2020

# Mean Field Optimal Control and Related Problems

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# MEAN FIELD OPTIMAL CONTROL AND RELATED PROBLEMS

by

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A dissertation submitted in partial fulfilment of the requirements
for the degree of Doctor of Philosophy
in the Department of Mathematics
in the College of Science
at the University of Central Florida
Orlando, Florida

Summer Term
2020

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# ABSTRACT

It has been decades since the first paper that mean field problems were studied. More and more problems are considered or solved as new methods and new concepts have been developed. In this dissertation, we will present a series of results on (recursive) mean field stochastic optimal control problems.

Comparing our results with those in the classical stochastic optimal control theory, there are following significant differences. First, the value function of a mean field optimal control problem is not Markovian any more, even when coefficient functions in the problem are deterministic. Second, the cost functional we considered is induced by a mean field backward stochastic differential equation. This leads to the value function to be random. Last but not the least, the backward stochastic differential equation we considered is of McKean-Vlasov form. The appearance of the distribution of its solution $Y$ at time $s$ leads to a new Hamiltion-Jacobi-Bellman equation.

To overcome these difficulties, we first introduce an auxiliary problem associated with the original optimal control problem, so that we can better analyze the dependence of the value function $V$ on the initial state $\xi$. We also give a description of optimal control by a necessary condition, which is derived from the Hamiltion-Jacobi-Bellman equation. About this new HJB equation, we will prove the verification theorem and introduce the notion of viscosity solution.

# ACKNOWLEDGMENTS

First I would like to express my sincere gratitude to my advisor, Professor Jiongmin Yong, for the guidance and encouragement he has given me during my entire time as a student at University of Central Florida.

Also, I would like to thank the rest of the professors in my dissertation committee, Professor Liqiang Ni, Yuanwei Qi and Jason Swanson, for their service on my dissertation committee.

Then, I want to thank my friends Jie, Jason, Jingmei, Nazar, Tong, Yuchen and many more who have made my life at UCF wonderful.

Last but not the least, I want to thank my family for the encouragement, support and company.

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# CHAPTER 1: INTRODUCTION

In this chapter, we review the main results in stochastic optimal control. The problem has been investigated in many literatures. See, for example, [17][18][53][43]. One powerful tool is the dynamic programming principle (DPP), also known as Bellman's principle of optimality, which was introduced by Bellman in 1950s, see [3][4][5]. With the help of dynamic programming principle, a partial differential equation (PDE), called *Hamliton-Jacobi-Bellman (HJB) equation*, can be derived to characterize value function in the sense that the value function is the unique viscosity solution of the HJB equation.

We will review the optimal control problem for which the cost functional is defined by a backward stochastic differential equation (BSDE). This is called a classical *recursive stochastic optimal control problem*. Further, we will also review the basics for the calculus in Wasserstein space, which is fundamentally important for discussions about equations of McKean-Vlasov type.

## 1.1 Recursive Stochastic Optimal Control

### *1.1.1 An Example*

First, we consider a classical example in reality, that is the optimal portfolio selection problem. Suppose that there are $n+1$ assets in market, which contain a bond, whose price process is denoted by $S_0$, and $n$ stocks with price processes denoted by $S_i$, $i = 1, ..., n$. We assume the bond price process follows

$$dS_0(s) = r(s)S_0(s)ds, \tag{1.1}$$

with $s \in [0, T]$ and initial condition $S_0(0) = s_0 \in \mathbb{R}$. In comparison, the investment in stocks is risky due to the randomness of stock prices. This is modeled by

$$dS_i(s) = \mu S_i(s)ds + \sigma S_i(s)dW(s), \tag{1.2}$$

with $s \in [0, T]$ and initial condition $S_i(0) = s_i \in \mathbb{R}$. The process $W$ is a standard Brownian motion in the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. For each asset $i$, let $\alpha_i(s)$ represent the portion of the wealth invested in asset $i$ at time $s$. Then, it is easy to see that the wealth process corresponding to the portfolio $\alpha(s)$ is $X(s) = \sum_{i=0}^{n+1} S_i(s)\alpha(s)$, for which the dynamics, under self-financing condition, is

$$\begin{cases} dX(s) = X(s)(r + \alpha(s)(\mu - r))dt + X(s)\alpha(s)\sigma dW(s), \\ X(t) = x. \end{cases} \tag{1.3}$$

Here each portfolio $\alpha : [t, T] \times \Omega \to A$ is regarded as a control, which is usually assumed to be a progressively measurable stochastic process and integrable in the sense that $\mathbb{E}[\int_t^T |\alpha(s)|^2 ds] < \infty$. It is called an *open-loop control* and we use $\mathcal{A}_{[t,T]}$ to denote the set of all such controls. The goal for an agent in financial market is to maximize the expected utility of terminal wealth at horizon $T$, that is, to find an optimal portfolio $\alpha^*$ such that,

$$\mathbb{E}[U(X(T; t, x; \alpha^*))] = \sup_{\alpha \in \mathcal{A}_{[t,T]}} \mathbb{E}[U(X(T; t, x; \alpha))] = V(t, x), \tag{1.4}$$

where $X(\cdot; t, x; \alpha)$ is the solution of (1.3), $U : x \mapsto U(x)$ is a utility function (assumed to be increasing and concave). A more strict and general description about optimal portfolio selection model as well as methods used to analyze it will be given in next section.

### 1.1.2 Mathematical Framework

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions. That is, $(\Omega, \mathcal{F}, \mathbb{P})$ is complete, $\mathcal{F}_0$ contains all $\mathbb{P}-$null sets in $\mathcal{F}$ and $\mathbb{F}$ and $\mathbb{F} = \{\mathcal{F}_t\}_{t \geqslant 0}$ is right continuous. Let $W$ be a $d$-dimensional standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. Here, suppose that $\mathcal{F}_t = \sigma\{W(s) : 0 \leq s \leq t\}$. Let $T > 0$. The wealth process is the solution of the following stochastic differential equation (SDE), which is usually called the *state process* and has the general form:

$$\begin{cases} dX(s) = b(s, X(s), u(s))ds + \sigma(s, X(s), u(s))dW(s), \qquad s \in [t, T], \\ X(t) = x, \end{cases} \tag{1.5}$$

where $t \in [0, T]$, $x \in \mathbb{R}^n$, $b : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}^n$, $\sigma : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}^{n \times d}$ are given (deterministic) maps, $U \subseteq \mathbb{R}^m$ is a non-empty set. Here $(t, x)$ is called an *initial pair* and $u$ is called a *control process*. To be more clear, $u$ here is chosen from the following *admissible control set*:

$$\mathcal{U}_{[t,T]} = \{u : \Omega \times [t, T] \to U : \ u \text{ is progressively measurable, } \mathbb{E}\Big[\int_t^T |u(s)|^2 ds\Big] < \infty\}. \tag{1.6}$$

Each $u \in \mathcal{U}_{[t,T]}$ is also called an *open-loop control*. It features that in this system, the controller does not make decisions based on the information of the states but choose the optimal one from a very (most) general pool of options, see [17].

To measure performance of each control, we introduce the following so-called recursive cost functional:

$$J(t, x, u) = Y(t; t, x; u), \tag{1.7}$$

where $(Y, Z)$ is the adapted solution of the following BSDE:

$$\begin{cases} dY(s) = -g(s, X(s), Y(s), Z(s), u(s))ds + Z(s)dW(s), \qquad s \in [t, T], \\ Y(T) = h(X(T)). \end{cases} \tag{1.8}$$

The advantage of considering a cost functional in the recursive form is that it takes into account of investor's attitudes: optimistic or pessimistic. When evaluating the current financial situations (portfolio of assets), the future utility should be taken into account, i.e., the current utility depends on the future utility, besides other dependence. *Recursive utility* was introduced to describe such a situation. In 1992, Duffie and Epstein introduced stochastic differential utility, see [15][16].

Now we introduce the following stochastic optimal control problem.

**Problem ($\mathbf{C_0}$).** For given $(t, x) \in [0, T] \times \mathbb{R}^n$, find a control $u^* \in \mathcal{U}_{[t,T]}$ such that

$$J(t, x, u^*) = \operatorname*{essinf}_{u \in \mathcal{U}_{[t,T]}} J(t, x, u) \equiv V(t, x). \tag{1.9}$$

The function $V(t, x)$ is called *value function*, and it satisfies the following dynamic programming principle:

**Proposition 1.1.1.** *For each $\tau \in [t, T]$, the value function $V(t, x)$ satisfies the following equation:*

$$V(t, x) = \inf_{u \in \mathcal{U}_{[t,\tau]}} \{\tilde{Y}(t; t, x; u)\}. \tag{1.10}$$

*where $(\tilde{Y}, \tilde{Z})$ is the adapted solution of the following BSDE:*

$$\begin{aligned} \tilde{Y}(t; t, x; u) = V(\tau, X(\tau; t, x; u)) - \int_t^\tau \tilde{Z}(r)dW(r) \\ + \int_t^\tau g(r, X(r; t, x; u), \tilde{Y}(r), \tilde{Z}(r), u(r))dr. \end{aligned} \tag{1.11}$$

*Proof.* See [42][52]. □

**Remark 1.1.2.** *The feature of Problem ($C_0$) is that the cost functional is induced by a BSDE, which means the cost functional $J(t,x,u)$, therefore, the value function $V(t,x)$, is random in general. However, it can be proved that $V(t,x)$ is actually a deterministic $\mathbb{R}-$valued function. This important observation is mentioned in [42], while some important details are not strictly proved there and we add it in Appendix A.*

By dynamic programming principle and Itô's formula, the HJB equation for the value function $V$ can be derived. Here are some results.

**Proposition 1.1.3.** *(Verification Theorem) If $\phi \in C^{1,2}([0,T]\times\mathbb{R}^n)$ is a classical solution to the following PDE:*

$$\begin{cases} \phi_t(t,x) + \inf\limits_{u\in U}\Big\{\phi_x(t,x)b(t,x,u) + \dfrac{1}{2}tr[\phi_{xx}(t,x)\sigma(t,x,u)\sigma^T(t,x,u)] \\ \qquad\qquad +g\big(t,x,\phi(t,x),\phi_x(t,x)\sigma(t,x,u),u\big)\Big\} = 0, \qquad (t,x)\in[0,T]\times\mathbb{R}^n, \\ \phi(T,x) = h(x), \end{cases} \tag{1.12}$$

*then $\phi = V$. Furthermore, suppose for each $(t,x)$, the set*

$$\arg\min_{u\in U}\Big\{V_x(t,x)b(t,x,u) + \frac{1}{2}tr[V_{xx}(t,x)\sigma(t,x,u)\sigma^T(t,x,u)] \\ +g\big(t,x,V(t,x),V_x(t,x)\sigma(t,x,u),u\big)\Big\}$$

*is a singleton and let*

$$\psi(t,x) = \arg\min_{u\in U}\Big\{V_x(t,x)b(t,x,u) + \frac{1}{2}tr[V_{xx}(t,x)\sigma(t,x,u)\sigma^T(t,x,u)] \\ +g\big(t,x,V(t,x),V_x(t,x)\sigma(t,x,u),u\big)\Big\},$$

*then $u^*(s) \equiv \psi(s,X^*(s;t,x,u^*))$ is an optimal control of Problem ($C_0$), provided that the state* *equation* (1.5) *admits a unique solution under* $u^*$.

*Proof.* The proof follows the same idea as the one for Theorem (3.4.1). □

If an HJB equation has classical solution then it can be proved that, this solution coincides with the value function. While, some examples tell us that value functions may not be differentiable, and the corresponding HJB equations might not have classical solutions. To overcome this difficulty, the notion of viscosity solution (due to Crandall–Lions 1980s) was introduced.

**Proposition 1.1.4.** *The value function* $V$ *is a viscosity solution of the PDE* (1.12).

*Proof.* See [42]. □

## 1.2 Mean Field Interactions and Wasserstein Space

Since papers like [22][23] [24][25][26] and [34][35][36], mean field game problems have attracted more and more attention. Mean field problems have actually been investigated since the mid of last century. It is not surprising that its first appearance was not in a mathematical paper but was about statistical mechanics and physics, see [28]. This important paper leads to the later works by McKean [39] about the Mckean-Vlasov type of stochastic differential equations, and Sznitman [45] about the propagation of chaos.

In this section, we first consider some examples about mean field stochastic differential equations and then review some mathematical prerequisites which are important in the research of mean field problems now.

### 1.2.1 Mean Field Stochastic Differential Equations

Mean field interactions can be expressed in different ways. Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space, on which $W$ is a standard Brownian motion. Let $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{\mathbb{P}})$ be a copy of $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. Here is an important mean field type:

$$\begin{cases} dX(s) = \bar{b}(s, X(s), \mathbb{E}[X(s)])ds + \bar{\sigma}(s, X(s), \mathbb{E}[X(s)])dW(s), \quad s \in [t, T], \\ X(t) = \xi, \end{cases} \tag{1.13}$$

or more generally,

$$\begin{cases} dX(s) = \tilde{\mathbb{E}}\Big[\bar{b}(s, X(s;t,\xi), \tilde{X}(s;0,x))\Big]ds \\ \qquad\qquad + \tilde{\mathbb{E}}\Big[\bar{\sigma}(s, X(s;t,\xi), \tilde{X}(s;0,x))\Big]dW(s), \quad s \in [t, T], \\ X(t) = \xi. \end{cases} \tag{1.14}$$

where $\tilde{X}$ is a copy of $X$. The other type is the so-called McKean-Vlasov type. The mean field interaction is expressed through distribution processes. The problem we considered is of McKean-Vlasov form. Here is a general form,

$$\begin{cases} dX(s) = b(s, X(s), \mathbb{P}_{X(s)})ds + \sigma(s, X(s), \mathbb{P}_{X(s)})dW(s), \quad s \in [t, T], \\ X(t) = \xi, \end{cases} \tag{1.15}$$

where $\xi$ is an $\mathcal{F}_t-$measurable random variable. The McKean-Vlasov framework is actually a more general case essentially. It is obvious that the frame work (1.13) can be written in terms of the form (1.15) by letting

$$b(s, x, \mu) = \bar{b}(s, x, \int_{\mathbb{R}^n} \bar{x}\mu(d\bar{x})).$$

Now we give an example to show that (1.15) is actually a more general case. That is, not every form in (1.13) can be written in the form (1.15).

**Example 1.2.1.** *Let $g : \mathbb{R} \to \mathbb{R}$ have continuous second order derivative and $h : \mathcal{P}^2 \to \mathbb{R}$ be defined by*

$$h(\mathbb{P}_\xi) = \int_{-\infty}^{0} g(\mathbb{P}(\xi \geq t) - 1)dt + \int_{0}^{\infty} g(\mathbb{P}(\xi \leq t))dt.$$

*It is easy to see that this function cannot be written in terms of expectations.*

### *1.2.2 Wasserstein Space*

From the formulation of the mean field optimal control problem we can tell that it needs calculus in the space of probability measures to help further exploring deeper results. Let $(M, d)$ be a metric space. For each $p \geq 1$, define the space $\mathcal{P}^p(M)$ by

$$\mathcal{P}^p(M) = \{\mu : M \to [0,1] \text{ is a measure} : \int_M d(x, x_0)^p \mu(dx) < \infty \text{ for some } x_0 \in M\}. \quad (1.16)$$

For any $\mu$, $\nu \in \mathcal{P}^p(M)$, the $p-$*Wasserstein distance* between them is defined by

$$\mathcal{W}_p(\mu, \nu) = (\inf_{\lambda \in \Lambda(\mu,\nu)} \int_{M \times M} d(x,y)^p \lambda(dx, dy))^{\frac{1}{p}}, \quad (1.17)$$

where $\Lambda(\mu, \nu) = \{\lambda : M \times M \to [0,1] \text{ is a probability measure} : \lambda(\cdot, M) = \mu(\cdot), \lambda(M, \cdot) = \nu(\cdot)\}$. $\lambda$ is called a coupling of $\mu$ and $\nu$. The space $(\mathcal{P}^p(M), \mathcal{W}_p(\cdot,\cdot))$ is called a *Wasserstein space*. In the rest of the dissertation, we adopt the notation $\mathcal{P}^2 = \mathcal{P}^2(\mathbb{R}^n)$.

**Example 1.2.2.** *Let $M = \mathbb{R}$, $d(x,y) = |x - y|$, $\mu = \delta_{\{x_0\}}$ and $\nu = \delta_{\{y_0\}}$ for some $x_0,\ y_0 \in \mathbb{R}$. Since*

$$\int_{\mathbb{R}} |x|^p \mu(dx) < \infty, \ \int_{\mathbb{R}} |x|^p \nu(dx) < \infty,$$

*for all* $p \geq 1$*. Then* $\mu,\ \nu \in \mathcal{P}^p(\mathbb{R})$*. Moreover, the Wasserstein distance between these two delta measures is*

$$\begin{aligned}\mathcal{W}_p(\mu,\nu) &= (\inf_{\lambda\in\Lambda(\mu,\nu)}\int_{\mathbb{R}\times\mathbb{R}}|x-y|^p\lambda(dx,dy))^{\frac{1}{p}}\\ &= |x_0-y_0|.\end{aligned}$$

**Example 1.2.3.** *Let* $M=[0,1]$*,* $d(x,y)=|x-y|$*,* $\mu$ *be the Lebesgue measure and* $\nu=\delta_{\{x_0\}}$ *for some* $x_0\in[0,1]$*. Since*

$$\int_{[0,1]}|x|^p\mu(dx)<1,$$

$$\int_{[0,1]}|x|^p\nu(dx)\leqslant 1,$$

*for all* $p \geq 1$*. Then* $\mu,\ \nu \in \mathcal{P}^p([0,1])$*. Moreover, the Wasserstein distance between Lebesgue measure and delta measure on* $[0,1]$ *is*

$$\begin{aligned}\mathcal{W}_p(\mu,\nu) &= (\inf_{\lambda\in\Lambda(\mu,\nu)}\int_{[0,1]\times[0,1]}|x-y|^p\lambda(dx,dy))^{\frac{1}{p}}\\ &= \int_0^1|x-x_0|^p dx\\ &= \frac{1}{p+1}\Big[(1-x_0)^{p+1}-x_0^{p+1}\Big].\end{aligned}$$

Let $\Omega$ be a Polish space, $\mathcal{G}$ be its Borel $\sigma-$algebra and $\mathbb{P}$ be an atomless probability measure. Then for each $\mu\in\mathcal{P}^2$, there exists a random variable $\xi\in L^2(\Omega,\mathcal{G};\mathbb{R}^n)$ such that $\mu=\mathbb{P}_\xi$. Let $f:\mathcal{P}^2\to\mathbb{R}$, the *lifting* of $f$ is defined to be the function $\tilde{f}:L^2(\Omega,\mathcal{G};\mathbb{R}^n)\to\mathbb{R}$ by letting

$$\tilde{f}(\xi)=f(\mathbb{P}_\xi).$$

**Definition 1.2.4.** *Function* $f:\mathcal{P}^2\to\mathbb{R}$ *is called differentiable at* $\mu$*, if there exists a random* *variable $\xi$ with $\mathbb{P}_\xi = \mu$ such that its lifting $\tilde{f} : L^2(\Omega, \mathcal{G}; \mathbb{R}^n) \to \mathbb{R}$ is Fréchet differentiable at $\xi$.*

By Proposition 5.25 in [12], we know that when $\tilde{f}$ is differentiable, there exists a deterministic function, denoted by $\partial_\mu f(\mu) : \mathbb{R}^n \to \mathbb{R}$, such that

$$D\tilde{f}(\cdot) = \partial_\mu f(\mu)(\cdot).$$

This function $\partial_\mu f(\mu)(\cdot)$ is defined to be the derivative of $f$. Since it is a deterministic function on $\mathbb{R}^n$, its derivative can be defined. We use the notation $\partial_x \partial_\mu f(\mu)(\cdot)$.

**Example 1.2.5.** *Let $f, g : \mathcal{P}^2(\mathbb{R}) \to \mathbb{R}$ be defined by*

$$f(\mu) = \int_{\mathbb{R}} x^2 \mu(dx), \;\; g(\mu) = (\int_{\mathbb{R}} x \mu(dx))^2.$$

*Then*

$$\partial_\mu f(\mu)(x) = 2x, \;\; \partial_x \partial_\mu f(\mu)(x) = 2,$$

*and*

$$\partial_\mu g(\mu)(x) = 2 \int_{\mathbb{R}} x' \mu(dx'), \;\; \partial_x \partial_\mu g(\mu)(x) = 0.$$

# CHAPTER 2: CLASSICAL MEAN FIELD OPTIMAL CONTROL

The objective in this chapter is to consider an optimal control problem with only forward McKean-Vlasov dynamics. Here we call it *classical mean field optimal control problem*. We point out that the control we considered in this chapter is the so-called *closed-loop strategy* or *control law*. In some literatures, see [44], it is also called feedback control, though there is a significant difference between the control law in there and the one in classical optimal control problem. The reason for the choice of closed-loop strategy, instead of open-loop control, should be the technical difficulties encountered when considering the later one. While, there are still some inspiring results obtained for this problem, see, for example, [2].

Our motivation of considering the mean field stochastic optimal control problems (both the classical and recursive cases) comes from our curiosity in the following question: if optimal control problems with mean field type of influences are time-consistent? Moreover, what type of mean field influence corresponds to time-inconsistency? Does it give any new insights on the reasons for occurrence of time-inconsistency? The famous mean-variance model and the problems considered in [51] are examples showing the time-inconsistency of a mean field optimal control problem. While, some other cases in recent paper, for example, [44] [49], showed the opposite. To answer or better understand these questions, we start the research on the recursive mean field optimal control problem. For more details regarding time-inconsistency, see [48][49][50][51][47].

The problem is also considered in [44]. In comparison, we consider a different admissible control set and use a different way to talk about viscosity solutions. We also cover more details in our proof. Another significant difference is about the time-consistency of this problem. Different from the remark in [44], we conclude the problem considered here is time-consistent based on dynamic programming principle.

## 2.1 Examples

**Example 2.1.1.** *Consider:*

$$\begin{cases} dX(s) = u(s)ds + X(s)dW(s), \qquad s \in [t,T] \\ X(t) = x, \end{cases} \tag{2.1}$$

*with cost functional*

$$J(t,x,u) = \mathbb{E}\Big[\int_t^T |u(s)|^2 ds + |\mathbb{E}[X(T)]|^2\Big]. \tag{2.2}$$

*It can be proved that the optimal control $u^*$ has feedback form:*

$$u^*(s,\mu) = -\frac{1}{T-s+1}\int_{\mathbb{R}^n} x'\mu(dx'). \tag{2.3}$$

Note that the independence of $u^*$ on $(t,x)$ means the problem is time-consistent. While, when consider conditional expectations, we have a different observation:

**Example 2.1.2.** *Consider a one-dimensional controlled linear SDE:*

$$\begin{cases} dX(s) = u(s)ds + X(s)dW(s), \qquad s \in [t,T] \\ X(t) = x, \end{cases} \tag{2.4}$$

*with cost functional*

$$J(t,x,u) = \mathbb{E}_t\Big[\int_t^T |u(s)|^2 ds + |\mathbb{E}_t[X(T)]|^2\Big]. \tag{2.5}$$

*It can be shown that the optimal control corresponding to initial condition $(t,x)$ is*

$$u^*(s;t,x) = -\hat{P}(s)\mathbb{E}_t[X^*(s)], \;\; s \in [t,T], \tag{2.6}$$

*where $\hat{P}$ is the solution of the following Ricatti equation:*

$$\begin{cases} \hat{P}'(s) - \hat{P}(s)^2 = 0, \qquad s \in [0,T], \\ \hat{P}(T) = 1, \end{cases} \tag{2.7}$$

*and $(X^*, u^*)$ is the optimal pair. Time-inconsistency of the optimal control is shown by*

$$u^*(s;t,x) = -\hat{P}(s)\mathbb{E}_t[X^*(s)] = -\frac{x}{T-t+1} \neq -\frac{X^*(\tau)}{T-\tau+1} = u^*(s;\tau, X^*(\tau)).$$

Let $h(t,x) = \mathbb{E}_{t,x}[X^*(s;t,x,u^*)]$. Then,

$$u^*(s;t,x) = -\hat{P}(s)h(t,x) = -\frac{x}{T-t+1},$$

and

$$u^*(s;\tau,X^*(\tau)) = -\hat{P}(s)h(\tau,X^*(\tau)) = -\frac{X^*(\tau)}{T-\tau+1}.$$

It can be seen that the optimal close-loop strategy corresponding to initial condition $(t,x)$ is $u^*(s,x',\mu;t,x) = -\hat{P}(s)h(t,x)$, which depends on the initial condition of the problem $(t,x)$. We can tell that the close-loop optimal control problem is also time-inconsistent.

## 2.2 The Problem Considered

The problem considered here is in the same form as in [44], while the discussions, especially the one about viscosity solutions, are different. To give a complete picture about the problem, we start from the following introduction.

In the rest of the paper, let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space on which a $d$-

dimensional standard Brownian motion $W$ is defined. Let $\mathcal{G}$ be a filtration that is independent of $W$ and large enough such that $\mathcal{P} = \{\mu \mid \mu = \mathbb{P}_\eta \text{ for some } \mathcal{G}-\text{measurable } \eta\}$. The natural filtration of $W$, augmented by all the $\mathbb{P}$-null sets and $\mathcal{G}$, is denoted by $\mathbb{F} = \{\mathcal{F}_t\}_{t\geqslant 0}$. Let $p, q \in [1, \infty)$, $t \in [0, T]$, we introduce the following spaces.

$$
\begin{aligned}
&\mathcal{D} = \Big\{(t,\xi) : t \in [0,T], \xi \in L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)\Big\},\\
&\mathcal{P}(\mathbb{R}^n) = \Big\{\mu : \mathbb{R}^n \to [0,1] \text{ is a measure}\Big\},\\
&\mathcal{P}^2(\mathbb{R}^n) = \Big\{\mu \in \mathcal{P}(\mathbb{R}^n) : \int_{\mathbb{R}^n} |x|^2 \mu(dx) < \infty\Big\},\\
&L^p_{\mathcal{F}_t}(\Omega;\mathbb{R}^n) = \Big\{\xi : \Omega \to \mathbb{R}^n : \xi \text{ is } \mathcal{F}_t\text{-measurable}, \mathbb{E}[|\xi|^p] < \infty\Big\},\\
&L^p_{\mathcal{F}_T}(\Omega; L^q(t,T;\mathbb{R}^n)) = \Big\{\phi : [t,T] \times \Omega \to \mathbb{R}^n : \phi(\cdot) \text{ is } \mathcal{B}[t,T] \otimes \mathcal{F}_T\text{-measurable},\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \mathbb{E}\Big[\int_t^T |\phi(s)|^q ds\Big]^{\frac{p}{q}} < \infty\Big\},\\
&L^p_{\mathbb{F}}(\Omega; L^q(t,T;\mathbb{R}^n)) = \Big\{\phi \in L^p_{\mathcal{F}_T}(\Omega; L^q(t,T;\mathbb{R}^n)) :\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \phi \text{ is } \mathbb{F}\text{-progressively measurable}\Big\},\\
&L^p_{\mathbb{F}}(\Omega; C([t,T];\mathbb{R}^n)) = \Big\{\phi : [t,T] \times \Omega \to \mathbb{R}^n : \phi \text{ is } \mathbb{F}\text{-adapted and has continuous paths},\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \mathbb{E}\Big[\sup_{s\in[t,T]} |\phi(s)|^p\Big] < \infty\Big\}.
\end{aligned}
$$

For $p = \infty$ and/or $q = \infty$, we can obviously define the corresponding spaces. We denote

$$
L^p_{\mathbb{F}}(\Omega; L^p(0,T;\mathbb{R}^n)) = L^p_{\mathbb{F}}(0,T;\mathbb{R}^n), \qquad 1 \leqslant p \leqslant \infty.
$$

Now consider the state dynamics:

$$
\begin{cases}
dX(s) = b(s, X(s), \mathbb{P}_{X(s)}, u(s, X(s), \mathbb{P}_{X(s)}))ds \\
\qquad\qquad +\sigma(s, X(s), \mathbb{P}_{X(s)}, u(s, X(s), \mathbb{P}_{X(s)}))dW(s), \quad s \in [t, T], \\
X(t) = \xi,
\end{cases}
\tag{2.8}
$$

where $b : [0,T] \times \mathbb{R}^n \times \mathcal{P}^2 \times U \to \mathbb{R}^n$, $\sigma : [0,T] \times \mathbb{R}^n \times \mathcal{P}^2 \times U \to \mathbb{R}^{n\times d}$ are given (deterministic) maps, $U \subseteq \mathbb{R}^m$ is a non-empty set. $(t, \xi) \in \mathcal{D}$ is called an *initial pair*. Under proper conditions, (2.8) has the unique solution $X(\cdot)$, it is called *state process*. $u(\cdot)$ is called a *control law*. Here is a detailed discussion.

For each $L > 0$, we introduce the *admissible control set* $\mathcal{U}_L$, which is the set of all functions $u : [0,T] \times \mathbb{R}^n \times \mathcal{P}^2 \to U$ such that

$$
\begin{aligned}
|u(s, x, \mu) - u(s, x', \mu')| \leqslant L(|x - x'| + \mathcal{W}_2(\mu, \mu')), \\
\forall s \in [0,T],\ x, x' \in \mathbb{R}^n,\ \mu, \mu' \in \mathcal{P}^2,
\end{aligned}
\tag{2.9}
$$

and

$$
\int_0^T |u(s, 0, \delta_0)|^2 ds < \infty. \tag{2.10}
$$

Define $\mathcal{U} \equiv \cup_{L>0}\mathcal{U}_L$. Let $u \in \mathcal{U}_L$, fix $(s, \mu) \in [0,T] \times \mathcal{P}^2$. Then $u(s, \cdot, \mu)$ is a function from $\mathbb{R}^n$ to $U$, which is uniformly Lipschitz continuous with Lipschitz constant $L$. We use the notation $u(s, \cdot, \mu) \in \mathbb{L}_L(\mathbb{R}^n; U)$.

**Remark 2.2.1.** *It is easy to see that for any $u \in \mathcal{U}$, and $X$ being the corresponding adapted solution of* (2.8)*, then $u(s, X(s), \mathbb{P}_{X(s)})$ is progressively measurable and*

$$
\mathbb{E}[\int_t^T |u(s, X(s), \mathbb{P}_{X(s)})|^2 ds] < \infty.
$$

*This means the process*

$$\alpha(s) \equiv u(s, X(s), \mathbb{P}_{X(s)})$$

*is an open-loop control. While, generally, an open-loop control may not be written in the feedback form.*

To measure the performance of each control, we introduce the following cost functional:

$$J(t,\xi,u) = \mathbb{E}\Big[h(X(T),\mathbb{P}_{X(T)}) + \int_t^T g(s,X(s),\mathbb{P}_{X(s)},u(s,X(s),\mathbb{P}_{X(s)}))ds\Big], \tag{2.11}$$

where $h : \mathbb{R}^n \times \mathcal{P}^2 \to \mathbb{R}$ and $g : [0,T] \times \mathbb{R}^n \times \mathcal{P}^2 \times U \to \mathbb{R}$, for which we introduce the following assumption:

**(H2)** The maps $g : [0,T] \times \mathbb{R}^n \times \mathcal{P}^2 \times U \to \mathbb{R}$ and $h : \mathbb{R}^n \times \mathcal{P}^2 \to \mathbb{R}$ are continuous and there exists a constant $M > 0$ such that

$$\begin{aligned} |g(t,x,\mu,u)| + |h(x',\nu)| \leqslant M\big(1 + |x|^2 + |x'|^2 + \|\mu\|_2^2 + \|\nu\|_2^2 + |u|^2\big), \\ \forall (t,x,x',u,\mu,\nu) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times U \times \mathcal{P}^2 \times \mathcal{P}^2. \end{aligned} \tag{2.12}$$

It is clear that for any $(t,\xi) \in \mathcal{D}$ and $u(\cdot) \in \mathcal{U}_L$, the state process $X(\cdot) \in L^2_{\mathbb{F}}(\Omega; C([t,T];\mathbb{R}^n))$. By **(H2)**,

$$\begin{aligned} &|g(s,X(s),\mathbb{P}_{X(s)},u(s,X(s),\mathbb{P}_{X(s)}))| + |h(X(T),\mathbb{P}_{X(T)})| \\ &\leqslant M_L(1 + |X(s)|^2 + \mathbb{E}[|X(s)|^2] + |X(T)|^2 + \mathbb{E}[|X(T)|^2] + |u(s,0,\delta_0)|^2), \end{aligned}$$

for some constant $M_L > 0$. Hence, the cost functional

$$J(t,\xi,u) < \infty.$$

Now we introduce the following stochastic optimal control problem:

**Problem (C)**. For given $(t, \xi) \in \mathcal{D}$, find a control $u^* \in \mathcal{U}$ such that

$$J(t, \xi, u^*) = \inf_{u \in \mathcal{U}} J(t, \xi, u) \equiv V(t, \xi). \tag{2.13}$$

Any such a control $u^* \in \mathcal{U}$ is called an *optimal control* of Problem (C). The map $V : (t, \xi) \mapsto V(t, \xi)$ is called the *value function* of **Problem (C)**. Note that $V$ is a function defined on $\mathcal{D}$. Generally, it is not in a Markovian form anymore, i.e.,

$$V(t, \xi) \neq V(t, x)|_{x=\xi},$$

or equivalently,

$$V(t, \xi)(\omega) \neq V(t, \xi(\omega)).$$

For each $L > 0$, we also consider the following stochastic optimal control problem:

**Problem ($\mathbf{C_L}$)**. For given $(t, \xi) \in \mathcal{D}$, find a control $u^* \in \mathcal{U}_L$ such that

$$J(t, \xi, u^*) = \inf_{u(\cdot) \in \mathcal{U}_L} J(t, \xi, u) \equiv V_L(t, \xi). \tag{2.14}$$

## 2.3 Main Properties

### *2.3.1 Solution of the Mean Field Controlled SDE*

To talk about solution results for equation (2.8), we first introduce following conditions on coefficients $b$ and $\sigma$:

**(H1)** The maps $b : [0,T] \times \mathbb{R}^n \times \mathcal{P}^2 \times U \to \mathbb{R}^n$ and $\sigma : [0,T] \times \mathbb{R}^n \times \mathcal{P}^2 \times U \to \mathbb{R}^{n\times d}$ are continuous and there exists a constant $C > 0$ such that

$$\begin{aligned} &|b(s,x,\mu,u) - b(s,x',\mu',u')| + |\sigma(s,x,\mu,u) - \sigma(s,x',\mu',u')| \\ &\leqslant C(|x-x'| + \mathcal{W}_2(\mu,\mu') + |u-u'|), \end{aligned} \tag{2.15}$$

for all $s \in [0,T], u, u' \in U,\ x, x' \in \mathbb{R}^n,\ \mu, \mu' \in \mathcal{P}^2$. And

$$|b(s,x,\mu,u)| + |\sigma(s,x,\mu,u)| \leqslant C(1 + |x| + \|\mu\|_2), \tag{2.16}$$

for all $s \in [0,T], u \in U,\ x \in \mathbb{R}^n,\ \mu \in \mathcal{P}^2$. Here $\|\mu\|_2 \equiv \mathcal{W}_2(\mu, \delta_0)$.

**Proposition 2.3.1.** *Under* **(H1)***, for any* $(t,\xi) \in \mathcal{D}$ *and any* $u(\cdot) \in \mathcal{U}$*, there exists a unique solution* $X(\cdot) = X(\cdot\,; t, \xi; u) \in L^2_{\mathbb{F}}(\Omega; C([t,T];\mathbb{R}^n))$ *to equations* (2.8)*. Moreover, the following estimates hold:*

$$\mathbb{E}\Big[\sup_{s\in[t,T]} |X(s;t,\xi;u)|^2\Big] \leqslant K(1 + \mathbb{E}[|\xi|^2]), \tag{2.17}$$

$$\mathbb{E}\Big[|X(s_1;t,\xi;u) - X(s_2;t,\xi;u)|^2\Big] \leqslant K(1 + \mathbb{E}[|\xi|^2])|s_2 - s_1|, \tag{2.18}$$

$$\begin{aligned} &\mathbb{E}\Big[|X(s;t_1,\xi_1;u) - X(s;t_2,\xi_2;u)|^2\Big] \\ &\leqslant K(1 + \mathbb{E}[|\xi_1|^2] + \mathbb{E}[|\xi_2|^2])(|t_1 - t_2| + \mathbb{E}[|\xi_1 - \xi_2|^2]), \end{aligned} \tag{2.19}$$

*for all* $(t_i, \xi_i) \in \mathcal{D}$*,* $s_i \in [t,T]$*,* $i = 1,2$*,* $s \in [t_1 \vee t_2, T]$*. Note that the constant* $K$ *here depends only on Lipschitz constants of* $b$*,* $\sigma$ *and* $u$*.*

*Proof.* Let $x(\cdot) \in L^2_{\mathbb{F}}(\Omega; C([t,T];\mathbb{R}^n))$, $(t,\xi) \in \mathcal{D}$, $u \in \mathcal{U}$, then

$$
\begin{aligned}
\Delta \equiv & \mathbb{E}\Big[|\xi + \int_t^T b(r, x(r), \mathbb{P}_{x(r)}, u(r, x(r), \mathbb{P}_{x(r)}))dr \\
& \quad + \int_t^T \sigma(r, x(r), \mathbb{P}_{x(r)}, u(r, x(r), \mathbb{P}_{x(r)}))dW(r)|^2\Big] \\
\leqslant & K_1 \mathbb{E}\Big[|\xi|^2 + (\int_t^T (1 + \sup_{t\le r\le T} |x(r)| + \sup_{t\le r\le T} \|\mathbb{P}_{x(r)}\|_2 + |u(r, x(r), \mathbb{P}_{x(r)})|)dr)^2 \\
& \quad + (\int_t^T (1 + \sup_{t\le r\le T} |x(r)|^2 + \sup_{t\le r\le T} \|\mathbb{P}_{x(r)}\|_2^2 + |u(r, x(r), \mathbb{P}_{x(r)})|^2)dr)\Big] \\
\leqslant & K_2 \mathbb{E}\Big[|\xi|^2 + (1 + \sup_{t\le r\le T} |x(r)|^2 + \sup_{t\le r\le T} \|\mathbb{P}_{x(r)}\|_2^2)(T-t)^2 + \int_t^T |u(t, 0, \delta_0)|^2 dr \\
& \quad + (1 + \sup_{t\le r\le T} |x(r)|^2 + \sup_{t\le r\le T} \|\mathbb{P}_{x(r)}\|_2^2)(T-t)\Big],
\end{aligned}
$$

where $\|\mathbb{P}_{x(r)}\|_2 \equiv \inf\{(\mathbb{E}[|\eta|^2])^{\frac{1}{2}} \mid \eta$ is $\mathcal{G}-$measurable and $\mathbb{P}_\eta = \mathbb{P}_{x(r)}\} = \mathcal{W}_2(\mathbb{P}_{X(r)}, \delta_{\{0\}})$, then

$$
\|\mathbb{P}_{x(r)}\|_2 \le (\mathbb{E}[|x(r)|^2])^{\frac{1}{2}}.
$$

So,

$$
\Delta \leqslant K\mathbb{E}\Big[|\xi|^2 + (1 + 2\|x(\cdot)\|_2^2)(T-t)(1 + (T-t)) + \int_t^T |u(t, 0, \delta_0)|^2 dr\Big] < \infty, \tag{2.20}
$$

for some constant $K > 0$. This enables us to define the map $\Psi : L^2_{\mathbb{F}}(\Omega; C([t,T];\mathbb{R}^n)) \to L^2_{\mathbb{F}}(\Omega; C([t,T];\mathbb{R}^n))$ by letting

$$
\begin{aligned}
\Psi(x)(\cdot) \equiv \xi + & \int_t^\cdot b(r, x(r), \mathbb{P}_{x(r)}, u(r, x(r), \mathbb{P}_{x(r)}))dr \\
& + \int_t^\cdot \sigma(r, x(r), \mathbb{P}_{x(r)}, u(r, x(r), \mathbb{P}_{x(r)}))dW(r).
\end{aligned} \tag{2.21}
$$

Note that $\Psi(x)$ has continuous paths by the continuity of $b$, $\sigma$ and $u$. Now, we show that $\Psi$ is a contraction. Let $x_i(\cdot) \in L^2_{\mathbb{F}}(\Omega; C([t,T];\mathbb{R}^n))$, $i = 1, 2$, then

$$
\begin{aligned}
& \|\Psi(x_1) - \Psi(x_2)\|_2^2 \\
= & \mathbb{E}\Big[\sup_{t \leqslant s \leqslant T} \Big| \int_t^s b(r, x_1(r), \mathbb{P}_{x_1(r)}, u(r, x_1(r), \mathbb{P}_{x_1(r)})) - b(r, x_2(r), \mathbb{P}_{x_2(r)}, u(r, x_2(r), \mathbb{P}_{x_2(r)}))dr \\
& + \int_t^s \sigma(r, x_1(r), \mathbb{P}_{x_1(r)}, u(r, x_1(r), \mathbb{P}_{x_1(r)})) - \sigma(r, x_2(r), \mathbb{P}_{x_2(r)}, u(r, x_2(r), \mathbb{P}_{x_2(r)}))dW(r)\Big|^2\Big] \\
\leqslant & K_1 \mathbb{E}\Big[\Big(\int_t^T |b(r, x_1(r), \mathbb{P}_{x_1(r)}, u(r, x_1(r), \mathbb{P}_{x_1(r)})) - b(r, x_2(r), \mathbb{P}_{x_2(r)}, u(r, x_2(r), \mathbb{P}_{x_2(r)}))|dr\Big)^2 \\
& + \Big(\int_t^T |\sigma(r, x_1(r), \mathbb{P}_{x_1(r)}, u(r, x_1(r), \mathbb{P}_{x_1(r)})) - \sigma(r, x_2(r), \mathbb{P}_{x_2(r)}, u(r, x_2(r), \mathbb{P}_{x_2(r)}))|^2 dr\Big)\Big] \\
\leqslant & K_2 \mathbb{E}\Big[\Big(\int_t^T \big(\sup_{t \leqslant r \leqslant T} |x_1(r) - x_2(r)| + \sup_{t \leqslant r \leqslant T} \mathcal{W}_2(\mathbb{P}_{x_1(r)}, \mathbb{P}_{x_2(r)}) \\
& \quad + |u(r, x_1(r), \mathbb{P}_{x_1(r)}) - u(r, x_2(r), \mathbb{P}_{x_2(r)})|\big)dr\Big)^2 \\
& \quad + \int_t^T \big(\sup_{t \leqslant r \leqslant T} |x_1(r) - x_2(r)|^2 + \sup_{t \leqslant r \leqslant T} \mathcal{W}_2^2(\mathbb{P}_{x_1(r)}, \mathbb{P}_{x_2(r)}) \\
& \quad + |u(r, x_1(r), \mathbb{P}_{x_1(r)}) - u(r, x_2(r), \mathbb{P}_{x_2(r)})|^2\big)dr\Big] \\
\leqslant & K \mathbb{E}\Big[2(T-t)^2 \big(\sup_{t \leqslant r \leqslant T} |x_1(r) - x_2(r)|^2\big) + 2(T-t)^2 \big(\sup_{t \leqslant r \leqslant T} |x_1(r) - x_2(r)|^2\big) \\
& \quad + 2(T-t) \big(\sup_{t \leqslant r \leqslant T} |x_1(r) - x_2(r)|^p\big) + 2(T-t) \big(\sup_{t \leqslant r \leqslant T} |x_1(r) - x_2(r)|^2\big)\Big] \\
\leqslant & K((T-t)^2 + (T-t))\|x_1 - x_2\|_2^2.
\end{aligned}
$$

When $(T-t)$ is small enough, $K(T-t)(1+(T-t)) < 1$. The corresponding equation (2.8) has a unique solution, $X(\cdot) = X(\cdot; t, \xi, u)$ on a small interval. Let $\tau \in [t, T]$ and $\tau - t$ and $T - \tau$ are small enough, then the process

$$
X(s) = X_1(s; t, \xi, u) 1_{[t,\tau]}(s) + X_2(s; \tau, X_1(\tau; t, \xi, u), u) 1_{(\tau, T]}(s) \tag{2.22}
$$

is a solution of (2.8) on the whole interval $[t,T]$ since

$$\begin{aligned}
X(s) &= X_1(s;t,\xi,u)1_{[t,\tau]}(s) + X_2(s;\tau,X_1(\tau;t,\xi,u),u)1_{(\tau,T]}(s)\\
&= \xi + \int_t^{\tau\wedge s} b(r,X_1(r),\mathbb{P}_{X_1(r)},u(r,X_1(r),\mathbb{P}_{X_1(r)}))dr\\
&\quad + \int_t^{\tau\wedge s} \sigma(r,X_1(r),\mathbb{P}_{X_1(r)},u(r,X_1(r),\mathbb{P}_{X_1(r)}))dW(r)\\
&\quad + \int_{\tau\wedge s}^{s} b(r,X_2(r),\mathbb{P}_{X_2(r)},u(r,X_2(r),\mathbb{P}_{X_2(r)}))dr\\
&\quad + \int_{\tau\wedge s}^{s} \sigma(r,X_2(r),\mathbb{P}_{X_2(r)},u(r,X_2(r),\mathbb{P}_{X_2(r)}))dW(r)\\
&= \xi + \int_t^{s} b(r,X(r),\mathbb{P}_{X(r)},u(r,X(r),\mathbb{P}_{X(r)}))dr\\
&\quad + \int_t^{s} \sigma(r,X(r),\mathbb{P}_{X(r)},u(r,X(r),\mathbb{P}_{X(r)}))dW(r).
\end{aligned}$$

So, we can get the global existence of a solution to (2.8).

Let $X_i(\cdot) = X(\cdot;t_i,\xi_i;u)$, $i=1,2$ be the solutions of (2.8) on $[t_i,T]$, respectively. Let $s \in [t_1\vee t_2,T]$, $i=1,2$. Without loss of generality, assume $t_2 \leqslant t_1$, then

$$\begin{aligned}
&\mathbb{E}\Big[|X(s;t_1,\xi_1;u)-X(s;t_2,\xi_2;u)|^2\Big]\\
&\leqslant K\mathbb{E}\Big[|\xi_1-\xi_2|^2 + |\int_{t_2}^{t_1} b(r,X_2(r),\mathbb{P}_{X_2(r)},u(r,X_2(r),\mathbb{P}_{X_2(r)}))dr|^2\\
&\quad +|\int_{t_2}^{t_1} \sigma(r,X_2(r),\mathbb{P}_{X_2(r)},u(r,X_2(r),\mathbb{P}_{X_2(r)}))dW(r)|^2\\
&\quad +|\int_{t_1}^{s} b(r,X_1(r),\mathbb{P}_{X_1(r)},u(X_1(r),\mathbb{P}_{X_1(r)})) - b(r,X_2(r),\mathbb{P}_{X_2(r)},u(X_2(r),\mathbb{P}_{X_2(r)}))dr|^2\\
&\quad +|\int_{t_1}^{s} \sigma(r,X_1(r),\mathbb{P}_{X_1(r)},u(X_1(r),\mathbb{P}_{X_1(r)})) - \sigma(r,X_2(r),\mathbb{P}_{X_2(r)},u(X_2(r),\mathbb{P}_{X_2(r)}))dW(r)|^2\Big]\\
&\leqslant K\Big\{\mathbb{E}\Big[|\xi_1-\xi_2|^2\Big] + C^2(1+L^2)(s-t_1+1)\int_t^s \mathbb{E}\Big[|X_1(r)-X_2(r)|^2\Big]dr\\
&\quad +C^2(t_1-t_2+1)(1+\mathbb{E}\Big[\sup_{r\in[t_2,t_1]}|X_2(r)|^2\Big])(t_1-t_2)\Big\}.
\end{aligned}$$

By Gronwall's lemma, we have

$$
\begin{aligned}
&\mathbb{E}\Big[|X(s;t_1,\xi_1,u)-X(s;t_2,\xi_2,u)|^2\Big]\\
\leqslant\,& K(1+\mathbb{E}\Big[\sup_{r\in[t_2,t_1]}|X_2(r)|^2\Big])((t_1-t_2)+\mathbb{E}\Big[|\xi_1-\xi_2|^2\Big])\\
\leqslant\,& K(1+\mathbb{E}[|\xi_2|^2])((t_1-t_2)+\mathbb{E}[|\xi_1-\xi_2|^2]).
\end{aligned}
$$

This shows the uniqueness of the global solution on $[t,T]$ as well as the estimate (2.19). Now we prove the estimate (2.18). Let $(t,\xi)\in\mathcal{D}$ and $s_i\in[t,T]$, $i=1,2$, without loss of generality, assume $s_1\leqslant s_2$, then

$$
\begin{aligned}
&\mathbb{E}\Big[|X(s_1;t,\xi,u)-X(s_2;t,\xi,u)|^2\Big]\\
\leqslant\,& K\mathbb{E}\Big[|\int_{s_1}^{s_2}b(r,X(r),\mathbb{P}_{X(r)},u(r,X(r),\mathbb{P}_{X(r)}))dr|^2\\
&+|\int_{s_1}^{s_2}\sigma(r,X(r),\mathbb{P}_{X(r)},u(r,X(r),\mathbb{P}_{X(r)}))dW(r)|^2\Big]\\
\leqslant\,& KC^2(s_2-s_1+1)(1+\mathbb{E}\Big[\sup_{r\in[s_1,s_2]}|X(r)|^2\Big])(s_2-s_1)\\
\leqslant\,& K(1+\mathbb{E}[|\xi|^2])|s_2-s_1|.
\end{aligned}
$$

□

**Proposition 2.3.2.** *(Flow Property) Let $X(\cdot)=X(\cdot;t,\xi,u)$ be the solutions to equations* (2.8)*. Then it satisfies the flow property:*

$$
X(s;t,\xi,u)=X(s;\tau,X(\tau;t,\xi,u_1),u_2); \tag{2.23}
$$

*for all $0\leq t\leq\tau\leq s\leq T$, where $u(\cdot)=u_1(\cdot)1_{[t,\tau)}+u_2(\cdot)1_{[\tau,T]}$.*

*Proof.* This is an easy corollary from the existence and uniqueness for solution of (2.8). □

**Remark 2.3.3.** *Note that the initial state $\xi$ being an $\mathcal{F}_t-$measurable random variable is necessary* *for the flow property of the dynamics $X(s)$. In comparison, $X(s;t,x,u)$ is not a flow, where $x \in \mathbb{R}^n$.*

### 2.3.2 About Value Functions

The dependence of value function $V$ ($V_L$) on $\xi$ makes trouble when deriving HJB equation. The main reason is: $V$ cannot be simplified to be a deterministic function on $[0,T]\times\mathbb{R}^n$, this makes the Itô's formula impossible to be applied. The following key observation helps to better understand the dependence of $V$ ($V_L$) on $\xi$.

**Proposition 2.3.4.** *Let $X(\cdot) = X(\cdot;t,\xi,u)$ be the solutions to equations* (2.8). *Then, for any $s \in [t,T]$, the distribution $\mathbb{P}_{X(s)}$ depends on $\xi$ only through its distribution $\mathbb{P}_\xi$.*

*Proof.* We will use this result in the next chapter and it is more natural to show the proof there. From this result, we can rewrite $\mathbb{P}_{X(s;t,\xi,u)}$ by adopting the notation: $\mathbb{P}_{X(s;t,\xi,u)} = \mathbb{P}_s^{t,\mathbb{P}_\xi,u}$. □

**Proposition 2.3.5.** *The dependence of $V$ on $\xi$ is through its distribution $\mathbb{P}_\xi$. That is, there exists function $\tilde{V} : [0,T]\times\mathcal{P}^2 \to \mathbb{R}$ such that $V(t,\xi) = \tilde{V}(t,\mathbb{P}_\xi)$. A similar result can be obtained for $V_L$.*

*Proof.* We first show that the dependence of cost functional $J(t,\xi,u)$ on $\xi$ is through $\mathbb{P}_\xi$. By definition,

$$
\begin{aligned}
&J(t,\xi,u)\\
&= \mathbb{E}\Big[h(X(T),\mathbb{P}_T^{t,\mathbb{P}_\xi,u}) + \int_t^T g(r,X(r),\mathbb{P}_r^{t,\mathbb{P}_\xi,u},u(r,X(r),\mathbb{P}_r^{t,\mathbb{P}_\xi,u}))dr\Big]\\
&= \int_{\mathbb{R}^n} h(x',\mathbb{P}_T^{t,\mathbb{P}_\xi,u})\mathbb{P}_T^{t,\mathbb{P}_\xi,u}(dx') + \int_t^T \int_{\mathbb{R}^n} g(r,x',\mathbb{P}_r^{t,\mathbb{P}_\xi,u},u(r,x',\mathbb{P}_r^{t,\mathbb{P}_\xi,u}))\mathbb{P}_r^{t,\mathbb{P}_\xi,u}(dx')dr.
\end{aligned}
$$

We use the notation $\tilde{J}(t, \mathbb{P}_\xi, u) \equiv J(t, \xi, u)$. Let

$$\tilde{V}(t, \mathbb{P}_\xi) \equiv \inf_{u \in \mathcal{U}} \tilde{J}(t, \mathbb{P}_\xi, u),$$

then

$$V(t, \xi) = \inf_{u \in \mathcal{U}} J(t, \xi, u) = \inf_{u \in \mathcal{U}} \tilde{J}(t, \mathbb{P}_\xi, u) = \tilde{V}(t, \mathbb{P}_\xi).$$

Follow similar discussion, we can find a function $\tilde{V}_L : [0, T] \times \mathcal{P}^2 \to \mathbb{R}$ such that

$$V_L(t, \xi) = \tilde{V}_L(t, \mathbb{P}_\xi).$$

□

Now we are going to talk about conditions under which we can get the well-posedness and regularity of $V$. Assume that besides **(H2)**, $g$ and $h$ also satisfy

**(H3)** $g$ and $h$ are lower bounded by a constant or a convex function, uniformly for all $u \in U$. And,

$$\begin{aligned} &|g(t, x_1, \mu, u) - g(t, x_1', \mu', u')| + |h(x_2, \nu) - h(x_2', \nu')| \\ &\leqslant M\big(|x_1 - x_1'|^2 + |x_2 - x_2'|^2 + \mathcal{W}_2^2(\mu, \mu') + \mathcal{W}_2^2(\nu, \nu') + |u - u'|^2\big), \end{aligned} \tag{2.24}$$

for all $(t, x_1, x_1', x_2, x_2', u, \mu, \mu') \in [0, T] \times (\mathbb{R}^n)^4 \times U \times \mathcal{P}^2 \times \mathcal{P}^2$.

**Proposition 2.3.6.** *Under conditions* **(H2)** *and* **(H3)**, *the value functions* $V,\ V_L : \mathcal{D} \to \mathbb{R}$, *equivalently,* $\tilde{V},\ \tilde{V}_L : [0, T] \times \mathcal{P}^2 \to \mathbb{R}$, *are a well-defined functions. Especially,* $V_L$ *and* $\tilde{V}_L$ *are continuous.*

*Proof.* Let $(t,\xi),\ (s,\eta)\in\mathcal{D}$, $t\leqslant s$, then

$$
\begin{aligned}
&|V(t,\xi)-V(s,\eta)|\\
\leqslant&\sup_{u\in\mathcal{U}}|J(t,\xi,u)-J(s,\eta,u)|\\
\leqslant&\sup_{u\in\mathcal{U}}\Big\{\mathbb{E}[\int_s^T|g(r,X(r;t,\xi),\mathbb{P}_r^{t,\xi,u},u(r,X(r;t,\xi),\mathbb{P}_r^{t,\xi,u}))\\
&\qquad\qquad -g(r,X(r;s,\eta),\mathbb{P}_r^{s,\eta,u},u(r,X(r;s,\eta),\mathbb{P}_r^{s,\eta,u}))|dr\\
&\quad+|h(X(T;t,\xi),\mathbb{P}_T^{t,\xi,u})-h(X(T;s,\eta),\mathbb{P}_T^{s,\eta,u})|\\
&\quad+\int_t^s|g(r,X(r;t,\xi),\mathbb{P}_r^{t,\xi,u},u(r,X(r;t,\xi),\mathbb{P}_r^{t,\xi,u}))|dr]\Big\}\\
\leqslant&K\sup_{u\in\mathcal{U}}\Big\{\mathbb{E}[\int_s^T|X(r;t,\xi)-X(r;s,\eta)|^2+\mathcal{W}_2^2(\mathbb{P}_r^{t,\xi,u},\mathbb{P}_r^{s,\eta,u})+|u(r,X(r;t,\xi),\mathbb{P}_r^{t,\xi,u})\\
&\quad-u(r,X(r;s,\eta),\mathbb{P}_r^{s,\eta,u})|^2dr+|X(T;t,\xi)-(X(T;s,\eta)|^2+\mathcal{W}_2^2(\mathbb{P}_T^{t,\xi,u},\mathbb{P}_T^{s,\eta,u})\\
&\quad+\int_t^s(1+|X(r;t,\xi)|^2+\|\mathbb{P}_r^{t,\xi,u}\|_2^2+|u(r,X(r;t,\xi),\mathbb{P}_r^{t,\xi,u}))|^2)dr]\Big\}\\
\leqslant&K\sup_{u\in\mathcal{U}}\Big\{\mathbb{E}[\sup_{s\leqslant r\leqslant T}|X(r;t,\xi)-X(r;s,\eta)|^2+\sup_{s\leqslant r\leqslant T}\mathcal{W}_2^2(\mathbb{P}_r^{t,\xi,u},\mathbb{P}_r^{s,\eta,u})\\
&\qquad+\tilde{K}_u(\sup_{s\leqslant r\leqslant T}|X(r;t,\xi)-X(r;s,\eta)|^2+\sup_{s\leqslant r\leqslant T}\mathcal{W}_2^2(\mathbb{P}_r^{t,\xi,u},\mathbb{P}_r^{s,\eta,u}))\\
&\qquad+(s-t)(1+\sup_{s\leqslant r\leqslant T}|X(r;t,\xi)|^2+\sup_{s\leqslant r\leqslant T}\|\mathbb{P}_r^{t,\xi,u}\|_2^2)\\
&\qquad+(s-t)\tilde{K}_u(1+\sup_{s\leqslant r\leqslant T}|X(r;t,\xi)|^2+\sup_{s\leqslant r\leqslant T}\|\mathbb{P}_r^{t,\xi,u}\|_2^2)]\Big\}\\
\leqslant&K\Big\{(\sup_{u\in\mathcal{U}}\tilde{K}_u)\mathbb{E}[|X(s;t,\xi)-\eta|^2]+(s-t)(1+\mathbb{E}[|\xi|^2])\Big\}\\
\leqslant&K((\sup_{u\in\mathcal{U}}\tilde{K}_u)\mathbb{E}[|\xi-\eta|^2]+(s-t)(1+\mathbb{E}[|\xi|^2])),
\end{aligned}
$$

Then,

$$
\begin{aligned}
&|\tilde{V}(t,\mathbb{P}_\xi)-\tilde{V}(s,\mathbb{P}_\eta)|\\
\leqslant&K((\sup_{u\in\mathcal{U}}\tilde{K}_u)\mathcal{W}_2^2(\mathbb{P}_\xi,\mathbb{P}_\eta)+(1+\|\mathbb{P}_\xi\|_2^2+\|\mathbb{P}_\eta\|_2^2)|s-t|).
\end{aligned}
$$

Thus, the value function $\tilde{V}(t,\mu)$ and $\tilde{V}_L(t,\mu)$ are continuous with respect to $t$. The $L-$value function $\tilde{V}_L(\cdot,\cdot)$ is continuous on $\mathcal{D}$. For each $t \in [0,T]$, $\tilde{V}(t,\cdot)$ is uniformly continuous on $\mathcal{P}^2$. □

## 2.4 Main Results

### *2.4.1 Dynamic Programming Principle*

**Theorem 2.4.1.** *(Dynamic Programming Principle) The value function $\tilde{V}(t,\mu)$ satisfies the following equation:*

$$\tilde{V}(t,\mu) = \inf_{u\in\mathcal{U}}\{\int_t^\tau \int_{\mathbb{R}^n} g(s,x,\mathbb{P}_s^{t,\mu,u}, u(s,x,\mathbb{P}_s^{t,\mu,u}))\mathbb{P}_s^{t,\mu,u}(dx)ds + \tilde{V}(\tau,\mathbb{P}_\tau^{t,\mu,u})\}. \tag{2.25}$$

*Proof.* By the definition of $\tilde{V}$, we have

$$\begin{aligned}
\tilde{V}(t,\mu) &\leqslant \int_t^T \int_{\mathbb{R}^n} g(s,x,\mathbb{P}_s^{t,\mu,u}, u(s,x,\mathbb{P}_s^{t,\mu,u}))\mathbb{P}_s^{t,\mu,u}(dx)ds + \int_{\mathbb{R}^n} h(x,\mathbb{P}_T^{t,\mu,u})\mathbb{P}_T^{t,\mu,u}(dx) \\
&= \int_t^\tau \int_{\mathbb{R}^n} g(s,x,\mathbb{P}_s^{t,\mu,u}, u(s,x,\mathbb{P}_s^{t,\mu,u}))\mathbb{P}_r^{t,\mu,u}(dx)ds \\
&\quad + \int_\tau^T \int_{\mathbb{R}^n} g(s,x,\mathbb{P}_s^{t,\mu,u}, u(s,x,\mathbb{P}_s^{t,\mu,u}))\mathbb{P}_r^{t,\mu,u}(dx)ds + \int_{\mathbb{R}^n} h(x,\mathbb{P}_T^{t,\mu,u})\mathbb{P}_T^{t,\mu,u}(dx) \\
&= \int_t^\tau \int_{\mathbb{R}^n} g(s,x,\mathbb{P}_s^{t,\mu,u}, u(s,x,\mathbb{P}_s^{t,\mu,u}))\mathbb{P}_r^{t,\mu,u}(dx)ds \\
&\quad + \int_\tau^T \int_{\mathbb{R}^n} g(s,x,\mathbb{P}_s^{\tau,\mathbb{P}_\tau^{t,\mu,u},u}, u(s,x,\mathbb{P}_s^{\tau,\mathbb{P}_\tau^{t,\mu,u},u}))\mathbb{P}_s^{\tau,\mathbb{P}_\tau^{t,\mu,u},u}(dx)ds \\
&\quad + \int_{\mathbb{R}^n} h(x,\mathbb{P}_T^{\tau,\mathbb{P}_\tau^{t,\mu,u},u})\mathbb{P}_T^{\tau,\mathbb{P}_\tau^{t,\mu,u},u}(dx) \\
&= \int_t^\tau \int_{\mathbb{R}^n} g(s,x,\mathbb{P}_s^{t,\mu,u}, u(s,x,\mathbb{P}_s^{t,\mu,u}))\mathbb{P}_s^{t,\mu,u}(dx)ds + \tilde{J}(\tau,\mathbb{P}_\tau^{t,\mu,u},u)
\end{aligned}$$

for all $u(\cdot) \in \mathcal{U}$. Then,

$$\tilde{V}(t,\mu) \leqslant \int_t^\tau \int_{\mathbb{R}^n} g(s,x,\mathbb{P}_s^{t,\mu,u},u(s,x,\mathbb{P}_s^{t,\mu,u}))\mathbb{P}_s^{t,\mu,u}(dx)ds + \tilde{V}(\tau,\mathbb{P}_\tau^{t,\mu,u}),$$

for all $u \in \mathcal{U}$. So,

$$\tilde{V}(t,\mu) \leqslant \inf_{u\in\mathcal{U}}\{\int_t^\tau \int_{\mathbb{R}^n} g(s,x,\mathbb{P}_s^{t,\mu,u},u(s,x,\mathbb{P}_s^{t,\mu,u}))\mathbb{P}_s^{t,\mu,u}(dx)ds + \tilde{V}(\tau,\mathbb{P}_\tau^{t,\mu,u})\}.$$

On the other hand, let $\varepsilon > 0$, then there exists some $u^{(\varepsilon)} \in \mathcal{U}$ such that

$$\tilde{V}(t,\mu) + \varepsilon > \tilde{J}(t,\mu,u^{(\varepsilon)}),$$

then

$$\begin{aligned}
\tilde{V}(t,\mu) + \varepsilon &> \int_t^\tau \int_{\mathbb{R}^n} g(s,x,\mathbb{P}_s^{t,\mu,u^{(\varepsilon)}},u^{(\varepsilon)}(s,x,\mathbb{P}_s^{t,\mu,u^{(\varepsilon)}}))\mathbb{P}_s^{t,\mu,u^{(\varepsilon)}}(dx)ds + \tilde{J}(\tau,\mathbb{P}_\tau^{t,\mu,u^{(\varepsilon)}},u^{(\varepsilon)}) \\
&\geqslant \int_t^\tau \int_{\mathbb{R}^n} g(s,x,\mathbb{P}_s^{t,\mu,u^{(\varepsilon)}},u^{(\varepsilon)}(s,x,\mathbb{P}_s^{t,\mu,u^{(\varepsilon)}}))\mathbb{P}_s^{t,\mu,u^{(\varepsilon)}}(dx)ds + \tilde{V}(\tau,\mathbb{P}_\tau^{t,\mu,u^{(\varepsilon)}}) \\
&\geqslant \inf_{u\in\mathcal{U}}\{\int_t^\tau \int_{\mathbb{R}^n} g(s,x,\mathbb{P}_s^{t,\mu,u},u(s,x,\mathbb{P}_s^{t,\mu,u}))\mathbb{P}_s^{t,\mu,u}(dx)ds + \tilde{V}(\tau,\mathbb{P}_\tau^{t,\mu,u})\}.
\end{aligned}$$

Let $\varepsilon \to 0$, we have

$$\tilde{V}(t,\mu) = \inf_{u\in\mathcal{U}}\{\int_t^\tau \int_{\mathbb{R}^n} g(s,x,\mathbb{P}_s^{t,\mu,u},u(s,x,\mathbb{P}_s^{t,\mu,u}))\mathbb{P}_s^{t,\mu,u}(dx)ds + \tilde{V}(\tau,\mathbb{P}_\tau^{t,\mu,u})\}.$$

□

It can be proved that dynamic programming principle also holds for $\tilde{V}_L$.

**Remark 2.4.2.** *Note that the dynamic programming principle still holds for the value function*

*$V(t,\xi)$ since $V(t,\xi) = \tilde{V}(t,\mu)$. In Remark 3.2 of [44], the authors mistakenly admitted that the appearance of nonlinear function of $\mathbb{E}[X]$ results in time-inconsistency. From examples in the first section and the discussion above about dynamic programming principle, we conclude that it is nonlinear functions of conditional expectation $\mathbb{E}_t[X]$ or conditional distribution that causes time-inconsistency.*

*The reason for introducing distributions functions here is that it helps giving an HJB equation that is easier for further discussions. While the trade-off is that the open-loop control problem cannot be dealt, if not impossible, so neatly as for the closed-loop case.*

### *2.4.2 Verification Theorem*

**Theorem 2.4.3.** *Suppose that the following PDE*

$$\partial_t \tilde{V}(t,\mu) + H(t,\mu,\partial_\mu \tilde{V}(t,\mu)(\cdot), \partial_x \partial_\mu \tilde{V}(t,\mu)(\cdot)) = 0, \tag{2.26}$$

*for all $t \in [0,T]$, $\mu \in \mathcal{P}^2$, where*

$$H(t,\mu,p,A) = \inf_{u \in \mathbb{L}(\mathbb{R}^n;U)} \{\mathbb{E}[\frac{1}{2} tr[A \cdot \sigma\sigma^T(t,\xi,\mu,u(\xi))] + p \cdot b(t,\xi,\mu,u(\xi)) + g(t,\xi,\mu,u(\xi))]\},$$

*has a classical solution $\psi$, and for each $(t,\mu)$, $u^*(t,\cdot,\mu) \in \mathbb{L}(\mathbb{R}^n;U)$ such that*

$$H(t,\mu,p,A) = \mathbb{E}\Big[\frac{1}{2} tr[A \cdot \sigma\sigma^T(t,\xi,\mu,u^*(t,\xi,\mu))] + p \cdot b(t,\xi,\mu,u^*(t,\xi,\mu)) + g(t,\xi,\mu,u^*(t,\xi,\mu))\Big],$$

*and $u^* \in \mathcal{U}$. Then $\psi$ is the value function, that is, $\psi = \tilde{V}$ and $u^*$ is an optimal control.*

*Proof.* Let $X^*(s) = X(s;t,\xi,u^*)$. By applying Itô's formula (see [10]) to the process $\psi(s,\mathbb{P}_{X(s)})$, we have

$$
\begin{aligned}
&\psi(s,\mathbb{P}_s^{t,\mu,u^*})\\
=&\,\mathbb{E}\Big[h(\bar{X}^*(T),\mathbb{P}_T^{t,\mu,u^*})\Big]-\int_s^T \partial_r\psi(r,\mathbb{P}_r^{t,\mu,u^*})\\
&+\mathbb{E}\Big[\partial_\mu\psi(r,\mathbb{P}_r^{t,\mu,u^*})(X^*(r))b(r,X^*(r),\mathbb{P}_r^{t,\mu,u^*},u^*(r,X^*(r),\mathbb{P}_r^{t,\mu,u^*}))\\
&+\frac{1}{2}\mathrm{tr}\,[\partial_\omega\partial_\mu\psi(r,\mathbb{P}_r^{t,\mu,u^*})(X^*(r))\sigma\sigma^T(r,\tilde{X}^*(r),\mathbb{P}_r^{t,\mu,u^*},u^*(r,\tilde{X}^*(r),\mathbb{P}_r^{t,\mu,u^*}))]\Big]dr.
\end{aligned}
$$

Since $\psi$ solves (3.50), we have

$$\psi(t,\mu)=\tilde{J}(t;t;\mu,u^*)\geqslant\tilde{V}(t,\mu).$$

On the other hand, let $u\in\mathcal{U}$. There exists $f:[0,T]\times\mathbb{R}^n\times\mathcal{P}^2\to[0,\infty)$, such that

$$
\begin{aligned}
\partial_t\tilde{V}(t,\mu)+\mathbb{E}\Big[&\frac{1}{2}\mathrm{tr}\,[\partial_\omega\partial_\mu\tilde{V}(t,\mu)(\xi)\cdot\sigma\sigma^T(t,\xi,\mu,u(t,\xi,\mu))]+\partial_\mu\tilde{V}(t,\mu)(\xi)\cdot b(t,\xi,\mu,u(t,\xi,\mu))\\
&+g(t,\xi,\mu,u^*(t,\xi,\mu))\Big]-f(t,\mu)=0.
\end{aligned}
$$

It's easy to see that

$$\psi(t,\mu)\leqslant\tilde{J}(t,\mu,u),$$

for all $u\in\mathcal{U}$. Then $\psi(t,\mu)\leqslant\tilde{V}(t,\mu)$. □

**Remark 2.4.4.** *Let*

$$\phi(s,x,\mu,p,A)=\arg\inf_{u\in U}\{\frac{1}{2}\mathrm{tr}\,[A\cdot\sigma\sigma^T(t,x,\mu,u)]+p\cdot b(t,x,\mu,u)+g(t,x,\mu,u)\},$$

*and*

$$u^*(s,x,\mu)=\phi(s,x,\mu,\partial_\mu\psi(t,\mu)(x),\partial_\omega\partial_\mu\psi(t,\mu)(x)).$$

*Suppose that* $u^*\in\mathcal{U}$*, then* $\inf_{u\in U}\geqslant\inf_{u\in\mathcal{U}}$*. On the other hand, it is generally correct that*

$\inf_{u\in U} \leqslant \inf_{u\in\mathcal{U}}$. *This shows that an optimal control could be found by taking*

$$u^*(t,x,\mu) = \arg\inf_{u\in U}\{\frac{1}{2}\text{tr}\,[A\cdot\sigma\sigma^T(t,x,\mu,u)] + p\cdot b(t,x,\mu,u) + g(t,x,\mu,u)\}.$$

### *2.4.3 Viscosity Solutions*

First we introduce the following definition:

**Definition 2.4.5.** *(Viscosity Solution) A continuous function $\psi : [0,T]\times\mathcal{P}^2 \to \mathbb{R}$ is called a viscosity supersolution of equation* (2.28) *if for any $\varphi \in C^{1,2}([0,T]\times\mathcal{P}^2)$, whenever $\psi-\varphi$ attains a local maximum at $(t,\mu)\in[0,T]\times\mathcal{P}^2$, we have*

$$\partial_t\varphi(t,\mu) + H(t,\mu,\partial_\mu\varphi(t,\mu)(\cdot),\partial_x\partial_\mu\varphi(t,\mu)(\cdot)) \geqslant 0. \tag{2.27}$$

*It is called a viscosity subsolution if in* (2.27) *the inequality "$\geqslant$" is replaced by "$\leqslant$" and "local maximum" is replaced by "local minimum". $\psi$ is called a viscosity solution if it is a viscosity supersolution and viscosity subsolution.*

**Theorem 2.4.6.** *For each $L>0$, the L-value function $\tilde{V}_L(t,\mu)$ is a viscosity solution to the following PDE:*

$$\partial_t\tilde{V}_L(t,\mu) + H_L(t,\mu,\partial_\mu\tilde{V}_L(t,\mu)(\cdot),\partial_x\partial_\mu\tilde{V}_L(t,\mu)(\cdot)) = 0, \tag{2.28}$$

*for all $t\in[0,T]$, $\mu\in\mathcal{P}^2$, where $H_L : [0,T]\times\mathcal{P}^2\times\mathbb{R}^n\times\mathbb{R}^{n\times n}\to\mathbb{R}$ is defined by*

$$\begin{aligned}&H_L(t,\mu,p,A)\\ =&\inf_{u\in\mathbb{L}_L(\mathbb{R}^n;U)}\Big\{\mathbb{E}\Big[\frac{1}{2}tr[A\cdot\sigma\sigma^T(t,\xi,\mu,u(\xi))] + p\cdot b(t,\xi,\mu,u(\xi)) + g(t,\xi,\mu,u(\xi))\Big]\Big\}.\end{aligned}$$

*Proof.* The continuity of $\tilde{V}_L$ is given by the discussion in the section 2.3.2. Now, we first prove $\tilde{V}_L$ is a viscosity supersolution of (2.28). Let $(t,\mu) \in [0,T] \times \mathcal{P}^2$, $\varphi \in C^{1,2}([0,T] \times \mathcal{P}^2)$ such that $\tilde{V}_L - \varphi$ has a local maximum at $(t,\mu)$, without loss of generality, assume that $\tilde{V}_L \leqslant \varphi$ on $[0,T] \times \mathcal{P}^2$. By Itô's formula, we have

$$\begin{aligned}
&\varphi(s, \mathbb{P}_s^{t,\mu,u}) - \varphi(t,\mu) \\
= &\int_t^s \partial_r \varphi(r, \mathbb{P}_r^{t,\mu,u}) + \mathbb{E}\Big[\partial_\mu \varphi(r, \mathbb{P}_r^{t,\mu,u})(X(r)) \cdot b(r, X(r), \mathbb{P}_r^{t,\mu,u}, u(r, X(r), \mathbb{P}_r^{t,\mu,u})) \\
&\quad + \frac{1}{2} \mathrm{tr}\,[\partial_x \partial_\mu \varphi(r, \mathbb{P}_r^{t,\mu,u})(X(r)) \cdot \sigma\sigma^T(r, X(r), \mathbb{P}_r^{t,\mu,u}, u(r, X(r), \mathbb{P}_r^{t,\mu,u}))]\Big] dr,
\end{aligned}$$

for all $t \leqslant s \leqslant T$. By $\tilde{V}_L(s, \mathbb{P}_s^{t,\mu,u}) - \tilde{V}_L(t,\mu) \leqslant \varphi(s, \mathbb{P}_s^{t,\mu,u}) - \varphi(t,\mu)$ and dynamic programming principle, we have

$$\begin{aligned}
0 &\leqslant -\tilde{V}_L(s, \mathbb{P}_s^{t,\mu,u}) + \tilde{V}_L(t,\mu) + \varphi(s, \mathbb{P}_s^{t,\mu,u}) - \varphi(t,\mu) \\
&\leqslant \int_t^s \partial_r \varphi(r, \mathbb{P}_r^{t,\mu,u}) + \mathbb{E}\Big[\partial_\mu \varphi(r, \mathbb{P}_r^{t,\mu,u})(X(r)) \cdot b(r, X(r), \mathbb{P}_r^{t,\mu,u}, u(r, X(r), \mathbb{P}_r^{t,\mu,u})) \\
&\quad + \frac{1}{2} tr[\partial_x \partial_\mu \varphi(r, \mathbb{P}_r^{t,\mu,u})(X(r)) \cdot \sigma\sigma^T(r, X(r), \mathbb{P}_r^{t,\mu,u}, u(r, X(r), \mathbb{P}_r^{t,\mu,u}))] \\
&\quad + g(r, X(r), \mathbb{P}_r^{t,\mu,u}, u(r, X(r), \mathbb{P}_r^{t,\mu,u}))\Big] dr.
\end{aligned}$$

Then it is easy to see that

$$\begin{aligned}
0 \quad \leqslant \quad & \partial_t \varphi(t,\mu) + \mathbb{E}\Big[\partial_\mu \varphi(t,\mu)(\xi) \cdot b(t, \xi, \mu, u(t,\xi,\mu)) \\
& + \frac{1}{2} tr[\partial_x \partial_\mu \varphi(t,\mu)(\xi) \cdot \sigma\sigma^T(t, \xi, \mu, u(t,\xi,\mu))] + g(t, \xi, \mu, u(t,\xi,\mu))\Big],
\end{aligned}$$

for all $u \in \mathcal{U}_L$. Since $u(t,\cdot,\mu) \in \mathbb{L}_L(\mathbb{R}^n; U)$ for each fixed $(t,\mu)$, we have

$$\begin{aligned}
0 \quad \leqslant \quad & \inf_{u \in \mathbb{L}_L(\mathbb{R}^n;U)} \Big\{ \partial_t \varphi(t,\mu) + \mathbb{E}[\partial_\mu \varphi(t,\mu)(\xi) \cdot b(t, \xi, \mu, u(t,\xi,\mu)) \\
& + \frac{1}{2} tr[\partial_x \partial_\mu \varphi(t,\mu)(\xi) \cdot \sigma\sigma^T(t, \xi, \mu, u(t,\xi,\mu))] + g(t, \xi, \mu, u(t,\xi,\mu))] \Big\}.
\end{aligned}$$

Then

$$\partial_t\varphi(t,\mu)+H_L(t,\mu,\partial_\mu\varphi(t,\mu)(\cdot),\partial_x\partial_\mu\varphi(t,\mu)(\cdot))\geqslant 0. \tag{2.29}$$

Now we show that $\tilde{V}_L$ is also a viscosity subsolution. Let $\varepsilon>0$, $\tau>t$ with $\tau-t$ small enough. Then there exists $u_{\varepsilon,\tau}\in\mathcal{U}_L$ such that

$$\begin{aligned}&\tilde{V}_L(t,\mu)-\tilde{V}_L(\tau,\mathbb{P}_\tau^{t,\mu,u_{\varepsilon,\tau}})\\ \geqslant&\int_t^\tau\mathbb{E}\Big[g(r,X_r^{t,\xi,u_{\varepsilon,\tau}},\mathbb{P}_r^{t,\mu,u_{\varepsilon,\tau}},u_{\varepsilon,\tau}(r,X_r^{t,\xi,u_{\varepsilon,\tau}},\mathbb{P}_r^{t,\mu,u_{\varepsilon,\tau}}))\Big]dr-\varepsilon(\tau-t).\end{aligned}$$

Let $(t,\mu)$ be a local minimum point of $\tilde{V}_L-\varphi$, then

$$\begin{aligned}0\geqslant&-\tilde{V}_L(\tau,\mathbb{P}_\tau^{t,\mu,u_{\varepsilon,\tau}})+\tilde{V}_L(t,\mu)+\varphi(\tau,\mathbb{P}_\tau^{t,\mu,u_{\varepsilon,\tau}})-\varphi(t,\mu)\\ \geqslant&-\varepsilon(\tau-t)+\int_t^\tau\partial_r\varphi(r,\mathbb{P}_r^{t,\mu,u_{\varepsilon,\tau}})\\ &+\mathbb{E}\Big[\partial_\mu\varphi(r,\mathbb{P}_r^{t,\mu,u_{\varepsilon,\tau}})(X(r;u_{\varepsilon,\tau}))\cdot b(r,X(r;u_{\varepsilon,\tau}),\mathbb{P}_r^{t,\mu,u_{\varepsilon,\tau}},u_{\varepsilon,\tau}(r,X(r;u_{\varepsilon,\tau}),\mathbb{P}_r^{t,\mu,u_{\varepsilon,\tau}}))\\ &+\frac{1}{2}tr[\partial_x\partial_\mu\varphi(r,\mathbb{P}_r^{t,\mu,u_{\varepsilon,\tau}})(X(r;u_{\varepsilon,\tau}))\cdot\sigma\sigma^T(r,X(r;u_{\varepsilon,\tau}),\mathbb{P}_r^{t,\mu,u_{\varepsilon,\tau}},X(r;u_{\varepsilon,\tau}),\mathbb{P}_r^{t,\mu,u_{\varepsilon,\tau}}))]\\ &+g(r,X(r;u_{\varepsilon,\tau}),\mathbb{P}_r^{t,\mu,u_{\varepsilon,\tau}},u_{\varepsilon,\tau}(r,X(r;u_{\varepsilon,\tau}),\mathbb{P}_r^{t,\mu,u_{\varepsilon,\tau}}))\Big]dr\\ \geqslant&-\varepsilon(\tau-t)+\int_t^\tau\inf_{u\in\mathbb{L}_L}\Big\{\partial_r\varphi(r,\mathbb{P}_r^{t,\mu,u})\\ &+\mathbb{E}\Big[\partial_\mu\varphi(r,\mathbb{P}_r^{t,\mu,u})(X(r;t,\xi;u))\cdot b(r,X(r;t,\xi;u),\mathbb{P}_r^{t,\mu,u},u(X(r;t,\xi;u)))\\ &+\frac{1}{2}tr[\partial_x\partial_\mu\varphi(r,\mathbb{P}_r^{t,\mu,u})(X(r;t,\xi;u))\cdot\sigma\sigma^T(r,X(r;t,\xi;u),\mathbb{P}_r^{t,\mu,u},u(X(r;t,\mu;u)))]\\ &+g(r,X(r;t,\xi;u),\mathbb{P}_r^{t,\mu,u},u(X(r;t,\xi;u)))\Big]\Big\}dr,\end{aligned}$$

where $X(r;u_{\varepsilon,\tau})=X(r;t,\xi;u_{\varepsilon,\tau})$. Divide by $(\tau-t)$ on both sides of above inequality and let $\tau\to t$. By the arbitrariness of $\varepsilon$, we have

$$\partial_t\varphi(t,\mu)+H_L(t,\mu,\partial_\mu\varphi(t,\mu)(\cdot),\partial_x\partial_\mu\varphi(t,\mu)(\cdot))\leqslant 0. \tag{2.30}$$

Thus, $\tilde{V}_L$ is a viscosity solution of (2.28). $\square$

**Proposition 2.4.7.** *For any* $(t, \mu) \in [0, T] \times \mathcal{P}^2$,

$$\lim_{L \to \infty} \tilde{V}_L(t, \mu) = \tilde{V}(t, \mu).$$

*Proof.* It is clear that $\lim\limits_{L \to \infty} \tilde{V}_L(t, \mu) \geqslant \tilde{V}(t, \mu)$. On the other hand, for each $n$, there exists some control $u_n \in \mathcal{U}$ such that

$$\tilde{J}(t, \mu, u_n) < \tilde{V}(t, \mu) + \frac{1}{n}.$$

While, $u_n \in \mathcal{U}_{L_n}$ for some $L_n > 0$, we have

$$\tilde{J}(t, \mu, u_n) \geqslant \tilde{V}_{L_n}(t, \mu).$$

Since $\tilde{V}_L(t, \mu)$ decreases as $L$ increases, we have

$$|\tilde{V}_L(t, \mu) - \tilde{V}(t, \mu)| \leqslant \frac{1}{n},$$

for all $L \geqslant L_n$, Thus, $\lim\limits_{L \to \infty} \tilde{V}_L(t, \mu) = \tilde{V}(t, \mu)$. $\square$

## 2.5 Linear-Quadratic Mean Field Stochastic Optimal Control Problem

It would be good to completely solve a mean field optimal control. That is, to find out optimal control and value function. Then HJB equation could be double checked that if it really has value function as its solution. While, this is usually difficult, if not impossible, to give a general example, especially in recursive case. Generally, linear quadratic problems are used as examples, since under proper and relatively very mild conditions, it can be solved completely. For example, it is considered in [49] the state dynamics:

$$
\begin{cases}
dX(s) = (A(s)X(s) + \bar{A}(s)\mathbb{E}[X(s)] + B(s)u(s) + \bar{B}(s)\mathbb{E}[u(s)])ds \\
\qquad\qquad +(C(s)X(s) + \bar{C}(s)\mathbb{E}[X(s)] + D(s)u(s) + \bar{D}(s)\mathbb{E}[u(s)])dW(s), \\
X(0) = x,
\end{cases} \tag{2.31}
$$

with the cost functional:

$$
\begin{aligned}
J(x,u(\cdot)) = \mathbb{E}\Big[ & \langle GX(T), X(T)\rangle + \big\langle \bar{G}\mathbb{E}[X(T), \mathbb{E}[X(T)]]\big\rangle \\
& + \int_t^T \langle Q(s)X(s), X(s)\rangle + \big\langle \bar{Q}\mathbb{E}[X(s)], \mathbb{E}[X(s)]\big\rangle \\
& + \big\langle \bar{R}\mathbb{E}[u(s)], \mathbb{E}[u(s)]\big\rangle + \langle R(s)u(s), u(s)\rangle\, ds\Big],
\end{aligned} \tag{2.32}
$$

The optimal control problem considered there is

**Problem (MF-LQ-0).** For given $x \in \mathbb{R}$, find a $u^* \in \mathcal{U}_{[t,T]}$ such that

$$
J(x, u^*) = \underset{u(\cdot)\in\mathcal{U}_{[t,T]}}{\operatorname{essinf}} J(x,u). \tag{2.33}
$$

To better compare with our result, we consider the problem with the more general initial condition $(t, \xi)$, where $\xi \in \mathcal{F}_t$ is a random variable. Consider the state dynamics:

$$
\begin{cases}
dX(s) = (A(s)X(s) + \bar{A}(s)\mathbb{E}[X(s)] + B(s)u(s))ds \\
\qquad\qquad +(C(s)X(s) + \bar{C}(s)\mathbb{E}[X(s)] + D(s)u(s))dW(s), \qquad s \in [t,T], \\
X(t) = \xi,
\end{cases} \tag{2.34}
$$

and cost functional

$$J(t,\xi;u(\cdot)) = \mathbb{E}\Big[\,\langle GX(T),X(T)\rangle + \big\langle \bar{G}\mathbb{E}[X(T),\mathbb{E}[X(T)]]\big\rangle \\ + \int_t^T \langle Q(s)X(s),X(s)\rangle + \big\langle \bar{Q}\mathbb{E}[X(s)],\mathbb{E}[X(s)]\big\rangle \\ + \langle R(s)u(s),u(s)\rangle\, ds\Big], \tag{2.35}$$

And we considered the following problem:

**Problem (MF-LQ-general).** For given $(t,\xi)\in\mathcal{D}$, find a $u^*\in\mathcal{U}_{[t,T]}$ such that

$$J(t,x,u^*) = \operatorname*{essinf}_{u(\cdot)\in\mathcal{U}_{[t,T]}} J(x,u) \equiv V(t,\xi). \tag{2.36}$$

**Remark 2.5.1.** *Note that the difference between the problem considered here and the one in [49] is not only the more general initial condition, but also the form of state dynamics and cost functional. That is the term containing $\mathbb{E}[u(s)]$ is not considered here. From the result below we can see that, it is enough to considered the simple form* (2.34) *and* (2.35)*, since the influence of $\mathbb{E}[u(s)]$ can be covered by $u(s)$ and $\mathbb{E}[X(s)]$.*

Here is our main result:

**Theorem 2.5.2.** *Let $(X^*,u^*)$ be an optimal pair. Then the following mean field backward SDE admits a unique adapted solution $(Y(\cdot),Z(\cdot))$:*

$$\begin{cases} dY(s) = -(A(s)^TY(s) + \bar{A}^T(s)\mathbb{E}[Y(s)] + C(s)^TZ(s)) \\ \qquad\qquad +Q(s)X^*(s) + \bar{Q}(s)\mathbb{E}[X^*(s)])ds + Z(s)dW(s), \qquad s\in[t,T], \\ Y(T) = GX^*(T) + \bar{G}\mathbb{E}[X^*(T)], \end{cases} \tag{2.37}$$

*such that*

$$R(s)u^*(s) + B(s)^T Y(s) + D(s)^T Z(s) = 0, \tag{2.38}$$

This condition is proved to be sufficient under certain convexity condition. Furthermore, we have the following result about decoupled mean field forward backward stochastic differential equations (MF-FBSDE for short) and Riccati equation.

**Theorem 2.5.3.** *Under proper conditions on the coefficients, the following Riccati equations admit unique solutions* $P$ *and* $\Pi$*, respectively:*

$$\begin{cases} P' + PA + A^T P + C^T PC + Q \\ \qquad -(PB + C^T PD)\Sigma^{-1}(PB + C^T PD)^T = 0, \qquad s \in [t, T], \\ P(T) = G, \end{cases} \tag{2.39}$$

$$\begin{cases} \Pi' + \Pi(A + \bar{A}) + (A + \bar{A})^T \Pi + C^T PC + Q + \bar{Q} \\ \qquad -(\Pi B + C^T PD)\Sigma^{-1}(\Pi B + C^T PD)^T = 0, \qquad s \in [t, T], \\ \Pi(T) = G + \bar{G}, \end{cases} \tag{2.40}$$

*where* $\Sigma = R + D^T PD$*. Further, the following closed-loop system admits a unique solution* $X^*$*:*

$$\begin{cases} dX^*(s) = ((A(s) - B(s)\Sigma^{-1}(s)(B^T(s)P(s) + D^T(s)P(s)C(s)))X^*(s) \\ \quad +(\bar{A}(s) + B(s)\Sigma^{-1}(s)(B^T(s)P(s) + D^T(s)P(s)C(s) + B^T(s)\Pi(s) \\ \quad +D^T(s)P(s)(C(s) + \bar{C}(s))))\mathbb{E}[X^*(s)])ds + ((C(s) - D(s)\Sigma^{-1}(s)(B^T(s)P(s) \\ \quad +D^T(s)P(s)C(s)))X^*(s) + (\bar{C}(s) + D(s)\Sigma^{-1}(s)(B^T(s)P(s) \\ \quad +D^T(s)P(s)C(s) + B^T(s)\Pi(s) + D^T(s)P(s)(C(s) + \bar{C}(s))))\mathbb{E}[X^*(s)])dW(s), \\ X^*(t) = \xi, \end{cases} \tag{2.41}$$

*and by defining*

$$\begin{cases} u^* = -\Sigma^{-1}(B^T P + D^T PC)(X^* - \mathbb{E}[X^*]) - (B^T\Pi + D^T P(C+\bar{C}))\mathbb{E}[X^*] \\ Y = P(X^* - \mathbb{E}[X^*]) + \Pi\mathbb{E}[X^*] \\ Z = (PC - PD\Sigma^{-1}(B^T P + D^T PC))(X^*\mathbb{E}[X^*]) \\ \quad +(P(C+\bar{C}) - PD\Sigma^{-1}(B^T\Pi + D^T P(C+\bar{C})))\mathbb{E}[X^*], \end{cases} \tag{2.42}$$

*the four-tuple* $(X^*, u^*, Y, Z)$ *is the adapted solution to the MF-FBSDE and* $(X^*, u^*)$ *is the optimal pair. Moreover,*

$$\operatorname*{essinf}_u J(t,\xi,u) = J(t,\xi,u^*) = p(t)\mathbb{E}[\langle \xi, \xi\rangle] + (\Pi(t) - p(t))\langle \mathbb{E}[\xi], \mathbb{E}[\xi]\rangle\,. \tag{2.43}$$

*Proof.*

$$
\begin{aligned}
& J(t,\xi,u) - \mathbb{E}[\langle P(t)(\xi-\mathbb{E}[\xi]) + \Pi(t)\mathbb{E}[\xi], \xi\rangle] \\
= & \mathbb{E}[\int_t^T \langle QX, X\rangle + \left\langle \bar{Q}\mathbb{E}[X], \mathbb{E}[X]\right\rangle + \langle Ru, u\rangle + \langle P'(X-\mathbb{E}[X]), X-\mathbb{E}[X]\rangle \\
& +2\langle P(A(X-\mathbb{E}[X]) + B(u-\mathbb{E}[u])), X-\mathbb{E}[X]\rangle + \left\langle P(CX+\bar{C}\mathbb{E}[X]+Du),\right. \\
& \left. CX+\bar{C}\mathbb{E}[X]+Du + \langle \Pi'\mathbb{E}[X], \mathbb{E}[X]\rangle + 2\left\langle \Pi(A+\bar{A})\mathbb{E}[X] + B\mathbb{E}[u], \mathbb{E}[X]\right\rangle ds \right. \\
& + \langle GX(T), X(T)\rangle + \left\langle \bar{G}\mathbb{E}[X](T), \mathbb{E}[X](T)\right\rangle - \langle P(T)(X(T)-\mathbb{E}[X(T)]), X(T) \\
& -\mathbb{E}[X(T)] - \langle \Pi(T)\mathbb{E}[X(T)], \mathbb{E}[X(T)]\rangle] \\
= & \mathbb{E}[\int_t^T \left\langle (P'+Q+2PA+C^TPC)(X-\mathbb{E}[X]), X-\mathbb{E}[X]\right\rangle \\
& + \left\langle (Q+\bar{Q}+(C+\bar{C})^TP(C+\bar{C}) + \Pi' + 2\Pi(A+\bar{A}))\mathbb{E}[X], \mathbb{E}[X]\right\rangle \\
& + \left\langle (R+D^TPD)(u-\mathbb{E}[u]), u-\mathbb{E}[u]\right\rangle + \left\langle (R+D^TPD)\mathbb{E}[u], \mathbb{E}[u]\right\rangle \\
& +2\left\langle u-\mathbb{E}[u], (B^TP+D^TPC)(X-\mathbb{E}[X])\right\rangle + 2\left\langle \mathbb{E}[u], B^T\Pi + D^TP(C+\bar{C})\mathbb{E}[X]\right\rangle ds] \\
= & \mathbb{E}[\int_t^T \left\langle (C^TPD+PB)\Sigma^{-1}(C^TPD+PB)^T(X-\mathbb{E}[X]), X-\mathbb{E}[X]\right\rangle + \langle \Sigma\mathbb{E}[u], \mathbb{E}[u]\rangle \\
& + \left\langle ((C+\bar{C})^TPD+\Pi B)\Sigma^{-1}((C+\bar{C})^TPD+\Pi B)^T\mathbb{E}[X], \mathbb{E}[X]\right\rangle \\
& + \langle \Sigma(u-\mathbb{E}[u]), u-\mathbb{E}[u]\rangle + 2\left\langle \Sigma^{\frac{1}{2}}(u-\mathbb{E}[u]), \Sigma^{-\frac{1}{2}}(C^TPD+PB)^T(X-\mathbb{E}[X])\right\rangle \\
& +2\left\langle \mathbb{E}[u], ((C+\bar{C})^TPD+\Pi B)^T\mathbb{E}[X]\right\rangle ds] \\
= & \mathbb{E}[\int_t^T \|\Sigma^{\frac{1}{2}}(u-\mathbb{E}[u]+\Sigma^{-1}(B^TP+D^TPC)(X-\mathbb{E}[X]))\|^2 \\
& +\|\Sigma^{\frac{1}{2}}(\mathbb{E}[u]+\Sigma^{-1}(B^T\Pi+D^TP(C+\bar{C})\mathbb{E}[X]))\|^2 ds] \\
\geqslant & \, 0,
\end{aligned}
$$

On the other hand, note that

$$
J(t,\xi,u^*) = \mathbb{E}[\langle Y(t), X^*(t)\rangle] = \mathbb{E}[\langle P(t)(\xi-\mathbb{E}[\xi]) + \Pi(t)\mathbb{E}[\xi], \xi\rangle],
$$

so we have

$$V(t,\xi) = \mathbb{E}[\langle P(t)(\xi - \mathbb{E}[\xi]) + \Pi(t)\mathbb{E}[\xi], \xi\rangle] = \mathbb{E}[\langle P(t)\xi, \xi\rangle + (\Pi(t) - P(t))\,\langle \mathbb{E}[\xi], \mathbb{E}[\xi]\rangle].$$

□

Now we verity that, the value function obtained by the discussion above is exactly the classical solution of the HJB we obtained in the previous section. Moreover, the optimal control obtained from the two methods coincide.

**Proposition 2.5.4.** *The Value function obtained in the LQ case satisfies the HJB equation*

$$\partial_t V(t,\mu) + \inf_{u\in U} H(t,x,\mu,u,\partial_\mu V(t,\mu), \partial_x\partial_\mu V(t,\mu)) = 0, \tag{2.44}$$

*for all* $t \in [0,T]$*,* $\mu \in \mathcal{P}^2$*,* $x \in \mathbb{R}^n$*, where*

$$H(t,x,\mu,u,y,z) = \frac{1}{2} tr[z \cdot \sigma\sigma^T(t,x,\mu,u)] + y \cdot b(t,x,\mu,u) + g(t,x,\mu,u).$$

*Proof.* The value function in the LQ case, when writing in terms of distribution, is

$$\tilde{V}(t,\mu) = \int_{\mathbb{R}^n} \langle P(t)x, x\rangle\,\mu(dx) + \left\langle (\Pi(t) - P(t)) \int_{\mathbb{R}^n} x\mu(dx), \int_{\mathbb{R}^n} x\mu(dx) \right\rangle,$$

and $V(t,\xi)$ is the lift of $\tilde{V}(t,\xi)$, where $\mathbb{P}_\xi = \mu$. It is easy to get that

$$\partial_\mu \tilde{V}(t,\mu)(x) = 2P(t)x + 2(\Pi(t) - P(t)) \int_{\mathbb{R}^n} x\mu(dx),$$

$$\partial_x\partial_\mu \tilde{V}(t,\mu)(x) = 2P(t),$$

$$
\begin{aligned}
H(t,x,\mu,u,p,) &= \frac{1}{2}\left\langle z(Cx+\bar{C}\int_{\mathbb{R}^n} x\mu(dx)+Du), Cx+\bar{C}\int_{\mathbb{R}^n} x\mu(dx)+Du\right\rangle \\
&\quad +\left\langle y,(Ax+\bar{A}\int_{\mathbb{R}^n} x\mu(dx)+Bu)\right\rangle+\left\langle \bar{Q}(\int_{\mathbb{R}^n} x\mu(dx)),\int_{\mathbb{R}^n} x\mu(dx)\right\rangle \\
&\quad +\langle Ru,u\rangle+\langle Qx,x\rangle
\end{aligned}
$$

Then, by plugging in all these terms, the value function $\tilde{V}$ is a (classical) solution of (2.44). Moreover, the minimum point of $H$ is

$$
\begin{aligned}
u^*(t,x,\mu) &= -\frac{1}{2}(\frac{1}{2}D^TzD+R)^{-1}(D^Tz(Cx+\bar{C}\int_{\mathbb{R}^n} x\mu(dx))+B^Ty) \\
&= -\frac{1}{2}(D^TPD+R)^{-1}(2D^TP(Cx+\bar{C}\int_{\mathbb{R}^n} x\mu(dx)) \\
&\quad +B^T(2Px+2(\Pi-P)\int_{\mathbb{R}^n} x\mu(dx))) \\
&= -\Sigma^{-1}((D^TPC+B^TP)x+(D^TP\bar{C}+B^T(\Pi-P)\int_{\mathbb{R}^n} x\mu(dx))),
\end{aligned}
$$

This gives a closed-loop form for the optimal control. It is true that

$$
u^*(s,X^*(s),\mathbb{P}_{X^*(s)}) = u^*(s).
$$

□

**Remark 2.5.5.** *Here are some remarks about the optimal control in linear quadratic mean field optimal control problem and the related time-(in)consistency:*

- *The optimal control $u^*$ depends on $\mu$, it is not the classical "feedback" form.*
- *In the explicit form of $u^*$,*

$$
u^*(t,x,\mu) = -\Sigma^{-1}((D^TPC+B^TP)x+(D^TP\bar{C}+B^T(\Pi-P)\int_{\mathbb{R}^n} x'\mu(dx'))), \quad (2.45)
$$

*we have* $u^*(s) = u^*(s, X(s), \mathbb{P}_{X(s)})$*, this can be seen from* (2.42)*. This shows that the open-loop optimal control can be represented in terms of close-loop form.*

- *Time-consistency of this problem can be seen from* (2.45)*.*

$$u^*(s;t,x) = F(s)X^*(s;t,x,u^*) + G(s)\mathbb{E}[X^*(s;t,x,u^*)],$$

*and*

$$u^*(s;\tau, X^*(\tau;t,x,u^*|_{[t,\tau]})) = F(s)X^*(s;\tau, X^*(\tau;t,x,u^*|_{[t,\tau]}), u^*|_{[\tau,T]}) + G(s)\mathbb{E}[X^*(s;...)],$$

*this is obtained by the flow property of* $X(s)$*.*

- *When dynamic programming principle can be derived for a problem, then the problem must be time-consistent. Conversely, when a problem is time-consistent, and there exists an optimal control, then a relation like dynamic programming principle is true.*

# CHAPTER 3: RECURSIVE MEAN FIELD OPTIMAL CONTROL PROBLEM

The objective of this chapter is to present the mean field recursive stochastic optimal control problem and establish the basic results including dynamic programming principle and HJB equations for it.

The research of BSDEs can be traced as early as 1970s, see [32][7][6][21]. Since the paper by Peng and Paradox [40], which initiated the study of nonlinear BSDE, the related theory has been explored extensively. See [31][38][41][54] and the reference therein. Many interesting applications of BSDEs have been found in finance. For example, when we consider asset allocation problems, people could be optimistic or pessimistic. In evaluating the current financial situations (portfolio of assets), when the future utility is taken account, i.e., the current utility depends on the future utility, besides other dependence. *Recursive utility* was introduced to describe such situations. In 1992, Duffie and Epstein introduced stochastic differential utility[15][16], which is in the form:

$$Y(t) = \mathbb{E}_t\Big[\eta + \int_t^T g(s, Y(s))ds\Big], \tag{3.1}$$

where $E_t[\,\cdot\,] = \mathbb{E}[\,\cdot\,|\mathcal{F}_t]$ represents expectation conditional on information at time $t$. (3.1) is the *recursive utility* of the payoff $\eta$ at $T$. Also see [53][43][14][46].

## 3.1 The Statement of the Problem

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space on which a $d$-dimensional standard Brownian motion $W(\cdot)$ is defined. Let $\mathcal{G}$ be a filtration that is independent of $W$. The filtration $\mathbb{F}$ is defined the same as in the previous chapter. Now we consider the state dynamics:

$$\begin{cases} dX(s) = b(s, X(s), \mathbb{P}_{X(s)}, u(s, X(s), \mathbb{P}_{X(s)}))ds \\ \qquad\qquad +\sigma(s, X(s), \mathbb{P}_{X(s)}, u(s, X(s), \mathbb{P}_{X(s)}))dW(s), \quad s \in [t, T], \\ X(t) = \xi, \end{cases} \tag{3.2}$$

where $\xi \in \mathcal{F}_t$. The state dynamics is the same as (2.8), while, the cost functional we considered here is defined by the adapted solution of the mean field backward stochastic differential equation (MF-BSDE):

$$J(t, \xi; u) \equiv Y(t; t, \xi, u), \tag{3.3}$$

where

$$\begin{cases} dY(s) = -g(s, X(s), \mathbb{P}_{X(s)}, Y(s), \mathbb{P}_{Y(s)}, Z(s), u(s, X(s), \mathbb{P}_{X(s)}))ds \\ \qquad\qquad +Z(s)dW(s), \quad s \in [t, T], \\ Y(T) = h(X(T), \mathbb{P}_{X(T)}). \end{cases} \tag{3.4}$$

The problem considered here is

**Problem (MF-R).** For given $(t, \xi) \in \mathcal{D}$, find a $u^* \in \mathcal{U}$ such that

$$J(t, \xi, u^*) = \operatorname*{essinf}_{u \in \mathcal{U}} J(t, \xi, u) = V(t, \xi), \tag{3.5}$$

where $\mathcal{U}$ is the set of *closed-loop strategies* defined in the previous chapter.

## 3.2 Properties

In this section, we will discuss the existence and uniqueness of the solution for the corresponding SDEs and BSDEs.

### *3.2.1 Mean Field Controlled Stochastic Differential Equation*

The existence and uniqueness of solution to (3.2) have been showed in the previous chapter. While, the main difficulty we encounter here is still that the initial condition $\xi$ is random, which leads to the value function $V(t,\xi)$ to be random. Moreover, unlike the case in the last chapter, the dependence of $V$ on $\xi$ here is not only through its distribution $\mathbb{P}_\xi$. That is,

$$V(t,\xi) \neq V(t,x)|_{x=\xi},$$

and

$$V(t,\xi) \neq \tilde{V}(t,\mathbb{P}_\xi),$$

generally. To overcome this difficulty, we consider the following auxiliary equation for (3.2):

$$\begin{cases} d\bar{X}(s) = b(s,\bar{X}(s),\mathbb{P}_{X(s)},u(s,\bar{X}(s),\mathbb{P}_{X(s)}))ds \\ \qquad\qquad +\sigma(s,\bar{X}(s),\mathbb{P}_{X(s)},u(s,\bar{X}(s),\mathbb{P}_{X(s)}))dW(s), \qquad s\in[t,T], \\ \bar{X}(t) = x, \end{cases} \tag{3.6}$$

**Proposition 3.2.1.** *Under* **(H1)***, for any* $(t,\xi)\in\mathcal{D}$*,* $x\in\mathbb{R}^n$ *and any* $u\in\mathcal{U}$*, there exist unique solutions* $X(\cdot)=X(\cdot\,;t,\xi,u(\cdot))\in L^2_{\mathbb{F}}(\Omega;C([t,T];\mathbb{R}^n))$ *and* $\bar{X}(\cdot)=\bar{X}(\cdot;t,x,\xi,u)\in L^2_{\mathbb{F}}(\Omega;C([t,T];\mathbb{R}^n))$ *to equations* (3.2) *and* (3.6) *respectively. Moreover, the following estimates hold for* $\bar{X}$*:*

$$\mathbb{E}\Big[\sup_{s\in[t,T]}|\bar{X}(s;t,x,\xi,u)|^2\Big] \leqslant K(1+x^2+\mathbb{E}[|\xi|^2]), \tag{3.7}$$

$$\begin{aligned} &\mathbb{E}\Big[\sup_{s\in[t,T]}|\bar{X}(s;t,x_1,\xi_1,u)-\bar{X}(s;t,x_2,\xi_2,u)|^2\Big] \\ &\leqslant K(|x_1-x_2|^2+\mathcal{W}_2^2(\mathbb{P}_{\xi_1},\mathbb{P}_{\xi_2})(T-t)). \end{aligned} \tag{3.8}$$

*Note that the constant $K$ here depends on $b$, $\sigma$ and $u$.*

*Proof.* Note that (3.6) is actually a classical SDE since $\mathbb{P}_{X(s)}$ can be considered as given. Then, the existence and uniqueness of the solution to (3.6) under **(H1)** and the property (3.7) are standard, see, for example, [27]. The proof of the property (3.8) needs also the following result. $\square$

Now, we rewrite the genuine dependence of $\mathbb{P}_{X(s)}$ and $\bar{X}(s)$ on initial condition $\xi$. The result is first proved in [10] without showing all important details. We give a complete proof here.

**Proposition 3.2.2.** *Let $X(\cdot) = X(\cdot; t, \xi, u)$ and $\bar{X}(\cdot) = \bar{X}(\cdot; t, x, \xi, u)$ be the solutions to equations* (3.2) *and* (3.6) *respectively. Then, for any $s \in [t, T]$, the distribution $\mathbb{P}_{X(s)}$ and the state $\bar{X}(s)$ depend on $\xi$ only through its distribution $\mathbb{P}_\xi$. We adopt the notation $\mathbb{P}_{X(s;t,\xi,u)} = \mathbb{P}_s^{t,\mathbb{P}_\xi,u}$. And it satisfies the following estimate:*

$$\begin{aligned}&\mathcal{W}_2(\mathbb{P}_{s_1}^{t_1,\mathbb{P}_{\xi_1},u}, \mathbb{P}_{s_2}^{t_2,\mathbb{P}_{\xi_2},u}) \\ &\leqslant K(1 + \|\mathbb{P}_{\xi_1}\|_2 + \|\mathbb{P}_{\xi_1}\|_2)(\mathcal{W}_2(\mathbb{P}_{\xi_1}, \mathbb{P}_{\xi_2}) + |t_2 - t_1|^{\frac{1}{2}} + |s_2 - s_1|^{\frac{1}{2}}),\end{aligned} \tag{3.9}$$

*for all $(t_i, \xi_i) \in \mathcal{D}$, $s_i \in [t_i, T]$, $i = 1, 2$.*

*Proof.* By the uniqueness of the SDE (3.2), it's easy to see that, for any $s \in [t, T]$,

$$\bar{X}(s; t, \xi, \xi, u) = \bar{X}(s; t, x, \xi, u)|_{x=\xi} = X(s; t, \xi, u).$$

First we show that, for any $\eta, \eta' \in L^2_{\mathcal{F}_t}(\Omega; \mathbb{R}^n)$ with $\mathbb{P}_{\eta'} = \mathbb{P}_\eta$, it is true that $\bar{X}(s; t, \eta', \xi, u)$ and $\bar{X}(s; t, \eta, \xi, u)$ have the same distribution.

When $\eta$ and $\eta'$ are both simple randome variables, say $\eta = \sum_{i=1}^m x_i 1_{E_i}$ and $\eta' = \sum_{i=1}^m x_i 1_{E_i'}$, where $x_i's$ are constants and the partition $E_i's$ are $\mathcal{F}_t-$measurable sets with $\mathbb{P}(E_i) = \mathbb{P}(E_i')$ for each

$i = 1, ..., m$. Since

$$\bar{X}(s;t,\eta,\xi,u) = \sum_{i=1}^{m} \bar{X}(s;t,x_i,\xi,u)1_{E_i};$$

$$\bar{X}(s;t,\eta',\xi,u) = \sum_{i=1}^{m} \bar{X}(s;t,x_i,\xi,u)1_{E_i'};$$

Let $A \in \mathcal{B}(\mathbb{R}^n)$, then

$$\begin{aligned}
&\mathbb{P}(\bar{X}(s;t,\eta,\xi,u) \in A) \\
&= \mathbb{P}(\sum_{i=1}^{m} \bar{X}(s;t,x_i,\xi,u)1_{E_i} \in A) \\
&= \sum_{i=1}^{m} \mathbb{P}(\bar{X}(s;t,x_i,\xi,u)1_{E_i} \in A) \\
&= \sum_{i=1}^{m} \mathbb{P}(\bar{X}(s;t,x_i,\xi,u) \in A \cap E_i) \\
&= \sum_{i=1}^{m} \mathbb{P}(\bar{X}(s;t,x_i,\xi,u) \in A) \cdot \mathbb{P}(E_i) \\
&= \sum_{i=1}^{m} \mathbb{P}(\bar{X}(s;t,x_i,\xi,u) \in A) \cdot \mathbb{P}(E_i') \\
&= \mathbb{P}(\bar{X}(s;t,\eta',\xi,u) \in A).
\end{aligned}$$

Generally, since $\mathbb{P}_{\eta'} = \mathbb{P}_{\eta}$, there exists sequence $\{\eta_n\}$ ($\{\eta_n'\}$) of simple $\mathcal{F}_t-$measurable random variables that converges to $\eta$ ($\eta'$) pointwisely as $n \to \infty$ and $\mathbb{P}_{\eta_n'} = \mathbb{P}_{\eta_n}$, for all $n$. By (3.8), for each $s \in [t, T]$,

$$\bar{X}(s;t,\eta_n,\xi,u) \xrightarrow{L^2} \bar{X}(s;t,\eta,\xi,u),$$

$$\bar{X}(s;t,\eta_n',\xi,u) \xrightarrow{L^2} \bar{X}(s;t,\eta',\xi,u)$$

as $n \to \infty$. By Theorem 5.5 in [12], it implies that

$$\lim_{n\to\infty} \mathcal{W}_2(\mathbb{P}_{\bar{X}(s;t,\eta_n,\xi,u)}, \mathbb{P}_{\bar{X}(s;t,\eta,\xi,u)}) = 0,$$

and

$$\lim_{n\to\infty}\mathcal{W}_2(\mathbb{P}_{\bar{X}(s;t,\eta_n',\xi,u)},\mathbb{P}_{\bar{X}(s;t,\eta',\xi,u)})=0.$$

Also, from the discussion for simple random variables, we have $\mathbb{P}_{\bar{X}(s;t,\eta_n,\xi,u)}=\mathbb{P}_{\bar{X}(s;t,\eta_n',\xi,u)}$ for all $n$. Since

$$\begin{aligned}&\mathcal{W}_2(\mathbb{P}_{\bar{X}(s;t,\eta,\xi,u)},\mathbb{P}_{\bar{X}(s;t,\eta',\xi,u)})\\ \leqslant\ &\mathcal{W}_2(\mathbb{P}_{\bar{X}(s;t,\eta,\xi,u)},\mathbb{P}_{\bar{X}(s;t,\eta_n,\xi,u)})+\mathcal{W}_2(\mathbb{P}_{\bar{X}(s;t,\eta_n',\xi,u)},\mathbb{P}_{\bar{X}(s;t,\eta',\xi,u)}),\end{aligned}$$

we have

$$\mathbb{P}_{\bar{X}(s;t,\eta,\xi,u)}=\mathbb{P}_{\bar{X}(s;t,\eta',\xi,u)}.$$

Now, it is easy to see that $\mathbb{P}_{\bar{X}(s;t,\xi',\xi,u)}=\mathbb{P}_{X(s;t,\xi,u)}$ wheneven $\mathbb{P}_{\xi'}=\mathbb{P}_{\xi}$.

By the discussion above and the definition of Warsserstein distance,

$$\begin{aligned}&\mathcal{W}_2^2(\mathbb{P}_{X(s;t,\xi_1,u)},\mathbb{P}_{X(s;t,\xi_2,u)})\\ =\ &\mathcal{W}_2^2(\mathbb{P}_{\bar{X}(s;t,\xi_1',\xi_1,u)},\mathbb{P}_{\bar{X}(s;t,\xi_2',\xi_2,u)})\\ \leqslant\ &\mathbb{E}[|\bar{X}(s;t,\xi_1',\xi_1,u)-\bar{X}(s;t,\xi_2',\xi_2,u)|^2]\\ \leqslant\ &K\mathbb{E}[|\xi_1'-\xi_2'|^2+\int_t^s\mathcal{W}_2^2(\mathbb{P}_{X(r;t,\xi_1,u)},\mathbb{P}_{X(r;t,\xi_2,u)})dr],\end{aligned}$$

for all $\xi_i'$ with $\mathbb{P}_{\xi_i'}=\mathbb{P}_{\xi_i}$, $i=1,2$. So,

$$\mathcal{W}_2^2(\mathbb{P}_{X(s;t,\xi_1,u)},\mathbb{P}_{X(s;t,\xi_2,u)})\leqslant K(\mathcal{W}_2^2(\mathbb{P}_{\xi_1},\mathbb{P}_{\xi_2})+\int_t^s\mathcal{W}_2^2(\mathbb{P}_{X(r;t,\xi_1,u)},\mathbb{P}_{X(r;t,\xi_2,u)})dr),$$

for all $s\in[t,T]$, where the constant $K$ depends only on $L$ and $s-t$. By Gronwall's inequality,

$$\mathcal{W}_2^2(\mathbb{P}_{X(s;t,\xi_1,u)},\mathbb{P}_{X(s;t,\xi_2,u)})\leqslant K\mathcal{W}_2^2(\mathbb{P}_{\xi_1},\mathbb{P}_{\xi_2}),$$

the distribution $\mathbb{P}_{X(s;t,\xi,u)}$ depends on $\xi$ only through its distribution $\mathbb{P}_\xi$. And we adopt the notation

$\mathbb{P}_{X(s;t,\xi,u)} = \mathbb{P}_s^{t,\mathbb{P}_\xi,u}$. Moreover,

$$
\begin{aligned}
&\mathbb{E}\Big[\sup_{s\in[t,T]} |\bar{X}(s;t,x,\xi_1,u_1) - \bar{X}(s;t,x,\xi_2,u_2)|^2\Big] \\
&\leqslant K\mathbb{E}[\int_t^T (\mathcal{W}_2^2 \mathbb{P}_{X(r;t,\xi_1,u)}, \mathbb{P}_{X(r;t,\xi_2,u)})dr] \\
&\leqslant K\mathcal{W}_2^2(\mathbb{P}_{\xi_1}, \mathbb{P}_{\xi_2})(T-t).
\end{aligned}
$$

So, $\bar{X}(s;t,x,\xi,u)$ depends on $\xi$ only through $\mathbb{P}_\xi$ and we adopt the notation $\bar{X}(s;t,x,\xi,u) = \bar{X}(s;t,x,\mathbb{P}_\xi,u)$. To prove the estimate (3.9), note that

$$
\begin{aligned}
&\mathcal{W}_2^2(\mathbb{P}_{s_1}^{t_1,\mathbb{P}_{\xi_1},u}, \mathbb{P}_{s_2}^{t_2,\mathbb{P}_{\xi_2},u}) \\
&\leqslant \mathbb{E}[|\bar{X}(s_1;t_1,\xi_1',\xi_1,u) - \bar{X}(s_2;t_2,\xi_2',\xi_2,u)|^2] \\
&\leqslant K(1+\mathbb{E}[|\xi_1|^2+|\xi_2|^2])(\mathbb{E}[|\xi_1'-\xi_2'|^2] + \mathcal{W}_2^2(\mathbb{P}_{\xi_1},\mathbb{P}_{\xi_2}) + |t_2-t_1| + |s_2-s_1|) \\
&\leqslant K(1+\|\mathbb{P}_{\xi_1}\|_2^2 + \|\mathbb{P}_{\xi_1}\|_2^2)(\mathcal{W}_2^2(\mathbb{P}_{\xi_1},\mathbb{P}_{\xi_2}) + |t_2-t_1| + |s_2-s_1|),
\end{aligned}
$$

where $K$ depends on the Lipschitz constant of $u$.

□

**Proposition 3.2.3.** *(Flow Property) Let $X(\cdot) = X(\cdot;t,\xi,u)$ and $\bar{X}(\cdot) = \bar{X}(\cdot;t,x,\xi,u)$ be the solutions to equations* (3.2) *and* (3.6) *respectively. Then they satisfy the flow property:*

$$
X(s;t,\xi,u) = X(s;\tau,X(\tau;t,\xi,u_1),u_2); \tag{3.10}
$$

*its distribution process also satisfies the flow property that*

$$
\mathbb{P}_s^{t,\mathbb{P}_\xi,u} = \mathbb{P}_s^{\tau,\mathbb{P}_\tau^{t,\mathbb{P}_\xi,u_1},u_2}, \tag{3.11}
$$

$$\bar{X}(s;t,x,\mathbb{P}_\xi,u) = \bar{X}(s;\tau,\bar{X}(\tau;t,x,\mathbb{P}_\xi,u_1),\mathbb{P}_\tau^{t,\mathbb{P}_\xi,u_1},u_2), \tag{3.12}$$

*for all* $0 \le t \le \tau \le s \le T$*, where* $u(\cdot) = u_1(\cdot)1_{[t,\tau)} + u_2(\cdot)1_{[\tau,T]}$.

*Proof.* The proof follows from the existence and uniqueness of solution to equations (3.2) and (3.6). □

### 3.2.2 Mean Field Controlled Backward Stochastic Differential Equation

The mean field BSDE we considered here is of McKean-Vlasov type. Similar to the discussion in previous section, we know this is a very general framework. The discussions on different types of mean field BSDE started in the last decade, see [8][9][37][20][1]. Note that the cost functional considered in our problem is defined by

$$J(t,\xi;u(\cdot)) \equiv Y(t;t,\xi,u), \tag{3.13}$$

where $Y$ is the solution to the mean field BSDE (3.4). Similar as the discussion for state dynamics, we also introduce the auxiliary BSDE for (3.4),

$$\begin{cases} d\bar{Y}(s) = -g(s,\bar{X}(s),\mathbb{P}_s^{t,\mathbb{P}_\xi,u},\bar{Y}(s),\mathbb{P}_{Y(s)},\bar{Z}(s),u(s,\bar{X}(s),\mathbb{P}_s^{t,\mathbb{P}_\xi,u}))ds \\ \qquad\qquad +\bar{Z}(s)dW(s), \\ \bar{Y}(T) = h(\bar{X}(T),\mathbb{P}_T^{t,\mathbb{P}_\xi,u}), \end{cases} \tag{3.14}$$

where $s \in [t,T]$, $\bar{X}(s) = \bar{X}(s;t,x,\mathbb{P}_\xi,u)$ and the coefficients $h$ and $g$ are deterministic functions, for which we introduce the following assumptions:

**(H2')** The map $g : [0,T] \times \mathbb{R}^n \times \mathcal{P}^2 \times \mathbb{R} \times \mathcal{P}^2(\mathbb{R}) \times \mathbb{R}^d \times U \to \mathbb{R}$ and $h : \mathbb{R}^n \times \mathcal{P}^2 \to \mathbb{R}$ are continuous and there exists constant $C > 0$ such that

$$
\begin{aligned}
&|g(t,x,\mu,y,\nu,z,u)| + |h(x',\mu')| \\
\leqslant\; & C\big(1 + |x| + \|\mu\|_2 + |y| + \|\nu\|_2 + |z| + |x'| + \|\mu'\|_2\big),
\end{aligned} \tag{3.15}
$$

for all $(t,x,\mu,y,\nu,z,u,x',\mu') \in [0,T] \times \mathbb{R}^n \times \mathcal{P}^2 \times \mathbb{R} \times \mathcal{P}^2(\mathbb{R}) \times \mathbb{R}^d \times U \times \mathbb{R}^n \times \mathcal{P}^2$, and

$$
\begin{aligned}
&|g(t,x_1,\mu_1,y_1,\nu_1,z_1,u_1) - g(t,x_2,\mu_2,y_2,\nu_2,z_2,u_2)| \\
&\quad + |h(x_1^{'},\mu_1^{'}) - h(x_2^{'},\mu_2^{'})| \\
\leqslant\; & C\big(|x_1 - x_2| + \mathcal{W}^2(\mu_1,\mu_2) + |x_1^{'} - x_2^{'}| + \mathcal{W}_2(\mu_1^{'},\mu_2^{'}) \\
&\quad + |u_1 - u_2| + |y_1 - y_2| + \mathcal{W}_2(\nu_1,\nu_2)\big),
\end{aligned} \tag{3.16}
$$

for all $(t,x_i,\mu_i,x_i^{'},\mu_i^{'},u_i) \in [0,T] \times \mathbb{R}^n \times \mathcal{P}^2 \times \mathbb{R}^n \times \mathcal{P}^2 \times U$, $i = 1,2$.

**Proposition 3.2.4.** *Under (H2'), for any $(t,\xi) \in \mathcal{D}$, $x \in \mathbb{R}^n$ and any $u \in \mathcal{U}$, there exists a unique solution $(Y(\cdot),Z(\cdot)) = (Y(\cdot;t,\xi,u),Z(\cdot;t,\xi,u)) \in L^2_{\mathbb{F}}(\Omega;C([t,T];\mathbb{R})) \times L^2_{\mathbb{F}}(\Omega;L^2([t,T];\mathbb{R}))$ to* (3.4) *and $(\bar{Y}(\cdot),\bar{Z}(\cdot)) \in L^2_{\mathbb{F}}(\Omega;C([t,T];\mathbb{R})) \times L^2_{\mathbb{F}}(\Omega;L^2([t,T];\mathbb{R}))$ to* (3.14) *with the following estimations:*

$$
\mathbb{E}\Big[\sup_{s\in[t,T]} |Y(s;t,\xi,u)|^2\Big] \leqslant C(1 + \mathbb{E}[|\xi|^2]). \tag{3.17}
$$

$$
\mathbb{E}\Big[\sup_{s\in[t,T]} |Y(s;t,\xi_1,u) - Y(s;t,\xi_2,u)|^2\Big] \leqslant C\mathbb{E}\Big[|\xi_1 - \xi_2|^2\Big]. \tag{3.18}
$$

$$
\mathbb{E}\Big[\sup_{s\in[t,T]} |\bar{Y}(s;t,x,\xi,u)|^2\Big] \leqslant C(1 + x^2 + \mathbb{E}[|\xi|^2]). \tag{3.19}
$$

$$
\begin{aligned}
&\mathbb{E}\Big[\sup_{s\in[t,T]}|\bar{Y}(s;t,x_1,\xi_1,u)-\bar{Y}(s;t,x_2,\xi_2,u)|^2\Big] \\
&\leqslant C(|x_1-x_2|^2+\mathcal{W}_2^2(\mu_1,\mu_2)+\int_t^T \mathcal{W}_2^2(\mathbb{P}_{Y_1(r)},\mathbb{P}_{Y_2(r)})dr),
\end{aligned}
\tag{3.20}
$$

*where* $\mu_i = \mathbb{P}_{\xi_i}$, $i = 1, 2$. *Moreover, note that*

$$
Y(s;t,\xi,u) = \bar{Y}(s;t,x,\xi,u)|_{x=\xi},
$$

*for all* $s \in [t, T]$.

First we recall the following result:

**Lemma 3.2.5.** *The BSDE*

$$
\begin{cases}
dY(s) = -g_0(s)ds + Z(s)dW(s), \qquad s\in[t,T], \\
Y(T) = \zeta.
\end{cases}
\tag{3.21}
$$

*has a unique adapted solution, provided that* $\mathbb{E}\big[(\int_t^T \big|g_0(r)\big|dr)^2\big] < \infty$ *and* $\zeta \in L^2(\Omega, \mathcal{F}_T^W, \mathbb{R})$. *Further, there exists a constant* $K_1$ *such that*

$$
\mathbb{E}_t\big[\sup_{t\le s\le T}\big|Y(s)\big|^2+\int_t^T\big|Z(s)\big|^2ds\big] \le K_1\mathbb{E}_t\big[\big|\zeta\big|^2+(\int_t^T\big|g_0(r)\big|dr)^2\big].
$$

*Proof.* The unique existence is the standard result of BSDE. We just show that the inequality holds.

$$\begin{aligned}
& \mathbb{E}_t\big[\sup_{t\leq s\leq T}\big|Y(s)\big|^2\big] \\
= \;& \mathbb{E}_t\big[\sup_{t\leq s\leq T}\big|\mathbb{E}_s\big[\zeta+\int_s^T g_0(r)dr\big]\big|^2\big] \\
\leq \;& \mathbb{E}_t\big[\sup_{t\leq s\leq T}\big|\mathbb{E}_s\big[\big|\zeta\big|+\int_t^T \big|g_0(r)\big|dr\big]\big|^2\big] \\
\leq \;& 4\mathbb{E}_t\big[(\big|\zeta\big|+\int_t^T \big|g_0(r)\big|dr)^2\big] \\
\leq \;& 4\mathbb{E}_t\big[\big|\zeta\big|^2+(\int_t^T \big|g_0(r)\big|dr)^2\big].
\end{aligned}$$

And,

$$\begin{aligned}
\mathbb{E}_t\big[\int_t^T \big|Z(s)\big|^2 ds\big] &\leq c\mathbb{E}_t\big[\sup_{t\leq s\leq T}\big|\int_t^s Z(r)dW(r)\big|^2\big] \\
&\leq c\mathbb{E}_t\big[\sup_{t\leq s\leq T}\big|\int_s^T Z(r)dW(r)\big|^2\big] \\
&\leq c\mathbb{E}_t\big[\big|\zeta\big|^2+\sup_{t\leq s\leq T}\big|Y(s)\big|^2+\sup_{t\leq s\leq T}\big|\int_s^T g_0(r)dr\big|^2\big].
\end{aligned}$$

Combine with the first result, we have

$$\mathbb{E}_t\big[\sup_{t\leq s\leq T}\big|Y(s)\big|^2+\int_t^T \big|Z(s)\big|^2 ds\big]\leq K_1\mathbb{E}_t\big[\big|\zeta\big|^2+(\int_t^T \big|g_0(r)\big|dr)^2\big],$$

for some constant $K_1 > 0$.

□

Now we give the proof of Proposition 3.2.4.

*Proof.* First of all, we consider the following BSDE:

$$\begin{cases} dY(s) = -f(s, Y(s), \mathbb{P}_{Y(s)}, Z(s))ds + Z(s)dW(s), \qquad s \in [t, T], \\ Y(T) = \zeta, \end{cases} \tag{3.22}$$

where $\zeta \in L^2(\Omega, \mathcal{F}_T^W, \mathbb{R})$, $f : \Omega \times [0, T] \times \mathbb{R} \times \mathcal{P}^2(\mathbb{R}) \times \mathbb{R}^d \to \mathbb{R}$ is $\mathbb{F}^W-$progressively measurable and satisfies: $\mathbb{P}-$a.s.

$$\begin{gathered} |f(s, y_1, \nu_1, z_1) - f(s, y_2, \nu_2, z_2)| \leqslant M\big(|y_1 - y_2| + \mathcal{W}_2(\nu_1, \nu_2) + |z_1 - z_2|\big), \\ \forall (s, y_i, \nu_i, z_i) \in [0, T] \times \mathbb{R} \times \mathcal{P}^2 \times \mathbb{R}^d, \ i = 1, 2. \end{gathered} \tag{3.23}$$

and

$$\mathbb{E}[(\int_0^T |f(s, 0, \delta_0, 0)|ds)^2] < \infty, \tag{3.24}$$

for some constant $M > 0$. Let $(y(\cdot), z(\cdot)) \in L^2_{\mathbb{F}^W}(\Omega; C([t, T]; \mathbb{R})) \times L^2_{\mathbb{F}^W}(\Omega; L^2([t, T]; \mathbb{R}))$, then $f(s, y(s), \mathbb{P}_{y(s)}, z(s))$ satisfies the condition in Lemma 3.2.5. There exists uniquely $(Y(\cdot), Z(\cdot)) \in L^2_{\mathbb{F}^W}(\Omega; C([t, T]; \mathbb{R})) \times L^2_{\mathbb{F}^W}(\Omega; L^2([t, T]; \mathbb{R}))$ such that,

$$Y(s) = \zeta + \int_s^T f(r, y(r), \mathbb{P}_{y(r)}, z(r))dr - \int_s^T Z(r)dW(r).$$

This leads to the definition of a map $\Phi : (y(\cdot), z(\cdot)) \mapsto (Y(\cdot), Z(\cdot))$, which is proved to be a contraction, since for any $(y_i, z_i) \in L^2_{\mathbb{F}^W}(\Omega; C([t, T]; \mathbb{R})) \times L^2_{\mathbb{F}^W}(\Omega; L^2([t, T]; \mathbb{R}))$, let $(Y_i, Z_i) = \Phi(y_i, z_i)$, for $i = 1, 2$, then

$$\begin{aligned} & \mathbb{E}\big[\sup_{t \leq s \leq T} \big|Y_1(s) - Y_2(s)\big|^2 + \int_t^T |Z_1(s) - Z_2(s)|^2 ds\big] \\ \leq \ & \mathbb{E}\big[(\int_t^T f(s, y_1(s), \mathbb{P}_{y_1(s)}, z_1(s)) - f(s, y_2(s), \mathbb{P}_{y_2(s)}, z_2(s))ds)^2\big] \\ \leq \ & K_1 \mathbb{E}\big[\sup_{t \leqslant s \leqslant T} |y_1(s) - y_2(s)|^2 (T - t)^2 + (\int_t^T |z_1(s) - z_2(s)|^2 ds)(T - t)\big]. \end{aligned}$$

So, there exists a unique solution to the equation (3.22).

Generally, for the solution to the SDE (3.2), we have $X(\cdot) \in L^2_{\mathbb{F}}(\Omega; C([t,T];\mathbb{R}^n))$, where $\mathcal{F}_s = \mathcal{F}^W_s \vee \mathcal{G}$. By the Theorem 5.1 of [29], for any $(y(\cdot), z(\cdot)) \in L^2_{\mathbb{F}}(\Omega; C([t,T];\mathbb{R})) \times L^2_{\mathbb{F}}(\Omega; L^2([t,T];\mathbb{R}))$, the following generalized BSDE

$$\begin{cases} d\tilde{Y}(s) = -g(s, X(s), \mathbb{P}^{t,\mathbb{P}_\xi,u}_s, y(s), \mathbb{P}_{y(s)}, z(s), u(s, X(s), \mathbb{P}^{t,\mathbb{P}_\xi,u}_s))ds \\ \qquad\qquad + \tilde{Y}(s)dW(s) - d\tilde{M}(s), \\ \tilde{Y}(T) = h(X(T), \mathbb{P}^{t,\mathbb{P}_\xi,u}_T), \end{cases} \tag{3.25}$$

has a unique solution $(\tilde{Y}, \tilde{Z}, \tilde{M})$, where $(\tilde{Y}, \tilde{Z}, \tilde{M}) \in L^2_{\mathbb{F}}(\Omega; C([t,T];\mathbb{R})) \times L^2_{\mathbb{F}}(\Omega; L^2([t,T];\mathbb{R})) \times L^2_{\mathbb{F}}(\Omega; L^2([t,T];\mathbb{R}))$ and $\tilde{M}$ is an $\mathbb{F}-$martingale orthogonal to $W$. Moreover, the following estimate holds:

$$\begin{aligned} &\mathbb{E}\Big[\sup_{s\in[t,T]} |\tilde{Y}(s)|^2 + \int_t^T |\tilde{Z}(s)|^2 ds + [\tilde{M}]_T\Big] \\ &\leqslant C\mathbb{E}\Big[\int_t^T |g(r, X(r), \mathbb{P}^{t,\mathbb{P}_\xi,u}_r, y(r), \mathbb{P}_{y(r)}, z(r), u(r, X(r), \mathbb{P}^{t,\mathbb{P}_\xi,u}_r))|^2)dr \\ &\qquad + |h(X(T), \mathbb{P}^{t,\mathbb{P}_\xi,u}_T)|^2\Big]. \end{aligned} \tag{3.26}$$

Now we show that the map $\Phi : (y, z) \mapsto (\tilde{Y}, \tilde{Z})$ is a contraction. Let $(y_i(\cdot), z_i(\cdot)) \in L^2_{\mathbb{F}}(\Omega; C([t,T];\mathbb{R})) \times L^2_{\mathbb{F}}(\Omega; L^2([t,T];\mathbb{R}))$, and $(\tilde{Y}_i, \tilde{Z}_i) = \Phi(y_i, z_i)$, $i = 1, 2$. By (3.26), we have

$$\begin{aligned} &\mathbb{E}\Big[\sup_{s\in[t,T]} |\tilde{Y}_1(s) - \tilde{Y}_2(s)|^2 + \int_t^T |\tilde{Z}_1(s) - \tilde{Z}_2(s)|^2 ds + [\tilde{M}_1 - \tilde{M}_2]_T\Big] \\ &\leqslant C\mathbb{E}\Big[\int_t^T |g(r, X(r), \mathbb{P}^{t,\mathbb{P}_\xi,u}_r, y_1(r), \mathbb{P}_{y_1(r)}, z_1(r), u(r, X(r), \mathbb{P}^{t,\mathbb{P}_\xi,u}_r)) \\ &\qquad - g(r, X(r), \mathbb{P}^{t,\mathbb{P}_\xi,u}_r, y_2(r), \mathbb{P}_{y_2(r)}, z_2(r), u(r, X(r), \mathbb{P}^{t,\mathbb{P}_\xi,u}_r))|^2)dr\Big] \\ &\leqslant C\mathbb{E}\Big[\sup_{s\in[t,T]} |y_1(s) - y_2(s)|^2 + \int_t^T |z_1(s) - z_2(s)|^2 ds\Big](T-t) \end{aligned}$$

Combine with the flow property of the BSDE, we can prove the existence and uniqueness of the equation:

$$\begin{cases} dY(s) = -g(s, X(s), \mathbb{P}_s^{t,\mathbb{P}_\xi,u}, Y(s), \mathbb{P}_{Y(s)}, Z(s), u(s, X(s), \mathbb{P}_s^{t,\mathbb{P}_\xi,u}))ds \\ \qquad\qquad +Z(s)dW(s) - dM(s), \\ Y(T) = h(X(T), \mathbb{P}_T^{t,\mathbb{P}_\xi,u}), \end{cases} \tag{3.27}$$

Now, we consider the auxiliary BSDE

$$\begin{cases} d\bar{Y}(s) = -g(s, \bar{X}(s), \mathbb{P}_s^{t,\mathbb{P}_\xi,u}, \bar{Y}(s), \mathbb{P}_{Y(s)}, \bar{Z}(s), u(s, \bar{X}(s), \mathbb{P}_s^{t,\mathbb{P}_\xi,u}))ds \\ \qquad\qquad +\bar{Z}(s)dW(s) - d\bar{M}(s), \\ \bar{Y}(T) = h(\bar{X}(T), \mathbb{P}_T^{t,\mathbb{P}_\xi,u}). \end{cases} \tag{3.28}$$

Since $\bar{X}(s)$ is $\mathcal{F}_s^W-$measurable, we have $\bar{M}(s) = \bar{M}(s; t, x, \mathbb{P}_\xi, u) = 0$. For any $\xi \in L^2_{\mathcal{F}_t}(\Omega; \mathbb{R}^n)$, there exists a sequence $\xi_n = \sum_{i=1}^{m_n} \eta_i 1_{E_i^n}$, where $\eta_i \in \mathcal{F}_t^W$ and $E_i^n \in \mathcal{G}$ which is independent of $W$. So, we have

$$\bar{M}_i(s) = \bar{M}(s; t, \eta_i, \mathbb{P}_\xi, u) = 0,$$

and

$$
\begin{aligned}
&\sum_{i=1}^{m_i} \bar{Y}_i(s) 1_{E_i^n} \\
= &\sum_{i=1}^{m_i} h(\bar{X}(T;\eta_i), \mathbb{P}_T^{t,\mathbb{P}_\xi,u}) 1_{E_i} - \sum_{i=1}^{m_i} \int_t^T \bar{Z}_i(s) dW(s) 1_{E_i^n} \\
&+ \sum_{i=1}^{m_i} \int_t^T g(s, \bar{X}(s;\eta_i), \mathbb{P}_s^{t,\mathbb{P}_\xi,u}, \bar{Y}_i(s), \mathbb{P}_{Y(s)}, \bar{Z}_i(s), u(s, \bar{X}(s;\eta_i), \mathbb{P}_s^{t,\mathbb{P}_\xi,u})) ds 1_{E_i^n} \\
= &h(\bar{X}(T; \sum_{i=1}^{m_i} \eta_i 1_{E_i^n}), \mathbb{P}_T^{t,\mathbb{P}_\xi,u}) - \int_t^T \sum_{i=1}^{m_i} \bar{Z}_i(s) 1_{E_i^n} dW(s) \\
&+ \int_t^T g(s, \bar{X}(s; \sum_{i=1}^{m_i} \eta_i 1_{E_i^n}), \mathbb{P}_s^{t,\mathbb{P}_\xi,u}, \sum_{i=1}^{m_i} \bar{Y}_i(s) 1_{E_i^n}, \mathbb{P}_{Y(s)}, \\
&\quad \sum_{i=1}^{m_i} \bar{Z}_i(s) 1_{E_i^n}, u(s, \bar{X}(s; \sum_{i=1}^{m_i} \eta_i 1_{E_i^n}), \mathbb{P}_s^{t,\mathbb{P}_\xi,u})) ds.
\end{aligned}
$$

Then, by (3.26) and (3.8), we can show that $(\bar{Y}(s;t,x,\xi,u)|_{x=\xi}, \bar{Z}(s;t,x,\mu,u)|_{x=\xi})$ solves the BSDE (3.27) in the sense that:

$$
\bar{Y}(s;t,x,\xi,u)|_{x=\xi} = \bar{Y}(s;t,\xi,\xi,u) = Y(s;t,\xi,\mu,u),
$$

$$
\bar{Z}(s;t,x,\xi,u)|_{x=\xi} = \bar{Z}(s;t,\xi,\xi,u) = Z(s;t,\xi,\mu,u),
$$

and

$$
M = 0.
$$

The proof of estimations (3.17)-(3.20) follows the idea of standard discussion for BSDE, while note that the proof of (3.20) needs the following result on $\mathbb{P}_{Y(s)}$.

□

**Proposition 3.2.6.** *Let* $Y(\cdot) = Y(\cdot;t,\xi,u)$ *and* $\bar{Y}(\cdot) = \bar{Y}(\cdot;t,x,\xi,u)$ *be the solutions to equations* (3.4) *and* (3.14) *respectively. Then, for any* $s \in [t,T]$*, the distribution* $\mathbb{P}_{Y(s)}$ *and the state* $\bar{Y}(s)$

*depend on $\xi$ only through its distribution $\mathbb{P}_\xi$. We adopt the notation $\mathbb{P}_{Y(s)} = \mathbb{P}^{t,\mu,u}_{Y,s}$ and $\bar{Y}(s) = \bar{Y}(s;t,x,\mu,u)$, where $\mu = \mathbb{P}_\xi$.*

*Proof.* Apply the discussion in Proposition 3.2.2 for $\bar{Y}$, we have for any $\eta, \eta' \in L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)$ with $\mathbb{P}_{\eta'} = \mathbb{P}_\eta$, it is true that $\bar{Y}(s;t,\eta',\xi,u)$ and $\bar{Y}(s;t,\eta,\xi,u)$ have the same distribution. Then, by (3.20), we have

$$\begin{aligned}
&\mathcal{W}_2^2(\mathbb{P}_{Y(s;t,\xi_1,u)}, \mathbb{P}_{Y(s;t,\xi_2,u)})\\
&= \mathcal{W}_2^2(\mathbb{P}_{\bar{Y}(s;t,\xi_1',\xi_1,u)}, \mathbb{P}_{\bar{Y}(s;t,\xi_2',\xi_2,u)})\\
&\leqslant \mathbb{E}[|\bar{Y}(s;t,\xi_1',\xi_1,u) - \bar{Y}(s;t,\xi_2',\xi_2,u)|^2]\\
&\leqslant K\mathbb{E}[|\xi_1' - \xi_2'|^2 + \mathcal{W}_2^2(\mu_1,\mu_2) + \int_s^T \mathcal{W}_2^2(\mathbb{P}_{Y(r;t,\xi_1,u)}, \mathbb{P}_{Y(r;t,\xi_2,u)}dr]
\end{aligned},$$

for all $\xi_i'$ with $\mathbb{P}_{\xi_i'} = \mathbb{P}_{\xi_i}$, $i = 1, 2$. By Burkholder-Davis-Gundy Inequality [27][54],

$$\mathcal{W}_2^2(\mathbb{P}_{Y(s;t,\xi_1,u)}, \mathbb{P}_{Y(s;t,\xi_2,u)}) \leqslant K(\mathcal{W}_2^2(\mathbb{P}_{\xi_1}, \mathbb{P}_{\xi_2}) + \int_t^s \mathcal{W}_2^2(\mathbb{P}_{Y(r;t,\xi_1,u)}, \mathbb{P}_{Y(r;t,\xi_2,u)})dr),$$

for all $s \in [t,T]$, where the constant $K$ depends only on $L$ and $s-t$. By Gronwall's inequality,

$$\mathcal{W}_2^2(\mathbb{P}_{Y(s;t,\xi_1,u)}, \mathbb{P}_{Y(s;t,\xi_2,u)}) \leqslant K\mathcal{W}_2^2(\mathbb{P}_{\xi_1}, \mathbb{P}_{\xi_2}),$$

the distribution $\mathbb{P}_{Y(s;t,\xi,u)}$ depends on $\xi$ only through its distribution $\mathbb{P}_\xi$. And we adopt the notation $\mathbb{P}_{T(s;t,\xi,u)} = \mathbb{P}^{t,\mathbb{P}_\xi,u}_{Y,s}$. Moreover,

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{s\in[t,T]} |\bar{Y}(s;t,x,\xi_1,u_1) - \bar{Y}(s;t,x,\xi_2,u_2)|^2\Big]\\
&\leqslant K\mathbb{E}[\mathcal{W}_2^2(\mathbb{P}_{\xi_1}, \mathbb{P}_{\xi_2}) + \int_t^T (\mathcal{W}_2^2\mathbb{P}_{Y(r;t,\xi_1,u)}, \mathbb{P}_{Y(r;t,\xi_2,u)})dr]\\
&\leqslant K\mathcal{W}_2^2(\mathbb{P}_{\xi_1}, \mathbb{P}_{\xi_2}).
\end{aligned}$$

So, $\bar{Y}(s;t,x,\xi,u)$ depends on $\xi$ only through $\mathbb{P}_\xi$ and we adopt the notation $\bar{Y}(s;t,x,\xi,u) = \bar{Y}(s;t,x,\mathbb{P}_\xi,u)$

□

**Proposition 3.2.7.** *(Flow Property) The solutions to* (3.27) *and* (3.28)*,* $Y(s;t,\xi,u)$ *and* $\bar{Y}(s;t,x,\mu,u)$ *satisfy the following property: for any* $\tau \in [t,T]$

$$Y(s) = \hat{Y}(s), \quad s \in [t,\tau], \tag{3.29}$$

*where* $(Y,Z)$ *is the adapted solution to* (3.4) *and* $(\hat{Y},\hat{Z})$ *is the adapted solution to*

$$\begin{cases} d\hat{Y}(s) = -g(s,X(s),\mathbb{P}_s^{t,\mathbb{P}_\xi,u},\hat{Y}(s),\mathbb{P}_{\hat{Y}(s)},\hat{Z}(s;\xi),u(s,X(s),\mathbb{P}_s^{t,\mathbb{P}_\xi,u}))ds \\ \qquad\qquad +\hat{Z}(s;\xi)dW(s), \\ \hat{Y}(\tau) = Y(\tau). \end{cases} \tag{3.30}$$

*And,*

$$\bar{Y}(s;t,x,\mu,u) = \tilde{Y}(s), \quad s \in [t,\tau]. \tag{3.31}$$

*where* $(\bar{Y},\bar{Z})$ *is the adapted solution to* (3.14) *and* $(\tilde{Y},\tilde{Z})$ *is the adapted solution to*

$$\begin{cases} d\tilde{Y}(s) = -g(s,\bar{X}(s),\mathbb{P}_s^{t,\mathbb{P}_\xi,u},\tilde{Y}(s),\mathbb{P}_{Y(s)},\tilde{Z}(s;\xi),u(s,\bar{X}(s),\mathbb{P}_s^{t,\mathbb{P}_\xi,u}))ds \\ \qquad\qquad +\tilde{Z}(s;\xi)dW(s), \\ \tilde{Y}(\tau) = \bar{Y}(\tau). \end{cases} \tag{3.32}$$

*Proof.* It can be proved by the uniqueness of the solution to equations (3.30) and (3.32). □

**Theorem 3.2.8.** *(Comparison Theorem) Let* $\zeta_i \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)$*,* $g_i(\cdot,0,\mathbb{P}_{\{0\}},0) \in L^2_{\mathbb{F}}(\Omega;C([t,T];\mathbb{R}^n))$*,* $i=1,2$*, satisfy:* $\zeta_1 \geqslant \zeta_2$*,* $g_1(s,y,\nu,z) \geqslant g_2(s,y,\nu,z)$*,* $d\mathbb{P}\times ds-$*a.s. Let* $(Y_i,Z_i) \in L^2_{\mathbb{F}}(\Omega;C([t,T];\mathbb{R}))\times$

$L^2_{\mathbb{F}}(\Omega; L^2([t,T];\mathbb{R}))$ *be the solution to the following BSDE:*

$$\begin{cases} dY_i(s) = -g_i(s, Y_i(s), \mathbb{P}_{Y_i(s)}, Z_i(s))ds + Z_i(s)dW(s), \\ Y_i(T) = \zeta_i, \end{cases} \tag{3.33}$$

$i = 1, 2$. *Furthermore,* $g_i : \Omega \times [0,T] \times \mathbb{R} \times \mathcal{P}^2(\mathbb{R}) \times \mathbb{R}^d \to \mathbb{R}$ *have bounded first-order partial derivatives with respect to* $(y, z)$ *and have bounded derivative with respect to* $\nu$*, that is, for all* $s \in [0,T]$

$$0 \leqslant \partial_\mu g_i(s, y, \nu, z)(\cdot) \leq C. \tag{3.34}$$

$\mathbb{P}-$*a.s., a.e. Then* $Y_1(t) \geqslant Y_2(t)$*,* $\mathbb{P}-$*a.s.*

*Proof.* See [20]. □

**Theorem 3.2.9.** *(Comparison Theorem) Let* $\zeta_i \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n)$*,* $g_i(\cdot, 0, \mathbb{P}_{\{0\}}, 0) \in L^2_{\mathbb{F}}(\Omega; C([t,T]; \mathbb{R}^n))$*,* $i = 1, 2$*, satisfy the same conditions as in* **Theorem 3.2.8**. *Let* $(\bar{Y}_i, \bar{Z}_i) \in L^2_{\mathbb{F}}(\Omega; C([t,T]; \mathbb{R})) \times L^2_{\mathbb{F}}(\Omega; L^2([t,T]; \mathbb{R}))$ *be the solution to the following BSDE:*

$$\begin{cases} d\bar{Y}_i(s) = -g_i(s, \bar{Y}_i(s), \mathbb{P}_{Y_i(s)}, \bar{Z}_i(s))ds + \bar{Z}_i(s)dW(s), \\ \bar{Y}_i(T) = \zeta_i, \end{cases} \tag{3.35}$$

*where* $Y_i$ *is given by* (3.33)*,* $i = 1, 2$*. Then* $\bar{Y}_1(t) \geqslant \bar{Y}_2(t)$*,* $\mathbb{P}-$*a.s.*

*Proof.* Let $\delta\bar{Y}(s)=\bar{Y}_1(s)-\bar{Y}_2(s)$, $\delta\bar{Z}(s)=\bar{Z}_1(s)-\bar{Z}_2(s)$, and $\delta\zeta=\zeta_1-\zeta_2$, then

$$\begin{aligned}\delta\bar{Y}(s)=\delta\zeta&+\int_s^T(g_1(r,\bar{Y}_1(r),\mathbb{P}_{Y_1(r)},\bar{Z}_1(r))-g_2(r,\bar{Y}_2(r),\mathbb{P}_{Y_2(r)},\bar{Z}_2(r)))dr\\&-\int_s^T\delta\bar{Z}(r)dW(r)\\=\delta\zeta&+\int_s^T(\bar{A}(r)\delta\bar{Y}(r)+\tilde{\mathbb{E}}[\bar{B}(r)\delta\tilde{Y}(r)]+\bar{C}(r)\delta\bar{Z}(r)+\phi(r))dr\\&-\int_s^T\delta\bar{Z}(r)dW(r),\end{aligned}$$

where

$$\bar{A}(r)=\int_0^1\partial_y g_1(r,\bar{Y}_2(r)+\lambda\delta\bar{Y}(r),\mathbb{P}_{Y_1(r)},\bar{Z}_1(r))d\lambda,$$

$$\bar{B}(r)=\int_0^1\partial_\mu g_1(r,\bar{Y}_2(r),\mathbb{P}_{Y_2(r)+\lambda\delta Y(r)},\bar{Z}_1(r))(\tilde{Y}_2(r)+\lambda\delta\tilde{Y}(r))d\lambda,$$

$$\bar{C}(r)=\int_0^1\partial_z g_1(r,\bar{Y}_2(r),\mathbb{P}_{Y_1(r)},\bar{Z}_2(r)+\lambda\delta\bar{Z}(r))d\lambda,$$

are all bounded and

$$\phi(r)=g_1(r,\bar{Y}_2(r),\mathbb{P}_{Y_2(r)},\bar{Z}_2(r))-g_2(r,\bar{Y}_2(r),\mathbb{P}_{Y_2(r)},\bar{Z}_2(r))\geqslant 0.$$

By the result of the last theorem, we have $\tilde{\mathbb{E}}[\bar{B}(r)\delta\tilde{Y}(r)]+\phi(r)\geqslant 0$. By the comparison for standard BSDE, we have the desired result. □

Now we introduce the following stochastic optimal control problem.

**Problem ($\mathbf{C_R}$).** For given $(t,\xi)\in\mathcal{D}$, find a $u^*\in\mathcal{U}$ such that

$$J(t,\xi;u^*)=\operatorname*{essinf}_{u(\cdot)\in\mathcal{U}}J(t,\xi;u)\equiv V(t,\xi). \tag{3.36}$$

Note that, by definition of essential infimum, $V(t, \xi)$ is a random variable such that,

$$J(t, \xi; u) \geq V(t, \xi), \ \mathbb{P} - a.s.,$$

and for any $\eta$ that satisfies the property above,

$$V(t, \xi) \geq \eta, \ \mathbb{P} - a.s..$$

Any $u^* \in \mathcal{U}$ satisfying (3.36) is called an *optimal strategy* of **Problem ($\mathbf{C_R}$)**, and $(t, \xi) \mapsto V(t, \xi)$ is called the *value function* of **Problem ($\mathbf{C_R}$)**. Note that $V$ is a function defined on $\mathcal{D}$, generally, it is not in a Markovian form, i.e., $V(t, \xi) \neq V(t, x)|_{x=\xi}$. Similarly, we introduce the auxiliary problem:

**Problem ($\mathbf{C_{au}}$).** For given $(t, x, \mu) \in [0, T] \times \mathbb{R}^n \times \mathcal{P}^2(\mathbb{R}^n)$, find a $u^* \in \mathcal{U}$ such that

$$\bar{J}(t, x, \mu; u^*) = \inf_{u \in \mathcal{U}} \bar{J}(t, x, \mu; u) \equiv \bar{V}(t, x, \mu), \tag{3.37}$$

where the auxiliary cost functional is

$$\bar{J}(t, x, \mu; u) = \bar{Y}(t; t, x, \mu, u), \tag{3.38}$$

$\bar{Y}$ is the solution to the auxiliary BSDE (3.14).

**Proposition 3.2.10.** *Suppose that there exists an optimal control $u^*$ such that*

$$\bar{J}(t, x, \mu; u^*) = \bar{V}(t, x, \mu), \tag{3.39}$$

*for all $x \in \mathbb{R}^n$, then*

$$V(t, \xi) = \bar{V}(t, x, \mu)|_{x=\xi}. \tag{3.40}$$

*Proof.* Since the adapted solution $(\bar{Y},\bar{Z})$ to the equation (3.14) is $\mathbb{F}^{\bar{X}}-$adapted, where $\mathcal{F}_s^{\bar{X}} = \sigma\{\bar{X}(r) : t \leqslant r \leqslant s\}$. So, $\bar{V} : [0,T]\times\mathbb{R}^n\times\mathcal{P}^2 \to \mathbb{R}$ is a deterministic function. By the definition of essential infimum, it's easy to show that

$$V(t,\xi) \geq \bar{V}(t,\xi,\mu).$$

On the other hand, by (3.39), we have

$$\bar{J}(t,\xi,\mu,u^*) = \bar{V}(t,\xi,\mu) = J(t,\xi,u^*) \geq V(t,\xi).$$

□

## 3.3 Dynamic Programming Principle

**Theorem 3.3.1.** *(Dynamic Programming Principle) For each $\tau \in [t,T]$, the value function $\bar{V}(t,x,\mu)$ satisfies the following equation:*

$$\bar{V}(t,x,\mu) \geqslant \inf_{u\in\mathcal{U}}\{\tilde{Y}(t;t,x,\mu,u)\}. \tag{3.41}$$

*where $\tilde{Y}$ is the solution of the following BSDE:*

$$\begin{aligned}\tilde{Y}(t;t,x,\mu,u) &= \bar{V}(\tau,\bar{X}(\tau;t,x,\mu,u),\mathbb{P}_\tau^{t,\mu,u}) - \int_t^\tau \tilde{Z}(r)dW(r)\\ &\quad + \int_t^\tau g(r,\bar{X}(r),\mathbb{P}_t^{t,\mu,u},\tilde{Y}(r),\mathbb{P}_{\hat{Y}(r)},\tilde{Z}(r),u(r,\bar{X}(r),\mathbb{P}_r^{t,\mu,u}))dr,\end{aligned} \tag{3.42}$$

*where $\hat{Y}$ is the solution to:*

$$\begin{aligned}\hat{Y}(t;t,\xi,\mu,u) &= \bar{V}(\tau, X(\tau;t,\xi,u), \mathbb{P}_{\tau}^{t,\mu,u})\} - \int_t^{\tau} \hat{Z}(r)dW(r) \\ &+ \int_t^{\tau} g(r, X(r), \mathbb{P}_t^{t,\mu,u}, \hat{Y}(r), \mathbb{P}_{\hat{Y}(r)}, \hat{Z}(r), u(r, X(r), \mathbb{P}_r^{t,\mu,u}))dr,\end{aligned} \tag{3.43}$$

*for $s \in [t,\tau]$, where $\tau \in [t,T]$. Suppose that for each $(t,\mu) \in [0,T] \times \mathcal{P}^2$, there exists an optimal control $u^*$ such that*

$$\bar{J}(t,x,\mu,u^*) = \bar{V}(t,x,\mu), \tag{3.44}$$

*for all $x \in \mathbb{R}^n$. Then,*

$$\bar{V}(t,x,\mu) \leqslant \inf_{u \in \mathcal{U}} \{\tilde{Y}(t;t,x,\mu,u)\}. \tag{3.45}$$

*In other words, the dynamic programming principle holds for* **Problem ($\mathbf{C_{au}}$).**

*Proof.* By definition, we have

$$\begin{aligned}
&\bar{Y}(t;t,x,\mu,u) \\
&= h(\bar{X}(T), \mathbb{P}_T^{t,\mu,u}) - \int_t^T \bar{Z}(r)dW(r) \\
&\quad + \int_t^T g(r, \bar{X}(r), \mathbb{P}_t^{t,\mu,u}, \bar{Y}(r), \mathbb{P}_{Y,r}^{t,\mu,u}, \bar{Z}(r), u(r, \bar{X}(r), \mathbb{P}_r^{t,\mu,u}))dr, \\
&= h(\bar{X}(T), \mathbb{P}_T^{t,\mu,u}) - \int_{\tau}^T \bar{Z}(r)dW(r)q \\
&\quad + \int_{\tau}^T g(r, \bar{X}(r), \mathbb{P}_r^{t,\mu,u}, \bar{Y}(r), \mathbb{P}_{Y,r}^{t,\mu,u}, \bar{Z}(r), u(r, \bar{X}(r), \mathbb{P}_r^{t,\mu,u}))dr, \\
&\quad - \int_t^{\tau} \bar{Z}(r)dW(r) + \int_t^{\tau} g(r, \bar{X}(r), \mathbb{P}_r^{t,\mu,u}, \bar{Y}(r), \mathbb{P}_{Y,r}^{t,\mu,u}, \bar{Z}(r), u(r, \bar{X}(r), \mathbb{P}_r^{r,\mu,u}))dr \\
&= \bar{Y}(\tau;\tau, \bar{X}(\tau;t,x,\mu,u_1), \mathbb{P}_{\tau}^{t,\mu,u_1}, u_2) - \int_t^{\tau} \bar{Z}(r)dW(r) \\
&\quad + \int_t^{\tau} g(r, \bar{X}(r), \mathbb{P}_t^{t,\mu,u_1}, \bar{Y}(r), \mathbb{P}_{Y,r}^{t,\mu,u_1}, \bar{Z}(r), u_1(r, \bar{X}(r), \mathbb{P}_r^{t,\mu,u_1}))dr,
\end{aligned}$$

where $u_1 = u|_{[t,\tau]}$, $u_2 = u|_{(\tau,T]}$. Since

$$\bar{V}(\tau, \bar{X}(\tau; t, x, \mu, u_1), \mathbb{P}_\tau^{t,\mu,u_1}) \leqslant \inf_{u_2 \in \mathcal{U}_{[\tau,T]}} \{\bar{Y}(\tau; \tau, \bar{X}(\tau; t, x, \mu, u_1), \mathbb{P}_\tau^{t,\mu,u_1}, u_2)\}, \tag{3.46}$$

and the comparison theorem 3.2.8, we have

$$\hat{Y}(s; t, \xi, u_1) \leqslant Y(s; t, \xi, u), \tag{3.47}$$

for $s \in [t, \tau]$. By applying the comparison theorem 3.2.9, we have

$$\tilde{Y}(t; t, x, \mu, u_1) \leqslant \bar{Y}(t; t, x, \mu, u_1), \tag{3.48}$$

for any $u_1 \in \mathcal{U}_{[t,\tau]}$. So,

$$\inf_{u_1 \in \mathcal{U}_{[t,\tau]}} \{\tilde{Y}(t; t, x, \mu, u_1)\} \leqslant \bar{V}(t, x, \mu).$$

On the other hand, by assumption, there exists a control $u_2^* \in \mathcal{U}_{[\tau,T]}$, such that

$$\bar{Y}(\tau; \tau, \bar{X}(\tau; t, x, \mu, u_1), \mathbb{P}_\tau^{t,\mu,u_1}, u_2^*) = \bar{V}(\tau, \bar{X}(\tau; t, x, \mu, u_1), \mathbb{P}_\tau^{t,\mu,u_1}),$$

then

$$Y(\tau; \tau, X(\tau; t, \xi, u_1), \mathbb{P}_\tau^{t,\mu,u_1}, u_2^*) = V(\tau, \bar{X}(\tau; t, \xi, u_1), \mathbb{P}_\tau^{t,\mu,u_1}).$$

So,

$$\bar{Y}(t; t, x, \mu, u) = \tilde{Y}(t; t, x, \mu, u_1) \geqslant \bar{V}(t, x, \mu),$$

for all $u_1 \in \mathcal{U}_{[t,T]}$, and

$$\bar{V}(t, x, \mu) \leqslant \inf_{u \in \mathcal{U}_{[t,\tau]}} \tilde{Y}(t; t, x, \mu, u_1). \tag{3.49}$$

□

## 3.4 HJB Equation

In this section, we introduce the following PDE:

$$\begin{aligned}\bar{V}_s(s,x,\mu)+\inf_{u\in\mathcal{U}}\{\int_{\mathbb{R}^n}&H(s,x,\bar{x},\mu,\bar{V}(s,x,\mu),\mathbb{P}_{\bar{V}(s,\xi,\mu)},\bar{V}_x(s,x,\mu),\partial_\mu\bar{V}(s,x,\mu)(\bar{x}),\\&\bar{V}_{xx}(s,x,\mu),\partial_w\partial_\mu\bar{V}(s,x,\mu)(\bar{x}),u,u')\mu(d\bar{x})\}=0,\end{aligned}\tag{3.50}$$

for $s\in[t,T]$, with terminal condition $\bar{V}(T,x,\mu)=h(x,\mu)$, where

$$H:[0,T]\times\mathbb{R}^n\times\mathbb{R}^n\times\mathcal{P}^2\times\mathbb{R}\times\mathcal{P}^2(\mathbb{R})\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^{n\times n}\times\mathbb{R}^{n\times n}\times U\times U\to\mathbb{R}$$

is defined by letting

$$\begin{aligned}&H(s,x,\bar{x},\mu,y,\nu,p,\bar{p},A,\bar{A},u,u')\\=&\,p\cdot b(s,x,\mu,u)+\bar{p}\cdot b(s,\bar{x},\mu,u')+\frac{1}{2}\text{tr}\left[A\cdot\sigma\sigma^T(s,x,\mu,u)+\bar{A}\cdot\sigma\sigma^T(s,\bar{x},\mu,u')\right]\\&+g(s,x,\mu,y,\nu,p\cdot\sigma(s,x,\mu,u),u).\end{aligned}\tag{3.51}$$

Define the map

$$\mathcal{H}:[0,T]\times\mathbb{R}^n\times\mathcal{P}^2\times\mathbb{R}\times\mathcal{P}^2(\mathbb{R})\times\mathbb{R}^n\times C^1(\mathbb{R}^n)\times\mathbb{R}^{n\times n}\times C(\mathbb{R}^n;\mathbb{R}^n)\times\mathcal{U}\to\mathbb{R}$$

by letting:

$$\begin{aligned}&\mathcal{H}(s,x,\mu,y,\nu,p,\bar{p}(\cdot),A,\bar{A}(\cdot),u)\\=&\int_{\mathbb{R}^n}H(s,x,\bar{x},\mu,y,\nu,p,\bar{p}(\bar{x}),A,\bar{A}(\bar{x}),u(s,x,\mu),u(s,\bar{x},\mu))\mu(d\bar{x}).\end{aligned}\tag{3.52}$$

Let

$$\mathbb{H}(s,x,\mu,y,\nu,p,\bar{p}(\cdot),A,\bar{A}(\cdot)) = \inf_{u\in\mathcal{U}}\{\mathcal{H}(s,x,\mu,y,\nu,p,\bar{p}(\cdot),A,\bar{A}(\cdot),u)\},$$

then (3.50) could be rewritten as

$$\begin{aligned}\bar{V}_s(s,x,\mu)+\mathbb{H}(s,x,\mu,\bar{V}(s,x,\mu),\mathbb{P}_{\bar{V}(s,\xi,\mu)},\bar{V}_x(s,x,\mu),\partial_\mu\bar{V}(s,x,\mu)(\cdot),\\ +\bar{V}_{xx}(s,x,\mu),\partial_w\partial_\mu\bar{V}(s,x,\mu)(\cdot))=0.\end{aligned} \tag{3.53}$$

### *3.4.1 Verification Theorem*

**Theorem 3.4.1.** *(Verification Theorem) Suppose the equation* (3.50) *has a classical solution $\psi$ and there exists $u^*\in\mathcal{U}$ such that*

$$\begin{aligned}&\mathcal{H}(s,x,\mu,\psi(s,x,\mu),\mathbb{P}_{\psi_s(s,\xi,\mu)},\psi_x(s,x,\mu),\partial_\mu\psi(s,x,\mu)(\cdot),\psi_{xx}(s,x,\mu),\\ &\qquad\partial_w\partial_\mu\psi(s,x,\mu)(\cdot),u^*)\\ &=\mathbb{H}(s,x,\mu,\psi(s,x,\mu),\mathbb{P}_{\psi_s(s,\xi,\mu)},\psi_x(s,x,\mu),\partial_\mu\psi(s,x,\mu)(\cdot),\psi_{xx}(s,x,\mu),\\ &\qquad\partial_w\partial_\mu\psi(s,x,\mu)(\cdot)),\end{aligned} \tag{3.54}$$

*for all $(s,x,\mu)\in[0,T]\times\mathbb{R}^n\times\mathcal{P}^2$. Then $\psi$ is the value function of the problem, i.e., $\psi=\bar{V}$. Furthermore, such defined $u^*$ is an optimal strategy.*

*Proof.* Let $X^*(s)=X(s;t,\xi,u^*)$ and $\bar{X}^*(s)=\bar{X}(s;t,x,\mu,u^*)$. By applying Itô's formula (see

[10]) to the process $\psi(s, \bar{X}(s), \mathbb{P}_{X(s)})$, we have

$$
\begin{aligned}
\psi(s, \bar{X}^*(s), \mathbb{P}_s^{t,\mu,u^*}) &= h(\bar{X}^*(T), \mathbb{P}_T^{t,\mu,u^*}) - \int_s^T \{\partial_r \psi(r, \bar{X}^*(r), \mathbb{P}_r^{t,\mu,u^*}) \\
&+\partial_x \psi(r, \bar{X}^*(r), \mathbb{P}_r^{t,\mu,u^*}) b(r, \bar{X}^*(r), \mathbb{P}_r^{t,\mu,u^*}, u^*(r, \bar{X}^*(r), \mathbb{P}_r^{t,\mu,u^*})) \\
&+\frac{1}{2}\text{tr}\left[\partial_{xx}\psi(r, \bar{X}^*(r), \mathbb{P}_r^{t,\mu,u^*})\sigma\sigma^T(r, \bar{X}^*(r), \mathbb{P}_r^{t,\mu,u^*}, u^*(r, \bar{X}^*(r), \mathbb{P}_r^{t,\mu,u^*}))\right] \\
&+\tilde{\mathbb{E}}\Big[\partial_\mu \psi(r, \bar{X}^*(r), \mathbb{P}_r^{t,\mu,u^*})(\tilde{X}^*(r; t, \tilde{\xi}, u^*)) b(r, \tilde{X}^*(r; t, \tilde{\xi}, u^*), \mathbb{P}_r^{t,\mu,u^*}, \\
&\quad u^*(r, \tilde{X}^*(r; t, \tilde{\xi}, u^*), \mathbb{P}_r^{t,\mu,u^*})) + \frac{1}{2}\text{tr}\left[\partial_\omega \partial_\mu \psi(r, \bar{X}^*(r), \mathbb{P}_r^{t,\mu,u^*})(\tilde{X}^*(r; t, \tilde{\xi}, u^*))\cdot\right. \\
&\quad \left.\sigma\sigma^T(r, \tilde{X}^*(r; t, \tilde{\xi}, u^*), \mathbb{P}_r^{t,\mu,u^*}, u^*(r, \tilde{X}^*(r; t, \tilde{\xi}, u^*), \mathbb{P}_r^{t,\mu,u^*}))\right]\Big]\}.
\end{aligned}
$$

Since $\psi$ solves (3.50), we have

$$
\psi(t, x, \mu) = \bar{Y}(t; t, x, \mu, u^*) \geqslant \bar{V}(t, x, \mu).
$$

On the other hand, let $u \in \mathcal{U}$. There exists $f : [0, T] \times \mathbb{R}^n \times \mathcal{P}^2 \to [0, \infty)$, such that

$$
\begin{aligned}
&\mathcal{H}(s, x, \mu, \psi(s, x, \mu), \mathbb{P}_{\psi_s(s,\xi,\mu)}, \psi_x(s, x, \mu), \partial_\mu \psi(s, x, \mu)(\cdot), \psi_{xx}(s, x, \mu), \\
&\partial_w \partial_\mu \psi(s, x, \mu)(\cdot), u) + \psi_s(s, x, \mu) - f(s, x, \mu) = 0.
\end{aligned}
$$

Combine with Ito's formula, we have

$$
\psi(t, x, \mu) = \hat{\hat{Y}}(t; t, x, \mu, u),
$$

where $\hat{\hat{Y}}(t; t, x, \mu, u)$ is the solution to the following BSDE

$$
\begin{aligned}
\hat{\hat{Y}}(t; t, x, \mu, u) &= h(\bar{X}(T; t, x, \mu, u), \mathbb{P}_T^{t,\mu,u}) - \int_t^T \hat{\hat{Z}}(r) dW(r) \\
&+ \int_t^T g(r, \bar{X}(r), \mathbb{P}_r^{t,\mu,u}, \hat{\hat{Y}}(r), \mathbb{P}_{\hat{Y}(r)}, \hat{\hat{Z}}(r), u(r, \bar{X}(r), \mathbb{P}_r^{t,\mu,u})) + f(r, \bar{X}(r), \mathbb{P}_r^{t,\mu,u}) dr,
\end{aligned}
\tag{3.55}
$$

and $\hat{Y}$ is the solution to the BSDE:

$$\begin{aligned}\hat{Y}(t;t,x,\mu,u) &= h(X(T;t,\xi,u),\mathbb{P}_T^{t,\mu,u}) - \int_t^T \hat{Z}(r)dW(r) \\ &+ \int_t^T g(r,X(r),\mathbb{P}_r^{t,\mu,u},\hat{Y}(r),\mathbb{P}_{\hat{Y}(r)},\hat{Z}(r),u(r,X(r),\mathbb{P}_r^{t,\mu,u})) + f(r,X(r),\mathbb{P}_r^{t,\mu,u})dr,\end{aligned} \tag{3.56}$$

By comparison theorem, we have

$$\psi(t,x,\mu) \leq \bar{Y}(t;t,x,\mu,u),$$

for all $u \in \mathcal{U}$. Then we have the desired result. □

**Remark 3.4.2.** *Note that the HJB equation related to the classical recursive optimal control problem is a special case of* (3.53). *That is, let* $b(s,x,\mu,u) = b(s,x,u)$, $\sigma(s,x,\mu,u) = \sigma(s,x,u)$, $g(s,x,\mu,y,\nu,z,u) = g(s,x,y,z,u)$ *and* $h(x,\mu) = h(x)$. *Then,* (3.53) *becomes* (1.12).

*Also, the problem stated in Chapter 2 can also be covered here. Let* $g(s,x,\mu,y,\nu,z,u) = g(s,x,\mu,u)$ *and* $V_F(t,\mu)$ *denote the value function in Chapter 2, then*

$$V_F(t,\mu) = \mathbb{E}\Big[\bar{V}(t,\xi,\mu)\Big].$$

*By definition,*

$$\partial_\mu V_F(t,\mu)(x) = \bar{V}_x(t,x,\mu) + \partial_\mu \bar{V}(t,\xi,\mu)(x),$$

*and*

$$\partial_\omega \partial_\mu V_F(t,\mu)(x) = \bar{V}_{xx}(t,x,\mu) + \partial_\omega \partial_\mu \bar{V}(t,\xi,\mu)(x).$$

*Let* $x = \xi$ *in* (3.53) *and apply* $\mathbb{E}$ *to both sides, we can recover the HJB equation* (2.28).

### *3.4.2 Necessary Condition*

**Theorem 3.4.3.** *(Necessary Condition for Optimal Strategy) Suppose $\bar{V} \in C^{1,2,2}$ and $\hat{u} \in \mathcal{U}$ is an optimal strategy, then it solves the following equation:*

$$F(x, \hat{u}(x)) + \int_{\mathbb{R}^n} G(\bar{x}, x, \hat{u}(x))\mu(d\bar{x}) = 0, \tag{3.57}$$

*where*

$$\begin{aligned} F(x, \hat{u}(x)) = &\bar{V}_x(s,x,\mu) \cdot b_u(s,x,\mu,\hat{u}(s,x,\mu)) + \bar{V}_{xx}(s,x,\mu) \cdot \sigma\sigma_u(s,x,\mu,\hat{u}(s,x,\mu)) \\ &+ g_z(s,x,\mu,\bar{V}(s,x,\mu),\mathbb{P}_{\bar{V}(s,\xi,\mu)},\bar{V}_x(s,x,\mu)\cdot\sigma(s,x,\mu,\hat{u}(s,x,\mu)), \\ &\quad \hat{u}(s,x,\mu)) \cdot \bar{V}_x(s,x,\mu) \cdot \sigma_u(s,x,\mu,\hat{u}(s,x,\mu)) \\ &+ g_u(s,x,\mu,\bar{V}(s,x,\mu),\mathbb{P}_{\bar{V}(s,\xi,\mu)},\bar{V}_x(s,x,\mu)\cdot\sigma(s,x,\mu,\hat{u}(s,x,\mu)),\hat{u}(s,x,\mu)), \end{aligned} \tag{3.58}$$

*and*

$$\begin{aligned} G(\bar{x}, x, \hat{u}(x)) = &\partial_\mu \bar{V}(s,\bar{x},\mu)(x) \cdot b_u(s,x,\mu,\hat{u}(s,x,\mu)) \\ &+ \partial_w \partial_\mu \bar{V}(s,\bar{x},\mu)(x) \cdot \sigma\sigma_u(s,x,\mu,\hat{u}(s,x,\mu)). \end{aligned} \tag{3.59}$$

*Proof.* Suppose that $\hat{u} \in \mathcal{U}$ is an optimal strategy, then $\bar{Y}(t;t,x,\mu,\hat{u}) = \bar{V}(t,x,\mu)$. Since $\bar{V} \in C^{1,2,2}$, we have

$$\begin{aligned} &\bar{V}_s(s,x,\mu) + \bar{V}_x(s,x,\mu)b(s,x,\mu,\hat{u}(s,x,\mu)) + \mathbb{E}\Big[\partial_\mu \bar{V}(s,x,\mu)(\xi)b(s,\xi,\mu,\hat{u}(s,\xi,\mu))\Big] \\ &+\frac{1}{2}tr\Big[\bar{V}_{xx}(s,x,\mu)\sigma\sigma^T(s,x,\mu,\hat{u}(s,x,\mu)) + \mathbb{E}\Big[\partial_w\partial_\mu \bar{V}(s,x,\mu)(\xi)\sigma\sigma^T(s,\xi,\mu,\hat{u}(s,\xi,\mu))\Big]\Big] \\ &+g(s,x,\mu,\bar{V}(s,x,\mu),\mathbb{P}_{\bar{V}(t,\xi,\mu)},\bar{V}_x(s,x,\mu)\sigma(s,x,\mu,\hat{u}(s,x,\mu)),\hat{u}(s,x,\mu)) = 0, \end{aligned} \tag{3.60}$$

$\bar{V}$ is the classical solution to (3.53) and $\hat{u}$ is a minimizer of

$$\int_{\mathbb{R}^n} H(s, x, \bar{x}, \mu, \bar{V}(s, x, \mu), \mathbb{P}_{\bar{V}(s,\xi,\mu)}, \bar{V}_x(s, x, \mu), \partial_\mu \bar{V}(s, x, \mu)(\bar{x}), \bar{V}_{xx}(s, x, \mu),$$
$$\partial_w \partial_\mu \bar{V}(s, x, \mu)(\bar{x}), \bar{V}_x(s, x, \mu) \cdot \sigma(s, x, \mu, u), u)\mu(d\bar{x}).$$

The first order variational condition gives:

$$F(x, \hat{u}(x)) \cdot \delta(x) + \int_{\mathbb{R}^n} G(x, \bar{x}, \hat{u}(\bar{x})) \cdot \delta(\bar{x})\mu(d\bar{x}) = 0, \tag{3.61}$$

for all $\delta \in \mathcal{U}$. Take integral with respect to $x$ on both sides, we have

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (F(\bar{x}, \hat{u}(\bar{x})) + G(x, \bar{x}, \hat{u}(\bar{x}))) \cdot \delta(\bar{x})\mu(d\bar{x})\mu(dx) = 0, \tag{3.62}$$

or,

$$\int_{\mathbb{R}^n} [F(\bar{x}, \hat{u}(\bar{x})) + \int_{\mathbb{R}^n} G(x, \bar{x}, \hat{u}(\bar{x}))\mu(dx)] \cdot \delta(\bar{x})\mu(d\bar{x}) = 0, \tag{3.63}$$

for all $\delta \in \mathcal{U}$. We have (3.57).

□

**Remark 3.4.4.** *For the classical recursive stochastic optimal control problem, we have $G \equiv 0$ and* (3.57) *becomes $F(x, \hat{u}(x)) = 0$, which is the first order variational condition for the minimizer of Hamiltonian.*

*For the mean field optimal control problem in Chapter 2, we have $F \equiv 0$ and* (3.57) *becomes $\int_{\mathbb{R}^n} G(\bar{x}, x, \hat{u}(x))\mu(d\bar{x}) = 0$. This also corresponds to the first order necessary condition for the minimizer of the Hamiltonian there.*

### *3.4.3 Viscosity Solution*

If an HJB equation has classical solution then this solution must be the value function. But, some examples tell us that value function may not be differentiable, thus it cannot be the classical solution of the HJB equation. Also, the HJB equation may have no classical solution in generaly. To overcome this difficulty, the notion of viscosity solution (due to Crandall–Lions 1980s [33]) is introduced. Also see [13][53][11]. Now, we introduce the following definition.

**Definition 3.4.5.** *(Viscosity Solution) A continuous function $\psi : [0,T] \times \mathbb{R}^n \times \mathcal{P}^2 \to \mathbb{R}$ is called a viscosity supersolution of equation* (3.53) *if,*

$$\psi(t,x,\mu) \geqslant h(x,\mu),$$

*and, for any test function $\varphi \in C^{1,2,2}([0,T] \times \mathbb{R}^n \times \mathcal{P}^2)$, whenever $\psi - \varphi$ attains a local minimum at $(t,x,\mu) \in [0,T] \times \mathbb{R}^n \times \mathcal{P}^2$, we have*

$$\begin{aligned} \varphi_t(t,x,\mu) + \mathbb{H}(t,x,\mu,\varphi(t,x,\mu),\mathbb{P}_{\varphi(t,\xi,\mu)},\varphi_x(t,x,\mu),\partial_\mu\varphi(t,x,\mu)(\cdot),\varphi_{xx}(t,x,\mu), \\ \partial_w\partial_\mu\varphi(t,\mu)(\cdot)) \leqslant 0. \end{aligned} \tag{3.64}$$

*It is called a viscosity subsolution if in* (3.64) *the inequality "$\leqslant$" is replaced by "$\geqslant$" and "local minimum" is replaced by "local maximum".*

We first talk about the continuity of the value function $\bar{V}$.

**Lemma 3.4.6.** *The value function $\bar{V}_L(t,x,\mu) \equiv \inf_{u \in \mathcal{U}_L} \bar{J}(t,x,\mu)$ is continuous. Also,*

$$\lim_{L\to\infty} \bar{V}_L(t,x,\mu) = \bar{V}(t,x,\mu).$$

*Proof.* The discussion is similar to Proposition 2.3.6. □

**Theorem 3.4.7.** *The value function $\bar{V}_L$ is a viscosity solution to* (3.53)*, for which the infimum is taken for $u \in \mathcal{U}_L$.*

*Proof.* Let $\phi \in C^{1,2,2}([0,T] \times \mathbb{R}^n \times \mathcal{P}^2(\mathbb{R}^n);\mathbb{R})$. Define the function $F : [0,T] \times \mathbb{R}^n \times \mathcal{P}^2(\mathbb{R}^n) \times \mathbb{R} \times \mathcal{P}^2(\mathbb{R}) \times \mathbb{R}^d \times \mathcal{U} \to \mathbb{R}$ by letting

$$\begin{aligned}
&F(s,x,\mu,y,\nu,z,u) \\
&= \phi_s(s,x,\mu) + \phi_x(s,x,\mu)b(s,x,\mu,u) + \mathbb{E}[\partial_\mu \phi(s,x,\mu)(\xi)b(s,\xi,\mu,u)] \\
&\quad + \frac{1}{2}tr[\phi_{xx}(s,x,\mu)\sigma\sigma^T(s,x,\mu,u)] + \frac{1}{2}\mathbb{E}[\partial_w \partial_\mu \phi(s,x,\mu)(\xi)\sigma\sigma^T(s,\xi,\mu,u)] \\
&\quad + g(s,x,\mu,y+\phi(s,x,\mu),\nu,z+\phi_x(s,x,\mu)\sigma(s,x,\mu,u),u) \\
&= \phi_s(s,x,\mu) + \mathcal{H}(s,x,\mu,y+\phi(s,x,\mu),\nu,\phi_x(s,x,\mu),\partial_\mu\phi(s,x,\mu)(\cdot), \\
&\quad \phi_{xx}(s,x,\mu),\partial_w\partial_\mu\phi(s,x,\mu)(\cdot),u).
\end{aligned} \tag{3.65}$$

Let $\delta > 0$.

Step 1: Let's consider following BSDE:

$$\begin{cases}
d\bar{Y}_1(s) = -F(s,\bar{X}(s),\mathbb{P}_{X(s)},\bar{Y}_1(s),\mathbb{P}_{Y_1(s)+\phi(s,X(s),\mathbb{P}_{X(s)})},\bar{Z}_1(s), \\
\qquad\qquad u(s,\bar{X}(s),\mathbb{P}_{X(s)}))ds + \bar{Z}_1(s)dW(s), \\
\bar{Y}_1(t+\delta) = 0,
\end{cases} \tag{3.66}$$

where $s \in [t,t+\delta]$, $Y_1$ is defined by the following BSDE:

$$\begin{cases}
dY_1(s) = -F(s,X(s),\mathbb{P}_{X(s)},Y_1(s),\mathbb{P}_{Y_1(s)+\phi(s,X(s),\mathbb{P}_{X(s)})},Z_1(s), \\
\qquad\qquad u(s,X(s),\mathbb{P}_{X(s)}))ds + Z_1(s)dW(s), \\
Y_1(t+\delta) = 0.
\end{cases} \tag{3.67}$$

Then $Y_1(s) = \tilde{Y}_1(s) - \phi(s, X(s), \mathbb{P}_{X(s)})$, where $\tilde{Y}_1$ is given by the BSDE:

$$\begin{cases} d\tilde{Y}_1(s) = -g(s, X(s), \mathbb{P}_{X(s)}, \tilde{Y}_1(s), \mathbb{P}_{\tilde{Y}_1(s)}, \tilde{Z}_1(s), u(s, X(s), \mathbb{P}_{X(s)}))ds \\ \qquad\qquad +\tilde{Z}_1(s)dW(s), \\ \tilde{Y}_1(t+\delta) = \phi(t+\delta, X(t+\delta), \mathbb{P}_{X(t+\delta)}). \end{cases} \tag{3.68}$$

And $\bar{Y}_1(s) = \tilde{\bar{Y}}_1(s) - \phi(s, \bar{X}(s), \mathbb{P}_{X(s)})$, where $\tilde{\bar{Y}}_1$ is given by the BSDE:

$$\begin{cases} d\tilde{\bar{Y}}_1(s) = -g(s, \bar{X}(s), \mathbb{P}_{X(s)}, \tilde{\bar{Y}}_1(s), \mathbb{P}_{\tilde{Y}_1(s)}, \tilde{\bar{Z}}_1(s), u(s, \bar{X}(s), \mathbb{P}_{X(s)}))ds \\ \qquad\qquad +\tilde{\bar{Z}}_1(s)dW(s), \\ \tilde{\bar{Y}}_1(T) = \phi(t+\delta, \bar{X}(t+\delta), \mathbb{P}_{X(t+\delta)}). \end{cases} \tag{3.69}$$

The above result can be proved by applying the Ito formula to $\phi(s, \bar{X}(s), \mathbb{P}_{X(s)})$ and $\phi(s, X(s), \mathbb{P}_{X(s)})$ and the uniqueness of the solution to mean field BSDEs.

Step 2: We show

$$|\bar{Y}_1(t) - \bar{Y}_2(t)| \leq C\delta\rho(\delta), \tag{3.70}$$

$$\mathbb{E}[|Y_1(t) - Y_2(t)|] \leq C\delta\rho(\delta), \tag{3.71}$$

where $\bar{Y}_2$ is the solution to the following BSDE:

$$\begin{cases} d\bar{Y}_2(s) = -F(s, x, \mu, \bar{Y}_2(s), \mathbb{P}_{Y_2(s)+\phi(s,\xi,\mu)}, \bar{Z}_2(s), u(s, x, \mu))ds \\ \qquad\qquad +\bar{Z}_2(s)dW(s), \\ \bar{Y}_2(t+\delta) = 0, \end{cases} \tag{3.72}$$

where $s \in [t, t+\delta]$, and $Y_2$ is defined by the following BSDE:

$$\begin{cases} dY_2(s) = -F(s, \xi, \mu, Y_2(s), \mathbb{P}_{Y_2(s)+\phi(s,\xi,\mu)}, Z_2(s), u(s, \xi, \mu))ds \\ \qquad\qquad +Z_2(s)dW(s), \\ Y_2(t+\delta) = 0, \end{cases} \tag{3.73}$$

and

*Proof.* (for step 2) By Lemma 3.2.5, we have

$$\begin{aligned}
&\mathbb{E}[|Y_1(t) - Y_2(t)|^2 + \int_t^{t+\delta} |Z_1(r) - Z_2(r)|^2 dr] \\
\leqslant\ & C\mathbb{E}\Big[(\int_t^{t+\delta} \phi_s(r, X(r), \mathbb{P}_{X(r)}) - \phi_s(r, \xi, \mu) \\
&\quad +\phi_x(r, X(r), \mathbb{P}_{X(r)})b(r, X(r), \mathbb{P}_{X(r)}, u(r, X(r), \mathbb{P}_{X(r)})) \\
&\quad -\phi_x(r, \xi, \mu)b(r, \xi, \mu, u(r, \xi, \mu)) \\
&\quad +\tilde{\mathbb{E}}[\partial_\mu\phi(r, X(r), \mathbb{P}_{X(r)})(\tilde{X}(r))b(r, \tilde{X}(r), \mathbb{P}_{X(r)}, u(r, \tilde{X}(r), \mathbb{P}_{X(r)})) \\
&\quad -\partial_\mu\phi(r, \xi, \mu)(\tilde{\xi})b(r, \tilde{\xi}, \mu, u(r, \tilde{\xi}, \mu))] \\
&\quad +\frac{1}{2}\text{tr}\,[\phi_{xx}(r, X(r), \mathbb{P}_{X(r)})\sigma\sigma^T(r, X(r), \mathbb{P}_{X(r)}, u(r, X(r), \mathbb{P}_{X(r)})) \\
&\qquad\qquad -\phi_{xx}(r, \xi, \mu)\sigma\sigma^T(r, \xi, \mu, u(r, \xi, \mu))] \\
&\quad +\frac{1}{2}\tilde{\mathbb{E}}\Big[\text{tr}\,[\partial_w\partial_\mu\phi(r, X(r), \mathbb{P}_{X(r)})(\tilde{X}(r))\sigma\sigma^T(r, \tilde{X}(r), \mathbb{P}_{X(r)}) \\
&\qquad\qquad -\partial_w\partial_\mu\phi(r, \xi, \mu)(\tilde{\xi})\sigma\sigma^T(r, \tilde{\xi}, \mu)]\Big] \\
&\quad +g(r, X(r), \mathbb{P}_{X(r)}, Y_1(r) + \phi(r, X(r), \mathbb{P}_{X(r)}), \mathbb{P}_{Y_1(r)}, \\
&\quad Z_1(r) + \phi_x(r, X(r), \mathbb{P}_{X(r)})\sigma(r, X(r), \mathbb{P}_{X(r)}), u(r, X(r), \mathbb{P}_{X(r)}))- \\
&\quad g(r, \xi, \mu, Y_2(r) + \phi(r, \xi, \mu, \mathbb{P}_{Y_2(r)}, Z_2(r) + \phi_x(r, \xi, \mu)\sigma(r, \xi, \mu, u(r, \xi, \mu)))))^2\Big] \\
\leqslant\ & C\mathbb{E}\Big[(\int_t^{t+\delta} L(r)|X(r) - \xi|dr)^2\Big],
\end{aligned}$$

for some $L(r)$ with $\int_t^{t+\delta}|L(r)|^2dr < \infty$. Since $\mathbb{E}\Big[|X(r)-\xi|^2\Big] \to 0$, as $r \to t$, we have the desired result (3.71). The proof for (3.70) follows a similar discussion.

□

Step 3: Now we prove that $\bar{Y}_0(t) = \inf\limits_{u\in\mathcal{U}}\{\bar{Y}_2(t)\}$, where

$$\begin{cases} d\bar{Y}_0(s) = -F_0(s, x, \mu, \bar{Y}_0(s), \mathbb{P}_{Y_0(s)+\phi(s,\xi,\mu)}, 0)ds, \\ \bar{Y}_0(t+\delta) = 0, \end{cases} \tag{3.74}$$

where $F_0(s, x, \mu, y, \nu, z) = \inf\limits_{u\in\mathcal{U}}\{F(s, x, \mu, y, \nu, z, u)\}$ and $Y_0(s)$ is the solution to the following BSDE:

$$\begin{cases} dY_0(s) = -F_0(s, \xi, \mu, Y_0(s), \mathbb{P}_{Y_0(s)+\phi(s,\xi,\mu)}, 0)ds, \\ Y_0(t+\delta) = 0. \end{cases} \tag{3.75}$$

By the definition of $F_0$ and comparison theorem, we have $\bar{Y}_0(t) \leqslant \inf_{u\in\mathcal{U}}\{\bar{Y}_2(t)\}$. On the other hand, let $u^* \in \mathcal{U}$ such that $F(s, x, \mu, y, \nu, z, u^*) = F_0(s, x, \mu, y, \nu, z)$, then we have

$$\bar{Y}_0(t) = \bar{Y}_2(t; u^*) \geqslant \inf_{u\in\mathcal{U}}\{\bar{Y}_2(t)\}.$$

Step 4: Let $(t, x, \mu)$ be a minimum point of $\bar{V} - \phi$, without loss of generality, suppose that $\bar{V}(s', x', \mu') \geqslant \phi(s', x', \mu')$, for any $(s', x', \mu') \in [0, T] \times \mathbb{R}^n \times \mathcal{P}^2$. By the dynamic programming principle and the comparison theorem, we have

$$\phi(t, x, \mu) \geqslant \inf_{u\in\mathcal{U}}\{\tilde{\bar{Y}}_1(t)\},$$

where $\tilde{\bar{Y}}_1(t)$ is decided by the BSDE (3.68). Since $\bar{Y}_1(t) = \tilde{\bar{Y}}_1(t) - \phi(t, \bar{X}(t), \mathbb{P}_{X(t)})$, we have

$\inf\limits_{u \in \mathcal{U}} \{\bar{Y}_1(t)\} \leqslant 0$. By (3.70), we have

$$\bar{Y}_0(t) = \inf_{u \in \mathcal{U}} \{\bar{Y}_2(t)\} \leqslant \delta\rho(\delta).$$

By letting $\delta \to 0$, we have

$$F_0(t, x, \mu, 0, \mathbb{P}_{\phi(t,\xi,\mu)}, 0) = \inf_{u \in \mathcal{U}} \{F(t, x, \mu, 0, \mathbb{P}_{\phi(t,\xi,\mu)}, 0, u)\} \leqslant 0.$$

So, $\bar{V}$ is a viscosity supersolution of the equation (3.50). By applying a similar discussion we can prove the other direction. □

# CHAPTER 4: FUTURE RESEARCH

## 4.1 Mean field Problems

Mean-field models can be used to describe many group activities in economy and psychology. I do have strong interests in building, solving and comparing mean-filed models that describe processes with mean-field interactions for a problem in reality. There are several works on this topic, while many other interesting topic to be considered.

## 4.2 Time-Inconsistency

The description for time-inconsistency is not completely clear, especially about risk preference. That is to find an accurate and applicable way to describe how the change of people's attitude towards risk and what is the equilibrium control related.

Another interesting question related is if equilibrium control is not unique, how do we find and characterize the optimal one among them? This natural and non-trivial problem is in my research plan.

One problem which can be studied is the time-inconsistent problem with conditional distribution and recursive cost functional. The difficulty lies is about the solution condition for the equilibrium HJB equation.

## 4.3 HJB Equation

There are many interesting technical problems concerning the HJB equations. For example, the classical solution to the equation (3.50). A similar result can be found in [12], for example. While, the PDE (3.50) is in a new form whose well-posedness problem is interesting, important and challenging. Especially, it will be interesting to consider the problem from a pure PDE point of view, without the help of the related stochastic optimal control problem. People have mentioned some seemingly potential ways several years ago, and more work needs to be done on it.

## 4.4 Deep Learning

Another topic that is attrative is to apply deep learning method for numerical results in mean field stochastic optimal control and related problems.

Since the paper [19], applying deep learning in solving PDEs and stochastic optimal control problems have attracted more and more attentions, as it overcomes many difficulties encountered when dealing in traditional methods. It would be interesting to show the numerical results for a non-trivial optimal control problem and it gives more insights for understanding new problems.

# APPENDIX A: A PROPERTY of VALUE FUNCTION IN PROBLEM ($C_0$)

For each fixed $t \in [0, T)$, define

$$\mathcal{F}_{t,s} \equiv \sigma\{W(r) - W(t) : t \leqslant r \leqslant s\}. \tag{A.1}$$

Note that for each $s \in [t, T]$,

$$\mathcal{F}_s = \mathcal{F}_{t,s} \vee \mathcal{F}_t. \tag{A.2}$$

And,

$$\mathbb{F}_t \equiv \{\mathcal{F}_{t,s}\}_{s \leqslant t}. \tag{A.3}$$

Corresponding to the two filtrations $\mathbb{F}$ and $\mathbb{F}_t$, it is natrual to consider two admissible control sets:

$$\mathcal{U}_{[t,T]} = \{u : [t.T] \times \Omega \to U \text{ is } \mathbb{F} - \text{progressive measurable}\}; \tag{A.4}$$

and

$$\hat{\mathcal{U}}_{[t,T]} = \{u : [t.T] \times \Omega \to U \text{ is } \mathbb{F}_t - \text{progressive measurable}\}, \tag{A.5}$$

where $U \subset \mathbb{R}^m$. It is easy to prove that:

**Proposition A.0.1.** *Let $\mathcal{A}$ and $\mathcal{B}$ be two sigma algebras, then*

$$\mathcal{P} = \{A \cap B : A \in \mathcal{A}, B \in \mathcal{B}\},. \tag{A.6}$$

*is a $\pi-$system that generates $\mathcal{A} \vee \mathcal{B}$.*

**Proposition A.0.2.** $(\mathcal{U}_{[t,T]}, \rho)$ *is a Polish space, where*

$$\rho(u_1, u_2) \equiv (\mathbb{E}[(\int_t^T |u_1(r) - u_2(r)|^2 dr)])^{\frac{1}{2}}.$$

*And the set* $\mathcal{U}_{step} \equiv \bigcup_{n\geqslant 1} \mathcal{U}_{cT}(I_n)$ *is a dense subset of* $\mathcal{U}_{[t,T]}$ *under* $\rho$*, where* $I_n$*:*

$$t = t_{n_0}^{(n)} < t_{n_1}^{(n)} < ... < t_{n_K}^{(n)} = T$$

*is a partition of* $[t, T]$ *with* $\|I_n\| \to 0$ *as* $n \to \infty$ *and*

$$\begin{aligned}\mathcal{U}_{cT}(I_n) = \{ &\sum_{i=0}^{n_K-1} (\sum_{j=1}^{N_i} a_{ji}^{(n)} 1_{E_{ji}^{(n)}}(\omega)) 1_{[t_i^{(n)}, t_{i+1}^{(n)})}(\cdot) : a_{ji}^{(n)} \in U_{dense}, \\ &\{E_{ji}^{(n)}\}_{j=1}^{N_i} \text{ generated by } \{E_l^{t_i}\}_{1\leqslant l\leqslant L}, \text{ is a partition of } (\Omega, \mathcal{F}_{t_i})\}.\end{aligned} \tag{A.7}$$

$U_{dense}$ *is a countable dense subset of* $U$ *and for each* $s \in [t, T]$*,* $\{E_l^{(s)}\}_{l\geqslant 1}$ *is a countable class of subset of* $\Omega$ *that generates* $\mathcal{F}_s$*.*

*Proof.* By Lemma 3.2.6 in [31]. □

Let $b$, $\sigma$, $g$ and $h$ be deterministic functions that satisfy proper conditions. Let$(t, x) \in [0, T] \times \mathbb{R}^n$. Consider the following decoupled FBSDE

$$\begin{cases} dX(s; t, x, u) = b(s, X(s), u(s))ds + \sigma(s, X(s), u(s))dW(s), \qquad s \in [t, T], \\ X(t; t, x, u) = x, \end{cases} \tag{A.8}$$

$$\begin{cases} dY(s; t, x, u) = -g(s, X(s), u(s), Y(s), Z(s))ds + Z(s)dW(s), \qquad s \in [t, T], \\ Y(T; t, x, u) = h(X(T)), \end{cases} \tag{A.9}$$

We introduce the cost functional as

$$J(t, x, u) = Y(t; t, x, u). \tag{A.10}$$

And two value functions:

$$V(t,x) \equiv \sup_{u \in \mathcal{U}_{[t,T]}} J(t,x,u), \quad \hat{V}(t,x) \equiv \sup_{u \in \hat{\mathcal{U}}_{[t,T]}} J(t,x,u). \tag{A.11}$$

With the help of the following theorem, we can prove that the value function $V(t,x)$ is a deterministic function.

**Theorem A.0.3.**

$$V(t,x) = \hat{V}(t,x). \tag{A.12}$$

*Proof.* It is obvious that $\hat{V} : [0,T] \times \mathbb{R}^n$ is a deterministic function. In fact, for each $u \in \hat{\mathcal{U}}_{[t,T]}$, $J(t,x,u)$ is a deterministic function of $(t,x)$, while, generally, it is an $\mathcal{F}_t-$measurable random variable when $u$ is in $\mathcal{U}_{[t,T]}$.

**Step 1:** Show that there exists a sequence $u_n \in \mathcal{U}_{[t,T]}$ such that $J(t,x,u_n) \uparrow V(t,x)$ a.s., as $n \to \infty$. For any $u_1$, $u_2 \in \mathcal{U}_{[t,T]}$, let $E \equiv \{J(t,x,u_1) > J(t,x,u_2)\}$, then $u \equiv u_1 1_E + u_2 1_{E^c}$ satisfies that

$$u \in \mathcal{U}_{[t,T]}, \quad J(t,x,u) \geqslant \max\{J(t,x,u_1), J(t,x,u_2)\}.$$

Together with the separability of space $(\mathcal{U}_{[t,T]}, \rho)$, we get the step 1 proved. Actually, this sequence can be selected in $\mathcal{U}_{step}$.

**Step 2:** Show that for each $n$, there exists a sequence in the form $\{\sum_i u_{m,i}^{(n)}(\cdot) 1_{E_{m,i}^{(n)}}\}_{m \geqslant 1}$, where $E_{n,i} \in \mathcal{F}_t$ and $u_{n,i}^{(n)}(s)$ is $\mathcal{F}_{t,s}-$measurable, such that $\sum_i u_{m,i}^{(n)}(\cdot) 1_{E_{m,i}^{(n)}} \to u_n$ a.s., as $m \to \infty$. Before the proof, we introduce the following lemma:

**Lemma A.0.4.** *Let $\mathcal{G}$ be a $\sigma-$algebra and $\Pi$ be a $\pi-$system that generates $\mathcal{G}$. Then for $\forall\ B \in \mathcal{G}$,* *there exists a sequence of sets* $\{A_i\} \subset \Pi$ *and a sequence of numbers* $l(i) \in \{0, 1\}$*, such that*

$$1_B = \sum_{i \geqslant 1} (-1)^{l(i)} 1_{A_i}.$$

*Proof.* For any $A,\ C \in \Pi$, $A \cap C \in \Pi$. And

$$1_A \cdot 1_C = 1_{A \cap C};$$

$$1_A + 1_C = 1_A + 1_C - 1_{A \cap C}.$$

Note that the set $B$ can be constructed through countable unions and intersections of sets in $\Pi$. This corresponds to its characteristic function $1_B$ can be written as a countable 'sum' of characteristic functions of sets in $\Pi$.

□

Note that every element in $\mathcal{U}_{cT}(I_n)$ could be written as

$$\sum_{j=1}^{N^{(n)}} (\sum_{i=0}^{n_K - 1} a_{j,i}^{(n)} 1_{E_{j,i}^{(n)} \cap E_{j0}^{(n)}} 1_{[t_i, t_{i+1})}) 1_{E_{j0}^{(n)}}, \tag{A.13}$$

where each $E_{j,i}^{(n)} \in \mathcal{F}_{t,t_i}$. By Lemma A.0.4 and Proposition A.0.2, each process in the form of (A.13) could be approximated by a sequence

$$\begin{aligned} &\sum_{j=1}^{N^{(n)}} (\sum_{i=0}^{n_K - 1} a_{j,i}^{(n)} (\sum_{r=1}^{M^{(n)}} (-1)^{l(i,j,r)} 1_{A_{i,j,r}^{(n)} \cap B_{i,j,r}^{(n)} \cap E_{j,0}^{(n)}}) 1_{[t_i, t_{i+1})}) 1_{E_{j0}^{(n)}}, \\ = &\sum_{j=1}^{N^{(n)}} \sum_{i=0}^{n_K - 1} a_{j,i}^{(n)} \sum_{r=1}^{M^{(n)}} (-1)^{l(i,j,r)} 1_{A_{i,j,r}^{(n)}} 1_{[t_i, t_{i+1})} 1_{E_{j0}^{(n)} \cap B_{i,j,r}^{(n)}} \end{aligned} \tag{A.14}$$

a.s. as $M^{(n)} \uparrow \infty$, where $A_{i,j,r}^{(n)} \in \mathcal{F}_{t,t_i}$ and $B_{i,j,r}^{(n)} \in \mathcal{F}_t$. Now, by the continuity of $J$ with respect to

$u$, we have shown that there exists a sequence $\{\hat{u}_m\} \in \mathcal{U}_{[t,T]}$ such that $J(t,x,\hat{u}_m) \uparrow V(t,x)$ a.s., as $m \to \infty$ and each $u_m$ is in the form:

$$u_m(s) = \sum_{i}^{K_m} u_{m,i}(s) 1_{E_{m,i}},$$

with $u_{m,i}(s)$ being $\mathcal{F}_{t,s}$ measurable and $E_{m,i} \in \mathcal{F}_t$.

**Step 3:** Show that (A.12). Without loss of generality, assume that, for each $m$, $J(t,x,u_{m,1}) = \max\limits_{1 \leqslant i \leqslant K_m} \{J(t,x,u_{m,i})\}$. Then

$$V(t,x) \geqslant J(t,x,u_{m,1}) \geqslant \sum_{i=1}^{K_m} J(t,x,u_{m,i}) 1_{E_{m,i}} = J(t,x,u_m) \uparrow V(t,x).$$

So, $V(t,x)$ is a deterministic function. Further, since

$$\hat{V}(t,x) \geqslant J(t,x,u_{m,1}) \uparrow V(t,x) \geqslant \hat{V}(t,x),$$

we can get (A.12). □

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