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VARIATIONAL INCLUSIONS WITH GENERAL OVER-RELAXED PROXIMAL POINT
AND VARIATIONAL-LIKE INEQUALITIES WITH DENSELY PSEUDOMONOTONICITY

by

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A dissertation submitted in partial fulfilment of the requirements
for the degree of Doctor of Philosophy
in the Department of Mathematics
in the College of Sciences
at the University of Central Florida
Orlando, Florida

Summer Term
2019

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ABSTRACT

This dissertation focuses on the existence and uniqueness of the solutions of variational inclusion and variational inequality problems and then attempts to develop efficient algorithms to estimate numerical solutions for the problems. The dissertation consists a total of five chapters. Chapter 1 is an introduction to variational inequality problems, variational inclusion problems, monotone operators, and some basic definitions and preliminaries from convex analysis. Chapter 2 is a study of a general class of nonlinear implicit inclusion problems. The objective of this study is to explore how to omit the Lipschitz continuity condition by using an alternating approach to the proximal point algorithm to estimate the numerical solution of the implicit inclusion problems. In chapter 3 we introduce generalized densely relaxed $\eta - \alpha$ pseudomonotone operators and generalized relaxed $\eta - \alpha$ proper quasimonotone operators as well as relaxed $\eta - \alpha$ quasimonotone operators. Using these generalized monotonicity notions, we establish the existence results for the generalized variational-like inequality in the general setting of Banach spaces. In chapter 4, we use the auxiliary principle technique to introduce a general algorithm for solutions of the densely relaxed pseudomonotone variational-like inequalities. Chapter 5 is the chapter concluding remarks and scope for future work.

To my wife Scarlet and my son Michael with love :)

ACKNOWLEDGMENTS

It is my privilege and pleasure to express my gratitude to my supervisor Dr. Ram Narayan Mohapatra, whose guidance with patience and generosity enlightened me in my research work and in writing of this dissertation.

I am deeply thankful to Professor Ouayl Chadli, Ibn Zohr University, Morocco, and Dr. Gayatri Pany, Singapore University of Technology and Design, for their precious time spent on reading, making comments and suggestions, and correcting the papers during my work on the researches and also in preparation of this dissertation.

I also would like to express my special gratefulness to Bijaya Kumar Sahu, India, for his important ideas and contributions on the second research paper. The research paper would not be possible without his work.

Finally, I owe my sincere thanks to my colleagues, my friends, and especially my family for their anonymous and unconditional supports. I would not have come to this stage without their helps and encouragements. From the bottom of my heart, I sincerely thank all of you.

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CHAPTER 1: INTRODUCTION

1.1 Introduction to the Variational Inequality Problems

In 1964, Stampacchia ([38]) introduced the classical variational inequality problem. Since then, variational inequalities have been the research of many authors such as Cohen ([16]), Blum and Oetti ([10]), Ding ([19]), Verma ([47]), Bai et al. ([6], [7]). Variational inequalities have a wide variety of problems in many fields such as mathematical programming, mechanics, physics, optimization and control theory, elasticity theory, economics and transportation equilibrium problems, game theory, and engineering sciences.

Basically, a variational inequality problem is a problem of solving inequalities of a functional in which all possible values of a given variable usually belong to a convex subset of topological vector spaces such as Hilbert spaces and Banach spaces. The scope of solving variational inequalities involves two aspects, that is, proving the existence and uniqueness of the solution of the problem and then develop numerical methods to approximate the solutions.

The definitions of variational inequality are distinguishable between Hilbert spaces and Banach spaces.

Definition 1.1.1 *Let X be a Hilbert space, and let K be a nonempty closed convex subset of X .*

Then the variational inequality problem defined on X is as follow:

Find $x \in K$ such that

$$\langle f(x), y - x \rangle \geq 0, \tag{1.1.1}$$

for all $y \in K$ and $f \in X$ where $\langle \cdot, \cdot \rangle$ stands for the inner product in Hilbert space.

Definition 1.1.2 *Let X be a real reflexive Banach space.*

Let X^* be its dual, and let K be a nonempty closed convex subset of X .

Let $T : X \rightarrow X^*$ be an operator. Then the variational inequality problem defined on X can be mentioned as the following:

Find $x \in K$ such that

$$\langle Tx, y - x \rangle \geq 0, \quad (1.1.2)$$

for all $y \in K$.

Example 1.1.3 Minimum problem over a closed bounded interval in \mathbb{R} .

Let $f : [a, b] \rightarrow \mathbb{R}$ be a real-valued function, then

$$f(x_0) = \min f(x) \Rightarrow f'(x_0)(x - x_0) \geq 0, \quad \forall x \in [a, b].$$

Example 1.1.4 Minimum problem over a closed convex set in \mathbb{R}^n .

Let $K \subset \mathbb{R}^n$ be a closed convex subset and $f : K \rightarrow \mathbb{R}$ be a real-valued function, then

$$f(x_0) = \min f(x) \Rightarrow (\nabla f(x_0))^T (x - x_0) \geq 0, \quad \forall x \in K.$$

The definition of variational inequalities defined on Hilbert spaces X can be interpreted geometrically in the following manner: find a point $x^* \in K$ such that for any point $x \in K$, the angle between $(x - x^*)$ and $f(x^*)$ is an acute angle as shown in figure 1.1.

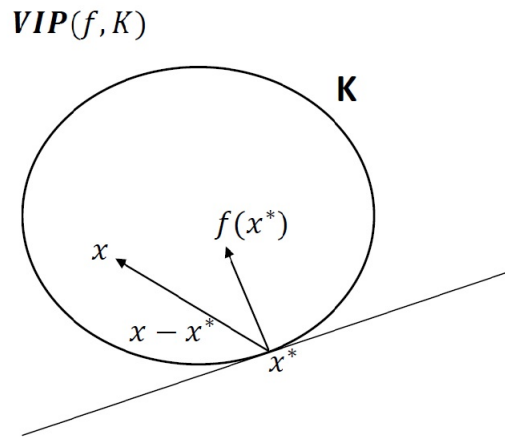


Figure 1.1: Geometrical Interpretation of Variational Inequality

In general, on Hilbert spaces the variational inequalities are generated in terms of bilinear forms as the following: given a function $f \in H$, find a point $x^* \in X$ such that

$$a(x^*, x) = (f, x),$$

where $a : X \times X \rightarrow \mathbb{R}$ is continuous and linear in both arguments.

1.2 Introduction to the Variational Inclusion Problems

Variational inclusion problem is one of the generalizations of interest and importance of variational inequality problem. In order to obtain an efficient and implementable algorithm to solve for such problem is a challenge. In 1976, Rockafellar ([34]) developed an algorithm for solving the variational inclusion problems: the proximal point algorithm. Since then, the algorithm has been used by many authors such as Agarwal ([1]), Verma ([39]-[46]), Lan ([27], Li ([29]), and the references therein.

The classical general class of inclusion problem is defined as follow:

Definition 1.2.1 *Let X be a vector space and let $M : X \rightarrow 2^X$ be a set-valued mapping on X .*

Then the variational inclusion problem is defined on X as follow: find a solution to

$$0 \in M(x), \tag{1.2.1}$$

where 2^X denotes the class of all subsets of X .

Recall that a cone in a vector space is defined as following:

Definition 1.2.2 *(Karamardian [26]) A set K in a vector space is a cone if and only if*

$$x \in K \Rightarrow \lambda x \in K \quad \forall \lambda > 0.$$

Example 1.2.3 *(Lan [27])*

Let $V : \mathbb{R}^n \rightarrow \mathbb{R}$ be a locally Lipschitz continuous real-valued function and $K \subset \mathbb{R}^n$ be a closed convex subset. If $x^ \in \mathbb{R}^n$ is a solution to the problem $\min_{x \in K} V(x)$, then*

$$0 \in \partial V(x^*) + \mathcal{N}_K(x^*),$$

where $\partial V(x^*)$ denotes the sub-differential of V at x^* , and $\mathcal{N}_K(x^*)$ is the normal cone of K at x^* .

In 1976, Rockafellar ([34]) studied the general convergence of the proximal point algorithm for solving problem (1.2.1). Rockafellar showed that, if M is a maximal monotone and for an initial point x_1 , we can construct a sequence $\{x_k\}$ that converges strongly to a solution of (1.2.1) with

$$x_{k+1} \approx P_k(x_k), \tag{1.2.2}$$

or equivalently,

$$\|x_{k+1} - P_k(x_k)\| \leq \epsilon_k, \quad (1.2.3)$$

for some $\epsilon_k \geq 0$, where $P_k = (I + c_k M)^{-1}$ is the resolvent operator and $\{c_k\}$ is a sequence of positive numbers that are bounded away from zero (i.e. $\exists \epsilon > 0$ such that $c_k \geq \epsilon \quad \forall k \in \mathbb{N}$). From (1.2.2), it follows that the approximation x_{k+1} is sufficiently accurate as the iteration proceeds and is an approximate solution to the inclusion problem

$$0 \in M(x) + \frac{1}{c_k}(x - x_k). \quad (1.2.4)$$

Since then, the notion of general maximal monotonicity developed and played a significant role in solving various types of variational problems such as variational inequality problems, minimax problems, minimization and maximization of functions, etc., which can be unified as problems of the form (1.2.1). The general maximal monotonicity provided a frame work to develop proximal point algorithms for estimating computational solutions of many variational problems mentioned above.

1.3 Monotone Operators

Monotone operators play as a key role on the study of the existence of solutions of variational inequality and variational inclusion problems. A monotone operator is generally defined as follow:

Definition 1.3.1 *Let X be a vector space and let $K \subset X$ be a non empty subset.*

Let X^ be the dual space of X .*

Let $T : K \rightarrow X^$ be an operator on K .*

Then T is said to be a monotone operator on K if for all $x, y \in K$,

$$\langle Tx - Ty, x - y \rangle \geq 0.$$

Gradually, the concept of monotonicity is generalized and extended into many other forms such as strict monotonicity, relaxed monotonicity, pseudomonotonicity, quasimonotonicity, and many other forms by various researchers.

Some basic definitions are introduced below:

Definition 1.3.2 Let X be a vector space and let $K \subset X$ be a non empty subset.

Let X^* be the dual space of X .

Let $T : K \rightarrow X^*$ be an operator on K .

Then T is said to be a

(i) strictly monotone operator on K if:

$$\langle Tx - Ty, x - y \rangle > 0$$

for all $x, y, \in K, x \neq y$;

(ii) r -strongly monotone operator on K if $\exists r > 0$:

$$\langle Tx - Ty, x - y \rangle \geq r\|x - y\|^2$$

for all $x, y, \in K$;

(iii) *pseudomonotone operator on K if :*

$$\langle Tx, y - x \rangle \geq 0 \implies \langle Ty, y - x \rangle \geq 0$$

for all $x, y, \in K$;

(iv) *quasimonotone operator on K if :*

$$\langle Tx, y - x \rangle > 0 \implies \langle Ty, y - x \rangle \geq 0$$

for all $x, y, \in K, x \neq y$.

Definition 1.3.3 (*Luc [30]*)

Let K be a convex set in X and K_0 a subset of K . The set K_0 is said to be segment-dense in K if for each $x \in K$, there exists $x_0 \in K_0$ such that x is a cluster point of the set $[x, x_0] \cap K_0$.

Definition 1.3.4 (*Definitions Bai et al. [7]*)

Let X be a vector space and let $K \subset X$ be a non empty subset.

Let X^ be the dual space of X .*

Let $T : K \rightarrow X^$ be an operator on K .*

Then T is said to be a

(i) *relaxed monotone operator on K if $\exists \mu > 0$ s.t. $\forall x, y \in K$:*

$$\langle Tx - Ty, x - y \rangle \geq -\mu \|x - y\|^2$$

for all $x, y, \in K, x \neq y$;

(ii) relaxed μ pseudomonotone operator on K if $\exists \mu > 0$ s.t. $\forall y \in K$:

$$\langle Tx, y - x \rangle \geq 0 \implies \langle Ty, y - x \rangle \geq -\mu \|y - x\|^2;$$

(iii) relaxed μ quasimonotone operator on K if $\exists \mu > 0$ s.t. $\forall y \in K$:

$$\langle Tx, y - x \rangle > 0 \implies \langle Ty, y - x \rangle \geq -\mu \|y - x\|^2;$$

(iv) densely relaxed μ pseudomonotone operator on K if there exists a segment-dense subset $K_0 \subset K$ such that T is relaxed μ pseudomonotone at every point of K_0 .

Definition 1.3.5 (Definitions Bai [6] and Arunchai [4])

Let X be a vector space and let $K \subset X$ be a non empty subset. Let X^* be the dual space of X .

Let $\eta : K \times K \rightarrow X$ be a mapping and $T : K \rightarrow X^*$ be an operator on K .

Let $\alpha : X \rightarrow \mathbb{R}$ be a real-valued function satisfying $\lim_{t \rightarrow 0^+} \sup \frac{\alpha(t\eta(x,y))}{t} = 0 \quad \forall (x, y) \in K \times K$.

Then T is said to be a

(i) strictly quasimonotone operator on K if:

$$\langle Tx, y - x \rangle > 0 \implies \langle Ty, y - x \rangle > 0$$

for all $x, y, \in K, x \neq y$;

(ii) strictly η quasimonotone operator on K if:

$$\langle Tx, \eta(y, x) \rangle > 0 \implies \langle Ty, \eta(y, x) \rangle > 0$$

for all $x, y, \in K, x \neq y$;

(iii) relaxed α pseudomonotone operator on K if $\forall x, y \in K$:

$$\langle Tx, y - x \rangle \geq 0 \implies \langle Ty, y - x \rangle \geq \alpha(y - x);$$

(iv) relaxed $\eta - \alpha$ monotone operator on K if $\forall x, y \in K$:

$$\langle Tx - Ty, \eta(x, y) \rangle \geq \alpha(\eta(x, y));$$

(v) relaxed $\eta - \alpha$ pseudomonotone operator on K if $\forall x, y \in K$:

$$\langle Tx, \eta(y, x) \rangle \geq 0 \implies \langle Ty, \eta(y, x) \rangle \geq \alpha(\eta(y, x)).$$

Next, we give simple examples of problems that can be put in the variational inequality set up.

Example 1.3.6 (Bai et al. [7])

Let $T : [0, +\infty) \rightarrow \mathbb{R}$ be a function defined by

$$T(x) = \begin{cases} x^2 - 2 & \text{if } x \geq \sqrt{3} - 1, \\ -2x & \text{if } 0 \leq x < \sqrt{3} - 1. \end{cases}$$

Then T is relaxed μ pseudomonotone with $\mu = 2$ but not pseudomonotone on $[0, +\infty)$, i.e., $T(x)$ is not pseudomonotone at $x = 0$.

Example 1.3.7 (Bai et al. [7])

Let $T : (-\infty, 0) \rightarrow (0, \infty)$ be a function defined by $T(x) = x^2$. Then T is relaxed μ pseudomonotone, but not relaxed monotone on $(-\infty, 0)$ since $\forall \delta > 0 \exists x_0 < 0, y_0 < 0$ with $x_0 + y_0 < -\delta < 0$ such that

$$\langle Tx_0 - Ty_0, x_0 - y_0 \rangle < -\delta \|x_0 - y_0\|^2.$$

So T is not relaxed monotone on $(-\infty, 0)$.

Example 1.3.8 (Luc [30])

Let $T : \mathbb{R} \rightarrow \mathbb{R}$ be a function defined by $T(x) = x^2$. Then T is quasimonotone, but not pseudomonotone on \mathbb{R} since T is not pseudomonotone at $x = 0$. Also, T is not relaxed μ pseudomonotone on \mathbb{R} .

Example 1.3.9 (Bai et al. [7])

Let $T : \mathbb{R} \rightarrow \mathbb{R}$ be a function defined by $T(x) = -x$. Then T is relaxed μ pseudomonotone, but T is neither pseudomonotone nor quasimonotone on \mathbb{R} .

Example 1.3.10 (Bai et al. [7])

Let $T : \mathbb{R} \rightarrow [-1, 1]$ be a function defined by $T(x) = \sin x$. Then T is relaxed μ quasimonotone on \mathbb{R} with $\mu = 1$, but not quasimonotone on \mathbb{R} . In fact,

$$\begin{aligned} |\sin y - \sin x| &= \left| 2 \cos \frac{y+x}{2} \sin \frac{y-x}{2} \right| \\ &\leq 2 \left| \sin \frac{y-x}{2} \right| \\ &\leq 2 \left| \frac{y-x}{2} \right| \\ &= 2|y-x|. \end{aligned}$$

Hence, if $\langle \sin x, y - x \rangle > 0$, then

$$\begin{aligned}
\langle \sin y, y - x \rangle &= \langle \sin y - \sin x + \sin x, y - x \rangle \\
&= \langle \sin y - \sin x, y - x \rangle + \langle \sin x, y - x \rangle \\
&> \langle \sin y - \sin x, y - x \rangle \\
&\geq -|\sin y - \sin x| \cdot |y - x| \\
&\geq -|y - x|^2.
\end{aligned}$$

Therefore, T is relaxed μ quasimonotone.

Example 1.3.11 (Bai et al. [6])

Let $T : \mathbb{R} \rightarrow \mathbb{R}$ be a function defined by

$$T(x) = \begin{cases} \frac{3}{2}x & \text{if } x \geq 0, \\ -\frac{1}{2}x & \text{if } x < 0, \end{cases}$$

and $\eta(x, y) = x - y$. Then T is relaxed $\eta - \alpha$ pseudomonotone with $\alpha(x) = -\frac{1}{2}x^2$, but T is not relaxed $\eta - \alpha$ monotone.

Example 1.3.12 (Arunchai [4])

Let $T : (-\infty, 0] \rightarrow [0, +\infty)$ be a function defined by $T(x) = x^2$ with $\eta(x, y) = c|x - y|$, where $c > 0$, and

$$\alpha(x) = \begin{cases} -\lfloor x \rfloor & \text{if } x > 0, \\ \lfloor x \rfloor & \text{if } x \leq 0, \end{cases}$$

Then T is relaxed $\eta - \alpha$ pseudomonotone, but T is not relaxed α pseudomonotone. In fact, if we let $y = -1$ and $x = 0$, then $\langle T(x), \eta(y, x) \rangle \geq 0$, but $\langle T(y), \eta(y, x) \rangle < \alpha(\eta(y, x))$, which is a contradiction.

Example 1.3.13 (Arunchai [4])

Let $T : (-\infty, 1) \rightarrow \mathbb{R}$ be a function defined by $T(x) = x^2 - 1$ and $\eta(x, y) = c(x - y)$, where $c < 0$.

Then T is strictly η -quasimonotone, but T is not strictly quasimonotone. In fact, if we let $y \in (-1, 1)$ and $x < -1$, then $\langle T(x), y - x \rangle > 0$, but $\langle T(y), y - x \rangle < 0$.

Remark 1.3.14 The implications of the monotone operators from the definitions and examples above can be summarized as in figure 1.2.

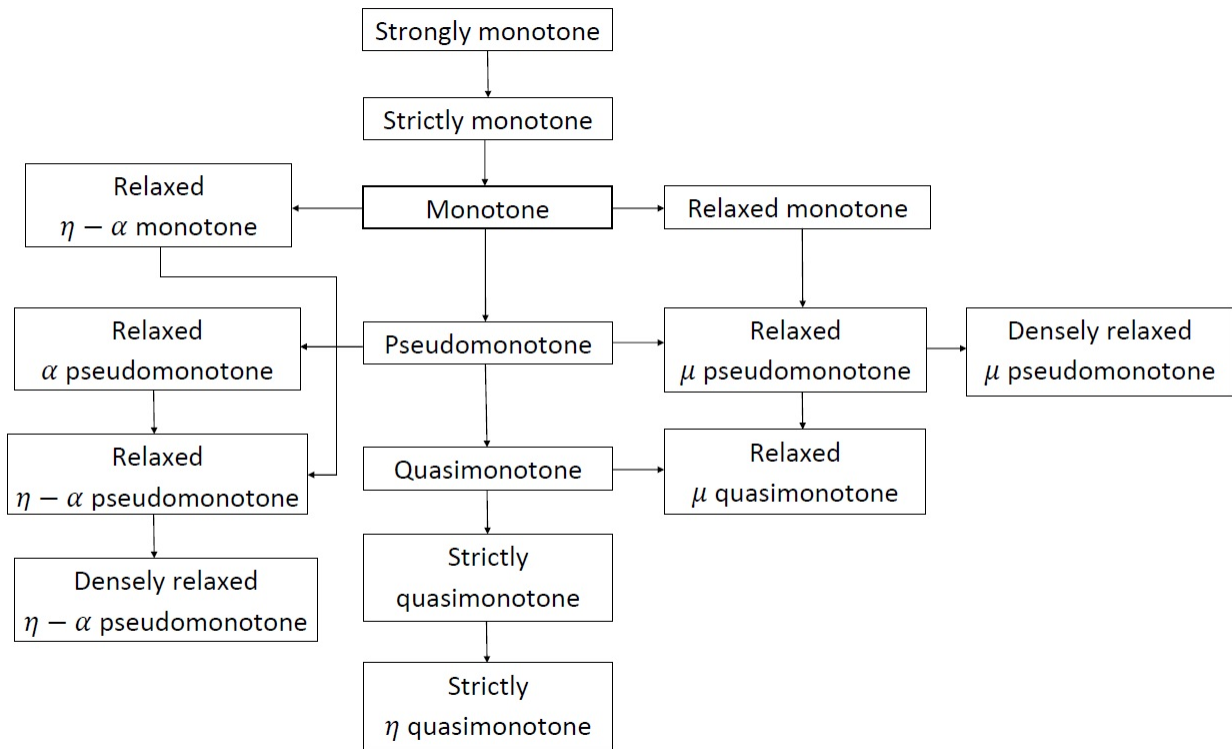


Figure 1.2: Relations between different types of monotonicity.

1.4 Some Basic Definitions and Preliminaries

In this section, we recall some basic definitions and theorems from convex analysis, which will be used in the sequel.

Let X be a real vector space. Let X^* be its dual and K be a non-empty subset of X .

Definition 1.4.1 *A subset $K \subset X$ is said to be convex if*

$$\forall x, y \in K, \forall \lambda \in [0, 1] \Rightarrow \lambda x + (1 - \lambda)y \in K.$$

Definition 1.4.2 *Let $K \subset X$ be a convex subset. The function $f : K \rightarrow \mathbb{R}$ is said to be convex if*

$$\forall x, y \in K, \forall \lambda \in [0, 1] : f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y).$$

Definition 1.4.3 *Let $A = \{x_1, x_2, \dots, x_n\}$ be a finite set.*

Then the convex hull of A , denoted by $\text{co}\{x_1, x_2, \dots, x_n\}$, is defined as

$$\text{co}\{x_1, x_2, \dots, x_n\} := \left\{ \sum_{i=1}^n \alpha_i x_i : \sum_{i=1}^n \alpha_i = 1, \forall \alpha_i \geq 0 \right\}.$$

Definition 1.4.4 *Let $T : K \rightarrow X^*$ be an operator.*

Then T is called Lipschitz continuous if there exists $\alpha > 0$ such that

$$\|Tx - Ty\| \leq \alpha \|x - y\|, \quad \forall x, y \in K.$$

Definition 1.4.5 *Let X be a Banach space and let $x_0 \in X$. A function $f : X \rightarrow (-\infty, +\infty]$ is called*

(i) lower semi-continuous at x_0 if

$$f(x_0) \leq \liminf_{x \rightarrow x_0} f(x);$$

(ii) upper semi-continuous at x_0 if

$$f(x_0) \geq \limsup_{x \rightarrow x_0} f(x).$$

Theorem 1.4.6 (Browder [12]) *Let K be a closed convex subset of a Banach space X and let $f : K \rightarrow \mathbb{R}$ be a convex function. Then f is lower semi-continuous if and only if f is weakly lower semi-continuous.*

Definition 1.4.7 (Brezis [11]) *Every closed bounded convex subset of a reflexive Banach space is weakly compact.*

Definition 1.4.8 *Let K be a convex subset of X .*

Let $f : K \rightarrow \mathbb{R}$ be a real-valued function.

Then f is said to be hemicontinuous if for each $x, y \in K$:

$$\lim_{t \rightarrow 0^+} f(tx + (1-t)y) = f(y).$$

CHAPTER 2: GENERAL OVER-RELAXED PROXIMAL POINT ALGORITHM TO GENERAL (A, η, m) -MONOTONE NONLINEAR INCLUSION FORMS

2.1 Introduction

Proximal point algorithms have been studied by many authors. See, for example, Rockafella [34], Agarwal et al. [1], [2], Verma [39], [40], [41], [42], [43], Lan [28], Li [29], and the references therein.

In [39], [42], and [43], Verma generalized the relaxed and over-relaxed proximal point algorithm based on the notions of A-maximal monotonicity for solving general problems in Hilbert spaces and Banach spaces. In [41], Verma also introduced the new relaxed algorithmic procedure based on the notions of A-maximal monotonicity for solving general inclusion problems in Hilbert spaces. On the other hand, Lan [27], [28] introduced a new concept of (A, η, m) -maximal monotone operators, which generalized the existing monotone operators such as A-monotonicity, (H, η) -monotonicity, and other monotone operators as special cases.

Furthermore, based on the studies of Verma and Lan, Li [29] introduced and studied a new class of over-relaxed proximal point algorithms for approximating solvability of the nonlinear operator equation $B(x) - R_{\rho, M}^{A, \eta} = 0$ (equation (1), Li [29]) in Hilbert spaces based on (A, η, m) -monotonicity framework.

Among these studies, Lipschitz continuity is always a necessary condition for the convergence of the sequence of the proximal point algorithm. Greatly motivated and inspired by the works mentioned above, we introduce an alternating approach to the relaxed algorithmic procedure based on the notion of A-maximal relaxed η -monotonicity where the Lipschitz continuity requirement for the monotone mapping A , under some conditions, can be omitted.

The objective of this chapter is to see how to remove the Lipschitz continuity requirement by using an alternating approach to the proximal point algorithm. It does not by any way take away the value and the importance of the original contributions given by Verma ([39]-[44]), Agawal et al. ([1]-[3]), Lan ([28]), Li ([29]), and many other authors who contributed their works on this subject. Instead, had these results not been proved, it may not be possible to see how to improve the proofs without Lipschitz continuity. In this sense, we think our work complements that of Verma [39]-[42].

Let X be a real Hilbert space with the norm $\|\cdot\|$ and the inner product $\langle \cdot, \cdot \rangle$. We consider a general class of nonlinear implicit inclusion problems of the form: find a solution to

$$0 \in M(x), \quad (2.1.1)$$

where $M : X \rightarrow 2^X$ is a set-valued mapping on X .

Equivalently, equation (2.1.1) can be written as: find a solution to

$$x - J_{\rho, A}^{M, \eta}(A(x)) = 0, \quad (2.1.2)$$

where $A : X \rightarrow X$ and $\eta : X \times X \rightarrow X$ are nonlinear operators; and $J_{\rho, A}^{M, \eta} = (A + \rho M)^{-1}$ is the generalized resolvent operator with $\rho > 0$.

Remark 2.1.1 *Let $g : X \rightarrow X$ be a single valued mapping.*

(i) *If $\eta(x, y) := x - y$, the operator $N : X \rightarrow 2^X$ is an A -maximal m -relaxed monotone, and $M(x) = N(g(x))$ with $\text{range}(g) \cap \text{dom}(N) \neq \emptyset$ on Hilbert spaces or Banach spaces, then problem (2.1.1) becomes problem (1) in Verma [39] and Verma [42].*

(ii) *In addition to (i) above, if $T : X \rightarrow X$ is a single valued mapping on X with $\text{range}(g) \cap$*

$dom(N) \neq \emptyset$ and $M(x) = T(x) + N(g(x))$, then problem (2.1.1) reduces to problem (1) in Agarwal et al. [1].

(iii) From (ii) above, if $\eta : X \times X \rightarrow X$ be a nonlinear operator and $T(x) := A(g(x)) - A(x)$ then problem (2.1.1) reduces to problem (2) in Li [29].

2.2 Preliminaries and Definitions

In order to reduce the uses of parentheses, whenever the context is clear, the notations of composite operators will be written without parentheses or composite notation. For example, $A(T(x))$ and $A \circ T(x)$ will be written as $AT(x)$ throughout this chapter.

In this section we recall some basic definitions and known results to prove the strong convergence in section 3.

Let $M, N : X \rightarrow 2^X$ be multi-valued mappings on X .

Let $A : X \rightarrow X$ and $\eta : X \times X \rightarrow X$ be single-valued mappings.

The graph of the mapping M is defined by $graph(M) := \{(x, y) | y \in M(x)\}$. In this section, we will denote both the mapping and its graph by M . It is equivalent to saying that a mapping is any subset of $X \times X$.

The following definitions can be found on [1]-[3], [39], [40], [44], and [45].

Definition 2.2.1

(i) *The domain of M :*

$$D(M) := \{x \in X | \exists y \in X \text{ s.t. } (x, y) \in M\} = \{x \in X | M(x) \neq \emptyset\};$$

(ii) *The range of M :*

$$R(M) := \{y \in X \mid \exists x \in X \text{ s.t. } (x, y) \in M\};$$

(iii) *The inverse M^{-1} of M :*

$$M^{-1} := \{(y, x) \in X \times X \mid (x, y) \in M\};$$

(iv) *Scalar multiplication:*

$$\rho M := \{(x, \rho y) \mid (x, y) \in M\};$$

(v) *Addition:*

$$M + N := \{(x, y + z) \mid (x, y) \in M, (x, z) \in N\};$$

(vi) *Composition:*

$$MN := \{(x, z) \mid (x, y) \in N, (y, z) \in M\}.$$

Definition 2.2.2

(i) *A is r -strongly monotone if $\forall u, v \in X, \exists r > 0$:*

$$\langle A(u) - A(v), u - v \rangle \geq r \|u - v\|^2;$$

(ii) *A is r -strongly η -monotone (a.k.a. (r, η) -monotone) if $\forall u, v \in X, \exists r > 0$:*

$$\langle A(u) - A(v), \eta(u, v) \rangle \geq r \|u - v\|^2;$$

(iii) A is r -Lipschitz continuous if $\forall u, v \in X, \exists r > 0 :$

$$\|A(u) - A(v)\| \leq r\|u - v\|;$$

(iv) η is r -Lipschitz continuous if $\forall u, v \in X, \exists r > 0 :$

$$\|\eta(u, v)\| \leq r\|u - v\|.$$

Definition 2.2.3

(i) M is monotone if $\forall (u, u^*) \in M, (v, v^*) \in M :$

$$\langle u^* - v^*, u - v \rangle \geq 0;$$

(ii) M is r -strongly monotone if $\forall (u, u^*) \in M, (v, v^*) \in M \exists r > 0 :$

$$\langle u^* - v^*, u - v \rangle \geq r\|u - v\|^2;$$

(iii) M is m -relaxed monotone if $\forall (u, u^*) \in M, (v, v^*) \in M \exists m > 0 :$

$$\langle u^* - v^*, u - v \rangle \geq -m\|u - v\|^2;$$

(iv) M is m -relaxed η -monotone if $\forall (u, u^*) \in M, (v, v^*) \in M \exists m > 0 :$

$$\langle u^* - v^*, \eta(u, v) \rangle \geq -m\|u - v\|^2.$$

Remark 2.2.4 Clearly, $(ii) \Rightarrow (i) \Rightarrow (iii)$, but not vice versa.

Also, if $\eta(x, y) = x - y$, then (iv) implies (iii).

Definition 2.2.5 Let $A : X \rightarrow X$ be a single-valued mapping. The map $M : X \rightarrow 2^X$ is said to be (A, η, m) -maximal monotone if

- (i) M is m -relaxed η -monotone for $m > 0$, and
- (ii) $(A + \rho M)(X) = X$ for $\rho > 0$.

Definition 2.2.6 Let $\eta : X \times X \rightarrow X$ be a single valued mapping.

Let $A : X \rightarrow X$ be an r -strongly η -monotone mapping.

Let $M : X \rightarrow 2^X$ be an (A, η, m) -maximal monotone mapping.

The generalized resolvent operator $J_{\rho, A}^{M, \eta} : X \rightarrow X$ is defined by $J_{\rho, A}^{M, \eta}(u) = (A + \rho M)^{-1}(u)$.

Proposition 2.2.7 (Sahu [37] Lemma 2.1)

Let $\eta : X \times X \rightarrow X$ be a single valued mapping.

Let $A : X \rightarrow X$ be an r -strongly η -monotone mapping.

Let $M : X \rightarrow 2^X$ be an (A, η, m) -maximal monotone mapping.

Then the generalized resolvent operator $J_{\rho, A}^{M, \eta} : X \rightarrow X$ is defined by $J_{\rho, A}^{M, \eta} = (A + \rho M)^{-1}$ is single-valued.

Proposition 2.2.8 (Verma [39] Proposition 2.1)

Let $\eta : X \times X \rightarrow X$ be a single valued mapping.

Let $A : X \rightarrow X$ be an r -strongly η -monotone mapping.

Let $M : X \rightarrow 2^X$ be an (A, η, m) -maximal monotone mapping.

Then $(A + \rho M)$ is a maximal monotone for $\rho > 0$.

Proposition 2.2.9 Let X be a real Hilbert space.

Let $\eta : X \times X \rightarrow X$ be a single valued mapping.

Let $A : X \rightarrow X$ be an r -strongly monotone mapping.

Let $M : X \rightarrow 2^X$ be an (A, η, m) -maximal monotone mapping.

Then the generalized operator associated with M and defined by $J_{\rho, A}^{M, \eta}(u) := (A + \rho M)^{-1}(u) \quad \forall u \in X$, satisfies

$$\langle J_{\rho, A}^{M, \eta} A(u) - J_{\rho, A}^{M, \eta} A(v), A(u) - A(v) \rangle \geq (r - \rho m) \|J_{\rho, A}^{M, \eta} A(u) - J_{\rho, A}^{M, \eta} A(v)\|^2,$$

where $r - \rho m > 0$.

Proof. $\forall u, v \in X \Rightarrow A(u), A(v) \in X$. By definition of resolvent operator $J_{\rho, A}^{M, \eta}$,

$$J_{\rho, A}^{M, \eta} A(u) = (A + \rho M)^{-1} A(u)$$

$$A(u) \in (A + \rho M) J_{\rho, A}^{M, \eta} A(u) \quad (2.2.1)$$

$$A(u) - A J_{\rho, A}^{M, \eta} A(u) \in \rho M J_{\rho, A}^{M, \eta} A(u).$$

Similarly,

$$A(v) - A J_{\rho, A}^{M, \eta} A(v) \in \rho M J_{\rho, A}^{M, \eta} A(v). \quad (2.2.2)$$

M is (A, η, m) -maximal monotone. Hence, M is m -relaxed η -monotone. Thus,

$$\begin{aligned} & \langle A(u) - A(v) - (A J_{\rho, A}^{M, \eta} A(u) - A J_{\rho, A}^{M, \eta} A(v)), J_{\rho, A}^{M, \eta} A(u) - J_{\rho, A}^{M, \eta} A(v) \rangle \\ &= \rho \langle M J_{\rho, A}^{M, \eta} A(u) - M J_{\rho, A}^{M, \eta} A(v), J_{\rho, A}^{M, \eta} A(u) - J_{\rho, A}^{M, \eta} A(v) \rangle \\ &\geq -\rho m \|J_{\rho, A}^{M, \eta} A(u) - J_{\rho, A}^{M, \eta} A(v)\|^2. \end{aligned} \quad (2.2.3)$$

Since A is r -strongly η -monotone, from (2.2.3) it follows that

$$\begin{aligned}
& \langle A(u) - A(v), J_{\rho,A}^{M,\eta} A(u) - J_{\rho,A}^{M,\eta} A(v) \rangle \\
& \geq \langle AJ_{\rho,A}^{M,\eta} A(u) - AJ_{\rho,A}^{M,\eta} A(v), J_{\rho,A}^{M,\eta} A(u) - J_{\rho,A}^{M,\eta} A(v) \rangle - \rho m \|J_{\rho,A}^{M,\eta} A(u) - J_{\rho,A}^{M,\eta} A(v)\|^2 \\
& \geq (r - \rho m) \|J_{\rho,A}^{M,\eta} A(u) - J_{\rho,A}^{M,\eta} A(v)\|^2.
\end{aligned} \tag{2.2.4}$$

This completes the proof. □

For convenience, from now on, let $d := r - \rho m$.

Definition 2.2.10 *Let X be a real Hilbert space.*

Let $\eta : X \times X \rightarrow X$ be a single valued mapping.

Let $A : X \rightarrow X$ be r -strongly η -monotone mapping.

Then for $x \in X$, define the function $R(x, \rho)$ by

$$R(x, \rho) := A(x) - AJ_{\rho,A}^{M,\eta} A(x). \tag{2.2.5}$$

Lemma 2.2.11 *$x^* \in X$ is a solution of (2.1.1) if and only if $R(x^*, \rho) = 0$.*

Proof.

$$\begin{aligned}
R(x^*, \rho) = 0 & \iff A(x^*) = AJ_{\rho,A}^{M,\eta} A(x^*) \\
& \iff x^* = J_{\rho,A}^{M,\eta} A(x^*) \\
& \iff A(x^*) \in (A + \rho M)(x^*) \\
& \iff 0 \in M(x^*).
\end{aligned} \tag{2.2.6}$$

□

2.3 Main Convergence Theorems

To prove the main theorem, we need the following lemma.

Lemma 2.3.1 *Let X be a real Hilbert space.*

Let $\eta : X \times X \rightarrow X$ be a single valued mapping.

Let $A : X \rightarrow X$ be r -strongly monotone.

Let $M : X \rightarrow 2^X$ be an (A, η, m) -maximal monotone.

Then $u \in X$ is a solution to (2.1.1) if and only if $u = J_{\rho, A}^{M, \eta} A(u)$.

Proof.

$$\begin{aligned}
 u = J_{\rho, A}^{M, \eta} A(u) &\iff A(u) \in (A + \rho M)(u) \\
 &\iff 0 \in \rho M(u) \\
 &\iff 0 \in M(u).
 \end{aligned}
 \tag{2.3.1}$$

□

Theorem 2.3.2 *Let X be a real Hilbert space.*

Let $\eta : X \times X \rightarrow X$ be a single valued mapping.

Let $A : X \rightarrow X$ be r -strongly monotone.

Let $M : X \rightarrow 2^X$ be (A, η, m) -maximal monotone.

Let $\{x^k\}$ be a sequence generated by

$$x^{k+1} = (1 - \alpha_k)x^k + \alpha_k J_{\rho_k, A}^{M, \eta} A(x^k), \quad (2.3.2)$$

where $J_{\rho_k, A}^{M, \eta} = (A + \rho_k M)^{-1}$, $\{\rho_k\} \subset [0, \infty)$, and $\alpha_k \in (0, 1]$ satisfying $0 < \epsilon = \inf \alpha_k < 1$.

Suppose that there exists at least one solution x^* of (2.1.1).

For any $u, v \in X$, if, in addition, $d_k > 1$ and the following inequality holds:

$$\langle AJ_{\rho_k, A}^{M, \eta} A(u) - AJ_{\rho_k, A}^{M, \eta} A(v), A(u) - A(v) \rangle \geq d_k \|AJ_{\rho_k, A}^{M, \eta} A(u) - AJ_{\rho_k, A}^{M, \eta} A(v)\|^2 \quad (2.3.3)$$

then

(i) the sequence $\{x^k\}$ converges strongly to a unique solution x^* of (2.1.1),

(ii) with the rate of convergence $\theta_k = \sqrt{1 - \alpha_k - \frac{\alpha_k}{2d_k - 1}} < 1$,

where $d_k := r - \rho_k m > 1$.

Proof of theorem 2.3.2.

We begin by proving the following claim:

Claim:

If $d_k := r - \rho_k m > 1$, then

$$\langle A(x^k) - A(x^*), R(x^k, \rho_k) \rangle \geq \frac{d_k - 1}{2d_k - 1} \|A(x^k) - A(x^*)\|^2 + \frac{d_k}{2d_k - 1} \|R(x^k, \rho_k)\|^2. \quad (2.3.4)$$

Proof of claim:

From (2.2.5) and (2.2.6), we have

$$AJ_{\rho,A}^{M,\eta}A(x) = A(x) - R(x, \rho)$$

and

$$AJ_{\rho,A}^{M,\eta}A(x^*) = A(x^*).$$

Then

$$\begin{aligned} \langle AJ_{\rho_k,A}^{M,\eta}A(x^k) - AJ_{\rho,A}^{M,\eta}A(x^*), A(x^k) - A(x^*) \rangle &= \langle A(x^k) - A(x^*) - R(x^k, \rho_k), A(x^k) - A(x^*) \rangle \\ &\geq d_k \|A(x^k) - A(x^*) - R(x^k, \rho_k)\|^2. \end{aligned} \quad (2.3.5)$$

That is,

$$\begin{aligned} \|A(x^k) - A(x^*)\|^2 - \langle A(x^k) - A(x^*), R(x^k, \rho_k) \rangle \\ \geq d_k (\|A(x^k) - A(x^*)\|^2 - 2\langle A(x^k) - A(x^*), R(x^k, \rho_k) \rangle + \|R(x^k, \rho_k)\|^2). \end{aligned} \quad (2.3.6)$$

Therefore,

$$(2d_k - 1)\langle A(x^k) - A(x^*), R(x^k, \rho_k) \rangle \geq (d_k - 1)\|A(x^k) - A(x^*)\|^2 + d_k\|R(x^k, \rho_k)\|^2. \quad (2.3.7)$$

Since $d_k > 1$, so this proves the claim.

Remark 2.3.3 *Technically we only need $d_k > \frac{1}{2}$ to prove the claim. However, the condition of $d_k > 1$ is necessary for the removal of Lipschitz continuity later on.*

Now, we begin to prove the theorem.

From (2.3.2), we have

$$\begin{aligned}
A(x^{k+1}) &= (1 - \alpha_k)A(x^k) + \alpha_k A J_{\rho_k, A}^{M, \eta} A(x^k) \\
A(x^{k+1}) &= A(x^k) - \alpha_k \left(A(x^k) - A J_{\rho_k, A}^{M, \eta} A(x^k) \right) \\
A(x^{k+1}) &= A(x^k) - \alpha_k R(x^k, \rho_k).
\end{aligned} \tag{2.3.8}$$

Hence,

$$\begin{aligned}
\|A(x^{k+1}) - A(x^*)\|^2 &= \|A(x^k) - A(x^*) - \alpha_k R(x^k, \rho_k)\|^2 \\
&= \|A(x^k) - A(x^*)\|^2 - 2\alpha_k \langle A(x^k) - A(x^*), R(x^k, \rho_k) \rangle + \|\alpha_k R(x^k, \rho_k)\|^2 \\
&\leq \|A(x^k) - A(x^*)\|^2 + \|\alpha_k R(x^k, \rho_k)\|^2 \\
&\quad - 2\alpha_k \left(\frac{d_k - 1}{2d_k - 1} \|A(x^k) - A(x^*)\|^2 + \frac{d_k}{2d_k - 1} \|R(x^k, \rho_k)\|^2 \right) \\
&= \left(1 - \alpha_k \frac{2d_k - 2}{2d_k - 1} \right) \|A(x^k) - A(x^*)\|^2 - \alpha_k \left(\frac{2d_k}{2d_k - 1} - \alpha_k \right) \|R(x^k, \rho_k)\|^2 \\
&= \left(1 - \alpha_k - \frac{\alpha_k}{2d_k - 1} \right) \|A(x^k) - A(x^*)\|^2 - \alpha_k \left(\frac{2d_k}{2d_k - 1} - \alpha_k \right) \|R(x^k, \rho_k)\|^2.
\end{aligned} \tag{2.3.9}$$

Since $d_k > 1$ and $\alpha_k \in (0, 1]$, it follows that

$$\begin{aligned}
0 &< \alpha_k < 1 + \frac{1}{2d_k - 2} = \frac{2d_k - 1}{2d_k - 2} \\
0 &< \alpha_k \frac{2d_k - 2}{2d_k - 1} < 1 \\
0 &< 1 - \alpha_k \frac{2d_k - 2}{2d_k - 1} < 1 \\
0 &< 1 - \alpha_k - \frac{\alpha_k}{2d_k - 1} < 1.
\end{aligned} \tag{2.3.10}$$

Therefore, from (2.3.9), since $\alpha_k \left(\frac{2d_k}{2d_k-1} - \alpha_k \right) > 0$, it follows that $\forall k \in \mathbb{N}$,

$$\begin{aligned} \|A(x^{k+1}) - A(x^*)\|^2 &\leq \left(1 - \alpha_k - \frac{\alpha_k}{2d_k-1} \right) \|A(x^k) - A(x^*)\|^2 \\ &\leq \|A(x^k) - A(x^*)\|^2 \\ &\leq \|A(x^1) - A(x^*)\|^2. \end{aligned} \tag{2.3.11}$$

Let $\theta_k = \sqrt{1 - \alpha_k - \frac{\alpha_k}{2d_k-1}}$. Since $d_k > 1$ and $0 < \epsilon = \inf \alpha_k \leq \alpha_k < 1$, we have

$$0 < \theta_k = \sqrt{1 - \alpha_k - \frac{\alpha_k}{2d_k-1}} \leq \sqrt{1 - \epsilon - \frac{\epsilon}{2d_k-1}} < 1. \tag{2.3.12}$$

Let $\rho = \inf \rho_k$, then $d_k = r - \rho_k m \leq d$ where $d := r - \rho m$. Hence, $0 < \theta_k \leq \theta < 1$, where $\theta = \sqrt{1 - \epsilon - \frac{\epsilon}{2d-1}}$.

Since A is r-strongly monotone, from (2.3.9) we have

$$\begin{aligned} r\|x^{k+1} - x^*\| &\leq \|A(x^{k+1}) - A(x^*)\| \\ &\leq \sqrt{1 - \alpha_k \frac{2d_k-2}{2d_k-1}} \|A(x^k) - A(x^*)\| \\ &= \theta_k \|A(x^k) - A(x^*)\| \\ &\leq \theta \|A(x^k) - A(x^*)\| \\ &\leq \theta^k \|A(x^1) - A(x^*)\| \rightarrow 0 \end{aligned} \tag{2.3.13}$$

as $k \rightarrow \infty$.

Therefore, $\lim_{k \rightarrow \infty} \|x^k - x^*\| = 0$. Moreover, from (2.3.11) it follows that the sequence $\{A(x^k)\}$ is bounded. Hence, $x^k \rightarrow x^*$ strongly. Now, we need to show that the solution is unique.

Assume that x_1^* and x_2^* are two solutions of (2.1.1).

From (2.3.11) we have

$$\|A(x^{k+1}) - A(x^*)\| \leq \|A(x^1) - A(x^*)\|. \quad (2.3.14)$$

Hence, let $a = \lim_{k \rightarrow \infty} \inf \|A(x^k) - A(x^*)\|$, then $0 \leq a < \infty$.

Let $a_1 = \lim_{k \rightarrow \infty} \inf \|A(x^k) - A(x_1^*)\|$ and $a_2 = \lim_{k \rightarrow \infty} \inf \|A(x^k) - A(x_2^*)\|$.

Then

$$\begin{aligned} \|A(x^k) - A(x_2^*)\|^2 &= \|A(x^k) - A(x_1^*) + A(x_1^*) - A(x_2^*)\|^2 \\ &= \|A(x^k) - A(x_1^*)\|^2 + 2\langle A(x^k) - A(x_1^*), A(x_1^*) - A(x_2^*) \rangle \\ &\quad + \|A(x_1^*) - A(x_2^*)\|^2. \end{aligned} \quad (2.3.15)$$

Since $A(x_1^*)$ is a limit point of $A(x^k)$, we have

$$0 = \lim_{k \rightarrow \infty} 2\langle A(x^k) - A(x_1^*), A(x_1^*) - A(x_2^*) \rangle = a_2^2 - a_1^2 - \|A(x_1^*) - A(x_2^*)\|^2. \quad (2.3.16)$$

Hence,

$$a_1^2 = a_2^2 - \|A(x_1^*) - A(x_2^*)\|^2. \quad (2.3.17)$$

Similarly,

$$a_2^2 = a_1^2 - \|A(x_1^*) - A(x_2^*)\|^2. \quad (2.3.18)$$

Therefore,

$$\begin{aligned}
\|A(x_1^*) - A(x_2^*)\| &= 0 \\
r\|x_1^* - x_2^*\| &\leq \|A(x_1^*) - A(x_2^*)\| = 0 \\
\|x_1^* - x_2^*\| &= 0.
\end{aligned} \tag{2.3.19}$$

It follows that $x_1^* = x_2^*$.

Finally, we find the estimate $z^{k+1} = (1 - \alpha_k)x^k + \alpha_k J_{\rho_k, A}^{M, \eta} A(x^k)$.

We have

$$\begin{aligned}
\|A(z^{k+1}) - A(x^*)\|^2 &= \|(1 - \alpha_k)A(x^k) + \alpha_k A J_{\rho_k, A}^{M, \eta} A(x^k) - A(x^*)\|^2 \\
&= \|A(x^k) - A(x^*) - \alpha_k (A(x^k) - A J_{\rho_k, A}^{M, \eta} A(x^k))\|^2 \\
&= \|A(x^k) - A(x^*) - \alpha_k R(x^k, \rho_k)\|^2 \\
&\leq \theta \|A(x^k) - A(x^*)\|,
\end{aligned} \tag{2.3.20}$$

where $0 < \theta = \sqrt{1 - \epsilon - \frac{\epsilon}{2d-1}} < 1$.

This completes the proof. □

Remark 2.3.4 *After defending our dissertation results, we found that if the equality conditions of y_n from the main theorems of the papers of Li ([29]), Verma ([39, 41, 42, 44, 48], and Agarwal et al. ([2]) are obtained, then we will be able to apply the method of this chapter to remove the Lipschitz continuity by using similar approaches:*

1. *Based on the specific conditions of each paper mentioned above, define the new function $R(x, \rho)$ that is similar to (2.2.5), then show that Lemma 2.2.11 and Lemma 2.3.1 also hold.*

2. Show that equation (2.3.4) from the proof of the claim in theorem 2.3.2 also holds.
3. Show that equation (2.3.8) also holds, i.e. $A(x^{k+1}) = A(x^k) - \beta_k R(x^k, \rho_k)$ for some $\beta_k \in [\epsilon, 1)$, with $0 < \epsilon < 1$.

With the new approach to the theorems mentioned above, we will need to adjust some conditions of the scalar sequences $\{\alpha_k\}$, $\{\delta_k\}$, or $\{\rho_k\}$ so that the sequence $\{\beta_k\}$ will be contained in $[\epsilon, 1)$. However, in return we will be able to simplify the restrictions of the parameters in the theorems. Moreover, we will also be able to remove the Lipschitz continuity requirement of the operator A from the theorems, which may not be necessary or difficult to compute in practice.

2.4 Conclusion

In this chapter, we restricted the conditions for the sequence $\{\alpha_k\}$ of positive real numbers to be between 0 and 1 and is bounded away from zero. We also need $d := r - \rho m > 1$. Under the given conditions, we showed that the Lipschitz continuity requirement for the monotone mapping A can be omitted. In 2009, Verma ([39]) generalized the over-relaxed proximal point algorithm and solved general implicit variational inclusion problems in Hilbert spaces. However in 2016, Huang and Noor ([25]) showed that the main result of Verma's paper was incorrect and also suggested that the Lipschitz continuity assumption of the monotone operator could be dropped. Inspired by the work of Huang and Noor, we showed that the Lipschitz continuity condition can actually be omitted from other papers such as Li ([29]), Verma ([39, 41, 42, 44, 48]), and Agarwal et al. ([2]). Once again, the objectives of this paper do not by any mean discredit or take away the original contribution given by Verma and many others. Instead, it would be interesting to examine if the Lipschitz continuity of the operators can be dropped and hence, it could open an alternating approach to study variational inclusion problems in the future.

CHAPTER 3: DENSELY RELAXED PSEUDOMONOTONE AND PROPERLY RELAXED QUASIMONOTONE VARIATIONAL-LIKE INEQUALITIES

3.1 Introduction

Let X be a real reflexive Banach space with dual space X^* and K be a nonempty closed convex subset of X . We shall denote by $\langle \cdot, \cdot \rangle$ the duality pairing between X^* and X , and by 2^{X^*} the family of all nonempty subsets of X^* . Let $\Phi : K \rightarrow 2^{X^*}$ be a set-valued mapping and $\eta : K \times K \rightarrow X$ be a mapping.

The *generalized variational-like inequality* defined by K , Φ and η , is the problem of finding $\bar{x} \in K$ such that

$$\text{(GVLIP)} \quad \exists x^* \in \Phi(\bar{x}), \quad \langle x^*, \eta(y, \bar{x}) \rangle \geq 0, \quad \forall y \in K. \quad (3.1.1)$$

If $\Phi(x) = \{T(x)\}$, where $T : K \rightarrow X^*$ be a single-valued mapping, then the problem (3.1.1) is called a *variational-like inequality* and it is reduced to finding a vector $\bar{x} \in K$ such that

$$\langle T(\bar{x}), \eta(y, \bar{x}) \rangle \geq 0, \quad \forall y \in K. \quad (3.1.2)$$

If we take $\eta(y, x) = y - x$, then the problems (3.1.1) and (3.1.2) reduce, respectively to classical generalized variational inequality and variational inequality problems which consist to find a vector $\bar{x} \in K$ such that

$$\exists x^* \in \Phi(\bar{x}), \quad \langle x^*, y - \bar{x} \rangle \geq 0, \quad \forall y \in K. \quad (3.1.3)$$

and,

$$\langle T\bar{x}, y - \bar{x} \rangle \geq 0, \forall y \in K. \quad (3.1.4)$$

The existence of solutions for the variational-like inequalities (3.1.2) was established by many authors, see for instance [6, 7, 31] and the reference therein. One can observe that the operators involved in the variational inequality problems play a leading role in the generalization of variational inequalities. The researchers in the literature extended the variational inequality (VI) problems into generalized variational inequality (GVI) problems, generalized quasi-variational inequality (GQVI) problems, variational-like inequality (VLI) problems and mixed variational-like inequality (MVLI) problems etc., with the proper generalizations of associated monotone operators and the underlying spaces. For instance Verma [39] defined p -monotone type maps and proved that the nonlinear variational inequality (NVI) problems have solutions, Sahu et al. [36] defined (A, η) -maximal monotonicity, Bai et al. [6] defined relaxed $\eta - \alpha$ pseudomonotonicity, Pany et al. [33] defined generalized weakly relaxed $\eta - \alpha$ monotonicity and proved that the variational inequality problems have solutions. The variational inequalities have many applications in mechanics, engineering and equilibrium problems etc. The researchers like Chadli et al. [14] and Sahu et al. [35], extended and applied the variational inequalities into equilibrium problems.

Hadjisavvas and Schaible [24] in 1996 defined inner points of reflexive Banach spaces and proved many existence results for the variational-like inequalities (3.1.4) by using quasimonotonicity of the associated operator. In 1999, Daniilidis and Hadjisavvas [18] defined properly quasimonotonicity and proved that the variational inequalities (3.1.4) has solution for T to be a multivalued mapping. In 2004, Aussel and Hadjisavvas [5] further generalized the results of Hadjisavvas and Schaible [24] by considering multivalued mapping T to be upper sign-continuous. In 2013, Chen and Luo [15] established the existence results for the variational-like inequalities (3.1.2) for relaxed $\eta - \alpha$ quasimonotonicity.

In 2007, Bai et al. [7] defined densely relaxed μ pseudomonotonicity and proved some interesting existence results for the variational inequalities (3.1.4). Again in 2006, Bai et. al. [6] defined relaxed $\eta - \alpha$ pseudomonotonicity as follows:

The operator $T : K \rightarrow X^*$ is said to be relaxed $\eta - \alpha$ pseudomonotone if there exists a function $\eta : K \times K \rightarrow X$ and a function $\alpha : X \rightarrow \mathbb{R}$ with $\alpha(tz) = t^p\alpha(z), \forall t > 0$ and $z \in X$ such that for any $x, y \in K$, we have

$$\langle Tx, \eta(y, x) \rangle \geq 0 \implies \langle Ty, \eta(y, x) \rangle \geq \alpha(y - x), \quad (3.1.5)$$

where $p > 1$ is a constant. They have used the above monotonicity and proved some important existence results for the variational-like inequalities (3.1.2).

Inspired and motivated by these works, in this chapter, we introduce generalized densely relaxed $\eta - \alpha$ pseudomonotone operator and generalized relaxed $\eta - \alpha$ properly quasimonotone operator as well as relaxed $\eta - \alpha$ quasimonotone operator. Using these generalized monotonicity notions, we establish the existence results for the generalized variational-like inequalities (3.1.1) and (3.1.2) in the general setting of Banach spaces. The results obtained in this chapter improve and generalize many existing results in literature, namely the results by Bai et. al. [7], Luc [30], Daniilidis and Hadjisavvas [18], Aussel and Hadjisavvas [5] and Chen et al. [15]. Furthermore, we give an alternative to the results obtained by Arunchai et al. in [4] which seem to be wrong results.

The chapter is organized as the following. In section 2, we introduce the notions of generalized densely relaxed $\eta - \alpha$ pseudomonotonicity, η -upper sign-continuity, generalized relaxed $\eta - \alpha$ proper quasimonotonicity and relaxed $\eta - \alpha$ quasimonotonicity for operators, and then give some definitions and preliminary results. Section 3 is devoted to the study of the existence and uniqueness of solutions for generalized variational-like inequalities with generalized densely relaxed $\eta - \alpha$

pseudomonotone operators in Banach spaces. In section 4, we focus our study on the existence of solutions for generalized variational-like inequalities associated to a generalized $\eta - \alpha$ quasimonotone operator. Finally, we end the chapter by some remarks and comments as well as some comparisons with existing results in the literature which show the interest of the approach developed in this study.

3.2 Preliminaries and Definitions

Assume X be a normed space with norm $\|\cdot\|$ and X^* be its dual. Let K be a nonempty subset of X and $\langle \cdot, \cdot \rangle$ denotes the pairing between X^* and X . We shall denote by $co(\{y_1, y_2, \dots, y_n\})$ the convex hull of a finite subset $\{y_1, y_2, \dots, y_n\}$ of K and by 2^X the family of all subsets of X . For $r > 0$, we shall denote by $\bar{\mathbb{B}}(0, r) := \{x \in X : \|x\| \leq r\}$ the closed ball in X .

The following definitions can be found on [4, 5, 6, 7, 8], [22], [30], and [31].

Definition 3.2.1 *Let K be a convex subset of X and $f : K \rightarrow \mathbb{R}$ be a real-valued function, then f is said to be*

(i) *hemicontinuous if for any $x, y \in K$ fixed,*

$$\lim_{t \rightarrow 0^+} f(x + t(y - x)) = f(x);$$

(ii) *upper semicontinuous at $x \in X$ if for any sequence $\{x_n\}_{n \in \mathbb{N}} \subset X$ converging to x , we have*

$$\limsup_{n \rightarrow \infty} f(x_n) \leq f(x);$$

(iii) *weakly lower semicontinuous at $x \in X$ if for any sequence $\{x_n\}_{n \in \mathbb{N}} \subset X$ converging weakly*

$$\text{to } x, \text{ we have } f(x) \leq \liminf_{n \rightarrow \infty} f(x_n).$$

Definition 3.2.2 An operator $T : K \rightarrow X^*$ is said to be

(i) *monotone on K if for each $x, y \in K$:*

$$\langle T(x) - T(y), x - y \rangle \geq 0;$$

(ii) *hemicontinuous (respectively, upper hemicontinuous) if for all $x, y, z \in X$, the functional $t \mapsto \langle T(x + t(y - x)), z \rangle$ is continuous (respectively, upper semicontinuous) at 0^+ .*

Definition 3.2.3 [30] An operator $T : K \rightarrow X^*$ is said to be

(i) *pseudomonotone at $x \in K$ if for each $y \in K$:*

$$\langle T(y), x - y \rangle \geq 0 \implies \langle T(x), x - y \rangle \geq 0,$$

if T is pseudomonotone for every $x \in K$, we say that T is pseudomonotone on K ;

(ii) *quasimonotone at $x \in K$ if for each $y \in K$:*

$$\langle T(y), x - y \rangle > 0 \implies \langle T(x), x - y \rangle \geq 0,$$

if T is quasimonotone for every $x \in K$, we say that T is quasimonotone on K .

Definition 3.2.4 [4] Let $T : K \rightarrow X^*$ and $\eta : K \times K \rightarrow X$ be mappings, and let $\alpha : X \rightarrow \mathbb{R}$ be a function such that $\lim_{t \rightarrow 0^+} \frac{\alpha(t\eta(x, y))}{t} = 0$, for all $(x, y) \in K \times K$. The operator T is said to be relaxed $\eta - \alpha$ pseudomonotone if for any $x, y \in K$, we have

$$\langle Tx, \eta(y, x) \rangle \geq 0 \implies \langle Ty, \eta(y, x) \rangle \geq \alpha(\eta(y, x)). \quad (3.2.1)$$

Definition 3.2.5 Let $\Phi : K \rightarrow 2^{X^*}$ be a set-valued mapping, $\eta : K \times K \rightarrow X$ be a mapping and $\alpha : X \rightarrow \mathbb{R}$ be a function such that $\lim_{t \rightarrow 0^+} \frac{\alpha(t\eta(x, y))}{t} = 0$, for all $(x, y) \in K \times K$. The set-valued mapping Φ is said to be

(i) *relaxed $\eta - \alpha$ pseudomonotone* if for any $x, y \in K$ and $x^* \in \Phi(x)$, $y^* \in \Phi(y)$, we have

$$\langle x^*, \eta(y, x) \rangle \geq 0 \implies \langle y^*, \eta(y, x) \rangle \geq \alpha(\eta(y, x)); \quad (3.2.2)$$

(ii) *properly $\eta - \alpha$ quasimonotone* if for all $\{x_1, x_2, \dots, x_n\} \subset K$ and $x \in \text{co}(\{x_1, x_2, \dots, x_n\})$, there exists $i \in \{1, \dots, n\}$ such that for all $x^* \in \Phi(x_i)$ we have

$$\langle x^*, \eta(x_i, x) \rangle \geq \alpha(\eta(x_i, x));$$

(iii) *$\eta - \alpha$ quasimonotone* if for any $x, y \in K$ and $x^* \in \Phi(x)$, $y^* \in \Phi(y)$, we have

$$\langle x^*, \eta(y, x) \rangle > 0 \implies \langle y^*, \eta(y, x) \rangle \geq \alpha(\eta(y, x)). \quad (3.2.3)$$

Remark 3.2.6

1. Suppose that Φ is relaxed $\eta - \alpha$ pseudomonotone and η satisfies the following properties:

(a) $\eta(x, x) = 0$ for all $x \in K$, (b) $\eta(tx + (1-t)y, z) = t\eta(x, z) + (1-t)\eta(y, z)$ for $t \in [0, 1]$ and $x, y, z \in K$. Then, Φ is properly $\eta - \alpha$ quasimonotone.

2. Suppose that Φ is properly $\eta - \alpha$ quasimonotone and η satisfies the following properties: (a)

$\eta(x, y) + \eta(y, x) = 0$ for all $x, y \in K$, (b) $\eta(tx + (1-t)y, z) = t\eta(x, z) + (1-t)\eta(y, z)$ for $t \in [0, 1]$ and $x, y, z \in K$, (c) $\lim_{t \rightarrow 0^+} \frac{\alpha(tz)}{t} = 0$ for any $z \in X$. Then, Φ is $\eta - \alpha$

quasimonotone. Indeed, let $x, y \in K$ and $x^* \in \Phi(x)$ such that

$$\langle x^*, \eta(y, x) \rangle > 0. \quad (3.2.4)$$

Since Φ is properly $\eta - \alpha$ quasimonotone, it follows that for any $w \in \text{co}(\{x, y\})$ there exists $z \in \{x, y\}$ such that for all $z^* \in \Phi(z)$ we have

$$\langle z^*, \eta(z, w) \rangle \geq \alpha(\eta(z, w)). \quad (3.2.5)$$

Since $\eta(z, w) + \eta(w, z) = 0$, we derive

$$\langle z^*, \eta(w, z) \rangle + \alpha(-\eta(w, z)) \leq 0. \quad (3.2.6)$$

Suppose that $z = x$. Then for $t \in]0, 1[$, let $w = (1 - t)x + ty \in \text{co}(\{x, y\})$. From (3.2.6) and by using the properties (a) and (b) of η , we deduce for all $x^* \in \Phi(x)$

$$t\langle x^*, \eta(y, x) \rangle + \alpha(-t\eta(y, x)) \leq 0.$$

Hence,

$$\langle x^*, \eta(y, x) \rangle + \frac{\alpha(-t\eta(y, x))}{t} \leq 0.$$

Since $\lim_{t \rightarrow 0^+} \frac{\alpha(tz)}{t} = 0$ for any $z \in X$, we deduce from the previous inequality that $\langle x^*, \eta(y, x) \rangle \leq 0$, which contradicts (3.2.4). Therefore, $z = y$ and the conclusion follows from (3.2.5) by taking $w = x$.

In the rest of the chapter, we shall call the relaxed $\eta - \alpha$ pseudomonotonicity given in the definition above and which was introduced by Arunchai et al. [4], by *generalized relaxed $\eta - \alpha$*

pseudomonotonicity to distinguish it from *relaxed $\eta - \alpha$ pseudomonotonicity* of Bai et al. [6].

Definition 3.2.7 Let $T : K \rightarrow X^*$ and $\eta : K \times K \rightarrow X$ be mappings, and let $\alpha : X \rightarrow \mathbb{R}$ be a function such that $\lim_{t \rightarrow 0^+} \frac{\alpha(t\eta(x, y))}{t} = 0$, for all $(x, y) \in K \times K$. The operator T is said to be *generalized relaxed $\eta - \alpha$ pseudomonotone* at $x \in K$ if for each $y \in K$

$$\langle T(y), \eta(x, y) \rangle \geq 0 \implies \langle T(x), \eta(x, y) \rangle \geq \alpha(\eta(y, x)). \quad (3.2.7)$$

Definition 3.2.8 Let $\Phi : K \rightarrow 2^{X^*}$ be a set-valued mapping, $\eta : K \times K \rightarrow X$ be a mapping and $\alpha : X \rightarrow \mathbb{R}$ be a function such that $\lim_{t \rightarrow 0^+} \frac{\alpha(t\eta(x, y))}{t} = 0$, for all $(x, y) \in K \times K$. The set-valued mapping Φ is said to be *relaxed $\eta - \alpha$ pseudomonotone* at $x \in K$ if for any $y \in K$ and $x^* \in \Phi(x)$, $y^* \in \Phi(y)$, we have

$$\langle y^*, \eta(x, y) \rangle \geq 0 \implies \langle x^*, \eta(x, y) \rangle \geq \alpha(\eta(x, y)). \quad (3.2.8)$$

Definition 3.2.9 [6] Let $T : K \rightarrow X^*$ and $\eta : K \times K \rightarrow X$ be two mappings. T is said to be *η -hemicontinuous* if, for any fixed $x, y \in K$, the mapping $f : [0, 1] \rightarrow \mathbb{R}$ defined by $f(t) = \langle T(x + t(y - x)), \eta(y, x) \rangle$ is continuous at 0^+ .

Definition 3.2.10 [30] Let K be a convex set in X and K_0 a subset of K . The set K_0 is said to be *segment-dense* in K if for each $x \in K$, there exists $x_0 \in K_0$ such that x is a cluster point of the set $[x, x_0] \cap K_0$.

Inspired and motivated by the works of Arunchai [4] and Luc [30], we introduce the definition of *generalized densely relaxed $\eta - \alpha$ pseudomonotone operator* as the following.

Definition 3.2.11 Let $T : K \rightarrow X^*$ and $\eta : K \times K \rightarrow X$ be mappings, and let $\alpha : X \rightarrow \mathbb{R}$ be a function such that $\lim_{t \rightarrow 0^+} \frac{\alpha(t\eta(x, y))}{t} = 0$, for all $(x, y) \in K \times K$. The operator $T : K \rightarrow X^*$ is said to be generalized densely relaxed $\eta - \alpha$ pseudomonotone mappings on K if there exists a segment-dense subset $K_0 \subset K$ such that T is generalized relaxed $\eta - \alpha$ pseudomonotone at every point of K_0 .

Definition 3.2.12 The set-valued mapping $\Phi : K \rightarrow 2^{X^*}$ is said to be generalized densely relaxed $\eta - \alpha$ pseudomonotone mappings on K if there exists a segment-dense subset $K_0 \subset K$ such that Φ is generalized relaxed $\eta - \alpha$ pseudomonotone at every point of K_0 .

Remark 3.2.13

1. If in Definition 3.2.11 we consider $\eta(x, y) = x - y$ and $\alpha(z) = -\mu\|z\|^2$, then the generalized densely relaxed $\eta - \alpha$ pseudomonotonicity reduces to the notion of densely relaxed μ pseudomonotonicity on K due to Bai et al. [7]. Thus the generalized densely relaxed $\eta - \alpha$ pseudomonotonicity generalizes the densely relaxed μ pseudomonotonicity given by Bai et al. [7].
2. If T is pseudomonotone at every point on K_0 , then T is said to be densely pseudomonotone on K by Luc [30]. Thus the generalized densely relaxed $\eta - \alpha$ pseudomonotonicity notion also extends the densely pseudomonotonicity concept introduced by Luc [30].

Definition 3.2.14 An operator $T : K \rightarrow X^*$ is said to be generalized relaxed $\eta - \alpha$ quasimonotone, if there exists a function $\eta : K \times K \rightarrow X$ and a function $\alpha : X \rightarrow \mathbb{R}$ with $\lim_{t \rightarrow 0^+} \frac{\alpha(t\eta(x, y))}{t} = 0, \forall (x, y) \in K \times K$ such that for any $x, y \in K$, we have

$$\langle Tx, \eta(y, x) \rangle > 0 \implies \langle Ty, \eta(y, x) \rangle \geq \alpha(\eta(y, x)). \quad (3.2.9)$$

Remark 3.2.15 *If $\alpha(\eta(y, x)) = \alpha(y - x)$, then (3.2.9) implies that T is relaxed $\eta - \alpha$ quasimonotone mapping established by Chen and Luo [15] in 2013:*

$$\langle Tx, \eta(y, x) \rangle > 0 \implies \langle Ty, \eta(y, x) \rangle \geq \alpha(y - x).$$

Again if $\eta(y, x) = y - x$ and $\alpha(u) = -\mu\|u\|^2$, then (3.2.9) implies that T is relaxed μ quasimonotone mapping given by Bai et al [7] in 2007:

$$\langle Tx, y - x \rangle > 0 \implies \langle Ty, y - x \rangle \geq -\mu\|y - x\|^2.$$

Definition 3.2.16 *An operator $T : K \rightarrow X^*$ is said to be generalized relaxed $\eta - \alpha$ properly quasimonotone, if there exists a function $\eta : K \times K \rightarrow X$ and a function $\alpha : X \rightarrow \mathbb{R}$ with $\lim_{t \rightarrow 0^+} \frac{\alpha(t\eta(x, y))}{t} = 0, \forall (x, y) \in K \times K$ such that for any $y_1, y_2, \dots, y_n \in K$ and $x \in \text{co}(\{y_1, y_2, \dots, y_n\})$, there exists $i \in \{1, 2, \dots, n\}$ such that*

$$\langle Ty_i, \eta(y_i, x) \rangle \geq \alpha(\eta(y_i, x)). \quad (3.2.10)$$

Remark 3.2.17

1. *If we take $\alpha(\eta(y, x)) = \alpha(y - x)$, then our generalized relaxed $\eta - \alpha$ properly quasimonotonicity reduces to the relaxed $\eta - \alpha$ properly quasimonotonicity given by Chen and Luo in [15].*
2. *If we take $\eta(y, x) = y - x$ and $\alpha(u) = 0$, then we will get properly quasimonotonicity given by Daniilidis and Hadjisavvas [18] in case of T single valued.*

Definition 3.2.18 [22] *The set-valued mapping $F : K \rightarrow 2^X$ is said to be a KKM mapping if for*

any finite subset $\{y_1, y_2, \dots, y_n\}$ of K , we have

$$\text{co}(\{y_1, y_2, \dots, y_n\}) \subset \bigcup_{i=1}^n F(y_i).$$

Definition 3.2.19 Let X be a Banach space with topological dual space X^* and K be a nonempty subset of X . A set-valued mapping $\Phi : K \rightarrow 2^{X^*}$ is said to be

- (i) *lower semicontinuous*, if $\Phi(x) \neq \emptyset$ for all $x \in K$ and for any $x \in K$, for any open set $\mathcal{O} \subset X^*$ such that $\Phi(x) \cap \mathcal{O} \neq \emptyset$, there exists a neighborhood U of x such that $\Phi(x) \cap \mathcal{O} \neq \emptyset$ for every $x \in U$.
- (ii) *upper semicontinuous*, if for any $x \in K$ and any open set $\mathcal{O} \subset X^*$ such that $\Phi(x) \subset \mathcal{O}$, there exists a neighborhood U of x such that $\Phi(x) \subset \mathcal{O}$ for every $x \in U$.
- (iii) *upper hemicontinuous*, if the restriction of Φ to every line segment of K is upper semicontinuous.

The following Lemma given by Fan [22] will be needed in the sequel.

Lemma 3.2.20 [22] Let M be a nonempty subset of a Hausdorff topological vector space X and let $F : M \rightarrow 2^X$ be a KKM mapping. If $F(y)$ is closed in X for all $y \in M$ and compact for some $y \in M$, then

$$\bigcap_{y \in M} F(y) \neq \emptyset.$$

Lemma 3.2.21 (Michael selection theorem [32]) Let X be a paracompact space and Y be a Banach space. Then every lower semicontinuous set-valued mapping $\Phi : X \rightarrow 2^Y$ such that $\Phi(x)$ is a nonempty, closed, convex subsets of Y admits a continuous selection, i.e. there exists a continuous function $\phi : X \rightarrow Y$ such that $\phi(x) \in \Phi(x)$ for each $x \in X$.

We end this section by the following result that will be needed in the sequel.

Lemma 3.2.22 [10] *Let D be a convex and compact set, and let K be a convex set. Let $p : D \times K \rightarrow \mathbb{R}$ be convex and lower semicontinuous in the first argument, and concave in the second argument. Assume that*

$$\min_{\xi \in D} p(\xi, y) \leq 0, \text{ for all } y \in K.$$

Then, there exists $\bar{\xi} \in D$ such that $p(\bar{\xi}, y) \leq 0$ for all $y \in K$.

3.3 Existence Results for (GVLIP) with Generalized Densely Relaxed $\eta - \alpha$ Pseudomonotone Mappings

In this section, we establish some existence results for the variational-like inequalities (3.1.2) with the generalized densely relaxed $\eta - \alpha$ pseudomonotonicity and compare our results with the existing results in literature.

Theorem 3.3.1 *Let K be a nonempty, convex and compact subset of a normed space X and $T : K \rightarrow X^*$ be an η -hemicontinuous and generalized densely relaxed $\eta - \alpha$ pseudomonotone on K . Suppose that*

- (i) $\eta(x, y) + \eta(y, x) = 0$, for all $x, y \in K$;
- (ii) $\eta(tx + (1 - t)z, y) = t\eta(x, y) + (1 - t)\eta(z, y)$, for all $x, y, z \in K$ and $t \in [0, 1]$;
- (iii) For each fixed $w, z \in K$, the mapping $y \in K \mapsto \alpha(\eta(y, z))$ is lower hemicontinuous and the mapping $x \in K \mapsto \alpha(\eta(w, x))$ is lower semicontinuous;
- (iv) For each fixed $x, z \in K$, the mapping $y \in K \mapsto \langle Tx, \eta(y, z) \rangle$ is lower semicontinuous.

Then variational-like inequalities (3.1.2) has a solution.

Proof. Since T is generalized densely relaxed $\eta - \alpha$ pseudomonotone on K , it follows that there exists a segment-dense subset $K_0 \subset K$ such that T is generalized relaxed $\eta - \alpha$ pseudomonotone at every point of K_0 . For any $y \in K_0$, define a set valued mapping $F : K_0 \rightarrow 2^K$ by

$$F(y) = \{x \in K : \langle Tx, \eta(y, x) \rangle \geq 0\}.$$

From (ii), we have that $y \in F(y)$ and hence $F(y) \neq \emptyset$ for each $y \in K_0$. We claim that F is a KKM mapping. Suppose by contradiction that F is not a KKM mapping, then there exists a subset $\{x_1, x_2, \dots, x_n\}$ of K_0 , such that

$$co(\{x_1, x_2, \dots, x_n\}) \not\subseteq \bigcup_{i=1}^n F(x_i).$$

That is there exists $x_0 \in co(\{x_1, x_2, \dots, x_n\})$, $x_0 = \sum_{i=1}^n t_i x_i$, where $t_i \geq 0$, $i = 1, 2, \dots, n$, $\sum_{i=1}^n t_i = 1$, such that $x_0 \notin \bigcup_{i=1}^n F(x_i)$. From the definition of F , we have

$$\langle T(x_0), \eta(x_i, x_0) \rangle < 0, \quad \forall i = 1, 2, \dots, n.$$

Since by (i) we have $\eta(x_0, x_0) = 0$, it follows by using (ii) that

$$\begin{aligned} 0 &= \langle T(x_0), \eta(x_0, x_0) \rangle \\ &= \langle T(x_0), \eta(\sum_{i=1}^n t_i x_i, x_0) \rangle \\ &= \sum_{i=1}^n t_i \langle T(x_0), \eta(x_i, x_0) \rangle \\ &< 0, \end{aligned}$$

which is a contradiction. Thus F is a KKM mapping. Let us consider the following set-valued mapping $G : K_0 \rightarrow 2^K$ defined by

$$G(y) = \{x \in K : \langle Ty, \eta(y, x) \rangle \geq \alpha(\eta(y, x))\}.$$

Since T is generalized relaxed $\eta - \alpha$ pseudomonotone on K_0 , it follows that $F(y) \subseteq G(y)$, for all $y \in K_0$. Therefore G is a KKM mapping as F is a KKM mapping. Now, let us verify that $G(y)$ is closed for each $y \in K_0$. To this aim, let $\{x_n\}_{n \in \mathbb{N}} \subset G(y)$ such that $x_n \rightarrow x$. Then from the definition of G , we have

$$\langle Ty, \eta(y, x_n) \rangle \geq \alpha(\eta(y, x_n)).$$

Hence, from (i) we get

$$\alpha(\eta(y, x_n)) + \langle Ty, \eta(x_n, y) \rangle \leq 0.$$

By using (iii) and (iv), it follows

$$\begin{aligned} \alpha(\eta(y, x)) + \langle Ty, \eta(x, y) \rangle &\leq \liminf \alpha(\eta(y, x_n)) + \liminf \langle Ty, \eta(x_n, y) \rangle \\ &\leq \liminf [\alpha(\eta(y, x_n)) + \langle Ty, \eta(x_n, y) \rangle] \\ &\leq 0. \end{aligned}$$

Thus, by using (i) once again, we get

$$\langle Ty, \eta(y, x) \rangle \geq \alpha(\eta(y, x)).$$

Therefore $x \in G(y)$ and hence $G(y)$ is closed for each $y \in K_0$. Since by assumptions we have that K is a nonempty compact subset of X , it follows that $G(y)$ is compact for each $y \in K_0$. Therefore

from Lemma 3.2.20, we have

$$\bigcap_{y \in K_0} G(y) \neq \emptyset. \quad (3.3.1)$$

Let $x^* \in \bigcap_{y \in K_0} G(y)$, then

$$\langle Tz, \eta(z, x^*) \rangle \geq \alpha(\eta(z, x^*)), \quad \forall z \in K_0. \quad (3.3.2)$$

Let y be an arbitrary element in K . Since K_0 is segment-dense in K , it follows that there exists $z_0 \in K$ such that y is a cluster point of $[y, z_0] \cap K_0$. Then, there exists a sequence $\{y_n\}_{n \in \mathbb{N}} \subset [y, z_0] \cap K_0$ such that $y_n \rightarrow y$. Hence, $y_n = y + t_n(z_0 - y) \in K_0$ with $t_n \in [0, 1]$ and $t_n \rightarrow 0$. By using relation (3.3.2), we obtain

$$\langle Ty_n, \eta(y_n, x^*) \rangle \geq \alpha(\eta(y_n, x^*)), \quad \forall n \in \mathbb{N}. \quad (3.3.3)$$

Assumption (ii) leads us to obtain

$$\begin{aligned} (1 - t_n) \langle T(y + t_n(z_0 - y)), \eta(y, x^*) \rangle + t_n \langle T(y + t_n(z_0 - y)), \eta(z_0, x^*) \rangle \\ \geq \alpha(\eta(y + t_n(z_0 - y), x^*)), \quad \forall n \in \mathbb{N}. \end{aligned} \quad (3.3.4)$$

Since T is η -hemicontinuous and $\alpha(\eta(\cdot, x^*))$ is lower hemicontinuous on K , it follows from relation (3.3.4) that

$$\langle Ty, \eta(y, x^*) \rangle \geq \alpha(\eta(y, x^*)), \quad \forall y \in K. \quad (3.3.5)$$

Now, for $t \in]0, 1[$ let us set $y_t := (1 - t)x^* + ty \in K$, where y is an arbitrary element in K . From (3.3.5), we have

$$\langle Ty_t, \eta(y_t, x^*) \rangle \geq \alpha(\eta(y_t, x^*)), \quad \text{for all } t \in]0, 1[. \quad (3.3.6)$$

By using assumption (ii) and the fact that $\eta(x^*, x^*) = 0$, we derive from (3.3.6)

$$t\langle Ty_t, \eta(y, x^*) \rangle \geq \alpha(t\eta(y, x^*)), \text{ for all } t \in]0, 1].$$

Hence,

$$\langle Ty_t, \eta(y, x^*) \rangle \geq \frac{\alpha(t\eta(y, x^*))}{t}, \text{ for all } t \in]0, 1].$$

By considering the limit when $t \rightarrow 0^+$ in the previous inequality and by taking account of the η -hemicontinuity of T and the fact that $\lim_{t \rightarrow 0^+} \frac{\alpha(t\eta(y, x^*))}{t} = 0$, we obtain

$$\langle Tx^*, \eta(y, x^*) \rangle \geq 0, \text{ for all } y \in K.$$

Consequently, x^* is a solution of the variational-like inequality (3.1.2). Which completes the proof.

□

Remark 3.3.1

1. *If in Theorem 3.3.1 we take $\eta(y, x) = y - x$ and $\alpha(u) = -\mu\|u\|^2$, then we obtain Theorem 3.1 of Bai et al. [7] as a particular case.*
2. *As every relaxed $\eta - \alpha$ pseudomonotone operator is densely relaxed $\eta - \alpha$ pseudomonotone operator, we see that Theorem 3.3.1 is the proper generalization of [6, Theorem 3.2].*
3. *Clearly our densely relaxed $\eta - \alpha$ pseudomonotonicity generalizes densely pseudomonotonicity of Luc [30]. Therefore, Theorem 3.3.1 extends Theorem 4.3 of Luc [30].*

We have the following existence result for the variational-like inequalities (3.1.2) when K is an unbounded subset of X .

Theorem 3.3.2 *Let K be a locally compact, convex unbounded subset of a normed space X and $T : K \rightarrow X^*$ be an η -hemicontinuous and generalized densely relaxed $\eta - \alpha$ pseudomonotone on K . Suppose that the conditions (i)-(iv) of Theorem 3.3.1 are satisfied. Furthermore, suppose that $\mathbf{0} \in K$ and that one of the following assumptions is satisfied: For every sequence $\{x_n\}_{n \in \mathbb{N}} \subset K$ with $\lim \|x_n\| = +\infty$,*

[A₁] $\exists n_0 \in \mathbb{N}^*$ such that $\langle Tx_{n_0}, \eta(\mathbf{0}, x_{n_0}) \rangle \leq 0$;

[A₂] $\exists n_0 \in \mathbb{N}^*$ and $y \in K$ with $\|y\| < \|x_{n_0}\|$ such that $\langle Tx_{n_0}, \eta(x_{n_0}, y) \rangle \geq 0$;

[A₃] $\exists n_0 \in \mathbb{N}^*$ and $y \in K$ such that $\langle Ty, \eta(y, x_n) \rangle < \alpha(\eta(y, x_n))$, $\forall n \geq n_0$.

Then the variational-like inequalities (3.1.2) has at least one solution.

Proof. Define the set $\mathbb{B}_n := K \cap \bar{\mathbb{B}}(0, n)$, for $n \in \mathbb{N}^*$. Clearly the sets \mathbb{B}_n , $n \in \mathbb{N}^*$, are compact and convex. Hence by Theorem 3.3.1, there exists $x_n \in \mathbb{B}_n$ such that

$$\langle Tx_n, \eta(x, x_n) \rangle \geq 0, \forall x \in \mathbb{B}_n, \text{ for every } n \geq 1. \quad (3.3.7)$$

If $\|x_n\| < n$, for some $n \in \mathbb{N}^*$, then x_n is local minimum of the function $\varphi(x) = \langle Tx_n, \eta(x, x_n) \rangle$ on K . Hence it is also a global minimum, that is

$$\langle Tx_n, \eta(x, x_n) \rangle \geq 0, \forall x \in K. \quad (3.3.8)$$

Hence x_n is a solution variational-like inequalities (3.1.2).

If $\|x_n\| = n$, for all $n \geq 1$. Assume that hypodissertation [A₁] holds. Let us verify that x_{n_0} is a solution of the variational-like inequalities (3.1.2). Since $\mathbf{0} \in K$, we have for any $x \in K$ there

exists $t \in]0, 1]$ such that $tx \in \mathbb{B}_{n_0}$. Hence,

$$\langle Tx_{n_0}, \eta(tx, x_{n_0}) \rangle \geq 0.$$

By using assumption (ii), it follows that

$$t\langle Tx_{n_0}, \eta(x, x_{n_0}) \rangle + (1-t)\langle Tx_{n_0}, \eta(\mathbf{0}, x_{n_0}) \rangle \geq 0.$$

Hence, from [A₁] we deduce that $\langle Tx_{n_0}, \eta(x, x_{n_0}) \rangle \geq 0$, and consequently x_{n_0} is a solution of the variational-like inequalities (3.1.2).

Assume that hypodissertation [A₂] holds, i.e. $\exists n_0 \in \mathbb{N}^*$ and $y \in K$ with $\|y\| < \|x_{n_0}\|$ such that

$$\langle Tx_{n_0}, \eta(x_{n_0}, y) \rangle \geq 0. \tag{3.3.9}$$

Since $\|y\| < \|x_{n_0}\| = n_0$, it follows that $y \in \mathbb{B}_{n_0}$ and hence

$$\langle Tx_{n_0}, \eta(y, x_{n_0}) \rangle \geq 0. \tag{3.3.10}$$

From (3.3.9), (3.3.10) and assumption (i), we deduce that

$$\langle Tx_{n_0}, \eta(y, x_{n_0}) \rangle = 0.$$

Hence, y is a local minimum of the function $\psi(x) = \langle Tx_{n_0}, \eta(x, x_{n_0}) \rangle$ on K . It follows that y is a global minimum of ψ on K . Thus,

$$\langle Tx_{n_0}, \eta(x, x_{n_0}) \rangle \geq \langle Tx_{n_0}, \eta(y, x_{n_0}) \rangle = 0, \text{ for all } x \in K.$$

Therefore, x_{n_0} is a solution of the variational-like inequalities (3.1.2).

Assume that hypodissertation $[A_3]$ holds, i.e. $\exists n_0 \in \mathbb{N}^*$ and $y \in K$ such that

$$\langle Ty, \eta(y, x_n) \rangle < \alpha(\eta(y, x_n)), \quad \forall n \geq n_0. \quad (3.3.11)$$

Let us fix $n \geq n_0$ and let us set $\tilde{z} = x_n$. Since K_0 is segment-dense in K , it follows that there exists $\tilde{z}_0 \in K_0$ such that \tilde{z} is a cluster point of $[\tilde{z}, \tilde{z}_0] \cap K_0$. Hence, there exists $\{\tilde{z}_k\}_{k \in \mathbb{N}} \subset K_0$ with $\tilde{z}_k = t_k \tilde{z}_0 + (1 - t_k) \tilde{z}$ where $\{t_k\}_{k \in \mathbb{N}} \subset [0, 1]$ and $t_k \rightarrow 0$, i.e. $\tilde{z}_k \rightarrow \tilde{z}$. Note that

$$\langle Ty, \eta(\tilde{z}_k, y) \rangle + \alpha(\eta(y, \tilde{z}_k)) = (1 - t_k) \langle Ty, \eta(\tilde{z}, y) \rangle + t_k \langle Ty, \eta(\tilde{z}_0, y) \rangle + \alpha(\eta(y, \tilde{z}_k)).$$

It follows, by using assumptions (i) and (iv),

$$\begin{aligned} & \liminf [\langle Ty, \eta(\tilde{z}_k, y) \rangle + \alpha(\eta(y, \tilde{z}_k))] \\ & \geq \lim [(1 - t_k) \langle Ty, \eta(\tilde{z}, y) \rangle + t_k \langle Ty, \eta(\tilde{z}_0, y) \rangle] + \liminf \alpha(\eta(y, \tilde{z}_k)) \\ & \geq \langle Ty, \eta(\tilde{z}, y) \rangle + \alpha(\eta(y, \tilde{z})) \end{aligned}$$

Since by (3.3.11) and assumption (i), we have that $\langle Ty, \eta(\tilde{z}, y) \rangle + \alpha(\eta(y, \tilde{z})) > 0$, it follows that

$$\liminf [\langle Ty, \eta(\tilde{z}_k, y) \rangle + \alpha(\eta(y, \tilde{z}_k))] > 0.$$

Hence, there exists a subsequence of $\{\tilde{z}_k\}_{k \in \mathbb{N}}$ also denoted by $\{\tilde{z}_k\}_{k \in \mathbb{N}}$ such that

$$\lim [\langle Ty, \eta(\tilde{z}_k, y) \rangle + \alpha(\eta(y, \tilde{z}_k))] > 0.$$

Therefore, there exists $k_0 \in \mathbb{N}^*$, such that

$$\langle Ty, \eta(\tilde{z}_k, y) \rangle + \alpha(\eta(y, \tilde{z}_k)) > 0, \text{ for all } k \geq k_0,$$

which implies, by using assumption (i),

$$\langle Ty, \eta(y, \tilde{z}_k) \rangle < \alpha(\eta(y, \tilde{z}_k)), \text{ for all } k \geq k_0.$$

Since $\tilde{z}_k \in K_0$ and T is generalized relaxed $\eta - \alpha$ pseudomonotone at \tilde{z}_k , it follows from relation (3.2.7) that

$$\langle T\tilde{z}_k, \eta(y, \tilde{z}_k) \rangle < 0, \text{ for all } k \geq k_0.$$

Thus, from assumption (i) we get

$$\langle T\tilde{z}_k, \eta(\tilde{z}_k, y) \rangle > 0, \text{ for all } k \geq k_0.$$

Consequently, by using (ii), we obtain

$$(1 - t_k)\langle T\tilde{z}_k, \eta(\tilde{z}, y) \rangle + t_k\langle T\tilde{z}_k, \eta(\tilde{z}_0, y) \rangle > 0, \text{ for all } k \geq k_0.$$

By considering the limit in the previous inequality and by taking account of the fact that T is η -hemicontinuous, we deduce

$$\langle T\tilde{z}, \eta(\tilde{z}, y) \rangle \geq 0.$$

Since $\tilde{z} = x_n$, where $n \geq n_0$ is arbitrary, we derive

$$\langle Tx_n, \eta(x_n, y) \rangle \geq 0, \text{ for all } n \geq n_0.$$

Let us consider $n \geq n_0$ such that $\|y\| < n$. Therefore, assumption $[A_2]$ is satisfied, and hence the variational-like inequality (3.1.2) has at least one solution. \square

Remark 3.3.2 *Theorem 3.3.2 generalizes the Corollary 3.1 of Bai et al. [7] and Theorem 3.3 of Bai et al. [6].*

Theorem 3.3.3 *Let K be a nonempty convex and compact subset of a normed space X . Let $\eta : K \times K \rightarrow X$ be a mapping and $\Phi : K \rightarrow 2^{X^*}$ be a set-valued mapping with nonempty closed and convex values. Suppose that conditions (i)-(iii) of Theorem 3.3.1 are satisfied and that the following assumptions hold:*

[H₁] Φ is lower semicontinuous where X^* is endowed with the strong topology;

[H₂] Φ is generalized densely relaxed $\eta - \alpha$ pseudomonotone;

[H₃] For each $x, z \in K$ and $\xi \in \Phi(x)$, the mapping $y \in K \mapsto \langle \xi, \eta(y, z) \rangle$ is lower semicontinuous.

Then the generalized variational-like inequality problem (3.1.1) has a solution.

Proof. Since every metrizable space is paracompact, then from the Michael's selection theorem (Lemma 3.2.21) we deduce that there exists a continuous mapping $F : K \rightarrow X^*$ such that $F(x) \in \Phi(x)$ for any $x \in K$. We can easily verify that F is generalized densely relaxed $\eta - \alpha$ pseudomonotone. Indeed, since Φ is generalized densely relaxed $\eta - \alpha$ pseudomonotone, there exists a segment-dense set $K_0 \subset K$ such that Φ is generalized relaxed $\eta - \alpha$ pseudomonotone on K_0 . Let $x_0 \in K_0$ and $y \in K$ such that $\langle F(y), \eta(x_0, y) \rangle \geq 0$. Since $F(y) \in \Phi(y)$ and Φ is generalized relaxed $\eta - \alpha$ pseudomonotone at x_0 , it follows for any $x^* \in \Phi(x_0)$ we have

$$\langle x^*, \eta(x_0, y) \rangle \geq \alpha(\eta(x_0, y)).$$

Particularly, if we take $x^* = F(x_0) \in \Phi(x_0)$ in the previous inequality, we obtain

$$\langle F(x_0), \eta(x_0, y) \rangle \geq \alpha(\eta(x_0, y)).$$

Therefore, F generalized densely $\eta - \alpha$ pseudomonotone. Hence, from Theorem 3.3.1 we deduce that there exists $\bar{x} \in F(\bar{x})$ such that

$$\langle F(\bar{x}), \eta(y, \bar{x}) \rangle \geq 0, \text{ for all } y \in K.$$

Thus, there exists $x^* = F(\bar{x}) \in \Phi(\bar{x})$ such that

$$\langle x^*, \eta(y, \bar{x}) \rangle \geq 0, \text{ for all } y \in K.$$

Which completes the proof of the theorem. □

Theorem 3.3.3 can be extended to the case where K is not necessarily bounded. We have the following result.

Theorem 3.3.4 *Let K be a locally compact, convex unbounded subset of a normed space X . Let $\eta : K \times K \rightarrow X$ be a mapping and $\Phi : K \rightarrow 2^{X^*}$ be a set-valued mapping with nonempty closed and convex values. Suppose that all the conditions of Theorem 3.3.3 are satisfied. Then each of the following conditions is sufficient for the generalized variational-like inequality problem (3.1.1) to have a solution: For every sequence $\{x_n\}_{n \in \mathbb{N}} \subset K$ with $\lim \|x_n\| = +\infty$,*

[C₁] $\exists n_0 \in \mathbb{N}^*$ such that $\langle \xi, \eta(\mathbf{0}, x_{n_0}) \rangle \leq 0$, for all $\xi \in \Phi(x_{n_0})$;

[C₂] $\exists n_0 \in \mathbb{N}^*$ and $y \in K$ with $\|y\| < \|x_{n_0}\|$ such that $\inf_{z^* \in \Phi(x_{n_0})} \langle z^*, \eta(x_{n_0}, y) \rangle \geq 0$;

[C₃] $\exists n_0 \in \mathbb{N}^*$ and $y \in K$ such that $\sup_{y^* \in \Phi(y)} \langle y^*, \eta(y, x_n) \rangle < \alpha(\eta(y, x_n))$, $\forall n \geq n_0$.

Proof. From the Michael's selection theorem, there exists a continuous mapping $F : K \rightarrow X^*$ such that $F(x) \in \Phi(x)$ for any $x \in K$. The operator F is generalized densely relaxed $\eta - \alpha$ pseudomonotone since Φ is generalized densely relaxed $\eta - \alpha$ pseudomonotone. Furthermore, F satisfies all the assumptions of Theorem 3.3.1. For $n \in \mathbb{N}^*$, define the set $\mathbb{B}_n := K \cap \bar{\mathbb{B}}(0, n)$. Clearly the sets $\mathbb{B}_n, n \in \mathbb{N}^*$, are compact and convex. Hence by Theorem 3.3.1, we deduce that for every $n \geq 1$, there exists $x_n \in \mathbb{B}_n$ such that

$$\langle F(x_n), \eta(x, x_n) \rangle \geq 0, \text{ for all } x \in \mathbb{B}_n. \quad (3.3.12)$$

Moreover, we can easily verify that if the condition $[C_1]$ (respectively, $[C_2]$, $[C_3]$) holds, then the operator $F : K \rightarrow X^*$ satisfies the assumption $[H_1]$ (respectively, $[H_2]$, $[H_3]$) of Theorem 3.3.3. Therefore, by using a similar development to the one considered in the proof of Theorem 3.3.3, we conclude that the generalized variational-like inequality problem (3.1.1) has a solution. \square

3.4 Existence Results for (GVLIP) with Generalized Relaxed $\eta - \alpha$ Quasimonotone Mappings

In this section, we study the existence of solutions for the generalized variational-like inequality (3.1.1) with Φ being $\eta - \alpha$ quasimonotone. In our development and rather than the one considered in the previous section, we consider different concepts of solutions for the problem (3.1.1). This is presented in the definition below.

Definition 3.4.1 *Let K be a nonempty set.*

- (i) *An element $x \in K$ is said to be a weak solution of the generalized variational-like inequality (3.1.1) if and only if $\forall y \in K, \exists x^* \in \Phi(x)$ such that $\langle x^*, \eta(y, x) \rangle \geq 0$. The set of weak solutions of (3.1.1) will be denoted by $\mathbb{S}_w(\Phi, K)$.*

(ii) An element $x \in K$ is said to be a solution (or a strong solution) of the generalized variational-like inequality (3.1.1) if and only if $\exists x^* \in \Phi(x)$ such that $\langle x^*, \eta(y, x) \rangle \geq 0, \forall y \in K$. The set of solutions (or strong solutions) of (3.1.1) will be denoted by $\mathbb{S}(\Phi, K)$.

Obviously we have $\mathbb{S}(\Phi, K) \subset \mathbb{S}_w(\Phi, K)$.

In our study, we need to consider the following dual problem, called also *Minty generalized variational-like inequality*: Find $x \in K$ such that for all $y \in K$ and $y^* \in \Phi(y)$, we have

$$\langle y^*, \eta(y, x) \rangle \geq 0. \quad (3.4.1)$$

We shall denote by $\mathbb{S}_d(\Phi, K)$ the solution set of the dual problem (3.4.1).

We consider also the following relaxed dual problem, called also *relaxed Minty generalized variational-like inequality*: Find $x \in K$ such that for all $y \in K$ and $y^* \in \Phi(y)$, we have

$$\langle y^*, \eta(y, x) \rangle \geq \alpha(\eta(y, x)). \quad (3.4.2)$$

We shall denote by $\mathbb{S}_{r,d}(\Phi, K)$ the solution set of the relaxed dual problem (3.4.2).

An element $x \in K$ is called *local solution* of the dual problem (3.4.1), if there exists a neighborhood U of x such that $x \in \mathbb{S}_d(\Phi, K \cap U)$. The set of all local solutions of the dual problem (3.4.1) will be denoted by $\mathbb{S}_{d,loc}(\Phi, K)$.

Definition 3.4.2 A set-valued mapping $S : K \rightarrow 2^{X^*}$ is called η -upper sign-continuous on K , if

for every $x, y \in K$ the following implication holds:

$$\left[\forall t \in]0, 1[, \inf_{x^* \in S(x_t)} \langle x^*, \eta(y, x) \rangle \geq 0 \right] \implies \left[\sup_{x^* \in S(x)} \langle x^*, \eta(y, x) \rangle \geq 0 \right],$$

where $x_t = (1 - t)x + ty$.

Remark 3.4.3 Suppose that $S : K \rightarrow 2^{X^*}$ is weak* upper hemicontinuous, i.e. the restriction of S to every line segment of K is upper semicontinuous with respect to the weak* topology of X^* , and $\eta : K \times K \rightarrow X$ satisfies the following properties: (a) $\eta(x, x) = 0$ for all $x \in K$, (b) $\eta(tx + (1 - t)y, z) = t\eta(x, z) + (1 - t)\eta(y, z)$, $\forall t \in [0, 1]$. Then S is η -upper sign-continuous.

In the following lemmas, we give some relations between the different concepts of solutions introduced above.

Lemma 3.4.4 Let K be a nonempty closed and convex subset of the Banach space X , $\Phi : K \rightarrow 2^{X^*}$ be a set-valued mapping and $\eta : K \times K \rightarrow X$ be a mapping such that $\eta(x, x) = 0$ and $\eta(tx + (1 - t)y, z) = t\eta(x, z) + (1 - t)\eta(y, z)$ for all $x, y, z \in K$ and $t \in [0, 1]$. Suppose that for every $x \in K$ there exists a convex neighborhood V of x and a set-valued mapping $\Phi_x : K \cap U \rightarrow 2^{X^*}$ with nonempty weak* compact values which is η -upper sign-continuous and satisfying $\Phi_x(y) \subset \Phi(y)$, for all $y \in K \cap V$. Then, $\mathbb{S}_{d,loc}(\Phi, K) \subset \mathbb{S}_w(\Phi, K)$. Furthermore, if for every $x \in K$ the set-valued mapping Φ_x is convex valued, then $\mathbb{S}_{d,loc}(\Phi, K) \subset \mathbb{S}_w(\Phi, K) = \mathbb{S}(\Phi, K)$.

Proof. Let $x \in \mathbb{S}_{d,loc}(\Phi, K)$. Then there exists a neighborhood U of x such that $x \in \mathbb{S}_d(\Phi, K \cap U)$. Since $\Phi_x(y) \subset \Phi(y)$ for each $y \in K \cap V$, it follows that $x \in \mathbb{S}_d(\Phi_x, K \cap U \cap V)$. Therefore, $\langle y^*, \eta(y, x) \rangle \geq 0$ for all $y \in K \cap U \cap V$ and $y^* \in \Phi_x(y)$. Let $y \in K$, then there exists $\tilde{y} \in]x, y[$

such that $[x, \tilde{y}] \subset K \cap U \cap V$. Hence, for $y_t = t\tilde{y} + (1-t)x$ with $t \in [0, 1]$, we have

$$\langle y^*, \eta(y_t, x) \rangle \geq 0, \quad \forall y^* \in \Phi_x(y_t).$$

By using the properties of the mapping η , we derive that

$$\langle y^*, \eta(\tilde{y}, x) \rangle \geq 0, \quad \forall y^* \in \Phi_x(y_t).$$

Thus $\inf_{y^* \in \Phi_x(y_t)} \langle x^*, \eta(\tilde{y}, x) \rangle \geq 0, \forall t \in [0, 1]$. By the η -upper sign continuity of Φ_x , we get

$$\sup_{y^* \in \Phi_x(x)} \langle y^*, \eta(\tilde{y}, x) \rangle \geq 0. \quad (3.4.3)$$

Since $\Phi_x(x)$ is weak* compact, it follows that there exists $\tilde{x}^* \in \Phi_x(x)$ such that

$$\langle \tilde{x}^*, \eta(\tilde{y}, x) \rangle \geq 0. \quad (3.4.4)$$

On the other hand, we have that $\tilde{y} = \lambda x + (1-\lambda)y$ for some $\lambda \in]0, 1[$ since $\tilde{y} \in]x, y[$. It follows, from (3.4.4) and the properties of η , that $\langle \tilde{x}^*, \eta(y, x) \rangle \geq 0$. Consequently, we have shown that for each $y \in K$, there exists $\tilde{x}^* \in \Phi_x(x) \subset \Phi(x)$ such that $\langle \tilde{x}^*, \eta(y, x) \rangle \geq 0$. Therefore, $x \in \mathbb{S}_w(\Phi, K)$. Now, let us suppose that $\Phi_x(x)$ is convex. To verify that $x \in \mathbb{S}(\Phi, K)$, it suffices to apply Lemma 3.2.22 with $D = \Phi_x(x)$ and $p(x^*, y) = \langle x^*, \eta(y, x) \rangle$ for $(x^*, y) \in D \times K$. Which completes the proof of the Lemma. \square

Lemma 3.4.5 *Let K be a nonempty closed and convex subset of the Banach space X . Let $\eta : K \times K \rightarrow X$ and $\alpha : X \rightarrow \mathbb{R}$ be mappings and $\Phi : K \rightarrow 2^{X^*}$ be a set-valued mapping with nonempty weak* compact values. Suppose that Φ is upper hemicontinuous with respect to the*

weak* topology of X^* and that the following properties hold:

(i) For all $x, y \in K$, $\eta(x, y) + \eta(y, x) = 0$;

(ii) $\eta(tx + (1 - t)y, z) = t\eta(x, z) + (1 - t)\eta(y, z)$ for all $t \in [0, 1]$ and $x, y, z \in K$;

(iii) For any $w \in X$, $\lim_{t \rightarrow 0^+} \frac{\alpha(tw)}{t} = 0$.

Then, $\mathbb{S}_{r,d}(\Phi, K) \subset \mathbb{S}_w(\Phi, K)$. Furthermore, if for every $x \in K$ the set-valued mapping Φ is convex valued, then $\mathbb{S}_{r,d}(\Phi, K) \subset \mathbb{S}_w(\Phi, K) = \mathbb{S}(\Phi, K)$.

Proof. Let $x \in \mathbb{S}_{r,d}(\Phi, K)$. Then,

$$\langle y^*, \eta(y, x) \rangle \geq \alpha(\eta(y, x)), \quad \forall y \in K, \quad \forall y^* \in \Phi(y). \quad (3.4.5)$$

For $t \in]0, 1]$ and $y \in K$, let us set $y_t = ty + (1 - t)x \in K$. Hence, from (3.4.5) we get

$$\langle y^*, \eta(y_t, x) \rangle \geq \alpha(\eta(y_t, x)), \quad \forall y^* \in \Phi(y_t), \quad \forall t \in]0, 1]. \quad (3.4.6)$$

Note that from (i), we have that $\eta(x, x) = 0$. By using (ii), we deduce from (3.4.6) that

$$\langle y^*, \eta(y, x) \rangle \geq \frac{\alpha(t\eta(y, x))}{t}, \quad \forall y^* \in \Phi(y_t), \quad \forall t \in]0, 1].$$

By (i), it follows that

$$\langle y^*, \eta(x, y) \rangle + \frac{\alpha(t\eta(y, x))}{t} \leq 0, \quad \forall y^* \in \Phi(y_t), \quad \forall t \in]0, 1].$$

Therefore,

$$\inf_{y^* \in \Phi(y_t)} \langle y^*, \eta(x, y) \rangle + \frac{\alpha(t\eta(y, x))}{t} \leq 0, \quad \forall t \in]0, 1]. \quad (3.4.7)$$

Let us set for $t \in [0, 1]$, $\phi(t) := \inf_{y^* \in \Phi(y_t)} \langle y^*, \eta(x, y) \rangle$. By the Berge's theorem [9, Théorème 2, p. 122], we deduce that the function $\phi : [0, 1] \rightarrow \mathbb{R}$ is lower semicontinuous. Hence, by considering the lower limit when $t \rightarrow 0^+$ in relation (3.4.7) and by taking account of (iii), we obtain

$$\phi(0) = \inf_{y^* \in \Phi(x)} \langle y^*, \eta(x, y) \rangle \leq 0.$$

Since $\Phi(x)$ is weak* compact, it follows that there exists $x^* \in \Phi(x)$ such that $\langle x^*, \eta(x, y) \rangle \leq 0$ and hence by (i), we have $\langle x^*, \eta(y, x) \rangle \geq 0$. This implies that $x \in \mathbb{S}_w(\Phi, K)$. Furthermore, if $\Phi(x)$ is convex, we use Lemma 3.2.22 with $D = \Phi(x)$ and $p : D \times K \rightarrow \mathbb{R}$ defined by $p(x^*, y) = \langle x^*, \eta(y, x) \rangle$ for $(x^*, y) \in D \times K$ to conclude that $x \in \mathbb{S}(\Phi, K)$. \square

We show the following result on the existence of solutions for the dual problem (3.4.1).

Theorem 3.4.1 *Let X be a Banach space with topological dual space X^* and K be a nonempty closed and convex subset of X . Let $\eta : K \times K \rightarrow X$ be a mapping and $\Phi : K \rightarrow 2^{X^*}$ be a properly $\eta - \alpha$ quasimonotone operator. Suppose that*

- (i) *For all $x \in K$, $\eta(x, x) = 0$;*
- (ii) *$\eta(tx + (1 - t)y, z) = t\eta(x, z) + (1 - t)\eta(y, z)$ for all $t \in [0, 1]$ and $x, y, z \in K$;*
- (iii) *For each $x \in K$ and $x^* \in \Phi(x)$, the mapping $y \in K \mapsto \langle x^*, \eta(x, y) \rangle$ is weakly upper semicontinuous;*
- (iv) *K is weakly compact, or alternatively there exists a weakly compact subset C of K and $x_0 \in C$ such that*

$$\forall x \in K \setminus C, \exists x_0^* \in \Phi(x_0) \text{ satisfying } \langle x_0^*, \eta(x_0, x) \rangle < \alpha(\eta(x_0, x)). \quad (3.4.8)$$

Then $\mathbb{S}_{r,d}(\Phi, K) \neq \emptyset$, i.e. the relaxed dual variational-like inequality problem (3.4.2) has a solution.

Proof. Let us consider the set-valued mapping $F : K \rightarrow 2^{X^*}$ defined by

$$F(x) := \{y \in K : \langle x^*, \eta(x, y) \rangle \geq \alpha(\eta(x, y)), \text{ for all } x^* \in \Phi(x)\}.$$

From condition (i), we have that $x \in F(x)$. Hence $F(x) \neq \emptyset$ for any $x \in K$. Furthermore, from condition (iii) we can easily verify that $F(x)$ is weakly closed for each $x \in K$. On the other hand, let $\{x_1, x_2, \dots, x_n\} \subset K$ and $x \in \text{co}(\{x_1, x_2, \dots, x_n\})$, proper $\eta - \alpha$ quasimonotonicity of Φ implies that $x \in \bigcup_{i=1}^n F(x_i)$. Hence, F is a KKM-mapping. Therefore, if K is weakly compact, then $F(x)$ is weakly compact for each $x \in K$, and from Lemma 3.2.20 we deduce that $\bigcap_{y \in K} F(y) \neq \emptyset$. Otherwise, from relation (3.4.8), we deduce that $F(x_0) \subset C$. Thus, $F(x_0)$ is weakly compact since it is a weakly closed subset of C . Hence, by using again Lemma 3.2.20 we deduce that $\bigcap_{y \in K} F(y) \neq \emptyset$. Which completes the proof. \square

Proposition 3.4.6 *Let K be a nonempty convex subset of the Banach space X , let $\eta : K \times K \rightarrow X$ be a mapping and $\Phi : K \rightarrow 2^{X^*}$ be an $\eta - \alpha$ quasimonotone operator. Suppose that*

- (i) $\eta(x, y) + \eta(y, x) = 0$ for all $x, y \in K$;
- (ii) $\eta(tx + (1 - t)y, z) = t\eta(x, z) + (1 - t)\eta(y, z)$ for all $t \in [0, 1]$ and $x, y, z \in K$;
- (iii) For each $x \in K$, the mappings $\eta(x, \cdot)$ and $\alpha(\eta(x, \cdot))$ are continuous.

Then one of the following holds:

- (a) Φ is properly $\eta - \alpha$ quasimonotone.

(b) $\mathbb{S}_{d,loc}(\Phi, K) \neq \emptyset$.

Proof. Suppose that Φ is not properly $\eta - \alpha$ quasimonotone. Then there exists $\{x_1, \dots, x_n\} \subset K$ and $x \in co(\{x_1, \dots, x_n\})$ such that for each $i \in \{1, \dots, n\}$ there exists $x_i^* \in \Phi(x_i)$ satisfying

$$\langle x_i^*, \eta(x_i, x) \rangle < \alpha(\eta(x_i, x)).$$

From condition (ii), we deduce that there exists a neighborhood U of x such that

$$\langle x_i^*, \eta(x_i, y) \rangle < \alpha(\eta(x_i, y)), \text{ for all } y \in K \cap U \text{ and } i = 1, \dots, n.$$

Since Φ is $\eta - \alpha$ quasimonotone, we deduce that

$$\langle y^*, \eta(x_i, y) \rangle \leq 0, \text{ for all } y^* \in \Phi(y) \text{ and } i = 1, \dots, n.$$

By using condition (ii), we get

$$\langle y^*, \eta(x, y) \rangle \leq 0, \text{ for all } y^* \in \Phi(y).$$

Hence, from (i) we obtain

$$\langle y^*, \eta(y, x) \rangle \geq 0, \text{ for all } y^* \in \Phi(y).$$

Therefore, $x \in \mathbb{S}_{d,loc}(\Phi, K)$. □

Now, we give the main result of this section.

Theorem 3.4.2 *Let K be a nonempty closed and convex subset of a Banach space with topological*

dual space X^* . Let $\eta : K \times K \rightarrow X$ and $\alpha : X \rightarrow \mathbb{R}$ be mappings and $\Phi : K \rightarrow 2^{X^*}$ be an $\eta - \alpha$ quasimonotone operator. Suppose that the following properties hold:

- (i) $\eta(x, y) + \eta(y, x) = 0$, for all $x, y \in K$;
- (ii) $\eta(tx + (1 - t)z, y) = t\eta(x, y) + (1 - t)\eta(z, y)$, for all $x, y, z \in K$ and $t \in [0, 1]$;
- (iii) For each $x \in K$ and $w \in X$, the mappings $\eta(x, \cdot)$ and $\alpha(\eta(x, \cdot))$ are continuous, and $\lim_{t \rightarrow 0^+} [\alpha(tw)/t] = 0$;
- (iv) For each $x \in K$ and $x^* \in \Phi(x)$, the mapping $y \in K \mapsto \langle x^*, \eta(x, y) \rangle$ is weakly upper semicontinuous;
- (v) Φ is upper hemicontinuous with nonempty weak* compact and convex values;
- (vi) (Coercivity condition) There exists $r_0 > 0$ such that for each $x \in K \setminus \bar{\mathbb{B}}(0, r_0)$, there exists $y \in K$ with $\|y\| < \|x\|$ satisfying: $\forall x^* \in \Phi(x), \langle x^*, \eta(x, y) \rangle \geq 0$.
Moreover, suppose that there exists $r_1 > r_0$ such that $K \cap \bar{\mathbb{B}}(0, r_1)$ is nonempty and weakly compact.

Then, the generalized variational-like inequality (3.1.1) has a solution, i.e. $\mathbb{S}(\Phi, K) \neq \emptyset$.

Proof. Let us set $K_{r_1} := K \cap \bar{\mathbb{B}}(0, r_1)$ which is a nonempty convex and weakly compact set. From Proposition 3.4.6 we have either Φ is properly $\eta - \alpha$ quasimonotone or $\mathbb{S}_{d,loc}(\Phi, K) \neq \emptyset$.

If $\mathbb{S}_{d,loc}(\Phi, K) \neq \emptyset$, we deduce from Remark 3.4.3 and Lemma 3.4.4 that $\mathbb{S}(\Phi, K) \neq \emptyset$.

If Φ is properly $\eta - \alpha$ quasimonotone, then from Theorem 3.4.1 we deduce that $\mathbb{S}_{r,d}(\Phi, K_{r_1}) \neq \emptyset$.

By using Lemma 3.4.5, we derive that $\mathbb{S}(\Phi, K_{r_1}) \neq \emptyset$. Let $x_0 \in \mathbb{S}(\Phi, K_{r_1})$, then there exists $x_0^* \in \Phi(x_0)$ such that $\langle x_0^*, \eta(y, x_0) \rangle \geq 0, \forall y \in K_{r_1}$. We have two possibilities:

- If $\|x_0\| < r_1$, then for any $y \in K$ we can find $t \in]0, 1[$ such that $y_t := ty + (1 - t)x_0 \in K_{r_1}$. Hence, $\langle x_0^*, \eta(y_t, x_0) \rangle \geq 0$. Which implies, by using (i) and (ii), that $\langle x_0^*, \eta(y, x_0) \rangle \geq 0$. Thus, $x_0 \in \mathbb{S}(\Phi, K)$.
- If $\|x_0\| = r_1$, then from condition (vi) we deduce that there exists $y_0 \in K$ with $\|y_0\| < \|x_0\|$ such that $\langle z^*, \eta(x_0, y_0) \rangle \geq 0$, for all $z^* \in \Phi(x_0)$. In particular for $z^* = x_0^*$, we get

$$\langle x_0^*, \eta(x_0, y_0) \rangle \geq 0. \quad (3.4.9)$$

On the other hand, since $y_0 \in K_{r_1}$, we obtain that $\langle x_0^*, \eta(y_0, x_0) \rangle \geq 0$. From (i), it follows

$$\langle x_0^*, \eta(x_0, y_0) \rangle \leq 0. \quad (3.4.10)$$

Hence from (3.4.9) and (3.4.10) we deduce that $\langle x_0^*, \eta(x_0, y_0) \rangle = \langle x_0^*, \eta(y_0, x_0) \rangle = 0$. Thus, y_0 is a minimum of the function $\theta(y) = \langle x_0^*, \eta(y_0, x_0) \rangle$ on K_{r_1} . This implies that y_0 is a global minimum of θ on K . Therefore, $y_0 \in \mathbb{S}(\Phi, K)$.

Which completes the proof of the theorem. □

3.5 Remarks and comments

Below, we give some remarks and comments on the results obtained in this chapter as well as some comparison with existing results in literature.

- (1) Theorem 3.3.1 extends Theorem 3.1 of Bai et al. [7] to the general case of generalized densely relaxed $\eta - \alpha$ pseudomonotone single-valued operators. Moreover, Theorem 3.3.2, considered with $\alpha(z) = -\mu\|z\|^2$ and $\eta(x, y) = x - y$, improves Corollary 3.1 in [7].

- (2) Theorems 3.3.1 and 3.3.2 extend the results obtained by Luc [30]. Furthermore, Theorems 3.3.3 and 3.3.4 give a set-valued version of the results obtained by Luc [30, Theorem 4.3, Corollary 4.5] as particular case.
- (3) Note that every generalized relaxed $\eta - \alpha$ pseudomonotone operator is a generalized densely relaxed $\eta - \alpha$ pseudomonotone operator. Therefore, Theorems 3.3.1, 3.3.2, 3.3.3 and 3.3.4 improve and extend to the general case of set-valued mapping the results obtained by Bai et al. [6]. In addition, the afore mentioned results of ours give an alternative to the results obtained by Arunchai et al in [4] which appear to be wrong results. In order to be more precise, in the proof of Theorem 3.2 in [4], the authors considered the set-valued mapping $T : K \rightarrow 2^X$ defined by

$$T(x) = \{y \in K \cap \bar{\Omega} : \langle F(y), \eta(x, y) \rangle \geq 0\},$$

where $\bar{\Omega}$ is the closure of the open ball Ω of X . In their proof, they showed that T is a KKM mapping, which means that $x \in T(x)$ for any $x \in K$. Hence, $K \subset K \cap \bar{\Omega}$. Therefore, K must be a bounded set. In this case condition (a) in [4, Theorem 3.2] is obviously obtained. The same remark is pointed out in the proof of Theorems 3.3, 4.2 and 4.3 in [4].

Furthermore, we point out that the strict η -quasimonotonicity notion introduced in [4] is nothing other than the generalized relaxed $\eta - \alpha$ pseudomonotonicity considered with $\alpha \equiv 0$ when the mapping η satisfies $\eta(x, y) + \eta(y, x) = 0$ for all $x, y \in K$. Therefore, the results obtained in the section 3 of this chapter give an alternative to the results in [4, Section 4] which are, by the same raison presented above, are wrong results.

- (4) The results obtained in Section 4 of this chapter generalize the ones obtained by Aussel and Hadjisavvas [5], as well as they extend and improve in a certain sense the results obtained by Bai et al. in [7] for relaxed μ quasimonotone, since in [7] the results are obtained on compacts

sets for single-valued μ quasimonotone operators while in this chapter we present more general results with more a general concept of quasimonotonicity for set-valued mappings.

- (5) We also point out that the results obtained in Section 4 improve considerably the results obtained by Chen and Luo in [15].

CHAPTER 4: FINDING ITERATIVE SOLUTIONS

4.1 General Algorithm and Convergence

In this section, we introduce a general algorithm using the auxiliary principal technique to approximate the solution of the problem (3.1.2).

Let $F : K \rightarrow (-\infty, +\infty]$ be a given differentiable proper convex functional, and let $\rho > 0$ be a given positive number. Consider the following auxiliary minimizing problem:

$$\min_{x \in K} [F(x) + \rho \langle Tx^*, \eta(x, x^*) \rangle - \langle F'(x^*), x \rangle]. \quad (4.1.1)$$

According to Ding [19], if $x \mapsto \langle Tx^*, \eta(x, x^*) \rangle$ is convex and $\forall x, y, z \in K, \eta(x, x) = 0$ and $\eta(x, y) + \eta(y, z) = \eta(x, z)$, then the solution of the auxiliary problem (4.1.1) can be characterized by the following variational inequality problem: find $x \in K$ such that for all $y \in K$:

$$\langle F'(x), y - x \rangle + \rho \langle Tx^*, \eta(y, x) \rangle \geq \langle F'(x^*), y - x \rangle. \quad (4.1.2)$$

Clearly, if $x = x^*$ then x^* is a solution of problem (3.1.2).

Based on this observation, we use the auxiliary principle technique to suggest the following general algorithm to compute the solution of the problem (3.1.2).

Algorithm 4.1.1 Given $\epsilon > 0$.

(i) *Step 1: Choose an arbitrary initial point $x_0 \in K$.*

(ii) *Step 2: Solve the auxiliary variational inequality (4.1.2) with $x^* = x_n$, and let x_{n+1} be the*

solution of problem (4.1.2).

(iii) Step 3: If $\|x_{n+1} - x_n\| < \epsilon$ then stop; otherwise, repeat Step 2.

Remark 4.1.2 Since the auxiliary problem (4.1.1) is a minimizing problem, there are a number of available methods for solving it. For instance, the computational algorithms including gradient, subgradient, and decomposition have been studied by Cohen [16]. In the Algorithm 4.1.1, we calculate the unique solution x_{n+1} of (4.1.2) and show that the sequence x_n is strongly convergent to x^* , which is the solution of problem (3.1.2).

The following lemma of Ding et al. will be needed in the sequel. Readers can find the proof of this lemma via Ding et al. (Ding [20] Corollary 1).

Lemma 4.1.3 (Ding [20] Corollary 1) Let K be a nonempty compact convex subset of a topological vector space X .

Let $h : K \times K \rightarrow [-\infty, +\infty]$ be a function such that for each $x \in K$, $h(x, y)$ is a lower semicontinuous function of y on K . Then for each $t \in \mathbb{R}$, there exists $y^* \in K$ such that $h(x, y^*) \leq t$ for all $x \in K$.

We are now ready to introduce and prove the main theorem.

Theorem 4.1.4 Let K be a nonempty convex compact subset of a normed space X .

Let $\eta : K \times K \rightarrow X$ and $T : K \rightarrow X^*$ be two mappings.

Let $F : X \rightarrow (-\infty, +\infty]$ be a differentiable proper convex functional. Assume that all the conditions from theorem 3.3.1 hold. In addition, suppose that

(i) the derivative F' of F is μ -strongly monotone;

(ii) the function η is δ -Lipschitz continuous;

(iii) the function α is η -strongly monotone, (i.e. $\exists \lambda > 0$ s.t. $\alpha(\eta(x, y)) \geq \lambda \|x - y\|^2 \forall x, y \in K$);

Then,

(a) there exists a solution $x^* \in K$ of problem (3.1.2);

(b) for each $\rho > 0$, there exists a unique solution $x_{n+1} \in K$ of the auxiliary problem (4.1.1) with $x^* = x_n$;

(c) if T is also β -Lipschitz continuous such that

$$0 < \mu < \min\{2\lambda - \rho\beta\delta - 1 + \frac{\rho\beta\delta}{2\lambda}, 2\lambda - \rho\beta\delta\}, \quad (4.1.3)$$

where $\lambda, \rho, \mu, \delta, \beta > 0$, then the sequence $\{x_n\}$ defined by Algorithm 4.1.1 is strongly convergent to the unique solution x^* of the problem (3.1.2).

Proof.

(a) Clearly from theorem 3.3.2, the unique solution $x^* \in K$ of problem (3.1.2) exists.

(b) For each fixed $\rho > 0$ and $x^* \in K$, define a functional $h : K \times K \rightarrow [-\infty, +\infty]$ by

$$h(x, y) = \langle F'(x^*) - F'(x), \eta(y, x) \rangle - \rho \langle Tx^*, \eta(y, x) \rangle,$$

then $h(x, y)$ satisfies all conditions of Lemma 4.1.3. Hence, let $t = 0$, there exists $y^* \in K$

such that $\forall x \in K$:

$$h(x, y^*) = \langle F'(x^*) - F'(x), \eta(x, y^*) \rangle - \rho \langle Tx^*, \eta(x, y^*) \rangle \leq 0. \quad (4.1.4)$$

Now we need to show that the solution is unique. Assume y_1 and y_2 are two solutions of (4.1.4). Since $y^* = y_1$ is a solution of (4.1.4), let $x = y_2$, we obtain

$$h(y_2, y_1) = \langle F'(x^*) - F'(y_2), \eta(y_2, y_1) \rangle - \rho \langle Tx^*, \eta(y_2, y_1) \rangle \leq 0. \quad (4.1.5)$$

Similarly,

$$h(y_1, y_2) = \langle F'(x^*) - F'(y_1), \eta(y_1, y_2) \rangle - \rho \langle Tx^*, \eta(y_1, y_2) \rangle \leq 0. \quad (4.1.6)$$

Adding (4.1.5) to (4.1.6) and notice that $\eta(y_1, y_2) + \eta(y_2, y_1) = 0$ we obtain

$$\langle F'(y_2) - F'(y_1), \eta(y_1, y_2) \rangle = 0. \quad (4.1.7)$$

Since F is a differentiable proper convex functional, it follows that $y_1 = y_2$. Hence, the solution of (4.1.4) is unique.

- (c) From part (b) above, let $x^* = x_n$, then equation (4.1.4) has a unique solution $y^* = x_{n+1}$, i.e. for each $x_n \in K$, there exists a unique $x_{n+1} \in K$ such that

$$h(x, x_{n+1}) = \langle F'(x_n) - F'(x_{n+1}), \eta(x, x_{n+1}) \rangle - \rho \langle Tx_n, \eta(x, x_{n+1}) \rangle \leq 0. \quad (4.1.8)$$

We need to show that the sequence $\{x_n\}$ converges strongly to 0 as $n \rightarrow \infty$.

Define a function $L : K \rightarrow (-\infty, +\infty]$ by

$$L(x) = F(x^*) - F(x) - \langle F'(x), \eta(x^*, x) \rangle.$$

Since F' is strongly monotone, we have

$$L(x) = F(x^*) - F(x) - \langle F'(x), \eta(x^*, x) \rangle \geq \frac{\mu}{2} \|x - x^*\|^2. \quad (4.1.9)$$

From (4.1.8) with $x = x^*$ we have

$$h(x^*, x_{n+1}) = \langle F'(x_n) - F'(x_{n+1}), \eta(x^*, x_{n+1}) \rangle - \rho \langle Tx_n, \eta(x^*, x_{n+1}) \rangle \leq 0.$$

It follows that

$$\langle F'(x_{n+1}) - F'(x_n), \eta(x^*, x_{n+1}) \rangle \geq \rho \langle Tx_n, \eta(x_{n+1}, x^*) \rangle.$$

Hence,

$$\begin{aligned}
& L(x_n) - L(x_{n+1}) \\
&= F(x_{n+1}) - F(x_n) - \langle F'(x_n), \eta(x^*, x_{n+1}) + \eta(x_{n+1}, x_n) \rangle + \langle F'(x_{n+1}), \eta(x^*, x_{n+1}) \rangle \\
&= F(x_{n+1}) - F(x_n) - \langle F'(x_n), \eta(x_{n+1}, x_n) \rangle + \langle F'(x_{n+1}) - F'(x_n), \eta(x^*, x_{n+1}) \rangle \\
&\geq \frac{\mu}{2} \|x_n - x_{n+1}\|^2 + \rho \langle Tx_n, \eta(x_{n+1}, x^*) \rangle \\
&= \frac{\mu}{2} \|x_n - x_{n+1}\|^2 + \rho \langle Tx_n - Tx_{n+1}, \eta(x_{n+1}, x^*) \rangle + \rho \langle Tx_{n+1}, \eta(x_{n+1}, x^*) \rangle \\
&= \frac{\mu}{2} \|x_n - x_{n+1}\|^2 + \rho \langle Tx_n - Tx_{n+1}, \eta(x_{n+1}, x_n) \rangle + \rho \langle Tx_n - Tx_{n+1}, \eta(x_n, x^*) \rangle \\
&\quad + \rho \langle Tx_{n+1}, \eta(x_{n+1}, x^*) \rangle \\
&\geq \frac{\mu}{2} \|x_n - x_{n+1}\|^2 - \rho\beta\delta \|x_n - x_{n+1}\|^2 - \rho\beta\delta \|x_n - x_{n+1}\| \cdot \|x_n - x^*\| + \alpha(\eta(x_{n+1}, x^*)) \\
&\geq \left(\frac{\mu}{2} - \rho\beta\delta\right) \|x_n - x_{n+1}\|^2 - \rho\beta\delta \|x_n - x^*\| \cdot \|x_n - x_{n+1}\| + \lambda \|x_{n+1} - x^*\|^2 \\
&\geq \left(\frac{\mu}{2} - \rho\beta\delta\right) \|x_n - x_{n+1}\|^2 - \rho\beta\delta \|x_n - x^*\| \cdot \|x_n - x_{n+1}\| + \lambda (\|x_{n+1} - x_n\| - \|x_n - x^*\|)^2 \\
&\geq \left(\lambda + \frac{\mu}{2} - \rho\beta\delta\right) \|x_n - x_{n+1}\|^2 + \lambda \|x_n - x^*\|^2 - (2\lambda + \rho\beta\delta) \|x_n - x^*\| \cdot \|x_n - x_{n+1}\| \\
&\geq \left(\lambda + \frac{\mu}{2} - \rho\beta\delta\right) \|x_n - x_{n+1}\|^2 + \lambda \|x_n - x^*\|^2 \\
&\quad - \frac{(2\lambda + \rho\beta\delta)}{4\left(\lambda + \frac{\mu}{2} - \rho\beta\delta\right)} \|x_n - x^*\|^2 - \left(\lambda + \frac{\mu}{2} - \rho\beta\delta\right) \|x_n - x_{n+1}\|^2 \\
&= \left(\lambda - \frac{(2\lambda + \rho\beta\delta)^2}{4\left(\lambda + \frac{\mu}{2} - \rho\beta\delta\right)}\right) \|x_n - x^*\|^2.
\end{aligned}$$

Together with the condition (4.1.3) it follows that

$$L(x_n) - L(x_{n+1}) \geq \left(\lambda - \frac{(2\lambda + \rho\beta\delta)^2}{4\left(\lambda + \frac{\mu}{2} - \rho\beta\delta\right)}\right) \|x_n - x^*\|^2 > 0.$$

Therefore, the sequence $\{L(x_n)\}$ is strictly decreasing (unless $x_n = x^*$).

Furthermore, by (4.1.9), $\{L(x_n)\}$ is a nonnegative sequence. Hence, it is a convergent sequence.

Therefore, $(L(x_n) - L(x_{n+1})) \rightarrow 0$ and so the sequence $\{x_n\}$ converges strongly to x^* as $n \rightarrow +\infty$.

This completes the proof. □

Remark 4.1.5 *The theorem provides a method to compute the solution of the variational inequality problem (3.1.2). It will be interesting to find some relaxation techniques to speed up the convergence.*

CHAPTER 5: CONCLUSION AND FUTURE RESEARCH

Our dissertation started with the study of general over-relaxed proximal point algorithm to approximate the solution of a general (A, η, m) -monotone nonlinear inclusion form. The study showed that, under some specific conditions of the coefficients, the Lipschitz continuity requirement of the monotone operator can be omitted. In chapter two, we showed that there were at least six papers with Lipschitz conditions that could be removed. Hence, there is an open research question for future study: can Lipschitz continuity condition also be removed from other variational inclusion papers?

Secondly, in chapter three we studied the existence and uniqueness of the solution of a variational-like inequality with densely relaxed pseudomonotone operators and relaxed quasimonotone operators. The study showed that the existence of the solution could be extended from relaxed pseudomonotone operator to densely relaxed pseudomonotone operators. Also, our study showed that the results obtained in Arunchai ([4], section 4) has some incorrect assertions and we have provided the correct version of those results in Chapter 3. It will be interesting to prove its vector analogue and devise a method to seek an iterated sequence which converges to the solution. Once that is established, then we can determine ways to speed up the convergence by using relaxation techniques.

Finally, in chapter 5 we introduced a general algorithm for estimating the numerical solution by applying the auxiliary principle technique, which introduced by Cohen and later on used by Noor, Pany, Mohapatra, and Pani. What all the above papers referred to and our work lacks is a determination of the degree of convergence of the iterated sequence to the solution. We need to seek Newton-like method to obtain solutions and, if possible, get quadratic convergence. For the future study, we believe that similar approach can be applied to relaxed quasimonotone operators to prove

the existence and uniqueness of the solution and to develop an algorithm for estimating a numerical solution.

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