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Spatial Models with Specific Error Structures

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by

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ABSTRACT

The purpose of this dissertation is to study the first order autoregressive model in the spatial context with specific error structures. We begin by supposing that the error structure has a long memory in both the i and the j components. Whenever the model parameters alpha and beta equal one, the limiting distribution of the sequence of normalized Fourier coefficients of the spatial process is shown to be a function of a two parameter fractional Brownian sheet. This result is used to find the limiting distribution of the periodogram ordinate of the spatial process under the null hypothesis that alpha equals one and beta equals one.

We then give the limiting distribution of the normalized Fourier coefficients of the spatial process for both a moving average and autoregressive error structure. Two cases of autoregressive errors are considered. The first error model is autoregressive in one component and the second is autoregressive in both components. We show that the normalizing factor needed to ensure convergence in distribution of the sequence of Fourier coefficients is different in the moving average case, and the two autoregressive cases. In other words, the normalizing factor differs in each of these three cases.

Finally, a specific case of the functional central limit theorem in the spatial setting is stated and proved. The assumptions made here are placed on the autocovariance functions. We then discuss some specific examples and provide a test statistics based on the periodogram ordinate.

To my wife Nathania and parents Johnson & Hannah

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CHAPTER 1: INTRODUCTION AND PRELIMINARIES

1.1 Time and Spatial Series

The first order autoregressive time series model $y_t = \alpha y_{t-1} + \mu_t,\ 1 \leq t \leq n$, has received considerable attention whenever α is either equal to or near one. Fuller (1976)[19] and Dickey and Fuller (1979, 1981)[15][16] developed a statistical test for detecting the presence of a unit root. Consider the case whenever y_0 is fixed and $\{\mu_t\}$ is an i.i.d. sequence of mean zero and finite variance innovations. Let $\hat{\alpha}_n$ denote the least-squares estimator of α_n. Whenever $|\alpha| < 1$, Mann and Wald (1943)[25] showed that $n^{\frac{1}{2}}(\hat{\alpha}_n - \alpha)$ has a limiting normal distribution. If $|\alpha| > 1$, White (1958, 1959)[35][36] proved that the limiting distribution of $|\alpha|^n(\alpha^2 - 1)^{-1}(\hat{\alpha}_n - \alpha)$ is Cauchy, and also showed that $n(\hat{\alpha}_n - 1)$ converges in distribution to a ratio of functionals of a Brownian motion process. Phillips and Magdalinos (2007)[30] and Magdalinos (2012)[24] proved that in the mildly explosive case $\alpha_n = 1 + \dfrac{c}{n^\alpha},\ \alpha \in (0, 1)$ and $c > 0$, $\dfrac{1}{2c} n^\alpha \alpha_n^n (\hat{\alpha}_n - \alpha_n)$ has a limiting Cauchy distribution. Several of the above results have been generalized by relaxing the requirements on the innovations. Near-integrated process obtain by replacing α with $\alpha_n = e^{c/n}$ has been worked on by Bobkoski (1983)[12], Cavanagh (1986)[13], Chan and Wei (1987)[14], Nabeya and Tanaka (1990a, b) [28], [29], and Phillips (1987)[31]. They considered the theoretical aspect of the limiting distribution of $\hat{\alpha}_n$. With weakly dependent errors, Phillips (1987)[31] showed that $n(\hat{\alpha}_n - \alpha_n)$ converges in distribution to a ratio of functionals of an Ornstein-Uhlenbeck process under appropriate mixing conditions on the sequence $\{\mu_t\}$.

Nabeya and Perron (1994)[27] considered the cases $\mu_t = \epsilon_t + \theta_n \epsilon_{t-1}$ (first order Moving Average) and $\mu_t = \rho_n \mu_{t-1} + \epsilon_t$ (first order Autoregressive), where $\{\epsilon_t\}$ is a sequence of i.i.d. normally distributed random variable. In the case of the moving average, they showed that if $\theta_n = -1 + \dfrac{\delta}{n^{1/2}}$ and $\epsilon_t \sim$ i.i.d.$(0, \sigma_\epsilon^2)$, then as $n \longrightarrow \infty$,

$$\hat{\alpha}_n \xrightarrow{D} \left\{ \delta^2 \int\limits_{[0,1]} J_c(r)^2 \, dr \right\} \left\{ 1 + \delta^2 \int\limits_{[0,1]} J_c(r)^2 \, dr \right\}^{-1},$$

where $J_c(r) = \int\limits_{[0,r]} e^{((r-s)c)} \, dW(s)$ and $W(s)$ is the unit Wiener process on $C[0,1]$.

In the autoregressive case, they showed that

$$n(\hat{\alpha}_n - \alpha_n) \xrightarrow{D} \frac{1}{2} \mathcal{Q}_c(J_d(1))^2 \left\{ \int\limits_{[0,1]} \mathcal{Q}_c(J_d(r))^2 \, dr \right\}^{-1} - c.$$

as $n \longrightarrow \infty$, where $\mathcal{Q}_c(J_d(r)) = \int\limits_{[0,r]} e^{((r-v)c)} \, J_d(v) \, dv$, and $J_d(v) = \int\limits_{[0,v]} e^{((v-s)d)} \, dW(s)$.

Most unit root tests proposed are from the time domain perspective due to the fact that the spectral density of the process fails to exist in the unit root case. Akdi (1995) [3] used the frequency domain to propose a unit root test in terms of the periodogram ordinate. Bhattacharyya and Richardson (1996)[7] gave a limiting distribution of a unit root test proposed by Akdi (1995)[3] under the local Pitman-type alternative of the form $\{\alpha_N = e^{c/N}\}$ by supposing that the $Y_t -$ process obeys the model $Y_t - \mu = \alpha_N(Y_{t-1} - \mu) + \epsilon_t, \ 1 \leq t \leq N$, where $\{\epsilon_t\}$ are i.i.d. each having mean zero and finite variance σ^2. Bhattacharyya, Richardson, and Flores (2006)[8] used the periodogram ordinate to define an asymptotic test for testing $H_0 : \alpha = 1$ vs $H_A : |\alpha| < 1$. They showed that the normalized periodogram ordinate converges in distribution to a linear combination of two independent χ^2 random variables each having one degree of freedom under appropriate assumptions.

Schwert (1987, 1989) [32], [33] cited several examples of economic data that can be approximated by the use of an autoregressive time series of order one.

Martin (1979)[26] extended the autoregressive time series model to the spatial context. He indicated that it is often desirable in practice for a process $\{Y_{ij}\}$ to have reflection symmetric autocorrelations $\rho_{ij} = \rho_{-i,j} = \rho_{i,-j} = \rho_{-i,-j}$ for lags i and j. This led Martin to use the following model to fit agriculture field data:

$$Y_{ij} = \alpha Y_{i-1,j} + \beta Y_{i,j-1} - \alpha\beta Y_{i-1,j-1} + \mu_{ij},\ 1 \leq i,\ j \leq N, \tag{1.1}$$

where μ_{ij} denotes the error at the (i,j) position. It is emphasized that all models considered here are on the regular rectangular lattice of nonnegative integers. Asymptotic normality results for the estimators of (α, β) have been obtained by Tjostheim (1978)[34], Khalil (1991)[23], and Basu and Reinsel (1992, 1994)[4][5], whenever $|\alpha| < 1,\ |\beta| < 1$, and $\{\mu_{ij}\}$ is an i.i.d. mean zero sequence with finite variance. These estimation methods include the Yule -Walker equations, maximum likelihood, and least squares procedures. Unlike the AR(1) time series process, Bhattacharyya, Khalil, and Richardson (1996)[6] have given an asymptotic normality result for a sequence of Gauss-Newton estimators of (α, β) whenever $\alpha = \beta = 1$ or either $\alpha = 1$ or $\beta = 1$ and the other has modulus less than one. As in the AR(1) time series case, the normalizing factors depend on whether the moduli of $\alpha,\ \beta$ are less than, equal to, or greater than one. Under the assumptions that $\alpha = \beta = 1$ and $\{\mu_{ij}\}$ is a mean zero, second order, stationary process having long range dependence, it is shown here that the limiting distribution of the sequence of normalized Fourier coefficients of the $Y-$ process is a function of a two parameter fractional Brownian motion process on $[0,1]^2$. Further, three models involving moving average and autoregressive errors are studied here, and stationarity is not a requirement. It is shown that the normalizing factors needed to ensure convergence in distribution of the sequence of Fourier coefficients differ in each of these three cases.

For local Pitman-type alternatives, α and β in model (1.1) are parameterized by $\alpha_N = e^{a/N}$ and $\beta_N = e^{b/N}$ in model (1.2) below:

$$Y_{ij}(N) = \alpha_N Y_{i-1,j} + \beta_N Y_{i,j-1} - \alpha_N \beta_N Y_{i-1,j-1} + \mu_{ij}, \tag{1.2}$$

where $1 \leq i \leq j \leq N$.

The limiting distribution of the normalized Fourier coefficients of the $Y-$ process obeying the near unit root model (1.2) is found for the following cases:

(E.1) $\mu_{ij} = \theta_N \epsilon_{i-1,j} + \epsilon_{ij}, \quad 1 \leq i, j \leq N$, where $\theta_N \longrightarrow -1$ and $N^\rho \left(1 + \dfrac{\theta_N}{\alpha_N}\right) \longrightarrow 1$ as $N \longrightarrow \infty$, for some $0 < \rho < \dfrac{1}{2}$

(E.2) $\mu_{ij} = \gamma_N \mu_{i-1,j} + \epsilon_{ij}$, where $\gamma_N = e^{c/N}$ and c is a parameter

(E.3) $\mu_{ij} = \gamma_N \mu_{i-1,j} + \delta_N \mu_{i,j-1} - \gamma_N \delta_N \mu_{i-1,j-1} + \epsilon_{ij}$, where $1 \leq i, j \leq N$, $\gamma_N = e^{c/N}$, $\delta_N = e^{d/N}$ and c and d are parameters.

1.2 Notations

The following notations are used throughout this work.

(N.1) $\mathbb{Z} =$ the set of all integers
$E_{ijN} = \left[\dfrac{i-1}{N}, \dfrac{i}{N}\right] \times \left[\dfrac{j-1}{N}, \dfrac{j}{N}\right]$
$E_{t_1 t_2 \ldots t_k} = [0, t_1] \times [0, t_2] \times \cdots \times [0, t_k]$

(N.2) $D_2 = D([0,1]^2)$ equipped with Skorohod's metric, where $[0,1]^2 = [0,1] \times [0,1]$. (See Billingsley (1999)[11] and Bickel and Wichura (1971)[9])

(N.3) $W(\underline{t})$ denotes a Brownian sheet on $[0,1]^2$; that is, $\{W(\underline{t}) : \underline{t} \in [0,1]^2\}$ is a mean zero, Gaussian process with $\text{cov}(W(\underline{s}), W(\underline{t})) = c(s_1 \wedge t_1) \cdot (s_2 \wedge t_2)$ where $\underline{s} = (s_1, s_2)$, for some $c \in \mathbb{R}$. (See Xiao (2009)[38])

(N.4) $U_N(\underline{t}) = \dfrac{1}{N} \sum_{i,j=1}^{[Nt_1],[Nt_2]} \epsilon_{ij},\ \underline{t} \in [0,1]^2$

(N.5) $J(\underline{t})$ denotes an Ornstein-Uhlenbeck process on $[0,1]^2$; in particular,

$$J(\underline{t}) = W(\underline{t}) + a \int\limits_{E_{t_1}} e^{a(t_1-x)}\ W(x,t_2)\ dx + b \int\limits_{E_{t_2}} e^{b(t_2-y)}\ W(t_1,y)\ dy$$
$$+\, ab \int\limits_{E_{t_1 t_2}} e^{a(t_1-x)} e^{b(t_2-y)}\ W(x,y)\ dxdy, \text{where } \underline{t} = (t_1,t_2) \in [0,1]^2$$

(N.6) $K(\underline{t}) = \int\limits_{E_{t_1}} e^{c(t_1-x)}\ W(x,t_2)\ dx$

(N.7) $L(\underline{t}) = \int\limits_{E_{t_1 t_2}} e^{c(t_1-x)} e^{d(t_2-y)}\ W(x,y)\ dxdy$

(N.8) $M(f)(\underline{t}) =$

$$f(\underline{t}) + a \int\limits_{E_{t_1}} e^{a(t_1-x)} f(x,t_2) dx + b \int\limits_{E_{t_2}} e^{b(t_2-y)} f(t_1,y) dy$$
$$+\, ab \int\limits_{E_{t_1 t_2}} e^{a(t_1-x)} e^{b(t_2-y)} f(x,y) dxdy,$$

where $f : [0,1]^2 \longrightarrow \mathbb{R}$

(N.9) A_N, B_N denotes the Fourier coefficients of the $Y-$ process ; that is,

$$A_N = \sum_{k,l=1}^{N} \cos \frac{2\pi}{N} (k+l)\, Y_{kl}(N)$$

$$B_N = \sum_{k,l=1}^{N} \sin \frac{2\pi}{N}(k+l) Y_{kl}(N)$$

(N.10) $I_N = A_N^2 + B_N^2$ is the periodogram ordinate of the $Y-$ process.

(N.11) $W_d(\underline{t})$ denotes a fractional Brownian sheet on $[0,1]^2$ (See Definition 1.3.1).

(N.12) $J_d(\underline{t})$ denotes a fractional Ornstein-Uhlenbeck process on $[0,1]^2$ (See Definition 1.3.2).

Suppose that the error structure $\{\mu_{ij} : i,j \in \mathbb{Z}\}$ is a mean zero second order $(E(\mu_{ij}^2) < \infty)$ process; then it is said to be stationary provided $\text{cov}(\mu_{ij}, \mu_{i+h,j+k})$ depends only on h and k, for all $i,j \in \mathbb{Z}$. From an asymptotic perspective, if $\frac{1}{N^{d_1+d_2+1}} \sum_{i,j=1}^{[Nt_1],[Nt_2]} \mu_{ij} \xrightarrow{D} W_d(\underline{t})$ on D_2, then the error structure is said to have a long memory in the i^{th} component if $0 < d_i < \frac{1}{2}$ and short memory whenever $d_i = 0, \; i = 1,2$. This definition permits long memory in one component of the error structure and short memory in the other. Observe that if $d_1 = d_2 = 0$ (short memory), then (N.11) and (N.12) coincide with (N.3) and (N.5) respectively.

Long memory of a stationary process exists whenever the covariance function decreases sufficiently slow. This means that, partial sums of such processes requires a larger normalizing factor in order to obtain convergence.

For the sake of easy reference, various conditions listed below are needed to prove the theorems that follows.

(A.0) $\alpha = \beta = 1$

(A.1) $Y_{ij} = \mu_{ij} = \epsilon_{ij} = 0$ whenever $i \wedge j \leq 0$

(A.2) $\alpha_N = e^{a/N}, \; \beta_N = e^{b/N}$, where $a < 0$ and $b < 0$

(A.3) $\{\epsilon_{ij} : i, j \in \mathbb{Z}\}$ is an independent and identically distributed , mean zero, finite variance sequence

(A.4) $\{\mu_{ij} : i, j \in \mathbb{Z}\}$ is a mean zero, second order, stationary process satisfying

$$\frac{1}{N^{d_1+d_2+1}} \sum_{i,j=1}^{[Nt_1],[Nt_2]} \mu_{ij} \xrightarrow{D} W_d(\underline{t}),$$

where $\underline{t} = (t_1, t_2) \in [0, 1]^2$ and $d = (d_1, d_2)$ with $0 \leq d_i < \frac{1}{2}$, $i = 1, 2$, and $\mu_{ij} = 0$ whenever $i \wedge j \leq 0$

The primary results of this work are listed below and proved in later chapters.

Theorem 1.2.1. *Let U_1 and U_2 denote independent chi-square random variables each having one degree of freedom. Assume that the $Y-$process satisfies*

(i) model (1.1), (A.0), (A.1), and (A.4). Then

$$\frac{1}{N^{2(d_1+d_2)+6}} I_N \xrightarrow{D} \sigma_{11} U_1 + \sigma_{22} U_2,$$

where σ_{11} and σ_{22} are given in (2.5).

(ii) model (1.2), (A.1), (A.2), and (A.4). Then

$$\frac{1}{N^{2(d_1+d_2)+6}} I_N \xrightarrow{D} \lambda_1(d) U_1 + \lambda_2(d) U_2,$$

where λ_1 and λ_2 are defined in (2.6).

Theorem 1.2.2. *Suppose that the $Y-$ process obeys model (1.2), (E.1), and (A.1)-(A.3). Then* $\frac{1}{N^{3-\rho}}(A_N, B_N) \xrightarrow{D} (A, B)$ *as* $N \longrightarrow \infty$ *on* $\mathbb{R}^2$*, where*

$$A = \int\limits_{[0,1]^2} \cos 2\pi(x+y)\ J(x,y)\ dxdy$$

and

$$B = \int\limits_{[0,1]^2} \sin 2\pi(x+y)\ J(x,y)\ dxdy.$$

Theorem 1.2.3. *Assume that the $Y-$ process obeys model (1.2), (E.2), and (A.1)-(A.3). Then* $\frac{1}{N^4}(A_N, B_N) \xrightarrow{D} (A, B)$ *as* $N \to \infty$ *on* $\mathbb{R}^2$*, where*

$$A = \int\limits_{[0,1]^2} \cos 2\pi(t_1+t_2)\ M(K(\underline{t}))\ d\underline{t}$$

and

$$B = \int\limits_{[0,1]^2} \sin 2\pi(t_1+t_2)\ M(K(\underline{t}))\ d\underline{t}.$$

Theorem 1.2.4. *If the $Y-$ process satisfies model (1.2), (E.3), and (A.1)-(A.3). Then* $\frac{1}{N^5}(A_N, B_N) \xrightarrow{D} (A, B)$ *as* $N \to \infty$ *on* $\mathbb{R}^2$*, where*

$$A = \int\limits_{[0,1]^2} \cos 2\pi(t_1+t_2)\ M(L(\underline{t}))\ d\underline{t}$$

and

$$B = \int\limits_{[0,1]^2} \sin 2\pi(t_1+t_2)\ M(L(\underline{t}))\ d\underline{t}.$$

An excellent treatment of convergence in distribution or weak convergence of a sequence of measurable functions from a probability space to the function space $D([0,1])$ can be found in Billingsley (1968) [10]. Bickel and Wichura (1971) [9] have extended these concepts to the function space D_2.

Fix $\underline{t} \in [0,1]^2$, and denote the four quadrant of $[0,1]^2$ having $\underline{t}$ as their origin by $Q_1(\geq, \geq)$, $Q_2(<, \geq)$, $Q_3(<, <)$, and $Q_4(\geq, <)$. Let D_2 denote the set of all real-valued functions f defined on $[0,1]^2$ for which $\lim_{\underline{s} \to \underline{t}} f(\underline{s})$ exists whenever $\underline{s}$ belongs to a single quadrant, and $\lim_{\underline{s} \longrightarrow \underline{t}} f(\underline{s}) = f(\underline{t})$ provided $\underline{s} \in Q_1$. Bickel and Wichura (1971) [9] show there is a metric on D_2 which makes it separable, complete, and whose Borel $\sigma-$ field coincides with that generated by the coordinate mappings. Further, this metric extends Skorohod's well-known metric on $D([0,1])$ to D_2 . An important result needed in this context is the Continuous Mapping Theorem. In particular, if $X_n,\ X$ are measurable functions from a probability space $(\Omega, \mathcal{F}, P)$ into D_2 , and $h : D_2 \longrightarrow \mathbb{R}$ is continuous (except possibly on a set of P^X measure zero), then $X_n \xrightarrow{D} X$ on D_2 implies that $h(X_n) \xrightarrow{D} h(X)$ on $\mathbb{R}$. In our application here, $h : D_2 \longrightarrow \mathbb{R}$ is defined using integration, $h(f) = \int\limits_{[0,1]^2} f(\underline{x})\, d\underline{x}$. Always $X_n \xrightarrow{D} X$ means $E\left(\phi(X_n)\right) \to E\left(\phi(X)\right)$ on $\mathbb{R}$, for each bounded continuous $\phi : D_2 \longrightarrow \mathbb{R}$.

Riemann-Stieltjes integration is another tool used extensively in proofs of theorems. Let $RS \int\limits_{A} f\ dg$ denote the Riemann-Stieltjes over a rectangular subset A of $[0,1]^2$. Recall that sufficient conditions for this to exist is for either f or g be continuous and the other be of bounded variation on $[0,1]^2$; moreover, an integration by parts formula is valid in this case. These and other results concerning Riemann-Stieltjes integration can be found in Hobson (1957)[21] and Yeh (1963)[39]. For easy reference, the Riemann-Stieltjes integration by parts formula for the subset A of $[0,1]^2$ shown below having boundary lines $L_i,\ 1 \leq i \leq 4$.

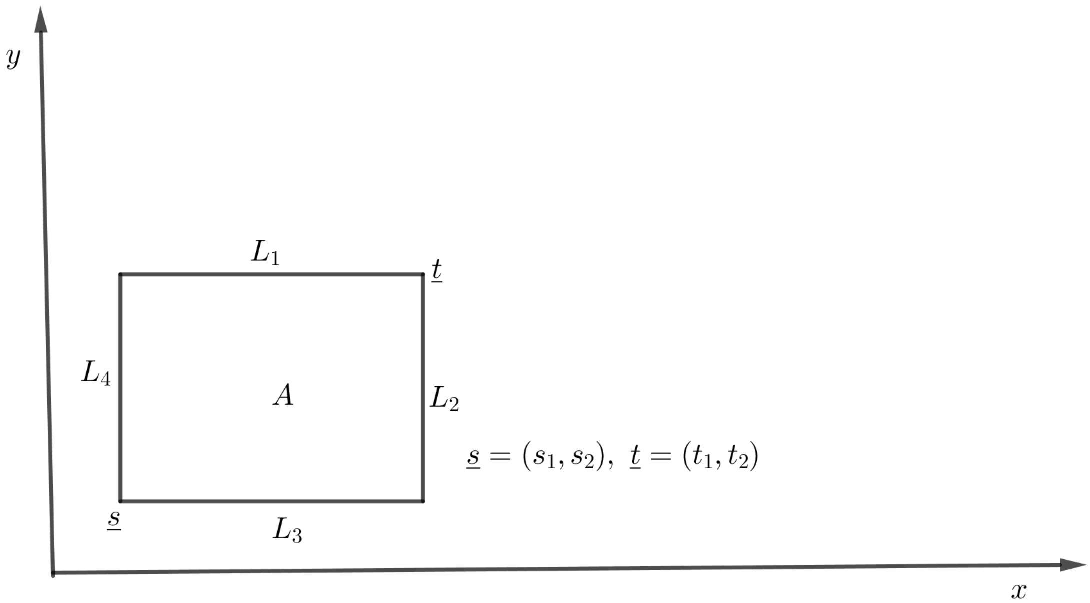

Figure 1.1: Subset of $[0, 1]^2$

Theorem 1.2.5. *Assume that the Riemann-Stieltjes integral of f with respect to g exists on the subset A as shown above. Then the Riemann-Stieltjes integral of g with respect to f exists. Moreover,*

$$\begin{aligned}\int_A g\ df =& f(\underline{t})g(\underline{t}) - f(s_1, t_2)g(s_1, t_2) - f(t_1, s_2)g(t_1, s_2) + f(\underline{s})g(\underline{s}) - \int_{[s_1,t_1]} f(x, t_2)\ dg(x, t_2) \\ & - \int_{[s_2,t_2]} f(t_1, y)\ dg(t_1, y) + \int_{[s_1,t_1]} f(x, s_2)\ dg(x, s_2) \\ & + \int_{[s_2,t_2]} f(s_1, y)\ dg(s_1, y) + RS\int_A f\ dg.\end{aligned}$$

Another tool which will be used in the proofs of theorems is the Cramér-Wold device. We will need the following theorem to prove the Cramér-Wold device.

Theorem 1.2.6. *(Lévy's Continuity Theorem) Let $\{X_n : n \geq 1\}$ be a sequence of $k-$*

dimensional random vectors with characteristic function ϕ_{X_n} and let X be a $k-$ dimensional random vector with characteristic function ϕ_X. Then $X_n \xrightarrow{D} X$ if and only if $\phi_{X_n}(t) \longrightarrow \phi_X(t)$ as $n \longrightarrow \infty$, for each fixed $t \in \mathbb{R}^k$.

Theorem 1.2.7. *(Cramér-Wold device)[17] Under the assumptions of Theorem 1.2.6, $X_n \xrightarrow{D} X$ iff $\lambda \cdot X_n \xrightarrow{D} \lambda \cdot X$ for all $\lambda \in \mathbb{R}^k$.*

1.3 Important Definitions

The definition of a fractional Brownian sheet was introduced by Kamont (1996)[22]. These and more general works on anisotropic Gaussian random fields can be found in Xiao (2009)[38].

Definition 1.3.1. **Fractional Brownian Sheet**([22]): Given $d = (d_1, d_2)$, $0 \leq d_i < \frac{1}{2}$, $i = 1, 2$. A mean zero, Gaussian process $\{W_d(\underline{t}) : \underline{t} \in [0,1]^2\}$ is called a fractional Brownian sheet provided that the $\operatorname{cov}(W_d(\underline{s}), W_d(\underline{t})) = c[s_1^{2d_1+1} + t_1^{2d_1+1} - |s_1 - t_1|^{2d_1+1}] \cdot [s_2^{2d_2+1} + t_2^{2d_2+1} - |s_2 - t_2|^{2d_2+1}]$ for some $c \in \mathbb{R}$, where $\underline{s} = (s_1, s_2)$ and $\underline{t} = (t_1, t_2) \in [0,1]^2$.

Rather than parameters d_1 and d_2, some authors use the Hurst indices $H_i = d_i + \frac{1}{2}$, $i = 1, 2$. For convenience, $d_1 = 0$ or $d_2 = 0$ is included in Definition 1.3.1. In particular, a Brownian sheet occurs whenever $d_1 = d_2 = 0$. In general each H_i lies between 0 and 1, since $0 \leq d_i < \frac{1}{2}$, it is obvious that we are considering only values of H_i between $\frac{1}{2}$ and 1 here, $i = 1, 2$.

Definition 1.3.2. **Fractional Ornstein-Uhlenbeck process:** Given $a, b \in \mathbb{R}$, let $\{W_d(\underline{t}) : \underline{t} \in [0,1]^2\}$ denote a fractional Brownian sheet. Define

$$\begin{aligned} J_d(\underline{t}) :=& W_d(\underline{t}) + a \int_{[0,t_1]} e^{a(t_1-x)} W_d(x, t_2) dx + b \int_{[0,t_2]} e^{b(t_2-y)} W_d(t_1, y) dy \\ &+ ab \int_{E_{t_1 t_2}} e^{a(t_1-x)} e^{b(t_2-y)} W_d(x, y) dx dy, \text{where } \underline{t} = (t_1, t_2) \in [0,1]^2. \end{aligned}$$

Then $\{J_d(\underline{t}) : \underline{t} \in [0,1]^2\}$ is called a fractional Ornstein-Uhlenbeck process on $[0,1]^2$. Whenever $\underline{d} = \underline{0}$, $\operatorname{cov}\Big(J_{\underline{0}}(s_1, s_2), J_{\underline{0}}(t_1, t_2)\Big) = \Big[\dfrac{e^{(s_1+t_1)a} - e^{(s_1-t_1)a}}{2a}\Big] \cdot \Big[\dfrac{e^{(s_2+t_2)b} - e^{(s_2-t_2)b}}{2b}\Big]$; that is, $J_{\underline{0}}$ has the same covariance structure as the product of two-independent one-parameter Ornstein-Uhlenbeck processes.

Definition 1.3.3. Fourier Coefficients and Periodogram Ordinate: Denote $\omega_k = \dfrac{2\pi k}{N}$. The Fourier coefficients of the $Y-$ process are defined as

$$A_{N,k,l} = \sum_{i,j=1}^{N} \cos(\omega_k i + \omega_l j) Y_{ij}$$

and

$$B_{N,k,l} = \sum_{i,j=1}^{N} \sin(\omega_k i + \omega_l j) Y_{ij}.$$

Remark 1.3.4. For ease of exposition, $k = l = 1$ is selected. The notation in Definition 1.3.3 is condensed to $\omega = 2\pi/N$, and $A_N = \sum\limits_{i,j=1}^{N} \cos\omega(i+j)Y_{ij}$, $B_N = \sum\limits_{i,j=1}^{N} \sin\omega(i+j)Y_{ij}$ denote the Fourier coefficients of the $Y-$ process. The periodogram ordinate of the $Y-$ process is given by

$$I_N = A_N^2 + B_N^2.$$

CHAPTER 2: UNIT ROOTS TEST: SPATIAL MODEL WITH LONG MEMORY ERRORS

Most of the results in this chapter have been published by this author in [1].

A stationary time series $\{X_t : \ t \in \mathbb{Z}\}$ obeying an autoregressive model has a covariance function satisfying $\gamma_X(h) \sim c|r|^h$ as $h \to \infty$, where $|r| < 1$ under suitable assumptions. In this case, the covariance approaches zero at a geometric rate as $h \longrightarrow \infty$. On the other hand, there has been some work done in time series whose covariance function decays to zero at a much slower rate. These processes are said to posses long memory provided the covariance function $\gamma_X(h) \sim c\dfrac{1}{h^\alpha}$ as $h \longrightarrow \infty$, where $0 < \alpha < 1$. In the spatial setting, recall that from an asymptotic perspective, if $\dfrac{1}{N^{d_1+d_2+1}} \sum_{i,j=1}^{[Nt_1],[Nt_2]} \mu_{ij} \xrightarrow{D} W_d(\underline{t})$ on D_2, then the error structure is said to have a <u>long memory</u> in the i^{th} component if $0 < d_i < \dfrac{1}{2}$ and <u>short memory</u> whenever $d_i = 0, \ i = 1, 2$. This extends the corresponding result $\dfrac{1}{N^{\frac{1}{2}+d}} \sum_{i=1}^{[Nt]} \mu_i \xrightarrow{D} W_d(t)$ on $D([0,1]), \ 0 < d < \dfrac{1}{2}$, in the time series setting.

We will state and prove the main theorem of this chapter below; however, the following lemma is verified first. The lemma establishes that the limiting distribution of the sequence of normalized Fourier coefficients of the $Y-$ process is a function of a two parameter fractional Brownian motion process on $[0,1]^2$ whenever $\alpha = \beta = 1$.

Lemma 2.0.1. *([1]) Suppose that $\{Y_{ij} : i, j \geq 1\}$ satisfies model (1.2), (A.1), (A.2), and the μ-process obeys (A.4). Let A_N and B_N denote the Fourier coefficients of the $Y-$ process defined in (N.9). Then*

$$\frac{1}{N^{d_1+d_2+3}}\Big(A_N(d), B_N(d)\Big) \xrightarrow{D} \Big(A(d), B(d)\Big)$$

in $\mathbb{R}^2$*, where* $A(d) := \int\limits_{[0,1]^2} \cos 2\pi(x+y) J_d(x,y)dxdy$ *and* $B(d) := \int\limits_{[0,1]^2} \sin 2\pi(x+y) J_d(x,y)dxdy$*.*

Proof. Denote $g = d_1 + d_2$. Iterating, using model (1.2) with $Y_{ij} = 0$ whenever $i \leq 0$ or $j \leq 0$,

$$Y_{kl} = \sum_{i,j=1}^{k,l} \alpha_N^{k-i} \beta_N^{l-k} \mu_{ij}.$$

Define

$$Z_N(\underline{t}) := \frac{1}{N^{g+1}} \cos \omega([Nt_1] + [Nt_2]) \sum_{i,j=1}^{[Nt_1],[Nt_2]} \alpha_N^{[Nt_1]-i} \beta_N^{[Nt_2]-j} \mu_{ij}. \tag{2.1}$$

The key steps in the proof are to show that $Z_N(\underline{t}) \xrightarrow{D} \cos 2\pi(t_1+t_2) \cdot J_d(\underline{t})$ and $\frac{1}{N^{g+3}} A_{N-1} = \int\limits_{[0,1]^2} Z_N(\underline{t}) d\underline{t}$.

According to (A.4), $X_N \xrightarrow{D} W_d$ in D_2, where $X_N(\underline{t}) = \frac{1}{N^{g+1}} \sum_{i,j=1}^{[Nt_1],[Nt_2]} \mu_{ij}$, whenever $\underline{t} = (t_1, t_2) \in [0,1]^2$.

Observe that

$$\begin{aligned} RS \int\limits_{E_{ijN}} 1 \cdot dX_N(x,y) &= X_N\Big(\frac{i}{N}, \frac{j}{N}\Big) - X_N\Big(\frac{i-1}{N}, \frac{j}{N}\Big) \\ &\quad - X_N\Big(\frac{i}{N}, \frac{j-1}{N}\Big) + X_N\Big(\frac{i-1}{N}, \frac{j-1}{N}\Big) \\ &= \frac{\mu_{ij}}{N^{g+1}}. \end{aligned} \tag{2.2}$$

Using (2.1), (2.2), $\alpha_N = e^{a/N}, \beta_N = e^{b/N}$, and the Mean Value Theorem,

$$
\begin{aligned}
Z_N(\underline{t}) &= \cos\omega([Nt_1]+[Nt_2])\cdot \\
&\quad \sum_{i,j=1}^{[Nt_1],[Nt_2]} RS\int\limits_{E_{ijN}} e^{a/N([Nt_1]-i)}e^{b/N([Nt_2]-j)}\ dX_N(x,y) \\
&= \cos 2\pi(t_1+t_2) \\
&\quad \times \sum_{i,j=1}^{[Nt_1],[Nt_2]} RS\int\limits_{E_{ijN}} \left[e^{a(t_1-x)}e^{b(t_2-y)} + O\Big(\frac{1}{N}\Big)\right]\ dX_N(x,y) \\
&= \cos 2\pi(t_1+t_2)\cdot RS\int\limits_{E_{t_1t_2}} e^{a(t_1-x)}e^{b(t_2-y)}\ dX_N(x,y) + \cos 2\pi(t_1+t_2)O\Big(\frac{1}{N}\Big)X_N(\underline{t}).
\end{aligned}
$$

Denote $V_N(\underline{t}) = \cos 2\pi(t_1+t_2)O\Big(\frac{1}{N}\Big)X_N(\underline{t})$, $\underline{t}\in[0,1]^2$, and note that $X_N \xrightarrow{D} W_d$ implies that $V_N \xrightarrow{D} 0$.

Integrating by parts (Theorem 1.2.5),

$$
\begin{aligned}
Z_N(\underline{t}) = \cos 2\pi(t_1+t_2)\cdot \Bigg[&X_N(\underline{t}) + a\int\limits_{[0,t_1]} e^{a(t_1-x)}X_N(x,t_2)\ dx \\
&+ b\int\limits_{[0,t_2]} e^{b(t_2-y)}X_N(t_1,y)\ dy + ab\int\limits_{E_{t_1t_2}} e^{a(t_1-x)}e^{b(t_2-y)}X_N(x,y)\ dx\ dy\Bigg] + o_p(1).
\end{aligned}
\tag{2.3}
$$

Define $h : D_2 \longrightarrow D_2$ by

$$
\begin{aligned}
h(f)(\underline{t}) := \cos 2\pi(t_1+t_2)\cdot \Big[f(\underline{t}) + a\int\limits_{[0,t_1]} e^{a(t_1-x)}f(x,t_2)\ dx + b\int\limits_{[0,t_2]} e^{b(t_2-y)}f(t_1,y)\ dy \\
+ ab\int\limits_{E_{t_1t_2}} e^{a(t_1-x)}e^{b(t_2-y)}f(x,y)\ dx\ dy\Big].
\end{aligned}
$$

Then $h \mid C([0,1]^2) \longrightarrow D_2$ is continuous, where $C([0,1]^2)$ denotes the set of all continuous real-valued functions defined on $[0,1]^2$. Since $X_N \xrightarrow{D} W_d$ in D_2 and $P_{W_d}\Big(C([0,1]^2)\Big) = 1$, it follows by the Continuous Mapping Theorem (Billingsley (1999), Theorem 2.7)[11] that $h(X_N) \xrightarrow{D} h(W_d)$ in D_2, where $h(W_d)(\underline{t}) = \cos 2\pi(t_1+t_2) \cdot J_d(\underline{t})$ for each $\underline{t} = (t_1,t_2) \in [0,1]^2$. Employing (2.3), $Z_N(\underline{t}) = h(X_N)(\underline{t}) + o_p(1)$, and thus

$$Z_N(\underline{t}) \xrightarrow{D} h(W_d)(\underline{t}) = \cos 2\pi(t_1+t_2) \cdot J_d(\underline{t}) \tag{2.4}$$

as $N \longrightarrow \infty$ in D_2.

Moreover, by (2.1),

$$\begin{aligned}
\int\limits_{[0,1]^2} Z_N(\underline{t}) d\underline{t} &= \frac{1}{N^{g+1}} \sum_{i,j=1}^{[Nt_1],[Nt_2]} \int\limits_{[0,1]^2} \cos\omega([Nt_1]+[Nt_2]) \alpha_N^{[Nt_1]-i} \beta_N^{[Nt_2]-j} \mu_{ij} \; d\underline{t} \\
&= \frac{1}{N^{g+1}} \sum_{k,l=1}^{N} \int\limits_{E_{klN}} \sum_{i,j=1}^{[Nt_1],[Nt_2]} \cos\omega([Nt_1]+[Nt_2]) \alpha_N^{[Nt_1]-i} \beta_N^{[Nt_2]-j} \mu_{ij} \; d\underline{t} \\
&= \frac{1}{N^{g+3}} \sum_{k,l=2}^{N} \sum_{i,j=1}^{k-1,l-1} \cos\omega(k+l-2) \alpha_N^{k-1-i} \beta_N^{l-1-j} \mu_{ij} \\
&= \frac{1}{N^{g+3}} \sum_{k,l=1}^{N-1} \sum_{i,j=1}^{k,l} \cos\omega(k+l) \alpha_N^{k-i} \beta_N^{l-j} \mu_{ij} \\
&= \frac{1}{N^{g+3}} A_{N-1}(d).
\end{aligned}$$

Since integration is continuous on $C([0,1]^2)$, it follows from (2.4) and the Continuous Mapping Theorem that $\frac{1}{N^{g+3}} A_{N-1}(d) = \int\limits_{[0,1]^2} Z_N(\underline{t}) \; d\underline{t} \xrightarrow{D} \int\limits_{[0,1]^2} \cos 2\pi(t_1+t_2) \cdot J_d(\underline{t}) \; d\underline{t} = A(d)$ in $\mathbb{R}$.

Likewise, $\frac{1}{N^{g+3}} B_{N-1}(d) \xrightarrow{D} \int\limits_{[0,1]^2} \sin 2\pi(t_1+t_2) \cdot J_d(\underline{t}) \; d\underline{t} = B(d)$ in $\mathbb{R}$, and the above argument extends to show that $\frac{1}{N^{g+3}}\Big(\lambda_1 A_N(d) + \lambda_2 B_N(d)\Big) \xrightarrow{D} \lambda_1 A(d) + \lambda_2 B(d)$ as $N \longrightarrow \infty$ in $\mathbb{R}$, for each $\lambda_1, \lambda_2 \in \mathbb{R}$. Hence, by the Cramér-Wold device (Theorem 1.2.7), $\frac{1}{N^{g+3}}\Big(A_N(d), B_N(d)\Big) \xrightarrow{D}$ $\Big(A(d), B(d)\Big)$ as $N \longrightarrow \infty$ in $\mathbb{R}^2$. $\square$

The main theorem establishes that the limiting distribution of the periodogram ordinate of the $Y-$ process under the null hypothesis that $\alpha = \beta = 1$ is a linear combination of two independent chi-square random variables.

Theorem 2.0.2. *([1]) Let U_1 and U_2 denote independent chi-square random variables each having one degree of freedom. Assume that the $Y-$process satisfies*

(i) model (1.1), (A.0), (A.1), and (A.4). Then

$$\frac{1}{N^{2(d_1+d_2)+6}} I_N \xrightarrow{D} \sigma_{11}U_1 + \sigma_{22}U_2,$$

where σ_{11} and σ_{22} are given in (2.5).

(ii) model (1.2), (A.1), (A.2), and (A.4). Then

$$\frac{1}{N^{2(d_1+d_2)+6}} I_N \xrightarrow{D} \lambda_1(d)U_1 + \lambda_2(d)U_2,$$

where λ_1 and λ_2 are defined in (2.6).

Proof. (i) : Observe that model (1.2) given by $Y_{ij}(N) = \alpha_N Y_{i-1,j} + \beta_N Y_{i,j-1} - \alpha_N \beta_N Y_{i-1,j-1} + \mu_{ij}$, reduces to model (1.1) which is $Y_{ij} = \alpha Y_{i-1,j} + \beta Y_{i,j-1} - \alpha\beta Y_{i-1,j-1} + \mu_{ij}$, with $\alpha = \beta = 1$ whenever $a = b = 0$. Moreover, when $a = b = 0$, $J_d = W_d$, and hence by Lemma 2.0.1, we obtain

$$A(d) = \int\limits_{[0,1]^2} \cos 2\pi(x+y) W_d(x,y)\,dxdy,$$

$$B(d) = \int\limits_{[0,1]^2} \sin 2\pi(x+y) W_d(x,y) dxdy.$$

Let $\Sigma_0 = \big(\sigma_{ij}(d)\big)$ denote the variance-covariance matrix of $\big(A(d), B(d)\big)$. Define, for each $\alpha > 0$, $L(\alpha) = \int\limits_{[0,1]} x^\alpha \cos 2\pi x dx - \frac{\alpha}{\alpha+1} \int\limits_{[0,1]} x^{\alpha+1} \cos 2\pi x dx$, and $M(\alpha) = \int\limits_{[0,1]} x^\alpha \cos 2\pi x dx - \frac{\alpha+2}{\alpha+1} \int\limits_{[0,1]} x^{\alpha+1} \cos 2\pi x dx$.
Straightforward calculations give the following results:

$$\begin{aligned} \sigma_{11}(d) &= b^2(L(2d_1+1)L(2d_2+1) + M(2d_1+1)M(2d_2+1)) \\ \sigma_{22}(d) &= b^2(M(2d_1+1)L(2d_2+1) + L(2d_1+1)M(2d_2+1)) \\ \sigma_{21}(d) &= 0. \end{aligned} \tag{2.5}$$

Since $\{W_d(\underline{t}) : \underline{t} \in [0,1]^2\}$ is a mean zero, Gaussian process, $\big(A(d), B(d)\big)$ is distributed as $N(\underline{0}, \Sigma_0)$. Applying Lemma 2.0.1, the normalized periodogram ordinate of the $Y-$ process satisfies

$$\frac{1}{N^{2g+6}} I_N = \frac{1}{N^{2g+6}} \Big(A_N^2(d) + B_N^2(d)\Big) \xrightarrow{D} \sigma_{11}(d)U_1 + \sigma_{22}(d)U_2,$$

where U_1 and U_2 are independent chi-square random variables each having one degree of freedom. Hence Theorem 2.0.2 (i) is valid.

(ii) : Consider model (1.2) with $\alpha_N = e^{a/N}$ and $\beta_N = e^{b/N}$, where a and b are negative real numbers. Let $\Sigma_1 = \big(\delta_{ij}(d)\big)$ denote the variance-covariance matrix of $\big(A(d), B(d)\big)$.

There exists an orthogonal matrix Q such that $Q\Sigma_1 Q' = \text{diag}(\lambda_1, \lambda_2)$, where

$$\lambda_1(d) \text{ and } \lambda_2(d) \tag{2.6}$$

are the eigenvalues of $\Sigma_1(d)$.

Since $\{W_d(\underline{t}) : \underline{t} \in [0,1]^2\}$ is a mean zero, Gaussian process, it follows that $\Big(A(d), B(d)\Big) \sim N\Big(\underline{0}, \Sigma_1(d)\Big)$, and thus by Lemma 2.0.1,

$$C_N := \frac{1}{N^{g+3}}\Big(A_N(d), B_N(d)\Big)Q' \xrightarrow{D} N\Big(\underline{0}, Q\Sigma_1 Q'\Big) = N(\underline{0}, \text{diag}(\lambda_1, \lambda_2)\Big).$$

Therefore,

$$\frac{1}{N^{2g+6}} I_N = \frac{1}{N^{2g+6}}\Big(A_N^2(d) + B_N^2(d)\Big) = C_N C_N' \xrightarrow{D} \lambda_1 U_1 + \lambda_2 U_2,$$

where U_1 and U_2 are independent chi-square random variables each having one degree of freedom.

$\square$

CHAPTER 3: UNIT ROOTS TEST: SPATIAL MODEL WITH MOVING AVERAGE ERROR STRUCTURE

Most of the results in this chapter have been published by this author in [2].

In this chapter, we establish the limiting distribution of the normalized Fourier coefficients of the $Y-$ process obeying the near unit root model

$$Y_{ij}(N) = \alpha_N Y_{i-1,j} + \beta_N Y_{i,j-1} - \alpha_N \beta_N Y_{i-1,j-1} + \mu_{ij}, \tag{3.1}$$

where μ_{ij} is a first order moving average of the form $\mu_{ij} = \theta_N \epsilon_{i-1,j} + \epsilon_{ij}$, $\theta_N \to -1$, $N^\rho \left(1 + \dfrac{\theta_N}{\alpha_N}\right) \longrightarrow 1$ as $N \longrightarrow \infty$, for some $0 < \rho < \dfrac{1}{2}$ and $1 \le i, j \le N$.

<u>Assumptions.</u>

The following assumptions are made about the $Y-$ process

(A.1) $Y_{ij} = \mu_{ij} = \epsilon_{ij} = 0$ whenever $i \wedge j \le 0$

(A.2) $\alpha_N = e^{a/N}$, $\beta_N = e^{b/N}$

(A.3) $\{\epsilon_{ij} : i, j \ge 0\}$ is an independent and identically distributed , mean zero, finite variance sequence

The main theorem of this chapter is stated and proved below.

Theorem 3.0.1. *[2] Suppose that the $Y-$ process obeys model (3.1), and (A.1)-(A.3). Then* $\frac{1}{N^{3-\rho}}(A_N, B_N) \xrightarrow{D} (A, B)$ *as* $N \longrightarrow \infty$ *on* $\mathbb{R}^2$*, where*

$$A = \int_{[0,1]^2} \cos 2\pi(x+y)\ J(x,y)\ dxdy$$

and

$$B = \int_{[0,1]^2} \sin 2\pi(x+y)\ J(x,y)\ dxdy.$$

Proof. Using (A.1) and iterating , $Y_{kl} = \sum_{i,j=1}^{k,l} \alpha_N^{k-i}\beta_N^{l-j}\mu_{ij} = \sum_{i,j=1}^{k,l} \alpha_N^{k-i}\beta_N^{l-j}(\epsilon_{ij} + \theta_N\epsilon_{i-1,j})$. The second equality is due to the fact that $\mu_{ij} = \theta_N\epsilon_{i-1,j} + \epsilon_{ij}$. Thus the Fourier coefficient A_N is given by

$$\begin{aligned} A_N &= \sum_{k,l=1}^{N} \cos\frac{2\pi}{N}(k+l)Y_{kl} \\ &= \sum_{k,l=1}^{N}\sum_{i,j=1}^{k,l} \cos\frac{2\pi}{N}(k+l)\alpha_N^{k-i}\beta_N^{l-j}\epsilon_{ij} \\ &+ \theta_N\sum_{k,l=1}^{N}\sum_{i,j=1}^{k,l} \cos\frac{2\pi}{N}(k+l)\alpha_N^{k-i}\beta_N^{l-j}\epsilon_{i-1,j}. \end{aligned}$$

Define $V_N = \sum_{k,l=1}^{N}\sum_{i,j=1}^{k,l} \cos\frac{2\pi}{N}(k+l)\alpha_N^{k-i}\beta_N^{l-j}\epsilon_{ij}$. Then

$$\begin{aligned} A_N =& V_N + \frac{\theta_N}{\alpha_N}\sum_{k,l=1}^{N}\sum_{i,j=1}^{k,l} \cos\frac{2\pi}{N}(k+l)\alpha_N^{k-i+1}\beta_N^{l-j}\epsilon_{i-1,j} \\ &= V_N + \frac{\theta_N}{\alpha_N}\sum_{k,l=1}^{N}\sum_{i=0,j=1}^{k-1,l} \cos\frac{2\pi}{N}(k+l)\alpha_N^{k-i}\beta_N^{l-j}\epsilon_{ij} \\ &= V_N + \frac{\theta_N}{\alpha_N}V_N + \frac{\theta_N}{\alpha_N}\sum_{k,l=1}^{N}\sum_{j=1}^{l}\cos\frac{2\pi}{N}(k+l)\alpha_N^{k}\beta_N^{l-j}\epsilon_{0j} \\ &- \frac{\theta_N}{\alpha_N}\sum_{k,l=1}^{N}\sum_{j=1}^{l}\cos\frac{2\pi}{N}(k+l)\beta_N^{l-j}\epsilon_{kj} = \left(1+\frac{\theta_N}{\alpha_N}\right)V_N - W_N, \end{aligned}$$

where $\frac{\theta_N}{\alpha_N}\sum_{k,l=1}^{N}\sum_{j=1}^{l}\cos\frac{2\pi}{N}(k+l)\alpha_N^k\beta_N^{l-j}\epsilon_{0j}=0$, since $\epsilon_{0j}=0$, and
$W_N=\frac{\theta_N}{\alpha_N}\sum_{k,l=1}^{N}\sum_{j=1}^{l}\cos\frac{2\pi}{N}(k+l)\beta_N^{l-j}\epsilon_{kj}$.
Write $W_N=\frac{\theta_N}{\alpha_N}\sum_{k=1}^{N}Z_{Nk}$, where $Z_{Nk}=\sum_{l=1}^{N}\sum_{j=1}^{l}\cos\frac{2\pi}{N}(k+l)\beta_N^{l-j}\epsilon_{kj}$; then
$\text{Var } W_N=\frac{\theta_N^2}{\alpha_N^2}\sum_{k=1}^{N}\text{Var } Z_{Nk}$ since $\{Z_{Nk}: 1\le k\le N\}$ is a set of independent random variables.
Note that $\text{Var } Z_{Nk}=\sum_{l_1,l_2=1}^{N}\text{cov}\left(\sum_{j=1}^{l_1}\cos\frac{2\pi}{N}(k+l_1)\beta_N^{l_1-j}\epsilon_{kj},\sum_{j=1}^{l_2}\cos\frac{2\pi}{N}(k+l_2)\beta_N^{l_2-j}\epsilon_{kj}\right)=$
$\sum_{l_1,l_2=1}^{N}\sum_{j=1}^{l_1\wedge l_2}\cos\frac{2\pi}{N}(k+l_1)\cos\frac{2\pi}{N}(k+l_2)\beta_N^{l_1-j}\beta_N^{l_2-j}\sigma^2\le M\sum_{l_1,l_2=1}^{N}l_1\wedge l_2=O(N^3)$. This implies that $\text{Var } W_N=\frac{\theta_N^2}{\alpha_N^2}\sum_{k=1}^{N}\text{Var } Z_{Nk}=O(N^4)$ and thus $W_N=O_p(N^2)$.

As shown above, $A_N=\left(1+\frac{\theta_N}{\alpha_N}\right)V_N-W_N$ and thus $\frac{1}{N^{3-\rho}}A_N=N^\rho\left(1+\frac{\theta_N}{\alpha_N}\right)\frac{1}{N^3}V_N-\frac{1}{N^{3-\rho}}W_N$ for some $0<\rho<\frac{1}{2}$. Since $W_N=O_p(N^2)$, $\frac{1}{N^{3-\rho}}W_N=o_p(1)$. Under assumption (A.3), we know from Lemma 2.0.1 that, $\frac{1}{N^3}V_N\xrightarrow{D}A$ as $N\to\infty$ on $\mathbb{R}$. Note that, $N^\rho\left(1+\frac{\theta_N}{\alpha_N}\right)\longrightarrow 1$ as $N\longrightarrow\infty$ and thus it follows that $\frac{1}{N^{3-\rho}}A_N\xrightarrow{D}A$ as $N\longrightarrow\infty$ on $\mathbb{R}$. Likewise $\frac{1}{N^{3-\rho}}B_N\xrightarrow{D}B$ as $N\longrightarrow\infty$ on $\mathbb{R}$. According to Lemma 2.0.1,

$$\frac{1}{N^3}\left(\sum_{k,l=1}^{N}\sum_{i,j=1}^{k,l}\left(\cos\frac{2\pi}{N}(k+l)\alpha_N^{k-i}\beta_N^{l-j}\epsilon_{ij},\sin\frac{2\pi}{N}(k+l)\alpha_N^{k-i}\beta_N^{l-j}\epsilon_{ij}\right)\right)\xrightarrow{D}(A,B)$$

on $\mathbb{R}^2$. Denote $V_N'=\sum_{k,l=1}^{N}\sum_{i,j=1}^{k,l}\sin\frac{2\pi}{N}(k+l)\alpha_N^{k-i}\beta_N^{l-j}\epsilon_{ij}$ and $W_N'=\frac{\theta_N}{\alpha_N}\sum_{k,l=1}^{N}\sum_{j=1}^{l}\sin\frac{2\pi}{N}(k+l)\beta_N^{l-j}\epsilon_{kj}$. Using above equations,
$\frac{1}{N^{3-\rho}}(A_N,B_N)=\frac{1}{N^{3-\rho}}\left(\left(1+\frac{\theta_N}{\gamma_N}\right)V_N-W_N,\left(1+\frac{\theta_N}{\gamma_N}\right)V_N'-W_N'\right)=$
$N^\rho\left(1+\frac{\theta_N}{\alpha_N}\right)\frac{1}{N^3}(V_N,V_N')-\frac{1}{N^{3-\rho}}(W_N,W_N')$. Since $N^\rho\left(1+\frac{\theta_N}{\alpha_N}\right)\longrightarrow 1$
and $\frac{1}{N^{3-\rho}}W_N=o_p(1)$, $\frac{1}{N^3}(V_N,V_N')\xrightarrow{D}(A,B)$ on $\mathbb{R}^2$ implies that $\frac{1}{N^{3-\rho}}(A_N,B_N)\xrightarrow{D}(A,B)$

as $N \longrightarrow \infty$ on $\mathbb{R}^2$. $\square$

3.1 Results on the boundary

Next we give a normalizing constants $\chi(N)$ and $\psi(N)$ in terms of $a = b = N$ and show that $\chi(N)\ A \xrightarrow{D} N(0,1)$, $\psi(N)\ B \xrightarrow{D} N(0,1)$, and $(\chi(N)A, \psi(N)B) \xrightarrow{D} N(\underline{0}, \Sigma_1)$ as $N \longrightarrow \infty$, where $\Sigma_1 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$.

Suppose that the assumptions made in Theorem 3.0.1 are fulfilled. Then $\dfrac{1}{N^{3-\rho}}(A_N, B_N) \xrightarrow{D}$ (A, B) as $N \longrightarrow \infty$ on $\mathbb{R}^2$, where

$$A = \int\limits_{[0,1]^2} \cos 2\pi(x+y)\ J(x,y)\ dxdy$$

and

$$B = \int\limits_{[0,1]^2} \sin 2\pi(x+y)\ J(x,y)\ dxdy.$$

Recall that $\operatorname{cov}\left(J(u,v), J(s,t)\right) = \dfrac{\left(e^{(u+s)a} - e^{|u-s|a}\right)}{2a} \cdot \dfrac{\left(e^{(v+t)b} - e^{|v-t|b}\right)}{2b}$. Assume that $a = b$. Then

$$\operatorname{cov}(A, B) = \int\limits_{[0,1]^4} \cos 2\pi(u+v) \sin 2\pi(s+t) \operatorname{cov}\left(J(u,v), J(s,t)\right)\ du\ dv\ ds\ dt.$$

Using Mathematica with $\theta = 2\pi$, one obtains

$$\text{Var A} = \frac{1}{2a^2\left(a^2+4\pi^2\right)^4}\Big((-12e^a+14e^{2a}-8e^{3a}+2e^{4a}+5)a^6-2(7e^a-6e^{2a}+2e^{3a}-3)a^5$$
$$+2(2(8\pi^2-1)e^a+32\pi^2e^{3a}-8\pi^2e^{4a}+(1-40\pi^2)e^{2a}+4\pi^2+1)a^4$$
$$+16\pi^4(-12e^a+14e^{2a}-8e^{3a}+2e^{4a}+5)a^2+32\pi^4(7e^a-6e^{2a}+2e^{3a}-3)a$$
$$+32\pi^4(e^a-1)^2)\Big).$$

$$\text{Var B} = \frac{1}{2a\left(a^2+4\pi^2\right)^4}\left(\left(-4e^a+2e^{2a}+3\right)a^5-2\left(e^a-1\right)a^4+\right.$$
$$8\pi^2\left(-20e^a+26e^{2a}-16e^{3a}+4e^{4a}+7\right)a^3$$
$$\left.+16\pi^2\left(\left(2\pi^2-1\right)e^{2a}+\left(2-4\pi^2\right)e^a+3\pi^2-1\right)a+32\pi^4\left(e^a-1\right)\right).$$

$$\text{cov}\,(A,B) = \frac{1}{(a^2+4\pi^2)^4}\left(4\pi\left(e^a-1\right)^3\left(-\left(e^a-1\right)a^3+a^2+4\pi^2\left(e^a-1\right)a+4\pi^2\right)\right).$$

Define $f(a) \sim g(a)$ provided $\dfrac{f(a)}{g(a)} \longrightarrow 1$ as $a \longrightarrow \infty$. Then one has Var $A \sim \dfrac{e^{4a}}{a^4}$, Var $B \sim \dfrac{4\theta^2e^{4a}}{a^6}$, and $\text{cov}(A,B) \sim \dfrac{-2\theta e^{4a}}{a^5}$. Denote $\chi(a) = \dfrac{a^2}{e^{2a}}$, $\psi(a) = \dfrac{a^3}{2\theta e^{2a}}$ and it follows that Var $\chi(N)\ A \longrightarrow 1$, Var $\psi(N)\ B \longrightarrow 1$, and $\text{cov}\left(\chi(N)\ A, \psi(N)\ B\right) \sim \dfrac{N^2}{e^{2N}}\cdot\dfrac{N^3}{2\theta e^{2N}}\cdot\dfrac{-2\theta e^{4N}}{N^5} = -1$ as $N \longrightarrow \infty$.

It follows that $\left(\dfrac{N^2}{e^{2N}}A, \dfrac{N^3}{2\theta e^{2N}}B\right) \xrightarrow{D} N(\underline{0},\Sigma_1)$ as $N \longrightarrow \infty$, where $\Sigma_1 = \begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$.

CHAPTER 4: UNIT ROOTS TEST: SPATIAL MODEL WITH AUTOREGRESSIVE ERROR STRUCTURE

Most of the results in this chapter have been published by this author in [2].

Two error models are studied in this chapter. It is shown that the normalizing factors needed to ensure convergence in distribution of the sequence of Fourier coefficients differ in each of these two cases. The following lemmas are needed to prove the two main theorems in this chapter.

Lemma 4.0.1. *Let* $X_N(x,y) = \dfrac{1}{N}\sum_{i,j=1}^{[Nt_1],[Nt_2]} \gamma_N^{[Nt_1]-i}\epsilon_{ij}$, $\underline{t} \in [0,1]^2$. *Then* $\Delta X_N = \dfrac{1}{N}\epsilon_{ij} + \dfrac{(\gamma_N - 1)}{N}\sum_{k=1}^{i-1}\gamma_N^{i-1-k}\epsilon_{kj}$.

Proof. Recall that $\Delta X_N = X_N\Big(\dfrac{i}{N},\dfrac{j}{N}\Big) - X_N\Big(\dfrac{i-1}{N},\dfrac{j}{N}\Big) - X_N\Big(\dfrac{i}{N},\dfrac{j-1}{N}\Big) + X_N\Big(\dfrac{i-1}{N},\dfrac{j-1}{N}\Big)$

$$
\begin{aligned}
X_N\Big(\frac{i}{N},\frac{j}{N}\Big) - X_N\Big(\frac{i-1}{N},\frac{j}{N}\Big) &= \frac{1}{N}\sum_{k,l=1}^{i,j}\gamma_N^{i-k}\epsilon_{kl} - \frac{1}{N}\sum_{k,l=1}^{i-1,j}\gamma_N^{i-1-k}\epsilon_{kl} \\
&= \frac{1}{N}\left(\gamma_N\sum_{k,l=1}^{i,j}\gamma_N^{i-1-k}\epsilon_{kl} - \sum_{k,l=1}^{i-1,j}\gamma_N^{i-1-k}\epsilon_{kl}\right) \\
&= \frac{1}{N}\left(\sum_{l=1}^{j}\epsilon_{il} + (\gamma_N - 1)\sum_{k,l=1}^{i-1,j}\gamma_N^{i-1-k}\epsilon_{kl}\right)
\end{aligned}
$$

Likewise,

$$
X_N\Big(\frac{i}{N},\frac{j-1}{N}\Big) + X_N\Big(\frac{i-1}{N},\frac{j-1}{N}\Big) = \frac{1}{N}\left(\sum_{l=1}^{j-1}\epsilon_{il} + (\gamma_N - 1)\sum_{k,l=1}^{i-1,j-1}\gamma_N^{i-1-k}\epsilon_{kl}\right).
$$

Thus

$$\begin{aligned}\Delta X_N =&\frac{1}{N}\left(\sum_{l=1}^{j}\epsilon_{il}+(\gamma_N-1)\sum_{k,l=1}^{i-1,j}\gamma_N^{i-1-k}\epsilon_{kl}\right)-\frac{1}{N}\left(\sum_{l=1}^{j-1}\epsilon_{il}+(\gamma_N-1)\sum_{k,l=1}^{i-1,j-1}\gamma_N^{i-1-k}\epsilon_{kl}\right)\\ =&\frac{1}{N}\epsilon_{ij}+\frac{(\gamma_N-1)}{N}\sum_{k=1}^{i-1}\gamma_N^{i-1-k}\epsilon_{kj}\end{aligned}$$

□

Lemma 4.0.2. *Let* $Q_N(\underline{t}) = \frac{\gamma_N\delta_N}{N}\sum_{i,j=1}^{[Nt_1],[Nt_2]}\gamma_N^{[Nt_1]-i}\delta_N^{[Nt_2]-j}\epsilon_{ij}$, $\underline{t}\in[0,1]^2$. *Then* $\Delta Q_N = \frac{1}{N}(\gamma_N-1)(\delta_N-1)\sum_{k,l=1}^{i,j}\gamma_N^{i-k}\delta_N^{j-l}\epsilon_{kl}+\frac{1}{N}(\gamma_N-1)\sum_{k=1}^{i}\gamma_N^{i-k}\epsilon_{kj}+\frac{1}{N}(\delta_N-1)\sum_{l=1}^{j}\delta_N^{j-l}\epsilon_{il}+\frac{1}{N}\epsilon_{ij}$.

Proof. Recall that

$$\Delta Q_N = \left(Q_N\Big(\frac{i}{N},\frac{j}{N}\Big)-Q_N\Big(\frac{i-1}{N},\frac{j}{N}\Big)\right)-\left(Q_N\Big(\frac{i}{N},\frac{j-1}{N}\Big)-Q_N\Big(\frac{i-1}{N},\frac{j-1}{N}\Big)\right).$$

$$\begin{aligned}Q_N\Big(\frac{i}{N},\frac{j}{N}\Big)-Q_N\Big(\frac{i-1}{N},\frac{j}{N}\Big) =&\frac{\gamma_N\delta_N}{N}\sum_{k,l=1}^{i,j}\gamma_N^{i-k}\delta_N^{j-l}\epsilon_{kl}-\frac{\gamma_N\delta_N}{N}\sum_{k,l=1}^{i-1,j}\gamma_N^{i-1-k}\delta_N^{j-l}\epsilon_{kl}\\ =&\frac{\gamma_N\delta_N}{N}\sum_{k,l=1}^{i,j}\gamma_N^{i-k}\delta_N^{j-l}\epsilon_{kl}-\frac{\gamma_N\delta_N}{N\gamma_N}\sum_{k,l=1}^{i-1,j}\gamma_N^{i-k}\delta_N^{j-l}\epsilon_{kl}\\ =&\frac{\gamma_N\delta_N}{N}\left(1-\frac{1}{\gamma_N}\right)\sum_{k,l=1}^{i,j}\gamma_N^{i-k}\delta_N^{j-l}\epsilon_{kl}+\frac{\gamma_N\delta_N}{N\gamma_N}\sum_{l=1}^{j}\delta_N^{j-l}\epsilon_{il}.\end{aligned}$$

Likewise,

$$\begin{aligned}Q_N\Big(\frac{i}{N},\frac{j-1}{N}\Big)-Q_N\Big(\frac{i-1}{N},\frac{j-1}{N}\Big) =&\frac{\gamma_N\delta_N}{N}\left(1-\frac{1}{\gamma_N}\right)\frac{1}{\delta_N}\sum_{k,l=1}^{i,j-1}\gamma_N^{i-k}\delta_N^{j-l}\epsilon_{kl}\\ &+\frac{\gamma_N\delta_N}{N\gamma_N\delta_N}\sum_{l=1}^{j-1}\delta_N^{j-l}\epsilon_{il}\end{aligned}$$

Thus

$$\begin{aligned}
\Delta Q_N = &\left(\frac{\gamma_N\delta_N}{N}\left(1-\frac{1}{\gamma_N}\right)\sum_{k,l=1}^{i,j}\gamma_N^{i-k}\delta_N^{j-l}\epsilon_{kl} + \frac{\gamma_N\delta_N}{N\gamma_N}\sum_{l=1}^{j}\delta_N^{j-l}\epsilon_{il}\right) - \\
&\left(\frac{\gamma_N\delta_N}{N}\left(1-\frac{1}{\gamma_N}\right)\frac{1}{\delta_N}\sum_{k,l=1}^{i,j-1}\gamma_N^{i-k}\delta_N^{j-l}\epsilon_{kl} + \frac{\gamma_N\delta_N}{N\gamma_N\delta_N}\sum_{l=1}^{j-1}\delta_N^{j-l}\epsilon_{il}\right) \\
&= \frac{\gamma_N\delta_N}{N}\left(1-\frac{1}{\gamma_N}\right)\left(1-\frac{1}{\delta_N}\right)\sum_{k,l=1}^{i,j}\gamma_N^{i-k}\delta_N^{j-l}\epsilon_{kl} + \frac{\gamma_N\delta_N}{N}\left(1-\frac{1}{\gamma_N}\right)\frac{1}{\delta_N}\sum_{k=1}^{i}\gamma_N^{i-k}\epsilon_{kj} \\
&+ \frac{\gamma_N\delta_N}{N\gamma_N}\left(1-\frac{1}{\delta_N}\right)\sum_{l=1}^{j}\delta_N^{j-l}\epsilon_{il} + \frac{\delta_N\gamma_N}{N\delta_N\gamma_N}\epsilon_{ij} \\
&= \frac{1}{N}(\gamma_N-1)(\delta_N-1)\sum_{k,l=1}^{i,j}\gamma_N^{i-k}\delta_N^{j-l}\epsilon_{kl} + \frac{1}{N}(\gamma_N-1)\sum_{k=1}^{i}\gamma_N^{i-k}\epsilon_{kj} \\
&+ \frac{1}{N}(\delta_N-1)\sum_{l=1}^{j}\delta_N^{j-l}\epsilon_{il} + \frac{1}{N}\epsilon_{ij}.
\end{aligned}$$

□

Now consider the $Y-$ process obeying the near unit root model

$$Y_{ij} = \alpha_N Y_{i-1,j} + \beta_N Y_{i,j-1} - \alpha_N\beta_N Y_{i-1,j-1} + \mu_{ij}, \tag{4.1}$$

where $\mu_{ij} = \gamma_N\mu_{i-1,j} + \epsilon_{ij}$, $\gamma_N = e^{c/N}$, c is a parameter, and $1 \le i,j \le N$.

Observe that the error term μ_{ij} is assumed to be a first order autoregressive model.

<u>Assumptions.</u>

As before, the following assumptions are made about the $Y-$ process

(A.1) $Y_{ij} = \mu_{ij} = \epsilon_{ij} = 0$ whenever $i \wedge j \le 0$

(A.2) $\alpha_N = e^{a/N}$, $\beta_N = e^{b/N}$

(A.3) $\{\epsilon_{ij} : i, j \geq 0\}$ is an independent and identically distributed , mean zero, finite variance sequence

Our aim is to extend the method used in the proof of Theorem 2.0.2 , where $U_N(\underline{t}) = \frac{1}{N}\sum_{i,j=1}^{[Nt_1],[Nt_2]} \epsilon_{ij} \xrightarrow{D} W(\underline{t})$ on D_2 and ϵ_{ij} denotes the error term. Observe that in this case $\Delta U_N = \frac{\epsilon_{ij}}{N}$. Extending this idea, an attempt is made here to find a process $X_N(\underline{t}),\ \underline{t} \in [0,1]^2$, such that ΔX_N approximates the model error term $\frac{\mu_{ij}}{N}$.

Now, let us consider $\mu_{ij} = \gamma_N \mu_{i-1,j} + \epsilon_{i,j}$. Iterating and employing (A.1), $\mu_{ij} = \sum_{k=1}^{i} \gamma_N^{i-k}\epsilon_{kj}$. Denote $U_N(\underline{t}) = \frac{1}{N}\sum_{i,j=1}^{[Nt_1],[Nt_2]} \epsilon_{ij}$ and define $X_N(\underline{t}) = \frac{\gamma_N}{N}\sum_{i,j=1}^{[Nt_1],[Nt_2]} (\gamma_N^{[Nt_1]-i} - 1)\epsilon_{ij},\ \underline{t} \in [0,1]^2$. It follows from Lemma 4.0.1 that $\Delta X_N = \frac{1}{N}(\gamma_N - 1)\sum_{k=1}^{i} \gamma_N^{i-k}\epsilon_{kj} - \frac{1}{N}(\gamma_N - 1)\epsilon_{ij} = \frac{(\gamma_N - 1)}{N}\mu_{ij} - \frac{1}{N}(\gamma_N - 1)\epsilon_{ij}$.

Now we are ready to state and prove one of the theorems. The following lemma is used to prove Theorem 4.0.4.

Lemma 4.0.3. *[2] Suppose the model satisfies (A.1)-(A.3) and* $\mu_{ij} = \gamma_N \mu_{i-1,j} + \epsilon_{ij}$. *Denote* $X_N(\underline{t}) = \frac{\gamma_N}{N}\sum_{i,j=1}^{[Nt_1],[Nt_2]} (\gamma_N^{[Nt_1]-i} - 1)\epsilon_{ij}$*; then* $X_N(\underline{t}) \xrightarrow{D} cK(\underline{t})$ *on* D_2*, where* $K(\underline{t})$ *is defined in (N.6).*

Proof. Note that

$$
\begin{aligned}
X_N(\underline{t}) =& \frac{\gamma_N}{N}\sum_{i,j=1}^{[Nt_1],[Nt_2]} (\gamma_N^{[Nt_1]-i} - 1)\epsilon_{ij} \\
=& \gamma_N \sum_{i,j=1}^{[Nt_1],[Nt_2]} (\gamma_N^{[Nt_1]-i} - 1)\int_{E_{ijN}} d\ U_N(x,y) \\
=& \gamma_N \sum_{i,j=1}^{[Nt_1],[Nt_2]} \int_{E_{ijN}} (\gamma_N^{[Nt_1]-i} - 1)\ d\ U_N(x,y)
\end{aligned}
$$

$$= \gamma_N \sum_{i,j=1}^{[Nt_1],[Nt_2]} \int\limits_{E_{ijN}} \left[e^{c(t_1-x)} - 1 + O\left(\frac{1}{N}\right)\right] \, d\,U_N(x,y)$$

$$= \gamma_N \int\limits_{E_{t_1t_2}} (e^{c(t_1-x)} - 1) \, d\,U_N(x,y) + o_p(1)$$

$$= \gamma_N \int\limits_{E_{t_1t_2}} e^{c(t_1-x)} \, d\,U_N(x,y) - \gamma_N U_N(\underline{t}) + o_p(1).$$

Integrating by parts, $X_N(\underline{t}) = \gamma_N U_N(\underline{t}) + c\gamma_N \int\limits_{E_{t_1}} e^{c(t_1-x)} \; U_N(x,t_2) \; dx - \gamma_N U_N(\underline{t}) + o_p(1)$.
Hence $X_N(\underline{t}) \xrightarrow{D} c \int\limits_{E_{t_1}} e^{c(t_1-x)} W(x,t_2) \; dx = \; cK(\underline{t})$ since $U_N \xrightarrow{D} W$ on D_2. □

The first main theorem is stated and proved below.

Theorem 4.0.4. *[2] Assume that the $Y-$ process obeys model (4.1), and (A.1)-(A.3). Then* $\frac{1}{N^4}(A_N, B_N) \xrightarrow{D} (A, B)$ *as* $N \to \infty$ *on* $\mathbb{R}^2$*, where*

$$A = \int\limits_{[0,1]^2} \cos 2\pi(t_1 + t_2) \; M(K(\underline{t})) \; d\underline{t}$$

and

$$B = \int\limits_{[0,1]^2} \sin 2\pi(t_1 + t_2) \; M(K(\underline{t})) \; d\underline{t}.$$

Proof. First, it is shown that $\frac{1}{N^4} A_N \xrightarrow{D} A$ on $\mathbb{R}$. Define $K(\underline{t}) = \int\limits_{E_{t_1}} e^{c(t_1-x)} W(x,t_2) \; dx$ and

$$Z_N(\underline{t}) = \frac{1}{N} \cos \frac{2\pi}{N}([Nt_1] + [Nt_2]) \sum_{i,j=1}^{[Nt_1],[Nt_2]} \alpha_N^{[Nt_1]-i} \beta_N^{[Nt_2]-j} \left((\gamma_N - 1)\mu_{ij} - (\gamma_N - 1)\epsilon_{ij}\right).$$

It was shown in Lemma 4.0.3 that $X_N(\underline{t}) \xrightarrow{D} \; cK(\underline{t})$ on D_2 and, moreover

$\Delta X_N = \dfrac{(\gamma_N - 1)}{N}\mu_{ij} - \dfrac{1}{N}(\gamma_N - 1)\epsilon_{ij}$. Hence

$$\begin{aligned} Z_N(\underline{t}) =& \cos\frac{2\pi}{N}([Nt_1]+[Nt_2]) \sum_{i,j=1}^{[Nt_1],[Nt_2]} \alpha_N^{[Nt_1]-i}\beta_N^{[Nt_2]-j} \int\limits_{E_{ijN}} d\ X_N(x,y) \\ &= \cos 2\pi(t_1+t_2) \sum_{i,j=1}^{[Nt_1],[Nt_2]} \int\limits_{E_{ijN}} \left[e^{a(t_1-x)}e^{b(t_2-y)} + O\left(\frac{1}{N}\right)\right] \ dX_N(x,y) \\ &= \cos 2\pi(t_1+t_2) \int\limits_{E_{t_1t_2}} e^{a(t_1-x)}e^{b(t_2-y)} \ dX_N(x,y) + o_p(1) \end{aligned}$$

since $X_N \xrightarrow{D} cK$ in D_2. Integrating by parts,

$$\begin{aligned} Z_N(\underline{t}) =& \cos 2\pi(t_1+t_2)[X_N(\underline{t}) + a\int\limits_{E_{t_1}} e^{a(t_1-x)}X_N(x,t_2)dx \\ &+ b\int\limits_{E_{t_2}} e^{b(t_2-y)}X_N(t_1,y)dy \\ &+ ab\int\limits_{E_{t_1t_2}} e^{a(t_1-x)}e^{b(t_2-y)}X_N(x,y)dxdy] + o_p(1). \end{aligned}$$

Hence

$$\begin{aligned} Z_N(\underline{t}) \xrightarrow{D} c\ \cos 2\pi(t_1+t_2)[K(\underline{t}) + a\int\limits_{E_{t_1}} e^{a(t_1-x)}K(x,t_2)dx + b\int\limits_{E_{t_2}} e^{b(t_2-y)}K(t_1,y)dy \\ + ab\int\limits_{E_{t_1t_2}} e^{a(t_1-x)}e^{b(t_2-y)}K(x,y)dxdy]. \end{aligned}$$

Therefore $Z_N(\underline{t}) \xrightarrow{D}\ c\ \cos 2\pi(t_1+t_2)M(K(\underline{t}))$ and thus $\int\limits_{[0,1]^2} Z_N(\underline{t})\ d\underline{t} \xrightarrow{D} c \int\limits_{[0,1]^2} \cos 2\pi(t_1+t_2)M(K(\underline{t}))\ d\underline{t}$ as $N \longrightarrow \infty$ on $\mathbb{R}$. Further ,

$$\int\limits_{[0,1]^2} Z_N(\underline{t})d\underline{t} = \frac{1}{N} \sum_{i,j=1}^{[Nt_1],[Nt_2]} \int\limits_{[0,1]^2} \cos\frac{2\pi}{N}([Nt_1]+[Nt_2])\alpha_N^{[Nt_1]-i}\beta_N^{[Nt_2]-j}$$
$$\Big((\gamma_N-1)\mu_{ij}-(\gamma_N-1)\epsilon_{ij}\Big)\ d\underline{t}$$
$$= \frac{1}{N}\sum_{k,l=1}^{N}\int_{E_{klN}} \sum_{i,j=1}^{[Nt_1],[Nt_2]} \cos\frac{2\pi}{N}([Nt_1]+[Nt_2])\alpha_N^{[Nt_1]-i}\beta_N^{[Nt_2]-j}$$
$$((\gamma_N-1)\mu_{ij}-(\gamma_N-1)\epsilon_{ij})\ \ d\underline{t}$$
$$= \frac{1}{N^3}\sum_{k,l=2}^{N}\sum_{i,j=1}^{k-1,l-1}\cos\frac{2\pi}{N}(k+l-2)\alpha_N^{k-1-i}\beta_N^{l-1-j}\left((\gamma_N-1)\mu_{ij}-(\gamma_N-1)\epsilon_{ij}\right)$$
$$= \frac{1}{N^3}\sum_{k,l=1}^{N-1}\sum_{i,j=1}^{k,l}\cos\frac{2\pi}{N}(k+l)\alpha_N^{k-i}\beta_N^{l-j}\left((\gamma_N-1)\mu_{ij}-(\gamma_N-1)\epsilon_{ij}\right).$$

Thus,

$$\int\limits_{[0,1]^2} Z_N(\underline{t})d\underline{t} = \Big(\frac{\gamma_N-1}{N^3}\Big)A_{N-1} - \Big(\frac{\gamma_N-1}{N^3}\Big)\sum_{k,l=1}^{N-1}\sum_{i,j=1}^{k,l}\cos\frac{2\pi}{N}(k+l)\alpha_N^{k-i}\beta_N^{l-j}\epsilon_{ij}.$$

Since $\Big(\frac{\gamma_N-1}{N^3}\Big) = \frac{c}{N^4}(1+o(1))$ and $\frac{1}{N^3}\sum_{k,l=1}^{N}\sum_{i,j=1}^{k,l}\cos\frac{2\pi}{N}(k+l)\alpha_N^{k-i}\beta_N^{l-j}\epsilon_{ij} \xrightarrow{D} \int\limits_{[0,1]^2}\cos 2\pi(t_1+t_2)J(\underline{t})\ d\underline{t}$, it follows that $\frac{c}{N^4}\ A_N \xrightarrow{D}\ c\int\limits_{[0,1]^2}\cos 2\pi(t_1+t_2)M(K(\underline{t}))\ d\underline{t}$.
Likewise, $\frac{c}{N^4}B_N \xrightarrow{D} c\int\limits_{[0,1]^2}\sin 2\pi(t_1+t_2)M(K(\underline{t}))\ d\underline{t}$ as $N\longrightarrow\infty$ on $\mathbb{R}$. An application of the Cramer-Wold device shows that $\frac{1}{N^4}(A_N,B_N)\xrightarrow{D}(A,B)$ on $\mathbb{R}^2$. □

In the next theorem, we consider a model which has an autoregressive error structure in both i and j components. In other words, consider the $Y-$ process obeying the near unit root model

$$Y_{ij}(N) = \alpha_N Y_{i-1,j} + \beta_N Y_{i,j-1} - \alpha_N \beta_N Y_{i-1,j-1} + \mu_{ij}, \tag{4.2}$$

where $\mu_{ij} = \gamma_N \mu_{i-1,j} + \delta_N \mu_{i,j-1} - \gamma_N \delta_N \mu_{i-1,j-1} + \epsilon_{ij},\ \ 1 \leq i,j \leq N,\ \gamma_N = e^{c/N},\ \delta_N = e^{d/N}$ and c and d are parameters.

Assumptions.

Just as before, the same assumptions are considered here. The $Y-$ process obeys

(A.1) $Y_{ij} = \mu_{ij} = \epsilon_{ij} = 0$ whenever $i \wedge j \leq 0$

(A.2) $\alpha_N = e^{a/N},\ \beta_N = e^{b/N}$

(A.3) $\{\epsilon_{ij} : i,j \geq 0\}$ is an independent and identically distributed , mean zero, finite variance sequence

Now let $\mu_{ij} = \gamma_N \mu_{i-1,j} + \delta_N \mu_{i,j-1} - \gamma_N \delta_N \mu_{i-1,j-1} + \epsilon_{ij}$. Employing (A.1), $\mu_{ij} = \sum\limits_{k,l=1}^{i,j} \gamma_N^{i-k} \delta_N^{j-l} \epsilon_{kl}$, and in this case define

$$\begin{aligned} X_N(\underline{t}) = & \frac{\gamma_N \delta_N}{N} \sum_{i,j=1}^{[Nt_1],[Nt_2]} (\gamma_N^{[Nt_1]-i} \delta_N^{[Nt_2]-j} - 1)\epsilon_{ij} \\ & - \frac{\gamma_N}{N} \sum_{i,j=1}^{[Nt_1],[Nt_2]} (\gamma_N^{[Nt_1]-i} - 1)\epsilon_{ij} - \frac{\delta_N}{N} \sum_{i,j=1}^{[Nt_1],[Nt_2]} (\delta_N^{[Nt_2]-j} - 1)\epsilon_{ij}. \end{aligned}$$

Using Lemma 4.0.2 and simplifying, $\Delta X_N = \frac{1}{N}(1-\gamma_N)(1-\delta_N)\mu_{ij} - \frac{(\gamma_N \delta_N - 1)}{N}\epsilon_{ij} - \frac{(1-\gamma_N)}{N}\epsilon_{ij} - \frac{(1-\delta_N)}{N}\epsilon_{ij}$ and thus $\int\limits_{E_{ijN}} d\ X_N(x,y) = \Delta X_N$.

This next lemma establishes the convergence of $X_N(\underline{t})$ defined above.

Lemma 4.0.5. *[2] Assume that the model obeys (A.1)-(A.3) and* $\mu_{ij} = \gamma_N \mu_{i-1,j} + \delta_N \mu_{i,j-1} - \gamma_N \delta_N \mu_{i-1,j-1} + \epsilon_{ij}$. *Define*

$$\begin{aligned} X_N(\underline{t}) =& \frac{\gamma_N \delta_N}{N} \sum_{i,j=1}^{[Nt_1],[Nt_2]} (\gamma_N^{[Nt_1]-i} \delta_N^{[Nt_2]-j} - 1)\epsilon_{ij} \\ &- \frac{\gamma_N}{N} \sum_{i,j=1}^{[Nt_1],[Nt_2]} (\gamma_N^{[Nt_1]-i} - 1)\epsilon_{ij} - \frac{\delta_N}{N} \sum_{i,j=1}^{[Nt_1],[Nt_2]} (\delta_N^{[Nt_2]-j} - 1)\epsilon_{ij}; \end{aligned}$$

then $X_N(\underline{t}) \xrightarrow{D} cd\ L(\underline{t})$ *as* $N \to \infty$ *on* D_2*, where* $L(\underline{t})$ *is defined in (N.7).*

Proof. Using the notations defined above,

$$\begin{aligned} X_N(\underline{t}) =& \gamma_N \delta_N \sum_{i,j=1}^{[Nt_1],[Nt_2]} (\gamma_N^{[Nt_1]-i} \delta_N^{[Nt_2]-j} - 1) \int_{E_{ijN}} d\ U_N(x,y) \\ &- \gamma_N \sum_{i,j=1}^{[Nt_1],[Nt_2]} (\gamma_N^{[Nt_1]-i} - 1) \int_{E_{ijN}} d\ U_N(x,y) \\ &- \delta_N \sum_{i,j=1}^{[Nt_1],[Nt_2]} (\delta_N^{[Nt_2]-j} - 1) \int_{E_{ijN}} d\ U_N(x,y) \\ &= \gamma_N \delta_N \int_{E_{t_1 t_2}} (e^{c(t_1-x)} e^{d(t_2-y)} - 1)\ d\ U_N(x,y) - \gamma_N \int_{E_{t_1 t_2}} (e^{c(t_1-x)} - 1)\ d\ U_N(x,y) \\ &- \delta_N \int_{E_{t_1 t_2}} (e^{d(t_2-x)} - 1)\ d\ U_N(x,y) + o_p(1) \\ &= \gamma_N \delta_N \int_{E_{t_1 t_2}} e^{c(t_1-x)} e^{d(t_2-y)}\ d\ U_N(x,y) - \gamma_N \int_{E_{t_1 t_2}} e^{c(t_1-x)}\ d\ U_N(x,y) \\ &- \delta_N \int_{E_{t_1 t_2}} e^{d(t_2-x)}\ d\ U_N(x,y) - \gamma_N \delta_N U_N(\underline{t}) + \gamma_N U_N(\underline{t}) + \delta_N U_N(\underline{t}) + o_p(1). \end{aligned}$$

Integrating by parts,

$$X_N(\underline{t}) = \gamma_N\delta_N[U_N(\underline{t}) + c\int\limits_{E_{t_1}} e^{c(t_1-x)}U_N(x,t_2)dx + d\int\limits_{E_{t_2}} e^{d(t_2-y)}U_N(t_1,y)dy$$

$$+ cd\int\limits_{E_{t_1t_2}} e^{c(t_1-x)}e^{d(t_2-y)}U_N(x,y)dxdy] - \gamma_N[U_N(\underline{t}) + c\int\limits_{E_{t_1}} e^{c(t_1-x)}U_N(x,t_2)dx]$$

$$- \delta_N[U_N(\underline{t}) + d\int\limits_{E_{t_2}} e^{d(t_2-y)}U_N(t_1,y)dy] - \gamma_N\delta_N U_N(\underline{t}) + \gamma_N U_N(\underline{t}) + \delta_N U_N(\underline{t}) + o_p(1)$$

$$= \gamma_N\delta_N\Big[c\int\limits_{E_{t_1}} e^{c(t_1-x)}U_N(x,t_2)dx + d\int\limits_{E_{t_2}} e^{d(t_2-y)}U_N(t_1,y)dy$$

$$+ cd\int\limits_{E_{t_1t_2}} e^{c(t_1-x)}e^{d(t_2-y)}U_N(x,y)dxdy\Big] - \gamma_N c\int\limits_{E_{t_1}} e^{c(t_1-x)}U_N(x,t_2)dx$$

$$- \delta_N d\int\limits_{E_{t_2}} e^{d(t_2-y)}U_N(t_1,y)dy + o_p(1)$$

$$= \gamma_N\delta_N cd\int\limits_{E_{t_1t_2}} e^{c(t_1-x)}e^{d(t_2-y)}U_N(x,y)dxdy + \gamma_N(\delta_N-1)c\int\limits_{E_{t_1}} e^{c(t_1-x)}U_N(x,t_2)dx$$

$$+ \delta_N(\gamma_N-1)d\int\limits_{E_{t_2}} e^{d(t_2-y)}U_N(t_1,y)dy + o_p(1).$$

Since $U_N \xrightarrow{D} W$ in D_2 and $\gamma_N(\delta_N-1) = O(\frac{1}{N})$, $\delta_N(\gamma_N-1) = O(\frac{1}{N})$, it follows that $X_N(\underline{t}) \xrightarrow{D} cd\ L(\underline{t})$ as $N \longrightarrow \infty$ on D_2. $\square$

The second theorem is stated and proved below.

Theorem 4.0.6. *[2] If the $Y-$ process satisfies model (4.2), and (A.1)-(A.3). Then* $\frac{1}{N^5}(A_N, B_N) \xrightarrow{D} (A,B)$ *as* $N \longrightarrow \infty$ *on* $\mathbb{R}^2$*, where*

$$A = \int\limits_{[0,1]^2} \cos 2\pi(t_1+t_2)\ M(L(\underline{t}))\ d\underline{t}$$

and

$$B = \int\limits_{[0,1]^2} \sin 2\pi(t_1+t_2)\ M(L(\underline{t}))\ d\underline{t}.$$

Proof. Recall that with $X_N(\underline{t})$ defined in Lemma 4.0.5, we obtain that

$$\Delta X_N = \frac{1}{N}(1-\gamma_N)(1-\delta_N)\mu_{ij} + \frac{(1-\gamma_N\delta_N)}{N}\epsilon_{ij} - \frac{(1-\gamma_N)}{N}\epsilon_{ij} - \frac{(1-\delta_N)}{N}\epsilon_{ij}$$

by Lemma 4.0.2.

Define $Z_N(\underline{t}) = \cos\frac{2\pi}{N}([Nt_1]+[Nt_2]) \sum_{i,j=1}^{[Nt_1],[Nt_2]} \alpha_N^{[Nt_1]-i}\beta_N^{[Nt_2]-j}\Delta X_N$; then

$$\begin{aligned} Z_N(\underline{t}) &= \cos\frac{2\pi}{N}([Nt_1]+[Nt_2]) \sum_{i,j=1}^{[Nt_1],[Nt_2]} \alpha_N^{[Nt_1]-i}\beta_N^{[Nt_2]-j} \int\limits_{E_{ijN}} d\ X_N(x,y) \\ &= \cos 2\pi(t_1+t_2) \int\limits_{E_{t_1t_2}} e^{a(t_1-x)}e^{b(t_2-y)}\ dX_N(x,y) + o_p(1). \end{aligned}$$

Integrating by parts,

$$\begin{aligned} Z_N(\underline{t}) &= \cos 2\pi(t_1+t_2)[X_N(\underline{t}) + a\int\limits_{E_{t_1}} e^{a(t_1-x)}X_N(x,t_2)dx \\ &\quad + b\int\limits_{E_{t_2}} e^{b(t_2-y)}X_N(t_1,y)dy \\ &\quad + ab\int\limits_{E_{t_1t_2}} e^{a(t_1-x)}e^{b(t_2-y)}X_N(x,y)dxdy] + o_p(1). \end{aligned}$$

Hence using Lemma 4.0.5, we obtain that $Z_N(\underline{t}) \xrightarrow{D} cd\ \cos 2\pi(t_1+t_2)M(L(\underline{t}))$ and since integration is continuous, we get $\int\limits_{[0,1]^2} Z_N(\underline{t})\ d\underline{t} \xrightarrow{D} cd \int\limits_{[0,1]^2} \cos 2\pi(t_1+t_2)M(L(\underline{t}))\ d\underline{t}$ as $N \longrightarrow \infty$ on $\mathbb{R}$.

Further,

$$\int\limits_{[0,1]^2} Z_N(\underline{t})d\underline{t} = \sum_{k,l=1}^{N} \int\limits_{E_{klN}} \sum_{i,j=1}^{[Nt_1],[Nt_2]} \cos\frac{2\pi}{N}([Nt_1]+[Nt_2])\alpha_N^{[Nt_1]-i}\beta_N^{[Nt_2]-j}\Delta X_N \; d\underline{t}$$
$$= \frac{1}{N^2}\sum_{k,l=1}^{N-1}\sum_{i,j=1}^{k,l} \cos\frac{2\pi}{N}(k+l)\alpha_N^{k-i}\beta_N^{l-j}\Delta X_N$$
$$= \frac{(\gamma_N-1)(\delta_N-1)}{N^3}A_{N-1} + o_p(1).$$

Moreover, $\dfrac{(\gamma_N-1)(\delta_N-1)}{N^3} = \dfrac{cd}{N^5}(1+o(1))$, and thus

$$\frac{cd}{N^5}A_N \xrightarrow{D} cd \int\limits_{[0,1]^2} \cos 2\pi(t_1+t_2)M(L(\underline{t}))\; d\underline{t}$$

as $N \longrightarrow \infty$ on $\mathbb{R}$.

Likewise,

$$\frac{cd}{N^5}B_N \xrightarrow{D} cd \int\limits_{[0,1]^2} \sin 2\pi(t_1+t_2)M(L(\underline{t}))\; d\underline{t}$$

on $\mathbb{R}$, and application of the Cramer-Wold device shows that $\dfrac{1}{N^5}(A_N, B_N) \xrightarrow{D} (A,B)$ on $\mathbb{R}$. $\square$

Just as in the case of the moving average errors, we verify some results on the boundary. Suppose that the hypothesis listed in Theorem 4.0.4 hold. Due to difficulty in computing variances, choose $a=b=0$. Only the parameter c in the error structure remains. In this case, Var A, Var B, and $\operatorname{cov}(A,B)$ are given in Chapter 6 below. It follows that Var $A \sim \dfrac{e^{2c}}{16\pi^2c^5}$, Var $B \sim \dfrac{3e^{2c}}{16\pi^2c^5}$, and $\operatorname{cov}(A,B) \sim \dfrac{e^{2c}}{4\pi c^6}$. Define $\phi(c) = \dfrac{c^{5/2}}{e^c}$ and note that Var $\phi(N)\ A \longrightarrow \dfrac{1}{16\pi^2}$, Var $\phi(N)\ B \longrightarrow \dfrac{3}{16\pi^2}$, and $\operatorname{cov}(\phi(N)\ A, \phi(N)\ B) \sim \phi^2(N)\dfrac{e^{2N}}{4\pi N^6} = \dfrac{1}{4\pi N} \longrightarrow 0$ as

$N \longrightarrow \infty$. It follows that $\dfrac{N^{5/2}}{e^N}(A, B) \xrightarrow{D} N(\underline{0}, \Sigma)$ as $N \longrightarrow \infty$, where $\Sigma = \begin{bmatrix} \dfrac{1}{16\pi^2} & 0 \\ 0 & \dfrac{3}{16\pi^2} \end{bmatrix}$.

CHAPTER 5: VERIFICATION OF FUNCTIONAL CENTRAL LIMIT THEOREM AND EXAMPLES

Donsker's Theorem 1951 is known as the **functional central limit theorem** since it extends the central limit theorem to random variables taking values in the Skorohod space $D[0,1]$. Sufficient conditions for a specific class of random variables taking values in D_2 and obeying the functional central theorem are discussed in this section.

The following assumptions are made on the error structure $\{\epsilon_{ij} : i, j \in \mathbb{Z}\}$ with autocovariance function γ :

(B.1) $\{\epsilon_{ij} : i, j \in \mathbb{Z}\}$ is a second order, mean zero, stationary Gaussian process

(B.2) $\gamma(i,j) = \gamma(i,-j)$ for all $i, j \in \mathbb{Z}$

(B.3) there exists a $d = (d_1, d_2), 0 < d_1, d_2 < \frac{1}{2}$, and $b \neq 0$ such that

(i) $\sum\limits_{k=1}^{N} \sum\limits_{i=1}^{k} \gamma(i, 0) = O(N^{2d_1+1})$ as $N \longrightarrow \infty, j \geq 0$ fixed

(ii) $\sum\limits_{l=1}^{N} \sum\limits_{j=1}^{l} \gamma(0, j) = O(N^{2d_2+1})$ as $N \longrightarrow \infty, i \geq 0$ fixed

(iii) $\sum\limits_{k,l=1}^{M,N} \sum\limits_{i,j=1}^{k,l} \gamma(i,j) \sim bM^{2d_1+1}N^{2d_2+1}$ as $M \wedge N \longrightarrow \infty$.

Sufficient conditions for assumption (B.3) to hold are given below.

Lemma 5.0.1. *Given that* $0 < d_1, d_2 < \dfrac{1}{2}$, *let* γ *denote the covariance function of a second order, mean zero, stationary process* $\{\epsilon_{ij} : i, j \in \mathbb{Z}\}$. *Assume that* γ *possesses the following properties:*

(G.1) $\gamma(i,j) \sim e_j(i^{2d_1-1})$ *as* $i \longrightarrow \infty$*, for each fixed* $j \geq 0$

(G.2) $\gamma(i,j) \sim f_i(j^{2d_2-1})$ *as* $j \longrightarrow \infty$*, for each fixed* $i \geq 0$

(G.3) $\gamma(i,j) \sim b i^{2d_1-1} j^{2d_2-1}$ *as* $i \wedge j \longrightarrow \infty, b \neq 0.$

Then γ *obeys condition (B.3) given above.*

Proof. Denote $b_{ij} = |b| i^{2d_1-1} j^{2d_2-1}, i \geq 1, j \geq 1$. It follows from (G.1) that $\sum_{k=1}^{N} \sum_{i=1}^{k} \gamma(i,j) \sim e_j \sum_{k=1}^{N} \sum_{i=1}^{k} i^{2d_1-1} \sim \dfrac{e_j N^{2d_1+1}}{2d_1(2d_1+1)}$ as $N \longrightarrow \infty$, for each fixed $j \geq 0$. Then (i) and (ii) of (B.3) are satisfied. It remains to verify (iii) of (B.3). Let $a_{ij} = |\gamma(i,j)|$ for $i \geq 1, j \geq 1$. First, it is shown that $\sum_{i,j=1}^{k,l} a_{ij} \sim \sum_{i,j=1}^{k,l} b_{ij}$ as $k \wedge l \longrightarrow \infty$. Given $0 < \delta < 1$, according to (G.3), there exists $c_0 > 0$ such that $1 - \delta < \dfrac{a_{ij}}{b_{ij}} < 1 + \delta$, and thus $(1-\delta) \sum_{i,j=c_0}^{k,l} b_{ij} < \sum_{i,j=c_0}^{k,l} a_{ij} < (1+\delta) \sum_{i,j=c_0}^{k,l} b_{ij}$ for all $i \wedge j \geq c_0$. Moreover, employing (G.1)-(G.3), $(1-\delta) < \sum_{i,j=1}^{k,l} a_{ij} \Big/ \sum_{i,j=c_0}^{k,l} b_{ij} < (1+\delta)$ for all k,l sufficiently large. Similarly,
$\dfrac{1}{1+\delta} < \sum_{i,j=c_0}^{k,l} b_{ij} \Big/ \sum_{i,j=1}^{k,l} a_{ij} < \dfrac{1}{1-\delta}$, and thus
$\dfrac{1}{1+\delta} < \sum_{i,j=1}^{k,l} b_{ij} \Big/ \sum_{i,j=1}^{k,l} a_{ij} < \dfrac{1}{1-\delta}$ for all $k \wedge l$ sufficiently large.
Therefore, $A_{kl} = \sum_{i,j=1}^{k,l} a_{ij} \sim \sum_{i,j=1}^{k,l} b_{ij} = B_{kl}$ as $k \wedge l \longrightarrow \infty$.

Again, given $\delta > 0$, there exist c_0 such that $1 - \delta < \dfrac{A_{kl}}{B_{kl}} < 1 + \delta$, and thus $(1-\delta) \sum_{k,l=c_0}^{M,N} B_{kl} < \sum_{k,l=c_0}^{M,N} A_{kl} < (1+\delta) \sum_{k,l=c_0}^{M,N} B_{kl}$ for all $M \wedge N \geq c_0$. Observe that $\sum_{k=1}^{M} A_{k1} = \sum_{k=1}^{M} \sum_{i=1}^{k} a_{i1} \sim e_1 \sum_{k=1}^{M} \sum_{i=1}^{k} i^{2d_1-1} \sim \dfrac{e_1 M^{2d_1+1}}{2d_1(2d_1+1)}$ as $M \longrightarrow \infty$, and thus it follows that $\sum_{k=1}^{M} A_{k1} \Big/ \sum_{k,l=c_0}^{M,N} B_{kl} \longrightarrow 0$ as $M \wedge N \longrightarrow \infty$. Continuing this process, $1 - \delta < \sum_{k,l=1}^{M,N} A_{kl} \Big/ \sum_{k,l=c_0}^{M,N} B_{kl} < 1 + \delta$ for all $M \wedge N$ sufficiently large. Likewise, $\dfrac{1}{1+\delta} < \sum_{k,l=1}^{M,N} B_{kl} \Big/ \sum_{k,l=c_0}^{M,N} A_{kl} < \dfrac{1}{1-\delta}$ for all $M \wedge N$ sufficiently large, and thus $\sum_{k,l=1}^{M,N} A_{kl} \sim \sum_{k,l=1}^{M,N} B_{kl} \sim \dfrac{|b|M^{2d_1+1}N^{2d_2+1}}{4d_1d_2(2d_1+1)(2d_2+1)}$ as $M \wedge N \longrightarrow \infty$. Hence condition (B.3) is valid. $\square$

This next result is an extension of Donsker's theorem from the time series context. Theorem 5.0.2 below is used to prove this specific functional central limit theorem in the spatial setting.

Theorem 5.0.2. *(Bickel and Wichura, 1971)[9]. Suppose that $\{V_N : N \geq 1\}$ is a sequence of random elements in D_2 which vanishes on the lower boundary of $[0,1]^2$, and let V be another random element in D_2. Moreover, assume that*

(i) the finite-dimensional distributions of $\{V_N\}$ converges in distribution to those of V

(ii) there exist constants $\gamma_1, \gamma_1, \beta_1, \beta_2$ and a finite measure μ on $[0,1]^2$ having continuous marginals such that for each pair $(\underline{s}, \underline{t}]$ and $(\underline{p}, \underline{q}]$ of neighbors,

$$E[|V_N(\underline{s}, \underline{t}]|^{\gamma_1}|V_N(\underline{p}, \underline{q}]|^{\gamma_2}] \leq (\mu(\underline{s}, \underline{t}])^{\beta_1}(\mu(\underline{p}, \underline{q}])^{\beta_2},$$

for all $N \geq 1$, where $\gamma_1 + \gamma_2 > 0$ and $\beta_1 + \beta_2 > 1$.

Then $V_N \xrightarrow{D} V$ in D_2.

Theorem 5.0.3. *Assume that the $\epsilon-$process obeys axioms (B.1)-(B.3) listed above, and define*

$$X_N(\underline{t}) = \frac{1}{N^{d_1+d_2+1}} \sum_{i,j=1}^{[Nt_1],[Nt_2]} \epsilon_{ij},\ \underline{t} \in [0,1]^2,$$

where $0 \leq d_i < \dfrac{1}{2}$, $i = 1, 2$. Then $X_N \xrightarrow{D} W_d$ in D_2, where W_d is a fractional Brownian sheet with constant $c = b$.

Proof. Given $d = (d_1, d_2)$, $0 < d_1, d_2 < \dfrac{1}{2}$. Denote $e_1 = 2d_1+1$ and $e_2 = 2d_2+1$ and let $X_N(\underline{t})$ be defined as above. Theorem 5.0.2 is used to show that $X_N \xrightarrow{D} W_d$ in D_2. First, it is shown that the finite-dimensional distributions of $\{X_N(s,t) : s, t \in I\}$ converge in distribution. Let $V_N = (X_N(s_1,t_1), X_N(s_1,t_2), \ldots, X_N(s_a,t_b))$ be a random vector in $\mathbb{R}^{ab}$. Since V_N is a mean zero, Gaussian random vector, its characteristic function is $\Phi_{V_N}(\theta_{11}, \theta_{12}, \ldots, \theta_{ab}) = e^{-\frac{1}{2}\underline{\theta}'\Sigma_N\underline{\theta}}$, where $\Sigma_N = \mathrm{Var}V_N$. Then $\{V_N\}$ converge in distribution to $N(\underline{0}, \Sigma)$ iff $\Sigma_N \longrightarrow \Sigma$ as $N \longrightarrow \infty$. In particular, it must be shown that $\{\mathrm{cov}(X_N(s_1,t_1), X_N(s_2,t_2)\}$ converges as $N \longrightarrow \infty$. It follows from (B.2) that

$$
\begin{aligned}
\sum_{i,i'=1}^{N_1}\sum_{j,j'=1}^{N_2}\gamma(i-i', j-j') &= N_1N_2\gamma(0,0) + 2N_1\sum_{l=1}^{N_2-1}\sum_{j=1}^{l}\gamma(0,j) \\
&\quad + 2N_2\sum_{k=1}^{N_1-1}\sum_{i=1}^{k}\gamma(i,0) + 4\sum_{k=1}^{N_1-1}\sum_{l=1}^{N_2-1}\sum_{i=1}^{k}\sum_{j=1}^{l}\gamma(i,j).
\end{aligned} \tag{5.1}
$$

Employing (B.3) and the assumption that $0 < d_1, d_2 < \dfrac{1}{2}$,

$$
\begin{aligned}
\frac{1}{N^{e_1+e_2}}\sum_{i,i'=1}^{[Ns]}\sum_{j,j'=1}^{[Nt]}\gamma(i-i', j-j') &= \frac{[Ns]^{e_1}[Nt]^{e_2}}{N^{e_1+e_2}}\cdot \\
&\quad \frac{1}{[Ns]^{e_1}}\frac{1}{[Nt]^{e_2}}\sum_{i,i'=1}^{[Ns]}\sum_{j,j'=1}^{[Nt]}\gamma(i-i', j-j') \longrightarrow 4bs^{e_1}t^{e_2}
\end{aligned} \tag{5.2}
$$

as $N \longrightarrow \infty$.

Observe that in order to apply (5.2), the upper bounds for i, i' (j, j') must be equal, respectively.

Given (s_1, t_1) and $(s_2, t_2) \in [0,1]^2$; assume that $s_1 \leq s_2$ and $t_1 \leq t_2$. Applying (B.1) and (B.2),

$$\begin{aligned} N^{e_1+e_2}\mathrm{cov}(X_N(s_1,t_1), X_N(s_2,t_2)) &= \sum_{i,i'=1}^{[Ns_1],[Ns_2]} \sum_{j,j'=1}^{[Nt_1],[Nt_2]} \gamma(i-i', j-j') \\ &= \frac{1}{2}\Bigg[\sum_{i,i'=1}^{[Ns_1]} \sum_{j,j'=1}^{[Nt_1],[Nt_2]} \gamma(i-i', j-j') \\ &+ \sum_{i,i'=1}^{[Ns_2]} \sum_{j,j'=1}^{[Nt_1],[Nt_2]} \gamma(i-i', j-j') \\ &- \sum_{i,i'=1}^{[Ns_2]-[Ns_1]} \sum_{j,j'=1}^{[Nt_1],[Nt_2]} \gamma(i-i', j-j') \Bigg] \\ &=: \frac{1}{2}(J_N + K_N + L_N). \end{aligned} \tag{5.3}$$

Likewise,

$$J_N = \frac{1}{2}\Bigg[\sum_{i,i'=1}^{[Ns_1]} \sum_{j,j'=1}^{[Nt_1]} \gamma(i-i', j-j') + \sum_{i,i'=1}^{[Ns_1]} \sum_{j,j'=1}^{[Nt_2]} \gamma(i-i', j-j') - \sum_{i,i'=1}^{[Ns_1]} \sum_{j,j'=1}^{[Nt_2]-[Nt_1]} \gamma(i-i', j-j') \Bigg],$$

and thus it follows from (5.2) that,

$$\frac{1}{N^{e_1+e_2}} J_N \longrightarrow 2b\Big[s_1^{e_1} t_1^{e_2} + s_1^{e_1} t_2^{e_2} - s_1^{e_1}(t_2 - t_1)^{e_2} \Big] \text{ as } N \longrightarrow \infty.$$

A similar argument shows that $\frac{1}{N^{e_1+e_2}} K_N \longrightarrow 2b\Big[s_2^{e_1} t_1^{e_2} + s_2^{e_1} t_2^{e_2} - s_2^{e_1}(t_2 - t_1)^{e_2} \Big]$ and

$$\frac{1}{N^{e_1+e_2}} L_N \longrightarrow 2b\Big[(s_2 - s_1)^{e_1} t_1^{e_2} + (s_2 - s_1)^{e_1} t_2^{e_2} - (s_2 - s_1)^{e_1}(t_2 - t_1)^{e_2} \Big] \text{ as } N \longrightarrow \infty.$$

Combining these results with (5.3), we obtain

$$
\begin{aligned}
\operatorname{cov}(X_N(s_1,t_1),X_N(s_2,t_2)) = \frac{1}{2N^{e_1+e_2}}(J_N+K_N-L_N) \longrightarrow \\
b[s_1^{e_1}t_1^{e_2}+s_1^{e_1}t_2^{e_2}-s_1^{e_1}(t_2-t_1)^{e_2} \\
+s_2^{e_1}t_1^{e_2}+s_2^{e_1}t_2^{e_2}-s_2^{e_1}(t_2-t_1)^{e_2} \\
-(s_2-s_1)^{e_1}t_1^{e_2}-(s_2-s_1)^{e_1}t_2^{e_2}+(s_2-s_1)^{e_1}(t_2-t_1)^{e_2}] \\
= b[s_1^{e_1}+s_2^{e_1}-(s_2-s_1)^{e_1}]\cdot[t_1^{e_1}+t_2^{e_2}-(t_2-t_1)^{e_2}]
\end{aligned}
$$

as $N \longrightarrow \infty$, whenever $s_1 \leq s_2$ and $t_1 \leq t_2$.

A similar argument is valid for the other orderings, and thus it follows that the finite-dimensional distributions of $\{X_N\}$ converge in distribution to those of W_d.

It remains to verify that $\{X_N\}$ satisfies the tightness condition listed in Theorem 5.0.2 (ii). Assume that $(\underline{s},\underline{t}]$ and $(\underline{p},\underline{q}]$ are neighbors in $[0,1]^2$, where
$\underline{s}=(s_1,s_2)$, $\underline{t}=(t_1,t_2)$, $\underline{p}=(p_1,p_2)$ and $\underline{q}=(q_1,q_2)$. Suppose that the line segment joining $\underline{p}$ and $\underline{t}$ is the common boundary of the neighbors as shown in Figure 5.1 below.

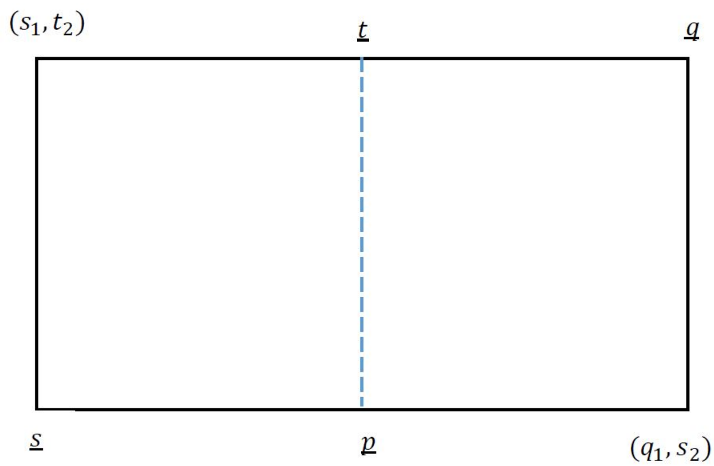

Figure 5.1: Increment

Observe that the increment of X_N over $(\underline{s}, \underline{t}]$ is $\frac{1}{N^{d_1+d_2+1}} \sum_{i=[Ns_1]+1}^{[Nt_1]} \sum_{j=[Ns_2]+1}^{[Nt_2]} \epsilon_{ij}$, and thus by the strict stationarity of the $\epsilon-$process,

$$X_N(\underline{s}, \underline{t}] \overset{D}{=} \frac{1}{N^{d_1+d_2+1}} \sum_{i=1}^{[Nt_1]-[Ns_1]} \sum_{j=1}^{[Nt_2]-[Ns_2]} \epsilon_{ij}.$$

Similarly, $X_N(\underline{p}, \underline{q}] \overset{D}{=} \frac{1}{N^{d_1+d_2+1}} \sum_{i=1}^{[Nq_1]-[Np_1]} \sum_{j=1}^{[Nq_2]-[Np_2]} \epsilon_{ij}$, and thus it follows from Cauchy's inequality that $E|X_N(\underline{s}, \underline{t}] \cdot X_N(\underline{p}, \underline{q}]| \leq \left(\mathrm{Var}X_N(\underline{s}, \underline{t}] \cdot \mathrm{Var}X_N(\underline{p}, \underline{q}]\right)^{\frac{1}{2}}$.

Employing (5.2) and the boundedness of (B.3), there exists an $M_1 > 0$ such that for all $N \geq 1$, $\sum_{i,i'=1}^{[Nt_1]-[Ns_1]} \sum_{j,j'=1}^{[Nt_2]-[Ns_2]} \gamma(i-i', j-j') \leq M_1([Nt_1]-[Ns_1])^{e_1}([Nt_2]-[Ns_2])^{e_2}$. Hence $\mathrm{Var}X_N(\underline{s}, \underline{t}] \leq M_1\left(\frac{[Nt_1]-[Ns_1]}{N}\right)^{e_1}\left(\frac{[Nt_2]-[Ns_2]}{N}\right)^{e_2}$ for all $N \geq 1$.

According to Bickel and Wichura (1971, p.1665)[9], it suffices to verify Theorem 5.0.2(ii) for each $T_N = \{(\frac{k}{N}, \frac{l}{N}) : 0 \leq k, l \leq N, k, l \text{ integers}\}$. However, if $\underline{s}, \underline{t} \in T_N$, then

$\frac{[Nt_1]-[Ns_1]}{N} = t_1 - s_1$, and thus,

$$\mathrm{Var} X_N(\underline{s}, \underline{t}] \leq M_1(t_1 - s_1)^{e_1}(t_2 - s_2)^{e_2} \leq M_1(\lambda(\underline{s}, \underline{t}])^{e_1 \wedge e_2},$$

where λ denotes the Lebesgue measure on $[0,1]^2$. It follows that there exists an $M > 0$ such that $E|X_N(\underline{s}, \underline{t}] \cdot X_N(\underline{p}, \underline{q}]| \leq M\Big[\lambda(\underline{s}, \underline{t}] \cdot \lambda(\underline{p}, \underline{q}]\Big]^{\frac{e_1 \wedge e_2}{2}}$, for all $N \geq 1$. Since $e_1 \wedge e_2 > 1$, Theorem 5.0.2(ii) is satisfied, and thus $X_N \xrightarrow{D} W_d$ in D_2. $\square$

5.1 Example

An illustration of an error process which satisfies (B.1)-(B.3) is given below.

The results in this section have been published by this author in [1].

Example 5.1.1. [1] Assume that $\{\delta_{ij} : i, j \in \mathbb{Z}\}$ is a two sided sequence of i.i.d. random variables with $\delta_{ij} \sim N(0,1)$. Further, suppose that $\{a_i : i \geq 0\}$ and $\{b_j : j \geq 0\}$ are two sequences of real numbers for which $\sum_{i=0}^{\infty} a_i^2 < \infty$ and $\sum_{j=0}^{\infty} b_j^2 < \infty$. For each integer $t \geq 0$, denote $S_t = \sum_{i,j=0}^{t} a_i b_j \delta_{m-i,n-j}$ and define $\epsilon_{mn} := \lim_{t \longrightarrow \infty} S_t = \sum_{i,j=0}^{\infty} a_i b_j \delta_{m-i,n-j}$, $m, n \in \mathbb{Z}$. It is shown below that the series $\sum_{i,j=0}^{\infty} a_i b_j \delta_{m-i,n-j}$ converges almost surely and thus ϵ_{mn} is well-defined. For each $i \geq 0$, $j \geq 0$, denote $X_{ij} = \delta_{m-i,n-j}$ and define $\mathcal{F}_t = \sigma(X_{ij} : 0 \leq i, j \leq t)$, $t \geq 0$. Observe that $E[S_{t+1} \mid \mathcal{F}_t] = S_t + E[\Big(\sum_{j=0}^{t} a_{t+1} b_j X_{t+1,j} + \sum_{i=0}^{t} a_i b_{t+1} X_{i,t+1} + a_{t+1} b_{t+1} X_{t+1,t+1}\Big) \mid \mathcal{F}_t] = S_t$ and thus $(S_t, \mathcal{F}_t, t \geq 0)$ is a martingale. Moreover, $E(S_t^2) \leq \sum_{i=0}^{\infty} a_i^2 \cdot \sum_{j=0}^{\infty} b_j^2 < \infty$ for each $t \geq 0$. Then $(S_t, \mathcal{F}_t, t \geq 0)$ is an L^2-bounded martingale and hence $S_t \longrightarrow \epsilon_{mn}$ almost surely and in L^2. Since each S_t is normally distributed, it follows that $\{\epsilon_{mn} : m, n \in \mathbb{Z}\}$ is a Gaussian process.

The $\epsilon-$ process is also stationary. Indeed, assume that k and l are fixed integers; it suffices to verify that $\operatorname{cov}(\epsilon_{m+k,n+l}, \epsilon_{mn})$ depends only on k, l, for all $m, n \in \mathbb{Z}$.

Shifting indices $i \longrightarrow i-k$, $j \longrightarrow j-l$,

$$\begin{aligned}\epsilon_{mn} &= \sum_{i,j=0}^{\infty} a_i b_j \delta_{m-i,n-j} \\ &= \sum_{i=k}^{\infty}\sum_{j=l}^{\infty} a_{i-k} b_{j-l} \delta_{m+k-i,n+l-j}.\end{aligned}$$

Since $\epsilon_{m+k,n+l} = \sum_{i,j=0}^{\infty} a_i b_j \delta_{m+k-i,n+l-j}$, it follows that

$$\operatorname{cov}(\epsilon_{m+k,n+l}, \epsilon_{mn}) = \sum_{i=0\vee k}^{\infty}\sum_{j=0\vee l}^{\infty} a_i a_{i-k} b_j b_{j-l},$$

which depends only on k and l.

Hence the $\epsilon-$ process is stationary.

It is shown that for each $m, n \in \mathbb{Z}$, $\gamma(-m, n) = \gamma(m, n)$.

Note that $\epsilon_{-m,n} = \sum_{i,j=0}^{\infty} a_i b_j \delta_{-m-i,n-j}$, $\epsilon_{0,0} = \sum_{i,j=0}^{\infty} a_i b_j \delta_{-i,-j}$, and shifting indices $i \longrightarrow i+m$, $j \longrightarrow j-n$ gives $\epsilon_{0,0} = \sum_{i=-m}^{\infty}\sum_{j=n}^{\infty} a_{i+m} b_{j-n} \delta_{-m-i,n-j}$. Hence

$$\operatorname{cov}(\epsilon_{-m,n}, \epsilon_{0,0}) = \sum_{i=0\vee -m}^{\infty}\sum_{j=0\vee n}^{\infty} a_{i+m} a_i b_{j-n} b_j,$$

and replacing $-m$ with m gives

$$\operatorname{cov}(\epsilon_{mn}, \epsilon_{0,0}) = \sum_{i=0\vee m}^{\infty}\sum_{j=0\vee n}^{\infty} a_{i-m} a_i b_{j-n} b_j.$$

Shifting indices $i \longrightarrow i+m$,

$$\begin{aligned}\operatorname{cov}(\epsilon_{mn}, \epsilon_{0,0}) &= \sum_{i=(0\vee m)-m}^{\infty} \sum_{j=0\vee n}^{\infty} a_i a_{i+m} b_{j-n} b_j \\ &= \sum_{i=0\vee -m}^{\infty} \sum_{j=0\vee n}^{\infty} a_i a_{i+m} b_{j-n} b_j.\end{aligned}$$

Therefore $\gamma(-m,n) = \gamma(m,n)$ for all $m,n \in \mathbb{Z}$. This implies that

$$\gamma(m,n) = \gamma(-m,n) = \gamma(m,-n) = \gamma(-m,-n)$$

for all $m,n \in \mathbb{Z}$.

Particular choices for $\{a_i\}$ and $\{b_j\}$ are as follows:

Choose $a_i = i^{d_1-1}$ and $b_j = j^{d_2-1}$, where $0 < d_1, d_2 < \frac{1}{2}, i \geq 1, j \geq 1$, and define $a_0 = b_0 = 1$.

As shown above, for $m \geq 0, n \geq 0$,

$$\begin{aligned}\gamma(m,n) &= \sum_{i=0\vee m}^{\infty} \sum_{j=0\vee n}^{\infty} a_{i-m} a_i b_{j-n} b_j \\ &= \sum_{i,j=0}^{\infty} a_i a_{i+m} b_j b_{j+n} \\ &= \sum_{i=0}^{\infty} a_i a_{i+m} \cdot \sum_{j=0}^{\infty} b_j b_{j+n}.\end{aligned}$$

According to Whitt (2002, p. 124)[37], $\sum_{i=1}^{\infty} i^{d_1-1}(i+m)^{d_1-1} \sim c_1 m^{2d_1-1}$ as $m \longrightarrow \infty$, and thus $\gamma(m,n) \sim C m^{2d_1-1} n^{2d_2-1}$ as $m \wedge n \longrightarrow \infty$.

Likewise, if $n \geq 0$ is fixed, $\gamma(m,n) = \sum_{i=0}^{\infty} a_i a_{i+m} \cdot \sum_{j=0}^{\infty} b_j b_{j+n} \sim e_n m^{2d_1-1}$ as $m \longrightarrow \infty$. Similarly, if $m \geq 0$ is fixed, $\gamma(m,n) \sim f_m n^{2d_2-1}$ as $n \longrightarrow \infty$. This shows that the ϵ-process above obeys assumptions (G.1)-(G.3). In particular, whenever $\{a_k\}$ and $\{b_k\}$ are chosen as above,

$$\frac{1}{N^{d_1+d_2+1}} \sum_{i,j=1}^{[Nt_1],[Nt_2]} \sum_{k,l=0}^{\infty} a_k b_l \delta_{i-k,j-l} \xrightarrow{D} W_d$$

in D_2 as $N \longrightarrow \infty$. $\square$

Remark 5.1.2. It should be mentioned that the proofs of Theorem 1.2.2, Theorem 1.2.3, and Theorem 1.2.4 are valid under assumptions weaker than (A.3). The proofs given of Theorems 1.2.3 and Theorem 1.2.4 are valid whenever (A.3) is replaced by $(A.3)'$: $\{\epsilon_{ij} : i,\ j \geq 0\}$ is a mean zero, finite variance sequence which satisfies $\frac{1}{N}\sum_{i,j=1}^{[N\underline{t}]} \epsilon_{ij} \xrightarrow{D} W(\underline{t})$ in D_2. In addition to $(A.3)'$, the assumption that $\{\epsilon_{ij} : i,\ j \geq 0\}$ is an uncorrelated sequence is needed in the proof of Theorem 1.2.2.

Example 5.1.3. [2] Based on Remark 5.1.2, an example is given to illustrate that the conclusions of Theorem 1.2.3 and 1.2.4 may still be valid whenever (A.3) fails. Let $\{\delta_{ij} : i, j \in \mathbb{Z}\}$ denote an i.i.d., mean zero sequence obeying $E|\delta_{ij}|^p < \infty$ for some $p > 4$. Choose any sequence $\{a_{kl} : k, l \in \mathbb{Z}\}$ of real numbers satisfying $\sum_{k,l\in\mathbb{Z}} |a_{kl}| < \infty$. Define the linear sequence $\epsilon_{kl} = \sum_{i,j\in\mathbb{Z}} a_{ij}\delta_{k-i,l-j},\ k \geq 1,\ l \geq 1$. It follows from Theorem 2(i) of Machkouri , Volný, and Wu (2013) [18] that $\{\epsilon_{kl} : k, l \in \mathbb{Z}\}$ obeys $\frac{1}{N}\sum_{i,j=1}^{[N\underline{t}]} \epsilon_{ij} \xrightarrow{D} \sigma^2 W(\underline{t})$ on D_2. In particular, $\frac{1}{N}\sum_{i,j=1}^{[N\underline{t}]} \frac{1}{\sigma^2}\epsilon_{ij} \xrightarrow{D} W(\underline{t})$. According to Remark 5.1.2, the conclusions of Theorem 1.2.3 and Theorem 1.2.4 are valid even though $\{\epsilon_{ij}/\sigma^2 : i,\ j \geq 0\}$ may not be i.i.d. Example 5.1.1 is a special case of Example 5.1.3 since $\{\delta_{ij} : i, j \in \mathbb{Z}\}$ was required to be i.i.d. with distribution $N(0, 1)$. $\square$

CHAPTER 6: CONCLUSION AND FUTURE WORK

Assuming that the error structure satisfies (E.1) - (E.3) and $\{Y_{ij} : i, j \geq 1\}$ obeys (1.2), consider the testing problem $H_0 : \alpha = \beta = 1$ vs $H_A : |\alpha| < 1, |\beta| < 1$. Define the test statistics $\Phi_{N,d} = \frac{1}{N^{2g+6}} I_N$, where $g = d_1 + d_2$ and $I_N = A_N^2 + B_N^2$ is the periodogram ordinate of the $Y-$process. Reject H_0 whenever $\Phi_{N,d}$ is sufficiently small. The critical region can be determined from the asymptotic result $\Phi_{N,d} \xrightarrow{D} \sigma_{11}(d)U_1 + \sigma_{22}(d)U_2$ proved in Theorem 2.0.2 (i) whenever H_0 is valid. Moreover, at a sequence of local Pitman-type alternatives $H_1 : \alpha_N = e^{a/N}, \beta_N = e^{b/N}$, where $a < 0$ and $b < 0$, Theorem 2.0.2 (ii) shows that $\Phi_{N,d} \xrightarrow{D} \lambda_1 U_1 + \lambda_2 U_2$, for eigenvalues $\lambda_1 = \lambda_1(d, a, b)$ and $\lambda_2 = \lambda_2(d, a, b)$ of Σ_1. Hence the asymptotic power of $\Phi_{N,d}$ at the sequence $\alpha_N = e^{a/N}$, $\beta_N = e^{b/N}$ is $P_{a,d,b}(x) = P\{\lambda_1 U_1 + \lambda_2 U_2 \leq x\}$, for $x > 0$. It is of course more difficult to attain a large value of the power function at a sequence of alternatives that approach H_0 than at a fixed alternative in H_A.

In practice, the long memory parameter $d = (d_1, d_2)$ needs to be estimated in the error structure. A regression method to estimate $d = (d_1, d_2)$ for model (1.2) is given by Ghodsi and Shitan (2009)[20] whenever the observable $Y-$process has long memory, and the errors form a white noise process. Based on simulation results, it is shown that the Mean Square Errors of estimates using the regression method are smaller than those obtained from Whittle's estimate. The regression method is based on using the observed Y_{ij}'s and assumed model to find the μ_{ij}'s.

Open Problem: Is the asymptotic power of test $\Phi_{N,d}$ at a sequence of $\alpha_N = e^{a/N},\ \beta_N = e^{b/N}$ of alternatives one?

An affirmative answer can be proved in the AR(1) time series model with independent and identically distributed errors.

Also considering Theorem 4.0.4, observe that $H_0 : \alpha = \beta = 1$ under the assumption of model (1.2) is equivalent to $a = b = 0$ in model (1.1). Since $a = b = 0$, $M(K(\underline{t})) = K(\underline{t}) = \int\limits_{E_{t_1}} e^{c(t_1-x)}\ W(x, t_2)\ dx$ and observe that $E\left(K(\underline{t})\right) = 0$. Using A and B defined in Theorem 4.0.4,

$$\begin{aligned} \mathrm{cov}(A, B) &= \mathrm{cov}\left(\int\limits_{[0,1]^2} \cos 2\pi(s_1 + s_2) K(\underline{s})\ d\underline{s}\ ,\ \int\limits_{[0,1]^2} \sin 2\pi(t_1 + t_2) K(\underline{t})\ d\underline{t} \right) \\ &= \int\limits_{[0,1]^4} \cos 2\pi(s_1 + s_2) \sin 2\pi(t_1 + t_2) \mathrm{cov}\left(K(\underline{s}), K(\underline{t})\right)\ d\underline{s}\ d\underline{t}. \end{aligned} \tag{6.1}$$

Further, $\mathrm{cov}\left(K(\underline{s}), K(\underline{t})\right) = \mathrm{cov}\left(\int\limits_{E_{s_1}} e^{c(s_1-x)}\ W(x, s_2)\ dx, \int\limits_{E_{t_1}} e^{c(t_1-y)}\ W(y, t_2)\ dy \right)$

$$= (s_2 \wedge t_2) \int\limits_{E_{s_1 t_1}} e^{c(s_1-x)} e^{c(t_1-y)} (x \wedge y)\ dx\ dy.$$

After calculations,

$$\begin{aligned} \mathrm{cov}\left(K(\underline{s}), K(\underline{t})\right) = &\frac{s_2 \wedge t_2}{c^2} \left(\frac{e^{c(s_1+t_1)} + e^{c|s_1-t_1|}}{2c} \right) + \frac{1}{c^2} (s_1 \wedge t_1)(s_2 \wedge t_2) \\ &+ \frac{1}{c^3} (s_2 \wedge t_2)(1 - e^{cs_1} - e^{ct_1}). \end{aligned} \tag{6.2}$$

Substituting (6.2) into (6.1) and using Mathematica to integrate, one obtains

$$\mathrm{cov}(A, B) = \frac{e^{-c}(-1 + e^c)^2 (2 + e^c)}{4c^2 \pi (c^2 + 4\pi^2)^2}. \text{ Likewise,}$$

$$\text{Var } A = \frac{e^{-c}(-1+e^c)^2(c^2(1+e^c)+12(-1+e^c)\pi^2)}{16c^3\pi^2(c^2+4\pi^2)^2} \qquad \text{and}$$

$$\text{Var } B = \frac{e^{-c}(-1+e^c)^2(3c^2(1+e^c)+4(-1+e^c)\pi^2)}{16c^3\pi^2(c^2+4\pi^2)^2}.$$

Since $\text{cov}(A,B) \neq 0$, let Q denote the orthogonal matrix such that $Q\Sigma Q' = \text{diag }(\lambda_1, \lambda_2)$, where $\lambda_1,\ \lambda_2$ are the eigenvalues of $\Sigma = \text{Var}\begin{bmatrix} A \\ B \end{bmatrix}$. Define $Z_N = Q\begin{bmatrix} A_N \\ B_N \end{bmatrix}$ and note that $\frac{1}{N^4}\ Z_N \xrightarrow{D} N(\underline{0}, \text{diag}(\lambda_1, \lambda_2))$ implies that $\frac{1}{N^8}\ I_N = \frac{1}{N^8}(A_N^2 + B_N^2) = \frac{1}{N^8} Z_N'\ Z_N \xrightarrow{D} \lambda_1 V_1 + \lambda_2 V_2$, where V_1 and V_2 are independent chi-square random variables each having one degree of freedom. The preceding limit can be used to form a test in terms of the periodogram ordinate by rejecting the null hypothesis whenever $\frac{1}{N^8} I_N$ is sufficiently small.

However, one needs to estimate c in the error structure $\mu_{ij} = \gamma_N \mu_{i-1,j} + \epsilon_{ij}$, where $\gamma_N = e^{c/N}$. Under the assumption of H_0, model (1.2) can be used to find $\mu_{ij},\ 1 \leq i,\ j \leq N$. The least squares estimator of γ_N is $\hat{\gamma}_N = \sum_{i=1}^{N} \mu_{ij}\mu_{i-1,j} \Big/ \sum_{i=1}^{N} \mu_{i-1,j}^2 = \gamma_N + \sum_{i=1}^{N} \epsilon_{ij}\mu_{i-1,j} \Big/ \sum_{i=1}^{N} \mu_{i-1,j}^2$. Since $\left(\frac{1}{N}\sum_{k=1}^{[Nt]} \gamma_N^{[Nt]-k}\epsilon_{kj}\right)^2 \xrightarrow{D} J^2(t) \in D([0,1])$, it follows that

$$\frac{1}{N^3}\sum_{s=1}^{N}\left(\sum_{k=1}^{s-1}\gamma_N^{s-1-k}\epsilon_{kj}\right)^2 \xrightarrow{D} \int_{[0,1]} J^2(t)\ dt.$$

Recall that $\mu_{i-1,j} = \sum_{k=1}^{i-1} \gamma_N^{i-1-k}\epsilon_{kj}$; then the above implies that $\frac{1}{N^3}\sum_{i=1}^{N} \mu_{i-1,j}^2 \xrightarrow{D} \int_{[0,1]} J^2(t)\ dt$ as $N \longrightarrow \infty$ on $\mathbb{R}$. It easily follows that the sequence $\{\epsilon_{ij}\mu_{i-1,j}\ :\ i,\ j \geq 1\}$ is uncorrelated and thus $\text{Var}\sum_{i=1}^{N} \epsilon_{ij}\mu_{i-1,j} = \sigma^4 \sum_{i=1}^{N}\sum_{k=1}^{i-1} \gamma_N^{2(i-1-k)} = O(N^2)$. Then $\sum_{i=1}^{N} \epsilon_{ij}\mu_{i-1,j} = O_p(N)$ and

$$\hat{\gamma}_N = \gamma_N + \frac{1}{N^3}\sum_{i=1}^{N} \epsilon_{ij}\mu_{i-1,j} \Big/ \frac{1}{N^3}\sum_{i=1}^{N} \mu_{i-1,j}^2$$

$$= \gamma_N + O_p\left(\frac{1}{N^2}\right) = 1 + \frac{c}{N} + O_p\left(\frac{1}{N^2}\right).$$

Hence $N(\hat{\gamma}_N - 1) \xrightarrow{P} c$ as $N \longrightarrow \infty$ and $N(\hat{\gamma}_N - 1)$ is consistent estimator of c. Since this estimator is based on a fixed $1 \leq j \leq N$, a more efficient estimator is formed by averaging over $1 \leq j \leq N$.

Finally, another open problem is to extend our results when considering an error structure having long range dependence in one component, but an alternative error structure such as a moving average or autoregressive in the other component.

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