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# Secondary And Postsecondary Calculus Instructors' Expectations Of Student Knowledge Of Functions: A Multiple-case Study

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# CALCULUS INSTRUCTORS' ASSUMPTIONS OF THEIR STUDENTS' PRIOR KNOWLEDGE OF FUNCTIONS: A MULTIPLE-CASE STUDY

by

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# ABSTRACT

This multiple-case study examines the explicit and implicit assumptions of six veteran calculus instructors from three types of educational institutions, comparing and contrasting their views on the iteration of conceptual understanding and procedural fluency of pre-calculus topics. There were three components to the research data recording process. The first component was a written survey, the second component was a "think-aloud" activity of the instructors analyzing the results of a function diagnostic instrument administered to a calculus class, and for the third component, the instructors responded to two quotations. As a result of this activity, themes were found between and among instructors at the three types of educational institutions related to their expectations of their incoming students' prior knowledge of pre-calculus topics related to functions. Differences between instructors of the three types of educational institutions included two identifiable areas: (1) the teachers' expectations of their incoming students and (2) the methods for planning instruction. In spite of these differences, the veteran instructors were in agreement with other studies' findings that an iterative approach to conceptual understanding and procedural fluency are necessary for student understanding of pre-calculus concepts.



*Keywords:* Student misconceptions of functions; Teacher expectations; Transition from high school to university mathematics; Diagnostic assessment; Error analysis; Conceptual understanding; Procedural fluency

This dissertation is dedicated to my family.

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# LIST OF ACRONYMS

AP = Advanced Placement

APICS = Atlantic Provinces Council on the Sciences (originally called the Atlantic Provinces Inter-University Committee on the Sciences)

CAS = Computer algebra system

CCSS-M = Common Core State Standards – Mathematics

CUPM = Committee on the Undergraduate Program in Mathematics

FOIL = First-Outside-Inside-Last (Often used by algebra instructors to help students remember a procedure for multiplying two binomial expressions.)

IRB = Internal Review Board

NCTM = National Council of Teachers of Mathematics

NSF = National Science Foundation

STEM = Science, Technology, Engineering and Mathematics

STEP = STEM Talent Expansion Program

# CHAPTER 1: INTRODUCTION

In the mid-1970s several mathematics educators posed the question, "Should we teach calculus in high school?" (Mann, 1976; Rash, 1977; Sorge & Wheatley, 1977) Ferrini-Mundy and Gaudard (1992) intended to answer that question through a study conducted in the 1987-88 academic year at a mid-sized university in a first semester calculus course with 751 college students. The study revealed that the highest level of mathematics one studies in secondary school has the strongest influence on subsequent completing of a bachelor's degree (p. 56). A better question before us now is not *should* we teach calculus in high school, but rather *how* should we teach calculus in high school. In an attempt to contribute to the mathematics education literature on how we should teach calculus in high school, I conducted a qualitative study of calculus instructors, but not just high school calculus instructors, rather instructors at the postsecondary levels as well. The aim of this research was to compare and contrast the similarities and differences between and among instructors of calculus at three types of educational institutions, high school, state college and university, where first semester college calculus is offered.

Although the Ferrini-Mundy and Gaudard (1992) study did not address affective matters, it is interesting to note that many college calculus instructors in the study noticed a "false confidence" among students with a year's calculus course in secondary school. The authors suggest that secondary school courses in calculus may predispose students to the procedural aspects of the college course and may be less open to the conceptual development of the derivative and integral (p. 68-9). Similarly, Orton (1985) states the crucial issue is not if or when calculus should be taught, but how should instructors promote the understanding of calculus and

pre-calculus concepts such that students can comprehend and retain the information presented. In order to examine the issue of *how* instructors promote the understanding of calculus, this study focused on the calculus instructors' assumptions of their incoming students' prior knowledge at the three educational institutions where calculus is taught.

Calculus is offered at three separate types of educational institutions: high schools, community/state colleges and four-year universities. The first semester of calculus, which is the focus of this study, is offered in most high schools through the Advanced Placement (AP) Calculus AB course. According to the College Board,

> An AP course in calculus consists of a full high school academic year of work that is comparable to calculus courses in colleges and universities. It is expected that students who take an AP course in calculus will seek college credit, college placement, or both, from institutions of higher learning…Calculus AB can be offered as an AP course by any school that can organize a curriculum for students with advanced mathematical ability (College Board AP, 2006, p. 3).

The major difference between AP Calculus AB and the first semester of Calculus in postsecondary institutions is the length of time of instruction. AP Calculus is designed as a year-long course while Calculus I at most postsecondary institutions is a semester course. In addition to the length of time from start to finish of the course, the amount of time the students spend with the instructor in the classroom is also worth noting. In a secondary school, students usually meet with their instructor on a daily basis for 45 minutes, or 90 minutes every other day if the school is on a "block" schedule. At the postsecondary institutions, classes typically meet for two 75 - 90 minute sessions per week. Although it states on the AP Calculus AB course description that "each AP course is modeled upon a comparable college course, and college and university faculty play a vital role in ensuring that AP courses align with college-level standards" (College Board AP, 2011, p. 2), surveys have shown that secondary and postsecondary mathematics

instructors tend to have differing views as to the importance of particular knowledge and skills that will lead to a student's success in college-level mathematics (ACT 2006, 2009).

Studies have compared high school and college faculty ratings of importance of specific content and pedagogy with respect to success in college (Artigue et al., 2007; Carlson, 1998; James, 1995; Stroumbakis, 2010); these studies were focused on the content of the calculus course itself. In contrast, this study focuses on the calculus instructors and their underlying assumptions about the iteration of conceptual understanding and procedural fluency. Specifically, the instructional decisions calculus instructors make based upon their assessment of their students' prior knowledge of functions. Topics in mathematics prior to calculus include arithmetic, algebra, geometry and trigonometry, but according to the Atlantic Universities Inter-university Council on the Sciences (APICS) Mathematics Committee, functions are the mathematical objects that link quantities and much of calculus involves manipulating functions (Dawson, 2007). "A strong understanding of the function concept is also essential for any student hoping to understand calculus – a critical course for the development of future scientists, engineers, and mathematicians" (Carlson & Oehrtman, 2005, p.1). Since functions are the fundamental objects of calculus, it is appropriate to begin a study of teaching calculus with students' prior knowledge of functions.

**Rationale**

Calculus is considered the first postsecondary course in a string of mandatory mathematics courses for students wishing to pursue careers in Science, Technology, Engineering and Mathematics (STEM). "Many students do not enter college prepared for mathematics at the level required for most STEM majors" (Cheatham, Rowell, Nelson, Stephens, & Tenpenny,

2012, p. 1). Cheatham et al. found that if students perform poorly in early mathematics courses in college, it may discourage students from pursuing a STEM-related degree, or worse, drop out of college, leading to a reduced STEM workforce. Organizations such as STEM Talent Expansion Program (STEP), funded by the National Science Foundation (NSF), were founded to improve recruitment and retention of STEM students based on "best practices" of prerequisite courses with typically high failure rates such as pre-calculus and calculus (STEP, n.d.). This and other such organizations believe a successful experience in calculus may lead to continued success in mathematics and further study in STEM fields (Cheatham et al., 2012).

The timeliness and relevance of this particular study was made clear by the March 2012 release of the joint position statement of the Mathematical Association of America (MAA) and the National Council of Teachers of Mathematics (NCTM). The statement reads as follows:

> Although calculus can play an important role in secondary school, the ultimate goal of the K–12 mathematics curriculum should not be to get students into and through a course in calculus by twelfth grade but to have established the mathematical foundation that will enable students to pursue whatever course of study interests them when they get to college. The college curriculum should offer students an experience that is new and engaging, broadening their understanding of the world of mathematics while strengthening their mastery of tools that they will need if they choose to pursue a mathematically intensive discipline (National Council of Teachers of Mathematics [NCTM], 2012, p. 1).

Along with the statement, NCTM and MAA listed several requirements and suggestions as to how this goal could be accomplished. This joint statement served as a background for the research questions, the initial surveys, and the subsequent interviews of both secondary and postsecondary calculus instructors in this study.

This was not the first joint statement issued by MAA and NCTM. In 1986 the presidents of these two professional organizations sent a letter to secondary mathematics teachers stating

two problems with teaching single variable calculus in the high school. "The first problem concerns the relationship between the calculus course offered in high school and the succeeding calculus courses in college" (The Mathematical Association of America [MAA], 1986, para.2). The MAA and NCTM presidents recommended that all students taking calculus in high school who are performing satisfactorily in the course should expect to place out of the comparable college calculus course and not use the high school course as an introductory course prior to taking Calculus 1 at the college. This recommendation spoke directly to the teachers expectations for their students. The teachers should expect students that successfully complete AP Calculus in high school to have the knowledge and skills necessary to successfully complete the follow-up Calculus course at the college-level. The second problem addressed in the 1986 letter concerned preparation for the calculus course, which was the focus of this study. The letter stated:

> MAA and NCTM recommend that students who enroll in a calculus course in secondary school should have demonstrated mastery of algebra, geometry, trigonometry, and coordinate geometry. This means that students should have at least four full years of mathematical preparation beginning with the first course in algebra. The advanced topics in algebra, trigonometry, analytic geometry, complex numbers, and elementary functions studied in depth during the fourth year of preparation are critically important for students' later courses in mathematics (The Mathematical Association of America [MAA], 1986, p. 1).

During the American Youth Policy Forum, Conley (2009) stated the importance of secondary and postsecondary educational institutions coming together to discuss and explicitly define the knowledge and skills necessary for a student to succeed in credit-bearing general education courses at the university level. Conley defined the term succeed as "completing entry-level courses at a level of understanding and proficiency that makes it possible for the student to consider taking the next course in the sequence or the next level of course in the subject area"

(p.5). When a student has reached this level, he or she is considered "college-ready." There is a larger proportion of high school students today aspiring for a college education than ever before; however, not all are "college-ready." According to the Chicago Public School Postsecondary Department and the Consortium's Research Project, 40% of all tenth-graders aspired to a 4-year degree in 1980 and that percentage jumped to 83% in 2005. Although the majority of tenth-grade students aspire for a 4-year degree, less than one-third actually attain that goal (Roderick, Nagaoka, Coca, & Moeller, 2008, p. 2). Although the percentage of students that begin work leading toward a college degree has increased dramatically, the percentage of students that actually accomplish their goal of *completing* a college degree has remained relatively unchanged (Conley, 2005). Waits and Demana (1988) found that only 28% of the freshmen entering Ohio State University with *five* or more years of preparatory mathematics were ready for calculus according to the college placement test. If we can clearly identify the differences in expectations between secondary and postsecondary instructors, a conversation can begin amongst and between the instructors of Calculus at the three educational institutions about how to best prepare students for the challenges they will face in postsecondary mathematics.

> The data about calculus readiness suggest a need for a careful look at the content of the college-preparatory mathematics courses. The current curriculum scratches the surface of too many concepts and fails to dig deeply enough for students to acquire needed understanding. The curriculum should stress problem-solving and place major emphasis on the fundamental concepts of functions and graphs, concepts that are so very important for the successful study of collegiate mathematics (Waits & Demana, 2008, p. 13).

The above quotation goes to the role of conceptual knowledge versus procedural knowledge in the teaching and learning of mathematics. Twenty-five years later, mathematics educators are continuing to state that the expectations of secondary and postsecondary calculus instructors

differ on their students' procedural knowledge and conceptual understanding of mathematics (Stoumbakis, 2010; Zelkowski, 2011).

**Conceptual Framework**

Debates on learning have often focused on which type of knowledge, conceptual or procedural, develops first or is more important (Baroody, 2003; Hiebert & LeFevre, 1986; Rittle-Johnson & Seigler, 1998; Star, 2005). However, Rittle-Johnson, Siegler, and Alibali (2001) believe that the efforts expended on debating which type of knowledge is more important may have overshadowed the importance of the interactions *between* the two knowledge types during development. Specifically, knowledge of concepts and procedures may develop iteratively, with increases in one type of knowledge leading to gains in the other type of knowledge, which in turn lead to increases in the first. More recent research by Rittle-Johnson and Koedinger (2009) seems to support the theory that an iterative sequencing of conceptual and procedural activities facilitates students' ability to learn and transfer new mathematical concepts.

An iterative perspective for the development of knowledge of concepts and procedures is also supported by the nation's largest professional organization of mathematics teachers and mathematics educators, the National Council of Teachers of Mathematics (NCTM). In the introduction to their document, *Principles and Standards for School Mathematics* (2000) it states: "… all students should learn important mathematical concepts and processes with understanding" (p. ix). And within the Learning Principle of this same document the authors state that the alliance of factual knowledge, procedural proficiency and conceptual understanding are vital to the learning of complex subjects such as mathematics.

This theory is reiterated in the more recent publication of Implementing the Common Core State Standards by the National Governor's Association Center for Best Practices and the Council of Chief State School Officers (2012). In this document, the authors pose the question, "But what does mathematical understanding look like?" (Standards for mathematical practice, n.d., p. 4) It is one thing for a student to be able to recall the mnemonic device such as FOIL (First-Outside-Inside-Last; see List of Acronyms), when they see (a+b)(c+d), but it is quite different thing for the same student to be able to explain the mathematics behind the mnemonic and why it works. If the student is able to do the latter, they are more likely to be able to extrapolate on that knowledge and be able to know what to do when confronted with a less familiar problem such as (a+b+c)(d+e+f). The authors of CCSS-M state the following: "Mathematical understanding and procedural skill are equally important, and both are assessable using mathematical tasks of sufficient richness" (p. 4).

This need for a balance of both types of understanding is not a new theory. Brownell (1956) warned against substituting the teaching of procedures to be memorized (skill) for the teaching of mathematical meaning (understanding). "Understanding and skill are not identical. A single instance of insight may lead to understanding but will hardly produce skill. For skill, practice is necessary" (p. 130). Skemp (1976) distinguished between two specific types of understanding: instrumental understanding, and relational understanding. Skemp also points out the advantages of both types of understanding. Within a limited context, instrumental mathematics is usually easier to understand and one can get correct answers quickly and reliably. On the other hand, relational mathematics is more adaptable to different contexts and easier to remember because you are not burdened with a series of steps or formulas to memorize.

The terms "procedural fluency" and "conceptual understanding" of mathematics have varied, though the ideas behind the terms, as related to mathematics education, have remained relatively unchanged. Skemp (1976) coined the term "instrumental understanding" and he defined the term to mean learning the rules without learning the reason behind those rules. He contrasted "instrumental understanding" with "relational understanding." He defined relational understanding as both the "what" and "why" of mathematics. Herscovics and Kieran (1980) distinguish between "mathematical form" and "mathematical content." They define mathematical content to be the concepts, rules and relationships of mathematics, much like Skemp's relational understanding, and how I am using the term conceptual understanding for this study. Regardless of the term being used, teaching for understanding in mathematics requires that the continuity of mathematical content (relational or conceptual understanding) be demonstrated to the student during, and prior to, the introduction of the new mathematical form in order for the attainment of instrumental understanding or procedural knowledge to occur (Byers & Herscovics, 1977).

In mathematics, Ben-Hur (2006) describes conceptual understanding as "a connected web of knowledge, a network in which the linking relationships are as prominent as the discrete bits of information" (p.2). By this definition, conceptual knowledge cannot be learned by rote, rather it must be learned by thoughtful and reflective learning. To have a conceptual understanding of the square root of 2, a student may think of the length of the hypotenuse of a right isosceles triangle with sides of one. By recalling the Pythagorean Theorem and applying it to the unit triangle the student may have a conceptual understanding of $\sqrt{2}$ as the length of the hypotenuse in proportion to the length of either side. Another student may provide a short proof of the irrationality of $\sqrt{2}$ from the Rational Root Theorem.

On the other hand, procedural knowledge "involves the ability to solve problems through the manipulation of mathematical skills with the help of pencil and paper, calculator, computer, and so forth" (Ben-Hur, 2006, p.5). An example of procedural knowledge for an approximation of the $\sqrt{2}$ a student would use an algorithm such as $\frac{a_n}{2} + \frac{1}{a_n}$ and iterate the answer until they received the necessary accuracy. Starting with substituting 1 in for $a_n$ , the student would use his procedural knowledge of mathematics to get $\frac{3}{2}$ or 1.5. If this level of accuracy was not sufficient, then the student would repeat the process by substituting the answer, 1.5, in the same algorithm, and get $\frac{17}{12}$ or 1.416. This process could be repeated indefinitely for an increasingly more accurate estimation of $\sqrt{2}$ . For a procedural knowledge of deriving the square root of two, the student may not know the formal definitions of terms they need for proofs or understand the relationship between the sides and hypotenuse of a right triangle which is necessary for a conceptual understanding of the same number.

**Research Questions**

Ernest (1989) stated that the practice of teaching mathematics depends upon three key elements. The first key element is the teacher's mental schema, or cognitive framework, that helps organize and interpret mathematics and its teaching and learning. The next is the social context of the teaching situation and the constraints and/or opportunities each provides and the final element is the teacher's level of thought processes and reflection. The selection of a multiple-case study was to describe the assumptions calculus instructors at three types of educational institutions have of their incoming students' procedural fluency and conceptual

understanding of the function concept and how these assumptions effect their instructional decisions concerning the teaching of calculus. Instructors' assumptions will be generally defined as beliefs they hold concerning their students' procedural fluency and conceptual understanding of functions prior to the students taking calculus. The instructors may either explicitly or implicitly state these assumptions.

The research questions for this study are:

1. How are secondary calculus *instructors' assumptions* of their students' prior knowledge of the function concept different from that of the postsecondary calculus instructors' assumptions?

2. How do secondary and postsecondary instructor assumptions of their students' prior knowledge of functions *impact instructional decision-making* for their calculus course?

3. How do secondary and postsecondary instructors differ on their views toward students' *conceptual understanding and procedural fluency* of functions?

The learning of calculus requires knowledge of algebra, analytic geometry, functions, and trigonometry. The task of explicating instructors' expectations in each of these areas would be a lifetime achievement for a mathematics educator. For the purpose of this study, research is limited to the specific concept of functions. This chapter presented a brief introduction, rationale, conceptual framework, and research questions for this study. In the next chapter, I will review the literature pertaining to instructor assessments including a section on teacher assumptions. I examined the function concept with a variety of definitions and the evolution of the function concept from various mathematicians' perspective. This section also presents

various research studies that have attempted to identify common student misconceptions of functions. It also incorporates misconceptions related to topics prior to pre-calculus to include algebra, variable, and interpreting and creating word problems.

Chapter 3 is a detailed description of the methodology used in order to conduct this study to include the research design, participant sampling strategies, and instrumentation. Chapter 4 presents the results of the six interviews that were conducted with veteran calculus instructors at three types of educational institutions. The intent of the interview process was to use the triangulation methods to answer the research questions posed at the beginning of this study. In this chapter, various tables were constructed to visually display the data and assist in the comparison and contrast of the six instructors' responses. Being as this is a qualitative study, there is an abundance of direct quotations from the calculus instructors. The final chapter is a discussion of the results of the multiple-case study as it pertains to the three research questions. The results are structured both by components as they occurred in the interview, and by research question.

# CHAPTER 2: LITERATURE REVIEW

In this literature review, I review and critique the research and scholarship on the differences between secondary and postsecondary mathematics instructors' assessments of their incoming calculus students' prior knowledge, specifically that of functions. Although studies in mathematics education have examined the differences between procedural and conceptual understandings of entering college freshman (Carlson, Oehrtman, & Engelke, 2010; Heibert & Lefevre, 1986; Rittle-Johnson et al., 2001; Stoumbakis, 2010), these studies have not asked instructors to analyze actual student results of a diagnostic instrument and then conduct follow-up interviews of the instructors to compare and contrast secondary and postsecondary instructors' instructional decision-making based on these results. As such, this literature review provides additional insight into the impact the instructors' expectations of incoming students have on the materials and methods that are chosen for instruction at each of these levels. The analytic focus on the communication between secondary and postsecondary instructors on clearly defining the necessary skills needed to ensure student success in college-level calculus provides another insight.

"Though it is possible, and even popular, to talk about teacher behavior, it is obvious that what teachers do is directed in no small measure by what they think" (National Institute of Education, 1975, p. 7). Recognizing the importance of teacher beliefs to teacher practice, the first part of the literature review is a sociological view as to the influence teacher expectations or assumptions have on student achievement. Since I used the results of a student diagnostic instrument to explicate these assumptions and how the assumptions teachers make influence their instructional decision-making, there is a section in the literature review about assessments. The

third part of the literature review is a detailed examination of the mathematical concept of a function from various mathematicians' perspective and its importance in the understanding of higher-level mathematics. In this section I have also clearly defined the specific aspects of the function concept that will be assessed in this study. Finally, the last part of this literature review is a survey of studies on students' misconceptions of functions.

**Teacher Assumptions/Expectations**

Cooper and Good (1983) define "teacher expectations" to be the presumptions that teachers make about their students' academic achievement. Academic expectations of teachers have shown to have an effect on student performance (Weinstein, 2002). The most publicized and controversial study concerning teacher expectation is Rosenthal and Jacobson's *Pygmalion in the Classroom* (1968). Prior to a new school year researchers told teachers that particular students scored high on a test for intellectual ability when in fact no such test was administered and the students were randomly selected for identification. Tests conducted at the end of the school year offered some evidence that the identified students did perform better than non-identified students. In their study, Rosenthal and Jacobson concluded that students' intellectual development is largely a response to the teachers' expectations of the students and how those expectations are communicated. This study sparked controversy over the ethics of the experiment and the age group of the students being studied, but this is consistent with the social-cognitive perspective that beliefs can affect classroom behavior (Good & Brophy, 1997). The Pygmalion study was conducted on first and second grade students. Some critics of the theory stated that the young age of the students was a major factor in teacher expectation influencing

student performance and therefore the same cannot be said of older students. When interviewed in 1999, Rosenthal replied,

> Oh, it applies. They're wrong. There've been experiments looking at college algebra classes at the Air Force Academy, a study of undergraduates in engineering; there've been lots of studies at the college level since the book came out confirming the findings," he continues. "In fact, the original research conducted when I was at the University of North Dakota was all done with graduate students and under-graduates (Rheem, 1999, para. 2).

Rosenthal's study examined how teachers influenced selected individuals within the classroom. Few have empirically examined the possibility that teachers can have expectations for a class, as a whole. This is in spite of the fact that in 1985, years after the Pygmalion study, Brophy stated, "differential teacher treatment of intact groups and classes may well be a much more widespread and powerful mediator of self-fulfilling prophecy effect on student achievement than differential teacher treatment of individual students within the same group or class" (p. 309).

Ernest (1989) also argues that teachers' beliefs have a powerful impact on the practice of teaching, specifically on the teaching of mathematics. He states there are three roles mathematics teachers assume, depending upon their intended outcome of instruction. Ernest uses the term "instructor" to describe the role a teacher assumes when the intended outcome is skills mastery with correct performance, he uses the term "explainer" when the intended outcome is conceptual understanding with unified knowledge and "facilitator" when the intended outcome is confident problem solving. When a teacher assumes the role of Ernest's "instructor" and expects their students to attain only a procedural understanding of mathematical concepts, the instructional decision-making for the class will reflect that expectation. When there is no expectation of conceptual development or problem solving, there is no impetus for the student to understand the concepts behind the procedures in order to pose and solve complex mathematical problems.

Mathematics could get reduced to a series of decontextualized steps to be memorized (Baroody, 2003; Ben-Hur, 2006; Heibert and Lefever, 1986; Herscovics, 1996; Skemp, 1976).

Gonzales Thompson (1984) stated that in mathematics, teachers' beliefs, views, and preferences about the subject matter are important factors in determining the role the teacher assumes between the subject and the learner.

> In some cases, these patterns may be manifestations of consciously held notions, beliefs, and preferences that act as 'driving forces' in shaping the teacher's behavior. In other cases, the driving forces may be unconsciously held beliefs or intuitions that may be evolved out of the teacher's experience (p. 105).

Gonzales Thompson conducted case studies of junior high school teachers in order to investigate the relationship between the teachers' beliefs about mathematics and their classroom practice. She concluded that the beliefs played a "significant, albeit subtle, role in shaping their instructional behavior" (p. 125). Whether these beliefs about the mathematics are conscious or unconscious, any attempt to improve the quality of mathematics teaching must begin with an understanding of the beliefs and assumptions held by the teachers and how these are manifested in their instructional decision-making.

**Diagnostic Assessment/Error Analysis**

Assessment is the process of gathering information about student learning and using that information to plan instruction (Ashlock, 2010). It is expected that teachers use their classroom experience to develop knowledge of their students' mathematical potential. What varies between teachers and among educational institutions is *how* the teacher conveys what mathematical knowledge or skills are important, and how the student interprets and expresses what the student thinks the teacher values.

Watson (2000) studied practices of 30 teachers of 10 -12 year old students as they assessed students' mathematics in the normal course of classroom work. The teachers of this age group were chosen because Watson was interested in teachers' assessment of students as they transitioned from primary to secondary school. Her study had three main parts: the identification of practices of mathematics teachers acting as informal assessors; a critical study of how two teachers developed their views of some of their students during their first term with them; and a brief inquiry into peer-examination of professional judgment in school-based moderation practices.

Watson found assessment to be complex and intimately related to every aspect of teaching and learning. She found that even teachers who had undergone some assessment training underestimated the role of interpretation of evidence. She raised questions of equity in the uses of teachers' judgments in relation to awareness and practice. Watson suggested that more care needed to be taken over the formation and use of professional judgments' within systems of assessment. Watson studied assessments that occurred *during* a mathematics course. At the *beginning* of a course, a teacher may opt to give a diagnostic assessment to determine their students' prior knowledge on a specific topic.

For this study, I will use a diagnostic assessment for the purpose of determining student prior knowledge of functions. "As the bridge between identification of students who may be at-risk for failure and delivery of carefully designed supplemental interventions, diagnosis provides valuable information about students' persistent misconceptions in the targeted domain" (Ketterlin-Geller & Yovanoff, 2009). I will refer to the definition for diagnostic assessment provided to students at the University of Exeter, "diagnostic assessment looks backwards rather

than forwards. It assesses what the learner already knows and/or the nature of difficulties that the learner might have." (UE, n.d. para. 3).

All calculus instructors do not choose to use a formal diagnostic assessment of functions as presented in this study. Factors such as the time involved administering, interpreting, and implementing changes based upon the results may cause many instructors to avoid diagnostic tests altogether. However, if such a diagnostic instrument is used, the results would provide information about students' level understanding of key concepts, as well as any misconceptions about the underlying concepts that could lead to confusion later in the course (Ketterlin-Geller and Yovanoff, 2009). Teachers that do choose to use a student diagnostic instrument, similar to the one used in this study, typically use this information to make instructional decisions based upon the results and adjust curricular plans by identifying which areas students have and have not mastered. Two major approaches to diagnostic assessment are: (a) deficit assessment, which focuses on weaknesses of the student, and (b) error analysis, which focuses on the kinds of errors the student commits (Bejar, 1984). Deficit assessment will occur as the teacher notices that a student chose not to answer a specific question on the diagnostic instrument. A teacher will employ error analysis when the student answers a diagnostic question incorrectly or incompletely.

Error analysis is not simply identifying when an incorrect answer is given to a mathematical problem. It is a first step, but further analysis needs to be conducted by the teacher in order to reveal if the error was due to a careless mistake that is easily corrected, or if the student has a misconception of the underlying mathematical concept. Ketterlin-Geller & Yovanoff (2009) refer to these two types of errors as "slips" and "bugs." Identifying bugs is the primary interest of diagnostic assessment. Error patterns in computation as described by

Ashlock (2010) often reveal these bugs or misconceptions our students have learned. He classified computational-skill bugs into three basic categories: (1) student uses an inappropriate operation when attempting to solve a math problem; (2) student uses the correct operation but makes an error involving number facts; (3) student makes a non-number fact error in one or more steps of applying the strategy or selects an incorrect strategy. Additional errors involve interpreting and applying the language of mathematics (Ketterlin-Geller & Yovanoff, 2009).

**Evolution of the Function Concept**

"Calculus is a branch of mathematics that deals with change and motion" (Stewart, 2010, p. 2). Its roots can be traced back at least 2500 years to ancient Greeks and China, but calculus as we know it today began in the 17th century with Newton and Leibniz (Rosenthal, 1951). Ideas of calculus that are included in first semester calculus are limit, derivative, and integral of a function. The derivative of a function is its instantaneous rate of change, with respect to something else. Thus, the derivative of height (with respect to position) is slope; the derivative of position (with respect to time) is velocity; and the derivative of velocity (with respect to time) is acceleration. The integral of a function can be thought of as the area under its graph, or as a sort of total over time. Thus, the integral of slope is (up to a constant) height; the integral of velocity is (up to a constant) position; and the integral of acceleration (with respect to time) is velocity (Stewart, 2010). Many functions studied in calculus can be represented by algebraic expressions. For instance, the area of a circle is related to its radius by the formula $A = r^2$; and the distance that a body falls in a time t, starting at rest, is given by $x = \frac{1}{2}at^2$. Given such an expression, calculus allows us to find expressions for the integral and derivative of the function, when they exist (Dawson, 2007).

Although the first stage of the concept of function was that of antiquity, it was toward the end of the 17th century before the word "function" appeared in mathematics literature (Youschkevitch, 1976). The word was first used by Leibniz to designate the dependence of geometrical quantities such as subtangents and subnormals on the shape of a curve. The words "constant," "variable," and "parameter" were also introduced at this time (Ponte, 1992). The mathematical definition of the word "function" as quantities that were dependent on one variable by means of an analytic expression was agreed upon through correspondence between Leibniz and Bernoulli right before the turn of the century (Youschkevitch, 1976). The term was first published in a mathematics lexicon in 1716. Euler, a former student of Bernoulli, later changed the definition of a function of a variable to be an analytic expression (as opposed to quantity) that is composed in some way from that variable and constants (Ponte, 1992). In the 19th century the definition was changed again to enlarge the concept of function to include a correspondence between two variables so that to any value of the independent variable, there is associated one and only one value of the dependent variable (Youschkevitch, 1976).

The function concept is a pre-calculus concept taught at most secondary educational institutions in preparation for the study of calculus. The importance of understanding the concept of function is foundational for the understanding of major concepts in advanced mathematics (Carlson, Smith & Persson, 2003; Rasmussen, 2000; Zandieh, 2000). The teaching of functions needs to include the definition of function as the correspondences between numerical sets and a balance of the three most important forms of the representation, namely the numerical, graphical, and algebraic forms. The "well-behaved" examples, for which there is a simple rule, must be clearly emphasized in school mathematics, but the focus should not stop

with the algebraic manipulation of the function. Students need to be provided with opportunities for the application of the functions being studied so as to ascertain the meaning of the concepts being presented (Ponte, 1992).

If one wants to teach functions, or topics dependent upon the function concept, it is important to know the starting point of their audience (Dreyfus & Eisenberg, 1982). This starting point would be an analysis of the stages a student passes through when they learn explicitly about functions. First they should learn about the subconcepts of domain, range and the rule of correspondence. This is usually done in a first year algebra course. Then they learn that functions can be represented in various forms such as mapping diagrams, tables (x-y charts), and graphical and algebraic representations. They also learn that the same function can be represented by each of these representations and they need to be able to go from one representation of a function to another. By the end of their first year of algebra, students should be introduced to the specific functions; linear and quadratic. (Markovits, Eylon, & Bruckheimer, 1986). In the follow-on algebra courses, preceding the study of calculus, students are introduced to higher-order polynomials, radical and exponential functions, the translation and composition of functions, inverse functions, and discontinuous functions such as piecewise and step functions. For the purposes of this study, the instructors being interviewed were asked to focus their analysis of student work to specific pre-calculus topics on functions that were identified as common student misconceptions by a review of the literature.

**Student Misconceptions about Functions**

All of us make mistakes from time to time, and an incorrect answer does not necessarily mean that the student does not understand the underlying mathematical concept behind the problem. However there is a difference between careless mistakes and misconceptions about mathematical ideas and procedures. Research has shown that many undergraduates that received a grade of A in a calculus course in high school still possessed a weak understanding of function (Breidenback, Dubinsky, Hawks, & Nichols, 1992; Carlson, 1998; Thompson, 1994).

Carlson (1998) conducted a study of college students that received a grade of A in College Algebra (a pre-calculus course) and the second semester of calculus, and their understanding of functions. She found that students who think about functions only in terms of procedural techniques are unable to comprehend a more general conceptual structure for modeling functions where the dependent variable changes continuously along with the continuous changes of the independent variable. She describes specific examples of college algebra students' work that reflected a lack of conceptual understanding of function. When students were asked to find $f(x+a)$, 43% of the A-students added "a" to the end of the expression for $f$ rather than substituting $x + a$ into the function. Only 7% of the A-students could produce a correct example of a function all of whose output values are equal to each other, and only 25% of A-students in second semester calculus produced $x = y$ as an example. "Such weak understandings and highly procedural orientations are often displayed in the inability to move fluidly between various function representations, such as the inability to construct a formula given a function situation described in words" (p. 114). In a more recent study of over 2000 pre-calculus students, at the end of the semester, only 17% were able to determine the

inverse of a function for a specific value given a small table of function values (Carlson, Oehrtman, and Engelke, 2010). Students who possess only a procedural understanding of functions will likely be unable to recognize even simple situations in which procedures such as those necessary to find the inverse of a function work (Carlson & Oehrtman, 2005).

In addition to these studies, many others have conducted studies to examine misconceptions of functions. Table 1:*Studies on Misconceptions of Functions* summarizes six studies that investigated misconceptions of functions over a thirty-year period from approximately 1980 – 2010. Although the number of participants, ages of participants, and education levels of the participants varied from study to study, an overview of all these studies informed decisions that were made as to which specific misconceptions would be used for this particular study. A detailed summary of each study follows the table.

Table 1:

*Studies on Misconceptions of Functions*

| **Author(s) of study** | **Year of study** | **# of participants** | **Age/Education of participants** | **Specific Difficulties/Misconceptions** |
|---|---|---|---|---|
| Markovits, Eylon, and Bruckheimer | 1986 | 400 | 9th graders (14-15 years old) | Neglect of domain and range<br>Difficulty with constant and piecewise functions<br>Difficulty with functions represented by a discrete set of points<br>Inability to transfer from graphic to algebraic form<br>Less successful with technically complex functions |
| Vinner and Dreyfus | 1989 | 271/36 | 1st year college students/jr high teachers | Difficulty with discontinuous functions<br>Difficulty with functions with split domains<br>Difficulty with functions with exceptional points |
| Becker | 1991 | 264 | College students enrolled in pre-calculus | Students do not recognize written forms of functions<br>Students think functions are linear and follow a pattern<br>Students think functions must include the variable "x"<br>Graphs of functions must be smooth, continuous curves |
| Doorman, et al. | 2012 | 155 | 8th grade (13-14 years old) | Difficulty integrating operational and structural aspects of the function concept. |
| Hitt | 1998 | 30 | Mathematics teachers | Difficulty with discontinuous functions<br>Misidentifying the domain and range of functions<br>Oversimplified definitions of function<br>Inability to interpret a graph to a physical context |
| Cansiz, Kucuk, and Isleyen | 2011 | 61 | Secondary students (9th-11th grade) | Difficulty identifying graphs of functions<br>Difficulty with verbal expressions of functions<br>Confusion with identifying algebraic functions |

The Vinner and Dreyfus (1989) study examined some aspects of the images and definitions for the concept of function held by college students and junior high school teachers in Israel. The students were in their first year of college and had not yet reviewed functions as part of a mathematics course, but were introduced to functions during their secondary mathematics courses prior to college. A seven question questionnaire was administered to the participants with six questions examining concept images and one question asking participants to define a function. Three questions referred to a graphical representation of a function and three referred to a verbal representation. There was no reference to algebraic representations of functions. The areas of difficulty included discontinuous functions, functions with split domains and functions with exceptional points. The authors emphasize that a concept is not acquired in one step and several stages precede the complete acquisition and mastery of a complex concept. The more mathematically-oriented the student, the more the student is aware of their thought processes and thus more likely to reflect upon them.

In order to investigate how students understand the components of the function concept, Markovits, Eylon, and Bruckheimer (1986) wrote a large variety of problems and administered them to approximately four hundred $9^{th}$ graders (ages 14-15). They limited their study to graphic and algebraic representation of numerical functions. They found three types of function caused difficulty: the constant function, a piecewise defined function, and a function represented by a discrete set of points. Transfer from the graphical form to the algebraic form of a function was more difficult than transfer from the algebraic to the graphical form of the same function. As the "complexity" of the technical manipulations increased, the students were less successful and when examples of functions were required, students tended to adhere to the linear function. They also noticed that students neglected the domain and range of a function.

Although the central question to the Doorman_Drijvers, Gravemeijer, Boon, and Reed. (2012) study focused on how computer tools foster the transition from a procedural calculation understanding to a conceptual understanding of functions, since their study did examine student misconceptions of functions, I chose to include it in this review of literature. The quantitative portion of their study used data from 155 students in 8$^{th}$ grade at two different schools. The participants were given paper-and-pencil tests consisting of open-ended questions to test the students' understanding of functions. Initial assessment revealed that students had difficulty integrating operational and structural aspects of the function concept. The study also showed that student learning of the operational aspects preceded the structural aspects of functions.

The purpose of the Becker (1991) study was to identify and then remediate pre-calculus students' major misconceptions about the function concept. A pre- and post-instruction questionnaire was administered to 227 students enrolled in one of eight pre-calculus classes at Illinois State University. A sub group of twenty volunteers then participated in a supplementary unit designed to remediate the misconceptions identified by the questionnaire. The questionnaire contained the same seven questions in the Vinner and Dreyfus (1989) study plus demographic questions and 14 additional questions pertaining to functions developed by the researcher. The Becker study examined the students' knowledge of graphic, tabular, written, and algebraic representations of the function concept. The Becker study revealed that students do not recognize written forms of functions. Students think functions are linear and must follow a recognizable pattern. Students think functions must include the variable "x," and the graphs of functions must be smooth, continuous curves.

The focus of the Hitt (1998) study was not on the students, but rather on the secondary school teachers who were beginning a postgraduate course on mathematics education and the

articulation of the concept of function. Thirty mathematics teachers were given two questionnaires per week for seven consecutive weeks for a total of 14 questions. The questions included verbal, algebraic, graphic, and written representations of functions. The results of the study revealed that the mathematics teachers had difficulty with identifying the domain and range of functions. They tended to oversimplify definitions of function to a rule of correspondence or a set of ordered pairs. The identification of functions did not pose a problem, but the construction of discontinuous or piecewise functions was a problem. Another problem identified was the interpretation of a graph to a physical context or vice versa. For example, if the teacher was asked to identify the graph of area vs. height of water being filled in a cylindrical beaker, they often were unable to identify the correct graph. The results of the Hitt study showed that the secondary school teachers in the study had difficulty coherently articulating between the various systems of representations involved with the concept of functions.

The purpose of the Cansiz, Kucuk, and Isleyen (2011) study was to detect secondary school students' misconceptions about functions. The study was conducted in the county of Turkey where the "introduction of the concept of function is made by taking set theory as a basis" (Cansiz et al., 2011, p. 3838). The Cansiz et al. study consisted of a 14 question function knowledge test given to 61 randomly selected secondary school students in grades 9-11. Of the 14 questions, six were from the Vinner and Dreyfus (1989) study, seven were from the Becker (1991) study and one question was developed by one of the researchers from a previous study. The researchers concluded that the students had difficulty identifying functions from written, graphic, and algebraic representations.

Throughout the history of our nation, education, generally, and mathematics education, specifically has been the lifework of many esteemed individuals. By familiarizing ourselves

with prior research through a literature review we can stand on the shoulders of those that have gone before us with the hope of making our contribution to the body of literature, thus improving mathematics education for subsequent generations. Calculus is a dominating presence in the preparation of students seeking STEM careers. Unlike other secondary disciplines, the studying of calculus is a "capstone for school mathematics, the culmination of study in the only subject (apart from reading) taught systematically all through K-12 education" (Steen, 1987, p. xi). For secondary students, calculus is often seen as the final course in mathematics, but for postsecondary STEM students, it is often the pre-requisite course for the majority of their programs of study.

This chapter presented a sociological view of teacher expectations, a description of how diagnostic assessment and error analysis influences instructional decisions, a closer look at functions as fundamental objects of calculus, and finally a review of studies that attempted to identify students' misconceptions of functions. In Chapter 3, I reviewed the literature to inform my decisions and describe the methods in which I studied the differences and similarities in vision amongst and between the instructors of calculus at the three educational institutions in which this subject is taught. Success in calculus requires the prior knowledge of many mathematical topics, but for the purpose of this study I focused specifically on these teachers' expectations of their students' knowledge of the pre-calculus topic of functions. Chapter 4 was written after the six interviews were conducted and contains various tables which display pertinent data and Chapter 5 is a discussion of the findings.

# CHAPTER 3: METHODOLOGY

The primary goal of this study was to take that which is implicit about one aspect of teaching calculus and make it explicit for the purpose of bringing the three educational institutions together for a discussion on how to best teach calculus in order to prepare our students for future STEM careers. As presented in the Rationale section of the Introduction chapter, there is a gap into which many students fall when transitioning from secondary mathematics (high school) to postsecondary mathematics (state/community colleges and universities). Since the primary goal of this study is identifying and describing that gap, I chose to conduct a qualitative, as opposed to a quantitative, study. I chose a multiple-case study with the intent of expressing the assumptions of instructors from each of the three types of institutions and comparing and contrasting their assumptions. I hypothesized the gap is related to teacher assumptions of prior knowledge of their students, and the communication of these expectations to their students among the three institutions teaching the same course.

## Multiple-case Study

When one sets out to study a complex system such as teaching, many variables may have an impact on what happens in a classroom. Some of these variables are explicit, and some are implicit (Wagner & Sternberg, 1985). Policies, procedures, and guidelines can have a positive influence on the explicit factors influencing the teaching of calculus and pre-calculus, but with this study I attempted to uncover the tacit influences that a teacher may not consciously realize are influencing his or her curricular decisions. The phenomenon of a teacher determining how to teach a calculus course cannot be separated from the context of the students' prior knowledge of mathematics. Yin and Davis (2007) state, "One strength of the case study method is its ability to

tolerate the real-life blurring between phenomenon and context" (p. 78). Baxter and Jack (2008) raise the question, "What is the difference between a holistic case study with embedded units and a multiple-case study?" (p. 550) and then answer their own question by stating the context is different for multiple-case studies as opposed to a holistic case study. If my goals were to just study the teaching of AP Calculus, or the teaching of first-year Calculus at a 4-year university, then I would conduct a holistic case study, but since I was attempting to explicate teacher expectations at various educational institutions in order to understand the similarities and differences between the cases, a multiple-case study was warranted.

**Participant Sampling Strategies**

Yin (2003) states that a multiple-case study enables the researcher to explore differences within and between cases. The goal is to replicate findings across cases. Because comparisons will be drawn, it is imperative that the cases are chosen carefully so that the researcher can predict similar results across cases, or predict contrasting results based on a theory. The six calculus instructors interviewed for my study were from the three types of institutions: high schools (secondary), community/state colleges and universities (postsecondary). In order to have a balance between the three educational institutions, I chose the same number of instructors from each type of institution. When choosing participants for this study, the intent was to interview effective, veteran teachers as opposed to novice teachers. Guarino, Santibanez & Daley (2006) state that teacher quality is an important variable when considering student academic success, but "evidence is not always clear regarding the observable characteristics of effective teachers" (p. 175). In order to select effective teachers for this study, names were suggested by peers and former graduate mathematics faculty for participation. From the list of names, the participant

sampling was purposive for a number of reasons. First, all persons interviewed were teaching calculus at the time of the interview, all instructors were identified as respected and trustworthy instructors by their peers, and all instructors had a minimum of 20 years teaching experience. Within the purposive parameters, the sampling was convenient sampling based upon the instructor's willingness to participate in the study. The secondary instructors were teaching calculus at different high schools in the same suburban school district. The state college instructors taught at the same state college located in the same large, metropolitan city and located relatively close to the university of the other postsecondary calculus instructors.

**Structure of Study**

I interviewed all instructors using the same protocol at both the secondary (high school) and postsecondary educational institutions (community/state college and university). Calculus is typically one of the last mathematics courses that a student will take in their secondary educational experience, but it is often the first course they take in a series of postsecondary mathematics courses if the student intends to pursue a STEM major. In this sense, Calculus serves as a type of bridge for these students from their secondary to postsecondary mathematics education experience and the reason why I chose to focus on Calculus for this study. It is also the reason why I chose to interview both secondary and postsecondary instructors at various educational settings. The mathematics a student typically takes prior to a calculus course is extensive. For the purpose of taking what is implicit and making it explicit I needed to limit the focus of the prerequisite mathematics to a fairly narrow topic. I chose the topic of functions based upon a review of the literature and my own experience of teaching secondary mathematics for over 20 years.

**Nine coherent arguments in regard to structure.**

Prior to conducting a qualitative study, it is recommended by Maxwell (2005) that the researcher address a series of nine arguments that were adapted by Creswell (2007) which need to be coherent in regard to the organization of the structure of a qualitative research study. I have taken those nine arguments, numerated them, and addressed each based upon my study.

1. *We need to better understand* the differences between how secondary and postsecondary calculus instructors teach calculus and if those differences can be explained within the conceptual framework of a balance between procedural fluency and conceptual understanding.

2. *We know little about* the gap between the increasing number of secondary students taking AP Calculus and the declining number of postsecondary students successfully completing advanced mathematics courses.

3. *I proposed to study* how teachers of calculus at the three types of educational institutions assess their calculus students' prior knowledge of the function concept and if those assessments include a balance between procedural fluency and conceptual understanding.

4. *The setting and participants were appropriate for this study* because I interviewed secondary and postsecondary calculus teachers at their work places.

5. *The methods I used provided the data I needed to answer the research questions* because I conducted an interview that consisted of three components. A 11-question written survey, a think-aloud as the teachers look at student responses to a functions diagnostic instrument and the instructors' responded to two quotations concerning assessment and the

balance between conceptual understanding and computational fluency. I asked the instructors to provide their course syllabus and a copy of any diagnostic tools that they use in their course.

6. *Analysis generated answers to these questions* by analyzing instructors' think-alouds and responses to the interview questions; I was able to identify how instructors assess their students' prior knowledge and how the results of that assessment impacted their instructional decision-making.

7. *The findings were validated by* peer review, triangulation, and member check.

8. *The study posed no serious ethical problems*. The participants were not identified by name or educational institution and the data were kept confidential by the researcher.

9. *Preliminary results supported the practicability and value of the study*. I conducted pilot studies with calculus instructors not used in the actual study. The time allotment was practical and by analyzing the interview I was able to assess some of the instructors' implicit assumptions about her students' prior knowledge of functions findings.

**Research questions.**

According to Creswell (2007) qualitative research questions are often one of four types: exploratory, explanatory, descriptive, and emancipatory. In this study, I attempted to explain or explicate teachers' assumptions concerning their students' procedural and conceptual understanding of functions, therefore my research questions are explanatory. According to the Merriam-Webster dictionary (n.d.) to explicate is to develop the implications of or analyze logically. In this study, the word explicate is similar to the word explain, but explicate contains the implication that the concept attempting to be explained is more complicated or detailed than it may initially appear. For example, while one may explain why they are late for work due to an

unexpected traffic jam, one would explicate their tardiness by examining the individual's behavior patterns, their psychological state of mind prior to leaving for work and their philosophy on tardiness. Explication is a process which is designed to uncover the implicit with the intention of revealing something which is more explicit. Implicit or tacit knowledge of a teacher's vision is that which is neither expressed nor declared openly but rather implied or simply understood and is often associated with intuition (Wagner & Sternberg, 1985). This kind of knowledge about teacher vision is difficult to transfer to another person by means of writing it down or verbalizing it. Although it is possible to distinguish between the idea of explicit and implicit pedagogical knowledge, an instructor does not separate their own types of knowledge when dealing with students in the classroom (Collins, 2010). The instructor comes into the classroom with some explicit knowledge of their students' prior knowledge from their studies in education and develops their implicit or tacit knowledge of their students' prior knowledge over time. One tenet of this study is to discover how the tacit knowledge of the secondary instructors compares to the tacit knowledge of the postsecondary calculus instructors with respect to their vision of the transition from secondary to postsecondary mathematics and preparation for mathematically intensive careers. Without clearly delineated visions of the various calculus instructors, there is the possibility that teachers of the same course are using different "sheets of music" when it comes to preparing their students for their futures.

I followed the guidelines of Stake (1995) for the formation of the research questions for this study. The three questions are (1) "How are secondary calculus instructors' assumptions of their students' prior knowledge of the function concept different from that of the postsecondary calculus instructors' assumptions? (2) "How do secondary and postsecondary instructor assumptions of their students' prior knowledge of functions impact instructional decision-making

for their calculus course?" and (3) "How do secondary and postsecondary instructors differ on their views toward students' conceptual understanding of functions and procedural fluency of functions?"

## Instrumentation

### Component 1: Written survey.

In order to maintain consistency, I wrote out and followed a script during the interviews (see Appendix A). The first component of data recording was an eleven question written questionnaire (see Appendix B). As stated in the review of literature, teachers have both implicit and explicit expectations of their students. In this study, I asked instructors to state their expectations of their incoming students in a variety of ways so that I could compare and contrast their statements in an attempt to reveal both the implicit and explicit expectations. In the first component, I asked instructors to write their answers to the eleven questions on the questionnaire.

Questions 1-3 pertained to specifics about their institution, course title, and textbook. Question 4 asked the instructor if they would provide me a copy of the syllabus they use for their calculus course. Parkes and Harris (2002) state that one purpose of a course syllabus is to state the expectations and "guide the behaviors" of both the instructor and the students during the course of the semester. I used the syllabi the instructors provided to compare and contrast the instructors' written expectations of both prior knowledge and student behavior to the oral expectations in the second and third components.

Question 5 asked if the instructor used a diagnostic assessment tool in their calculus class. The use of an assessment tool is discussed in Chapter 2. If the instructor stated that they

did use an assessment tool, I asked if the diagnostic instrument was research-based and how the results influenced their instructional plans. I also asked the instructor to provide me a copy of the diagnostic instrument so that I could compare and contrast the various tools instructors use at different institutions. The intent of Question 6 was to determine how familiar the instructor was with how and where their incoming students attained their pre-calculus knowledge. Question 7 was in the form of a chart where the instructors were given a list of 8 pre-calculus topics and asked to check if they felt their students needed "review", "clarification", "both", or "neither" for each topic. Question 8 then asked if the instructor determined that their students did need review or clarification of a pre-calculus topic, where would the student receive that service. Question 9 asked the instructor to estimate the amount of time they spent in their calculus class reviewing pre-calculus topics.

For question 10 I asked the instructors to write their understanding of the terms "procedural fluency" and "conceptual understanding." I wanted to compare and contrast the instructors' written definitions in Component 1 to their oral response to the same question in Component 3, after analyzing student work in Component 2, and reacting to two quotations in Component 3. The last question asked the instructor to take a few minutes to look at the textbook diagnostic instrument that was used for Component 2 and rate the diagnostic on a scale from 1-10 on the adequacy of the instrument for assessing their incoming students' prior knowledge of functions. The answers to the diagnostic were also provided (see Appendix C).

**Component 2: Student answers.**

The use of a student diagnostic instrument was vital for my study because it allowed me to ask the same questions during the interview of each instructor based upon the teacher's

expectations of their students' prior knowledge of functions. Because the instructors were not teaching at the same educational institution, there was no expectation of consistency between, nor among, instructors as to how, or even it, they assessed their students' prior knowledge of functions. In order to ensure consistency among the participants, I had all instructors assess the same results of the same diagnostic instrument regardless of the type of institution the instructor teaches. This means that the teachers did not assess their own students' work. By using the same student results with the six instructors at different institutions, I was able to compare and contrast comments made by the instructors during the analysis of the student responses in Component 2 of the interview.

For the student diagnostic instrument, I surveyed instruments from numerous Calculus textbooks and online sources for an instrument that could be administered in one typical class period (45 minutes) and focused specifically on functions (see Appendix D). I looked for a diagnostic which contained questions that could be identifiable as testing a students' procedural knowledge or conceptual knowledge. Since it is difficult to determine if a student is using their conceptual or procedural knowledge (Hiebert & Lefevre, 1986) without having the student explain what they are doing verbally, I looked specifically for an instrument with fewer questions and multiple parts to each question. The multiple parts helped to delineate if the student employed procedural knowledge or conceptual understanding in order to arrive at a solution. I found and received permission to use a seven-question diagnostic calculus exam on functions from a major textbook publisher (see Appendix E).

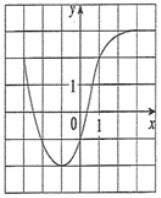
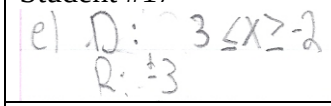
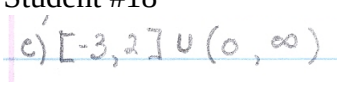
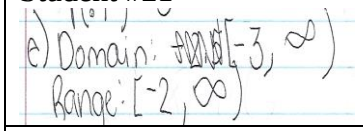
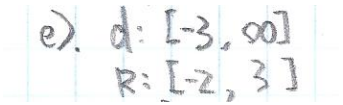
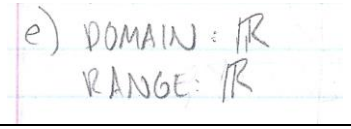
The textbook diagnostic instrument that was administered to students has 7 questions with most questions having multiple parts, for a total of 26 individual questions (see Appendix C for student answers to the functions diagnostic instrument). Initially, I intended for the

instructors to analyze a problem from the diagnostic representative of each of the student misconceptions listed in Table 1: *Studies on Misconceptions of Functions*. After an initial pilot study, I realized that the time it would take for a teacher to analyze student answers representative of each misconception was too time-consuming so I narrowed the misconceptions down to the ones that appeared most often in the literature. From those 26 questions on the diagnostic, I selected four questions for the instructors to analyze student work. The questions for the study included the following subtopics: determining the domain and range of the graph of a function, finding the domain of a rational function, given the equation, describing the translation of a function from an equation, and sketching the graph of a piecewise function.

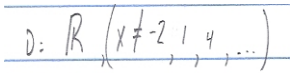
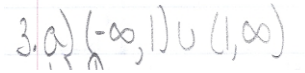
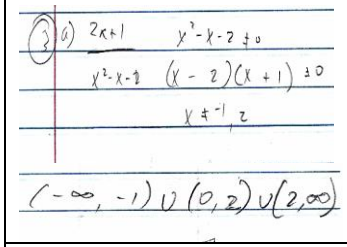
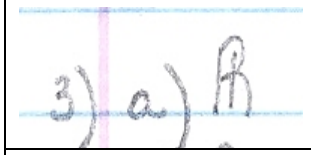
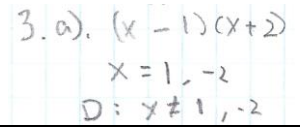
I selected five examples of student work from the class set of student answers for each of the four questions. The examples of student work were selected to reflect student misconceptions as presented in the review of literature and common errors as seen by the researcher with over 20 years of secondary education experience. In addition to misconceptions and errors, I selected one student answer for each question that exemplified a student's understanding of the particular pre-calculus subtopic. Prior to the study, mathematics education doctoral students that had previous experience teaching calculus reviewed the student work and agreed that they would give full-credit to the answers with the asterisk (*) symbol after the student number (see Table 2: *Four Questions and Five Student Answers Selected for Instructor Interviews*). The entire student answer sheet for each of the students listed on Table 2 is found in Appendix C.

Table 2:

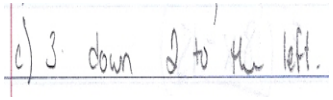
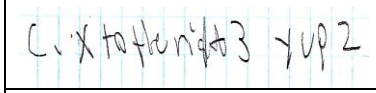
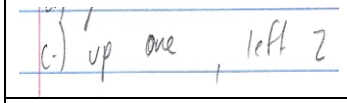
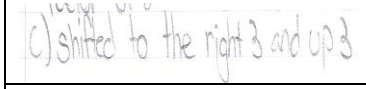
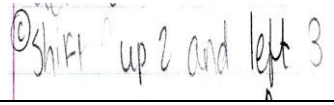
*Four Questions and Five Student Answers Selected for Instructor Interviews*

| | Student Answer | Misconception/Error | Study related to misconception/error |
|---|---|---|---|
|  <br><br>**Question #1:**<br><br>**The graph of a function f is given at the [left] above.**<br><br>**(e) State the domain and range of *f*.** | Student #17<br>  | Domain: Improper inequality notation<br>Range: Not written as an interval | Interval notation versus inequality notation discussed in member check. |
| | Student #18<br>  | Improper use of union symbol and misidentifying the domain and range | Hitt (1998) |
| | Student #21<br>  | Student assumes graph continues indefinitely | Becker (1991) |
| | Student #24*<br>  | Although considered correct, student used bracket instead of parenthesis after infinity symbol | Discussed in member check as common error. |
| | Student #27<br>  | Student does not appear to understand domain and range. | Markovits, Eylon, and Bruckheimer (1986) |

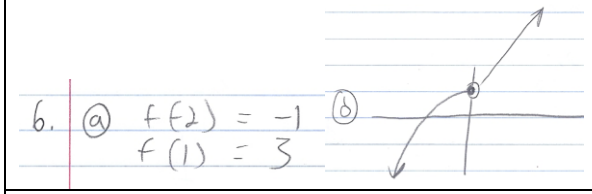
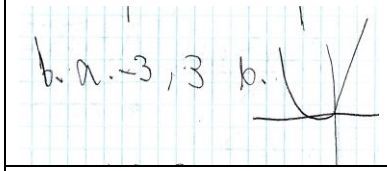
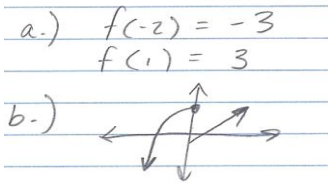
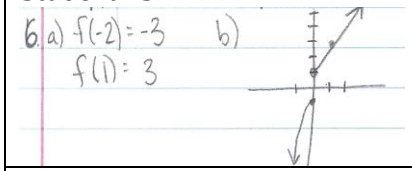
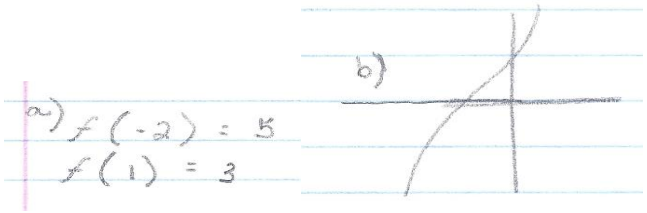
*Note*. The asterisk (*) identifies a student answer that educators agreed would receive full-credit.

| $f(x)=\frac{2x+1}{x^2+x-2}$<br><br>**Question #3(a):**<br><br>**Find the domain of the function.** | **Student Answer** | **Misconception/Error** | **Study related to misconception** |
|---|---|---|---|
| | Student #7<br>  | Student appears to believe there is a continuous pattern of values† which the function is not defined‡. | †Vinner and Dreyfus (1979)<br>‡Becker (1991) |
| | Student #14<br>  | Student only found one point of discontinuity | Hitt (1998) |
| | Student #16*<br>  | Although student copied the problem incorrectly, the work and answer would be considered correct. | Discussed during member check. |
| | Student #18<br>  | Student does not appear to understand the function is not defined for values that make the denominator equal zero. | Markovits, Eylon, and Bruckheimer (1986) |
| | Student #24<br>  | Student understands where the function is not defined, but does not seem to understand where the function is defined. | Hitt (1998) |

*Note*. The asterisk (*) identifies a student answer that educators agreed would receive full-credit.

| **Question #4:**<br><br>**How are the graphs of the functions obtained from the graph of *f*?**<br><br>**(c) y= *f*(x-3) + 2** | **Student Answer** | **Misconception/Error** | **Study related to misconception** |
|---|---|---|---|
| | Student #3<br>  | Incorrectly remembered how the values in the function effect the graph. | Doorman, et al. (2012) |
| | Student #4*<br>  | Although considered correct, answer should be more explicit | Cansiz, Kucuk, and Isleyen ( 2011)<br>Discussed during member check. |
| | Student #7<br>  | Student does not seem to know how the values in the algebraic form of the function effect the graph of the function. | Carlson (1998) |
| | Student #14<br>  | Student possibly did not read question carefully. | Discussed during member check. |
| | Student #20<br>  | Incorrectly remembered how the values in the function effect the graph. | Doorman, et al. (2012) |

*Note*. The asterisk (*) identifies a student answer that educators agreed would receive full-credit.

| $f(x)=\begin{cases}1-x^2, if x\le 0\\ 2x+1, if x\rangle 0\end{cases}$<br><br>**Question #6:**<br><br>**(a) Evaluate *f*(-2) and *f*(1)**<br><br>**(b) Sketch the graph of *f*.** | **Student Answer** | **Misconception** | **Study related to misconception** |
|---|---|---|---|
| | Student #2*<br>  | *f*(-2) was calculated incorrectly, but graph appeared to be correct. | Discussed in member check. |
| | Student #4<br>  | Student seems to believe graph should be smooth and continuous. | Becker (1991) |
| | Student #7<br>  | Student not relating algebraic answers in part (a) to graphic representation of same function in part (b). | Doorman, et al. (2012) |
| | Student #9<br>  | Difficulty graphing piecewise function. | Vinner and Dreyfus (1979) |
| | Student #18<br>  | Not relating algebraic and graphic representation of same function†, and believing graph should be smooth and continuous‡ | †Doorman, et al. (2012)<br>‡Becker (1991) |

**Component 3: Quotations.**

From the research on the use of assessments for instructional decision-making and the iteration of conceptual and procedural understanding, I selected two quotations to elicit the instructors' explicit assumptions. The first quotation is, "In the absence of research-based curricular instruments, teachers tend to rely on their own opinions about what students need to learn as they plan instruction" (Carlson, Oehrtman, Engelke, 2010, p. 114). The quotation was followed by two questions about their agreement or disagreement with this statement and then instructors were asked specific resources the instructor used to plan instruction for their calculus class.

The second quotation refers to the iteration of conceptual and procedural understanding in the teaching of mathematics,

> Developing fluency requires a balance and connection between conceptual understanding and computational fluency. On the one hand computational methods that are over-practiced are often forgotten or remembered incorrectly…On the other hand, understanding without fluency can inhibit the problem solving process" (*Principles and Standards*, 2000, p. 35).

This question was followed up by four questions. The first question asked the instructor to verbalize their meaning of conceptual understanding and computational fluency. Their verbal response to this question was compared and contrasted to their written response in the questionnaire. The intent of the next two questions were to get concrete examples from the instructor that support or refute the author of the quotation's position and then finally see if the instructor agrees or disagrees with this statement.

## Data Recording

Glesne (2011) suggests the phrase "data recording" or "data production" versus data collection when conducting a qualitative research study because often researchers have an active role in producing the data they record. There were three components to the research data recording process. Two of the three components were audio recorded and transcribed after the interview. The interviews for the secondary instructors were conducted in the teachers' classrooms and the interviews for the postsecondary instructors were conducted in the teachers' offices at the respective educational institutions, as opposed to having the instructors come to me, in order to immerse myself in the respective environments (Patton, 1990).

### Verbal reports and "think-aloud" technique

In order to ensure consistency between the interviews, a script was used and read verbatim (see Appendix A) for each interview. The data were recorded using verbal reports and "think-aloud" techniques as described by Someren, Barnard, and Sandberg (1994). I adopted Ericsson and Simon's central assumption of protocol analysis that it is possible to get the subject to verbalize their thoughts in a manner consistent with the sequencing of thoughts while performing a given task. The given task in this case was the instructor establishing the expectations of his or her students' prior knowledge of functions, thus making it possible to explicate the instructor's expectations.

For the "think-aloud" method to be valid, the subject must verbalize their thoughts in as much detail as possible, which is time-consuming. The amount of time the instructor is willing to be interviewed may shorten the interview and not be as thorough as if time was not an issue. "Verbal reports are only one indicator of the thought processes that occur during problem

solving. Other indicators include reaction times (RTs), error rates, patterns of brain activation, and sequences of eye fixations" (Kuusela & Paul, 2000, p. 390). Wilson (1994) contends that the protocol method cannot trace cognitive processes that never reach consciousness. Much of how an instructor bases their expectations of students' prior knowledge is not conscious, rather it is based on instinctual or unconscious processes. Wilson recommends including other methods such as reaction time and eye fixations, which do not seem pertinent for this study.

**Possible sources for error.**

Rip (1980) presents various possible sources for error for such data. The first is the error of transmission or communication. In this study, the instructors at the three institutions may not share the same vocabulary for the same process or they may use the same vocabulary to mean different things. The second is commission, in which the subject may misreport their cognitive processes and lastly, omission in which the subject leaves out particular elements. Considering the nature of this study, instructors may commit or omit information about the expectations of his or her students' prior knowledge in order to avoid showing themselves or their institutions in an unfavorable light. As the researcher, I was cognizant of these potential errors of communication, especially during the interview component. I made sure all participants were aware that their responses would be collected in strict confidentiality and they would not be traced back to themselves or to their particular institution. I asked participants to restate any terms that may be used differently in different institutions, for example the word, "homework." The amount of time and effort expected in a secondary institution can be quite different from postsecondary institutions. I used a system of comparing written responses from the survey from Component 1 to the oral responses in the remaining two components to check for inconsistencies. When

inconsistencies were noted, I asked the participant to clarify those statements during the interview.

**Component 1: Written Survey**

The instructors that participated in this study began the interview by filling out a written questionnaire (Appendix B) about their particular calculus course, their use of an assessment tool and were asked to evaluate the student answers to a diagnostic instrument on functions that will be used in the next component of this study. I also asked the instructors to provide me with a copy of any assessment tool they use to determine their students' prior knowledge of functions and a copy of their syllabus for their calculus course. Since I did not ask the instructor to think-aloud during this component, it was not audio-recorded. Some instructors did think-aloud while evaluating the diagnostic instrument for this first components and I noted their comments and addressed these comments during the recorded components.

**Component 2: Student answers.**

The intent of the second component of the data recording process was to uncover any implicit assumptions the instructors might have about their students' prior knowledge of functions. I showed the calculus teachers selected answers to specific questions from an actual student diagnostic instrument (see Appendix B). The intent of showing the teachers the diagnostic instrument results was to elicit the calculus instructor's explicit and implicit assumptions of student prior knowledge of functions. The actual student diagnostic instrument is discussed in more detail in the Instrumentation section of this chapter. The students that took this diagnostic were not the students of any of the instructors being interviewed. They were secondary students in their fourth month of AP Calculus and they took the diagnostic for the sole

purpose of this study. I chose to use the same diagnostic results with all instructors, regardless of their educational institutions, in order to standardize the information being presented to the instructors.

After the instructor evaluated a blank copy of the diagnostic instrument, they were asked to review five selected student answers to four specific questions. During the interview, I used Table 3: *Selected Student Answers for Think-Aloud* as a reference as I showed the instructor the student answers to the diagnostic. Four of the five student answers are student misconceptions or common errors and the answers with the asterisk were considered to be correct. The instructors did not see this table during the interview, nor did they have any knowledge as to why particular questions or student answers were selected.

Table 3:

*Selected Student Answers for Think-Aloud*

| Student Answers | Question Number | | | |
|---|---|---|---|---|
| | #1e | #3a | #4c | #6 |
| 1st | 17 | 7 | 3 | 2* |
| 2nd | 18 | 14 | 4* | 4 |
| 3rd | 21 | 16* | 7 | 7 |
| 4th | 24* | 18 | 14 | 9 |
| 5th | 27 | 24 | 20 | 18 |

During Component 2 of the interview, I showed the instructor one student's paper and pointed to the specific question that the instructor was to analyze using the "think-aloud" method. When the instructor finished analyzing the particular question for the selected student, I showed the

instructor another student's paper for the same question. The entire student answer sheet for each of the students listed on Table 3 is found in Appendix C. I provided the instructors with the answer key provided by the textbook company for the diagnostic instrument (see Appendix F). Following guidelines for verbal reports (Ericsson & Simon, 1983), I asked the instructor to "think-aloud" as he or she reviewed the set of student responses to the diagnostic instrument.

**Component 3: Quotations.**

The intent of the third component was to elicit the instructor's explicit assumptions about student prior knowledge and conceptual versus procedural understanding. I selected two quotations from the review of literature that speak directly to the research questions (see Appendix A). One quotation was on the use of assessments and instructional decision-making, and the other was on conceptual understanding and procedural fluency. For each quotation I developed follow-up questions. If after asking these questions, and I still did not have a clear idea as to the instructor's explicit assumptions about student prior knowledge and conceptual versus procedural understanding, I asked more questions such as, "Anything else?" "Can you tell me more," or "How do you mean that?" (Fraenkel & Wallen, 1990, p. 444).

## Analysis of Data

In order to analyze the data, I used an inductive analysis of calculus instructors' expectations of their students' prior knowledge of functions. According to Patton (1990), "The first decision to be made in analyzing interviews is whether to begin with case analysis or cross-case analysis" (p. 376). In order to analyze the written surveys, the information from all six participants was put in a grid by topic and participant so that the data could be compared and contrasted. The data were initially analyzed by case, and then afterward analyzed by cross-case.

As Patton suggests I immersed myself in the details and specifics of the data with the intent of discovering important categories of expectations, identifying dimensions of expectations that may not otherwise be apparent with the hope of finding interrelationships amongst instructors from the three educational institutions. The tables in Chapter 4 were constructed as a result of the hand-written grids that were developed during the analysis of the data. Once the grids were developed, the audio tape recordings were listened to while reading the written transcriptions from each interview. From this inductive analysis process, it was apparent that the initial eight pre-calculus topics listed in Table 7: *Review/Clarification of Pre-calculus Topics* needed to be narrowed down to the three main topics listed in Table 8: *Instructor Expectations of Prior Knowledge of Functions*.

**Validity/Reliability**

Since the intent of this qualitative study is to describe an individual's expectations it would not be appropriate to conduct traditional research methods for validity and reliability such as the test-retest method or the equivalent forms method as described by Fraenkel and Wallen (1990). In order to insure validity and reliability of this study, I used peer review as described by Saldana (2009), triangulation as described by Guion (2012) and Thurmond (2001) and a member check as described by Stake (1995) and Glesne (2011).

**Peer review.**

"Sometimes we need an outside pair of eyes or ears to respond to our work in progress" (Saldaña, 2009, p. 190). Peer review was used extensively in this study. I used peer review before, during, and after data collection. Prior to conducting research, mathematics education faculty and doctoral students from this and other institutions were consulted in order to validate

the research questions and the instruments that were used during the study. The feedback received during this peer review helped determine the format of the instruments and the structure of the study. Pilot studies were conducted with calculus instructors that did not participate in the study in order to get feedback on the interview protocol. Several changes were made to the protocol as a result of these initial pilot studies. Once the interviews were conducted, peer review was used during the analysis portion of the study. Education faculty and fellow doctoral students from both within and outside mathematics education, helped to validate the codes and emerging themes found in Component 2. During the member check, a fellow doctoral candidate that is familiar with qualitative research methods as described by Saldaña, (2009) assisted in the video recording and analyzing of the session.

**Triangulation.**

Triangulation is a method used by qualitative researchers to check and establish validity in their studies by analyzing a research question from multiple perspectives (Guion, 2012). Since this data recording often comes from multiple methods, qualitative researchers have borrowed the phrase *triangulation* from surveying to describe this practice. Triangulation in surveying followed the work of mathematician Willebrord Sness in 1615 – 1617, who showed how a point could be located from the angles subtended from three known points (O'Connor & Robertson, n.d.)

In this study, triangulation of data from the three data sources was used in order to answer the research questions. The written questionnaire in the first component was used to record the instructor's explicit responses to their expectations of incoming students' pre-calculus abilities. The instructor's syllabus was also requested and analyzed in order to compare the

explicit expectations given to the students with the expectations written on the survey. The instructors' "think-alouds" as they analyzed student work from the second component were used to uncover any possible implicit expectations on student expectations, and finally, analyzing the responses to questions about specific quotations on the use of a diagnostic assessment and the iteration of conceptual understanding and computational fluency during the third component established triangulation in this study. In Table 4: *Triangulation of Research Methods by Research Question and Component*, the three research questions of this study are listed on the left and the specific components of the interview, the instrument or data recording procedures, and corresponding analysis procedures that were used in order to triangulate the data are listed in the columns to the right.

Table 4:

*Triangulation of Research Methods by Research Question and Component*

| Research Questions | Component | Instrument or Data Recording Procedures | Analysis Procedures |
|---|---|---|---|
| 1. How are secondary calculus instructors' assumptions of their students' prior knowledge of the function concept different from that of the postsecondary calculus instructors' assumptions? | Component 1: Written Survey | Appendix B: Written Survey Questions #5-7, | Compare and contrast responses within and between types of institutions |
| | Component 2: Student Answers | Appendix A: Script for Instructor Interview, Question #1 | Inductive Analysis (Patton, 1990) |
| | Component 3: Quotations | Appendix A: Script of Instructor Interview, Quotation 2, Question #3 | Inductive Analysis (Patton, 1990) |
| 2. How do secondary and postsecondary instructor assumptions of their students' prior knowledge of functions impact instructional decision-making for their calculus course? | Component 1: Written Survey | Appendix B: Written Survey Questions #1-4 | Compare and contrast responses within and between types of institutions |
| | Component 2: Student Answers | This component did not directly relate to this research question. | |
| | Component 3: Quotations | Appendix A: Script for Instructor Interview, Quotation 1, Question1 #1-2 | Compare and contrast responses within and between types of institutions |
| 3. How do secondary and postsecondary instructors differ on their views toward students' conceptual understanding of functions and procedural fluency of functions? | Component 1: Written Survey | Appendix B: Written Survey Question #10 | Compare and contrast written response with verbal response of same instructor |
| | Component 2: Student Answers | Appendix A: Script for Instructor Interview, Question #2 | Inductive Analysis (Patton, 1990) |
| | Component 3: Quotations | Appendix A: Script for Instructor Interview, Quotation 2, Question #3 | Compare and contrast written response with verbal response of same instructor |

Thurmond (2001) stated that the benefits of triangulation include "increasing confidence in research data, creating innovative ways of understanding a phenomenon, revealing unique findings, challenging or integrating theories, and providing a clearer understanding of the problem" (p. 254). A disadvantage of triangulation is the time involved in data recording and analysis and reconciling any possible incongruence that may be uncovered through the triangulation process. In spite of the possible disadvantages, I used triangulation in this study to give depth to my understanding of the Calculus instructors' expectations of their students' prior knowledge of functions and to maximize confidence in the findings of the research.

**Member check.**

Upon completing analysis, the participants were invited to attend a member checking session. Lincoln and Guba (1985) consider member checking to be "the most critical technique for establishing credibility" (p. 314) when conducting qualitative research. For the member check, a focus group of the instructors participating in the study was convened, and after a presentation of the study, the instructors were asked to reflect on the accuracy of the account as suggested by Stake (1995). Although all instructors were invited, the four postsecondary instructors participated in the member check. The two secondary instructors were not able to attend. The results of the member check are recorded in Chapter 4. The use of a member check ensured the researcher accurately reflected the thoughts of the respondents. These are a few of the strategies suggested for researcher credibility as summarized by Glesne (2011).

The next two chapters present the results of the study and discuss the impact of the study on the teaching of calculus. Both of the next two chapters are organized according to the three

components of the study as discussed in this chapter. Each of these components was described in detail in the Instrumentation section of this chapter. In Component 2, the interviews were transcribed and analyzed in attempt to explicate each instructor's expectation of their students' prior knowledge of functions. In many cases, as I analyzed the interviews, I constructed tables that facilitated the comparison and contrast of the interviewee's responses. The use of the tables allowed me to look for common themes of expectations and use the themes to compare and contrast the explications. The final chapter was the result after immersing myself in the transcriptions, tables, and analysis of the interviews. It includes a summary of the study, conclusions that can be reached and recommendations for further studies

# CHAPTER 4: RESULTS

In this chapter, the results of the instructor interviews are revealed. The chapter is divided into three parts to reflect the three components of the instructor interview which were described in detail in the Methodology section. Within each component, the subsections address various aspects of the research questions of the study which were:

1. How are secondary calculus instructors' assumptions of their students' prior knowledge of the function concept different from that of the postsecondary calculus instructors' assumptions?

2. How do secondary and postsecondary instructor assumptions of their students' prior knowledge of functions impact instructional decision-making for their calculus course?

3. How do secondary and postsecondary instructors differ on their views toward students' conceptual understanding of functions and procedural fluency of functions?

The research questions, along with the review of the literature, drove the development of the instructor interview protocol, the analysis of the data, and the reporting of the data which is found in this section. The goal of the study was to answer the research questions, but due to the nature of qualitative studies, the questions are not addressed linearly as may be the case in a quantitative study. Each of the three research questions are addressed and readdressed throughout the three components of the interview in order to reveal implicit, as well as explicit, answers to the questions. Table 4: *Triangulation of Research Methods* in the Methodology section of this study identifies where in the structure of the interview each research question was addressed. The analysis generated answers to the research questions by analyzing instructors'

think-alouds and individual responses to the written and verbal interview questions. I was able to identify how these instructors assess their students' prior knowledge and how the results of that assessment impact their unique decision-making processes for the instruction of calculus. The findings were validated by comparing and contrasting the three components of the interview between and among the three types of educational institutions and then conducting a member check for the participants. After the results are presented in this chapter, the next and final chapter is a discussion of the results, also organized by the three components of the interview, and addressed potential limitations of the study along with suggestions for further study.

## Component 1: Written Survey

### Questions 1-3: Course and textbook.

The first three questions on the Written Survey were as follows:

1. Where are you currently employed and teaching calculus?
2. What is the title of your calculus course?
3. What is the title/publisher of the textbook you are using?

For the first question, the secondary instructors named two different secondary schools in the same east coast school district. The same state college was listed by both of the state college instructors. It is located in a major city and approximately 50 miles from the school district of the high school instructors. The same university was listed by both of the university instructors. It is located in the same major city as the state college and is in close proximity to the state college.

The secondary instructors referred to their course as AP Calculus AB. The course name and content is approved by the state department of education for all public secondary institutions.

The postsecondary instructors at both the state college and university use the nomenclature MAC 2311 Calculus with Analytic Geometry I. Similarly, this numbering system is used by all public postsecondary institutions within the state in order to facilitate the transfer of courses between institutions.

Although the nomenclature of the course for the two postsecondary institutions was the same, the textbooks used for the postsecondary courses were different. In fact, the state college instructors used the same textbook series for their course as did the secondary instructors, which was the Stewart (2010) Calculus Series, either the $6^{th}$ or $7^{th}$ edition, published by Cengage Learning Brooks/Cole. The two university instructors both used Briggs & Cochran (2011) *Calculus* published by Pearson. The textbooks used for the courses were consistent within the same institutions (see Table 5: *Textbook and Course Title*)

Table 5:

*Textbook and Course Title*

| **Instructor** | **Course title** | **Length of course** | **Textbook** |
|---|---|---|---|
| Sec #1 | AP Calculus AB | 1 year | *Calculus: Early Transcendental* from the Stewart's Calculus Series |
| Sec #2 | AP Calculus AB | 1 year | *Calculus: Early Transcendental* from the Stewart's Calculus Series |
| State #1 | MAC 2311 Calculus with Analytic Geometry | 1 semester | *Calculus, 7th Edition* from the Stewart Calculus Series |
| State #2 | MAC 2311 Calculus with Analytic Geometry | 1 semester | *Calculus, 7th Edition* from the Stewart's Calculus Series |
| Univ #1 | MAC 2311 Calculus with Analytic Geometry | 1 semester | *Calculus* by Briggs/Cochran |
| Univ #2 | MAC 2311 Calculus with Analytic Geometry | 1 semester | *Calculus* by Briggs/Cochran |

**Question 4: The syllabi.**

The fourth question on the Written Survey was as follows:

4. Can you provide me a copy of the syllabus for your calculus course?

When examining the instructors' syllabi I was looking specifically for any formal, written evidence of individual instructors' expectations of their incoming calculus students' prior knowledge of functions. One of the secondary instructor's syllabi was four pages in length with one page addressing Course Overview, Course Objectives and the Textbook. Three pages listed an outline of the course in bulleted format arranged by four units. Evidence of the instructor's expectations was found in a number of places throughout the syllabus. The first was in the Course Objectives section of the syllabus. It stated, "The course will provide students with the opportunity to work with functions represented in a variety of ways – graphically, numerically, analytically, and verbally; and emphasizes the connections among these representations." In the Course Outline section of the syllabus the instructor stated the first unit would include a section on the four ways to represent a function (verbally, numerically, visually, and algebraically) and within that unit the subtopics of domain and range, and piece-wise functions would be included. The following section reviewed basic families of functions and their graphs. Subtopics included polynomial functions, rational functions, transformations and composition of functions.

In addition to formal, written evidence of individual instructors' expectations of their incoming students' prior knowledge of functions, I was also looking for written evidence of the instructors' expectations of successful student behavior. This was evident in the secondary instructor's second paragraph which stated:

> Students are expected to complete all assignments and read the textbook daily. Students are expected to come to class prepared and to focus on the material being discussed in class. You will put out your best effort in keeping up with your assignments, in participating in class discussions/activities, and in preparing for tests and quizzes. You must devote **at least one hour** [the bold and underline of this phrase is in the syllabus] every day to studying and completing your assignments for this class.

The other secondary instructor stated that there was not a requirement to provide students with a syllabus by the school, and therefore the instructor did not write one.

All postsecondary instructors did provide a copy of their syllabi. The state college instructors' syllabi were very similar in structure, two to three pages in length, and did not specifically mention expectations of students' prior knowledge other than stating a minimum grade of C in the Pre-calculus Algebra and Trigonometry prerequisite courses or an appropriate score on the college's approved assessment test was required for the course. Written evidence of the instructors' expectations of successful student behavior were found in one state college instructor's syllabus in the Course Description, Attendance, Tardiness, and Cell Phones sections of the syllabus. In the Course Description section the instructor stated, "To be successful in this class **it is important to set aside time between each class meeting to work on the assignment** [the bold font is in the syllabus]**.** It is suggested that you discuss homework ideas with other students in the class." In the Attendance section, the instructor stated, "The student will be responsible of obtaining and doing any assignment that is made during their absence." In the Tardiness section of the same syllabus the instructor stated,

> Being late to class, leaving early, or leaving and returning, is a disruption to the class and is discourteous to the professor and the other students. All students are expected to be on time and to stay for the entire class period. Please inform the instructor if you know you will need to leave the class early.

In the Cell Phones section, the instructor stated,

> **Cell phones are a distraction!** [Bold font in syllabus] Students should turn the ringer off on all electronic devices and **THEY SHOULD NOT BE IN SIGHT** [Bold and capitalized font in syllabus]. There should be no texting during class time. If a student's phone rings or he/she is found texting during class time, then there will be an automatic pop quiz on a topic of the instructor's choice. Multiple infractions will result in expulsion from the class.

Written evidence of the instructors' expectations of successful student behavior was found in the other state college instructor's syllabus in the Attendance, Cell Phone, and Homework sections of the syllabus. In the Attendance section, the syllabus stated, "Students are responsible for all class materials and any announcements made in class whether they are present or not." In the Cell Phones and Other Disruptions section the instructor stated,

> Students are expected to turn off cell phones at the start of class unless the instructor is notified of a possible emergency call. Being late to class or leaving early is a disruption to the class and is discourteous to the professor and other students. All students are expected to be on time to class and to stay in class for the entire period. Students are expected to behave in a manner that is conducive to learning both for themselves and others in the class. Student may be asked to leave if their behavior is deemed a disruption by the instructor.

In the Homework section the instructor stated,

> Students are expected to make an honest attempt to complete all assigned problems prior to the next class. It is recommended that you keep all homework neatly organized in a notebook. The key for success in this course is to do the homework. Depending on your understanding of the material, you may wish to do more than or fewer than the suggested number of problems. Selected homework problems may be collected for a grade.

In addition to the instructor expectations stated above, both state college instructors included a section in their syllabi which outlined the competencies expected of a graduate of their particular educational institution that were defined by the faculty of that institution. The four interrelated competencies included Think, Value, Communicate, and Act. Students were directed to a website for more details concerning these institutional expectations.

The syllabi provided by the university instructors were the most lengthy and detailed. Like the state college instructors' syllabi, the university instructors did not specifically mention expectations of students' prior knowledge other than stating a minimum grade of C in the Pre-calculus Algebra and Trigonometry prerequisite courses or an appropriate score on the university's approved assessment test was required for the course. In addition to the course prerequisite statement, one of the university instructors added, "**If you have not passed the prerequisites with a "C" or better, you must retake the prerequisite course before you enroll in this course.** Note that the grade of "NC" does NOT constitute a passing grade in a course." [All bold, underlined, and capitalized fonts appeared in syllabus].

Although they did not mention specific mathematical expectations for their course, both university instructors' syllabi mentioned a requirement of the students to spend time in the computer lab located in the Mathematics Building on the main campus of the university. The computer lab requirements between the two university instructors varied. In one instructor's syllabus it stated that prior to the first test grade; all students were required to spend 4 hours per week in the computer lab. The amount of time after the first test decreased to zero hours, stayed 4 hours, or increased to 6, or 8 hours of required computer lab time depending upon the test score of the first test. In the other university instructor's syllabus it stated that the students were required to take four skills tests according to a stated time table throughout the semester. The instructor then gave an explanation of the computer lab's scheduling policy in general, how to schedule a skills test in particular, and rules to access the lab and the skills test. In addition to including a paragraph in his syllabus about the policies of the computer lab, he also attached a page of very detailed explanation of the policies and procedures of the computer lab at the end of his syllabus.

In addition to the required lecture given by the instructor, which is 50 minutes in length three times per week, one of the university instructors had the additional requirement for students to attend "Application Sessions" twice a week taught by teaching assistants. The syllabi included a section which delineated the section number, office hours, and email addresses of the three sections and their assigned teaching assistant. For the 50-minute lecture, the students were required to purchase and bring a remote student response device called an "iClicker2" to the lecture sessions. This device is used for attendance as well as participation during the lecture. The syllabus includes detailed instructions about the purchase and use of this device. This instructor also included a two page, week-by-week, outline of the sections from the textbook that would be covered and the due dates for the Skills Tests and written test dates for the semester.

One instructor stated these expectations in the Attendance/Etiquette section of the syllabus. He stated the following:

> Attendance of all the lectures and application sessions is **mandatory**: Past experience indicates that the students who will succeed in the class are the ones who attend. Observe common rules of courtesy. Once inside the classroom you must turn off all cell-phones and laptops, as they are not to be used during class. You should plan on staying the entire 50 minutes. Avoid leaving early or arriving late as it is a distraction to your classmates and your instructor.

There was also a section on the calculator usage, academic honesty, and online homework assignments. Students were permitted to use a non-graphing and non-programmable calculator for the in-class tests, but not for the computer-based tests. Cell phone usage and the sharing on calculators were not permitted. The homework was graded online, therefore students were expected to have access to a computer. All assignments had posted due dates and these due dates would not be extended under any circumstances and personal computer issues, including login

errors, would NOT be a reason to offer any type of extension. For academic honesty, the instructor stated, "The work submitted in this class is expected to be your own."

The instructor also included a detailed explanation of the scheduling policy for the skills tests. There were three more sections on the "No-Credit" (NC) Grade Policy for the university, the instructor's grading policy with two options for how grades could be calculated, and the expectation that all students will check their university email account on a regular basis for important messages from the instructor.

The other university instructor included expectations about calculator, cell phone, laptop computer, and music player usage. For Calculators, she stated the students are permitted to use only the TI-30XA scientific calculator for tests and quizzes and that lap top computers, iPods and other music players may not be used in class. She also stated that cell phones must be turned off before coming to class. She goes on to clarify that cell phones are strictly prohibited during tests and quizzes and the use of a cell phone will be viewed as academic dishonesty. She adds, "**Thus, do not touch your cell phone during a test or quiz. Wait until after you have left the room and are finished with the test/quiz to use it.**" She also contained a section on homework and academic dishonesty. She explained there will be graded and ungraded homework assignments that are to be completed online. Very similar to the other university instructor, she stated "As these assignments must be completed online, students will be expected to have access to a computer. Students were permitted to use a computer in one of the computer labs on the main campus." In addition, "All assignments will have posted due dates and these **due dates will not be extended** so please plan accordingly. Personal computer issues, including login errors, will NOT be a reason to offer any type of extension." Again, very similarly to the other university instructor, for academic honesty she stated, "The work you submit in this class is

expected to be your own." The tables below show the number of pages of each instructor's syllabus, and the written mathematical and behavioral expectations of the students.

Table 6:

*Written Expectations in Instructor Syllabi*

| **Instructor** | **Length # pages** | **Instructor's Written Expectations** | |
|---|---|---|---|
| | | **Mathematical** | **Behavioral** |
| Sec #1 | No syllabus provided | | |
| Sec #2 | 4 (3 pages of course content) | +Work with functions represented in a variety of ways – graphically, numerically, analytically, and verbally; and emphasizes the connections among these representations."<br>+Subtopics of domain and range, and piece-wise functions would be included. | +Complete all assignments and read the textbook daily.<br>+Come to class prepared and to focus on the material being discussed in class.<br>+Put out your best effort in keeping up with your assignments, in participating in class discussions/activities, and in preparing for tests and quizzes.<br>+Must devote **at least one hour** every day to studying and completing your assignments for this class. |
| State #1 | 2 | Minimum grade of C in the Pre-calculus Algebra and Trigonometry prerequisite courses or an appropriate score on the college's approved assessment test was required for the course | +Responsible for all class materials and any announcements made in class whether they are present or not<br>+Expected to turn off cell phones at the start of class unless the instructor is notified of a possible emergency call.<br>+Being late to class or leaving early is a disruption to the class and is discourteous to the professor and other students.<br>+Expected to be on time to class and to stay in class for the entire period.<br>+Expected to behave in a manner that is conducive to learning both for themselves and other in the class.<br>+Expected to make an honest attempt to complete all assigned problems prior to the next class.<br>+The key for success in this course is to do the homework. |

| Instructor | Length # pages | Instructor's Written Expectations | |
|---|---|---|---|
| | | **Mathematical** | **Behavioral** |
| State #2 | 3 | Minimum grade of C in the Pre-calculus Algebra and Trigonometry prerequisite courses or an appropriate score on the college's approved assessment test was required for the course | +To be successful in this class **it is important to set aside time between each class meeting to work on the assignment**<br>+Responsible of obtaining and doing any assignment that is made during their absence.<br>+Expected to be on time and to stay for the entire class period.<br>+ Inform the instructor if you know you will need to leave the class early.<br>+ Turn the ringer off on all electronic devices and **THEY SHOULD NOT BE IN SIGHT**<br>+No texting during class time. |
| Univ #1 | 5 | +MAC 1140 and MAC 1114 or equivalent.<br>+**If you have not passed the prerequisites with a "C" or better, you must retake the prerequisite course before you enroll in this course.** Note that the grade of "NC" does NOT constitute a passing grade in a course.<br>+ prior to the first test grade; all students were required to spend 4 hours per week in the computer lab<br>+The amount of time after the first test decreased to zero hours, stayed 4 hours, or increased to 6, or 8 hours of required computer lab time depending upon the test score of the first test. | +Students are permitted to use only the TI-30XA scientific calculator for tests and quizzes and that lap top computers,<br>+ iPods and other music players may not be used in class.<br>+Cell phones must be turned off before coming to class.<br>+Cell phones are strictly prohibited during tests and quizzes and the use of a cell phone will be viewed as academic dishonesty<br>+Do not touch your cell phone during a test or quiz. Wait until after you have left the room and are finished with the test/quiz to use it."<br>+Graded and ungraded homework assignments that are to be completed online. As these assignments must be completed online, students will be expected to have access to a computer. Students may use a computer in one of the computer labs on the main campus.<br>+All assignments will have posted due dates and these due dates will not be extended so please plan accordingly.<br>+Personal computer issues, including login errors, will NOT be a reason to offer any type of extension.<br>+The work you submit in this class is expected to be your own." |

| Instructor | Length # pages | Instructor's Written Expectations | |
|---|---|---|---|
| | | **Mathematical** | **Behavioral** |
| Univ #2 | 8 | +Appropriate score on the [college] Math Placement Exam, or MAC 1140 and MAC 1114 with a C or better.<br>+In addition to the required lecture given by the instructor, which is 50 minutes in length three times per week, one of the university instructors has the additional requirement for students to attend "Application Sessions" twice a week taught by teaching assistants. | +Students are required to purchase and bring a remote device called an "iClicker2" for attendance as well as participation during the lecture.<br>+Attendance of all the lectures and application sessions is mandatory:<br>+Observe common rules of courtesy.<br>+Once inside the classroom you must turn off all cell-phones and laptops, as they are not to be used during class.<br>+Plan on staying the entire 50 minutes. Avoid leaving early or arriving late as it is a distraction to your classmates and your instructor.<br>+Students may use a non-graphing and non-programmable calculator for the in-class tests, but not for the computer-based tests.<br>+Cell phone usage and the sharing on calculators will not be permitted.<br>+The homework is graded online homework, therefore students will be expected to have access to a computer.<br>+All assignments have posted due dates and these due dates will not be extended under any circumstances and personal computer issues, including login errors, will NOT be a reason to offer any type of extension.<br>+The work submitted in this class is expected to be your own." |

**Question 5: Use of diagnostic.**

The fifth question on the Written Survey was as follows:

5. Do you use a diagnostic tool to assess your incoming calculus students' prior knowledge of functions?

Only one instructor, a state college instructor, stated he or she used a diagnostic tool other than the placement test that is given to all students. When the instructors were asked why they did not use a diagnostic instrument to assess their incoming student prior knowledge of functions, the two university instructors did not answer the question. One secondary instructor stated, "I generally have taught the students for one to two years in a row prior to entering calculus (the majority)." The other instructor stated that she provides a summer assignment to be completed by her incoming students prior to the beginning of the course. The assignment covers pre-calculus material and the instructor reviews the assignment during the first week of Calculus and gives a test at the end of the first week of class. The state college instructor that does not use a diagnostic stated, "I feel that I can address review topics as they arise."

For the state college instructor that did provide a diagnostic instrument to her students during the course of the calculus class, the instructor stated that she developed her diagnostic instrument from years of teaching the course and having a sense as to the types of pre-calculus topics with which students had difficulty. The state college instructor's diagnostic instrument contained 11 questions. The first question asked students to factor a cubic trinomial, the next question had four parts, and asked students to simplify radical expressions with negative and fractional exponents. The next question was three parts and asked students to simplify rational expressions with a denominator. The next questions asked students to factor and simplify two

polynomial expressions. They need to recognize the difference of two squares and recognize common factors. The next question referred back to the rational polynomial expression they simplified in the previous question, and asked where the expression would be zero and where it would be undefined. The next question asked students to describe the graph of a quadratic and linear function. The remainder of the diagnostic covered logarithmic functions and trigonometric functions. The instructor stated that she uses the results of the diagnostic to determine how much class time needed to be spent on specific pre-calculus topics.

**Question 6: Learning of pre-calculus concepts.**

The sixth question on the Written Survey was as follows:

6. Where do the majority of your students learn the pre-calculus concept of function prior to your class?

Secondary instructors' responses:

> #1: "Functions are introduced in Honors Algebra 1 and taught more thoroughly in Honors Algebra 2, taught a third time in Pre-calculus."
> #2: "Pre-calculus at [this institution], different instructor. Book – Recall Stewart (same author as Calc book)

State college instructors' responses:

> #1: "[This institution]'s Precalc. MAC 1140 MAC 1114 Trig
> #2: "3/4 are from [this institution] and ¼ are transients." By transients, the instructor stated that she meant the students took the prerequisite courses at the nearby university.

University instructors' responses:

> #1: "Ask [name of another faculty member]
> #2: "I teach in the [name removed for anonymity] Program in the fall semester and they take Calc I with me in the spring.

**Question 7: Review/clarification of pre-calculus topics**

The seventh question on the Written Survey was as follows:

7. Based on your experience from teaching calculus, which, if any, of the following pre-calculus topics do you feel that many of your students need clarification/review prior to starting your class?

A chart was provided for the teachers to complete (see Appendix B). They were asked to put an X in the appropriate box for each pre-calculus concept. If asked by the instructor, I explained that by "review" I meant the students came to their course with a misconception of the subtopic and by "clarification" I meant that the students did not remember the concept incorrectly, but needed to be reminded of the concept. Table 7: *Review/Clarification of Pre-calculus Topics* reflects the instructors' responses by type of institution. It contains a color-coded legend. Red X's denote a secondary instructor's response, green X's denote a state college instructor's response and blue X's denote a university instructor's response.

Table 7:

*Review/Clarification of Pre-calculus Topics*

| | **"Many of my incoming students need"…** | | | |
|---|---|---|---|---|
| **Pre-calculus Topic** | **Review** | **Clarification** | **Both** | **Neither** |
| a. Arithmetic computation used to evaluate a function at a single numerical value | | | X | XXXXX |
| b. Subconcepts of function such as domain, range and correspondence | XXXX | XX | | |
| c. Definition of function | XXXXX | | | X |
| d. Graphing/evaluating piecewise functions | XXXXX | | X | |
| e. Graphing/evaluating discontinuous functions such as step functions or rational functions | XX | X | XX | X |
| f. Identifying points of discontinuity in rational functions | XXXX | | X | X |
| g. Graphing/identifying functions that have been translated | XXXX | | XX | |
| h. Problem solving using function models | XX | X | XXX | |

**Secondary Instructors (red)**
**State College Instructors (green)**
**University Instructors (blue)**

**Question 8: Opportunities for pre-calculus review.**

The eighth question on the Written Survey was as follows:

8. If you determine that your students need clarification or review of functions (excluding trigonometric, exponential, and logarithmic) in order to be successful in your calculus class, how and/or where do students receive this service?

The two secondary instructors stated that they provide review of the pre-calculus topics themselves either during class or by the use of worksheets and extra assignments. The state college instructors also stated that they provided some review in class and one state college instructor refers her students to an online tutoring site that was developed by herself and colleagues at the state college. One university instructor also mentioned that she provides review in class, "when those topics come up." The other university instructor stated that his teaching assistants provided the necessary review, clarification, or both.

**Question 9: Class days spent on functions.**

The ninth question on the Written Survey was as follows:

9. Approximately how much class time do you spend on functions (excluding trigonometric, exponential, and logarithmic) in your calculus class? For example, "I spend 5 class days on functions."

The secondary instructors ranged from 2 days to one week during the first week of the course. One state college instructor stated, "Initially one class day (75 minutes) and then intermittently throughout the semester." The other state college instructor stated that the functions were reviewed in approximately 10 minute segments throughout the entire course as opposed to one entire class day. The university instructors stated one day and two days.

**Question 10: Procedural fluency and computational understanding**

The tenth question on the Written Survey was in two parts. The first part was:

a) What is your understanding of the term "procedural fluency" as it relates to the pre-calculus concept of functions?

Secondary instructors' responses:

#1: "Students can identify domain, range, evaluate, find inverses and graph various functions."
#2: "knowing algorithm"

State college instructors' responses:

#1: "Not familiar"
#2: "Procedural is the "crunching" of functions – how to use them symbolically."

University instructors' responses:

#1: "I am not sure."
#2: left blank by instructor

The second part was:
(b) What is your understating of the term "conceptual understanding" as it relates to the pre-calculus concept of functions?

Secondary instructors' responses:

#1: "Students understand the definition of a function, why it is a function.
#2: "Students should have a visual picture of the function in their minds and know the properties (asymptotes, zeros, etc.)

State college instructors' responses:

#1: "Basic understanding"
#2: "Those students understand functions, function notation (graphically and symbolically) with ease – moving between representations with ease."

University instructors' responses:

#1: "Students understand what they are doing and why."
#2: "Students understand what a function is as well as related properties."

**Question 11: Adequacy of function diagnostic.**

For the last question on the written survey, the instructors were asked to take a few minutes and review the questions on the diagnostic instrument that was going to be used in the second component of the interview. A copy of the diagnostic instrument was provided (see Appendix B).

The eleventh question on the Written Survey was as follows:

11. On a scale of 1-10 with **1** being ***"completely inadequate"*** and **10** being ***"completely adequate"*** how would you rate the overall adequacy of this instrument in assessing your students' prior knowledge of functions?

Sec #1: 9, Sec #2: 10

State #1: 8, State #2: 9

Univ #1: 9, Univ #2: 7

In Component 1, after being shown the function diagnostic instrument, instructors were asked to give a rating from one to ten of the overall adequacy of the instrument for determining their incoming students' prior knowledge of functions. For the Likert-type scale, 1 was defined as "completely inadequate" and 10 being "completely adequate." The instructors were instructed to give one number for the entire diagnostic instrument which was composed of seven questions with multiple parts. The range of instructor scores were from 7 – 10 with the secondary instructors rating the instrument as 9 and 10, the state college instructors' ratings were 8 and 9 and the university instructors' ratings were 7 and 9. The mean average of these ratings was $8.\overline{6}$ and the mode was 9.

That completed the first component of the interview. The instructors handed me their written responses and I turned on the tape recorder to audio record the remainder of the

interview. During component two, I encouraged the interviewees to "think-aloud" as they responded to seeing 5 students' answers to 4 of the questions from the diagnostic instrument they just evaluated. I asked them to respond as if they had administered this instrument to their incoming calculus students.

**Component 2: Student Answers**

In Component 2, the instructors were asked to look at each question on the function diagnostic instrument individually and comment on how well they anticipated their incoming students would do on each question. Table 8: *Instructor Expectations of Prior Knowledge of Functions* shows the results of the teacher responses to the questions that were identified as specific problem areas in the literature review. The plus sign (+) was recorded when the instructors stated that they expected their students to be able to answer that specific question correctly. The instructors made statements such as, "They should be able to answer (question number) correctly" or "all students will answer (question number) with no problem." The tilde sign ( ~) was recorded when the instructors stated that some of their students would be able to answer that specific question correctly. The instructors made statements such as, "I hope the majority would be able to do it, but probably 50-50" or "It is important, but I am not too optimistic here, I would say 50-50." For the last symbol, the minus sign ( - ) was recorded when the instructors expected that the majority of their students would not be able to answer a question correctly. The instructors made statements such as, "I would not believe that they would do very well at all" or "I have to say that I don't anticipate my students being able to do that one correctly."

Table 8:

*Instructor Expectations of Prior Knowledge of Functions*

| Question # | Mathematical Concept | Sec #1 | Sec #2 | State #1 | State #2 | Univ #1 | Univ #2 |
|---|---|---|---|---|---|---|---|
| 1e | Domain/Range of Graph | + | + | + | + | ~ | + |
| 3a | Domain of Rational Function | + | + | - | + | ~ | + |
| 4c | Translation of Function | + | + | + | + | - | + |
| 6a | Evaluating Piecewise Function | + | + | + | + | + | - |
| 6b | Graphing Same Piecewise Function | ~ | - | - | - | ~ | - |

**+** = Instructor expects most students to answer correctly

**~** = Instructor expects about half to answer correctly

**-** = Instructor expects few, if any, to answer correctly

**Domain and range themes.**

For both questions 1e and 3a the students were asked about the domain of a given function (see Table 2: *Four Questions and Student Answers Selected for Instructor Interview*). In question 1e the unspecified function was given in graphical form and in question 3a the rational function was given in algebraic form. In 1a, the student was asked about the range as well as the domain while in 3a the student was asked only to find the domain. In both problems the following three themes for domain and range emerged when analyzing the instructors' responses to student answers:

*Domain/Range Theme 1*: Students may have the idea of domain and range, but they don't know how to communicate what they know.

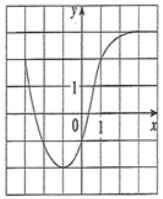
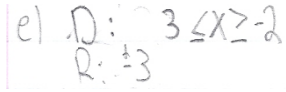
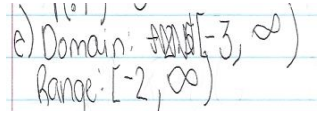
*Domain/Range Theme 2*: Sometimes the questions we ask, as teachers, are ambiguous to the students.

*Domain/Range Theme 3*: Students put "all real numbers" when they don't know the answer to a domain/range question.

Tables 9 and 10 below show the actual selected student responses to the diagnostic question on the left side of the table. The right side of the table displays the instructors' responses to the student work from which these themes emerged.

Table 9:

*Instructor Responses to Question 1e*

| | | |
|---|---|---|
|  | **Question #1: The graph of a function f is given at the left**<br><br>**(e) State the domain and range of *f*.** | |
| Domain/Range Theme 1: Students may have the idea of domain and range, but they don't know how to communicate what they know. | | |
| Student #17:<br>  | Sec#1 | They do not know how to properly show an interval |
| | Sec#2 | Inequality symbol is switched around. |
| | State#1 | This is a common error…they realize what they want to say, but they don't understand the inequality notation…they don't understand how to communicate it well. |
| | State#2 | It may be easier to explain this in interval notation rather that inequality notation. |
| | Univ#1 | The student seems to be confused |
| | Univ#2 | General idea of what he/she is doing, but doesn't know the notation to describe it. |
| Domain/Range Theme 2: Sometimes the questions we ask, as teachers, are ambiguous. | | |
| Student #21:<br>  | Sec#1 | Understands what domain and range are. |
| | Sec#2 | I could see how domain could be a possible answer based at what they are looking at. |
| | State#1 | It could be the student didn't understand the graph…I could see how you could misinterpret that. |
| | State#2 | I think it is our fault as teachers by not making it clear. |
| | Univ#1 | I think it would be better if a thicker line. I can see why they think that. |
| | Univ#2 | It could be the student didn't understand the graph ended and I could see how you could misinterpret that. |

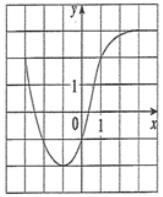
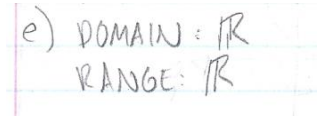
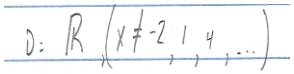
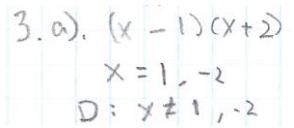
<table>
<tr><td>  </td><td colspan="2">Question #1: The graph of a function f is given at the left<br><br>(e) State the domain and range of f.</td></tr>
<tr><td colspan="3">Domain/Range Theme 3: Students put “all real numbers” when they don’t know the answer to a domain/range question.</td></tr>
<tr><td rowspan="6">Student #27:<br>  </td><td>Sec#1</td><td>Doesn’t understand domain and range. Not looking at graph.</td></tr>
<tr><td>Sec#2</td><td>They just don’t have the concept of what domain and range is.</td></tr>
<tr><td>State#1</td><td>I would say they don’t have any understanding of domain and range.</td></tr>
<tr><td>State#2</td><td>All R, that is the default if you don’t know what the domain and range are.</td></tr>
<tr><td>Univ#1</td><td>Student thinks that is a common answer. It is a good guess.</td></tr>
<tr><td>Univ#2</td><td>That student didn’t really understand much and just guessed.</td></tr>
</table>

Table 10:

*Instructor Responses to Question 3a*

| $f(x)=\frac{2x+1}{x^2+x-2}$ | **Question #3(a): Find the domain of the function.** | |
|---|---|---|
| Domain/Range Theme 1: Students may have the idea of domain and range, but they don't know how to communicate what they know. | | |
| Student #7:  | Sec#1 | This student knows there are restrictions, but not sure how to find them. |
| | Sec#2 | I think just the notation and the 4. Student knows 2 and -1. |
| | State#1 | I don't know where the 4 is coming from and the dot-dot-dot. It looks like they are continuing a pattern, but there is no pattern. |
| | State#2 | They obviously know there are several places undefined. I don't know where they got 4 or what dot-dot-dot means. |
| | Univ#1 | The student got two points but that 4 and dot-dot-dot. Communication is lacking. |
| | Univ#2 | The first two are correct, but I don't know why those other numbers are there. I'm not sure why they think it goes on indefinitely. |
| Domain/Range Theme 2: Sometimes the questions we ask, as teachers, are ambiguous. | | |
| Student #24:  | Sec#1 | Some teachers would accept this, but they should write intervals. |
| | Sec#2 | It should have specified to put it in interval form. |
| | State#1 | I think that is an acceptable answer. I don't feel they need to put it in interval form. |
| | State#2 | Some algebra teachers are perfectly fine with that answer. That may be just a notational thing. They understand the idea. |
| | Univ#1 | I think the student just isn't comfortable with how to write down the answer. It is clear that they know what to do. |
| | Univ#2 | Maybe not the notation, but it looks good. |

| | | |
|---|---|---|
| $f(x)=\frac{2x+1}{x^2+x-2}$ | **Question #3(a): Find the domain of the function.** | |

Domain/Range Theme 3: Students put "all real numbers" when they don't know the answer to a domain/range question.

| | | |
|---|---|---|
| Student #18:<br>  <br> | Sec#1 | Has no idea. |
| | Sec#2 | Definitely not right and no work shown. |
| | State#1 | They don't have a clue there. That person doesn't have an idea of domains and undefined. |
| | State#2 | Just a default. I think most would understand that you can't divide by zero. So it is just a default. |
| | Univ#1 | It is just the wrong thing and there is nothing there to explain it. |
| | Univ#2 | Maybe R is often the right answer. So maybe they are choosing something they see most often. |

**Translation of functions themes.**

For question 4c, students were asked to describe the translation of a graph from its parent function given the algebraic form of the function. In this particular problem, the function had either a horizontal and a vertical shift, but no reflection, rotation, or dilation. Two themes emerged for the translation of functions when analyzing the instructors' responses to student answers:

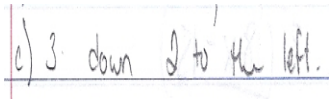
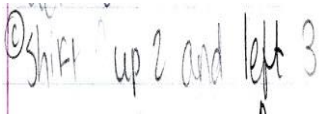
*Translation Theme 1:* Students remember something, they just remember it wrong.

*Translation Theme 2*: Students may have the idea of translation, but they don't know how to communicate what they know.

For the first translation theme, the instructors gave very similar responses to two different student's answers. Table 11: *Instructor Responses to Question 4c* shows the actual student answers and subsequent instructor responses to one of the translation questions on the function diagnostic instrument.

Table 11:

*Instructor Responses to Question 4c*

| **Question #4: How are the graphs of the functions obtained from the graph of *f*?<br>(c) y= *f*(x-3) + 2** | | |
|---|---|---|
| Translation Theme 1: Students remember something, they just remember it wrong. | | |
| Student #3:<br>  <br> | Sec#1 | This student has horizontal and vertical reversed. |
| | Sec#2 | I wouldn't give credit because it is vague. |
| | State#1 | This person has the vertical and horizontal shifting reversed. |
| | State#2 | They are backwards. They knew one was supposed to go the opposite, they're just backwards. |
| | Univ#1 | They remember something but they remember it wrong. They're just trying to memorize, and switched it around. |
| | Univ#2 | Student sort of has an idea, but has forgotten the details and why things work. He has messed up the vertical and horizontal shift. |
| | | |
| Student #20:<br>  <br> | Sec#1 | Backwards. |
| | Sec#2 | Shifting up is correct but not left 3. |
| | State#1 | They knew which to move horizontal and vertical but got it wrong. Memorizing instead of understanding. |
| | State#2 | This is a typical wrong answer. They don't understand that you have to set equal to zero to see the shift. |
| | Univ#1 | Well almost right. Some partial understanding. |
| | Univ#2 | Seems to understand the vertical shift but seems to have forgotten the horizontal shift. |

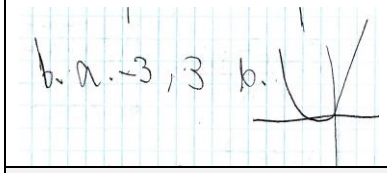
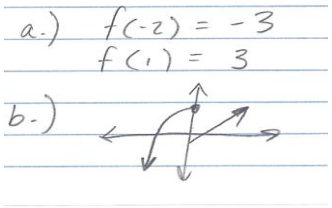
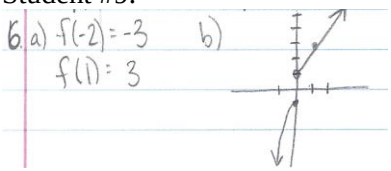
<table>
<tr><th colspan="3">Question #4: How are the graphs of the functions obtained from the graph of f?<br>(c) y= f(x-3) + 2</th></tr>
<tr><td colspan="3">Translation Theme 2: Students may have the idea of translation, but they don't know how to communicate what they know.</td></tr>
<tr><td rowspan="6">Student #4:<br>  </td><td>Sec#1</td><td>Needs reinforcement review. I'd tell them to plot some points.</td></tr>
<tr><td>Sec#2</td><td>They realize horizontal and vertical. Maybe lazy or slang. I think it is a communication thing.</td></tr>
<tr><td>State#1</td><td>Communication error. They understand it, they are just not saying it in a way that is good.</td></tr>
<tr><td>State#2</td><td>They have the translation idea right. We would have a discussion about it's the entire graph that move.</td></tr>
<tr><td>Univ#1</td><td>It's a funny way of saying that, but I think this student understands what is going on.</td></tr>
<tr><td>Univ#2</td><td>That student has the idea correct but does not express it correctly.</td></tr>
</table>

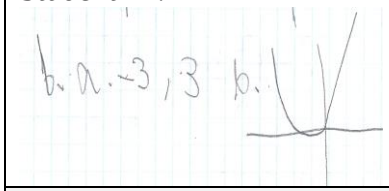
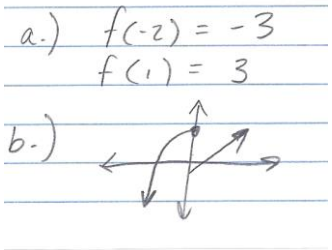
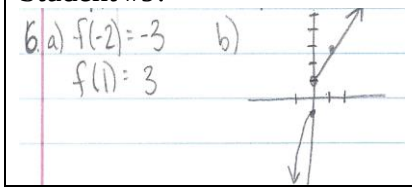
For Translation Theme 1, two of the instructors stated that the shifting is "reversed" but don't elaborate on why they believe the students reversed the two translations. One instructor just stated that the answer is "too vague" for Student #3, while another instructor just replies, "Backward" for Student #20. The other three instructors stated the reason for the students' vagueness or reversal is due to remembering incorrectly or not memorizing correctly. The students' lack of ability to communicate what they knew came up as a concern in both the domain and range problems and also the translation of functions problem.

The two themes pertaining to piecewise functions emerged differently than the other two sets of themes. For piecewise functions, the first of the two themes emerged from the secondary and university instructors and the second theme emerged from the two state college instructors. For that reason, Table 12: *Instructor Responses to Question 6*, is structured slightly different from Tables 9, 10, and 11. The same student answers are displayed but the themes are separated by the instructors rather than the diagnostic questions as they are in Tables 9, 10 and 11.

Table 12:

*Instructor Response to Question 6*

| $f(x)=\begin{cases}1-x^2, if x\le 0\\ 2x+1, if x>0\end{cases}$ | **Question #6: (a) Evaluate *f*(-2) and *f*(1)**<br>**(b) Sketch the graph of *f*.** | |
|---|---|---|
| Piecewise Function Theme 1: Students are better at evaluating a piecewise function than sketching a piecewise function. | | |
| Student #4:<br>  | Sec#1 | Letter (a) correct. No clue how to draw a graph. |
| | Sec#2 | (a) part is correct (b) is wrong. |
| | | |
| | Univ#1 | Here the numbers are right, the graph is not. |
| | Univ#2 | Student correctly found the answers in (a), but sketch is incorrect. |
| | | |
| Student #7:<br>  | Sec#1 | Part (a) right, part (b) I'm not sure. |
| | Sec#2 | (a) is correct, (b) is not correct. |
| | | |
| | Univ#1 | Almost right, correctable understanding. |
| | Univ#2 | The student correctly does the work in part (a). General idea but trouble in the details. |
| | | |
| Student #9:<br>  | Sec#1 | General idea. |
| | Sec#2 | Part (a) is OK but part (b) is not correct because they have discontinuity. They didn't know how to graph the parabola. |
| | | |
| | Univ#1 | The numbers are good. For the graph the placement is wrong. |
| | Univ#2 | Part (a) is correct. Part (b), problem with the details. |

| $f(x)=\begin{cases}1-x^2, if x\le 0\\ 2x+1, if x>0\end{cases}$ | **Question #6: (a) Evaluate *f*(-2) and *f*(1)**<br>**(b) Sketch the graph of *f*.** | |
|---|---|---|
| Piecewise Function Theme 2: Students don't see the connection between the answers they wrote in part (a) and the sketch they drew in part (b). | | |
| Student #4:<br>  | State#1 | Procedural understanding, but not conceptual. They understand how to evaluate but they don't seem to have a connection between the graph. |
| | State #2 | They have (a) right but they don't have (b) right. They did know there were two parts but they didn't know what the two parts looked like. |
| | | |
| Student #7:<br>  <br> | State#1 | They can do the procedural but there is no tie to the graph. Such a huge problem that they can't tie rule to graph. |
| | State#2 | They can evaluate. I would point out that part (a) should match part (b). |
| | | |
| Student #9:<br>  <br> | State #1 | Again, no connection between symbolic rule and the graph. No conceptual connection between these two pieces. |
| | State#2 | This one ignored the domain restrictions. I expected the wrong answer. |

### Piecewise function themes.

The theme that emerged from analyzing the *secondary and university* instructors' responses was:

*Piecewise Function Theme 1:* Students are better at evaluating a piecewise function than sketching a piecewise function.

The theme that emerged from analyzing the *state college* instructors' responses was:

*Piecewise Function Theme 2*: Students don't see the connection between the answers they wrote in part (a) and the sketch they drew in part (b).

All of these themes will be discussed in detail in Chapter 5.

## Component 3: Quotations

Component 3 consisted of asking the instructors to respond to two quotations from mathematics education research studies. One quotation is from the Carlson, Oehrtman, and Engleke (2010) study first described in Chapter 1, and the second quotation is from NCTM's *Principles and Standards for School Mathematics* (2000) and addresses conceptual understanding and computational fluency.

### Quotation 1: Resources to plan instruction.

Quotation 1 was as follows:

*"In the absence of research-based curricular instruments, teachers tend to rely on their own opinions about what students need to learn as they plan instruction" (Carlson, Oehrtman, & Engelke, 2010, p .114).*

The quotation was followed by the following two questions:

1. Do you agree or disagree with this statement?

2. What specific resources do you use to plan instruction for your calculus class?

One of the two secondary instructors stated that she both agreed and disagreed with the quotation. She stated that it depended upon the experience level of the teacher. Speaking from personal experience, she stated, "The more I taught, the more I learned." The second secondary instructor disagreed with the quotation. She stated, "I don't think we just rely on our own opinion if we are AP-trained. I follow everything I am taught. I don't rely on my own opinion. I use what the experts tell me. I think all teachers do because they want their students to do well."

One of the two state college instructors also stated that she both agreed and disagreed with the quotation. She stated the following,

> I think the second part is true about teachers use their own opinion about what students need to learn. I think they rely on opinion, but also what the [institutional mathematics] department has laid out, you have to follow a list of outcomes…own opinions but also requirements for the class. But I don't think the reason is because of research-based curricular instruments. I think there is an absence of research-based curricular instruments but I'm not sure that even if they were available teachers would rely on them anyway. There are too many variables. Well-designed tests are almost impossible. Too many things affect a classroom, time of the day, the semester, even class to class. Teachers see themselves as being unique and what happens in other classes with other teachers has nothing to do with them as a teacher and their students.

The second state college instructor just said, "I would agree with that." The two university instructors also agreed with the quotation. One stating, "Yeah, we really don't have any other option." The other simply stating, "I agree."

In reply to the question on specific resources the instructors used to plan instruction, both of the secondary instructors listed many AP resources including the AP syllabus, AP former questions, Barron's AP Prep Guide, the AP website, old resources from former AP workshops, the College Board website (see Table 13: *Resources Used to Plan Instruction*). One secondary instructor also mentioned the AP wiki, other AP Calculus teachers including local teachers, and teachers that are active on the AP listserv.

One state college instructor listed the Internet, online applets, the "fancy bells and whistles that come with the book, and 24 years of experience." She went on to mention that she also discusses teaching with colleagues and her husband whom also taught mathematics at the same institution for many years. The other state college instructor also mentioned how long she had been teaching calculus, she said, "I have been doing this a very long time, 25 years." She stated that she used the resources provided by the textbook company, she had "a million calculus books," and that she was a "worksheet" teacher. She developed her own activities and practice problems. She stated that calculus was the first time students had to determine how to make decisions based upon the problem. She said that when she was a new instructor, her main resource was the instructor she was replacing because "the books cover so much." She also mentioned that she "borrowed stuff" from her colleagues and from conferences.

In reply to the question about resources instructors used to prepare one of the university instructors stated, "My own understanding of the subject. I open the book and say, 'Oh!' I close the book and teach. With large lectures, I do have to prepare." He mentioned two websites in particular that are useful for preparing his classes. He stated that he does not use the material that comes with the textbook. The other university instructor stated that she has a set curriculum that she needs to cover that comes from the [institution's mathematics] department. She went on to say, "I have some other textbooks, especially for the recitation problems. Then we use the Math Lab which is the online component. Plus years of experience, just knowing where students are going to have difficulty."

Table 13:

*Resources Used to Plan Instruction*

| Instructor | Agree/Disagree with Quotation | Specific Resources Used to Plan Instruction |
|---|---|---|
| Sec #1 | Disagree | AP syllabus, AP former questions, Barron's AP Prep Guide, the AP website, older resources from former AP workshops, the College Board |
| Sec #2 | Disagree | the AP website, AP workshops, AP wiki, other AP Calculus teachers to include local teachers, and teachers that are active on the AP listserv |
| | | |
| State #1 | Agree | The internet, online applets, the "fancy bells and whistles that come with the book, and 24 years of experience, collaboration with colleagues and her husband whom also taught mathematics at the same institution for many years. |
| State #2 | Agree | 25 years of experience, resources provided by the textbook company, older calculus books, instructor-created worksheets with activities and practice problems, resources "borrowed" from her colleagues and from professional conferences. |
| | | |
| Univ #1 | Agree | set curriculum from the [institution's mathematics] department, other calculus textbooks, the online component of the course, years of experience |
| Univ #2 | Agree | Own understanding of the subject, two mathematics two websites |

**Quotation 2: Balance between conceptual understanding and computational fluency.**

The second quotation was selected in order for the instructors' written responses on the survey in Component 1, to be compared to the same instructor's oral responses to a very similar question in Component 3, after the instructors were able to analyze actual student work in Component 2. The quotation and the follow-up questions are below:

*"Developing fluency requires a balance and connection between conceptual understanding and computational fluency. On the one hand computational methods that are over-practiced and are often forgotten or remembered incorrectly…On the other hand, understanding without fluency can inhibit the problem solving process" (Principles and Standards, 2000, p. 35).*

1. What do you interpret the author of this quotation to mean by the terms "conceptual understanding" and "computational fluency"?

2. Do you agree or disagree with this statement? Why or why not?

3. If you had to choose between computational fluency and conceptual understanding for your incoming calculus students, which would you choose for the following pre-calculus topics:

State either "computational fluency" or "conceptual understanding".

a. Identifying the domain and range of functions

b. Graphing and evaluating piecewise functions

c. Graphing and evaluating discontinuous functions such as step functions or rational functions

d. Identifying points of discontinuity in rational functions

e. Graphing and identifying functions that have been translated

f. Problem solving using function models

The table below shows the instructors' written response in Component 1 and their oral response in Component 3.

Table 14:

*Written (Component 1) versus Oral (Component 3) Definitions*

| **Instructor** | **Type of Understanding** | **Instructor's Response** | |
|---|---|---|---|
| | | **Written in Component 1** | **Oral in Component 3** |
| Sec #1 | Conceptual | Students understand the definition of a function, why it is a function. | They can tell you, "When we do ___ we get ___" they can explain the process and why we are doing it |
| | Procedural | Students can identify domain, range, evaluate, find inverses and graph three functions. | Doing the problem. Step-by-step from beginning to end. They can work through a problem by steps. |
| | | | |
| Sec #2 | Conceptual | Students should have a visual picture of the function in their minds and know the properties (asymptotes, zeros, etc.) | Understanding whatever concepts need to be done. |
| | Procedural | Knowing algorithm | The working out of the problem |

| Instructor | Type of Understanding | Instructor's Response | |
|---|---|---|---|
| | | **Written in Component 1** | **Oral in Component 3** |
| State #1 | Conceptual | Students understand functions, function notation (graphically and symbolically) with ease – moving between representations with ease. | So important, but so under-valued. Students understand why things are happening and not how. The big picture in relation to the graph. |
| | Procedural | "Crunching" of functions – how to use them symbolically | Procedural fluency is doing it. Put -1 in the equation, number crunching, there's a rule. Details. |
| | | | |
| State #2 | Conceptual | Basic understanding | You actually understand what you are trying to do and its connection to any other topic. |
| | Procedural | Not familiar [with term] | You can actually do the analytic work to get the answer. One is you understand how to it and the other is you can actually do it. |

| Instructor | Type of Understanding | Instructor's Response | |
|---|---|---|---|
| | | **Written in Component 1** | **Oral in Component 3** |
| Univ #1 | Conceptual | Students understand what they are doing and why. | It is difficult to memorize without understanding. In calculus you can't survive without understanding it. |
| | Procedural | I am not sure [the meaning of this term]. | Things are remembered incorrectly. Students lack computational fluency. |
| | | | |
| Univ #2 | Conceptual | Students understand what a function is as well as related properties. | Understanding what you are doing and why you are doing it |
| | Procedural | (Instructor left question blank.) | Make computations and calculate whatever it is that you are calculating. |

When asked if the instructors agreed or disagreed with the author of this quotation, one secondary instructor replied, “I don’t think I agree with ‘overpracticed’ but forgotten or remembered incorrectly, yes!” The other secondary instructor replied, “I do agree because I tell them [calculus students], ‘Don’t use your calculator for homework!’ They use it and they find it hard to take a test without a calculator. It bogs them down.” The state college instructor said, “They see math as algorithms that need to be memorized.” The other state college instructor said, “I agree.” One of the university instructors replied, “Yes, yes, yes. Absolutely!” and the other said, “I agree with it. You need a balance.”

For the last question pertaining to this quotation I asked the instructors if they would prefer that their incoming students to have a conceptual understanding or computational fluency with the specific pre-calculus topics. Table 15: *Instructor Preference for Conceptual Understanding or Computational Fluency* was color-coded in order to visually display the instructors’ preferences. The blue box was used if an instructor stated that they prefer their incoming students had a conceptual understanding of a specific pre-calculus topic. A red box was used if the instructor stated that they prefer their students to have computational fluency of a specific pre-calculus topic and a purple box was used if the instructor preferred their students to have both conceptual understanding and computational fluency for a specific pre-calculus topic.

Table 15:

*Instructor Preference for Conceptual Understanding or Computational Fluency*

| Pre-calculus Concept | Sec #1 | Sec #2 | State #1 | State #2 | Univ #1 | Univ #2 |
|---|---|---|---|---|---|---|
| a. Identify domain and range of functions | | | | | | |
| b. Graph and evaluate piecewise functions | | | | | | |
| c. Graph and evaluate discontinuous functions | | | | | | |
| d. Identify points of discontinuity rational functions | | | | | | |
| e. Graph and identify translated functions | | | | | | |
| f. Problem solve using function models | | | | | | |

**Blue box = conceptual understanding**

**Red box = computational fluency**

**Purple box = both conceptual understanding and computational fluency**

**Member Check**

As stated in Chapter 3, member checking is considered by some to be the most critical technique for establishing credibility in a qualitative study (Lincoln & Guba, 1985). In this study, all six instructors were invited to attend a focus group in which a presentation of the study was given, and the instructors were asked to reflect on the accuracy of the account as suggested by Stake (1995). Four of the six instructors, were able to attend. The member check was video recorded by another doctoral student familiar with member checks. After the member check, the video recording was replayed and analyzed separately by myself and this outside reviewer. We met to discuss our analysis of the video recording to insure accuracy and validity.

During the member check, the issue of students successfully transitioning from high school to college was of immediate concern to all instructors. One university instructor stated that his understanding was that secondary instructors were required to focus on standardized state tests and therefore do not have time to focus on conceptual understanding of pre-calculus concepts. A state instructor stated that her understanding was that at the secondary level most learning happens in the classroom while at the postsecondary level, most learning happened outside of the classroom. She hypothesized that this could be an explanation for the difficulty students have in transitioning from high school to college. The other university instructor mentioned that she belonged to a local organization that meets once per semester with university and local state college instructors to discuss transition issues. She stated that K-12 instructors have recently been invited to these biannual meetings and have been in attendance.

When reviewing the Results chapter of this study, the state college instructors made a clarification of the name of the textbook that they were currently using (see Table 5: *Textbook*

*and Course Title*). During this clarification, the university instructors stated that they chose their textbook based on the online resources available for that particular textbook. The university instructors stated that they used the book from the series currently used by the secondary and state instructors until two years ago and decided to change textbooks because of their need for additional online resources. Although the textbook they are currently using provides the necessary online resources, the two university professors agreed that they prefer the actual textbook being used by the other instructors. The postsecondary instructors agreed with the codes, themes and conclusions of this study.

As the participants were reviewing Table 6: *Written Expectations in Instructor Syllabi,* comments were made by the instructors about successful student behaviors in the calculus classroom. All of the participants agreed that it was necessary to state expectations of student behavior in the syllabus at the postsecondary level. It was mentioned that many of these behaviors could be addressed by school administrators at the secondary level in the form of student handbooks. It was mentioned that the presence of such administrators as the Dean of Students at the secondary schools could help to explain the shorter syllabi of the secondary instructors.

Toward the end of the member check, the issue of preparing students for the transition from secondary to postsecondary resurfaced for a second time as a major concern to all participants. The comment was made that preparing students for postsecondary mathematics courses involved more than academics, but also instilling the need for students to take responsibility for their own education. The research of Conley (2005) and the "Habits of Mind" mentioned in his work were also discussed at this time.

In Chapter 4 I presented the results of the six interviews that were conducted with calculus instructors with 20 or more years of teaching experience at three types of educational institutions. The structure of Chapter 4 followed the structure of the interview which was divided into three components. The intent of each component was to answer the research questions for the study by means of triangulation as described in Chapter 3. When specific pre-calculus topics were addressed, the topics were chosen based upon a thorough review of the literature on student misconceptions of functions as described in Chapter 2. I constructed tables when I felt the visual display of the data would assist in the comparison and contrast of the various instructors. I used direct quotations from the interviews as often as possible. Since the study focused on veteran teachers, I felt it was important to provide a platform for the instructors to address their individual concerns with mathematics education, in general. In the final chapter of this study there will be a discussion of the interviews, with comparison and contrast among and between educational institutions being the focus of the discussion.

# CHAPTER 5:
# DISCUSSIONS AND RECOMMENDATIONS

During the beginning stages of this study the Mathematical Association of America (MAA) and the National Council of Teachers of Mathematics (NCTM), two of the largest professional organizations of mathematicians and mathematics educators, jointly released a position statement on the teaching of calculus. Within that statement, it says the ultimate goal of K-12 mathematics curriculum is to establish the mathematical foundation necessary for students to pursue further study (NCTM, 2012). With that in mind, the aim of this research was to pool the expertise of instructors from three types of educational institutions in order to compare and contrast the similarities and differences between and among the institutions that are currently teaching calculus in order to discuss how to best teach calculus in order to prepare our students for future STEM careers. As presented in the Rationale section of the Introduction chapter, many undergraduates that received an A in their calculus course in high school still possess a weak understanding of function (Breidenback, Dubinsky, Hawks, & Nichols, 1992; Carlson, 1998; Thompson, 1994) thus leaving a gap in the transition from secondary to postsecondary mathematics. In an attempt to describe and eventually bridge that gap, I used triangulation methods as described by Guion (2012) in both designing and conducting the study in order to address the research questions posed at the beginning of the study.

The three components used to structure this study are described in detail in the Methodology chapter. The ultimate goal of the study was to answer the research questions, but due to the nature of a qualitative study, the questions were not addressed linearly throughout the data collection process as may be the case in a quantitative study. Each of the questions were

addressed and readdressed throughout three components of the interview to reveal implicit as well as explicit answers to the questions (see Table 1 *Studies on Misconceptions of Functions).* Although, the last two chapters are not organized according to the research questions, I did include a section at the end of Chapter 5, which does answer each research question, in order. In Chapter 4, I immersed myself in the details of the written components and transcripts of the oral components in order to identify themes which were compared and contrasted between and among veteran Calculus instructors currently teaching at the three types of educational institutions. While reviewing the data for this study, I noticed a stark contrast in the instructors' syllabi and list of resources. Although interesting, since these findings did not directly address the research questions of this particular study, I included these findings in the last section of this chapter as suggestions for future research.

**Component 1: Written Survey**

In order to explicate both the instructors' implicit and explicit expectations of their incoming students, the research questions of this study were not asked and answered in numeric order. Each question was asked in a variety of ways in three components of the interview in order to reveal any possible implicit expectations. Table 4: *Triangulation of Research Methods by Research Question and Component* states each research question and where within the interview that particular question was addressed. It also states the instrument or data recording procedure used for that particular component of the interview. The first four questions on the written survey were intended to gather information from the instructors in order to address the second research question: "How do secondary and postsecondary instructor assumptions of their students' prior knowledge of functions impact instructional decision-making for their calculus

course?" Although information was drawn in order to address this question, while analyzing the data, an additional question for another study arose from the data. Since this data did not directly answer the research questions of this study, I have reserved the discussion of the results from these first four questions of Component 1 for the Suggestions for Future Research section of this chapter. There will be additional questions later in the interview that will further attempt to answer this same research question.

**Use of a diagnostic assessment.**

The next three questions on the written survey were used to gather information from the instructors in order to address the research question: How are secondary calculus instructors' assumptions of their students' prior knowledge of the function concept different from that of the postsecondary calculus instructors' assumptions? They were asked if and how they used a diagnostic tool in their calculus classes to assess their incoming students, and their expectations of their students' prior knowledge of functions. Again, this one component will not answer this research questions completely. This same research question will be readdressed in later components.

As stated in the literature review, not all calculus instructors use a diagnostic assessment like the one in this study. Only one instructor, a state college instructor, stated that they used a function diagnostic instrument similar to the one used in this study in her calculus course. One secondary instructor stated that she did not use a diagnostic instrument because she taught the pre-calculus course prior to AP Calculus and was very aware of her individual student's strengths and weaknesses. The other secondary instructor stated that she spends the first week reviewing pre-calculus concepts to include the concepts covered by the diagnostic instrument in

this study. She then administers a test on all pre-calculus topics in order to identify any misconceptions. The fact the secondary instructors have a year-long course with more contact hours per week with the students than the postsecondary instructors would have to be a consideration in the administration of a diagnostic assessment.

The other state and the two university instructors gave one or both of the following reasons for not using a diagnostic assessment instrument: (1) There was not enough time built into the course to administer and review/clarify misconceptions identified by analyzing the results of a diagnostic assessment and/or (2) The results would not have a significant impact on how the course was taught; therefore, there was not a need to assess the students' prior knowledge of functions prior to direct instruction.

**Awareness of pre-calculus instruction.**

The intent of Question 6 was to see how "aware" the calculus instructors were of the teaching practices of the instructors that were responsible for their students' pre-calculus concepts, prior to their course. The responses varied greatly among the three types of educational institutions. The secondary instructor that stated that she teaches the pre-calculus course at her high school stated that she sometimes even teaches the Honors Algebra 2 course which precedes the pre-calculus course. She also stated that it is not unusual for the same instructor to have taught the same student for three years in a row. Since she taught the courses, she was very aware of her own teaching practices. The other secondary instructor stated that the students learn the pre-calculus concept of function in the pre-calculus class at the same school. She stated the pre-calculus course is taught by a different instructor but they use the same series of textbook and she was very familiar with the instructor and how the pre-calculus concepts were

presented to her students. For the two secondary institutions, the instructors were very aware of how the pre-calculus concepts were presented to her current calculus students.

The secondary instructors' responses to this question were in contrast to the state college instructors' responses. The state college instructors did not know which of the instructors taught their students in prior mathematics courses, but they both stated that their students learned the pre-calculus concept of function at the same state college in which they taught. They seemed to have an awareness as to the content of the pre-calculus courses, but not specific details about which pre-calculus concepts were emphasized, or if the instructors emphasized computational fluency over conceptual understanding, or vice versa.

The two university instructors' responses to this question were quite different from each other. One university instructor stated that he did not know the answer to the question and directed me to another faculty member who could look the information up on the university data base. He also stated that there were too many students in the calculus course for him to have a sense of where the majority of the students learned the pre-calculus concept of function. He seemed to have little to no awareness of where or how his students were instructed in the pre-calculus concepts prior to his course.

This is in contrast to the other university instructor's response to the same question. Like the first secondary instructor, this university instructor stated that she was part of a special program at her university in which she taught her students pre-calculus the semester before teaching them calculus. She stated that her students enrolled in pre-calculus with her as the instructor in the fall semester and then took Calculus I with her in the spring semester. She mentioned that the middle half (second and third quartiles) of students that take the college placement test at the university are placed into this program. One of the goals of this program is

to have the same instructor and use the same series of textbook for both the Pre-calculus and Calculus courses. It is the program director's belief that this continuity will improve the success rate for this group of students.

**Preference for conceptual understanding or procedural fluency.**

For Question 7, the instructors completed a chart on their preference for conceptual understanding and computational fluency for their incoming students (see Table 15: *Instructor Preference for Conceptual Understanding or Computational Fluency*). All but one instructor felt that their students were coming into their calculus course not needing either a review or clarification of how to compute the value of a function at a single numerical value. This would suggest that all but one of the instructors expected their incoming students to enter Calculus with computational fluency. In contrast, the one university instructor that marked that his incoming students needed both a review and clarification stated his expectation was that his incoming students did not have enough practice of "algebra" prior to taking his course and were not computationally fluent.

For the pre-calculus topics domain, range, correspondence; definition of function; graphing/evaluation piecewise functions; identifying points of discontinuity in rational functions and graphing/identifying functions that have been translated all but one of the instructors interviewed agreed that their incoming students needed a review of these topics. There was very little agreement between or among instructors as to the subtopic of graphing/evaluating discontinuous functions such as step functions or rational functions. As for the last subtopic of problem solving using function models, all instructors agreed that their incoming students needed "something". Two felt as if they just needed a review, one felt that they just needed

clarification and three instructors felt that they needed both. Since word problems require conceptual understanding, the instructors' unanimous responses to this question suggest that all of the instructors expect incoming students to enter calculus without a conceptual understanding of pre-calculus topics.

**Review/clarification of pre-calculus topics.**

Questions 8 and 9 were follow-up questions to Question 7. If the instructors expected their incoming students to need review and/or clarification of a specific pre-calculus topic, where was that additional instruction to come from? All but one of the instructors stated that they would provide the review and/or clarification themselves during the course of the Calculus class. The means of providing that instruction varied from the use of worksheets, extra assignments, and online resources developed by the instructor. Only one of the instructors stated that he, personally, would not provide the additional review and/or clarification. He stated that his teaching assistants would provide any necessary review and/or clarification of pre-calculus topics. The disconnection between the presentation of the calculus material and the review and/or clarification of common student misconceptions of pre-calculus topics by this one instructor stood out as a stark contrast to the other instructors interviewed.

Question 9 asked the instructors how much time in their calculus class was spent on reviewing pre-calculus topics. All but one of the instructors stated they spent either one to two class days at the beginning of the course, or 10 minutes per class intermittently throughout the course. For most instructors interviewed, there is a fairly small percentage of the calculus class time spent on pre-calculus topics. The exception was with one of the secondary instructors. She stated that she spent the first week of class reviewing pre-calculus concepts. Her students are

given a packet of pre-calculus problems to be worked independently over the summer and for the first week of Calculus; the instructor answers questions and clarifies any misconceptions. At the end of the first week, the instructor gives the students a test over pre-calculus topics prior to starting with the calculus material.

**Conceptual understanding versus procedural fluency of functions**

Question 10 was the first attempt to answer the third research question, "How do secondary and postsecondary instructors differ on their views toward students' conceptual understanding of functions and procedural fluency of functions?"

I asked the instructors to write down their understanding of the terms "procedural fluency" and "conceptual understanding." I wanted to compare their understanding of these terms before they analyzed student work in Component 2 to their understanding after they analyzed student work. A direct comparison to each instructor's two responses can be found on Table 14: *Written (Component 1) versus Oral (Component 3) Definitions* and a discussion of the comparison between their written response in Component 1 and their oral response can be found in Component 3 of this chapter. When comparing just the instructors' written responses in Component 1 to each other, half of the instructors were not sure about the meaning of the term "procedural fluency." Two instructors actually stated that they were unfamiliar with the term and one instructor chose to leave that question blank. Two of the three that attempted to explain their understanding of the term, gave short answers and mentioned "algorithm", or "crunching numbers." One of the secondary instructors mentioned the pre-calculus concepts of domain, range, inverse functions, and graphing of various functions.

For the phrase "conceptual understanding" all of the instructors gave a more lengthy answer. Although the instructors seemed more familiar with this term their responses varied as to what the students should understand in order to attain "conceptual understanding." One secondary instructor stated that students should know the definition of a function; the other stated that the students should have a visual picture of the function and know the properties. For the state college instructors, one stated that the students should have a "basic understanding" while the other emphasized the ability to move between the graphic and symbolic representations of a function. One university instructor stated that the students needed to understand what they are doing and why, while the other stated that in addition to knowing what a function is the students needed to know the related properties.

**Adequacy of diagnostic instrument used in study.**

The final question on the written survey was to have the instructors rate the adequacy of the functions diagnostic instrument that was going to be used in component two of the interview. This question was important because if the instructors felt that the instrument was not adequate, the results of the student work would not be meaningful to the instructor. On the other hand, if the instructors believed the instrument was adequate, they would be more likely to value the results of the assessment. Although only one instructor considered the instrument *completely* adequate by giving it a score of 10 out of 10, all of the scores seemed to reflect that the instrument was adequate for assessing student knowledge of functions with a mean score of $8.\overline{6}$. This high score help to give credibility to the instrument used in the study.

**Component 2: Student Answers**

A major difference between Component 1 and Component 2 was that instructors were asked to write their responses to the questions in Component 1 and they were asked to state their responses in Component 2. Some of the same questions were asked in both Component 1 and Component 2 so that I could compare the instructors' written response to their oral response in order to uncover implicit as well as explicit expectations. After having the instructors give an overall rating of the functions diagnostic instrument in writing in Component 1, I asked the instructors to look at each question on the diagnostic individually and comment on their expectation of their incoming students' ability to provide an adequate answer to the question in order to readdress the first research question: "How are secondary calculus instructors' assumptions of their students' prior knowledge of the function concept different from that of the postsecondary calculus instructors' assumptions?:

Although the instructors commented on each question, I focused on the questions that were common misconceptions of functions based upon the literature review (see Table 1: *Studies on Misconceptions of Functions*). The questions that related directly to these misconceptions were #1e, 3a, 4c, and 6. Based upon the instructors' responses, I initially created a hand-written grid, and then subsequently converted that to a table to visualize the instructor's expectations of these common misconceptions (see Table 8*: Instructor Expectations of Prior Knowledge of Functions*).

Both secondary instructors anticipated their incoming students doing well on the functions diagnostic instrument, with the only exception being the graphing of a piecewise function. One instructor stated that her incoming students had trouble with graphing piecewise

functions and that she has to spend time in calculus going over that concept. One state college instructor was in agreement with the secondary instructors and the other stated she anticipated her students would also have trouble with stating the domain of rational functions. The biggest difference between calculus instructors at the same educational institution was with the university instructors. One instructor was in agreement with the secondary and state college instructors that her students would do well on the instrument, with the only exception being that she anticipated her students having trouble with both evaluating and graphing piecewise functions. She stated, "I think they would have trouble with both 6a and b. I think they have trouble with piecewise functions … although we go over it; there is a gap in time from when it is introduced in pre-calculus and when it occurs again in calculus." This is in contrast to the other university instructor that stated that the only question that his students would not have difficulty with was 6a. He stated that although his hope was that most of his students would be able to answer these questions correctly, he expected at least half to not be able to answer the questions correctly. He repeated the phrase, "It is important, but I am not too optimistic."

After the instructors familiarized themselves with the functions diagnostic instrument and stated their expectation of the incoming students' prior knowledge for each question, I then showed the instructors five different students' answers to the questions that related directly to the common student misconceptions of functions which were Questions 1e, 3a, 4c, and 6 (see Table 2: *Four Questions and Student Answers Selected for Instructor Interviews*) The second question in this component addressed the third research question, "How do secondary and postsecondary instructors differ on their views toward students' conceptual understanding of functions and procedural fluency of functions?"

**Domain and range themes.**

For both diagnostic questions 1e and 3a the students had to state the domain of a function. In question 1e they also stated the range of the function that was given in graphical form. In question 3a, the function was given in algebraic form. Tables 9 and 10 in Chapter 4 show the question, student answer and the instructor's responses. The first theme that emerged from the instructors' responses pertaining to these two questions was:

*Domain/Range Theme 1*: Students may have the idea of domain and range, but they don't know how to communicate what they know.

Calculus is considered the first postsecondary course in a string of mandatory mathematics courses for students wishing to pursue careers in Science, Technology, Engineering and Mathematics (STEM). Organizations such as STEM Talent Expansion Program (STEP), funded by the National Science Foundation (NSF), were founded to improve recruitment and retention of STEM students based on "best practices" of prerequisite courses with typically high failure rates such as pre-calculus and calculus (Cheatham et al., 2012). According to the Common Core State Standards, one of the eight mathematical practices necessary to attain mathematical proficiency is for students to be able to construct viable arguments and critique the reasoning of others (Standards for mathematical practice, n.d.). If our students are not able to communicate what they know from pre-calculus to calculus, it will be difficult for them to construct these arguments and critique the reasoning of others. Communication pertaining to the domain and range of functions needs to be considered when evaluating a students' understanding of pre-calculus topics.

The second theme was:

*Domain/Range Theme 2*: Sometime the questions we ask, as teachers, are ambiguous to the students.

This theme is addressing the ability of the teacher to ask a good question. A good question could also be ambiguous, if that is the intent. A question may be left intentionally ambiguous in order for the student to practice constructing viable arguments for multiple answers to the same question. But if this is the case, short answers and phrases such as those given by the students on this functions diagnostic instrument would not be acceptable. The teaching of mathematics must move away from checking to ensure student answers match the answers in the back of the book and move toward instructors being open to not only showing one algorithm for the solution to a specific type of mathematical problem, but explaining the concepts behind the algorithm and not only holding students accountable for getting the right answer, but knowing why he or she got that answer.

And the third theme was:

*Domain/Range Theme 3:* Students put "all real numbers" when they don't know the answer to a domain/range question.

At times students learn lessons that we as teachers never intended to teach. Such as, if you don't know the domain or range of a function, write all real numbers. Ashlock (2010) states, "… we need to remember that our students are not necessarily learning what we think we are teaching; we need to keep our eyes and ears open to find out what our students are actually learning" (p. 14). It is very possible that the students that wrote these answers don't even know what real numbers are, not to mention what the words domain and range of a function mean.

Some students get through the system with learning just enough of what we do intend to teach, and just enough of what we don't intend to teach to be passed on. Short answers to complicated questions need to be looked at with a certain amount of skepticism. Does the student really know what they are saying when they say "all real numbers" or even worse just use the all real numbers symbol when stating the domain and range, or have they instead learned that if they put that answer down, they usually get it right?

For these three themes, all six of the instructors appear to be in agreement. There does not seem to be a stark contrast in responses when analyzing secondary and postsecondary instructors. They only discrepancy based upon the types of educational institutions for which the instructors are employed, appears to occur for Question 3a for Student 24. The secondary instructors believe the students should write the answer in interval form, while the postsecondary instructors at both the state college and university are comfortable that the students have the correct idea of the domain, regardless of the form in which the student chooses to use for the answer.

**Translation of functions themes.**

Question 4c on the functions diagnostic instrument pertained to the translation of functions. The following two themes emerged when analyzing the instructors' responses to the student answers.

*Translation Theme 1*: Students remember something, they just remember it wrong.

*Translation Theme 2:* Students may have the idea of translation, but they don't know how to communicate what they know.

In 1998, Carlson found that students who think about functions only in terms of procedural techniques are unable to comprehend a more general conceptual structure for modeling functions where the dependent variable changes continuously along with the continuous changes of the independent variable. That seems to be the case with these students in this study as well. When students are taught about the translation of functions only in terms of procedural techniques they do not understand how a specific change in the domain affects the entire function. Without the conceptual understanding, the translation of functions is oversimplified to the students attempting to memorize a set of rules which are often times memorized incorrectly. That appears to be the case with these students. As one instructor stated, "They remember something, they just remember it wrong."

As evidenced by this second theme, the inability of students to communicate their knowledge is not limited to domain and range. We see this again with the pre-calculus concept of translation of functions as well. As was stated earlier, when teachers accept short phrases or symbols for answers to rather complex conceptual questions, they may be instilling a sense of "false confidence" in their students' mathematical ability that Ferrini-Mundy and Gaudard (1992). If the student is able to give a simple answer to a rather complex question and this answer is satisfactory to the instructor, the student will not realize the complexity of the question being asked. The result could be a "false confidence" in thinking that his or her surface level understanding of a concept is "good enough" and not persevere in order to develop a deeper understanding of what is being presented.

**Piecewise function themes.**

For Question 6 the students were asked to evaluate and sketch the graph of a piecewise function (see Table 12: *Instructor Responses to Question 6*). This is the question that most of the instructors expected their students to have difficulty with which is in line with the research on students' misconceptions of functions (see Table 8: *Instructor Expectations of Prior Knowledge of Functions*). If we include discontinuous and split domain functions in piecewise functions, four of the six research studies in the literature review (see Table 1: *Studies on Misconceptions of Functions*) specifically stated this as an area of concern. The first of two themes that emerged from this question was:

*Piecewise Function Theme 1*: Students are better at evaluating a piecewise function than sketching a piecewise function.

This theme emerged from both the secondary and university instructors. Some of the students that evaluated the function correctly had difficulty sketching the graph of the same function. Some of the student answers supported Becker's (1991) findings that students mistakenly think that all functions must be smooth and continuous. When students have to sketch a discontinuous function, they just "connect the parts" to force the continuity of the function.

A second theme emerged from the state college instructors' responses:

*Piecewise Function Theme 2*: Students don't see the connection between the answers they wrote in part (a) and the sketch they drew in part (b).

This seems to be the most troubling theme, not only with these instructors, but by the researchers mentioned in the literature review as well. Markovits, et al. (1986) found that students were

unable to transfer from graphic to algebraic forms of functions and Doorman (2012) found that students had difficulty integrating operational and structural aspects of the function concept.

For the piecewise functions all instructors seem to be in agreement that students have more difficulty graphing piecewise functions than evaluating a piecewise function given a function in algebraic form. The major discrepancy amongst the instructors in this case was when the instructors attempted to identify the cause of the problem with piecewise functions. The secondary instructors and the university instructors did not give a reason as to why students had more difficulty evaluating piecewise functions than graphing; they simply stated that the students were wrong. In contrast, the state college instructors commented that the students were not connecting the numerical answers in part (a) to the graph in part (b). That observation led one instructor to conclude that the students in this study had a procedural understanding of piecewise functions, but not a conceptual understanding. The same instructor stated that the disconnection between the algebraic manipulation and the resultant graph of the same function is a "huge problem" for her incoming calculus students.

As stated in the literature review, teachers use diagnostic assessment instruments to look for "slips," careless miscalculations, and "bugs," persistent misconceptions that interfere with students' demonstration of their abilities. Ashlock (2010) classified computational-skill bugs into three basic categories: (1) student uses an inappropriate operation when attempting to solve a math problem; (2) student uses the correct operation but makes an error involving number facts; (3) student makes a non-number fact error in one or more steps of applying the strategy or selects an incorrect strategy. Ketterlin-Geller and Yovanoff (2009) added that errors also involve misinterpreting and misapplying the language of mathematics. The two errors revealed in this

component as areas of concern for the calculus instructors were students selecting an incorrect strategy, and misinterpreting or misapplying the language of mathematics.

**Component 3: Quotations**

The intent of the third component was to have the calculus instructors clarify their expectations of their incoming students' prior knowledge of functions by having them respond to two quotations from mathematics educators. In light of the activity the instructors did in Component 2, I chose one quotation that dealt with the use of research-based diagnostic instruments in the mathematics classroom, and the other question was selected with the intent of having the instructors restate their definitions of conceptual understanding and computational fluency when it comes to mathematics education, in general, and the teaching of calculus, specifically.

**Quotation 1: Resources to plan instruction.**

Questions 1 and 2 of this first quotation readdress the second research question, "How do secondary and postsecondary instructor assumptions of their students' prior knowledge of functions impact instructional decision-making for their calculus course?"

Quotation 1 was as follows:

*"In the absence of research-based curricular instruments, teachers tend to rely on their own opinions about what students need to learn as they plan instruction" (Carlson, Oehrtman, Engelke, 2010, p. 114).*

The quotation was followed by the following two questions:

1. Do you agree or disagree with this statement?

2. What specific resources do you use to plan instruction for your calculus class?

In response to the whether the instructors agreed or disagreed with this quotation the reactions were quite varied. The two university instructors and one state college instructor stated that they agreed with the quotation and provided little to no explanation as to their agreement. One secondary instructor stated that she disagreed with the quotation. She stated that she does not believe calculus instructors like her that are "AP trained" use their own opinion in planning instruction. She explained that she follows the guidelines of the "experts" and the recommendations set forth by the College Board in multiple AP resources.

The other state college instructor, along with one of the secondary instructors agreed with one part of the quotation, but disagreed with other parts. Interestingly, the parts that they agreed with were different parts. The secondary instructor stated that the part she disagreed with was teachers using their own opinion. Like the secondary instructor that disagreed with the quotation, the instructor stated that she relied on AP workshops and other AP calculus instructors as resources to plan instruction, not her own opinion. In contrast, the state college instructor agreed that teachers use their own opinions about what students learn. She further explained that this opinion is formed from years of experience in teaching calculus, conversations with other instructors, and workshops at professional conferences. The part of the quotation that the state college instructor disagreed with was the part about research-based curricular instruments. She stated that instructors don't use research-based curricular instruments because there is an absence of quality, well-designed instruments available. Echoing this sentiment, the state college instructor that agreed with the quotation commented, "Yeah, we don't really have any other option."

The intent of the second question for the first quotation was to compare and contrast the resources instructors at the three types of educational institutions use to plan instruction. Ernest

(1989) stated that whether an instructor chooses to use or not use curricular materials in the teaching of mathematics reflects the teacher's mental model of the learning of mathematics. Teachers typically follow into one of three patterns, (1) strict following of text or scheme; (2) modification of the textbook approach, enriched with additional problems and activities; (3) teacher or school construction of the mathematics curriculum. The responses to this question fit nicely into these three patterns. The secondary instructors recited a long list of AP resources that they use to plan instruction (see Table 13: *Resources Used to Plan Instruction*). It appeared that the secondary instructors had a common goal of preparing the students taking Calculus to make a passing score on the nationally standardized calculus test prepared by the College Board. The secondary instructors appeared to fit into the first pattern of strictly following the text or scheme provided by the national testing agency.

This is in contrast to the postsecondary instructors. Not one of the four postsecondary instructors mentioned the AP test or AP resources in response to this question. All but one of the postsecondary instructors stated they used the textbook, accompanying teacher resources, and departmental guidelines for the course as resources to plan instruction. Additional resources were mentioned by these three instructors to include conversations with colleagues, self-developed activities, additional Calculus textbooks, online math tutorials, and information obtained at professional conferences. The state college instructors and one of the two university instructors fit into the second pattern of modifying the textbook with older versions of Calculus textbooks and with additional problems and activities.

One university instructor stated that he used the textbook only for quick reference and that he did not use any teacher resources that came with the textbook. He stated that he used mainly online resources and his own understanding of Calculus to prepare instruction. He

appeared to be the only instructor that fit the third pattern in which the teacher constructed the mathematics curriculum.

**Quotation 2: Balance between conceptual understanding and computational fluency.**

Unlike the first quotation, the main reason the second quotation was selected was to compare instructor responses in Component 1 to the same instructor's responses in Component 3. Table 14: *Written (Component 1) versus Oral (Component 3) Definitions* in Chapter 4 displays the two responses by the individual instructor to facilitate the comparison. In order to further explicate the instructor's understanding of conceptual understanding and computational fluency, additional questions were asked in reference to this second quotation. Similarly to the first quotation, the instructors were asked if they agree or disagree with the quotation, and they were asked to state whether they prefer their incoming students to have conceptual understanding or computational fluency in the pre-calculus topics identified as common misconceptions. A color chart was used to display the results of that question (see Table 15: *Instructor Preference for Conceptual Understanding or Computational Fluency*). The intent of this question was to help answer the first and third research questions. The first research question pertains to the teacher's expectations of the incoming students' prior knowledge of functions and the third question pertains to the students' conceptual and procedural fluency of functions.

By examining Table 14*: Written (Component 1) versus Oral (Component 3) Definitions* and comparing the written responses of the instructors' understanding to the terms "conceptual understanding" and "computational fluency" it becomes evident that after the instructors analyzed student work they provided more detailed responses to their understanding of these terms. Component 2 appears to have helped the instructors solidify their thoughts on the

meaning of these terms. Even the instructors that initially stated that they were not familiar with the term "procedural fluency" before the activity were able to give a definition of the term after Component 2. When comparing responses among the types of educational institutions, there is very little difference in the instructors' understanding of the terms. All instructors, regardless of the type of institution, stated their understanding of the terms conceptual understanding and procedural fluency were in alignment with the definition of the terms found in the review of literature as presented in Chapter 2.

When asked if the instructors agreed or disagreed with this quotation, there was unanimous agreement with the part of the quotation concerning "a balance and connection between conceptual understanding and computational fluency." This agreement supports the conceptual framework of this study which supports an iterative perspective for the development of knowledge of concepts and procedures. The only disagreements instructors had with this quotation was the use of the phrase "…computational methods that are over-practiced…" Two instructors (one secondary and one university) pointed out that they believe their incoming students have not practiced computational methods enough prior to their course, and do not see "over-practiced" methods as a problem.

The third question for this quotation asked the instructors to state if they would prefer their incoming students to have conceptual understanding or computational fluency with each of the pre-calculus topics identified as common misconceptions of functions (see Table 15: *Instructor Preference for Conceptual Understanding or Computational Fluency*). Although the majority of instructors prefer conceptual understanding to computational fluency on most of the topics, there were some exceptions between and among instructors from the three types of

educational institutions. For the secondary instructors, the preference for computational fluency was with discontinuous functions. One of the state college instructors was in agreement about her preference for computational fluency for identifying points of discontinuity but added a preference for computational fluency for identifying the domain and range of functions. This is in contrast to the other state college instructor that did not prefer computational fluency over conceptual understanding for any of the pre-calculus topics. Both state college instructors agreed that for graphing and identifying translated functions they preferred that their incoming students had both conceptual understanding and computational fluency.

One of the university instructors preferred that her incoming students had both conceptual understanding and computational fluency for all the pre-calculus topics with the exception of graphing and evaluating piecewise functions. For this topic, the instructor mentioned a preference for conceptual understanding over computational fluency. The other university instructor preferred his incoming students have a conceptual understanding of the topics except for graphing and evaluating piecewise functions for which he preferred both. Like the state college instructor, this university instructor preferred that his students had computational fluency for identifying the domain and range of functions.

**Answering the Research Questions**

The structure of Chapters 4 and 5 followed the structure of the study which was organized by components. For the next section, I reorganized the discussion by research question. The same information is being presented but in a manner that may be preferred by some readers. While reading the next section, the reader may find it helpful to refer to Table 4:

*Triangulation of Research Methods by Research Question and Component*. This table displays each research question and where, within the structure of the study, each question is answered.

**Research question 1.**

The first research question for this study was:

1. How are secondary calculus instructors' assumptions of their students' prior knowledge of the function concept different from that of the postsecondary calculus instructors' assumptions?

When examining the instructor's responses to the Written Survey in Component 1 to their "think-aloud" activity in Component 2 and finally their responses to the Quotations in Component 3 we can see how secondary calculus instructors' assumptions of their students' prior knowledge of the function concept differ from that of the postsecondary calculus instructors. The most striking contrast between secondary and postsecondary instructor expectations of their student prior knowledge of functions was displayed in Table 8*: Instructor Expectations of Prior Knowledge of Functions*. Prior to analyzing actual student answers to specific questions, the secondary instructors were confident that their incoming students would do well on most aspects of the functions diagnostic instrument. The only question in which the secondary instructors were either unsure or did not expect the students to answer correctly pertained to the graphing of piecewise functions. Similarly the state college instructors and one of the university instructors were fairly confident in their incoming students' ability with the additional concern for students being able to find the domain of a rational function and evaluating a piecewise function.

This is in contrast to one of the university instructors' assumptions that half or fewer of his incoming students would be able to answer the questions correctly. The only question that he expected his students to be able to answer correctly was the one pertaining to evaluating piecewise functions. In addition, in Question 7 on the Written Survey when the instructors were asked if many of their incoming students needed review and/or clarification on these same pre-calculus topics, this is the only instructor that replied that his students need review, clarification or both on all of the topics, but when asked in Question 8 where the students would receive this necessary review and/or clarification he was the only instructor to state that he did not provide this review/clarification, personally. He stated that it was provided by his teaching assistants (see Table 7: *Review/Clarification of Pre-calculus Topics*).

In Component 2, during the "think-aloud" of student answers to the functions diagnostic instrument, there was a difference in themes by educational institutions but surprisingly the contrast was not with the secondary and university instructors, but rather with the state college instructors. When comparing the instructor responses to Question 6 of the diagnostic instrument (see Table 12: *Instructor Responses to Question 6*) both the secondary and university instructors mentioned that students are better at evaluating a piecewise function than sketching the same function, but the two state college instructors went further to explain this observation. They agreed that students don't see the connection between graphical representation of the function in part (b) and the algebraic representation of the same function in part (a). In Component 3, there did not appear to be stark contrasts, neither between nor among instructors at the three types of educational institutions for the first research question.

**Research question 2.**

2. How do secondary and postsecondary instructor assumptions of their students' prior knowledge of functions impact instructional decision-making for their calculus course?

This question was addressed specifically in the first and third components of the interview process. Within the four questions of the Written Survey it is revealed that the course objectives of the secondary instructors teaching AP Calculus to high school students are quite different from the course objectives of the postsecondary instructors. The secondary instructors are specifically "teaching to the test." Their objective is to prepare their students to take and pass a standardized test in order to receive college credit in high school. There was no mention of a standardized end-of-course exam by the postsecondary instructors. The other interesting difference in instructional decision-making that was revealed in this study is that secondary instructors rely mainly on materials generated by the testing company for their instructional decision-making. None of the postsecondary instructors mentioned using resources from this company.

Another interesting contrast between secondary and postsecondary instructors was that all of the postsecondary instructors mentioned collaborating with colleagues for the preparation of their course, while the secondary instructors did not seem to have colleagues in their institution with which to collaborate. In the secondary educational institutions, AP Calculus is often the highest level mathematics course offered at the school and there is usually only one instructor at the school that teaches the course. In contrast, at the postsecondary institutions, there are other higher level mathematics courses offered and there are often many other instructors at the same

institution teaching calculus with which the instructors can collaborate for instructional decision-making.

In Component 3, for the first Quotation, the instructors' responses to the first and second questions again show the contrast between the secondary and postsecondary instructors' instructional decision-making. The secondary instructors disagree with the part of the quotation that stated "teachers tend to rely on their own opinions about what students need to learn…" The secondary instructors emphasized the point that they rely on the materials provided by the testing company, not their own opinions, while the postsecondary instructors unanimously agreed with this part of the quotation. This difference was again obvious when examining the responses to the second question to this quotation which asks the instructors to list the resources they use to plan instruction. The secondary instructor's list consisted mainly of resources provided by the testing company, while all but one of the postsecondary instructors listed the required textbook and teacher's resources provided by the textbook company.

**Research question 3.**

3. How do secondary and postsecondary instructors differ on their views toward students' conceptual understanding of functions and procedural fluency of functions?

The data in this study were triangulated from the three components of the interview as suggested by Guion (2012). When examining the instructors' responses to the Written Survey in Component 1 to their "think-aloud" activity in Component 2 and finally their responses to the quotations in Component 3, it is not clear as to how secondary and postsecondary calculus instructors differ on their views toward students' conceptual understanding of functions and

procedural fluency of functions. It appears that the instructors of the three types of educational institutions are in agreement as to the views. There are some slight differences in response to Question 10 in the first component in that the postsecondary instructors seemed less comfortable providing a definition for the term procedural fluency, initially, but when this question was readdressed in the third component, all of the instructors' definitions for this term were within the definitions discussed in the literature review.

In Component 2, themes were able to be identified for the questions pertaining to student misconceptions because similar statements were found by all instructors, regardless of educational institution. The only exception to this was Question 6 and even then, the secondary and postsecondary (university) instructors' responses were similar. The difference in themes was not detected based upon the differences between secondary and postsecondary institutions.

**Naturalistic Generalizations/Implications**

When describing case study analysis and representation, Creswell (2007) states that as a final step, "the researcher develops *naturalistic generalizations* from analyzing the data, generalizations that people can learn from the case either for themselves or to apply to a population of cases" (p. 163). This section is intended to take the data from Chapter 4 and develop these naturalistic generalizations that would be of benefit to teachers of calculus, pre-calculus, and mathematics educators of secondary mathematics. The generalizations from this study include communication about mathematics and the disconnection between the algebraic and graphic representations of functions.

### Communication about mathematics

When the calculus instructors in this study were asked to analyze student work, a theme of communication about mathematics emerged in both of the pre-calculus subconcepts of domain and range, and translation of functions. Wood (1998) describes communication in the mathematics classroom as either having a *univocal* function, one voice transmitting information to students, or a *dialogic* function, interactive conversation intended to generate new meanings for the receiver. When one instructor stated, "I think it is our fault as teachers by not making it clear" she is referring to the univocal function of the teacher and the need to be clear in how and what we say during this type of communication so as not to create confusion when explaining a new mathematical concept or procedure. Even if the mathematics instructor is the most eloquent of speakers, the need for a more dialogic view of communication is advocated by the NCTM (1989, 2000) and more recently, the Common Core State Standards (2012).

It is through this dialog that the instructor can informally assess the students' comprehension of the material being transmitted and make sense of the information for them. As teachers, we need to convey to convey to our students the importance of communication and the language of mathematics. It is through this mathematical language that students express their mathematical understanding. Instructors need to know and be able to communicate clearly, and expect students to communicate clearly, so the transmission and reception of the mathematics is without ambiguity. Pirie (1998) states, "All we can ever work from when trying to access the understanding being constructed by pupils is their language – of whatever form, verbal or symbolic" (p. 28).

As the emphasis on test-driven data to determine a teacher's effectiveness in a classroom increases, the idea of language and communication in the mathematics classroom cannot get "lost in the shuffle." Communication and the use of proper mathematical language need to be emphasized, not overlooked, when assessment instruments are being developed. Multiple-choice tests with numeric-only answers will hopefully be a thing of the past and not acceptable as a means of assessing a student's mathematical understanding in the future. Assessments should include written problems that require students to read and understand a mathematical problem, devise a plan to solve the problem, carryout the plan, and then examine the solution obtained to either justify or refute the solution. This suggestion is not new, in fact the famous mathematician, George Polya, recommended this four-step approach to mathematics education almost 70 years ago. Polya (1945) also stated, "One of the most important tasks of the teacher is to help his students. This task is not quite easy; it demands time, practice, devotion, and sound principles" (p.1).

**Operational aspects precede structural aspects of functions**

I started this study with the hypothesis that the gap between students desiring to pursue a STEM career and students actually attaining a STEM degree was related to teacher expectations of student prior knowledge, and the communication of these expectations among the three institutions teaching the same course. Throughout the study I found that both the secondary and postsecondary instructors have similar expectations of their students' conceptual and procedural understanding of function with slightly more pessimistic expectations from secondary to state college to university. Interestingly, all six instructors interviewed for this study agreed that students' understanding of function requires a balance of both types of knowledge. All of the

instructors also agreed that different representations of functions (algebraic, numeric, and graphic) offer a different view of the function and students need to see the connection between these different representations.

Rather than procedural fluency and conceptual understanding, Sfard (1991) distinguishes between *operational concepts* concerning mathematical processes and *structural concepts* concerning mathematical objects. He states that a function may *operationally* be seen as a computational process, and *structurally* thought of a set of ordered pairs. "Sfard's theory on this dual nature of mathematical conceptions plays a central role and is exemplified by the transition from a calculation view to an object view on function" (Doorman et al., p1245). This dual nature of the piecewise function appears difficult for students to comprehend and may help to explain the disconnect, described by the state college instructors, between the two parts of question 6 in this study.

The students seemed to be fine with the operational concept of this particular function, thus being able to correctly answer part a, but were less comfortable with the structural concept when asked to sketch a graph of the same function. In order to promote the understanding of the dual nature of functions, suggestions have been made to emphasize the coordinations of the dynamics of input-output dependency relationships (Oehrtman et al., 2008). The idea of presenting the operational concept prior to the structural concept was suggested in this study by one of the state college instructors when she stated, "I would have them make an x-y chart before they sketched the graph."

**Importance of assessments, reviews, and clarifications**

A stark contrast between the secondary and postsecondary instructors interviewed in this study was their awareness of, and importance placed upon, diagnosing the incoming students on prerequisite concepts. Three instructors (two secondary and one university) were *very familiar* with their students' prior knowledge of functions while three instructors (two state college and one university) ranged from *less familiar* to *not familiar at all*. The reasons the instructors gave that did not use a diagnostic assessment instrument during class were : (1) There was not enough time built into the course to administer and review/clarify misconceptions identified by analyzing the results of a diagnostic assessment and/or (2) The results would not have a significant impact on how the course was taught; therefore, they did not see a need to assess the students' prior knowledge of functions prior to direct instruction.

One has to consider the difference in the purposes of secondary education versus postsecondary education when making implications based upon these results. Because AP Calculus is taught at the high school, it is assumed that the secondary instructors are still preparing their students for the rigors of postsecondary mathematics. In contrast to the postsecondary instructors, that assume their incoming students have been properly prepared for the rigors of postsecondary education. David Bressoud (2013) mentions these concerns and states that as secondary instructors teaching first semester calculus:

> 1. We should ensure that students who take calculus in high school are prepared for the further study of mathematics.
> 2. We should address the particular needs of those students who arrive in college with credit for calculus.
> 3. We should recognize that the students who take first-semester calculus in college may need more support and be less likely to continue with further mathematics than those of a generation ago (Bressoud, 2013, para 3.)

**"I don't know, and I don't *want* to know."**

It is interesting to note that the same university instructor that stated he does not review and/or clarify pre-calculus concepts during his calculus class also does not expect his incoming students to do well on the functions diagnostic instrument. In other words, this instructor does not expect the students to do well on the questions pertaining to misconceptions of functions and also does not review and/or clarify these pre-calculus concepts in class. When asked earlier where his students would receive their review and/or clarification if it is needed, his response was that his teaching assistants would provide that instruction. This is also the only instructor that did not know where his students received their pre-calculus instruction prior to his course. It appeared as if the instructor had accepted a position of "I don't know, and I don't *want* to know" when it came to his students' prior knowledge. When this finding was raised during the member check, a discussion ensued about students taking responsibility for their own education. The university and state college instructors agreed that a difference in secondary and postsecondary education is the expectation at the postsecondary level that students are more accountable for finding additional assistance with review and/or clarification of prerequisite material, outside of class, than at the secondary level. In the same article, Bressoud (2013) addresses university instructors of first semester Calculus by stating,

> But there still remain far too few university-level mathematicians who are willing to assist in the task of preparing and supporting high school teachers. At the very least, all mathematicians have a responsibility to be aware of the AP Calculus program: its course expectations and the nature of its examinations. Every department should encourage at least one individual to attend the annual AP Reading (the grading of the free response questions), to work with local AP Calculus teachers, or to help prepare and support those who will teach calculus in high school (Bressoud, 2013, para. 11)

**Future research: Title, Textbook, and Syllabi**

During my analysis of the data from the first four questions in Component 1, I noticed some interesting observations about the course titles, textbooks and stark differences among the syllabi used by the calculus instructors at the three types of educational institutions. Although these questions were asked with the intent of answering the research questions of this study, the data did not directly answer the questions posed, rather they suggested new research questions for possible future research studies. I am including the discussion based upon these first four questions in this section of the chapter with the hope that the data can be used at a later time.

As presented in the Rationale section of the Introduction chapter, there is a gap into which many students fall when transitioning from secondary mathematics to postsecondary mathematics. However, Conley (2005) refers to a more general educational "gap" described by first year college students' between their high school experiences and college instructors' expectations of student behavior. "College courses require students to be independent, self-reliant learners who recognize when they are having problems and know how to seek help from professors, fellow students, or other sources" (Conley, 2007, p. 23). The need for further research on this gap, as described by Conley, of successful student behavior was mentioned during the member check of this study. All four postsecondary instructors commented on their concern for the lack of student preparation or maturity for college-level courses and recommended further research to be conducted in this area.

***Course title and textbook.*** In both the secondary and postsecondary institutions the name of the course and the textbook used for the course appeared to be determined by persons other than the instructor of the course. The flexibility the instructors had was the extent to which they

chose to use the required text and accompanying teacher resources. All of the instructors, except for one of the university instructors, stated that they did use the textbook and accompanying resources. The university instructor stated that he preferred to rely on his own familiarity with the topics being presented than the content by the textbook authors. The instructor mentioned that he would defer to the required textbook for names of theorems that had multiple names in various textbooks, so as not to confuse the students.

***Instructor syllabi.*** Conley (2007) states:

> Research suggests that the syllabi in high school courses are different from those found in college courses. High school syllabi rarely undergo external review, as all college syllabi do. The content of high school syllabi tends to be eclectic, with teachers selecting class content largely on the basis of their own interests and skills rather than on what students need to succeed in college. The format and structure may differ dramatically from teacher to teacher (p. 25).

My findings supported the research Conley refers to in this statement. There were stark contrasts between syllabi between the three types of educational institutions. Because one secondary instructor did not write a syllabus, and three out of four pages of the other secondary instructor's syllabus consisted of a reprint of the table of contents of the textbook, it appears as if the use of the syllabus itself is viewed differently at the three educational institutions. Although what is meant by a "syllabus" varies greatly among individuals, Parkes and Harris (2002) presume that "every college professor realizes the necessity of preparing a syllabus for each course taught" (p. 55). While attempting to define an agreed upon purpose for the syllabus by their review of over 200 syllabi, Parkes and Harris state the purpose of a syllabus is to set forth what is expected and to guide the behavior of the both the instructor and student. The syllabi of the postsecondary instructors of this study seem to do just that.

While the one secondary instructor was the only instructor to specifically address her expectations of her students incoming students' prior knowledge of functions, all of the postsecondary instructors used the majority of the syllabus to address student behavior. All postsecondary instructors clearly defined their expectations for attendance, homework assignments, and cell phone usage during class. A reason for these topics not to be mentioned in the syllabus of the secondary instructor may have to do with the type of educational institution. Most secondary institutions have institutional rules governing attendance, and cell phone usage during class and the instructor may not have seen a need to repeat those rules in her individual syllabus. That is in contrast to most postsecondary institutions that do not enforce violations of the stated attendance and cell phone usage regulations and the instructors feel it is necessary to include these items in their syllabus in case a student attempts to challenge their final grade in the course. Another major difference between the types of educational institutions, specifically the state college and university syllabi, were the detailed instructions explaining the requirements of students to utilize the university's computer lab located in the Mathematics building on the university. These sections were unique to the university because the other institutions did not have a dedicated mathematics computer laboratory.

In addition to homework, attendance and cell phone usage, the individual instructors emphasized multiple expectations of student behavior necessary for success in their course. Recommendations such as reading the textbook daily; participating in class discussions; time devoted outside of class for study; and keeping all homework neatly organized. Although there was some variance as to which student behaviors would lead to success in the courses, the inclusion of these items in the syllabus distinguishes the veteran teacher's syllabus from a novice teacher's syllabus.

## Potential Limitations of the Study

### Limited geographical area.

One of the potential limitations of this study is the fact that the participants were from the same geographical area. A multiple-case study methodology was chosen in this study to explore differences within and between cases, but if differences are found within and between secondary and postsecondary calculus instructors' expectations, the differences could be associated with a local issue within the particular region the study was conducted. Another potential limitation is the aspect of convenience sampling. I solicited instructors to voluntarily participate in this study. The willingness of the instructors to take the time and effort to participate may be due to strong views toward the topic of the study. The possibility of sampling instructors with strong views regarding either the transition from secondary to postsecondary or the instruction of calculus at the three institutions may skew the results. For the purpose of this study I limited calculus instructors to brick-and-mortar educational institutions. Today calculus is also being taught online. How to best teach calculus virtually is a topic that should be included in later studies of this type.

### Use of a student diagnostic.

Although as the researcher I felt the functions diagnostic instrument helpful to compare and contrast instructor expectations of their incoming student expectations at the three educational institutions, an argument could be made that the use of such an instrument is another potential limitation of the study. A specific diagnostic instrument was chosen for the purpose of this study in order to standardize the information being presented to the calculus instructors at the three institutions, but it is recognized that different purposes for educational assessment

require different levels and models of assessment as is pointed out by Snow (1989). Student diagnostic instruments may be useful for "macro adaptations" of instruction at the beginning of a course, but should be followed-up with "micro adaptations" as the course develops.

> Deep understanding, higher order skill, strategic flexibility, adaptive control, and achievement motivation are exhibited when students have to generate explanations, assemble skilled performances, and persist through learning problem solving and problem finding. Student weaknesses are exhibited in the degree and kind of help they need to do these things. Conventional tests are limited because they cannot assess these end states explicitly (Snow, 1989, p. 14).

Watson (2000) pointed out an interesting parallel (with parallel limitations) of a teacher assessing his/her students' mathematical ability and a researcher attempting to determine a teacher's intentions. The teacher is attempting to understand what the student knows about a mathematical topic and make a judgment about the student's ability through interpretation, into a description, using the assessment tool. The researcher attempts to do something similar through the interviews. There is interplay of conjecture and reality about the links and themes and issues identified. I believe Watson's advice of keeping an open mind should be heeded by both mathematics teachers and mathematics education researchers, alike:

> A more realistic approach might be to accept that the best a teacher can do is to behave as if her interpretation of students' responses gives her adequate but tentative, ephemeral information for teaching purposes, retaining an open mind and avoiding irrevocable decisions such as tracking, stereotyping and labeling (Watson, 2000, p. 88).

**Potential Contribution of the Study**

Surveys have shown that high school mathematics instructors and postsecondary mathematics instructors tend to have different views about the importance of particular knowledge and skills as prerequisites for success for continued study in college-level mathematics (ACT 2006, 2009). Surveys are a good start, but to address the issues we face as

mathematics educators there needs to be more. Other research studies have been done to compare high school and college faculty ratings of importance of specific content and pedagogy with respect to importance for success in college calculus (Artigue, et.al, 2007; Carlson, 1998; James, 1995; Stroumbakis, 2010). These studies all contribute to identifying the issues facing students when making the transition from secondary to postsecondary mathematics, but these studies were only focused on the calculus course itself without examining their underlying instructor expectations of their students' preparation prior to taking the calculus course.

Weinstein (2002) states that instructor expectations of their students' academic preparation for a particular course have been shown to have an effect on student performance within that course. This study focused on explicating the explicit and implicit expectations of calculus instructors. The different views from veteran instructors about the importance of particular knowledge and skills as prerequisites for success were highlighted. This qualitative study adds to the work begun by others by explicating instructor expectations for one specific pre-calculus topic, functions. This study is a step, not a destination. The work begun by this study could be extrapolated to many more areas of instructor expectations that need to be explicated, to include basic algebra, analytic geometry, trigonometry, series and sequences.

**Conclusion**

On the first page of the first volume of the fledging journal, *The Arithmetic Teacher*, William A. Brownell (1954) describes the mathematics classroom of 1900:

> Teachers relying pretty much upon what was in the textbook, showed pupils what to do and then relied upon abundant bodies of practice to produce mastery. Homework assignments were heavy, and many parents were called upon to revive, temporarily at least, skills that they had forgotten (p. 1).

Over a hundred years later, we need to ask ourselves how far we have come. Some aspects of this study, unfortunately, illustrate that we have not sufficiently progressed in the way we teach mathematics. Although the title of his article was "The Revolution of Arithmetic," Brownell clarifies that the process of change is more *evolutionary* than *revolutionary*. "Each modification has emerged from a given status and has led to the next modification. The steadying and stabilizing influence in this period of evolution has been what I have called the search for a functional curriculum" (p. 3). At the time of the article, Brownell was advocating for a mathematics curriculum that emphasized not only training the mind to calculate the correct answer, "faculty," but also understanding the meaning of the mathematics behind the answer, "function." He went on to state, "Both aims are essential to a functional curriculum in arithmetic, and both are attainable. Indeed, both are now being attained under the conditions of good instruction" (p. 5).

The conceptual framework for this study is that there needs to be a balance between procedural fluency and conceptual understanding and this balance is attained if both are taught iteratively. In this study, I was looking for evidence of instructors' explicit and/or implicit verification or rejection of this theory and how their assumptions of their students' prior knowledge, either procedural or conceptual, influenced how they taught calculus. Contrary to the Rosenthal and Jacobson (1968) Pygmalion study mentioned in the literature review, the participants in my study were not manipulated by providing false information about their students. Nor do I say secondary teachers have too low of expectations and postsecondary teachers have too high of expectations of their incoming calculus students. For this study I chose to show calculus instructors at the three different types of educational institutions the results of a

class set of student answers to a functions diagnostic instrument and asked them to analyze the student work as if they had given their own students this assessment. Through this process I was able to compare and contrast teacher assumptions of their students' prior procedural and conceptual knowledge of functions.

Findings from this study affirm the importance of iterating conceptual understanding and procedural fluency when teaching pre-calculus concepts. This research complements other studies highlighting the benefits to student understanding when mathematical concepts are taught. When there is no expectation of conceptual development, there is no impetus for the student to understand the concepts behind the procedures. Mathematics could get reduced to a series of decontextualized steps to be memorized (Baroody, 2003; Ben-Hur, 2006; Heibert and Lefever, 1986; Herscovics, 1996; Skemp, 1976). In this study, although differences of teacher expectations of their incoming students and methods for planning instruction varied between educational institutions, the calculus instructors were in agreement that a balanced approach to conceptual understanding and computational fluency was key to a students' understanding of pre-calculus concepts.

In Component 1, by examining the explicit instructors' expectations it was revealed that there were differences between types of educational institutions on the instructors' expectations of how well their incoming students would do on a diagnostic instrument of common misconceptions of functions. The expectations of the instructors seemed to become more pessimistic from high school to state college and then finally to a university instructor that repeated the stated, "I am not too optimistic" many times when speaking of his incoming students' prior knowledge of functions.

In Component 2, when the instructors used the "think-aloud" protocol to analyze student work, there were many similarities between instructors' responses for two of the three misconceptions of functions. The three themes all instructors agreed upon for domain and range of functions were (a) Students may have the idea of domain and range, but they don't know how to communicate what they know; (b) Sometimes the questions we ask, as teachers, are ambiguous to the students; (c) Students put "all real numbers" when they don't know the answer to a domain/range question. For translation of functions, the two themes were (a) Students remember something, they just remember it wrong; and similar to a domain and range theme, (b) Students may have the idea of translation, but they don't know how to communicate what they know. For piecewise functions, differences between institutions were noticed. The secondary and university instructors agreed that (a) Students are better at evaluating a piecewise function than sketching a piecewise function; in contrast, the state college instructors went further and stated (b) Students don't see the connection between the algebraic representation of the function and the graphical representation which is in keeping with the findings from other researchers discussed in the literature review.

In Component 3, there were differences between educational institutions as to the resources instructors used to plan instruction. The secondary instructors mainly used resources produced by the company that generates the national standardized exam used by many colleges for determining college credit for calculus, while the postsecondary instructors did not mention these resources, rather they relied upon their prior experience teaching the course, conversations with colleagues, online mathematics websites, professional conferences, and with the exception of one university instructor, the textbook and materials supplied by the textbook company. Only

one instructor stated that he used his own knowledge of the subject matter in order to prepare lectures. There were also similarities found as a result of the data recorded in Component 3. The instructors were in agreement that there needs to be a balance of both conceptual understanding and computational (or procedural) fluency for student understanding of pre-calculus topics, which was the basis for conceptual framework of this study.

This study provided a platform for instructors of calculus at three educational institutions to discuss their expectations concerning their incoming students. It provided a common language for secondary and postsecondary instructors to engage in conversation about differences and commonalities as recommended by both the NCTM and MAA in their joint position statements (NCTM, 1986, 2012). The themes that were revealed by this study can serve as a springboard for future studies. By conducting further studies of this type, mathematics educators representing both the secondary and postsecondary educational institutions are provided the opportunity to join forces to accomplish the common goal of preparing the next generation to face the STEM challenges of their future.

# APPENDIX A:
# SCRIPT FOR INSTRUCTOR INTERVIEW

## Appendix A: Script for Instructor Interview

Interview Questions for Calculus Instructor

There are three components to this interview.

The first component is a survey pertaining to you as a calculus instructor. This component will not be recorded. The second and third components will be audio recorded and transcribed at a later time. All information provided during this interview will be kept confidential.

For the second component I will be asking you some general questions pertaining to your students' prior knowledge of functions. Next I will ask you to analyze a diagnostic of functions found in a calculus textbook. Then, I will ask you to assess actual student answers to specific questions from the diagnostic.

For the third component, I am going to read you two quotations concerning students' understanding of mathematics. Afterwards I will ask you questions pertaining to the quotations.

Do you have any questions before we begin?

### First Component: Written Survey of Calculus Instructor

(Hand survey to instructor)

### Second Component: Diagnostic of Functions

(Hand diagnostic to instructor)

1. I ask you to relook at this diagnostic and this time, please comment on which questions you anticipate your incoming students being able to answer correctly and which questions you anticipate your incoming students to not answer correctly.

For example, you may say, "I expect that all of my students would be able to answer #1a correctly, but I think some will have trouble answer #1b because…"

(Give instructor time to complete this task)

2. I am going to show you actual calculus students' answers when they took this diagnostic. I would like you to respond to each of these students' answers as if these students were your incoming calculus students and you had administered this diagnostic.

Do you have any questions before we begin?

a. These are five students' answers to question #1e. Please comment on the students work as if they were your students and you were attempting to assess their prior knowledge of functions.

For example, you may say, "I think the student wrote this answer because…"

(Show answers to question #1e)

b. These are five students' answers to question #3a. Please comment on the students work as if they were your students and you were attempting to assess their prior knowledge of functions.

(Show answers to question #3a)

c. These are five students' answers to question #4c. Please comment on the students work as if they were your students and you were attempting to assess their prior knowledge of functions.

(Show answers to question #4c)

d. These are five students' answers to question #6. Please comment on the students work as if they were your students and you were attempting to assess their prior knowledge of functions.

(Show answers to question #6)

That completes the second component of this interview. For the third component I am going to be asking you to give your opinion about some quotations regarding mathematics education.

## Third component: Quotations about Mathematics Education

The first quotation is addressing research-based diagnostic instruments for mathematics:

***"In the absence of research-based curricular instruments, teachers tend to rely on their own opinions about what students need to learn as they plan instruction" (Carlson, Oehrtman, Engelke, 2010, p. 114).***

1. Do you agree or disagree with this statement?

2. What specific resources do you use to plan instruction for your calculus class?

The second quotation is referring to mathematics education, in general:

***"Developing fluency requires a balance and connection between conceptual understanding and computational fluency. On the one hand computational methods that are over-practiced and are often forgotten or remembered incorrectly…On the other hand, understanding without fluency can inhibit the problem solving process" (NCTM, 2000, p. 35).***

1. What do you interpret the author of this quotation to mean by the terms "conceptual understanding" and "computational fluency"?

2. This quotation refers to "computational methods." Can you give me specific "computational methods" from your past years of teaching calculus of pre-calculus concepts of functions that are "over-practiced" and often forgotten or remembered incorrectly?

3. Can you give me specific examples where a student's understanding of a function concept "without fluency inhibited their problem solving process"?

4. Overall, do you agree or disagree with this statement? Why or why not?

5. If you had to choose between computational fluency or conceptual understanding for your incoming calculus students, which would you choose for the following pre-calculus topics:

State either "computational fluency" or "conceptual understanding" for each of the following:

a. Identifying the domain and range of functions

b. Graphing and evaluating piecewise functions

c. Graphing and evaluating discontinuous functions such as step functions or rational functions

d. Identifying points of discontinuity in rational functions

e. Graphing and identifying functions that have been translated

f. Problem solving using function models

Those are all the questions that I have for you.

5. Is there any other information you would like to share concerning your students' prior knowledge of functions?

6. Is there any other information you would like to share concerning conceptual understanding versus computational fluency of mathematics?

Thank you for agreeing to participate in my study. I will be conducting a member-check with all the participants once I have completed my analysis, probably toward the end of May. I will be sending you an email and would appreciate it if you could attend and give me feedback as to how well I was able to incorporate your perspective into my study.

Thank you, again, for your time and honesty.

**References for Interview**


Becker, A. B. (1991). The concept of function: Misconceptions and remediation at the college level (Doctoral dissertation). Retrieved from ProQuest Dissertations & Theses.

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National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

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# APPENDIX B:
# WRITTEN SURVEY (COMPONENT 1)

Written Survey of Calculus Instructors

***Please answer the following questions in the spaces provided.***

1. Where are you currently employed and teaching calculus?

2. What is the title of your calculus course?

3. What is the title/publisher of the textbook you are using?

4. Can you provide me a copy of the syllabus for your calculus course? ***Circle one.*** Yes or No

(If yes, I will send you an email requesting this information after the interview.)

5. Do you use a diagnostic tool to assess your incoming calculus students' prior knowledge of functions? ***Circle one***. Yes or No

If No, please state why not and continue on to question #6.

If Yes, please answer the following questions:

a. Where do you get the diagnostic tool?

b. Do you know if the diagnostic is research-based? ***Circle one***. Yes or No

c. Do the results of the diagnostic influence your instructional plans? ***Circle one.*** Yes or No

If Yes, how?

d. Is it possible for me to get a copy of the diagnostic tool you use? ***Circle one.*** Yes or No

(If yes, I will send you an email requesting this information after the interview.)

***Please continue on to next page.***

6. Where do the majority of your students learn the pre-calculus concept of function prior to your class?

7. Based on your experience from teaching calculus, which, if any, of the following pre-calculus topics do you feel that many of your students need clarification/review prior to starting your class?

***Please put one X on the appropriate box for each pre-calculus topic.***

| | **Many of my incoming students need…** | | | |
|---|---|---|---|---|
| **Pre-calculus Topic** | **Review** | **Clarification** | **Both** | **Neither** |
| a. Arithmetic computation used to evaluate a function at a single numerical value | | | | |
| b. Subconcepts of function such as domain, range and correspondence | | | | |
| c. Definition of function | | | | |
| d. Graphing/evaluating piecewise functions | | | | |
| e. Graphing/evaluating discontinuous functions such as step functions or rational functions | | | | |
| f. Identifying points of discontinuity in rational functions | | | | |
| g. Graphing/identifying functions that have been translated | | | | |
| h. Problem solving using function models | | | | |

8. If you determine that your students need clarification or review of functions (excluding trigonometric, exponential and logarithmic) in order to be successful in your calculus class, how and/or where do students receive this service?

***Please continue on to next page.***

9. Approximately how much class time do you spend on functions (excluding trigonometric, exponential and logarithmic) in your calculus class? For example, "I spend 5 class days on functions."

***Please fill in the blanks to this statement:***

*"I spend _____ (numerical answer) class _____ (unit of measure, i.e. days, weeks, hours) of my calculus class on functions.*

10. a) What is your understanding of the term "procedural fluency" as it relates to the pre-calculus concept of functions?

(b) What is your understating of the term "conceptual understanding" as it relates to the pre-calculus concept of functions?

***Please take a few minutes to look at the individual questions on the diagnostic test of functions found on the next page of this survey. Refer to that diagnostic to answer the next question.***

11. On a scale of 1-10 with **1** being ***"completely inadequate"*** and **10** being ***"completely adequate"*** how would you rate the overall adequacy of this instrument in assessing your students' prior knowledge of functions?

***This completes the first component. The next two components will be audio recorded.***

***Do you have any questions?***

## C DIAGNOSTIC TEST: FUNCTIONS

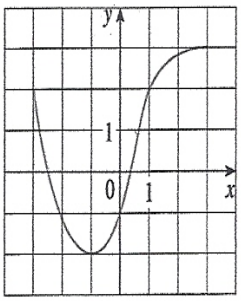



FIGURE FOR PROBLEM 1

1. The graph of a function $f$ is given at the left.
   (a) State the value of $f(-1)$.
   (b) Estimate the value of $f(2)$.
   (c) For what values of $x$ is $f(x) = 2$?
   (d) Estimate the values of $x$ such that $f(x) = 0$.
   (e) State the domain and range of $f$.

2. If $f(x) = x^3$, evaluate the difference quotient $\dfrac{f(2+h) - f(2)}{h}$ and simplify your answer.

3. Find the domain of the function.
   (a) $f(x) = \dfrac{2x+1}{x^2 + x - 2}$ (b) $g(x) = \dfrac{\sqrt[3]{x}}{x^2 + 1}$ (c) $h(x) = \sqrt{4 - x} + \sqrt{x^2 - 1}$

4. How are graphs of the functions obtained from the graph of $f$?
   (a) $y = -f(x)$ (b) $y = 2f(x) - 1$ (c) $y = f(x - 3) + 2$

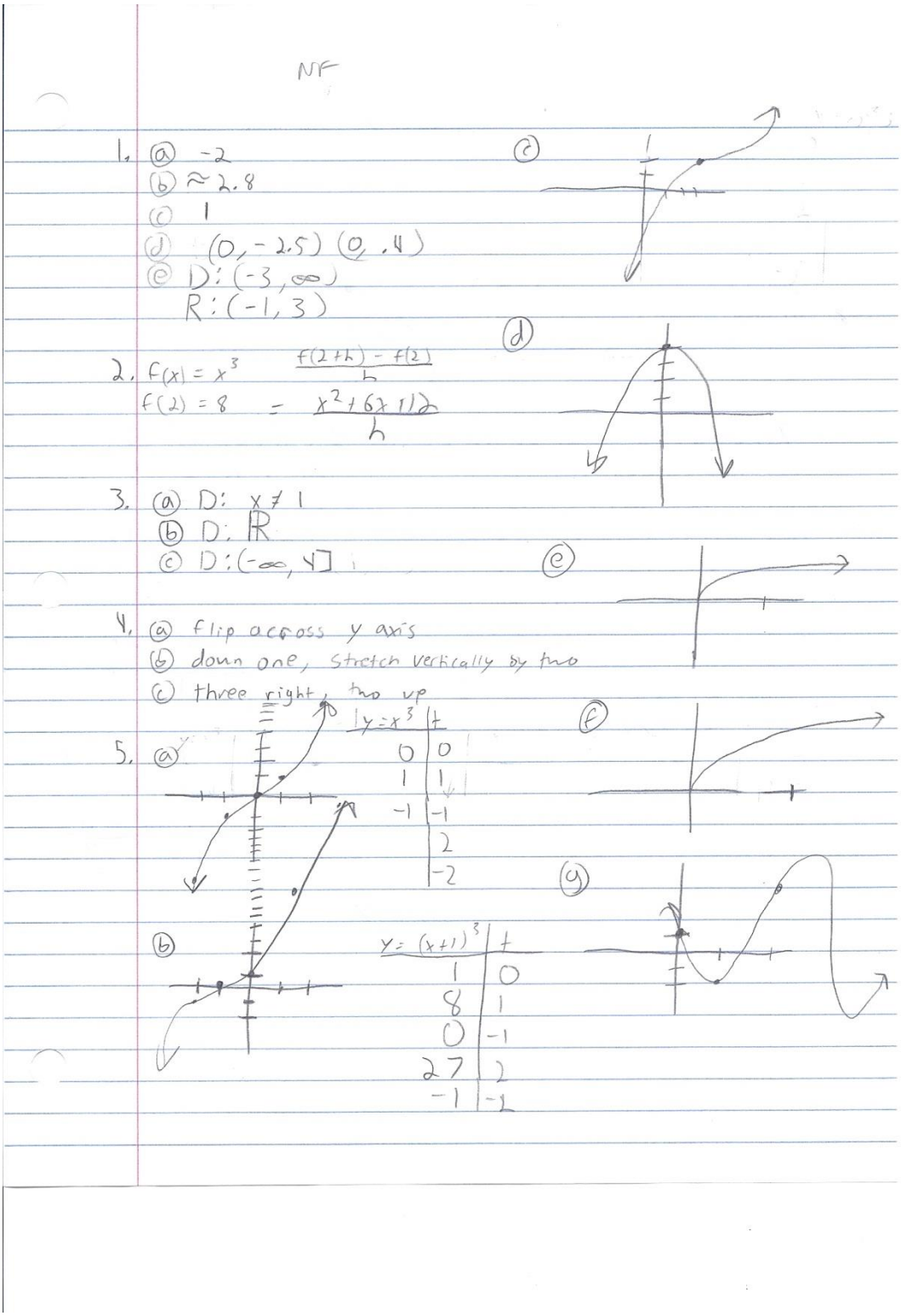
5. Without using a calculator, make a rough sketch of the graph.
   (a) $y = x^3$ (b) $y = (x + 1)^3$ (c) $y = (x - 2)^3 + 3$
   (d) $y = 4 - x^2$ (e) $y = \sqrt{x}$ (f) $y = 2\sqrt{x}$
   (g) $y = -2^x$ (h) $y = 1 + x^{-1}$

6. Let $f(x) = \begin{cases} 1 - x^2 & \text{if } x \leq 0 \\ 2x + 1 & \text{if } x > 0 \end{cases}$
   (a) Evaluate $f(-2)$ and $f(1)$. (b) Sketch the graph of $f$.

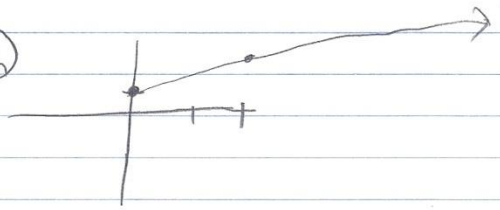
7. If $f(x) = x^2 + 2x - 1$ and $g(x) = 2x - 3$, find each of the following functions.
   (a) $f \circ g$ (b) $g \circ f$ (c) $g \circ g \circ g$

# APPENDIX C:
# STUDENT ANSWERS TO DIAGNOSTIC (COMPONENT 2)

Student #2

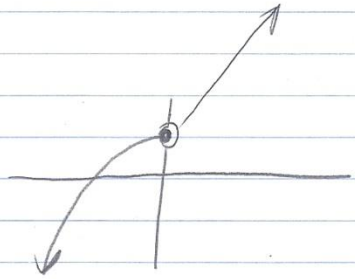

NF
1. (a) -2
(b) ≈ 2.8
(c) 1
(d) (0, -2.5) (0, 4)
(e) D: (-3, ∞)
R: (-1, 3)
2. f(x) = x³
f(2) = 8
= (f(2+h) - f(2))/h
= (x² + 6x + 12)/h
3. (a) D: x ≠ 1
(b) D: ℝ
(c) D: (-∞, 4]
4. (a) flip across y axis
(b) down one, stretch vertically by two
(c) three right, two up
5. (a)
y = x³ | t
0 | 0
1 | 1
-1 | -1
2
-2
(b)
y = (x+1)³ | t
1 | 0
8 | 1
0 | -1
27 | 2
-1 | -2
(c)
(d)
(e)
(f)
(g)

(b)



6. (a) $f(2) = -1$
$f(1) = 3$

(b)

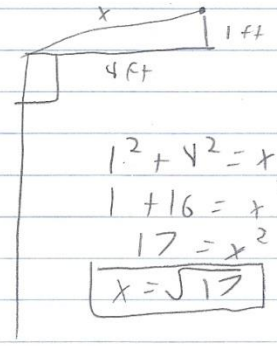


7. (a) $(2x-3)^2 + 2(2x-3) - 1$
$(2x-3)(2x-3)$
$4x^2 - 12x + 9 + 4x - 7$
$4x^2 - 8x + 2$
$2(2x^2 - 4x + 1)$
$(2x-1)(x+3)$
$x = -3, 1$

(b) $2(x^2 + 2x - 1) - 3$
$2x^2 + 4x - 2 - 3$
$2x^2 + 4x - 5$
$(2x+4)(2x-1)$
$x = 2 \quad x = \frac{1}{2}$

(c) $2(2(2x-3)-3)-3$
$2(4x-6)-3)-3$
$(8x-12)-6-3$
$8x - 12 - 9$
$8x - 21$
$8x = 21$
$x = \frac{21}{8}$

8. (a)




$1^2 + 4^2 = x^2$
$1 + 16 = x^2$
$17 = x^2$
$x = \sqrt{17}$

(b) 7

Student #3

1) a) -2
b) 2.8
c) 1, 3
d) -1
e) $[3, \infty)$

2. $\frac{(2+h)^3 - 8}{h}$ $(4+4h+h^2)(2+h)$

$8 + 8h + 2h^2 + 4h + 4h^2 + h^3$

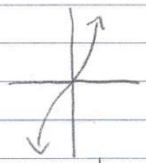
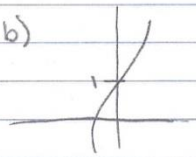
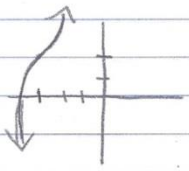
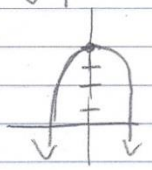
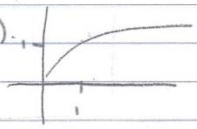
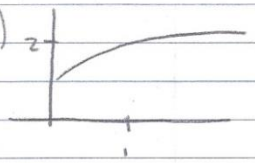
$8 + 12h + 6h^2 + h^3 - 8$

$\frac{12h + 6h^2 + h^3}{h}$

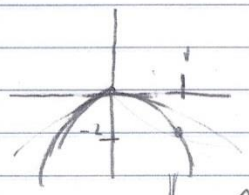
$\frac{h(12 + 6h + h^2)}{h} = 12 + 6h + h^2$

3 a) $(\infty, 1) \cup [1, \infty]$
b) $\mathbb{R}$
c) $\mathbb{R}$

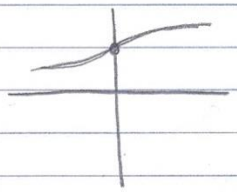
4. a) flip over x axis
b) one to the Right and 2x wider.
c) 3 down 2 to the left.

5. a) 
b) 
c) 
d) 
e) 
f) 

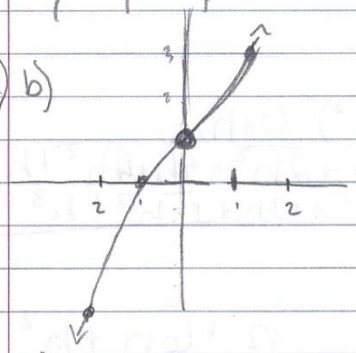
g)



h)



(a) b)



a) $f(-2) = -3$
$f(1) = 3$

7. a) $f(x) = x^2 + 2x - 1$ $\quad g(x) = 2x - 3$

$(2x-3)^2 + 2(2x-3) - 1$

$4x^2 - 12x + 9 + 4x - 6 - 1$

$4x - 8x + 2$

b) $2(x^2 + 2x - 1) - 3$

$2x^2 + 4x - 2 - 3 = 2x^2 + 4x - 5$

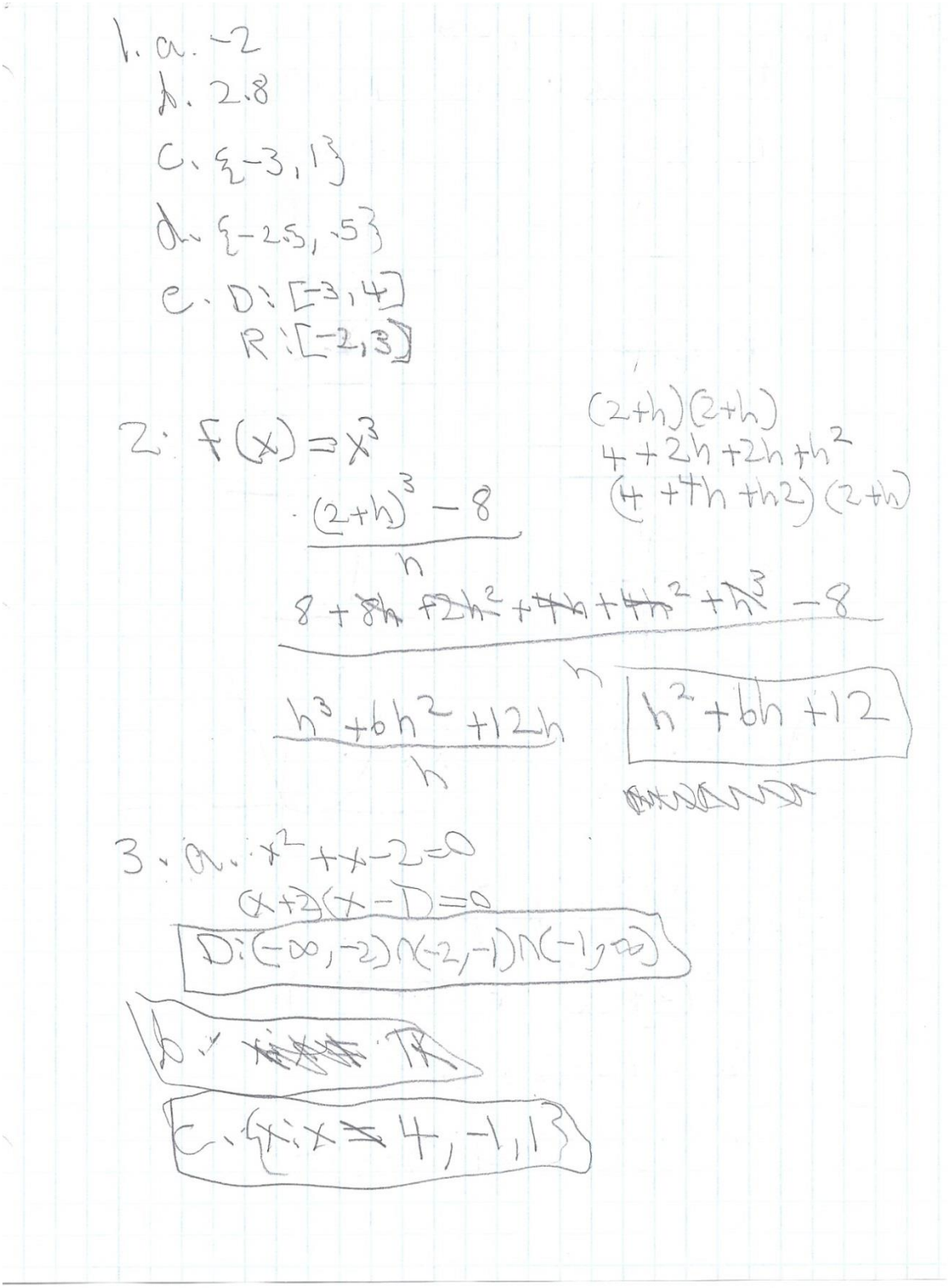
c) $2(2(2x-3) - 3) - 3$

$2(4x - 6 - 3.) - 3$

$8x - 18 - 3. = 8x - 15$

8. no

Student #4

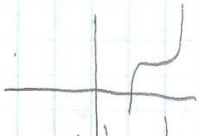

1. a. -2
b. 2.8
c. {-3, 1}
d. {-2.5, -5}
e. D: [-3, 4]
R: [-2, 3]
2. f(x) = x³
(2+h)³ − 8 / h
(2+h)(2+h)
4 + 2h + 2h + h²
(4 + 4h + h²)(2+h)
8 + 8h + 2h² + 4h + 4h² + h³ − 8
h³ + 6h² + 12h / h
h² + 6h + 12
3. a. x² + x − 2 = 0
(x+2)(x−1) = 0
D: (−∞, −2) ∩ (−2, −1) ∩ (−1, ∞)
b.
c. {x: x ≠ 4, −1, 1}

4. a. flip over y axis

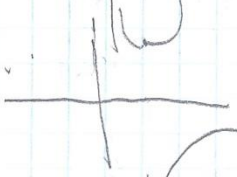
b. y values double & stretch one

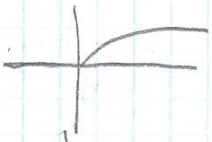
c. x to the right 3 y up 2

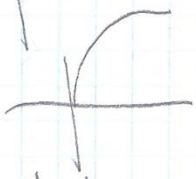
5. a. 

c. 

b. 

d. 

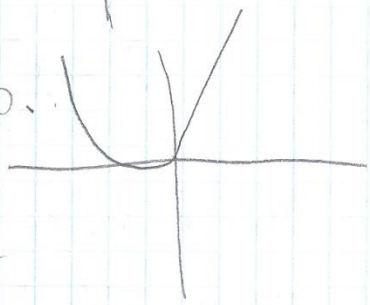
e. 

f. 

g. 

h. 

b. a. -3, 3

b. 

7. a. ~~$(x^2+2x-1)$~~

$((2x-3)^2+2(2x-3)-1)$

b. $2(x^2+2x+1)-3$

8. ?

Student #7

NF

Diagnostic Test

1. a.) $-2$ d.) $-2.5$, $.45$
b.) $2.8$ e.) $D: [-3, \infty)$ $R: [-2, 3)$
c.) @ $f(-3)$, $f(1)$

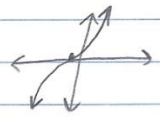
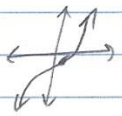
2. $\frac{f(2+h)-f(2)}{h} \rightarrow \frac{(2+h)^3-(2)^3}{h^3} = \boxed{\frac{x^2+6x+12}{x^2}}$

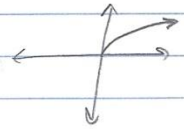
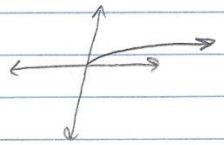
3. a.) $f(x) = \frac{2x+1}{x^2+x-2}$ $D: \mathbb{R}, (x \neq -2, 1, 4, \ldots)$

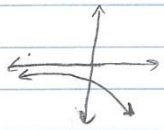
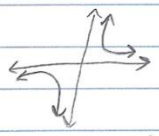
b.) $f(x) = \frac{\sqrt[3]{x}}{x^2+1}$ $D: \mathbb{R}$ $(-\infty, \infty)$

c.) $h(x) = \sqrt{4-x} + \sqrt{x^2-1}$ $D: (-\infty, \infty)$

4. a.) $y = -f(x)$ flipped over $X$
b.) $y = 2f(x)-1$ stretched, moved down 1
c.) up one, left 2

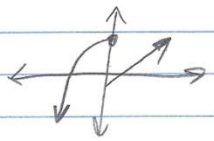
5. a.)  b.)  c.) 

d.)  e.)  f.) 

g.)  h.) 

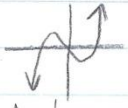
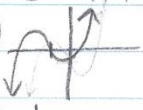
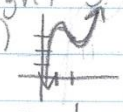
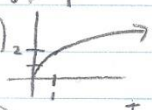
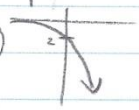
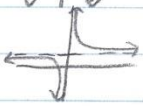
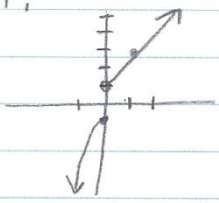
6. $f(x) = \begin{cases} 1-x^2 & x \leq 0 \\ 2x+1 & x > 0 \end{cases}$

a.) $f(-2) = -3$
$f(1) = 3$

b.) 

Student #9

NF

1. a) $y=-2$ d) $x=-2.5, 0.3$
   b) $y=2.8$ e) Domain: $[-3,3]$
   c) $x=-3,1$ Range: $[-2,3]$
2. $\frac{f(2+h)-f(2)}{h}$ $a=2 \Rightarrow f(2)=2^3=8$
3. a) $f(x)=\frac{2x+1}{x^2+x-2}=\frac{2x+1}{(x-1)(x+2)}$ Domain $\{x \mid x \neq -2, 1\}$
   b) $g(x)=\frac{\sqrt[3]{x}}{x^2+1}$ Domain: $x=\mathbb{R}$
   c) $h(x)=\sqrt{4-x}+\sqrt{x^2-1}$ Domain: $[-1,4)$
   $4-x \geq 0$ $x \geq -1$
   $x < 4$ $-1 \leq x < 4$
4. a) reflected over x axis
   b) shifted down 1 unit and stretched vertically by a factor of 2
   c) shifted up 2 and shifted right 3
5. a)  b)  c)  d) 
   e)  f)  g)  h) 
6. a) $f(-2)=-3$
   $f(1)=3$
   b) 
7. a) $f(g(x))=(2x-3)^2+2(2x-3)-1$
   $=(2x-3)^2+4x-6-1$
   $=4x^2-12x+9+4x-7$
   $=4x^2-8x+2$
   b) $g(F(x))=2(x^2+2x-1)-3$
   $=2x^2+4x-2-3$
   $=2x^2+4x-5$

Student #14

1. a) -2
   b) 2.9
   c) -3, 1
   d) -2.5, √3
   e) domain: [-3, 3]
   range: [-2, 3]

2. $\frac{(2+h)^3 - 8}{h}$

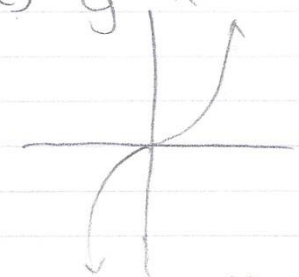
$\frac{h^3 + 6h^2 + 12h + 8 - 8}{h}$

$= h^2 + 6h + 12$

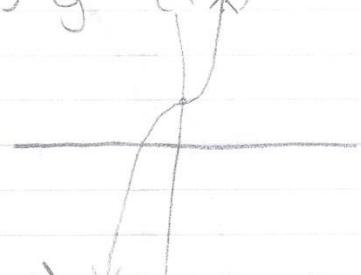
3. a) $(-\infty, 1) \cup (1, \infty)$
   b) $\mathbb{R}$
   c) $x \leq 4$ $(-\infty, 4]$

4. a) reflected over the x-axis
   b) shifted down 1, compressed by a factor of 2
   c) shifted to the right 3 and up 3

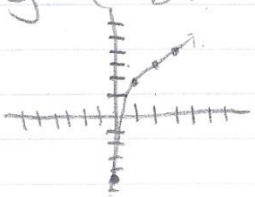
5. a) $y = x^3$



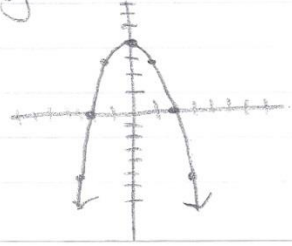
b) $y = (x+1)^3$



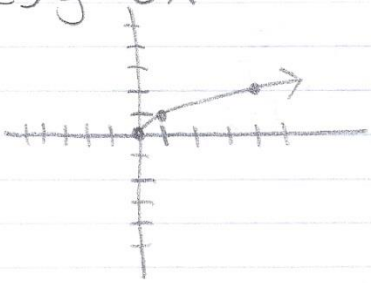
c) $y = (x-2)^3 + 3$



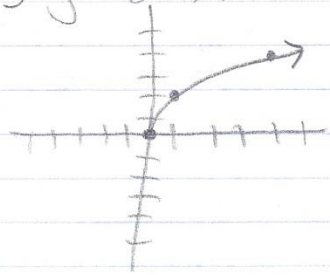
d) $y = 4 - x^2$



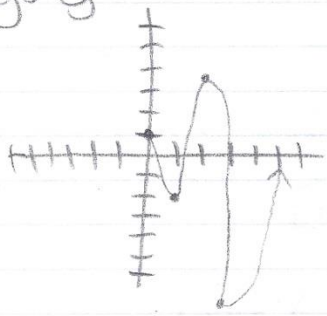
e) $y = \sqrt{x}$



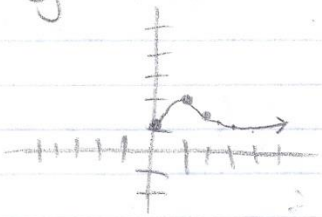
f) $y = 2\sqrt{x}$



g) $y = -2^x$

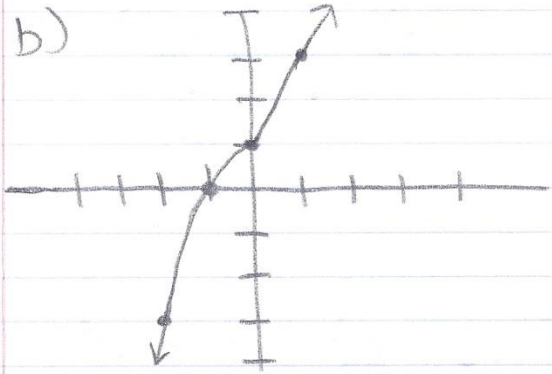


h) $y = 1 + x^{-1}$



6) a) $f(-2) = -3$
$f(1) = 3$

b)



Student #16

# Diagnostic Test

1) a) -2
b) 2.9
c) 1
d) .4, -2.5
e) Domain $[-3, \infty]$
Range $[-2, 3]$

2) $f(x) = x^3$ $\quad \frac{f(2+h) - f(2)}{h}$

$$\frac{(2+h)^3 - (2)^3}{h} = \frac{(4 + 4h + h^2)(2+h) - 8}{h}$$

$$= \frac{8 + 8h + 2h^2 + 4h + 4h^2 + h^3 - 8}{h} = \frac{h^3 + 6h^2 + 12h}{h} =$$

$$= \frac{h(h^2 + 6h + 12)}{h} = h^2 + 6h + 12$$

3) a) $\frac{2x+1}{x^2 - x - 2}$ $\quad x^2 - x - 2 \neq 0$

$(x - 2)(x + 1) \neq 0$ $\quad (-\infty, -1) \cup (0, 2) \cup (2, \infty)$

$x \neq -1, 2$

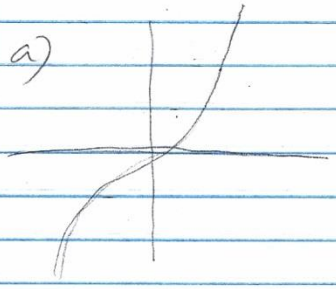
b) $x^2 + 1 \neq 0$

$x^2 = -1$ $\quad \mathbb{R}$ $\quad (-\infty, \infty)$

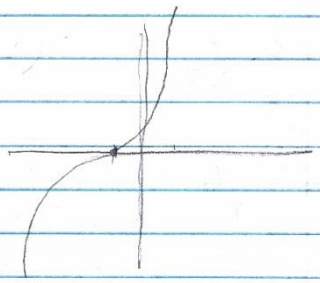
c) $(-\infty, \infty)$ $\mathbb{R}$

4) a) odd symmetry (origin)
b) stretched & down 1
c) right 3 up 2

(5) a)

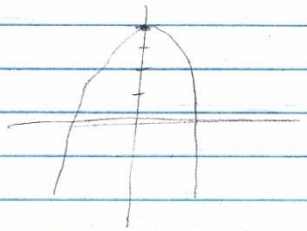


b)

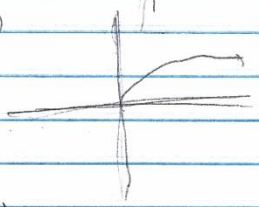


c)

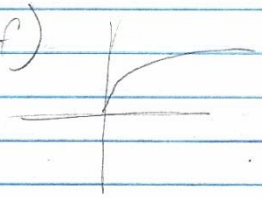
d)



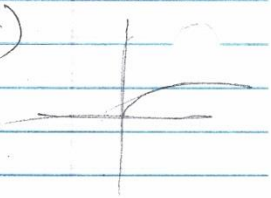
e)



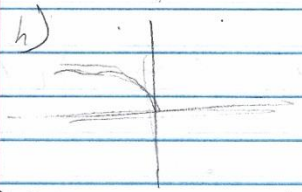
f)



g)

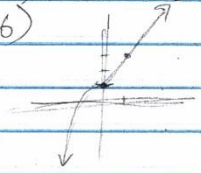


h)



(6) a) $f(-2) = -3$ $f(1) = 3$

b)



Student #17

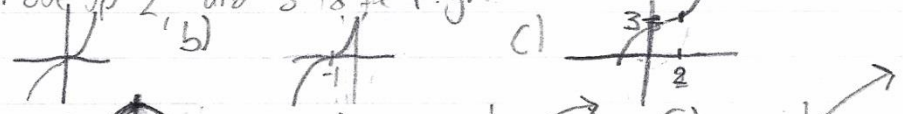
1 a) -2
b) 3. 9
c) 1, -3
d) -2.5, 1.3
e) D: $3 \leq x \geq -2$
R: $\pm 3$

2) $3x^2 - 8$

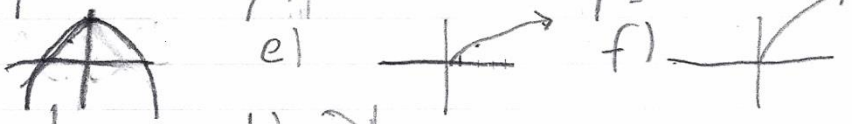
3) a) $x^2+x-2 \quad x \neq -2, 1$
$(x+2)(x-1)$
b) $x \geq 0, \quad x \geq -1$
c) $4-x \geq 0$ $(x \leq 4)$ $x^2-1 \geq 0$ $(x \geq 1)$

4) a) Symmetrical on y-axis
b) Move graph down 1; Stretch by factor of 2
c) Move graph up 2, and 3 to the r. gh

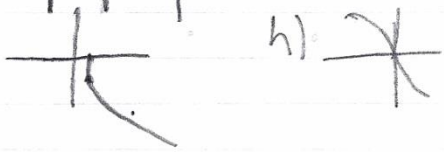
5 a) b) c)




d) e) f)



g) h)



6) $1-(-2)^2 \quad 1-4 = -3$
$2(1)+1 = 3$

7) a) $(2x-3)^2+2(2x-3)-1$
$4x^2-12x+9+4x-6-1$
$4x^2-8x+2 \quad 2(2x^2-4(x+1)$

b) $2(x^2+2x+1)-3$
$2x^2+4x-2-3$
$2x^2+4x-5$

c) $2(2x-3)-3 \quad 4x-6-3 \quad 4x-9 \quad 2(4x-9)-3 \quad 8x-18-3$
$= 8x-21$

Student #18

Diagnostic Test : Functions

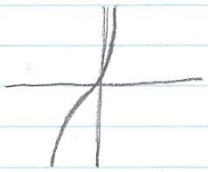
1) a) -2
b) 2.8
c) 1
d) -1
e) $[-3, 2] \cup (0, \infty)$

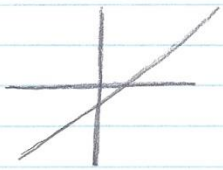
2) $f(x) = x^3$ $\quad \frac{f(2+h) - f(2)}{h}$

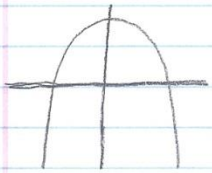
3) a) $\mathbb{R}$
b) $\mathbb{R}$
c) $(\infty, -1] \cup [1, 4]$

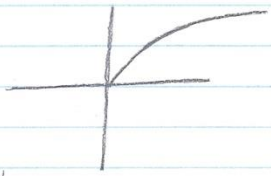
4) a)
b)
c)

5) a) $y = x^3$ 

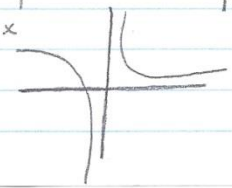
b) $y = (x+1)^3$ 

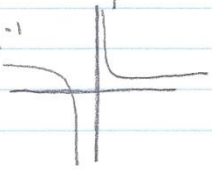
c) $y = f(x-3)+2$ 

d) $y = 4 - x^2$ 

e) $y = \sqrt{x}$ 

f) $y = 2\sqrt{x}$ 

g) $y = -2^x$ 

h) $y = 1 + x^{-1}$ 

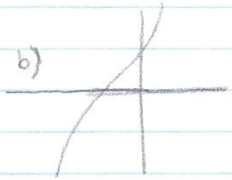
6) $f(x) = \begin{cases} 1-x^2 & x \leq 0 \\ 2x+1 & x > 0 \end{cases}$

a) $f(-2)$ $f(1)$

a) $f(-2) = 5$

$f(1) = 3$

b)



7) $f(x) = x^2 + 2x - 1$ $\quad g(x) = 2x - 3$

$f(g)$

$f(g) = (2x-3)^2 + 2(2x-3) - 1$

$= 4x^2 - 6 + 4x - 6 - 1$

$=$

Student #20

1) a) $f(-1) = -2$
b) $f(2) \approx 2.8$
c) $f(x) = 2$ when $x = 1$ and $x = -3$
d) $f(x) = 0$ when $x \approx .4$ and $x \approx -2.5$
e) $D: [-3, 3]$ $R: [-2, 3]$

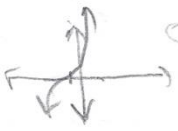
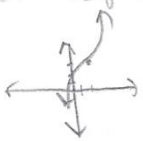
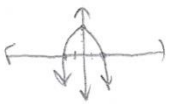
2) $\frac{f(2+h) - 8}{h}$

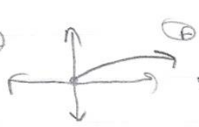
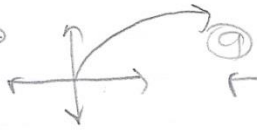
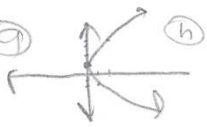
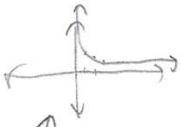
3) a) $x^2 + x - 2 = 0$ $D: (-\infty, -2) \cup (1, \infty)$
$(x-1)(x+2)$
$x \neq 1$ $x \neq -2$

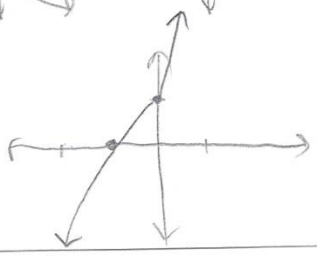
b) $D: \mathbb{R}$

c) $4 - x \geq 0$ $x^2 - 1 \geq 0$ $D: (-\infty, -1] \cup [1, 4]$
$x \leq 4$ $-1 \geq x \geq 1$

4) a) reflect about the x axis
b) shift down 1 and vertically stretch by a factor of 2
c) shift up 2 and left 3

5) a)  b)  c)  d) 

e)  f)  g)  h) 

6) a) $f(-2) = -3$ $f(1) = 3$ b) 

Student #21

1. a) $-2$
b) $2.75$
c) $x = 1$
d) $f(2.5) = 0$
$f(.4) = 0$
e) Domain: $[-3, \infty)$
Range: $[-2, \infty)$

2. $\frac{f(2+h) - f(2)}{h}$ $\quad f(x) = x^3$ $\quad f'(x) = 3x^2$

$\frac{LH' - HL'}{(h)^2}$

$\frac{h[(f(2+h))^3 - f(2)]' - f(2+h) - f(2)}{h^2}$

3. a) Domain: $f(x) = (-\infty, -3) \cup (-1, 0) \cup (2, \infty)$

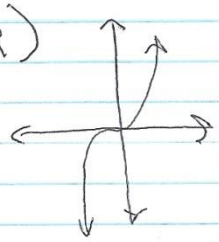
b) $g(x) =$ Domain: $[0, \infty)$

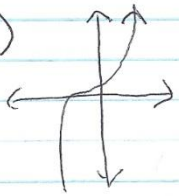
c) $h(x) =$ Domain: $(-\infty, -1] \cup [1, 5]$

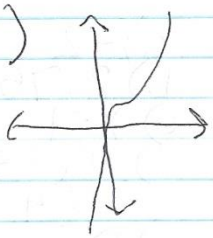
4. a) odd function

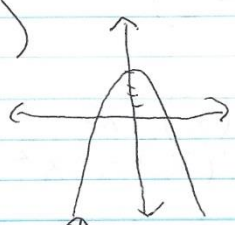
b) down on
multiply by 2

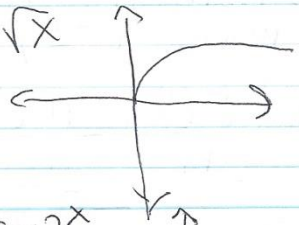
c) 6 to the right 3
up 2

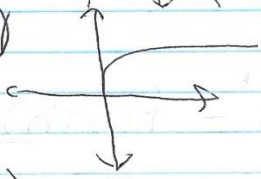
5. a) 

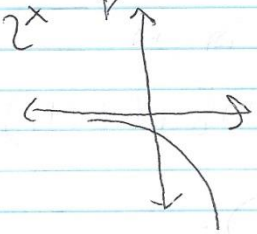
b) 

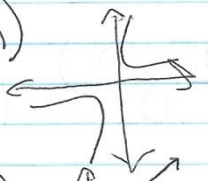
c) 

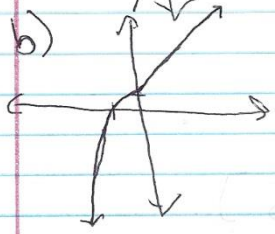
d) 

e) $y = \sqrt{x}$ 

f) 

g) $y = -2^x$ 

h) 

6. b) 

a) $f(-2) = -3 \quad x \leq 0$
$f(2) = -3 \quad x > 0$
$f(1) = 0 \quad x \leq 0$
$f(1) = 3 \quad x > 0$

7. a) $2x - 3 = 0$
$x = 3/2$
$(3/2)^2 + 2(3/2) - 1$
4.25

b) $x^2 + 2x - 1 = 0$
$x = \pm\sqrt{1/2}$
$x = -1.586$
$x = -4.414$

c) ~~$x = 3/2$~~ $x = 0$
-3

8.

Student #24

1. a). -2
   b). 2.8
   c). -3, 1
   d). 0.3, -2.5
   e). d: $[-3, \infty]$
   R: $[-2, 3]$

2. $\frac{(2+h)^3 - 8}{h}$

$(2+h)(2+h)(2+h)\,h$

$(4+4h+h^2)(h+2)$

$(8+8h+2h^2)(h)$

$\frac{8h+8h^2+2h^2-8}{h}$

$2h^2+8h$

$2h(h+4)$

3. a). $(x-1)(x+2)$

$x = 1, -2$

$D: x \neq 1, -2$

b). $D: \mathbb{R}$

c). $x \leq 4, x \geq 1$

$D: [1, 4]$

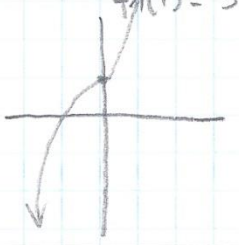
4. a). reflected by y-axis

b). down 1, value is 2 times greater

c). right 3, up 2.

5. a). ∪
   b). ∼
   c). ∼
   d). ∩
   e). ∨
   f). ∨
   g).
   h).

6. a). $f(-2) = -3$
   $f(1) = 3$



7. a). $(2x-3)^2 + (4x-6) - 1$

$4x^2 - 12x + 9 + 4x - 7$

$4x^2 - 8x + 2$

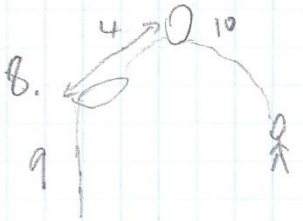
b). $2x^2 + 4x - 5$

c). $4x - 6 - 3$

$4x - 9$

$8x - 33$

8.



Student #27

1. a) $f(-1) = -2$
b) $f(2) = 2.8$
c) $f(x) = 2$
$x = 1$
d) $f(x) = 0$
$x = -2.5$
$x = .4$
e) DOMAIN: $\mathbb{R}$
RANGE: $\mathbb{R}$

2. $f(x) = x^3$ $\dfrac{f(2+h) - f(2)}{h}$

$= \dfrac{(2+h)^3 - 8}{h}$ $(2+h)(2+h)(2+h)$

$4 + 2h + 2h + h^2 + 4 + 2h + 2h + h^2$

$8 + 8h + 2h^2$

$= \dfrac{8 + 8h + 2h^2 - 8}{h} = \dfrac{8h + 2h^2}{h} = 8 + 2h$

3. a) $f(x) = \dfrac{2x+1}{x^2+x-2}$ b) $g(x) = \dfrac{\sqrt[3]{x}}{x^2+1}$

$\mathbb{R}$

$x^2 + x - 2 \neq 0$

$(x+2)(x-1)$

$x \neq -2, 1$

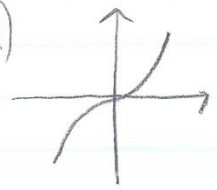
c) $h(x) = \sqrt{4-x} + \sqrt{x^2-1}$

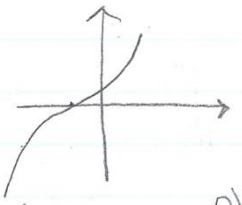
$x \leq 4$ $x \neq 0$ →

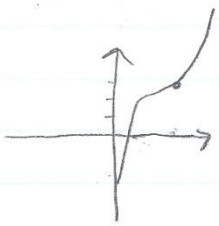
4. a) $y = -f(x)$
FLIP ACROSS y AXIS

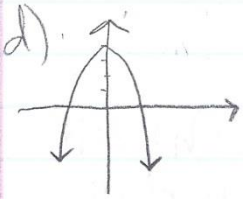
b) $y = 2f(x) - 1$
VERTICAL STRETCH 2; DOWN 1

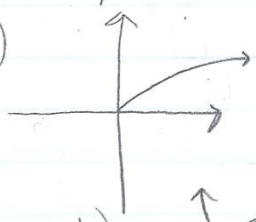
c) $y = f(x-3) + 2$
HORIZONTAL RIGHT 3; UP 2

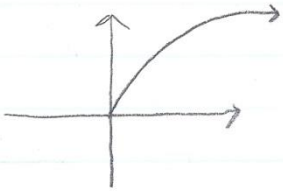
5. a) 

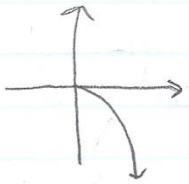
b) 

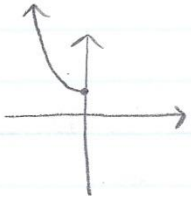
c) 

d) 

e) 

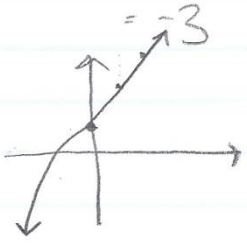
f) 

g) 

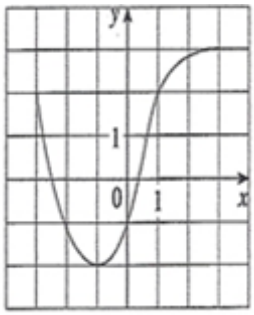
h) 

6. $f(x) = \begin{cases} 1 - x^2 & \text{IF } x \leq 0 \\ 2x + 1 & \text{IF } x > 0 \end{cases}$

a) $f(-2) = 1 - (-2)^2$ $f(1) = 2(1) + 1$
$= 1 - 4$ $= 3$
$= -3$

b) 

# APPENDIX D:
# ANALYSIS OF MATHEMATICS ASSESSED ON DIAGNOSTIC

| # | Information given in problem | Mathematics being assessed | Understanding being assessed |
|---|---|---|---|
| 1. | The graph of a function   | | |
| (a) | $f(-1)$ | Evaluate a function at a point | Procedural |
| (b) | $f(2)$ | Estimate a function at a point | Procedural/ Conceptual |
| (c) | $f(x) = 2$ | Determine a value for x, given $f(x)$. | Procedural/ Conceptual |
| (d) | $f(x) = 0$ | Estimate more than one value for x, given $f(x)$. | Procedural/ Conceptual |
| (e) | | State the domain and range. | Conceptual |
| | | | |
| 2. | $\frac{f(2+h)-f(2)}{h}$ | Evaluate the difference quotient for $f(x)=x^3$ | Procedural/ Conceptual |
| | | Simplify the expression. | Procedural |
| | | | |
| 3.(a) | $f(x)=\frac{2x+1}{x^2+x-2}$ | Find the domain of rational function with polynomial expressions | Conceptual |
| (b) | $g(x)=\frac{\sqrt[3]{x}}{x^2+1}$ | Find the domain of a rational function with a radical and polynomial expression | Conceptual |
| (c) | $h(x)=\sqrt{4-x}+\sqrt{x^2-1}$ | Find the domain of a sum of radical and polynomial expressions | Conceptual |
| | | | |
| 4.(a) | y = -f(x) | Recognize/describe the negative reflects the function over the x-axis | Conceptual |
| (b) | y=2f(x)-1 | Recognize/describe the stretch and translation | Conceptual |
| (c) | y= f(x-3) + 2 | Recognize/describe the two translations | Conceptual |

| # | Information given in problem | Mathematics being assessed | Understanding being assessed |
|---|---|---|---|
| 5.(a) | $y = x^3$ | Recognize/sketch a cubic function | Conceptual |
| (b) | $y = (x+1)^3$ | Recognize/sketch a translation | Conceptual/ Procedural |
| (c) | $y = (x-2)^3 + 3$ | Recognize/sketch a combination of translations | Conceptual/ Procedural |
| (d) | $y = 4 - x^2$ | Recognize/sketch a quadratic with translation/reflection | Conceptual |
| (e) | $y = \sqrt{x}$ | Recognize/sketch a radical function | Conceptual |
| (f) | $y = 2\sqrt{x}$ | Recognize/sketch a translation | Conceptual/ Procedural |
| (g) | $y = -2^x$ | Recognize/sketch an exponential function | Conceptual |
| (h) | $y = 1 + x^{-1}$ | Recognize/sketch a negative exponent as a rational function | Conceptual |
| 6. | $f(x) = \begin{cases} 1 - x^2, if x \le 0 \\ 2x + 1, if x \rangle 0 \end{cases}$ | | |
| (a) | f(-2) and f(1) | Evaluate a piecewise function at two points | Procedural |
| (b) | | Sketch the graph of a piecewise function | Conceptual |
| 7. | $f(x) = x^2 + 2x - 1$<br>$g(x) = 2x - 3$ | | |
| (a) | $f \circ g$ | Compose one function with another function, square binomial, simplify expression | Conceptual/ Procedural |
| (b) | $g \circ f$ | Order of composition, simplify expression | Procedural |
| (c) | $g \circ g \circ g$ | Compose a function with itself | Conceptual/ Procedural |

# APPENDIX E:
# LETTER OF PERMISSION TO USE DIAGNOSTIC

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# APPENDIX F:
# ANSWER KEY TO DIAGNOSTIC

**Answer Key to Student Diagnostic: Functions**

1. (a) -2

   (b) 2.8

   (c) -3,1

   (d) -2.5, 0.3

   (e) $[-3,3],[-3,3]$

2. $12+6h+h^2$

3. (a) $(-\infty,-2)\cup(-2,1)\cup(1,\infty)$

   (b) $(-\infty,\infty)$

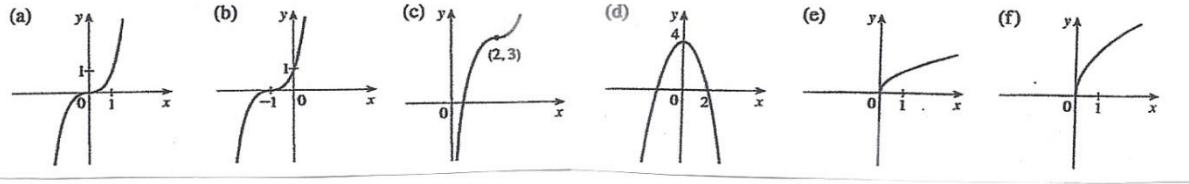
   (c) $(-\infty,-1]\cup[1,4]$

4. (a) Reflect about x-axis

   (b) Stretch vertically by a factor of 2, then shift 1 unit downward

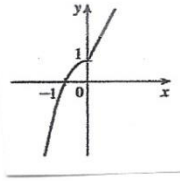
   (c) Shift 3 units to the right and 2 units upward

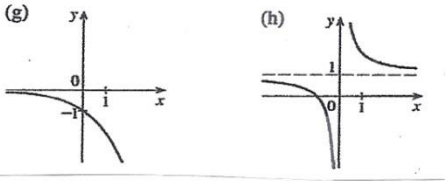
5.




6. (a) -3,3

   (b)





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