\varphi)} \operatorname{im} L_Q(\varphi) \longrightarrow 0$$

where $L_Q(\varphi) = \psi \circ \alpha$.

Definition 6.1.1. *We say that M and N are **directly linked by the** C-**quasi-Gorenstein module** Q if there are $\varphi, \psi \in Epi(Q)$ such that*

(i) $M = im\ \varphi$, $N = im\ \psi$

(ii) $M \cong im\ L_Q(\psi)$, $N \cong im\ L_Q(\varphi)$

It may not immediately be apparent that this is a generalization of the module linkage defined in [49], but if we restrict which types of modules we are allowed to be linked by then it becomes clear, see [53, Remark 3.20].

Now it is easy to ask when we would run across rings with semidualizing modules C such that there are nontrivial C-quasi-Gorenstein modules. Then the results that follow would be useful as

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they would give information about such modules that was not previously apparent.

The following construction allows one to build up a family of such rings.

Example 6.1.2. *Let k be a field and consider the ring $R = k[X,Y]/(X,Y)^2$. Then R is a local Cohen-Macaulay ring which is not Gorenstein. It is clear then that R is a free k-module. R has two semidualizing modules, R and $\omega_R = Hom_k(R,k)$. In fact, ω_R is dualizing and $\omega_R \ncong R$.*

Clearly, R is an R-quasi-Gorenstein module. It is then straightforward to see that k is a ω_R-quasi-Gorenstein module as

$$Hom_R(k, \omega_R) = Hom_R(k, Hom_k(R,k)) \cong Hom_k(k \otimes_R R, k) \cong Hom_k(k,k) \cong k$$

Therefore, one has that R is an ω_R-quasi-Gorenstein module, as R is a free k-module of rank 3. Therefore k and R are directly linked by the ω_R-quasi-Gorenstein module $k \oplus R$. However, k is not an R-quasi-Gorenstein module and so it is not easy to see if k and R are linked using some R-quasi-Gorenstein module. Thus, k and R may not be directly linked through some R-quasi-Gorenstein module, but they are through some ω_R-quasi-Gorenstein module. So we can see that the extension of linkage to semidualizing modules has given new information about linkage classes.

Now, if we take (R, m, k) as above and construct $S = R[U,V]/(U,V)^2$ (just as R is constructed above) then S is a local Cohen-Macaulay ring with residue field k. Then S has four distinct semidualizing modules S, $C_1 = Hom_R(S,R)$, $C_2 = S \otimes_R \omega_R$, and $\omega_S = Hom_R(S, \omega_R)$. In fact, ω_S is dualizing and $\omega_S \ncong S$. Once again, using S-quasi-Gorenstein modules it is not clear if S,

R, and k are directly linked to each other. However, R is a C_1-quasi-Gorenstein module as

$$Hom_S(R, C_1) = Hom_S(R, Hom_R(S, R)) \cong Hom_R(R \otimes_S S, R) \cong Hom_R(R, R) \cong R$$

and k is a ω_S-quasi-Gorenstein module as

$$Hom_S(k, \omega_S) = Hom_S(k, Hom_R(S, \omega_R)) \cong Hom_R(k \otimes_S S, \omega_R) \cong Hom_R(k, \omega_R) \cong k.$$

Then, as S is a free R-module (of rank 3) we have that S and R are directly linked by the C_1-quasi-Gorenstein module $S \oplus R$. Further, S is a free k-module (of rank 9) and so S and k are directly linked by the ω_S-quasi-Gorenstein module $S \oplus k$. As above, R and k are directly linked as well.

One could continue such a construction and obtain a ring with 2^n distinct semidualizing modules for any n. The idea to take away from this example is that using different semidualizing modules for linkage may allow one to find links between modules that may have not been there before, i.e. shift around the linkage classes of modules. So in what follows we are not only presenting new results concering linkage of modules but also presenting it in a way that concerns all such linkage classes with different semidualizing modules.

We will now show how this is an application of the theory for categorical linkage. Here we set $\mathcal{X} = \mathcal{Y} = R\text{-Mod}$ and $S = T = \text{Ext}_R^g(-, C)$ for a fixed g. When discussing R-modules we will assume their grade is g. Then $\mathcal{B}$ is the subcategory of R-Mod for with $\text{Ext}_R^g(B, C) \cong B$ and $\text{Ext}_R^i(B, C) = 0$ for $i \neq g$. These are exactly the C-quasi-Gorenstein R-modules which form the Fossum category for $\text{Ext}_R^g(-, C)$ and R-Mod. Also, the functors $F_\mathcal{X}$ and $F_\mathcal{Y}$ are the inclusion functors of $\mathcal{B}$ into R-Mod and $X_\mathcal{B} = Y_\mathcal{B}$ are the identity functors on $\mathcal{B}$.

The S and T-perfect objects then become the G_C^g-dimension zero modules over R. Further the natural map $M \to \text{Ext}_R^g(\text{Ext}_R^g(M, C), C)$ for any R-module M shows that R-Mod is a perfect category using $S = T = \text{Ext}_R^g(-, C)$. Further, $\text{Ext}_R^g(-, C)$ is a perfect functor using ideas and results from [11]. Therefore we get the following results

Corollary 6.1.3. *Suppose that $M \in \mathcal{G}_C^g$. Then $M' \in \mathcal{G}_C^g$ for any $M' \in [M]$.*

Proof : See Lemma 5.1.8.

$\square$

Corollary 6.1.4. *For any R-module we have an exact sequence*

$$0 \to Ext_R^{g+1}(D_C^g M, C) \to M \to Ext_R^g(Ext_R^g(M, C), C) \to Ext_R^{g+2}(D_C^g M, C) \to 0$$

Proof : See Proposition 5.1.11.

$\square$

Corollary 6.1.5. *Even linkage preserves G_C^g-dimension.*

Proof : See Corollary 5.3.21.

$\square$

6.1.1 Horizontal Linkage

We will follow the ideas in [49] to develop some properties of modules in $\mathcal{G}_C^g$ under linkage. We will now use Q to represent C-quasi-Gorenstein R-modules and $\mathcal{L}_Q(-) = \mathcal{L}_B^{\text{Ext}_R^g(-, C)}(-)$ in this case.

Definition 6.1.6. *Suppose that M is an R-module of grade g. We say that M is **horizontally-linked by** $\mathbf{Q}$ if $M \cong \mathcal{L}_Q^2(M) := \mathcal{L}_Q(\mathcal{L}_Q(M))$.*

Horizontal linkage is a concept explored by Martsinkovsky and Strooker in [49]. Horizontal linkage is equivalent to saying that $M \underset{Q}{\rightsquigarrow} \mathcal{L}_Q(M)$. Therefore it is clear that M is horizontally linked by some Q if and only if $||[M]|| \neq 0$. Recall by Corollary 5.1.17 that if M is not stable then we can find a stable representative of M in its linkage class, provided the stable representative has a nonempty linkage class. This is a generalization of [53, Lemma 3.11]. For this example this states the following

Corollary 6.1.7. *Suppose that M is not stable, i.e. $M \cong M' \oplus Q$ where Q is C-quasi-Gorenstein. Then M' is evenly linked to M.*

Proof : See Corollary 5.1.17 or [53, Lemma 3.11].

$\square$

Now we will combine this with the following statement, which is a special case of Proposition 5.1.15 (i).

Corollary 6.1.8. *Suppose that M is a stable R-module. Then $\mathrm{im}\,(\delta_C^g(M)) \cong \mathcal{L}_Q^2(M)$ and we have a short exact sequence*

$$0 \longrightarrow Ext_R^{g+1}(D_C^g M, C) \longrightarrow M \xrightarrow{\delta_C^g(M)} \mathcal{L}_Q^2(M) \longrightarrow 0.$$

Proof : See Proposition 5.1.15 (i).

$\square$

Proposition 6.1.9. *Suppose that M is horizontally linked by Q. Then so is $\mathcal{L}_Q(M)$.*

Proof : Since $M \cong \mathcal{L}_Q^2(M)$ we know that $\mathcal{L}_Q^2(\mathcal{L}_Q(M)) \cong \mathcal{L}_Q(\mathcal{L}_Q^2(M)) \cong \mathcal{L}_Q(M)$.

$\square$

Notice that for a stable R-module horizontal linkage is independent of the C-quasi-Gorenstein R-module Q. From now on we will say that M is horizontally linked if there is some Q such that $M \cong \mathcal{L}_Q^2(M)$. These results lead to the characterization of horizontally-linked modules

Theorem 6.1.10. *A finitely generated R-module M is horizontally-linked if and only if $Ext_R^{g+1}(D_C^g M, C) = 0$.*

Proof : Suppose that M is horizontally linked. If M is stable, then by Proposition 6.1.8 $\mathrm{Ext}_R^{g+1}(D_C^g M, C) = 0$. If M is not stable then M is evenly linked to a stable R-module M' by Corollary 6.1.7 and so by Theorem 5.1.18 $\mathrm{Ext}_R^{g+1}(D_C^g M, C) \cong \mathrm{Ext}_R^{g+1}(D_C^g M', C)$. Now as $|[M']| \neq 0$ M' is horizontally linked by some Q' and thus $\mathrm{Ext}_R^{g+1}(D_C^g M', C) = 0$.

If M is stable the converse follows by Corollary 6.1.8. If M is not stable then $M \cong M' \oplus Q'$ where Q' is C-quasi-Gorenstein and therefore $|[M']| \neq 0$ and so by Theorem 5.1.18 we have that $\mathrm{Ext}_R^{g+1}(D_C^g M', C) \cong \mathrm{Ext}_R^{g+1}(D_C^g M, C) = 0$. So by Proposition 6.1.8 we have that M' is horizontally linked by some C-quasi-Gorenstein R-module Q. Therefore M is horizontally linked by $Q \oplus Q'$.

$\square$

6.1.2   Serre-like Conditions

With this result in hand, we now turn to understanding how module linkage and G_C^g-dimension are related. We will prove Proposition 6.1.12 and Corollary 6.1.13 generalizations of Proposition 2.6 and Corollary 2.8 in [18], respectively. First, recall the Local Duality Theorem [14, Corollary 3.5.9]

Theorem 6.1.11. *Let $(R, \mathfrak{m}, k)$ be a Cohen-Macaulay local ring of dimension d with a canonical module ω_R. Then for all finitely generated R-modules M and all integers i there exists natural isomorphism*

$$H_{\mathfrak{m}}^i(M) \cong Hom_R(Ext_R^{d-i}(M, \omega_R), E_R(k)),$$

where $E_R(k)$ is the injective envelope of k.

We begin by relating the local cohomology of $M \otimes_R \omega_R$ in a Cohen-Macaulay local ring to properties of $\mathcal{L}_Q(M)$ when M is horizontally linked by Q.

Proposition 6.1.12. *Let R be a Cohen-Macaulay local ring of dimension d with canonical module ω_R and M an R-module. Suppose that M is stable and horizontally linked by Q with $\mathrm{grade}_R(M) = g$, and that $Ext_R^{g+1}(D_R^g M, \omega_R) = 0$. Then, for a positive integer $n \geq g$, the following statements are equivalent:*

(i) $\mathcal{L}_Q(M)$ *satisfies* $\tilde{S}_n^g$

(ii) $H_{\mathfrak{m}}^i(M \otimes_R \omega_R) = 0$ *for all i, $d - n + g < i < d$*

Proof : $\mathcal{L}_Q(M)$ satisfies $\tilde{S}_n^g$ if and only if $D_R^g M$ satisfies $\tilde{S}_{n-1}^g$ because $\mathcal{L}_Q(M)$ is a first $\mathcal{G}_R^g$-syzygy of $D_R^g M$ and $Ext_R^{g+1}(D_R^g M, \omega_R) = 0$. Thus by Theorem 4.2.5 statement (i) is equivalent to $D_R^g M$

being $(\omega_R)^g_{n-g-1}$-torsionless, i.e.

$$\text{Ext}_R^{g+i}(D^g_{\omega_R}(D^g_R M), \omega_R) = 0 \text{ for all } i, 1 \le i \le n - g - 1.$$

However, $D^g_{\omega_R}(D^g_R M) \cong D^g_R D^g_R M \otimes_R \omega_R$, and as M is stable $D^g_R D^g_R M \cong M$ and so $D^g_R D^g_R M \otimes_R \omega_R \cong M \otimes_R \omega_R$. Hence $\mathcal{L}_Q(M)$ satisfies $\tilde{S}^g_n$ if and only if $\text{Ext}_R^{g+i}(M \otimes_R \omega_R, \omega_R) = 0$ for all $i, 1 \le i \le n - g - 1$, which is equivalent to

$$H^i_\mathfrak{m}(M \otimes_R \omega_R) = 0 \text{ for all } i, d - n + g < i < d$$

by the Local Duality Theorem.

$\square$

Corollary 6.1.13. *Let R be a Cohen-Macaulay local ring of dimension d with canonical module ω_R and M an R-module. Suppose that M is stable and horizontally linked by Q with $\text{grade}(M) = g$, and that $\text{Ext}_R^{g+1}(D^g_R M, \omega_R) = 0$. Then, for a positive integer $n \ge g$, the following statements are equivalent:*

(i) $M \otimes_R \omega_R$ *satisfies* $\tilde{S}^g_n$

(ii) $H^i_\mathfrak{m}(\mathcal{L}_Q(M)) = 0$ *for* $d - n + g < i < d$

Proof : This is clear using Proposition 6.1.12 and Local Duality.

$\square$

Recall the Auslander class of a semidualizing module C is the collection $\mathcal{A}_C$ of R-modules M which satisfy:

(i) The natural map $M \to \mathrm{Hom}_R(C, M \otimes_R C)$ is an isomorphism.

(ii) $\mathrm{Tor}_i^R(M, C) = 0 = \mathrm{Ext}_R^i(C, M \otimes_R C)$ for all $i > 0$.

This collection was defined by Foxby, see [25], and studied by Avramov and Foxby, see [5]. With this definition, Proposition 4.2.8, and Theorem 6.1.10 we get the following generalization of [18, Corollary 2.14].

Theorem 6.1.14. *Let R be a Cohen-Macaulay ring, C a semidualizing R-module, and M a stable R-module with $\mathrm{grade}_R(M) = g$. Suppose that $n \geq g$, $M \in \mathcal{A}_C$, and $G_C^g\text{-}\dim_R(M) <_{loc} \infty$. Then the following are equivalent:*

(i) *M satisfies $\tilde{S}_n^g$*

(ii) *M is horizontally linked by some C-quasi-Gorenstein R-module Q and $\mathrm{Ext}_R^{g+i}(\mathcal{L}_Q(M), C) = 0$ for $0 < i < n - g$.*

Proof : Clear by Proposition 4.2.8 and Theorem 6.1.10.

$\square$

To summarize some of these results we give some properties that modules which are linked to C-quasi-Gorenstein R-modules have, which helps give new information concerning licci ideals.

Corollary 6.1.15. *Let $(R, \mathfrak{m}, k)$ be a local Cohen-Macaulay ring of dimension d with canonical module ω_R and M an R-module of grade g. Suppose M is directly linked to an ω_R-quasi-Gorenstein module by the module Q. Then the following hold*

(i) *$M \in \mathcal{G}_{\omega_R}^g$*

(ii) M *is horizontally linked by* Q

(iii) $depth_{R_p}(M_p) \geq depth(R_p) - g$ *for all* $p \in Spec(R)$

(iv) $H^i_{\mathfrak{m}}(M \otimes_R \omega_R) = 0$ *for* $g < i < d$.

6.2   Auslander and Bass Classes

In this section we give an application of the linkage theory for covariant functors which will give new proofs and results concerning the Auslander and Bass classes with respect to a semidualizing module.

We will be in the category R-Mod and let C be a semidualizing R-module. The two functors we will consider are $\text{Hom}_R(C, -)$ and $C \otimes_R -$. $\text{Hom}_R(C, -)$ is a left exact covariant functor and $C \otimes_R -$ is a right exact covariant functor. These two functors form a linkage functor pair for R-Mod where $\mathcal{B}$ is Fossum subcategory of R-Mod associated with these two functors in R-Mod.

Suppose that $M \in \text{Pres}(\mathcal{B})$ is directly linked to $N \in \text{Copres}(\mathcal{B})$. Then, by Lemma 5.2.4, there are short exact sequences

$$0 \to N \to Q \to C \otimes_R M \to 0$$

$$0 \to \text{Hom}_R(C, N) \to Q \to M \to 0$$

where $Q \in \mathcal{B}$ is a Fossum object. We also have short exact sequences

$$0 \to \text{Tor}^R_1(C, M) \to C \otimes_R (\text{Hom}_R(C, N)) \to N \to 0$$

$$0 \to M \to \mathrm{Hom}_R(C, C \otimes_R M) \to \mathrm{Ext}^1_R(C, N) \to 0.$$

This shows that if either M is flat or N is injective, we get certain isomorphisms that are useful. In fact the following holds for these two functors.

Proposition 6.2.1. *Let M be an R-module. The following holds*

    (i) *If M is injective then*

       (a) *There is an isomorphism $C \otimes_R (Hom_R(C, M)) \to M$.*

       (b) *$Ext^i_R(C, M) = 0$ for $i > 0$.*

       (c) *$Tor^R_i(C, Hom_R(C, M)) = 0$ for $i > 0$.*

    (ii) *If M is flat then*

       (a) *There is an isomorphism $M \to Hom_R(C, C \otimes_R M)$.*

       (b) *$Tor^R_i(C, M) = 0$ for $i > 0$.*

       (c) *$Ext^i_R(C, C \otimes_R M) = 0$ for $i > 0$.*

Proof : For a proof see [61, Lemma 2.5] and [36, Lemma 5.1]. $\qquad\qquad\square$

Therefore injective and flat R-modules are perfect modules for $\mathrm{Hom}_R(C, -)$ and $C \otimes_R -$, respectively. Given an R-module M, there are natural evaluation maps

$$\theta_M : C \otimes_R (\mathrm{Hom}_R(C, M)) \to M$$

and

$$\mu_M : M \to \mathrm{Hom}_R(C, C \otimes_R M)$$

where $\theta_M(c \otimes_R \varphi) = \varphi(c)$ and $\mu_M(m)(c) = c \otimes m$, see [62]. These natural maps show that R-Mod is a perfect category for the functors $C \otimes_R -$ and $\text{Hom}_R(C, -)$. Then we can discuss certain exact sequences and the associated homological dimensions for $C \otimes_R -$ and $\text{Hom}_R(C, -)$. First we determine what the class of perfect objects is for each of these functors. Recall the definition of the Auslander and Bass classes of C.

Definition 6.2.2. *Let M be an R-module. We say that*

    (i) *M is in the **Auslander class** with respect to C, Aus(C), if*

        (a) *$Tor_i^R(C, M) = 0$ for $i > 0$,*

        (b) *$Ext_R^i(C, C \otimes_R M) = 0$ for $i > 0$,*

        (c) *the map μ_M is an isomorphism.*

    (ii) *M is in the **Bass class** with respect to C, Bass(C), if*

        (a) *$Ext_R^i(C, M) = 0$ for $i > 0$,*

        (b) *$Tor_i^R(C, Hom_R(C, M)) = 0$ for $i > 0$,*

        (c) *the map θ_M is an isomorphism.*

Therefore the perfect objects for $\text{Hom}_R(C, -)$ are exactly the objects in Bass(C) and the perfect objects for $C \otimes_R -$ are exactly the objects in Aus(C). This leads to a new characterization of modules in the Auslander or Bass classes with respect to C.

Theorem 6.2.3. *Let M be an R-module. Then*

    (i) *$M \in Aus(C) \Leftrightarrow Tor_i^R(C, M) = 0 = Ext_R^i(C, D_{C \otimes_R -}M)$ for $i > 0$.*

    (ii) *$M \in Bass(C) \Leftrightarrow Ext_R^i(C, M) = 0 = Tor_i^R(C, D^{Hom_R(C,-)}M)$ for $i > 0$.*

We also get corresponding results about the even linkage classes using these functors and the Auslander and Bass classes with respect to C.

Proposition 6.2.4. *Given an R-module M where $M \in Aus(C)$ we have that $M' \in Aus(C)$ for any $M' \in [M]_e$ and $N \in Bass(C)$ for any $N \in [M]_o$.*

Proof : See the proof of Lemma 5.2.5.

$\square$

Proposition 6.2.5. *Let M be an R-module. Then there are exact sequences*

$$0 \longrightarrow Ext_R^1(C, D_{C \otimes_R -}M) \longrightarrow M \xrightarrow{\ \mu_M\ } Hom_R(C, C \otimes_R M) \longrightarrow Ext_R^2(C, D_{C \otimes_R -}M) \longrightarrow 0$$

$$0 \rightarrow Tor_2^R(C, D^{Hom_R(C,-)}M) \rightarrow C \otimes_R (Hom_R(C, M)) \xrightarrow{\ \theta_M\ } M \longrightarrow Tor_1^R(C, D^{Hom_R(C,-)}M) \rightarrow 0$$

Proof : See the proof of Proposition 5.2.8.

$\square$

The second sequence is [61, Proposition 3.2] where $D^{\mathrm{Hom}_R(C,-)}M$ is called the cotranspose of the R-module M. This generalizes results found in [61, 62] and sheds light on why these natural maps and semidualizing modules are important homologically.

As R-Mod is a perfect category for these two covariant functors we have associated homological dimensions. We will call these two dimension the Auslander and Bass dimensions of an R-module M, and denote them by $\mathrm{Aus}(C)$-$\dim_R(M)$ and $\mathrm{Bass}(C)$-$\dim_R(M)$, respectively. Then if these

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dimensions are finite they are equal to

$$\text{Aus}(C)\text{-dim}_R(M) = \sup\{i : \text{Tor}_i^R(C, M) \neq 0\}$$

$$\text{Bass}(C)\text{-dim}_R(M) = \sup\{i : \text{Ext}_R^i(C, M) \neq 0\}$$

by Theorem 5.3.15. Since every flat module is in the Auslander class with respect to C and every injective modules is in the Bass class with respect to C we have the following immediately.

Proposition 6.2.6. *For any R-module M we have the following inequalities*

$$\text{Aus}(C)\text{-dim}_R(M) \leq \text{fd}_R(M)$$

$$\text{Bass}(C)\text{-dim}_R(M) \leq \text{id}_R(M)$$

In fact, we see that using [36, Theorem 1] and [36, Lemma 5.1] that every R-module of the form $C \otimes_R N$ where N is flat is flat and every R-module of the form $\text{Hom}_R(C, N)$ where N is injective is injective. Such R-modules are call C-flat and C-injective and they form the basis of resolutions used to define Gorenstein flat and Gorenstein injective dimension, see [15, 20, 36, 61, 62]. These also fall into the Auslander and Bass classes and give us the following inequalities

Theorem 6.2.7. *For any R-module M we have the following inequalities*

$$\text{Aus}(C)\text{-dim}_R(M) \leq \text{Gfd}_R(M) \leq \text{fd}_R(M)$$

$$\text{Bass}(C)\text{-dim}_R(M) \leq \text{Gid}_R(M) \leq \text{id}_R(M)$$

Note that since we can iteratively construct resolutions of these modules using flat and injective modules we have that $C \otimes_R -$ and $\text{Hom}_R(C, -)$ are perfect functors in R-Mod. This gives the following result.

Corollary 6.2.8. *For any R-module, the Auslander dimension and the Bass dimension with respect to a semidualizing R-module C are preserved through even linkage using the functors $C \otimes_R -$ and $Hom_R(C, -)$.*

Proof : See the proof of Corollary 5.3.21.

6.3 Local Homology and Cohomology

We begin this section by defining a few concepts dual to regular sequences, depth, and dimension.

Let $(R, \mathfrak{m}, k)$ be a local ring and M an R-module. We define the **Noetherian dimension**, or $\mathrm{Ndim}_R(M)$, of M in the following manner. If $M = 0$ then $\mathrm{Ndim}_R(M) = -1$, and for $M \neq 0$ the Noetherian dimension $\mathrm{Ndim}_R(M)$ is the least integer i such that $(0 :_M (x_1, \ldots, x_r)R)$ has finite length for some $x_1, \ldots, x_r \in \mathfrak{m}$. Next, we say that $x \in R$ is a **coregular element** an R-module M if $xM = M$, i.e. there is a short exact sequence

$$0 \to (0 :_M xR) \to M \xrightarrow{\cdot x} M \to 0.$$

Then we say that a sequence $x_1, x_2, \ldots, x_n$ is a **coregular sequence** on M, or M-coregular sequence, if the mapping

$$(0 :_M (x_1, \ldots, x_{i-1})R) \xrightarrow{\cdot x_i} (0 :_M (x_1, \ldots, x_{i-1})R)$$

is surjective for $1 \leq i \leq n$. Then we define the **width** of M, $\mathrm{width}_R(M)$, as the length of a maximal M-coregular sequence in $\mathfrak{m}$. It follows that for any Artinian R-module M we have that

width$_R(M) \leq$ Ndim$_R(M) < \infty$. For more information concerning Noetherian dimension, coregular sequences, and width see [27, 56]

In the rest of this section we let $(R, \mathfrak{m}, k)$ be a Noetherian local ring which is the homomorphic image of a Gorenstein ring. This allows us to guarantee the existence of certain resolutions which will be expanded upon later.

Given an ideal I in R the section functor $\Gamma_I(-)$ between R-modules is a covariant left exact functor which takes a module M to the module $\Gamma_I(M) = \varinjlim \operatorname{Hom}_R(R/I^n, M)$. Its right derived functors are called the local cohomology modules of M and are denoted by $H_I^i(M) = \varinjlim \operatorname{Ext}_R^i(R/I^n, M)$. There is an enormous amount of literature concerning the local cohomology of modules and schemes. A few appropriate references are [1, 13, 19, 31, 33, 55].

Similarly one can define the I-adic completion using the functor $\Lambda_I(-) = \varprojlim R/I^n \otimes_R -$. This is a right exact covariant functor. In this case however, we end up with a module $\Lambda_I(M)$ over the completion of R, $\hat{R}$, with respect to the ideal I. The left derived functors of this functor are called the local homology modules of a module M and are denoted by $H_i^I(M) = \varprojlim \operatorname{Tor}_i^R(R/I^n, M)$. Local homology has not had as much of a focus as local cohomology because the duality theorems concerning local homology are a recent discovery. For more information on local homology see [1, 17, 30, 33, 34, 50]

So, over a Noetherian ring we set $I = \mathfrak{m}$ and then we can consider the functors $H_\mathfrak{m}^i(-)$ and $H_i^\mathfrak{m}(-)$ for $i \geq 0$. If we compose $H_i^\mathfrak{m}(-)$ with the forgetful functor $U : \hat{R}$-Mod$\to R$-Mod, then $H_\mathfrak{m}^i(-)$ and $U \circ H_i^\mathfrak{m}(-)$ form a linkage functor pair for $i \geq 0$. As neither of the vanishing of these functors

or their derived functors depends upon the completeness of R, we will consider R to be a complete Noetherian ring and forget the use of U.

In this situation, $\mathcal{X}$ will be the Noetherian R-modules, which we denote by $\mathcal{N}_R$ and $\mathcal{Y}$ will be the Artinian R-modules which we denote by $\mathcal{A}_R$. Then $\mathcal{B} = \{M \in \mathcal{N}_R \cap \mathcal{A}_R : H^i_{\mathfrak{m}}(M) \cong M \cong H^{\mathfrak{m}}_i(M)\}$. Then we can perform linkage using local cohomology and local homology.

Suppose that $M \in \mathrm{Pres}(\mathcal{B}) \cap \mathcal{N}_R$ is directly linked to $N \in \mathrm{Copres}(\mathcal{B}) \cap \mathcal{A}_R$. Then, by Lemma 5.2.4, there are short exact sequences

$$0 \to N \to Q \to H^{\mathfrak{m}}_i(M) \to 0$$

$$0 \to H^i_{\mathfrak{m}}(N) \to Q \to M \to 0$$

in $\mathcal{A}_R$ and $\mathcal{N}_R$, respectively, where $Q \in \mathcal{B}$ is a Fossum object. There are also short exact sequences

$$0 \to H^{\mathfrak{m}}_{i+1}(M) \to H^{\mathfrak{m}}_i(H^i_{\mathfrak{m}}(N)) \to N \to 0$$

$$0 \to M \to H^i_{\mathfrak{m}}(H^{\mathfrak{m}}_i(M)) \to H^{i+1}_{\mathfrak{m}}(N) \to 0$$

Recently, there has been a focus on the connection between local cohomology and local homology. In fact, $H^i_{\mathfrak{m}}(-)$ and $H^{\mathfrak{m}}_i(-)$ form an adjoint pair of functors for each $i \geq 0$ [1]. Further, if we assume that R is a homorphic image of a Gorenstein ring, then we will show that $\mathcal{A}_R = \mathrm{Pres}(\mathrm{Per}(H^{\mathfrak{m}}_i(-)))$ and $\mathcal{N}_R = \mathrm{Copres}(\mathrm{Per}(H^i_{\mathfrak{m}}(-)))$ by determining exactly which R-modules are perfect. Thus, $\mathcal{N}_R$ is a perfect $H^i_{\mathfrak{m}}(-)$-category and $\mathcal{A}_R$ is a perfect $H^{\mathfrak{m}}_i(-)$-category. We will now find out what the perfect objects in this scenario are. We have the following results

concerning Noetherian and Artinian R-modules.

Proposition 6.3.1. *Let $(R, \mathfrak{m}, k)$ be a local ring and M a Noetherian R-module. Then $H^i_{\mathfrak{m}}(M) = 0$ if $i < depth(M)$ or $i > \dim(M)$ and $H^i_{\mathfrak{m}}(M) \neq 0$ if $i = depth(M)$ or $i = \dim(M)$.*

Proposition 6.3.2. *Let $(R, \mathfrak{m}, K)$ be a local ring and M an Artinian R-module. Then $H_i^{\mathfrak{m}}(M) = 0$ if $i < width(M)$ or $i > Ndim(M)$ and $H_i^{\mathfrak{m}}(M) \neq 0$ if $i = width(M)$ or $i = Ndim(M)$.*

Recall that a Noetherian R-module M is Cohen-Macaulay if $depth(M) = \dim(M)$. We also define an Artinian R-module to be **co-Cohen-Macaulay** if $width(M) = Ndim(M)$. These two classes of R-modules form the collection of perfect objects for $H^i_{\mathfrak{m}}(-)$ and $H_i^{\mathfrak{m}}(-)$ by Hellus [33] which states the following:

Proposition 6.3.3. *Let $(R, \mathfrak{m}, k)$ be a complete Noetherian local ring. Then*

(i) *If M is a Noetherian Cohen-Macaulay R-module with $depth(M) = d$, then $H^d_{\mathfrak{m}}(M)$ is an Artinian co-Cohen-Macaulay R-module with width d.*

(ii) *If M is an Artinian co-Cohen-Macaulay R-module with $width(M) = d$, then $H_d^{\mathfrak{m}}(M)$ is a Noetherian Cohen-Macaulay R-module with depth d.*

(iii) *If M is a Noetherian Cohen-Macaulay R-module with $depth(M) = d$, then*

$$H_d^{\mathfrak{m}}(H^d_{\mathfrak{m}}(M)) \cong M.$$

(iv) *If M is an Artinian co-Cohen-Macaulay R-module with $width(M) = d$, then*

$$H^d_{\mathfrak{m}}(H_d^{\mathfrak{m}}(M)) \cong M.$$

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Therefore in this situation the perfect objects are the Cohen-Macaulay and co-Cohen-Macaulay R-modules. So we only need to check for the vanishing of $H_{\mathfrak{m}}^i(-)$ or $H_i^{\mathfrak{m}}(-)$ to see if an object is perfect. Thus through linkage using local cohomology and homology we see the following:

Proposition 6.3.4. *Let $(R, \mathfrak{m}, k)$ be a complete Noetherian ring. Given a Cohen-Macaulay R-module M we have that M' is Cohen-Macaulay for any $M' \in [M]_e$ and N is co-Cohen-Macaulay for any $N \in [M]_o$.*

Proof : See the proof of Lemma 5.2.5.

$\square$

Recall that the dual of an object is found by using a perfect presentation, and if we have an R-module we can copresent it by a Cohen-Macaulay R-module if the ring is a homorphic image of a Gorenstein ring, see [4]. Given an R-module M of depth d we can find a minimal Cohen-Macaulay copresentation

$$0 \to M \to C_0 \to C_1$$

which leads to the exact sequence

$$0 \to H_{\mathfrak{m}}^d(M) \to H_{\mathfrak{m}}^d(C_0) \to H_{\mathfrak{m}}^d(C_1) \to D^{H_{\mathfrak{m}}^d(-)}(M) \to 0.$$

Similarly we can find the dual using local homology. In fact, using the duals of these modules constructed by the local cohomology and local homology we get the following exact sequences

Proposition 6.3.5. *Let $(R, \mathfrak{m}, k)$ be a complete Noetherian local ring. For an R-module M we have the following:*

(i) *If M is Noetherian then there is an exact sequence*

$$0 \longrightarrow H^{\mathfrak{m}}_{i+2}(D^{H^i_{\mathfrak{m}}(-)}(M)) \longrightarrow H^{\mathfrak{m}}_i(H^i_{\mathfrak{m}}(M)) \longrightarrow M \longrightarrow H^{\mathfrak{m}}_{i+1}(D^{H^i_{\mathfrak{m}}(-)}(M)) \longrightarrow 0$$

(ii) *If M is Artinian then there is an exact sequence*

$$0 \longrightarrow H^{i+1}_{\mathfrak{m}}(D_{H^{\mathfrak{m}}_i(-)}(M)) \longrightarrow M \longrightarrow H^i_{\mathfrak{m}}(H^{\mathfrak{m}}_i(M)) \longrightarrow H^{i+2}_{\mathfrak{m}}(D_{H^{\mathfrak{m}}_i(-)}(M)) \longrightarrow 0.$$

Proof : See the proof of Proposition 5.2.8.

$\square$

This leads to another characterization of Cohen-Macaulay and co-Cohen-Macaulay R-modules.

Theorem 6.3.6. *Let $(R, \mathfrak{m}, k)$ be a complete Noetherian local ring and M be an R-module.*

(i) *Suppose $depth(M) = d$. Then M is Cohen-Macaulay if and only if $H^{\mathfrak{m}}_i(D^{H^i_{\mathfrak{m}}(-)}(M)) = 0$ for all $i \neq d$.*

(ii) *Suppose $width(M) = d$. Then M is co-Cohen-Macaulay if and only if $H^i_{\mathfrak{m}}(D_{H^{\mathfrak{m}}_i(-)}(M)) = 0$ for all $i \neq d$.*

In other words, an R-module is Cohen-Macaulay (co-Cohen-Macaulay) if and only if its dual through local cohomology (homology) is co-Cohen-Macaulay (Cohen-Macaulay).

Since we are in a perfect category pair with linkage, we have certain homological dimensions that can be define. We can find a coresolution of a Noetherian R-module M by Cohen-Macaulay

R-modules of depth i such as

$$0 \to M \to C_0 \to C_1 \to C_2 \to \cdots,$$

and define a $H_{\mathfrak{m}}^i(-)$-dimension, denoted by $H_{\mathfrak{m}}^i$-$\dim_R(M)$. For $i = 0$ we will denote $H_{\mathfrak{m}}^0$-$\dim_R(M)$ by $\Gamma_{\mathfrak{m}}$-$\dim_R(M)$.

Similarly we can find a resolution of an Artinian R-module M by co-Cohen-Macaulay R-modules of width i such as

$$\cdots \to N_2 \to N_1 \to N_0 \to M \to 0,$$

and define a $H_i^{\mathfrak{m}}(-)$-dimension, denoted by $H_i^{\mathfrak{m}}$-$\dim_R(M)$. For $i = 0$ we will denote $H_0^{\mathfrak{m}}$-$\dim_R(M)$ by $\Lambda_{\mathfrak{m}}$-$\dim_R(M)$. Therefore, for each $i \geq 0$ we have homological dimensions associated to local cohomology and local homology. We must be careful in knowing which dimensions we can consider for certain modules. Given an R-module M, if it is Noetherian then the $H_{\mathfrak{m}}^i$-$\dim_R(M)$ is defined when $i \leq \operatorname{depth}_R(M)$ and if it is Artinian then the $H_i^{\mathfrak{m}}$-$\dim_R(M)$ is defined when $i \leq \operatorname{width}_R(M)$. In these cases we get the following result:

Theorem 6.3.7. *Let $(R, \mathfrak{m}, k)$ be a complete Noetherian local ring. For an R-module M we have the following*

$$H_{\mathfrak{m}}^j\text{-}dim_R(M) = \sup\{i : H_{\mathfrak{m}}^i(M) \neq 0\} - j$$

$$H_j^{\mathfrak{m}}\text{-}dim_R(M) = \sup\{i : H_i^{\mathfrak{m}}(M) \neq 0\} - j$$

provided that these dimensions are finite.

Proof : See the proof of Theorems 5.3.5 and 4.1.9.

$\square$

We immediately get the following corollary:

Corollary 6.3.8. *Let $(R, \mathfrak{m}, k)$ be a complete Noetherian local ring. Given an R-module M we have the following:*

(i) *If $\Gamma_\mathfrak{m}$-$dim_R(M) < \infty$ then $\Gamma_\mathfrak{m}$-$dim_R(M) = \dim_R(M)$.*

(ii) *If $\Lambda_\mathfrak{m}$-$dim_R(M) < \infty$ then $\Lambda_\mathfrak{m}$-$dim_R(M) = Ndim_R(M)$.*

Proof : Follows by Theorem 6.3.7 and Propositions 6.3.1 and 6.3.2.

$\square$

These results can give us information about the ring R. Since it is Noetherian we have that $\operatorname{depth}(R) \leq \dim(R) < \infty$. This leads us to the following characterization of Cohen-Macaulay rings which are homomorphic images of Gorenstein rings.

Corollary 6.3.9. *Let $(R, \mathfrak{m}, k)$ be a complete Noetherian local ring which is the homomorphic image of a Gorenstein ring. Let $d = depth_R(R)$. Then the following are equivalent:*

(i) *R is Cohen-Macaulay*

(ii) *R is evenly linked to a Cohen-Macaulay R-module*

(iii) *R is oddly linked to a co-Cohen-Macaulay R-module*

(iv) *$\Gamma_\mathfrak{m}$-$dim_R(R) = d$, i.e. there is an exact sequence*

$$0 \to R \to C_0 \to C_1 \to C_2 \to \cdots \to C_d \to 0$$

where C_i is Cohen-Macaulay of depth 0 for $0 \leq i \leq d$

(v) $H_\mathfrak{m}^d(R)$ *is a co-Cohen-Macaulay module*

(vi) $\Lambda_\mathfrak{m}$-*$dim_R(H_\mathfrak{m}^d(R)) = d$, i.e. there is an exact sequence*

$$0 \to N_d \to \cdots \to N_2 \to N_1 \to N_0 \to H_\mathfrak{m}^d(R) \to 0$$

where N_i is co-Cohen-Macaulay of width 0 for $0 \le i \le d$.

LIST OF REFERENCES

[1] Leovigildo Alonso Tarrí o, Ana Jeremí as López, and Joseph Lipman, *Local homology and cohomology on schemes*, Ann. Sci. École Norm. Sup. (4) **30** (1997), no. 1, 1–39. MR 1422312

[2] Paolo Aluffi, *Algebra: chapter 0*, Graduate Studies in Mathematics, vol. 104, American Mathematical Society, Providence, RI, 2009. MR 2527940

[3] Maurice Auslander and Mark Bridger, *Stable Module Theory*, Memoirs of the American Mathematical Society, No. 94, American Mathematical Society, Providence, RI, 1969. MR 0269685 (42 #4580)

[4] Maurice Auslander and Ragnar-Olaf Buchweitz, *The homological theory of maximal Cohen-Macaulay approximations*, Mém. Soc. Math. France (N.S.) (1989), no. 38, 5–37, Colloque en l'honneur de Pierre Samuel (Orsay, 1987). MR 1044344

[5] Luchezar L. Avramov and Hans-Bjørn Foxby, *Ring homomorphisms and finite Gorenstein dimension*, Proc. London Math. Soc. (3) **75** (1997), no. 2, 241–270. MR 1455856

[6] Hyman Bass, *On the ubiquity of Gorenstein rings*, Math. Z. **82** (1963), 8–28. MR 0153708

[7] Andrew Berget, Andrew Manion, Molly Maxwell, Aaron Potechin, and Victor Reiner, *The critical group of a line graph*, Ann. Comb. **16** (2012), no. 3, 449–488. MR 2960015

[8] Christian Böhning, *L. Szpiro's conjecture on Gorenstein algebras in codimension 2*, J. Algebra **288** (2005), no. 2, 545–555. MR 2146144

[9] _____, *Gorenstein algebras in codimension 2 and Koszul modules*, Comm. Algebra **36** (2008), no. 6, 2014–2022. MR 2418373

[10] Francis Borceux and Dominique Bourn, *Mal'cev, protomodular, homological and semi-abelian categories*, Mathematics and its Applications, vol. 566, Kluwer Academic Publishers, Dordrecht, 2004. MR 2044291

[11] Joseph P. Brennan and Alexander York, *Linkage and Intermediate C-Gorenstein Dimensions*, Unpublished.

[12] Joseph P. Brennan and Alexander York, *An extension of a theorem of frobenius and stickelberger to modules of projective dimension one over a factorial domain*, Journal of Pure and Applied Algebra (2018).

[13] M. P. Brodmann and R. Y. Sharp, *Local cohomology*, second ed., Cambridge Studies in Advanced Mathematics, vol. 136, Cambridge University Press, Cambridge, 2013, An algebraic introduction with geometric applications. MR 3014449

[14] Winfried Bruns and Jürgen Herzog, *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics, vol. 39, Cambridge University Press, Cambridge, 1993. MR 1251956

[15] Lars Winther Christensen, *Gorenstein dimensions*, Lecture Notes in Mathematics, vol. 1747, Springer-Verlag, Berlin, 2000. MR 1799866

[16] Lars Winther Christensen and Srikanth Iyengar, *Gorenstein dimension of modules over homomorphisms*, J. Pure Appl. Algebra **208** (2007), no. 1, 177–188. MR 2269838

[17] Nguyen Tu Cuong and Tran Tuan Nam, *The I-adic completion and local homology for Artinian modules*, Math. Proc. Cambridge Philos. Soc. **131** (2001), no. 1, 61–72. MR 1833074

[18] Mohammad T. Dibaei and Arash Sadeghi, *Linkage of modules and the Serre conditions*, J. Pure Appl. Algebra **219** (2015), no. 10, 4458–4478. MR 3346501

[19] David Eisenbud, *Commutative algebra*, Springer-Verlag, New York, 1995. MR 1322960 (97a:13001)

[20] Edgar E. Enochs and Overtoun M. G. Jenda, *Gorenstein injective and flat dimensions*, Math. Japon. **44** (1996), no. 2, 261–268. MR 1416263

[21] Carl Faith, *Algebra. II*, Springer-Verlag, Berlin-New York, 1976, Ring theory, Grundlehren der Mathematischen Wissenschaften, No. 191. MR 0427349

[22] Hubert Flenner, Bernd Ulrich, and Wolfgang Vogel, *On limits of joins of maximal dimension*, Math. Ann. **308** (1997), no. 2, 291–318. MR 1464904

[23] Robert Fossum, *Duality over Gorenstein rings*, Math. Scand. **26** (1970), 165–176. MR 0263806 (41 #8406)

[24] Hans-Bjørn Foxby, *Gorenstein modules and related modules*, Math. Scand. **31** (1972), 267–284 (1973). MR 0327752

[25] _____ , *Quasi-perfect modules over Cohen-Macaulay rings*, Math. Nachr. **66** (1975), 103–110. MR 0376663

[26] Georg Frobenius and Ludwig Stickelberger, *Ueber Gruppen von vertauschbaren Elementen.*, J. Reine Angew. Math. **86** (1878), 217–262. MR JFM 10.0075.03

[27] J. R. Garcí a Rozas, Inmaculada López, and Luis Oyonarte, *Relative attached primes and coregular sequences*, Taiwanese J. Math. **17** (2013), no. 3, 1095–1114. MR 3072278

[28] Evgenii S. Golod, *G-dimension and generalized perfect ideals*, Trudy Mat. Inst. Steklov. **165** (1984), 62–66, Algebraic geometry and its applications. MR 752933

[29] Michele Grassi, *Koszul modules and Gorenstein algebras*, J. Algebra **180** (1996), no. 3, 918–953. MR 1379218

[30] J. P. C. Greenlees and J. P. May, *Derived functors of I-adic completion and local homology*, J. Algebra **149** (1992), no. 2, 438–453. MR 1172439

[31] Robin Hartshorne, *Local cohomology*, A seminar given by A. Grothendieck, Harvard University, Fall, vol. 1961, Springer-Verlag, Berlin-New York, 1967. MR 0224620

[32] ——, *Liaison with Cohen-Macaulay modules*, Rend. Semin. Mat. Univ. Politec. Torino **64** (2006), no. 4, 419–432. MR 2295448

[33] Michael Hellus, *Local Cohomology and Matlis Duality*, eprint arXiv:math/0703124v1, March 2007.

[34] Michael Hellus, *Local homology, Cohen-Macaulayness and Cohen-Macaulayfications*, Algebra Colloq. **15** (2008), no. 1, 63–68. MR 2371578

[35] Jürgen Herzog and Dorin Popescu, *Finite filtrations of modules and shellable multicomplexes*, Manuscripta Math. **121** (2006), no. 3, 385–410. MR 2267659

[36] Henrik Holm and Diana White, *Foxby equivalence over associative rings*, J. Math. Kyoto Univ. **47** (2007), no. 4, 781–808. MR 2413065

[37] Brent Holmes, *A Generalized Serre's Condition*, arXiv:1710.02631v1, October 2017.

[38] Livia Hummel and Thomas Marley, *The Auslander-Bridger formula and the Gorenstein property for coherent rings*, J. Commut. Algebra **1** (2009), no. 2, 283–314. MR 2504937

[39] Craig Huneke and Bernd Ulrich, *The structure of linkage*, Ann. of Math. (2) **126** (1987), no. 2, 277–334. MR 908149

[40] ——, *Algebraic linkage*, Duke Math. J. **56** (1988), no. 3, 415–429. MR 948528

[41] Nathan Jacobson, *Basic algebra. I*, W. H. Freeman and Co., San Francisco, Calif., 1974. MR 0356989

[42] ——, *Basic algebra. II*, W. H. Freeman and Co., San Francisco, Calif., 1980. MR 571884

[43] Steven Kleiman and Bernd Ulrich, *Gorenstein algebras, symmetric matrices, self-linked ideals, and symbolic powers*, Trans. Amer. Math. Soc. **349** (1997), no. 12, 4973–5000. MR 1422609

[44] Andrew R. Kustin, Matthew Miller, and Bernd Ulrich, *Linkage theory for algebras with pure resolutions*, J. Algebra **102** (1986), no. 1, 199–228. MR 853240

[45] Lionel Levine, *The sandpile group of a tree*, European J. Combin. **30** (2009), no. 4, 1026–1035. MR 2504661 (2010m:05145)

[46] Dino Lorenzini, *Smith normal form and laplacians*, J. Combin. Theory Ser. B **98** (2008), no. 6, 1271–1300. MR 2462319 (2010d:05092)

[47] Dino J. Lorenzini, *A finite group attached to the laplacian of a graph*, Discrete Math. **91** (1991), no. 3, 277–282. MR 1129991 (93a:05091)

[48] Heath M. Martin, *Linkage and the generic homology of modules*, Comm. Algebra **28** (2000), no. 9, 4285–4301. MR 1772507

[49] Alex Martsinkovsky and Jan R. Strooker, *Linkage of modules*, J. Algebra **271** (2004), no. 2, 587–626. MR 2025542

[50] Eben Matlis, *The Koszul complex and duality*, Comm. Algebra **1** (1974), 87–144. MR 0344241

[51] Hideyuki Matsumura, *Commutative ring theory*, second ed., Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, Cambridge, 1989, Translated from the Japanese by M. Reid. MR 1011461

[52] Juan C. Migliore, *Introduction to liaison theory and deficiency modules*, Progress in Mathematics, vol. 165, Birkhäuser Boston, Inc., Boston, MA, 1998. MR 1712469

[53] Uwe Nagel, *Liaison classes of modules*, J. Algebra **284** (2005), no. 1, 236–272. MR 2115013

[54] Uwe Nagel and Stephen Sturgeon, *Combinatorial interpretations of some Boij-Söderberg decompositions*, J. Algebra **381** (2013), 54–72. MR 3030509

[55] Tran Tuan Nam, *Some properties of local homology and local cohomology modules*, Studia Sci. Math. Hungar. **50** (2013), no. 1, 129–141. MR 3181478

[56] Akira Ooishi, *Matlis duality and the width of a module*, Hiroshima Math. J. **6** (1976), no. 3, 573–587. MR 0422243

[57] Christian Peskine and Lucien Szpiro, *Liaison des variétés algébriques. I*, Invent. Math. **26** (1974), 271–302. MR 0364271

[58] Sean Sather-Wagstaff, Tirdad Sharif, and Diana White, *Comparison of relative cohomology theories with respect to semidualizing modules*, Math. Z. **264** (2010), no. 3, 571–600. MR 2591820

[59] Sean Sather-Wagstaff and Siamak Yassemi, *Modules of finite homological dimension with respect to a semidualizing module*, Arch. Math. (Basel) **93** (2009), no. 2, 111–121. MR 2534418

[60] Peter Schenzel, *Notes on liaison and duality*, J. Math. Kyoto Univ. **22** (1982/83), no. 3, 485–498. MR 674606

[61] Xi Tang and Zhaoyong Huang, *Homological aspects of the dual Auslander transpose*, Forum Math. **27** (2015), no. 6, 3717–3743. MR 3420357

[62] _____, *Homological aspects of the dual Auslander transpose, II*, Kyoto J. Math. **57** (2017), no. 1, 17–53. MR 3621778

[63] Charles A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994. MR 1269324