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Hilbert Series of Graphs, Hypergraphs, and Monomial Ideals

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HILBERT SERIES OF GRAPHS, HYPERGRAPHS, AND MONOMIAL IDEALS

by

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ABSTRACT

In this dissertation, identities for Hilbert series of quotients of polynomial rings by monomial ideals are explored, beginning in the contexts of graph and hypergraph rings and later generalizing to general monomial ideals. These identities are modeled after constructive identities from graph theory, and can thus be used to construct Hilbert series iteratively from those of smaller algebraic structures.

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CHAPTER 1: INTRODUCTION

This dissertation discusses identities useful for the computation of Hilbert series of quotients of polynomial rings by monomial ideals, particularly those monomial ideals which arise as edge ideals of graphs or hypergraphs.

1.1 Background

We begin by defining some necessary graph terminology, operations and invariants. In this dissertation, graphs and hypergraphs, defined below, will be assumed to never have repeated edges.

Definition 1.1.1. A graph $G = (V, E)$ is an ordered pair of a vertex set V and a corresponding edge set E. A vertex set is a set of indeterminants, known as vertices. Given some vertex set V, an edge set of a graph is a set of multisets, each of which is size two and contains two (perhaps nondistinct) elements of V. Each element of the edge set is called an edge, and vertices jointly contained in some edge are said to be adjacent.

An edge of a graph is said to be a loop if it contains a single vertex with multiplicity two. A graph is simple if it contains no loops.

A hypergraph $H = (V, E)$ is an ordered pair of a vertex set V and a corresponding edge set E. A vertex set is a set of indeterminants, known as vertices, whereas the edge set of a hypergraph is a subset of the power set of V such that all elements of the edge set, known as edges, have size at least two and no edge is strictly contained within another.

Some authors define a hypergraph more loosely and allow edge inclusion, and refer to our definition as that of a simple hypergraph or clutter. This definition more readily allows us to translate properties between graph theoretic terms and algebraic ones.

One of the main properties that will be translated is that of the edge induced sub-graph (or sub-hypergraph) enumerating polynomial. The subgraph of H induced by some subset E of its edge set is the smallest subgraph (in terms of both vertices and edges) of H containing every edge in E. The edge induced subgraph enumerating polynomial of H counts and organizes how many such subgraphs there are, separated by their numbers of vertices and edges, as defined below.

Definition 1.1.2. Let H be a graph or hypergraph. Then

$$S_H(u,v) = \sum_{i,j} C_{ij} u^i v^j \tag{1.1}$$

is the edge induced subgraph enumerating polynomial of H, where C_{ij} is the number of edge induced sub-graphs or sub-hypergraphs of H with i vertices and j edges.

In order to present a significant result related to subgraph enumerating polynomials that will motivate much of the results of this work, we first must present some graph operations and corresponding notation.

Definition 1.1.3 ([4])**.** Let $G = (V, E)$ be a graph and let $a \in E$. Then

(1) G_a is the graph formed by deleting the edge a by removing it from the edge set E.

(2) $G_{[a]}$ is the graph formed by contracting the edge a and ignoring any multiple edges. This is done by deleting the edge a and then identifying all of the vertices formerly in a.

(3) $G_{\langle a\rangle}$ is the graph formed by deleting the vertices in the edge a and adding an isolated vertex, often denoted by δ in this and other works, and is referred to as the extraction. Note that this also deletes any edges incident to a vertex in a.

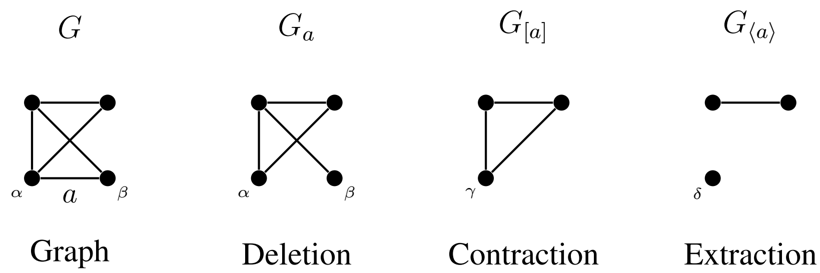

Figure 1.1: Graph Operations

The given definition for deletion is standard, as is the definition for contraction - though some authors resolve multiple edges differently. The extraction is less commonly defined, but may be found alongside the other definitions in [4]. For a visual example of these operations, see Figure 1.1. The vertices in Figure 1.1 are labeled in a manner consistent with the conventions in [4].

Many of the results in this thesis involve various generalizations and translations of a theorem found in [4], which we present here.

Theorem 1.1.4 ([4])**.** *For any graph $G = (V, E)$, and for every $a \in E$, we have:*

(1) if a is not a loop:

$$S_G = S_{G_a} + uvS_{G_{[a]}} + uv(u-1)S_{G_{\langle a \rangle}} \tag{1.2}$$

(2) if a is a loop:

$$S_G = (1+v)S_{G_a} + v(u-1)S_{G_{\langle a \rangle}} \tag{1.3}$$

1.2 Hilbert Series

The results in this dissertation center around the Hilbert series of a graded vector space with finite dimension at each grading, which we define below in the specific case of a finitely generated graded commutative algebra. For more general treatments of Hilbert and Poincaré series, see [2] and [8].

Definition 1.2.1. Let S be a finitely generated graded commutative algebra over a field k.

Then, the Hilbert function of S is defined by

$$HF_S : n \mapsto dim_k S_n. \tag{1.4}$$

The Hilbert series of S is the formal power series associated to the Hilbert function of S, that is,

$$HS(S) = \sum_{n=0}^{\infty} HF_S(n) t^n. \tag{1.5}$$

As a formal power series, convergence is typically ignored, and the series may sometimes be expressed as a rational function. Fortunately, the Hilbert-Serre theorem, presented below, guarantees a rational expression in the case of an ideal of a polynomial ring over a field. This is sufficient to also guarantee rational expressions for quotients of polynomial rings by their ideals.

Theorem 1.2.2 ([8])**.** *Let $I \subseteq k[x_1, \ldots, x_n]$ be a homogeneous ideal of a polynomial ring over a field k. Then the Hilbert series has the form*

$$HS(I) = \frac{a_0 + a_1 t + \cdots + a_m t^m}{(1-t)^{n+1}}, \tag{1.6}$$

where m is a nonnegative integer and $a_0, a_1, \ldots, a_n \in \mathbb{Z}$.

1.3 Edge Ideals

The primary structure under examination will be the edge ideal of a graph or hypergraph. We will use an altered version of the definition presented in [15] that allows for hypergraphs and for loops in graphs.

Definition 1.3.1 ([15])**.** Let $A = k[x_1, x_2, \ldots, x_n]$ be a polynomial ring over a field k. The edge ideal $I(G)$ associated to the graph G is the ideal of A generated by the set of all monomials $x_i x_j$ such that x_i is adjacent to x_j, allowing for the case where $x_i = x_j$ due to a loop in the graph.

The edge ideal associated to the hypergraph H is the ideal of A generated by the set of all monomials $x_{i_1} x_{i_2} \ldots x_{i_m}$, where $\{x_{i_1}, x_{i_2}, \ldots, x_{i_m}\}$ is an edge in the hypergraph.

The graph ring associated to an edge ideal I is the quotient ring A/I.

Some authors refer to the graph ring associated to a graph as the edge ring associated to that graph, but that term is also used for an algebra generated by the edges, and so we will exclusively use the term graph ring.

The properties of the edge ideal and associated graph ring have been a subject of much recent study (see, for instance [11]), with many results connecting the algebraic structure of the graph ring with the graph theoretic structure of its defining (hyper)graph.

For instance, the Hilbert series of graph and hypergraph rings have been connected to edge induced subgraph enumerating polynomials, first by Renteln [12] for graphs, then by Goodarzi [7]. As Goodarzi's version covers a more general case, we refrain from presenting Renteln's.

Theorem 1.3.2 ([7])**.** *Let $I \subset A = k[x_1, ..., x_n]$ be a squarefree monomial ideal and H its associated hypergraph. Then*

$$HS(A/I) = \frac{S_H(t, -1)}{(1-t)^n}, \tag{1.7}$$

where $HS(A/I)$ is the Hilbert Series of the ring A/I.

This relationship has been previously explored by multiple authors. In [6], Emtander explicitly noted the relationship between this result and the Betti numbers of Stanley-Reisner rings (which can be understood as hypergraph rings), with particularly concise results if the ring has a linear resolution. Moving in a different direction, Tadesse [14] extended the reconstruction results of Borzacchini and Pulito [4] to show that subgraph enumerating polynomials for hypergraphs are reconstructible, which naturally extends via the above theorem to Hilbert Series and Betti numbers. The central results of this work will explicitly extend Theorem 1.1.4 to corresponding theorems for Hilbert series in various contexts, culminating in Theorem 2.3.3. These extensions, while inspired by the above theorem, avoid its direct use so that we may consider cases involving nonsquare-free ideals - some of which differ from the results which would otherwise be predicted by direct application of Theorem 1.3.2 outside its allowed context.

CHAPTER 2: RESULTS

Most of the results in this section deal with the behavior of the Hilbert series of graphs and hypergraphs when deletion, contraction, and extraction operations are applied. The results will be presented first for graphs, then for hypergraphs, after which further generalizations and similar techniques will be discussed.

2.1 Hilbert Series of Graphs

Most of the results in this section and the next were previously published by this author in [5].

The theorems of Renteln [12] and Goodarzi [7] relate the subgraph enumerating polynomial of a graph to the Hilbert series of the associated graph ring. Accordingly, we may use Theorem 1.3.2 to translate the first half of Theorem 1.1.4 into a result for Hilbert series of simple graphs. However, we present a slightly generalized version that allows for loops within the graph through the use of a more algebraic proof.

Theorem 2.1.1 ([5]). *Let G be a graph without multiple edges with an edge $a = \{\alpha, \beta\}$ which is not a loop. Let $R, R_a, R_{[a]}$, and $R_{\langle a \rangle}$ be the graph rings corresponding to $G, G_a, G_{[a]}$, and $G_{\langle a \rangle}$, respectively. Then,*

If there is a loop at at most one of α and β,

$$HS(R) = HS(R_a) - t\left(\frac{HS(R_{[a]})}{1-t} - HS(R_{\langle a \rangle})\right). \tag{2.1}$$

If there are loops at both α and β,

$$HS(R) = HS(R_a) - t\left(HS(R_{[a]}) - (1-t)HS(R_{\langle a\rangle})\right). \tag{2.2}$$

Before beginning the proof in earnest, it is useful to establish a connecting lemma.

Lemma 2.1.2 ([5]). *Let G be a graph without multiple edges with an edge $a = \{\alpha, \beta\}$ which is not a loop (i.e. $\alpha \neq \beta$), and assume that there is a loop at at most one of α and β. Let R_a and $R_{[a]}$ be the graph rings corresponding to G_a and $G_{[a]}$, respectively. Define γ to be the vertex created by the identification of α and β in the contraction, and let δ be an indeterminate with no relations in R. Then, the ring homomorphism*

$$\frac{R_{[a]}[\delta]}{(0 :_{R_{[a]}[\delta]} \gamma)} \xrightarrow[\delta\mapsto\beta]{\gamma\mapsto\alpha} \frac{R_a}{(0 :_{R_a} \alpha\beta)} \tag{2.3}$$

is a (grade preserving) isomorphism.

Alternatively, if there are loops at both α and β,

$$\frac{R_{[a]}}{(0 :_{R_{[a]}[\delta]} \gamma)} \xrightarrow{\gamma\mapsto\alpha} \frac{R_a}{(0 :_{R_a} \alpha\beta)} \tag{2.4}$$

is a (grade preserving) isomorphism.

Proof. The reader may wish to note that, while δ solely serves as an extra isolated vertex within this lemma, it will later refer to the isolated vertex resulting from the extraction of the edge a. The map is understood to be the identity on any elements containing neither γ nor δ.

First, we note that, given the structure of the edge ideal, the colon ideal $(0 :_{R_{[a]}[\delta]} \gamma)$ is generated by the neighbors of γ in $G_{[a]}[\delta]$. If some vertex v is adjacent to γ in $G_{[a]}[\delta]$, then $\{v, \gamma\}$ is an edge in $G_{[a]}[\delta]$, so $v\gamma = 0$ in the corresponding ring. Thus $v \in (0 :_{R_{[a]}[\delta]} \gamma)$. If a monomial $m \in (0 :_{R_{[a]}[\delta]} \gamma)$, then $m\gamma$ is equal to zero in $R_{[a]}[\delta]$. Then, $m\gamma$ must be divisible by some degree two monomial equal to zero in that ring, as $I_{[a]}$, an ideal of an integral domain, is generated by degree two monomials. If that degree two monomial is divisible by γ, then it must correspond to an edge incident to γ, implying that m is divisible by an indeterminate corresponding to a neighbor of γ, and thus m is in the ideal generated by those neighbors. If that degree two monomial is not divisible by γ, then m is already equal to zero in $R_{[a]}[\delta]$ prior to multiplication by γ, and trivially also in the ideal generated by the neighbors of γ.

As γ can be understood as the identification of α and β in G_a, a vertex x is adjacent to γ in $G_{[a]}$ if and only if x is adjacent to α or β in G_a. If there is a loop at a vertex, it is considered adjacent to itself.

Since the generators of the edge ideal of a graph are of degree two, and α and β are not neighbors in the edge-deleted graph G_a, the colon ideal $(0 :_{R_a} \alpha\beta) = (0 :_{R_a} \alpha) + (0 :_{R_a} \beta)$ is generated by the neighbors of α and β.

The quotients in the domain and codomain, then, correspond to deletion or isolation of all neighbors of γ, α, and β, either deleting them or leaving them as isolated vertices depending on whether there is a loop at each vertex. As the graphs $G_{[a]}$ and G_a differ only in whether or not the identification of α and β has been made and related differences in edges incident to α, β, or γ, this deletion results in graphs that are only not isomorphic in that one of the two graphs may have one more isolated vertex than the other. This occurs exactly when there is a loop at at most one of α and β. Adjoining an indeterminate to $R_{[a]}$ is tantamount to adding an extra isolated vertex, solving this problem in that case.

As the map, then, can be understood to be the identity up to relabeling of isolated vertices, it is a grade-preserving isomorphism. $\square$

We may now prove the preceding theorem.

Proof of Theorem 2.1.1. The bulk of the proof consists of noting that the following diagrams connect two short exact sequences via the appropriate isomorphism from the above lemma.

In the case where there is a loop at at most one of α and β:

$$\begin{tikzcd} 0 \arrow[r] & \frac{R_{[a]}[\delta]}{(0:\gamma)}(-2) \arrow[r, hook, "\cdot\gamma"] \arrow[d, "\cong"] & R_{[a]}[\delta](-1) \arrow[r, two heads, "\gamma \mapsto 0"] & R_{\langle a \rangle}(-1) \arrow[r] & 0 \\ 0 \arrow[r] & \frac{R_a}{(0:\alpha\beta)}(-2) \arrow[r, hook, "\cdot\alpha\beta"] & R_a \arrow[r, two heads, "\alpha\beta \mapsto 0"] & R \arrow[r] & 0 \end{tikzcd} \tag{2.5}$$

In the case where there is a loop at both α and β,

$$\begin{tikzcd} 0 \arrow[r] & \frac{R_{[a]}[\delta]}{(0:\gamma)}(-2) \arrow[r, hook, "\cdot\gamma"] \arrow[d, "\cong"] & R_{[a]}[\delta](-1) \arrow[r, two heads, "\gamma \mapsto 0"] & R_{\langle a \rangle}(-1) \arrow[r] & 0 \\ 0 \arrow[r] & \frac{R_a[\delta]}{(0:\alpha\beta)}(-2) \arrow[r, hook, "\cdot\alpha\beta"] & R_a[\delta] \arrow[r, two heads, "\alpha\beta \mapsto 0"] & R[\delta] \arrow[r] & 0 \end{tikzcd} \tag{2.6}$$

A shift is made in the first sequence in order to make the isomorphism a degree zero homomorphism.

The only difficulty in the construction of these sequences is the map from $R_{[a]}[\delta]$ to $R_{\langle a \rangle}$. Graphically, the map deletes the vertex γ created from the contraction, leaving the remainder of the graph and an extra isolated vertex δ. As this is exactly how graph extraction is defined, the surjectivity is clear.

First, assume that there is a loop at at most one of α and β, including the case where there is a loop at neither.

Appealing to the fact that Hilbert series satisfy an additive relation on exact complexes, we can collect the following equations:

$$t^2 HS\left(\frac{R_{[a]}[\delta]}{(0:\gamma)}\right) = tHS\left(R_{[a]}[\delta]\right) - tHS\left(R_{\langle a\rangle}\right), \text{ and} \tag{2.7}$$

$$t^2 HS\left(\frac{R_a}{(0:\alpha\beta)}\right) = HS\left(R_a\right) - HS\left(R\right). \tag{2.8}$$

Invoking the grade perserving isomorphism,

$$\begin{aligned} HS\left(R\right) &= HS\left(R_a\right) - t\left(HS\left(R_{[a]}[\delta]\right) - HS\left(R_{\langle a\rangle}\right)\right) &(2.9)\\ &= HS\left(R_a\right) - t\left(\frac{HS\left(R_{[a]}\right)}{1-t} - HS\left(R_{\langle a\rangle}\right)\right). &(2.10)\end{aligned}$$

The proof follows in a similar manner in the case where there are loops at both α and β.

$\square$

Theorem 1.1.4 included an identity for subgraph enumerating polynomials in the case where the specified edge a was a loop. However Theorem 1.3.2 is unable to directly translate this theorem into an identity for Hilbert series, as the edge ideal of a graph with loops is not squarefree. In fact, the identity that Theorem 1.3.2 would predict is false - substituting -1 for v in Equation 1.3 completely eliminates any dependence on the edge deleted graph. Considering, for example, the graph consisting of a single vertex with a single loop quickly shows that identity to be incorrect. However, we can imitate the techniques of the previous proof in order to reproduce a similar result in the case where a is a loop.

Theorem 2.1.3. *Let G be a graph without multiple edges with a loop a at the vertex α. Let R, R_a, and $R_{\langle a \rangle}$ be the graph rings corresponding to G, G_a, and $G_{\langle a \rangle}$, respectively. Then,*

$$HS(R) = (1-t)HS(R_a) + t(1-t)HS(R_{\langle a \rangle}) \tag{2.11}$$

Proof. Note that, for a loop, contraction and deletion are identical by our definitions, and thus induce the same graph rings. This justifies the isomorphism in the following diagram, and also allows the contraction to be replaced by the deletion within the final identity.

$$\begin{tikzcd}
0 \arrow[r] & \frac{R_{[a]}[\delta]}{(0:\alpha)}(-2) \arrow[r, hook, "\cdot\gamma"] \arrow[d, "\cong"] & R_{[a]}[\delta](-1) \arrow[r, two heads, "\gamma \mapsto 0"] & R_{\langle a \rangle}(-1) \arrow[r] & 0 \\
0 \arrow[r] & \frac{R_a[\delta]}{(0:\alpha)}(-2) \arrow[r, hook, "\cdot\alpha^2"] & R_a[\delta] \arrow[r, two heads, "\alpha^2 \mapsto 0"] & R[\delta] \arrow[r] & 0
\end{tikzcd} \tag{2.12}$$

The rest follows in a manner similar to the previous proof. □

2.2 Hilbert Series of Hypergraphs

In order to extend the results of the previous section to our notion of hypergraphs (which some authors would refer to as clutters or simple hypergraphs), the edges of which may contain multiple vertices, we must first extend our definitions accordingly.

Definition 2.2.1. Let $H = (V, E)$ be a hypergraph and let $a \in E$. Then

(1) H_a is the hypergraph formed by deleting the edge a by removing it from the edge set E.

(2) $H_{[a]}$ is the hypergraph formed by contracting the edge a and ignoring any multiple edges. This is done by deleting the edge a and identifying all vertices formerly in a. In the case that

this introduces edges strictly contained within one another, we only keep the edges minimal under inclusion so that the result remains a hypergraph.

(3) $H_{\langle a \rangle}$ is the hypergraph formed by deleting all of vertices in the edge a and adding an isolated vertex, and is referred to as the extraction. Note that this also deletes any edges incident to a vertex in a.

As before, we will first establish an isomorphism lemma, then use that to create a version of Theorem 1.1.4 for Hilbert series, but for hypergraphs. The intuition behind this lemma is that, if vertex contraction (see [13]) is applied to each vertex within an edge, the result would be the same even if the edge had been contracted or deleted first.

Lemma 2.2.2 ([5]). *Let H be a hypergraph with an edge $a = \{\delta_1, \delta_2, \ldots, \delta_n\}$. Let R_a and $R_{[a]}$ be the hypergraph rings corresponding to H_a and $H_{[a]}$, respectively. Let the vertex created by the contraction of the edge a retain the name δ_1. Then, the ring homomorphism*

$$\frac{R_{[a]}[\delta_2, \ldots, \delta_n]}{(0 :_{R_{[a]}} \delta_1)} \xrightarrow{\quad i \quad} \frac{R_a}{(0 :_{R_a} \delta_1 \cdots \delta_n)} \tag{2.13}$$

is a (grade preserving) isomorphism.

Proof. Let S be a polynomial ring over a field with an indeterminate for each vertex of H, and let I_a and $I_{[a]}$ be the edge ideals for R_a and $R_{[a]}$. Then

$$\frac{R_{[a]}[\delta_2, \ldots, \delta_n]}{(0 :_{R_{[a]}} \delta_1)} \cong \frac{\frac{S}{I_{[a]}}}{(0 :_{R_{[a]}} \delta_1)} \cong \frac{\frac{S}{I_{[a]}}}{\frac{(I_{[a]} :_S \delta_1)}{I_{[a]}}} \cong \frac{S}{(I_{[a]} :_S \delta_1)} \tag{2.14}$$

and

$$\frac{R_a}{(0 :_{R_a} \delta_1 \cdots \delta_n)} \cong \frac{\frac{S}{I_a}}{(0 :_{R_a} \delta_1 \cdots \delta_n)} \cong \frac{\frac{S}{I_a}}{\frac{(I_a :_S \delta_1 \cdots \delta_n)}{I_a}} \cong \frac{S}{(I_a :_S \delta_1 \cdots \delta_n)}. \tag{2.15}$$

Thus, it suffices to show that $(I_{[a]} :_S \delta_1) = (I_a :_S \delta_1 \cdots \delta_n)$.

We will proceed by constructing generating sets for both ideal quotients, and noting that they are the same generating set.

As S is an integral domain, note that

$$(I_{[a]} :_S \delta_1) = \frac{1}{\delta_1}\left(I_{[a]} \cap (\delta_1)\right) \text{, and} \tag{2.16}$$

$$(I_a :_S \delta_1 \cdots \delta_n) = \frac{1}{\delta_1 \cdots \delta_n}\left(I_{[a]} \cap (\delta_1 \cdots \delta_n)\right) \tag{2.17}$$

Division by δ_1, etc., is here defined to be reduction of the power of δ_1 in a given monomial by one, so long as it is divisible by δ_1. It remains undefined otherwise.

An intersection of monomial ideals of a polynomial ring over a field is generated by the least common multiples of the generators of the ideals being intersected. In the case of $I_{[a]}$ and (δ_1), this is straightforward: take the typical list of generators of $I_{[a]}$ formed from the edges of $H_{[a]}$ and multiply any generators not yet in (δ_1) by δ_1. We then find a generating set for $(I_{[a]} : \delta_1)$ by dividing each of these by δ_1. The combination of these processes results in taking the edges in $I_{[a]}$ and removing δ_1 from them if necessary. Finally, remove any monomials which are products of a smaller degree monomial, and thus unnecessary.

Similarly, to find a generating set for $(I_a :_S \delta_1 \dots \delta_n)$, we apply the same technique and arrive at a generating set of monomials constructed by removing any copies of $\delta_1, \delta_2, \dots, \delta_n$ present in the edges of I_a. As before, remove any unnecessary monomials.

Since the edges of the contraction $H_{[a]}$ are simply the edges of the edge deletion H_a after identifying the vertices in the specified edge a, removing the vertices incident to a (or contracted to via a) removes any differences in the edge lists, up to removing edges that strictly contain another edge. Therefore, we have constructed two identical generating sets, and the ideal quotients in question are indeed equal.

□

We may now proceed to the main theorem of this section.

Theorem 2.2.3 ([5])**.** *Let H be a hypergraph with an edge $a = \{\delta_1, \delta_2, \dots, \delta_n\}$. Let $R, R_a, R_{[a]}$, and $R_{\langle a \rangle}$ be the hypergraph rings corresponding to $H, H_a, H_{[a]}$, and $H_{\langle a \rangle}$, respectively. Let the vertex created by either the contraction or extraction of the edge a retain the name δ_1. Then,*

$$HS(R) = HS(R_a) - \frac{t^{n-1}}{(1-t)^{n-2}} \left(\frac{HS(R_{[a]})}{1-t} - HS(R_{\langle a \rangle}) \right). \tag{2.18}$$

Proof.

$$\begin{tikzcd}
0 \arrow[r] & \frac{R_{[a]}[\delta_2,\dots,\delta_n]}{(0:\delta_1)}(-n) \arrow[r, hook, "\cdot\delta_1"] \arrow[d, "\cong"] & R_{[a]}[\delta_2, \dots, \delta_n](1-n) \arrow[r, two heads, "\delta_1 \mapsto 0", "\delta_n \mapsto \delta_1"'] & R_{\langle a \rangle}[\delta_2, \dots, \delta_{n-1}](1-n) \arrow[r] & 0 \\
0 \arrow[r] & \frac{R_a}{(0:\delta_1 \dots \delta_n)}(-n) \arrow[r, hook, "\cdot\delta_1 \dots \delta_n"] & R_a \arrow[r, two heads, "\delta_1 \dots \delta_n \mapsto 0"] & R \arrow[r] & 0
\end{tikzcd} \tag{2.19}$$

Again, the Hilbert series result follows after some simple computation. The rows of the diagram are the usual short exact sequences for multiplication by a ring element, after noting that deleting the contracted vertex in $G_{[a]}[\delta_2, \ldots, \delta_n]$ results in the extraction with a few extra isolated vertices, after borrowing one of the isolated vertices already present. The map connecting the rows is from Lemma 2.2.2. □

For an example using this theorem to iteratively build the Hilbert series of a hypergraph, see Example 3.1.3.

2.3 Hilbert Series of Monomial Ideals

In this section, the ideas of the preceding sections are generalized beyond graph theoretic contexts to the quotient of a polynomial ring by any (perhaps non-squarefree) monomial ideal. As the Hilbert series associated to a quotient by a polynomial ideal is equal to that of the quotient by its initial ideal [9], this functionally gives a generalization to polynomial ideals as well.

As before, we will make definitions inspired by those from the preceding sections.

Definition 2.3.1. Let $I \subset k[x_1, \ldots, x_n]$ be a monomial ideal with a minimal generating set of monomials that contains the monomial m. Then,

1. The deletion $\tilde{I}_m$ of I by m is the ideal of $k[x_1, \ldots, x_n]$ created by removing m from the list of generators of I. That is, $\tilde{I}_m$ is the smallest ideal such that $\tilde{I}_m + (m) = I$
2. The extraction $\tilde{I}_{\langle m \rangle}$ of I by m is the ideal of $k[x_1, \ldots, x_n]$ such that

$$\tilde{I}_{\langle m \rangle} = \left(\tilde{I}_m \cap k[x_i : x_i \notin \text{supp}(m)]\right) k[x_1, \ldots, x_n] \tag{2.20}$$

3. The contraction $\tilde{I}_{[m]}$ of I by m is the ideal of $k[x_1, \ldots, x_n, \gamma]$ such that

$$\tilde{I}_{[m]} = \tilde{I}_{\langle m \rangle} k[x_1, \ldots, x_n, \gamma] + \gamma \left(\left(\tilde{I}_m : m \right) k[x_1, x_2, \ldots, x_n, \gamma] \right) \tag{2.21}$$

To avoid unnecessary clutter, when the ideal I is understood, R, $\tilde{R}_m$, $\tilde{R}_{[m]}$, and $\tilde{R}_{\langle m \rangle}$ will refer to the quotient of the appropriate polynomial ring by I, $\tilde{I}_m$, $\tilde{I}_{[m]}$, and $\tilde{I}_{\langle m \rangle}$, respectively.

Intuitively, the deletion by m simply removes m from the list of generators, the contraction by m removes any common factors with m from remaining generators of the deletion (leaving behind an extra indeterminate γ when there is a nontrivial common factor to remove), and the extraction by m entirely removes any monomials that have nontrivial common factors with m. These definitions largely agree with the definitions in [5] and [4] when the ideal I is squarefree, with the exception that indeterminates are not removed from the polynomial ring when contracting or extracting. This only leads to extra factors of $(1 - t)$ in the denominator of the Hilbert series, so the change is minor. In particular, the generators of the ideal agree exactly. While this leads to some ambiguity in naming conventions (deletion, contraction, and extraction), the notation corresponding to the definitions from this section has been slightly altered for clarity.

In addition, the definitions here similarly almost agree (up to extra indeterminates with no relations) with the results of polarizing the ideal (defined in [10]) and then using the definitions from [5] on the resultant squarefree monomial ideal before removing the polarization. However, polarization is typically inefficient and overall impractical as a tool for computation [10].

Lemma 2.3.2. *The ring homomorphism*

$$\frac{k[x_1, \ldots, x_n, \gamma]}{(\tilde{I}_{[m]} : \gamma)} \xrightarrow{\quad i \quad} \frac{k[x_1, \ldots, x_n]}{(\tilde{I}_m : m)}[\gamma] \tag{2.22}$$

is a degree zero isomorphism.

Proof. Since the polynomial rings only differ in that one includes γ as an indeterminate and one does not, it is sufficient to show that

$$\left(\tilde{I}_{[m]} : \gamma\right) = \left(\tilde{I}_m : m\right) k[x_1, x_2, \ldots, x_n, \gamma]. \tag{2.23}$$

Recall that $\tilde{I}_{[m]} = \tilde{I}_{\langle m\rangle} k[x_1, \ldots, x_n, \gamma] + \gamma \left(\left(\tilde{I}_m : m\right) k[x_1, x_2, \ldots, x_n, \gamma]\right)$, and that $\tilde{I}_{\langle m\rangle}$ contains no interactions with γ and consequently $(\tilde{I}_{\langle m\rangle} : \gamma) = \tilde{I}_{\langle m\rangle}$. Therefore,

$$\begin{aligned}
(\tilde{I}_{[m]} : \gamma) &= \left(\left[\tilde{I}_{\langle m\rangle} k[x_1, \ldots, x_n, \gamma] + \gamma \left(\left(\tilde{I}_m : m\right) k[x_1, x_2, \ldots, x_n, \gamma]\right)\right] : \gamma\right) && (2.24)\\
&= \left(\tilde{I}_{\langle m\rangle} + (\tilde{I}_m : m)\right) k[x_1, x_2, \ldots, x_n, \gamma] && (2.25)\\
&= (\tilde{I}_m : m) k[x_1, x_2, \ldots, x_n, \gamma], && (2.26)
\end{aligned}$$

where the last step holds as $\tilde{I}_{\langle m\rangle} \subseteq \tilde{I}_m$.

□

Note that this proof is much simpler than the equivalents in the preceding sections, and reproduces their results in a more general case up to a few small changes in definition.

Theorem 2.3.3. *Let m be a member of a minimal generating set of monomials for a monomial ideal $I \subset k[x_1, \ldots, x_n]$. Let deg$(m)$ represent the degree of the monomial m. Then,*

$$HS(R) = HS(\tilde{R}_m) - (1-t)t^{deg(m)-1} \left(HS(\tilde{R}_{[m]}) - HS(\tilde{R}_{\langle m\rangle})\right) \tag{2.27}$$

Proof. The bulk of the proof involves examining a pair of short exact sequences connected by the previous lemma.

$$\begin{tikzcd}
0 \arrow[r] & \frac{k[x_1,\ldots,x_n,\gamma]}{(\tilde{I}_{[m]}:\gamma)}(-\deg(m)) \arrow[r, hook, "\cdot\gamma"] \arrow[d, "\cong"] & \tilde{R}_{[m]}(1-\deg(m)) \arrow[r, two heads, "\gamma\mapsto 0"] & \tilde{R}_{\langle m\rangle}(1-\deg(m)) \arrow[r] & 0 \\
0 \arrow[r] & \frac{k[x_1,\ldots,x_n]}{(\tilde{I}_m:m)}[\gamma](-\deg(m)) \arrow[r, hook, "\cdot m"] & \tilde{R}_m[\gamma] \arrow[r, two heads, "m\mapsto 0"] & R[\gamma] \arrow[r] & 0
\end{tikzcd} \tag{2.28}$$

Note that the top sequence has undergone a shift in grading and the bottom sequence has had the indeterminate γ adjoined solely to facilitate the isomorphism from the lemma.

As in previous sections, since Hilbert Series satisfy an additive relation on exact sequences and are preserved under degree zero isomorphisms, the following equations hold:

$$t^{\deg(m)}HS\left(\frac{k[x_1,\ldots,x_n,\gamma]}{(I_{[m]}:\gamma)}\right) = t^{\deg(m)-1}HS(R_{[m]}) - t^{\deg(m)-1}HS(R_{\langle m\rangle}), \text{ and} \tag{2.29}$$

$$t^{\deg(m)}HS\left(\frac{k[x_1,\ldots,x_n]}{(I_m:m)}[\gamma]\right) = \frac{1}{1-t}HS(R_m) - \frac{1}{1-t}HS(R) \tag{2.30}$$

Using Lemma 2.3.2, the equations can be combined and simplified.

$$t^{\deg(m)-1}HS(R_{[m]}) - t^{\deg(m)-1}HS(R_{\langle m\rangle}) = \frac{1}{1-t}HS(R_m) - \frac{1}{1-t}HS(R) \tag{2.31}$$

$$HS(R) = HS(R_m) - t^{\deg(m)-1}(1-t)\left(HS(R_{[m]} - HS(R_{\langle m\rangle})\right), \text{ as desired.} \tag{2.32}$$

$\square$

For an example using the preceding theorem to construct the Hilbert series associated to a family of quotients by monomial ideals, see Examples 3.2.4 and 3.2.5.

2.4 Subgraph Contraction

In the preceding two sections, an identity for graphs was generalized by moving into more complicated structures - hypergraphs and general monomial ideals. In this section, we generalize in a different manner. In particular, we delete, contract, and extract entire vertex-induced subgraphs simultaneously. Consequently, we first define what these operations mean in this context and fix notation accordingly.

Definition 2.4.1. Let G be a simple graph with a vertex-induced subgraph H. Let R be the graph ring of G. Then

(1) G_H is the graph formed by deleting the edges of G which are also in its subgraph H, and R_H is its associated graph ring.

(2) $G_{[H]}$ is the graph formed by contracting every edge in H, thus identifying all vertices in H to a single vertex, and $R_{[H]}$ is its associated graph ring.

(3) $G_{\langle H\rangle}$ is the graph formed by deleting all of vertices in the graph G which are shared with its subgraph H, referred to as the extraction, and $R_{\langle H\rangle}$ is its associated graph ring. This is equivalent to deleting the contracted vertex in $G_{[H]}$.

This generalization, however, has a nontrivial cost to the simplicity of the results, which will also require the following definitions

Definition 2.4.2. Let G be a simple graph with a vertex-induced subgraph H. Let H have vertex set $V(H) = \{\delta_0, \delta_1, \ldots, \delta_n\}, n \geq 1$. Then,

(1) J_H is the ideal of R_H defined by

$$J_H = \langle \delta_i\delta_j : \delta_i \text{ and } \delta_j \text{ are adjacent in } H\rangle \tag{2.33}$$

(2) K_H is the ideal of R_H defined by

$$K_H = \langle \delta_i - \delta_0 : \delta_i \in V(H)\rangle \cap J_H \tag{2.34}$$

Note that this definition of J_H is similar to that of the edge ideal for H, but it is an ideal of a different ring.

Theorem 2.4.3. *Let G be a simple graph with a vertex-induced subgraph H. Assume that H contains at least two vertices and label the vertices of G within H as $\{\delta_0, \delta_1, \ldots, \delta_n\}, n \geq 1$. Let the vertex created by the contraction of H retain the name δ_0. Then,*

$$HS(R) = HS(R_H) - tHS(R_{[H]}) + tHS(R_{\langle H\rangle}) - HS(K_H). \tag{2.35}$$

Proof.

$$\begin{tikzcd}
 & K_H \arrow[d, hook, "i"] & & & \\
0 \arrow[r] & J_H \arrow[r, hook, "i"] \arrow[d, two heads, "\forall i\leq n"', "\delta_i-\delta_0\mapsto 0"] & R_H \arrow[r, two heads, "J_H\mapsto 0"] & R \arrow[r] & 0 \\
0 \arrow[r] & \delta_0^2 R_{[H]} \arrow[r, hook, "\phi"] & R_{[H]}(-1) \arrow[r, two heads, "\delta_0\mapsto 0"] & R_{\langle H\rangle}(-1) \arrow[r] & 0
\end{tikzcd} \tag{2.36}$$

A few of the listed maps need some explanation or justification. First, the map ϕ reduces the exponent of the indeterminate δ_0 by one. Since G and H are simple, both $(0 :_{R_{[H]}} \delta_0)$ and $(0 :_{R_{[H]}} \delta_0^2)$ are generated by the neighbors of δ_0 outside of H, and therefore they are equal. Consequently,

ϕ is just the typical inclusion accompanied by a shift in grading. Secondly, since identifying the vertices of H within G_H graphically results in $G_{[H]}$, the codomain of the map connecting the two short exact sequences is appropriate. K_H is the kernel of the map connecting the two short exact sequences simply because it contains exactly what is simultaneously within the domain of the map and what is explicitly sent to zero by the map. The Hilbert Series result follows from rote computation. □

This result simplifies notably in the case where H is connected to the remainder of G by a bridge, as discussed below:

Corollary 2.4.4. *Let G be a simple graph composed of two subgraphs connected by a bridge, and let H be one of the subgraphs. Assume that H contains at least two vertices and label the vertices of G within H as $\{\delta_0, \delta_1, \ldots, \delta_n\}, n \geq 1$ with the bridge incident to δ_0. Let the vertex created by the contraction of H retain the name δ_0. Then,*

$$HS(R) = (1 - t(1-t)^n)\, HS(R_H) + tHS(R_{\langle H \rangle}) - HS(K_H). \tag{2.37}$$

Proof. The proof amounts to noting that contraction of the subgraph H to an endpoint of the bridge only differs in result from the deletion of H in that the deletion contains n isolated vertices: all of the vertices of H excepting δ_0. Consequently,

$$HS(R_{[H]}) = (1-t)^n HS(R_H). \tag{2.38}$$

Substituting this into the result of Theorem 2.4.3 yields the result. □

2.5 Hilbert Series of Graph Complements

The following theorem gives a brief, elementary description of how Hilbert series behave with regards to graph complements.

Theorem 2.5.1. *Let G be a simple graph on n vertices with graph ring R, and let $\overline{R}$ represent the graph ring of the complement of G. Let K_n and $\overline{K}_n$ represent the graph rings of the complete and empty graphs on these vertices, respectively. Define I to be a function which maps a graph ring to its corresponding edge ideal understood as a module of its vertex ring. Then,*

$$HS(R) + HS(\overline{R}) = HS(K_n) + HS(\overline{K}_n) + HS(I(R) \cap I(\overline{R})). \tag{2.39}$$

Alternatively, this is equivalent to

$$HS(R) + HS(\overline{R}) = HS(I(R) \cap I(\overline{R})) + \frac{1 + (1-t)^{n-1}(1 + (n-1)t)}{(1-t)^n} \tag{2.40}$$

Proof. Consider the following exact complex:

$$0 \longrightarrow I(R) \cap I(\overline{R}) \xhookrightarrow{i} \overline{K}_n \xrightarrow{\phi} R \oplus \overline{R} \xrightarrow{\pi} K_n \longrightarrow 0, \tag{2.41}$$

where

$$\phi : x \mapsto (x, x) + I(R) \oplus I(\overline{R}) \tag{2.42}$$

$$\pi : (x, y) + I(R) \oplus I(\overline{R}) \mapsto x - y + I(K_n). \tag{2.43}$$

Note that π is well defined, as $I(R) \subseteq I(K_n)$ and $I(\overline{R}) \subseteq I(K_n)$.

Exactness is clear by inspection except perhaps at $R \oplus \overline{R}$. Let $(x, y) + I(R) \oplus I(\overline{R}) \in \ker \pi$.

Then there exists some $k \in I(K_n)$ such that $x = y + k$. As $I(K_n)$ is generated by monomials corresponding to all possible edges on the vertices of G, we can write k as a linear combination of these generators. By perhaps reordering these generators so they are grouped by correspondence to either edges in G or $\overline{G}$, we can write $k = k_R + k_{\overline{R}}$, where $k_R \in I(R)$ and $k_{\overline{R}} \in I(\overline{R})$. Then

$$
\begin{aligned}
(x, y) + I(R) \oplus I(\overline{R}) &= (y + k, y) + I(R) \oplus I(\overline{R}) && (2.44)\\
&= (y + k_R + k_{\overline{R}}, y) + I(R) \oplus I(\overline{R}) && (2.45)\\
&= (y + k_{\overline{R}}, y + k_{\overline{R}}) + (k_R, -k_{\overline{R}}) + I(R) \oplus I(\overline{R}) && (2.46)\\
&= (y + k_{\overline{R}}, y + k_{\overline{R}}) + I(R) \oplus I(\overline{R}) && (2.47)\\
&= \phi(y + k_{\overline{R}}). && (2.48)
\end{aligned}
$$

Thus the kernel of π is within the range of ϕ. The reverse inclusion is clear.

The Hilbert series result follows from rote computation.

□

CHAPTER 3: EXAMPLES

3.1 A Hypergraph Example

In [13], Schrijver notes an example, denoted J_n, of a minimally non-ideal hypergraph, minimal under vertex contraction and vertex deletion. In this section, we will use the main result of Section 2.2 to compute the Hilbert series for this example. This example was also published as part of [5].

Definition 3.1.1. Let J_n refer to the hypergraph with vertex set $V = \{\delta_0, \delta_1, \ldots, \delta_n\}$ and edge set $E = \{\{\delta_1, \delta_2, \ldots, \delta_n\}, \{\delta_0, \delta_1\}, \{\delta_0, \delta_2\}, \ldots, \{\delta_0, \delta_n\}\}$.

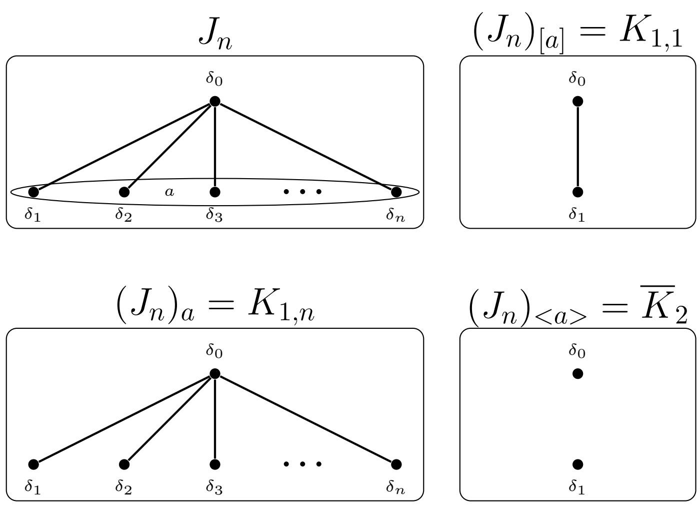

Figure 3.1: J_n and a contraction, deletion, and extraction.

Note, then, that if we pick the edge $a = \{\delta_1, \delta_2, \ldots, \delta_n\}$,

$$(J_n)_a = S_n = K_{1,n} \tag{3.1}$$

$$(J_n)_{[a]} = K_2 = K_{1,1} \tag{3.2}$$

$$(J_n)_{\langle a\rangle} = \overline{K}_2 \tag{3.3}$$

For the sake of notational brevity, we will define the Hilbert series of a hypergraph to be the Hilbert series of its associated hypergraph ring.

The Hilbert series for $K_{1,1}$ and $\overline{K}_n$ are easily understood from the definition of the Hilbert series, and thus their formal calculations are omitted. In particular, $HS(K_{1,1}) = \frac{1+t}{1-t}$ and $HS(\overline{K}_n) = \frac{1}{(1-t)^n}$.

We begin by building the Hilbert series of $K_{1,n}$ inductively to help illustrate the results of this thesis, though it could also be determined combinatorially.

Lemma 3.1.2. *Let $n \geq 1$. The Hilbert series of the hypergraph ring associated to $K_{1,n}$, the star with n edges, is*

$$HS(K_{1,n}) = \frac{1 + t(1-t)^{n-1}}{(1-t)^n}. \tag{3.4}$$

Proof. The case when $n = 1$ is trivial, as mentioned above. Assume the Lemma holds at $n = k$. By Theorem 2.2.3,

$$HS(K_{1,k+1}) = \frac{HS(K_{1,k})}{1-t} - t\left(\frac{HS(K_{1,k})}{1-t} - HS(\overline{K}_{k+1})\right) \tag{3.5}$$

$$= \frac{1 + t(1-t)^{k-1}}{(1-t)^{k+1}} - t\left(\frac{1 + t(1-t)^{k-1}}{(1-t)^{k+1}} - \frac{1}{(1-t)^{k+1}}\right) \tag{3.6}$$

$$HS(K_{1,k+1}) = \frac{1 + t(1-t)^k}{(1-t)^{k+1}}, \text{ as desired.} \tag{3.7}$$

Thus, the lemma holds by induction. $\square$

Now that we know the Hilbert series of the contraction, deletion, and extraction of a specified edge of J_n, we can easily find the Hilbert series of J_n itself:

Example 3.1.3. Let $n \geq 1$. The Hilbert series of the hypergraph ring associated to J_n is

$$HS(J_n) = \frac{1 - t^n + t(1-t)^{n-1}}{(1-t)^n} \tag{3.8}$$

Proof. By Theorem 2.2.3,

$$\begin{aligned} HS(J_n) &= HS(K_{1,n}) - \frac{t^{n-1}}{(1-t)^{n-2}} \left(\frac{HS(K_{1,1})}{1-t} - HS(\overline{K}_2) \right) && (3.9) \\ &= \frac{1 + t(1-t)^{n-1}}{(1-t)^n} - \frac{t^{n-1}}{(1-t)^{n-2}} \left(\frac{1+t-1}{(1-t)^2} \right) && (3.10) \\ &= \frac{1 - t^n + t(1-t)^{n-1}}{(1-t)^n} && (3.11) \end{aligned}$$

$\square$

3.2 A Monomial Ideal Example

In [3], an example of a family of monomial ideals with maximal decomposition cost for the algorithm then used by CoCoA[1] to compute Hilbert series is given. Our techniques allow for the quick computation of a recursive formula for the Hilbert series of these ideals, which can then be used to inductively prove a direct formula.

Definition 3.2.1. Let m and n be positive integers.

If n is even, define

$$I_{m,n} = (x_1^0 x_2^m x_3^0 x_4^m \dots x_{n-1}^0 x_n^m, \dots, x_1^i x_2^{m-i} x_3^i x_4^{m-i} \dots x_{n-1}^i x_n^{m-i}, \dots, x_1^m x_2^0 x_3^m x_4^0 \dots x_{n-1}^m x_n^0) \tag{3.12}$$

If n is odd, define

$$I_{m,n} = (x_1^0 x_2^m x_3^0 x_4^m \dots x_n^0, \dots, x_1^i x_2^{m-i} x_3^i x_4^{m-i} \dots x_n^i, \dots, x_1^m x_2^0 x_3^m x_4^0 \dots x_n^m) \tag{3.13}$$

To better organize the computation, it is useful to first consider the results of deleting, extracting, and contracting $I_{m,n}$ by the first monomial listed in its definition.

Lemma 3.2.2. *Let $n > 1$ be even and define $u = x_1^0 x_2^m x_3^0 x_4^m \dots x_{n-1}^0 x_n^m$. Then,*

$$(I_{m,n})_u = x_1 x_3 \dots x_{n-1} I_{m-1,n} \tag{3.14}$$

$$(I_{m,n})_{\langle u \rangle} = (x_1^m x_2^0 x_3^m x_4^0 \dots x_{n-1}^m x_n^0) \tag{3.15}$$

$$(I_{m,n})_{[u]} = (x_1 x_3 \dots x_{n-1} \gamma, x_1^m x_2^0 x_3^m x_4^0 \dots x_{n-1}^m x_n^0) \tag{3.16}$$

Let $n > 1$ be odd and define $u = x_1^0 x_2^m x_3^0 x_4^m \dots x_n^0$. Then,

$$(I_{m,n})_u = x_1 x_3 \dots x_n I_{m-1,n} \tag{3.17}$$

$$(I_{m,n})_{\langle u \rangle} = (x_1^m x_2^0 x_3^m x_4^0 \dots x_n^m) \tag{3.18}$$

$$(I_{m,n})_{[u]} = (x_1 x_3 \dots x_n \gamma, x_1^m x_2^0 x_3^m x_4^0 \dots x_n^m) \tag{3.19}$$

Proof. Let $n > 1$ be even and define u accordingly. Then,

$$(I_{m,n})_u = (x_1^1 x_2^{m-1} \dots x_n^{m-1}, \dots, x_1^i x_2^{m-i} \dots x_n^{m-i}, \dots, x_1^m x_2^0 \dots x_n^0) \tag{3.20}$$

$$= x_1 x_3 \dots x_{n-1}(x_1^0 x_2^{m-1} \dots x_n^{m-1}, \dots, x_1^{i-1} x_2^{m-i} \dots x_n^{m-i}, \dots, x_1^{m-1} x_2^0 \dots x_n^0) \tag{3.21}$$

$$= x_1 x_3 \dots x_{n-1} I_{m-1,n} \tag{3.22}$$

Also, since supp$(u) = \{x_2, x_4, \dots, x_n\}$ and the only monomial in $(I_{m,n})_u$ divisible by only indeterminates outside the support of u is $x_1^m x_2^0 x_3^m x_4^0 \dots x_{n-1}^m$, we have that

$$(I_{m,n})_{\langle u \rangle} = (x_1^m x_2^0 x_3^m x_4^0 \dots x_{n-1}^m). \tag{3.23}$$

Finally,

$$(I_{m,n})_{[u]} = (I_{m,n})_{\langle u \rangle} + \gamma\left((I_{m,n})_u : f\right) \tag{3.24}$$

$$= (x_1^m x_2^0 x_3^m x_4^0 \dots x_{n-1}^m) + \gamma(x_1 x_3 \dots x_{n-1}) \tag{3.25}$$

$$= (x_1 x_3 \dots x_{n-1}\gamma, x_1^m x_2^0 x_3^m x_4^0 \dots x_{n-1}^m x_n^0) \tag{3.26}$$

The proofs when n is odd are identical up to small changes in indices. □

Note that the Hilbert series for both the extraction and the contraction become easily available via direct computation, regardless of whether n is even or odd. To handle the deletion, we make use of its relation to $I_{m-1,n}$ in the following lemma.

Lemma 3.2.3. *Let $n > 1$ and let $R_{m,n}$ refer to the quotient of the polynomial ring $k[x_1, \dots, x_n]$ by $I_{m,n}$. Then,*

$$HS((R_{m,n})_u) = t^{\lceil \frac{n}{2} \rceil} HS(R_{m-1,n}) + \frac{1 - t^{\lceil \frac{n}{2} \rceil}}{(1-t)^n} \tag{3.27}$$

Proof. As before, we will provide the proof in the case when n is even. The only difference in the case when n is odd is a change in a shift in grading.

Define $v = x_1x_3 \dots x_{n-1}$ and consider the following short exact sequence:

$$0 \to \frac{k[x_1,\dots,x_n]}{((I_{m,n})_u : v)}(-\tfrac{n}{2}) \xhookrightarrow{\cdot v} (R_{m,n})_u \xrightarrow{v \mapsto 0} \frac{k[x_1,x_2,\dots,x_n]}{(v)} \to 0 \tag{3.28}$$

Since $(I_{m,n})_u = x_1x_3 \dots x_{n-1}I_{m-1,n} = vI_{m-1,n}$, it is clear that $((I_{m,n})_u : v) = I_{m-1,n}$, so we can alter our sequence accordingly. This yields

$$0 \to R_{m-1,n}\,(-\tfrac{n}{2}) \xhookrightarrow{\cdot v} (R_{m,n})_u \xrightarrow{v \mapsto 0} \frac{k[x_1,x_2,\dots,x_n]}{(v)} \to 0 \tag{3.29}$$

Therefore, we can construct the following Hilbert series relation

$$HS((R_{m,n})_u) = t^{\frac{n}{2}}HS(R_{m-1,n}) + HS\left(\frac{k[x_1,x_2,\dots,x_n]}{(v)}\right) \tag{3.30}$$

The result for when n is even then follows from the fact that the degree of v is $\frac{n}{2}$. $\square$

We may now continue to form a recursive relation for the Hilbert series of $R_{m,n}$ and use it to inductively verify a direct computation of the Hilbert series. We will proceed separately in the cases where n is even and odd.

Example 3.2.4. Let $n > 1$ be even, m a positive integer, and let $R_{m,n}$ refer to the quotient of the polynomial ring $k[x_1,\dots,x_n]$ by $I_{m,n}$. Then,

$$HS(R_{m,n}) = t^{\frac{n}{2}}HS(R_{m-1,n}) + \frac{(1-t^{\frac{n}{2}})(1-t^{\frac{mn}{2}})}{(1-t)^n} \text{ when m} > 1 \tag{3.31}$$

and

$$HS(R_{m,n}) = \frac{1-(m+1)t^{\frac{mn}{2}} + mt^{\frac{(m+1)n}{2}}}{(1-t)^n}. \tag{3.32}$$

Proof. We begin by proving the recursive relation. By applying Theorem 2.3.3 to $R_{m,n}$ using the monomial $u = x_1^m x_2^0 x_3^m x_4^0 \dots x_{n-1}^m$ and then performing some calculations and simplifications using Lemma 3.2.2 and Lemma 3.2.3, we can see that

$$HS(R_{m,n}) = HS((R_{m,n})_u) - (1-t)t^{\frac{mn}{2}-1}\left(HS((R_{m,n})_{[u]}) - HS((R_{m,n})_{\langle u\rangle})\right) \tag{3.33}$$

$$= t^{\frac{n}{2}} HS(R_{m-1,n}) + \frac{1-t^{\frac{n}{2}}}{(1-t)^n} \tag{3.34}$$

$$- (1-t)t^{\frac{mn}{2}-1}\left(HS\left(\frac{k[x_1,\dots,x_n,\gamma]}{(x_1x_3\dots x_{n-1}\gamma, x_1^m x_3^m \dots x_{n-1}^m)}\right) - HS\left(\frac{k[x_1,\dots,x_n]}{(x_1^m x_3^m \dots x_{n-1}^m)}\right)\right)$$

$$= t^{\frac{n}{2}} HS(R_{m-1,n}) + \frac{1-t^{\frac{n}{2}}}{(1-t)^n} - (1-t)t^{\frac{mn}{2}-1}\left(\frac{1-t^{\frac{n}{2}+1} - t^{\frac{mn}{2}} + t^{\frac{mn}{2}+1}}{(1-t)^{n+1}} - \frac{1-t^{\frac{mn}{2}}}{(1-t)^n}\right) \tag{3.35}$$

$$= t^{\frac{n}{2}} HS(R_{m-1,n}) + \frac{(1-t^{\frac{n}{2}})(1-t^{\frac{mn}{2}})}{(1-t)^n} \tag{3.36}$$

Now, we use this relation to inductively build the direct formula. Note that for any positive, even n,

$$HS(R_{1,n}) = HS\left(\frac{k[x_1,\dots,x_n]}{(x_1x_3\dots x_{n-1}, x_2x_4\dots x_n)}\right) \tag{3.37}$$

$$= \frac{(1-t^{\frac{n}{2}})(1-t^{\frac{n}{2}})}{(1-t)^n} \tag{3.38}$$

$$= \frac{1-2t^{\frac{n}{2}}+t^n}{(1-t)^n}, \tag{3.39}$$

so the result holds for $m = 1$. Next, assume that the result holds when $m = i$ for some fixed $i \geq 1$.

We will consider $m = i + 1$ and apply the recursion relation.

$$HS(R_{i+1,n}) = t^{\frac{n}{2}} HS(R_{i,n}) + \frac{(1 - t^{\frac{n}{2}})(1 - t^{\frac{(i+1)n}{2}})}{(1-t)^n} \tag{3.40}$$

$$= t^{\frac{n}{2}} \frac{1 - (i+1)t^{\frac{in}{2}} + it^{\frac{(i+1)n}{2}}}{(1-t)^n} + \frac{1 - t^{\frac{n}{2}} - t^{\frac{(i+1)n}{2}} + t^{\frac{(i+2)n}{2}}}{(1-t)^n} \tag{3.41}$$

$$= \frac{1 - (i+2)t^{\frac{(i+1)n}{2}} + (i+1)t^{\frac{(i+2)n}{2}}}{(1-t)^n}, \tag{3.42}$$

as desired. Therefore the result holds for all positive integers m when n is an even positive integer.

□

Now, we imitate the above steps when n is odd. However, the reader will note that this case provides extra complications. This largely occurs due to the lack of symmetry in $I_{m,n}$ as compared to when n is even.

Example 3.2.5. Let $n \geq 1$ be odd, m a positive integer, and let $R_{m,n}$ refer to the quotient of the polynomial ring $k[x_1, \ldots, x_n]$ by $I_{m,n}$. Then,

$$HS(R_{m,n}) = t^{\frac{n+1}{2}} HS(R_{m-1,n}) + \frac{(1 - t^{\frac{n+1}{2}})(1 - t^{\frac{m(n-1)}{2}})}{(1-t)^n} \text{ when m} > 1 \tag{3.43}$$

and

$$HS(R_{m,n}) = \frac{1 - t - t^{\frac{m(n-1)}{2}} + t^{\frac{m(n+1)}{2}+1} + t^{\frac{(m+1)(n-1)}{2}+1} - t^{\frac{(m+1)(n+1)}{2}}}{(1-t)^{n+1}}. \tag{3.44}$$

Proof. As before, we begin by constructing the recursive relation using Theorem 2.3.3 alongside Lemmas 3.2.2 and 3.2.3.

$$
\begin{aligned}
HS(R_{m,n}) &= HS((R_{m,n})_u) - (1-t)t^{\frac{m(n-1)}{2}-1}\left(HS((R_{m,n})_{[u]}) - HS((R_{m,n})_{\langle u\rangle})\right) && (3.45)\\
&= t^{\frac{n+1}{2}} HS(R_{m-1,n}) + \frac{1-t^{\frac{n+1}{2}}}{(1-t)^n} && (3.46)\\
&\quad - (1-t)t^{\frac{m(n-1)}{2}-1}\left(HS\left(\frac{k[x_1,\dots,x_n,\gamma]}{(x_1x_3\dots x_n\gamma, x_1^m x_3^m \dots x_n^m)}\right) - HS\left(\frac{k[x_1,\dots,x_n]}{(x_1^m x_3^m \dots x_n^m)}\right)\right)\\
&= t^{\frac{n+1}{2}} HS(R_{m-1,n}) + \frac{1-t^{\frac{n+1}{2}}}{(1-t)^n} && (3.47)\\
&\quad - (1-t)t^{\frac{m(n-1)}{2}-1}\left(\frac{1-t^{\frac{n+1}{2}+1} - t^{\frac{m(n+1)}{2}} + t^{\frac{m(n+1)}{2}+1}}{(1-t)^{n+1}} - \frac{1-t^{\frac{m(n+1)}{2}}}{(1-t)^n}\right)\\
&= t^{\frac{n+1}{2}} HS(R_{m-1,n}) + \frac{(1-t^{\frac{n+1}{2}})(1-t^{\frac{m(n-1)}{2}})}{(1-t)^n} && (3.48)
\end{aligned}
$$

Therefore, the recursive formula holds.

Let n be a positive odd number. Then,

$$
\begin{aligned}
HS(R_{1,n}) &= HS\left(\frac{k[x_1,\dots,x_n]}{(x_1x_3\dots x_n, x_2x_4\dots x_{n-1})}\right) && (3.49)\\
&= \frac{(1-t^{\frac{n+1}{2}})(1-t^{\frac{n-1}{2}})}{(1-t)^n} && (3.50)\\
&= \frac{1-t-t^{\frac{n-1}{2}} + t^{\frac{n+1}{2}+1} + t^n - t^{n+1}}{(1-t)^{n+1}}, && (3.51)
\end{aligned}
$$

so the direct result holds when n is odd and $m = 1$. Now assume that it holds for some odd n and some fixed $m = i$. Then,

$$HS(R_{i+1,n}) = t^{\frac{n+1}{2}} HS(R_{i,n}) + \frac{(1 - t^{\frac{n+1}{2}})(1 - t^{\frac{(i+1)(n-1)}{2}})}{(1-t)^n} \tag{3.52}$$

$$= t^{\frac{n+1}{2}} \frac{1 - t - t^{\frac{i(n-1)}{2}} + t^{\frac{i(n+1)}{2}+1} + t^{\frac{(i+1)(n-1)}{2}+1} - t^{\frac{(i+1)(n+1)}{2}}}{(1-t)^{n+1}} \tag{3.53}$$

$$+ \frac{1 - t^{\frac{n+1}{2}} - t^{\frac{(i+1)(n-1)}{2}} + t^{\frac{(i+2)(n-1)}{2}+1}}{(1-t)^n}$$

$$= \frac{1 - t - t^{\frac{(i+1)(n-1)}{2}} + t^{\frac{(i+1)(n+1)}{2}+1} + t^{\frac{(i+2)(n-1)}{2}+1} - t^{\frac{(i+2)(n+1)}{2}}}{(1-t)^{n+1}}, \tag{3.54}$$

as desired. Thus, the result holds for all $m \geq 1$ by induction. $\square$

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