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A Multiple Case Study Investigating The Effects Of Technology On Students' Visual And Nonvisual Thinking Preferences Comparing Paper-pencil And Dynamic Software Based Strategies Of Algebra Word Problems

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A MULTIPLE CASE STUDY INVESTIGATING THE EFFECTS OF TECHNOLOGY ON
STUDENTS' VISUAL AND NONVISUAL THINKING PREFERENCES: COMPARING
PAPER-PENCIL AND DYNAMIC SOFTWARE BASED STRATEGIES OF ALGEBRA
WORD PROBLEMS

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ABSTRACT

In this multiple-case study, I developed cases describing three students' (Mary, Ryan and David) solution methods for algebra word problems and investigated the effect of technology on their solution methods by making inferences about their preferences for visual or nonvisual solutions. Furthermore, I examined the students' solution methods when presented with virtual physical representations of the situations described in the problems and attempted to explain the effect of those representations on students' thinking preferences. In this study, the use of technology referred to the use of the dynamic software program Geogebra. Suwarsono's (1982) Mathematical Processing Instrument (MPI) was administered to determine their preferences for visual and nonvisual thinking. During the interviews, students were presented with paper-and-pencil-based tasks (PBTs), Geogebra-based tasks (GBTs) and Geogebra-based tasks with virtual physical representations (GBT-VPRs). Each category included 10 algebra word problems, with similar problems across categories. (i.e., PBT 9, GBT 9 and GBT-VPR 9 were similar). By investigating students' methods of solution and their use of representations in solving those tasks, I compared and contrasted their preferences for visual and nonvisual methods when solving problems with and without technology.

The comparison between their solutions of PBTs and GBTs revealed how dynamic software influenced their method of solution. Regardless of students' preferences for visual and nonvisual solutions, with the use of dynamic software students employed more visual methods when presented with GBTs. When visual methods were as accessible and easy to use as nonvisual methods, students preferred to use them, thus demonstrating that they possessed a more complete knowledge of problem-solving with dynamic software than their work on the PBTs.

Nowadays, we can construct virtual physical representations of the problems in technology environments that will help students explore the relationships and look for patterns that can be used to solve the problem. Unlike GBTs, GBT-VPRs did not influence students' preferences for visual or nonvisual methods. Students continued to rely on methods that they preferred since their preferences for visual or nonvisual solutions regarding GBT-PRs were similar to their solution preferences for the problems on MPI that was administered to them to determine their preferences for visual or nonvisual methods. Mary, whose MPI score suggested that she preferred to solve mathematics problems using nonvisual methods, solved GBT-VPRs with nonvisual methods. Ryan, whose MPI score suggested that he preferred to solve mathematics problems using visual methods, solved GBT-VPRs with visual methods. David, whose MPI score suggested that he preferred to solve mathematics problems using both visual and nonvisual methods, solved GBT-VPRs with both visual and nonvisual methods.

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LIST OF ACCRONYMS/ ABBREVIATIONS

GBT	Geogebra- Based Task
GBT-VPR	Geogebra Based Task with Virtual Physical Representations
MPI	Mathematical Processing Instrument
PBT	Paper-and-pencil Based Task

CHAPTER ONE: INTRODUCTION

Technology plays an important role in mathematics education. With emergent, innovative technologies, perceptions of mathematics have changed, especially during the last three decades. Improvements in technological tools open doors to new methods of teaching and learning mathematics.

The effect of technology can be observed in the teaching and assigning of content/tasks; mathematical activity; working with symbols; conceptualizing; representing; generalizing and developing mathematical concepts, procedures, and skills. According to Heid and Blume (2008), technology affects mathematical activity by impacting the content and tasks of any field of mathematics (eg., algebra) as well as its teaching. In addition to that, technology directly affects mathematical activity by extending or inhibiting student approaches to mathematical tasks. Eventually, mathematical activity and reflection on that activity by working with symbols, conceptualizing, representing and generalizing result in development of procedural and conceptual understanding.

The new technological tools in mathematics education have changed the nature of opportunities for mathematical activities, as well as students' solution methods while they attempt to solve mathematics problems. The content taught in mathematics classes, mathematical activity, and representations are a few components involved in this continuous change (Pierce & Stacey, 2010). Research studies (Janvier, 1987; Kaput, 2002; Lesh, Post & Behr, 1987; Yerushalmy, 2006) showed that students use visual (e.g., graphic) and nonvisual (e.g., algebraic, verbal, and numeric) methods while solving mathematics problems. Visual methods involve visual imagery and nonvisual methods do not involve visual imagery (Moses, 1980; Presmeg, 1985; Suwarsono, 1982). By visual imagery, Suwarsono meant graphic

representations either on paper or in the mind. In this study, methods mostly including graphic representations are considered visual methods; and methods mostly including numeric, algebraic and verbal representations are considered nonvisual solution methods (see Figure 1). The new technological tools support visualization by providing easy access to visual images of mathematical ideas; they also support nonvisual methods by organizing data, as well as computing efficiently and accurately (NCTM, 2000).

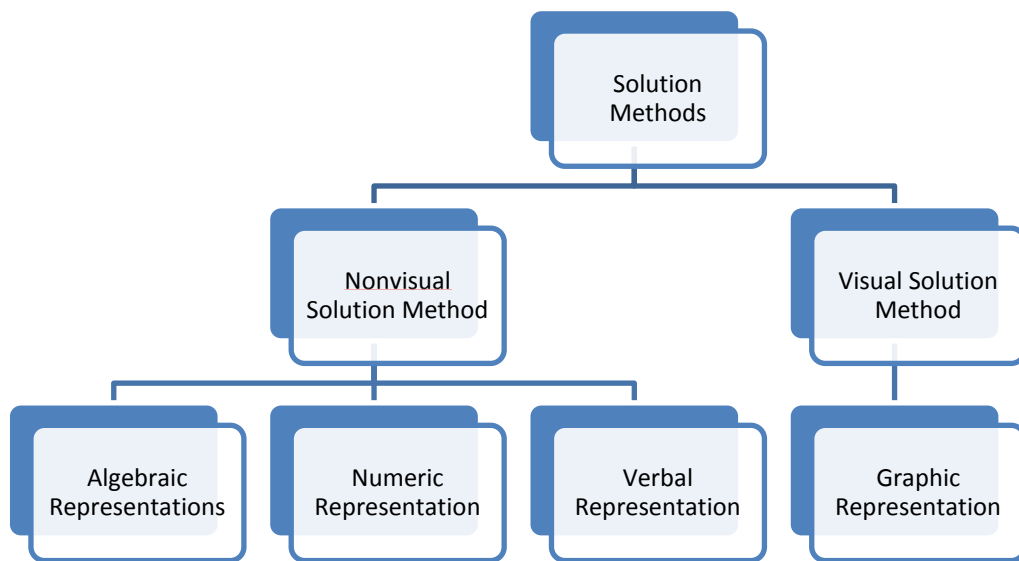


Figure 1. Categorization of students' solutions

By investigating students' solution methods of mathematical problems (visual vs. nonvisual), researchers contend that students have cognitive preferences for visual or nonvisual solution methods (Krutetskii, 1976; Presmeg, 1985a; Suwarsono, 1982). These cognitive preferences determine the methodological type to which students belong. Krutetskii (1976) defines three types of thinkers, namely geometric, analytic, and harmonic, according to students' preferences between verbal-logical and visual-pictorial components of mathematical abilities. In my study, I describe students' thinking based on preferences for visual and nonvisual solution

methods and there are three types of thinkers: visualizers who use mostly visual methods, nonvisualizers who use mostly nonvisual methods, and harmonic thinkers who use both visual and nonvisual methods.

The question “how does using technology to solve algebra word problems influence students’ preferences for visual or nonvisual thinking?” is the question that needs to be answered to determine the effect of technology on students’ mathematical thinking especially in solving algebra word problems. In my study, I attempted to answer this question with specific references to students’ preferences for various representations and solution methods. This study investigates how students with different preferences for solution methods solve algebra word problems with and without technology.

Rationale

Visualization has become an important part of mathematics education with innovative technology. Curriculum reform efforts in mathematics education attest to the importance of visual processes and the use of visual representations (NCTM, 2000). Moreover, connecting visual representations with nonvisual representations is expected to result in deeper understanding of mathematics (Ainsworth, Bibby, & Wood, 1997; Seufert, 2003; van Labeke & Ainsworth, 2001). It is claimed that technology helps build this connection with different computer programs and offers students a broad perspective on mathematics with visual support (Borba & Villarel, 2005). Researchers (Haciomeroglu, 2007; Krutetskii, 1976; Presmeg 1986, Suwarsono, 1982) propose that students have preferences for solution methods (visual and nonvisual) when solving mathematics problems. In their studies, researchers determined students’ preferences for thinking based on visual/nonvisual solution methods by presenting

mathematics problems and asking students to solve them by using paper and pencil. They observed that students differed in methods used to solve mathematics problems.

Haciomeroglu, Aspinwall and Presmeg (2010) observed that students' responses strongly correlated with one representation and weakly associated with the other representations. Due to one-sided thinking and over-reliance on one mathematical representation, students had difficulties in solving tasks that were presented in the interviews. As previously mentioned, researchers claim that technology offers students opportunities to use multiple representations and to connect those representations in mathematics (Kaput, Noss & Hoyles, 2002; O'Collaghan, 1998). Since technology supports multiple representations, there is a need to investigate preferences for students' methods of solution and how students use multiple representations to produce different methods. Knowing the methods of students' solutions and the ways students use the software inform teachers and researchers about students' thinking patterns. Since software that supports multiple representations is new to educators as well as to students, teachers need to better understand how students might think and learn in a technology enhanced environment. This study seeks to investigate the relationship between students' preferences for visual or nonvisual thinking and students' trains of thought when they attempt to solve algebra word problems with and without technology. Therefore, educators will benefit from the results of this study by recognizing the ways students with different cognitive preferences think and solve problems with the given technological tool. The purpose of this study was to reveal whether or not the tendencies students have when solving algebra problems with technology are related to cognitive preferences so that teachers would know students' tendencies as they impacted by technology use in their classroom.. In order help students construct deeper understanding for all students, it is important to know their thinking patterns and problem-

solving tendencies. By knowing their thinking patterns and problem solving tendencies, teachers can diagnose students' learning difficulties when students experience difficulties in solving mathematics problems by using technology.

In this study, I used a software program called Geogebra because it is open source and available to the students and schools with which I worked. This program only requires one time Internet access and can be downloaded or used through a web-browser. Moreover, Geogebra software supports multiple representations by offering multiple tools to graph functions, show algebraic expressions, do numeric calculations, and tools to transfer data from numeric to visual representation. Researchers contend that being able to use each representation and translate between representations will result in an in-depth understanding of mathematics (de Jong & van Joolingen, 1998; Lesh, Behr & Post, 1987).

I worked with high school students that were enrolled in a Pre-Calculus course, so they had experiences with different representations (i.e., graphic, algebraic, numeric and verbal solution) when they attempt to solve algebra problems. I presented algebra word problems and made inferences about students' thinking process as they attempted to solve algebra word problems that are open to multiple solution methods. Additionally, there is a commonly held belief that algebra word problems are notoriously difficult for students (Koedinger & Nathan, 2004). Knowing this common belief, I started to think how students would solve algebra word problems with technology. Since technology offers opportunities for using multiple representations and translating between representations, I attempted to investigate the effect of the technology on students' problem solving strategies for algebra word problems. For those reasons, I chose to focus on algebra word problems in my study.

Before the research study, I conducted a pilot study to field test the tasks that I created in Geogebra. According to pilot study results, I revised, added and eliminated some tasks. Moreover, the results of pilot study showed that students' solutions differed between paper-pencil-based and technology-based environment.

Previous studies (Kaput, Noss & Hoyles, 2002; O'Collaghan, 1998) stated that technology provides tools for students to use different representations in their solution methods; however, there were not a lot of studies that investigated the differences or similarities between their solution methods with and without technology. Examining students' solution methods and comparing and contrasting their solutions performed in different environments provided insight into students' thinking as they solve problems. This study also explained the possible reasons for change in students' solution methods for the problems presented in different media. By knowing the technological tools' influences on student thinking, teachers will be able to make informed decisions about how to integrate those tools into their classrooms.

Research Questions

- What is the nature of students' thinking when solving algebra word problems?
- How do students with different preferences for visual or nonvisual thinking solve algebra word problems with using technology compared to without using technology?

CHAPTER TWO: LITERATURE REVIEW

The literature review for this study includes two parts: (a) a review of the literature on visualization and its relationship with technology and (b) a review of the literature on representation in mathematics education. These concepts play significant roles in understanding students' solution methods of algebra problems. The literature on these concepts and related issues are discussed in this chapter. Since the scope of this study is investigating students' thinking, I start the literature review with visualization. Visualization is an important process that students go through when they are thinking about a solution process. After discussing significant definitions related with visualization, I briefly mentioned the literature on preferences for thinking that will explain differences in students' thinking in terms of visual and nonvisual methods. In the last section, I focused on representations, especially external representations that allows me to make inferences about students' thinking and identify their preferences in visual/nonvisual thinking when they attempt to solve algebra word problems.

Visualization

Research conducted about visualization in mathematics education is widespread. Recent researchers have examined the merits of visualization in teaching and learning mathematics (Arcavi, 2003; Presmeg, 2006). Moreover, researchers have investigated the relationships between visualization and mathematical performance and between visualization and mathematical giftedness (Krutetskii, 1976; Presmeg, 1986a).

Definitions

Visualization is defined differently by different researchers. Ben-Chaim, Lappan, and Houang (1989) characterize *visualization* as the ability to interpret and understand figural information, and the ability to conceptualize and translate abstract relationships and non-figural

information into visual terms. Eisenberg and Dreyfus (1989) emphasize the second part of their definition and state that most of the concepts and processes in school mathematics can be tied to visual representations.

Zimmerman and Cunningham (1991) use *visualization* in mathematics to mean “the process of producing or using geometrical or graphic representations of mathematical concepts, principles or problems, whether hand drawn or computer generated” (p. 1). Gutierrez (1996) considers *visualization* as “the kind of reasoning activity based on the use of visual or spatial elements, either mental or physical, performed to solve problems or prove properties” (p. 9). Zazkis, Dubinsky, and Dautermann (1996) propose a broad definition of *visualization*, which is valid for other contexts as well as mathematics; however, it restricts *visualization* to constructions transforming between mental and external media. In this definition, Nemirovsky and Noble remark that Zazkis, Dubinsky, and Dautermann (1996) define *visualization* as a means of traveling between the student’s mind:

Visualization is an act in which an individual establishes a strong connection between an internal construct and something to which access is gained through the senses. Such a connection can be made in either of two directions. An act of visualization may consist of any mental construction of objects or processes that an individual associates with objects or events perceived by her or him as external. Alternatively, an act of visualization may consist of the construction, on some external medium such as paper, chalkboard or computer screen, of objects or events that the individual identifies with object(s) or process(es) in her or his mind. (Nemirovsky & Noble, 1997, p. 441)

Nemirovsky and Noble (1997) state that this definition disallows the possibility of objects, representations, and graphs present either inside or outside, or both inside and outside, at the

same time. Hence, they object to this dichotomy and introduce a new concept: *lived in space*. They state that “a lived in space is not carried by the individual but created in an ongoing process that involves memories, intentions, and the situation at hand” (Nemirovsky & Noble, 1997, p. 105).

Researchers have varied definitions for much of the terminology that they use: *spatial ability*, *imagery*, and *visual image*, as well as *visualization*. Lohman (1979) defines *spatial ability* as “the ability to generate, retain, and manipulate abstract visual images” (p. 188). *Imagery* is defined by Hebb (1972, cited in Suwarsono, 1985, p. 270) as the occurrence of mental activity corresponding to the perception of an object when the object is not present. Suwarsono (1985, p. 128) defines *visual images* as pictorial representations, either on paper or in the mind. Presmeg (1985, p.174) broadens this definition by defining *visual image* as “a mental scheme depicting visual or spatial information.” Presmeg’s definition includes different types of images such as models, shapes, and pictures in the mind. It also allows for the possibility that a person could spatially arrange verbal, numeric, or mathematical symbols to form an image. Presmeg (1986b) defines five types of visual imagery: *concrete pictorial imagery*, *pattern imagery*, *memory images of formulae*, *kinesthetic imagery*, and *dynamic imagery*. *Concrete pictorial imagery* is the detailed memory images of objects such as memory images of special triangles. *Pattern imagery* is the type of imagery in which a person disregards concrete details and designates pure relationships, such as chess masters’ remembering the places of pieces on a chessboard for a given unfamiliar situation. *Memory images of formulae* are the bearers of abstract information, such as remembering a formula written on a blackboard or notebook. *Kinesthetic imagery* is imagery involving muscular activity, such as tracing out a shape or pattern or making ‘walking’ movements with two fingers. *Dynamic imagery* is the type of

images that are efficient in movement, such as mental transformation of a rectangle into a parallelogram.

Preferences for thinking

Krutetskii (1969, 1976) investigates the relationship between students' spatial abilities and mathematical abilities; his reports emphasize cases of gifted students. He categorizes modes of thought as visual-pictorial and verbal-logical. He identifies *geometric*, *analytic*, and two types of *harmonic thinkers*, according to students' preferences for thinking and the correlation between verbal-logical and visual-pictorial components of mathematical abilities. He characterizes *analytic thinkers*, who have an obvious predominance of a very well-developed verbal-logical component over a weak visual-pictorial one. They tend to operate easily within abstract schemes. The representatives of this group are not marked by ability for visual-pictorial conceptualization. For this end, they use a less efficient and more complicated logical-analytical method of solution, when relying on an image would be a much simpler and more efficient solution.

The second type is the *geometric thinkers* whose representatives are characterized by a very well-developed visual-pictorial component, and it has predominance over a well-developed verbal logical component. These representatives need to visually interpret an expression of an abstract mathematical relationship. Their failure to create a visual support results in difficulty operating with abstract schemes. They show perseverance in operating with visual schemes, even if a problem is easily solved with reasoning.

The third type is the *harmonic thinkers*. They have a relative equilibrium of well-developed verbal-logical and visual-pictorial components. They mostly have well-developed spatial concepts and are successful at implementing both an analytic and pictorial-geometric

approach in solving many problems. Krutetskii defines two types of *harmonic thinkers*, namely *abstract-harmonic* and *pictorial-harmonic*. The former has developed verbal-logical and visual-pictorial components in equilibrium but has an inclination for mental operations without the use of pictorial means. The latter also has equilibrium between the two but has an inclination to mental operations with the use of visual-pictorial schemes.

Krutetskii (1963) also categorizes low-achieving students according to their modes of thinking. He examines the factors that affect poor performance in mathematics. He suggests that high-level development of analytic thinking does not determine mathematical ability, but a low level of development of analytic thinking does result in an incapacity for mathematics. Krutetskii (1976) also concludes that there is a correlation between the ability to visualize abstract relationships and the ability for spatial geometric concepts; however, they are not essential components of mathematical abilities. He states that the strengths or weaknesses of analytic or visual thinking do not determine the extent of mathematical giftedness; however, it reveals its type, which means a person can be mathematically capable with a different correlation between the two components previously mentioned (visual-pictorial, verbal-logical). This correlation determines the type (analytic, geometric, and harmonic) to which a person belongs.

Krutetskii (1976) is the first person who proposed preferences (preferred mode of cognitive processing) rather than ability in mathematics. He posits that ability and preferences are not the same. For example, students might be able solve a problem with visual methods; however, they might not prefer to solve it with visual methods. Following his framework, Moses (1977) defines *degree of visuality* as the extent to which the subject uses visual solution processes to solve a given problem or set of problems. She conducted a study with fifth graders and measured *degree of visuality* by asking ten mathematical problems in the Problem-Solving

Inventory. Students' visuality scores were based on the number of visual solution processes in the written work. She studied the relationships among fifth graders' spatial abilities, problem-solving performance, and *degree of visuality*. Moses found that both problem-solving performance and *degree of visuality* correlated with spatial ability and that the strongest relationship was between spatial-ability and problem solving. Another important finding is the almost-zero correlation between successful problem-solving performance and the use of visual solution processes. Moses offers a possible explanation for this finding and proposes that individuals with high spatial ability may not find the need to write down their solution processes as they attempt to solve problems. The method that Moses uses in this study is criticized by Lean and Clements (1981). They find her method unreliable, since it only considers the written responses. Moses counts the number of visual methods in written responses and determines their *degree of visuality*. Determining *degree of visuality* by examining the students' written responses to the problems may be problematic, since some students may not express their solution processes in their written responses. Therefore, students' responses may not provide enough information on how students obtain results to the mathematics problems. It creates a limitation within the study. In order to resolve this limitation, Suwarsono (1982) developed the Mathematical Processing Instrument (MPI) that includes a second part, which offers possible visual and nonvisual solutions for the given problems. The first part includes thirty mathematical word problems, and the second part consists of written descriptions of different methods commonly used by pupils attempting to solve word problems in Part I. First, the students are asked to solve the problems in Part I, and then they need to indicate which methods they use in Part II. If their methods are different from those mentioned in Part II, the researcher instructs them to say so and describe their methods. Therefore, it offers an opportunity to

understand the solution methods of the students who do not express their solutions in their written responses, which partially solves the problem in Moses's method. It solves the problem partially, since we never know what is exactly happening in students' minds when they are solving problems. In the study of Lean and Clements (1981), 15 out of 30 original questions were used from Suwarsono's instrument. This study provides further evidence of the reliability and validity. In my study, to determine the students' preferences for visual or nonvisual solutions, I used Suwarsono's MPI. In order to analyze students' solutions for paper-pencil and Geogebra-based tasks, I took MPI as a reference point and identified solutions as visual or nonvisual. To investigate the effect of technology, students' solution methods of GBTs were compared with PBTs. In the next section, I will illustrate some examples for the possible visual and nonvisual solutions for both PBTs, GBTs and GBT-VPRs.

Suwarsono defines *visual methods* as the methods that involve visual imagery and *non-visual methods* as the methods that do not involve visual imagery. By *visual imagery*, he means pictorial representations, either on paper or in the mind. Suwarsono argues that even if the pictorial representations were drawn on paper, *visual imagery* is also involved, since before the pictorial representations are put on paper, they first must be imagined in the mind. Thus, a visual method of solution is one that involves any pictorial representation. I used those definitions of visual and nonvisual imagery in my study as they were defined by him.

Different from Krutetskii, Suwarsono objects to classifying students in the three categories of analytic, geometric, and harmonic thinkers. He introduces the term *mathematical visuality* in order to describe the extent (degree) to which someone prefers to use visual methods when attempting mathematical problems that are solvable by both visual and nonvisual methods. Although he rejects Krutetskii's classification, Suwarsono believes that some students do have

preferences for particular modes of thinking (e.g., visual) and that some students are more visual or nonvisual than others.

I analyzed the data collected through interviews, students' work and written responses, and field notes by using Krutetskii's framework (1963, 1969, 1976). In my study, I used the term "nonvisual" to describe analytic students and "visual" to describe geometric students. I also used Krutetskii's harmonic type of thinkers, as defined by him. Since I analyzed students' solution methods and made inferences about their thought processes according to their responses to the tasks, work, and representations, it was relevant to use Krutetskii's framework in my study.

Presmeg (1985, 1986b) refined Suwarsono's Mathematical Processing Instrument (MPI). Her instrument includes 3 parts, A, B, and C, in increasing degrees of difficulty. Parts A and B are intended for high school students, and Parts B and C are intended for mathematics teachers. Presmeg (1985) also conducted interviews with students, in addition to implementing the MPI. After analyzing interviews, she proposed the five types of visual imagery mentioned before.

Kozhevnikov, Hegarty, and Mayer (2002) suggested that the visual-verbalizer cognitive dimension needed to be revised to include two groups of visualizers. Their study revealed that visualizers are not homogeneous groups with respect to their spatial abilities. Some of them have high spatial ability, and some of them have low spatial ability. Moreover, people with low spatial ability primarily have pictorial imagery, and people with high spatial ability primarily have spatial imagery. For this reason, Kozhevnikov, et al. characterize group one with high spatial ability (the spatial type) and group two with low spatial ability (the iconic type).

Their findings help explain why previous studies (e.g., Krutetskii, 1976; Lean & Clements, 1981; Presmeg, 1986a, 1986b, 1992) could not find correlation between use of visual-

spatial representations and problem solving. Since visual and spatial abilities are different than each other. Another study conducted by Kozhevnikov, Kosslyn and Shephard (2005) supports the findings of Kozhevnikov, et al. (2002). Kozhevnikov, et al., (2005) reject the visualizers-verbalizers dichotomy and posit three cognitive style dimensions: verbalizers, object visualizers and spatial visualizers. Object visualizers and spatial visualizers visualize words in different ways. According to their theory, spatial visualizers are more accurate and faster in generating and manipulating dynamic images; whereas object visualizers are more accurate and faster in generating static images. They found that object visualizers have difficulty interpreting graphs but that they are better than spatial visualizers at generating images and recognizing individual shapes.

Furthermore, the relationship between object and spatial visualization abilities is investigated by Kozhevnikov, Blazhenkova, and Becker (2010). Their findings suggest a trade-off, rather than independence between object and spatial visual abilities. In their study, people with above-average object visualization abilities have below-average spatial visualization abilities, and the inverse is also true. There are no groups that show both above-average object- and above-average spatial-visualization abilities.

Visualization and school performance in Mathematics

Different studies report success of visualizers in school mathematics compared to non-visualizers (Lean & Clements, 1981) and the relationship between visualization and giftedness (Krutetskii, 1976; Presmeg, 1986a). According to Lean and Clements's study results, the first-year, non-visual engineering students tended to outperform visual students on spatial tests. The results of their study are criticized by Kozhevnikov, et al. (2005), who posit that being a visual thinker and having spatial ability are separate issues and that visualizers are categorized among

themselves as high and low spatial-ability visualizers. The results of Krutetskii's extensive study (1976) suggest that among the gifted students, there is no dearth of visualizers. Presmeg (1986a) reports that visualizers are underrepresented among mathematical high achievers.

Contradictory results on this relationship might stem from how performance is measured and how understanding mathematics is defined. According to Lesh, Behr, and Post (1987) *understanding an idea* is defined as the ability to recognize and translate the idea embedded in different representations. Researchers (Eisenberg & Dreyfus, 1991; Presmeg, 1986a; Zazkis, Dubinsky & Dautermann, 1996) propose that in order for students to construct rich understandings of concepts, both visual and nonvisual reasoning must be presented and integrated. Therefore, instruction in mathematics needs to support both visual and nonvisual understanding. Moses (1977) claims that instruction on certain visual processes has a significant effect on spatial ability. Suwarsono (1982) found that visual instruction increased students' *degree of visibility*. This finding suggests that teaching methods used by mathematics teachers can significantly influence their students' modes of mathematical thinking. Moreover, developments in technology accelerate efforts in visual instruction and create differences between technologically-aided and traditional visual instruction.

Visualization and technology

Visualization came into prominence in the last two decades, and it has been one of the most discussed topics (Presmeg, 2006), with the popularization of computers and graphing calculators. Many researchers have discussed the effect of technological tools on students' visualization and performance. Kaput (1989) states that new technologies present a world of new dynamic possibilities—external systems where representations can be manipulated with mouse clicks or the drag of a cursor and linked to one another. Visual mathematical topics, such

as geometry, have been attracting much attention by showing technology's effects on visualization.

In Dixon's (1995) study, she found that students who experienced instruction with Geometer's Sketchpad (dynamic software), which supports visualization, outperform students who experience traditional instruction with two-dimensional visualization. Another study conducted by Hollebrans (2007) investigated the methods students employed for using Geometer's Sketchpad when they were engaged in a unit of instruction focused on geometric transformation. In this study, students used dragging and measuring for different purposes. These purposes were affected by students' mathematical understandings. The author made inferences about students' understanding by investigating their reasoning processes about the virtual physical representations, the types of abstractions they made, and the reactive and proactive strategies employed. When students do not know what the next action will be without first seeing the information on the screen, this strategy is called reactive because the choice of action is in response to what the computer produces. However, students using the tool in a proactive manner have certain expectations of what they want to do with this technology.

Parzysz (1999) showed that visualization is not only powerful for apparently visual mathematical topics (e.g., geometry and trigonometry) but also in algebra. In the study by Ferrara, Pratt, and Robutti (2006), which discusses the ideas from *Psychology of Mathematics Education* over the last 30 years, they mention that, in this time frame, technology shaped the way algebra has been perceived. They emphasized that algebra's symbol system has been linked more powerfully to tabular, geometric, and graphic contexts with the help of technology. In Yerushalmy's (1997) study, students experienced positive effects of using technology in learning algebra, particularly in their thinking about symbols, equations, and problems in context. Further

in 1999, Yerushalmy, Shternberg and Gilead focused on some of the special advantages of computer software that encourages visualization.

Yerushalmy (2006) studied ways less successful mathematics students use graphic software when solving algebra word problems. In this study, Yerushalmy found that there is a difference between the methods used by less successful students and the traditional problem-solving patterns of successful students. The aims of less successful students' uses of the software, when solving algebra problems, include obtaining a broader view, confirming conjectures, and completing difficult operations. Their attempts to solve problems focused mostly on numeric and graphic representations, and they delayed using symbolic formalism. Moreover, solving problems took a long time for them, and they did not use the graphic software at all because it did not support symbolic formulation and manipulations. In contrast, successful students used the tool at the early stages of the solution. They used software immediately to complete their solutions and confirm their mental plan.

In another study, Doerr and Zangor (2000) also investigated how students used technology in a classroom, specifically Pre-Calculus students using graphing calculators. They concluded that there were five patterns and modes of graphing calculator use which emerged in their study: a computational tool, transformational tool, data collection and analysis tool, visualizing tool, and checking tool. Their results suggest that the graphing calculator is a rich and multi-dimensional tool.

Technological tools not only support visual representations, but they also show the connections between different representations. NCTM (2000) attests to the importance of students' abilities in selecting, applying, and translating among mathematical representations. As mentioned before, *understanding the idea* is the ability to recognize the idea embedded in

different representations, to manipulate the idea within given representations, and to translate the idea from one representation to another. According to Aspinwall and Shaw (2002), students gain deep understandings of the concepts once they investigate and synthesize the relationships between graphic and analytic representations. The production of effective mathematics is possible through coordination of graphic representations with numeric and algebraic representations (Borba & Villarreal, 2005). Technology helps to coordinate different representations, and there are computer packages specifically designed to serve the purpose of this coordination such as Geogebra (Hohenwarter, 2002), and The Geometer's Sketchpad (Jackiw, 1995). In my study, I used software that was created for supporting multiple representations and translations between representations, namely Geogebra.

Preferences for thinking and their relation to technology

The impact of preferences for thinking in students' learning and performance while using technology has become an important subject in the last decade (Parkinson & Redmond, 2002). The studies on this topic reveal different results. Pitta-Pantazi and Christou (2009) studied the impact of instruction with a dynamic geometry environment on students' performance on measurement tasks, such as determining the areas of triangles and parallelograms. They also investigated whether using dynamic geometry reduces the cognitive load of students by accommodating different preferences for thinking. The results of the study showed that preferences for thinking are not related to students' performance on measurement tasks. In this study, verbalizers benefited more from visual instruction with dynamic geometry software. However, those results contradicted the findings of previous research studies in mathematics education (Bishop, 1989) that stated that optimum learning occurs when people are taught in a way that matches their preferences for thinking. Riding and Douglas (1993) also found that

verbalizers performed better than imagers in a text-based environment, while imagers outperform verbalizers whenever the presentation is graphic and visual. In another study that compares learning with achievement, in the context of computer-aided learning and traditionally-taught environments, Atkinson (2004) found differences in the learning and performing when different teaching methods were employed in different teaching situations. In this study, verbalizers showed a more positive attitude toward computers, performed better, and gained a greater benefit compared to analytic/imagers, who gained less from computer-aided instruction in chemistry. Riding and Grimley (1999) conducted a similar study that compared traditional instruction with CD-ROM multi-media packages on science topics. They investigated the relationship between preferences for thinking and performance by teaching students with both traditional and multi-media-based instruction. Riding and Grimley use the definitions Riding developed with Cheema (1991) to describe preferences for thinking in two dimensions: (a) the Wholist-Analytic style in which an individual tends to process information in wholes or parts, and (b) the Verbal-Imagery style in which an individual is inclined to represent information by thinking verbally or in mental images. Riding and Grimley reported the results of this comparison as follows:

a) In terms of overall science performance (traditional and multi-media), Wholist-Imagers and Analytic-Verbalisers were superior to Wholist-Verbalisers and Analytic-Imagers, and (b) Analytics did better on traditional work than multi-media, with the reverse for the Wholists. With the multi-media materials there were three modes of presentation—picture and sound (PS), picture and text (PT), and picture, text and sound (PTS). Female Wholist-Imagers and Analytic-Verbalisers were better with PS than PT, with the reverse for Wholist-Verbalisers and Analytic-Imagers, while the opposite applied for males. For all style and gender groups performance was best with PTS (p. 1).

Grimley also independently studied multi-media use for learning (2007). He explored whether or not the principles of cognitive load and the multimedia theory are mediated by preferences for thinking and gender. According to the cognitive load theory (Sweller, 1988), people divide information they process in working memory into two units that are differentiated by modality, thus utilizing the capacity of both a visuo-spatial scratch pad that manipulates the visuo-spatial information and a phonological loop that processes verbal material and the central executive controls. In this way, more cognitive capacity should be available whenever information is split between the auditory system and the visual system. Mayer (1999, 2001) extends this theory and posits a theory of multi-media learning in terms of an information-processing model. Grimley (2007) investigates the effects of preferences for thinking on those two theories. He tests these effects by creating two conditions. In Condition 1, children are presented with diagrams supported by printed textual format, and in Condition 2, the same diagrams are supported by narrated text. According to the results, male imagers' and female verbalizers' performances are higher in Condition 2 than in Condition 1. However, male verbalizers' and female imagers' performances are lower in Condition 2 than in Condition 1.

The relationship between preference for thinking and the use of technology has started to take an important place in educational technology. Researchers have investigated this relationship as technology has advanced to offer different tools for thinking and application in mathematics. In the following section, I provide an explanation of the types of representations, which is the main idea behind creating different tools for technology users.

Representation

Introduction

Representation is an important topic for this study, and researchers have studied it more in the last three decades (Presmeg, 2006). Researchers defined, categorized, and examined representation and developed models of representation. For example, Davis, Young and McLoughlin (1982) define *representation* as a combination of something written on the paper, something existing in the form of physical objects, and a carefully constructed arrangement of an idea in one's mind. Basically their definition of *representation* is a combination of external and internal representations. Since the main purpose of this study is to examine students' solution methods, which are external representations, it is important to make the distinction between internal and external representations.

Goldin and Shteingold (2001) distinguish external systems of representation from internal representations. They define *external systems* as a conventional symbol system of mathematics (e.g., base-ten numeration, formal algebraic notation, the real number line, Cartesian coordinate representation) and structured learning environments (e.g., those involving concrete manipulative materials or computer-based microworlds). *Internal systems* include students' personal symbolization constructs and assignments of meanings to mathematical notations, as well as their language, their visual imagery and spatial representation, their problem-solving strategies and heuristics, and their effects in relation to mathematics. Goldin and Shteingold's examination of the interaction between internal and external representations determine that internal representations are the result of external representations that a teacher uses while teaching. Kaput (1987a) uses another categorical system for representations and outlines four categories: (1) cognitive and perceptual representations, (2) explanatory representations involving models,

(3) representations within mathematics, and (4) external symbol representations. He uses the term *cognitive and perceptual representations* instead of *internal representations*.

Internal representations are not observed directly; however, we make inferences about students' internal representations on the basis of their interactions with, discourse about, or production of *external representations*. In this study, I investigated students' *external representations* and made inferences about their *internal representations* to examine the nature of their understanding.

Models for representation

According to Kaput (1987a), mathematics is a study in the representation of one structure by another, and the focus is usually to determine what structure is preserved in that representation. Palmer (1978) defines *representation* as something that stands for something else. His description implies the existence of two related but functionally separate entities, which are called the *represented world* and the *representing world*. Those two worlds should have some type of correspondence between some of their aspects. However, not all aspects of the *represented world* need to be modeled and not all aspects of the *representing world* need to model an aspect of the *represented world*. Preserving information about the *represented world* is the main function of the *representing world*. Palmer defines *information contained in the two worlds* as the set of operational relations among objects. For example, if a represented relation, R , holds for ordered pairs of represented objects, $\langle x, y \rangle$, then the representational mapping requires that a corresponding relation, R' , holds for ordered pairs of represented objects, $\langle x', y' \rangle$.

Palmer claims that any particular specification of *representation* should describe the following five entities: (1) the *represented world*, (2) the *representing world*, (3) which aspects

of the *represented world* are represented, (4) which aspects of the *representing world* are represented, and (5) the correspondence between the two worlds.

Janvier (1987a) considers *representation* as a combination of three components that are symbols (written), real objects, and mental images. However, Vergnaud (1998) stresses that *representations* are dynamic processes, as opposed to static things. Therefore, he does not accept the metaphor of the triangle, as illustrated in Figure 2.

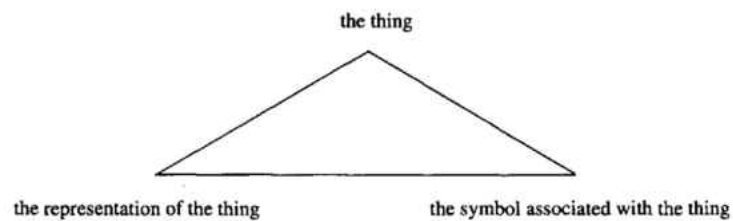


Figure 2. The metaphor of the triangle.

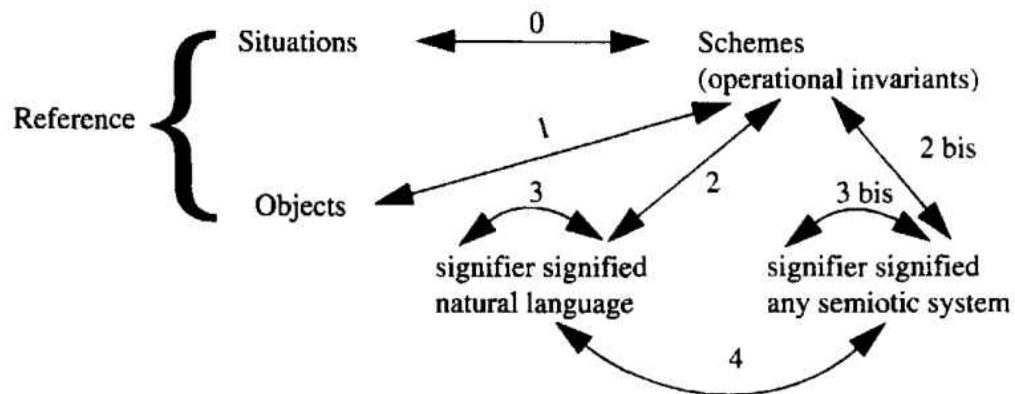
From "A Comprehensive Theory of Representation for Mathematics Education," by G. Vergnaud, 1998, *The Journal of Mathematical Behavior*, 17, p.168. Copyright 1998 by *The Journal of Mathematical Behavior*.

Vergnaud analyzes the concept of scheme, which is an invariant organization of behavior for a certain class of situations. According to him, theorems-in-action and concepts-in-action are operational invariants and essential components of schemes. He states:

Concepts-in-action: In every action, we select a very small part of the information available. Nevertheless we need a wide variety of categories for this selection to take place, if one takes the word "category" to figure the wide meaning of object, class, predicate, condition, etc. Concepts-in-action are relevant, or not relevant, or more or less relevant, to identifying and selecting information... Theorems-in-action can be true or not. This is a strong property as it offers the only possibility of making more concrete the idea of computability and computable representation. If a theory of representation is to be at all useful, it must contain the idea that representation offers possibilities for some

inferences to take place. Representation enables us to anticipate future events, and to generate behavior to reach some positive effect or avoid some negative one (p. 173).

Vergnaud suggests a more comprehensive theory for representation in comparison with the triangle. The reality may convey two different ideas: reference to objects and reference to situations. The connection between knowledge-in-action and knowledge-in-texts is only constructed by operational invariant, and the transformation of operational invariants into words and texts is not straightforward. This action requires the learning and practicing of natural language. However, those words and texts do not express exactly what each individual has in mind when he or she is facing a situation, selecting information, and processing it. This phenomenon creates a gap between what is represented in an individual's mind and the meanings of the words. Therefore, one cannot just consider operational invariants to be the same thing as the signified side of language or any other semiotic system. He suggested the model illustrated in *Figure 3*.



- Arrow 0: the relationship between situations and schemes is the first source of representation, and therefore conceptualization.
- Arrow 1: the formation of concepts implies the identification of objects, with their properties, relationships, and transformations. This is the main function of operational invariants, which are essential components of schemes.
- Arrows 3 and 3bis: a language is a system of signifier/signified; a semiotic system also. The relationship between signifier and signified is not usually a one-to-one correspondence.
- Arrows 2 and 2bis: the relationship between operational invariants and the signified side of a particular linguistic or semiotic instance is not a one-to-one correspondence either.
- Arrow 4: natural language is a metalanguage for all semiotic systems. Again there is no one-to-one correspondence.

Figure 3. An alternative representation to the triangle.

From “A Comprehensive Theory of Representation for Mathematics Education,” by G. Vergnaud, 1998, *The Journal of Mathematical Behavior*, 17, p. 177. Copyright 1998 by *The Journal of Mathematical Behavior*.

Goldin (1987) calls a representational system to higher-level language. Each higher-level language needs to have the following:

- A reasonable degree of complexity
- Some rules analogous to “rules of grammar”
- The capability of being used to symbolize a reasonably wide variety of things
- Use or potential use by human beings to communicate with each other

Goldin also suggests a model for competence in mathematical problem solving based on five “higher level languages.” These are listed as follows:

- 1) A verbal/syntactic system

- 2) Nonverbal systems for imagistic (visual-spatial, auditory, kinesthetic) processing
- 3) Formal notational systems of representation
- 4) A system for heuristic planning and executive control abbreviated as planning language
- 5) An effective system that monitors and evaluates problem-solving processes.

There is an interaction between those languages in order to have a mental representation of problem solving. These interactions and a suggested model for high-level languages are illustrated in *Figure 4*.

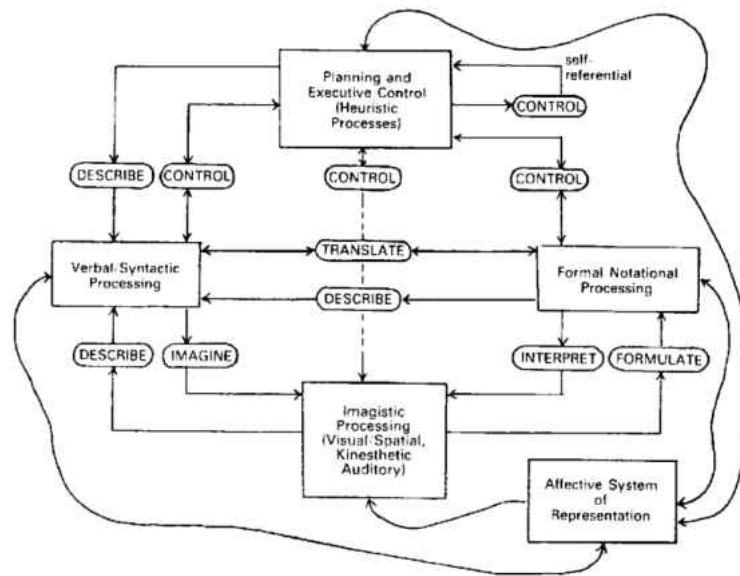


Figure 4. A model for competency in mathematical problem solving.

From *Problems of Representations in the Teaching and Learning of Mathematics* by G. Goldin, 1987, p. 136. Copyright 1987 by Lawrence Erlbaum Associates.

Cifarelli (1998) also examines the development of mental representations in problem solving by incorporating a constructivist perspective, which posits that mental representations evolve as mathematical conceptions. He claims that students develop three increasing levels of structural knowledge while solving mathematical problems (See Figure 5).

	Level of Activity	Attributes	Examples
<div style="display: flex; align-items: center;"> <div style="margin-right: 10px;"> <div style="text-align: center;">High</div> <div style="text-align: center;">↑</div> <div style="text-align: center;">Low</div> </div> </div>	Structural Abstraction	Solver can “run through” potential solution activity in thought and operate on its results	Solver can draw inferences from results of potential activity without the need to carry out solution activity
	Re-Presentation	Solver can “run through” prior activity in thought	Solver can anticipate potential difficulties
	Recognition	Solver encounters new situation and identifies activity from previous tasks as relevant for solving current task	Solver recognizes diagrammatic analysis activity as appropriate for solving Tasks 2–9

Figure 5. Levels of conceptual structure.

From “The Development of Mental Representations as a Problem Solving Activity,” by V. V. Cifarelli, 1998, *The Journal of Mathematical Behavior*, 17, p. 246. Copyright 1998 by *The Journal of Mathematical Behavior*.

The results of his study suggest that traditional views of representation need to be reconsidered. Moreover, a new perspective that acknowledges both the constructive function of representation in the development of conceptual knowledge and the resulting mental objects, which solvers can then reflect on and transform as they interpret problem situations, need to be adopted. The issue with cognitive theories is that they adopt a single perspective to study representations, and according to this perspective, they examine the students’ abilities to recognize similar problem types as a sophisticated problem-solving ability. Second, the nature of the representation is much more dynamic than it was considered to be in traditional theories of mathematics learning. Third, the students in this study performed increasingly abstract levels of solution activity while solving the problems; therefore, there is a need to address qualitative aspects of mathematical performance.

Goldin and Shteingold (2001) define *representation* as a sign or configuration of signs, characters, or objects. A *representation* can stand for (symbolize, depict, encode or represent) something other than itself. The thing represented can vary according to the context or the use of

representation. For example, the numeral 5 can represent a set of five objects, or it can stand for an equivalence class of such sets. Goldin and Shteingold also stated that systems of external representation are structured by the conventions that underlie them. When the system is established, patterns in the system are no longer arbitrary. They are ready to be discovered. For instance, a mathematician creates a system of rules (e.g. orthogonal pair of real number lines and identification of a point in the plane with a pair of real numbers) to construct Cartesian graphs.

Another important property of *representation* is its dual nature. The representing relation can go in either direction. A circle of radius 1 centered at the origin of a Cartesian plane could provide a geometrical representation of an equation in two variables. The other way, an equation of the circle ($x^2 + y^2 = 1$) could provide an algebraic symbolization of the circle in a Cartesian plane.

Goldin and Shteingold also summarize kinds of systems of internal cognitive representations. Verbal/syntactic representational systems include individual's natural language capabilities. Imagistic systems of representation consist of the visual and spatial cognitive configurations or "mental images." They also include kinesthetic encoding, related to actual or imagined hand gestures and body movements. Auditory and rhythmic constructs are also important internal cognitive constructs. Formal notational representations are internal constructs whenever students mentally manipulate numerals, perform arithmetic operations, or visualize the symbolic steps in solving an algebraic equation.

The mentioned models include different aspects of representations and play important roles in the formation of the definition of *representation*. The relationships between different representations also play crucial roles in the topic of representations. These are explained in the next section.

Translation models and difficulties

Janvier (1987b) examines the translation processes in mathematics education. By *translation processes*, he means the psychological processes involved in going from one type of representation to another type of representation. He also labels the processes of translating between two modes of representation. He limits the modes of representations to four: verbal description, table, graph, and formula (equation). *Figure 6* lists these modes.

To From	Situations, Verbal Description	Tables	Graphs	Formulae
Situations, Verbal Description		Measuring	Sketching	Modelling
Tables	Reading		Plotting	Fitting
Graphs	Interpretation	Reading off		Curve fitting
Formulae	Parameter Recognition	Computing	Sketching	

Figure 6. Translation processes.

From *Problems of Representations in the Teaching and Learning of Mathematics* by C. Janvier, 1987, p. 30. Copyright 1987 by Lawrence Erlbaum Associates. Reprinted with permission.

Although the table lists names for the translation between two modes of representation, it is not always the path that students use while solving problems. Janvier (1987b) states that these translations are not always directly carried out. For example, the translation “table → formula” is often carried out as “table → graph → formula.”

According to Lesh, Behr, and Post (1987) *representations* are external embodiments of students’ internal conceptualizations. They define five types of *representations* (see Figure 7), listed as follows : (1) experience based “scripts”—in which knowledge is organized around “real world” events that serve as general contexts for solving and interpreting and solving other kinds of problem situations; (2) manipulatable models—like Cuisenaire rods, arithmetic blocks, fraction bars, number lines; (3) pictures or diagrams—static figural models that can be

internalized as “images”; (4) spoken languages—including specialized sublanguages related to domains like logic; and (5) written symbols that, like spoken languages, can involve specialized sentences and phrases.

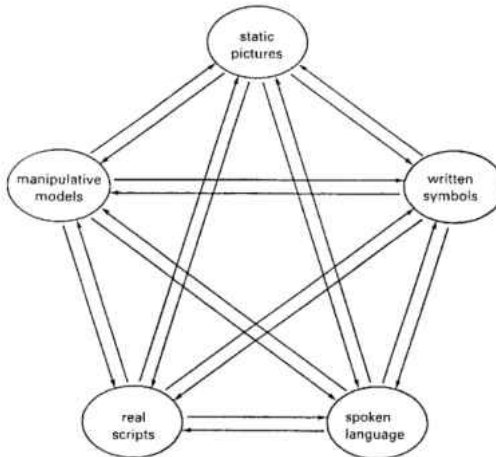


Figure 7. Types of representation systems.

From *Problems of Representations in the Teaching and Learning of Mathematics* by R. Lesh, T. Post, and M. Behr, 1987, p. 34. Copyright 1987 by Lawrence Erlbaum Associates.

Lesh, Behr, and Post suggest that *representations* tend to be plural, unstable, and evolving. The act of representation is plural, since solutions are characterized by several partial mappings from components of the given situation to the parts of several representational systems (see Figure 8).

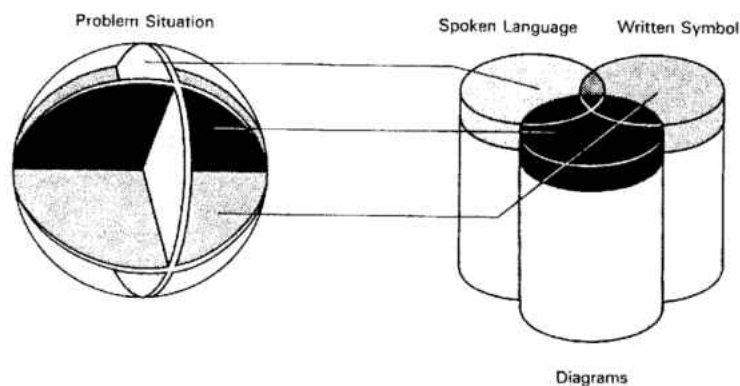


Figure 8. The act of representation.

From *Problems of Representations in the Teaching and Learning of Mathematics* by R. Lesh, T. Post, and M. Behr, 1987, p. 38. Copyright 1987 by Lawrence Erlbaum Associates.

Representational systems are plural in another sense, since a student may start to solve a problem by translating it to one representational system and may then map from this system to yet another system (see Figure 9).

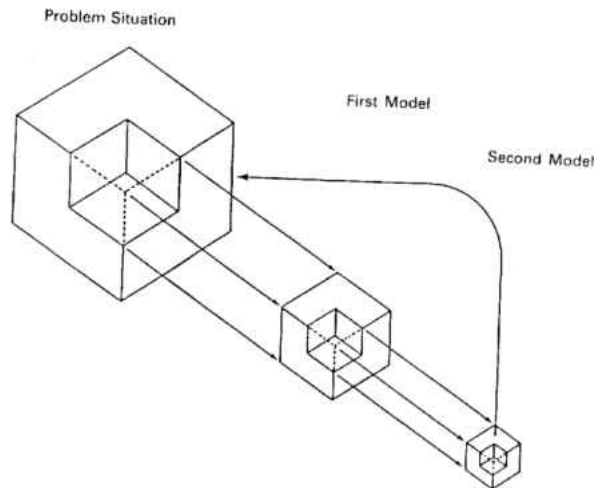


Figure 9. Problem situation.

From *Problems of Representations in the Teaching and Learning of Mathematics* by R. Lesh, T. Post, and M. Behr, 1987, p. 39. Copyright 1987 Lawrence Erlbaum Associates.

Janvier (1987a) widens the concept of translation by using *schematization* to discuss a more specific process within *representation*. According to him, a translation between schematizations is performed within *representation*. For representations, he suggests a star-like iceberg model that will show one point at a time, as illustrated in Figure 10. In this model, a translation would consist of going from one point to another.

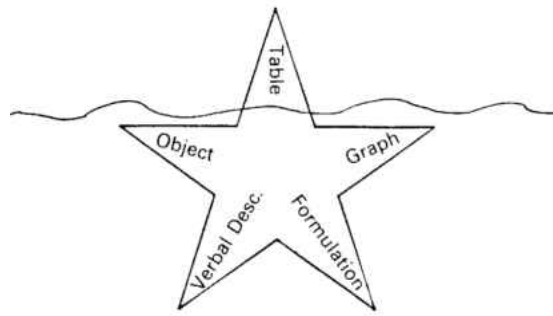


Figure 10. Representation as a star-shaped iceberg.

From *Problems of Representations in the Teaching and Learning of Mathematics* by C. Janvier, 1987, p. 69. Copyright 1987 by Lawrence Erlbaum Associates. Reprinted with permission.

Janvier (1987a) investigates students' major difficulties, which include contaminations coming from close schematizations. These translations are from verbal to formula, from graph to picture, and from verbal to graph. An example of contamination from verbal to formula is one of the results from the study of Clement and Kaput (1979). They found that students have difficulty with the following problem: "At a certain university, there are six times as many students as there are professors." More than twenty percent of the freshman engineering students wrote $6S = P$, and the percentage is higher than fifty percent when the nontrivial ratio (when the ratio is not whole number or unit fraction such as $2/3$) is used. The second contamination from graph to picture might be illustrated with the racing-car test item in Janvier's study (1978). The study shows that students tend to see in a graph a total or partial picture of some situation involving the variables with which they deal. The third contamination, which is from verbal to graph, might be observed in children's difficulty of "expurgating" the expression "grow fast" from the idea of being tall that it insidiously contains.

Vergnaud (1987) emphasizes that the translation problem from one symbolic system to another, from natural language to algebra and back for instance, could not possibly be solved if natural language did not refer to the real world and did not convey ideas about properties,

relationships, and transformations. Therefore, in order to understand what subjects do and what they say, the representing/represented duality or the referent/referred distinction is not enough to model the situation. Vergnaud (1987) suggests a new model and identifies two types of translation problems. The number 1 problem is between the referent and the signified, and the number 2 problem is between the signified and the signifier (see Figure 11).

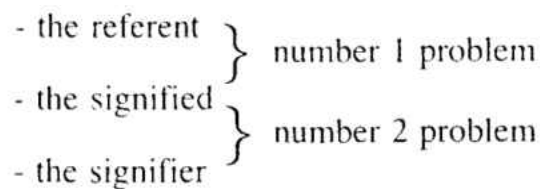


Figure 11. Translation problems.

From *Problems of Representations in the Teaching and Learning of Mathematics* by G. Vergnaud, 1987, p. 228. Copyright 1987 by Lawrence Erlbaum Associates.

In this model, the referent is the real world as it appears to the subject from his experience. The signified level is where invariants are recognized, inferences are drawn, actions are generated, and predictions are made. The signifier level includes different symbolic systems such as the number 1 problem is the problem of adequacy between the signified level of representation and the real world. There are two types of the number 2 problem. These problems are the correspondence problem between the signifier and the signified and the existence or nonexistence of a symbol expressing a cognitive entity.

In my study, the external representations are analyzed in four categories namely, algebraic, numerical, verbal, and graphic representations. The first three of them are considered as nonvisual and the last one is considered as visual representations. By examining these external representations, I made inferences about students' internal representations.

The importance of representation in Mathematics

Lesh, et al. (1987) emphasize the importance of representations in mathematics. They state that students have deficient understandings of the models and languages needed to represent (describe and illustrate) and manipulate these ideas. Those deficiencies have an effect on mathematical learning and problem solving performance. They define understanding an idea as a person (1) recognizing the idea embedded in a variety of qualitatively different representational systems, (2) flexibly manipulating the idea within given representational systems, and (3) accurately translating the idea from one system to another. Lesh, et al. mention that good problem solvers are flexible in their use of different representational systems and that they switch to the most convenient representation at any point in the solution process. Yerushalmy (2006) also supports this idea by stating that developing competence by solving real-world problems means learning to move freely along the tetrahedral path described in Figure 12. Therefore, complete understanding of an idea requires competence in using representations and translating among representations.

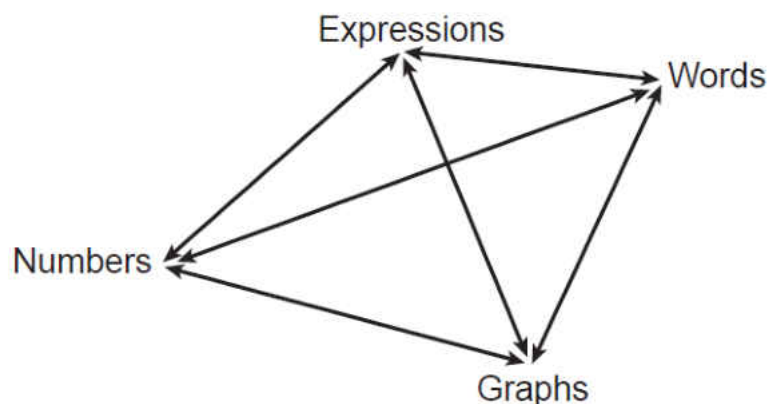


Figure 12. The tetrahedral relations of function representations.
 From “Slower Algebra Students Meet Faster Tools: Solving Algebra Word Problems with graphing software,” by M. Yerushalmy, 2006, *Journal for Research in Mathematics Education*, 37, p. 358. Copyright 2006 by the *Journal for Research in Mathematics Education*.

NCTM (2000) also gives special importance to *representation* and includes it in the *Principles and Standards for School Mathematics*. According to the standards, students do the following:

- Create and use representations to organize, record, and communicate mathematical ideas
- Select, apply, and translate among mathematical representations to solve problems
- Use representations to model and interpret physical, social, and mathematical phenomena

Abrams (2001) focuses on the importance of mathematical modeling, or using mathematics to study questions outside the discipline. In the modeling process, he suggests that the abilities to analyze units, make choices among alternative forms of representations, and recognize common structures should not be ignored. These abilities become important when we teach mathematics as a tool both for solving important problems from other disciplines and for making beautiful abstract discoveries. He mentions that modeling is a cyclical process. By using mathematical modeling, students construct a connection between the real world and the mathematical world. He summarizes the modeling process in Figure 13.

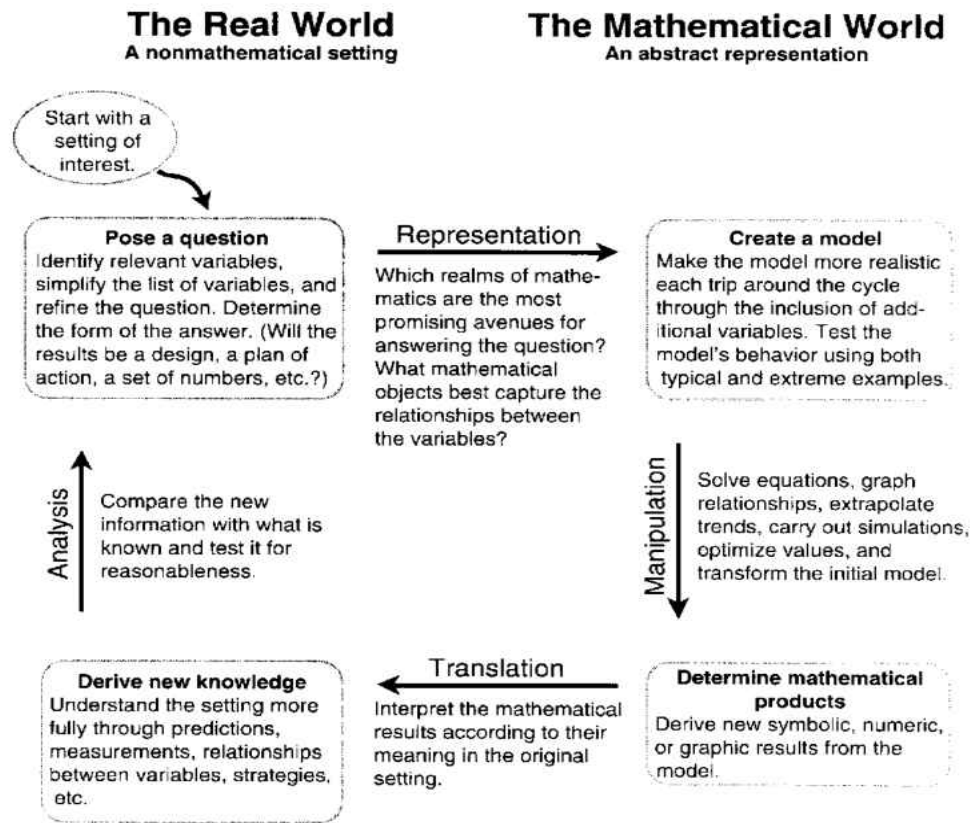


Figure 13. The mathematical modeling cycle.
 From *The Roles of Representation in School Mathematics* by J.P. Abrams, 2001, p. 272.
 Copyright 2001 by the National Council of Teachers of Mathematics.

In the first stage, which is the creation of a mathematical representation, students are challenged with an unfamiliar problem. Students need to translate an imprecise, complex, multivariate situation into a simpler and more clearly defined mathematical structure. In this process, students need to determine what they know and what they need to know about their problem. When they choose a specific type of representation, they must think about the important relationships and structures between and among variables in selecting the mathematical realm that offers the best possibilities for expressing all these features and mathematically capturing the meanings.

Other than connecting the real and mathematical world, the results from the study conducted by Dufour-Janvier, Bednarz and Belanger (1987) suggest the motives for using external representations in mathematics teaching. According to them, representations are characterized by the following:

- Inherent parts of mathematics
- Multiple concretizations of a concept
- Used locally to mitigate certain difficulties
- Intended to make mathematics more attractive and interesting. (pp. 110-111)

As stated before, *representation* is an important aspect of understanding mathematics. For this study, my purpose is to examine the nature of students' understanding of mathematics. I fulfilled this purpose by investigating their external representations. The nature of students' understanding and internal representations are not observable phenomenon. A person's thinking cannot be completely explained by others who examine that person's representation; however, there is a connection between external and internal representations. Students' external representations may offer clues about their internal representations and the nature of their understanding.

TechnologyThe effect of technology on mathematics education is one of the famous research topics for the last three decades. There are a lot of technological tools used in mathematics classrooms (e.g., smart boards, projectors); however, the scope of this study is the technological tools that can create graph of functions, solve equations, and connect visual representations of mathematical expressions with nonvisual representations since the software I used in this study is specially designed to support different representations and connect visual

representations with nonvisual representations. Hence the literature review is based on the effect of graphing technology on students' mathematical understanding.

It is also important to mention that mathematical understanding (proficiency) is a broad concept. In my study, the focus is the effect of technology especially on improvement in representational fluency (being able to translate between representations) and having a wider repertoire of solution methods.

Representational Fluency

According to a report of the National Research Council (NRC, 2001), conceptual understanding constitutes the important part of mathematical proficiency. NRC (2001) identifies the significant indicator of conceptual understanding as the ability to use different representations for mathematical situations and knowing how different representation forms can be utilized for different purposes. Hence, there is a close relationship between representational fluency and conceptual understanding and increasing representational fluency results in development of conceptual understanding (Lesh, Post & Behr, 1987). Multiple representations theories contend that the most important factor for understanding a mathematical idea is the ability to link representations and to gain representational fluency and interpret mathematical ideas in distinct representations (Pierce & Stacey, 2010).

Ruthven (1990) investigated the effect of graphing calculators on students' representational fluency especially translating from graphic to symbolic forms of functions. He compared the mathematical performance of students that use graphing calculator as a standard tool with the students of similar background but not having regular access to graphing technology. The mathematical performance test included two types of items. The first type was symbolization items, asking to write the algebraic forms of the functions given the graphic form.

The second type was the interpretation items asking to interpret the graph in the context of the problem and extract the information from the graphic representation of the problem. The study results showed the difference between two groups especially in symbolization items. The experimental group (used graphing calculator in their mathematics classes) outperformed the control group on symbolization items; however, there were no significant differences between the two groups on interpretation items.

According to study results, Ruthven claimed two reasons that might explain the results of the study. The first claim was that the students in experimental group improved representational fluency by using the graphing calculator in their mathematics classes. The second claim was that availability of graphing calculators in the exam for the experimental group might affect the performance on the symbolization items since the students in the experimental group could check their conjectures by using graphing calculators. The second claim does not support the results; however, it reveals the limitation of the study. In Ruthven's study, since there is no significant differences in interpretation items between the two groups, and the success in the symbolization items might be attributed to access of graphing technology, the results of the study do not confirm that the effect of graphing calculator on representational fluency results in conceptual understanding.

Choi-Koh (2003) also investigated the effect of graphing calculator use on student's learning. His study results revealed the improvement of a student's performance in translating between graphic and algebraic representations of trigonometric functions even though the graphing calculator was absent while solving problems of trigonometric functions. The findings of the study are very important; however, he could not generalize the results since he conducted

tutoring series with only one student. Hence, methodological concerns limit the results of those studies.

Hennesy, Fung, and Scanlon (2001) conducted a study with calculus students to investigate the effect of graphing calculators on learning calculus. They indicated that using graphing calculators encouraged translation between different representations. Since there was no control group to compare the results with, the generalizability of the results is limited. In another research study conducted by Huntley, Rasmussen, Villarubi, Sangtong and Fey (2000) compared the impact of new curriculum (Core-Plus Mathematics Project or CPMP) with the integrated graphing calculator use and the traditional curricula on growth of students' understanding of algebra especially functions. They tested students' understanding by using problems which include realistic contexts and context-free symbolic manipulations. The results indicated that students experienced with calculator use and taught with new curriculum outperformed the students who do not have access to graphing calculators on realistic context problems. In addition to that graphing calculator experienced group showed better performance in representational fluency than the control group. However, students taught with traditional curricula and no access to graphing calculators outperformed the other group on context-free symbolic manipulations. For this end, the study showed both advantages and pitfalls of technology. Since there was also a difference in the curricula between the two groups, the results cannot be attributed to only the effect of technology.

Another striking result of the study is even though there was a statistically significant difference in representational fluency, both groups showed low performance. Graphing calculators might increase the representational fluency; however, the effective use of the calculator also depends on the ability to translate between representations. Therefore the lack of

understanding in representational fluency might cause inefficient use of calculators, so using calculators might not create an expected effect on the performances of students. Micheltore and Cavanaugh (2000) indicated that students' inefficient calculator use and misinterpretation of the displayed results on the calculator screen is the result of deficiency in understanding the relations between multiple representations. Marchand, McDevitt, Bosse and Nandakumar's study (2007) also revealed that graphing software does not always show the correct graph. Hence, the efficient use of graphing calculators requires the knowledge about the family of functions or an estimation of the behavior of functions' graphs from given algebraic or numeric representations.

Slavit (1998) conducted a multiple case study with three students. His study results indicated that students have problems in translating between algebraic and graphic representation. Moreover, students mostly tended to work with algebraic representations, although they were taught with graphical approach and using graphing calculators. When students solved the problems in the test, they used graphing calculators only when they were asked to solve with graph. Slavit concluded that students interpreted problems not from a multi-representational perspective. They viewed each representation as a different entity.

Porzio (1999) indicated that the graphing calculator is not a panacea for improving representational fluency. Mathematics instruction needs to focus on representational fluency supported by graphing calculators. His study results showed that the group using graphing calculators and instructed with a special emphasis on representational fluency outperformed the students instructed with traditional curriculum and without the use of technology.

The literature on the effect of graphing technology on students' learning and success reveals different results. Differences in the curriculum besides the use of graphing calculator affect findings; however, it will be biased to attribute the results only to curriculum changes or

graphing calculator use. In this study, I attempted to investigate the effect of technology without any changes in curriculum. Some studies did not use control group or mechanism to compare the results in the absence of technology. In my study, since I compared solutions for PBTs and GBTs, the role of technology on representational fluency was more explicit. Moreover, previous studies did not focus on how graphing technology affect representational fluency. I also focused on the way that technology impacts students' translations between different representations.

Solution Methods

The graphing technology also impacts students' solution methods and the representations they use while they are solving mathematical problems. Graphing software and calculators make different representations more available to students by offering on scale and automatic drawing of the functions. It is cumbersome for most of the students to use graphical strategies when they solve the problem with paper and pencil, since they need to draw the coordinate system and also the scaling issue might hinder the accuracy of the results. There are not only the graphic and algebraic representations but also numeric representations supported by graphing technology. Embedded spreadsheets to graphing technology allow creating functions from given number patterns and also automatically calculates the rest of the number pattern in the spreadsheet. Therefore graphing technologies enhance algebraic, graphic, and numeric representations of mathematical concepts.

Yerushalmy (2006) studied the effect of graphing software on students' solution strategies. The findings of the study revealed that there was a difference between the work low achieving students and traditional problem-solving strategies of low achieving students in terms of solution strategies. The experimental group used the graphing software to confirm their conjectures, to gain a broader perspective, and to complete difficult operations. Their solution

strategies included mostly graphic and numeric methods and they delayed symbolic formalism. Harskamp, Suhre, and Van Streaun (2000) found that the use of the graphing calculator increases the number of attempts to solve problems and students' test scores. Moreover, they reported that low-achieving students benefited most from the graphic strategies which explains the reasons of especially low-achieving students' increase in the performances. Overall they stated that more students from the experimental group used graphic strategies compared to the control group. However, there were no differences in the use of heuristic (guess and check) and algorithmic strategies.

Merriweather and Tharp (1999) investigated how the instruction with graphing calculators' affected the students' solutions of algebra word problems. The findings of the study show that there were no students who attempted to solve algebra word problems by using the graphing calculator. One of the limitations of the study is the two week duration of instruction on using graphing calculators might have an effect on the results. Another limitation was the type of task asked on the interview which requires no benefit when students solve it by using graphing software. Therefore, methodological flaws need to be carefully considered before interpreting the results.

Ruthven (1990) reported the difference in solution strategies of students on the items when asked to find algebraic representations of the function from given graphic representation. The solution strategies of the students using graphing calculators were graphic-trial approach which is basically graph, revise and re-graph of the algebraic representations of the functions. However, the students who were not using calculators attempted to symbolize the equation by using the information to translate from algebraic representations of the functions to graphic representations. Except Merriweather and Tharp (1999), all studies mentioned in this section

reported the differences in students' solution strategies and they explained the way graphing technology affected students' solution strategies. However, there are not sufficient numbers of studies to describe how the solution strategies of students with graphing technology are different from their paper-pencil-based solutions. In my study, I compared students' solutions of PBTs and GBTs for the same type of algebra word problems in detail. Since the participants in my study were proficient with the use of Geogebra, there is not a methodological concern about the duration of training.

CHAPTER THREE: METHODOLOGY

Qualitative Research

According to Patton (2002), “Qualitative methods facilitate study of issues in depth and detail and enable researchers to approach fieldwork or problems without being constrained by predetermined categories of analysis” (p. 14). Qualitative research delves in-depth into complexities and processes (Marshall & Rossman, 2010). To gain an in-depth understanding of students’ thinking patterns regarding algebra problems as well as their representations and methods, qualitative methods were employed in my study.

A Multiple-case Study

This project involves a multiple-case study of high school students who were enrolled in a Pre-Calculus course. In a multiple-case study, two or more cases are analyzed. According to Merriam (1998), including more cases in a study increases the variation across the cases and results in a more compelling interpretation. The first reason behind choosing a multiple-case study was that in order to investigate students’ solutions based on their preferences for visual or nonvisual methods, I included at least one person from each group (visualizer, nonvisualizer, and harmonic) as measured by Mathematical Processing Instrument (Suwarsono, 1982) to represent those three types of thinkers in my study. The second reason for choosing a multiple-case study was to be able to compare and contrast the cases. The third reason was that including multiple participants maximized the information about students’ solution processes.

Selection of Cases

Students for this study were high school students who were enrolled in a Pre-Calculus class in the Fall Semester of 2010. I contacted pre-calculus teachers at the Central Florida high schools and invited them to participate in my study. After finding teachers, I observed the

students in the classroom. Introducing myself to the class and spending time with students in the classroom to allow them to get to know me before I started interviews so they would feel comfortable while they were solving problems. Classroom observations helped identify students who could express their solutions comfortably. It was important to my study work with students who did not have problems expressing their solution methods, since I made inferences about their thinking according to their external representations. After observations I asked students to participate in my study. There were 8 students who volunteered to study with me.

After finding participants, I administered Suwarsono's MPI which has two parts to determine students' thinking based on preferences for visual/nonvisual methods. Students were asked to attempt to solve the problems in Part I and indicated their methods of solutions which were given in Part II. Suwarsono's instrument consists of 30 problems. For each problem, a score of 2 was given if the student used a visual solution method, a score of 1 was given if the student did not indicate any visual or nonvisual method, and a score of 0 was given if the student used a nonvisual solution method. Therefore the students' visuality score ranged between 0 and 60. I chose students with different visuality scores to generate and collect rich data.

The method of selection of students in this study was purposeful sampling. According to Patton (2002), the logic and power of purposeful sampling is selecting information-rich cases that will illuminate the questions under study. In Patton's study, he identified 15 types of purposeful sampling. Among those, the most appropriate strategies for my study are theory-based, or operational construct, sampling, and maximum-variation sampling.

Theory-based sampling is defined by Patton as follows: "The researcher samples incidents, slices of life, time periods, or people on the basis of their potential manifestation or representation of important theoretical constructs" (p. 238). I used theory-based sampling, since

in the student selection process, I used the theoretical construct of mathematical visuality in selecting my sample. After the students were selected according to their MPI visuality, they were presented with paper-pencil and Geogebra-based tasks. As previously mentioned, I selected at least one student for each group: visualizers, nonvisualizers, and harmonic thinkers. Therefore, the selection of students for the interviews was dependent on the results of the MPI.

Patton (2002) emphasizes the results of using maximum variation sampling:

The data collection and analysis will yield two kinds of findings: (1) high-quality, detailed descriptions of each case, which are useful for documenting uniqueness, and (2) important shared patterns that cut across cases and derive their significance from having emerged out of heterogeneity (p. 235).

In this study, the sample included students who were visualizers, nonvisualizers, and harmonic thinkers to ensure maximum variation.

Pilot Study

I worked with 5 Pre-Calculus students (2 females and 3 males) at a technology-enhanced charter school in Georgia during the Spring 2010 semester. I conducted interviews with them to investigate and compare these students' solution methods for algebra word problems when using Geogebra and when using paper and pencil. These students had previously taken Algebra I and Algebra II and received A grades in those courses. The interviews included eight paper-and-pencil-based tasks (PBTs) and eight Geogebra-based tasks with virtual physical representations (GBT-VPRs). PBTs and GBT-VPRs have similar methods of solution; however, their contexts are different. Therefore, I had an ideal opportunity to compare student solution methods for algebra word problems. Students attended Geogebra workshops prior to the interviews. The workshop lasted three hours in two sessions. The training focused especially on using multiple

representations on Geogebra and translation between those representations. They were given homework to solve some word problems on Geogebra. Following the workshop, the interviews started. First they worked on PBTs, and the following day for a class period, they worked on GBT-VPRs. I presented PBTs only in written format and GBT-VPRs both in written format and as a Geogebra file created for each problem with virtual physical representations.

I conducted these interviews especially to field test Geogebra files created with virtual physical representations of the situations described in the problems. Since these Geogebra files were unique and had not been tested before by other researchers, I had to test them and revise them according to feedback from students' solution methods. Suwarsono's (1982) Mathematical Processing Instrument (MPI) is a paper test to determine students' preferences for visual or nonvisual thinking regarding algebra word problems. Following Suwarsono's instrument I designed a questionnaire consisting of visual and nonvisual solution methods for each problem and I field tested them with the students in the pilot study.

One of the important reasons to test these tasks was to ensure that the given representations would not lead students to solve the tasks only in one specific way. After I had presented the tasks to the students, I realized that there were two tasks which lead students to use visual methods. For one of them, I revised the given representations and added nonvisual representations to balance the visual and nonvisual aspects of the problem. I excluded the other problem from my study because I believed that the task influenced student preferences and led them to choose only visual methods.

I encountered the opposite issue with two other tasks, GBT-VPR 6 and 7 (presented in Chapters 4, 5 and 6). The ease of solving these tasks with nonvisual methods led students in the pilot study to select nonvisual methods. In order to test whether the easiness overdetermined

their preferences, I created GBT-VPR 8 (presented in Chapters 4, 5 and 6), which was a similar problem but for which Geogebra did not automatically calculate the dependent variables (in this case the amount of money they saved in their piggy banks depended on the number of days) for the given independent variables. Students would have to calculate the dependent variables by themselves if they preferred to solve it by using nonvisual means. Hence, if students still preferred to solve the task by nonvisual methods, I would assume their choice of solution method was not based on choosing the method that took least time but on their thinking preferences.

Two of the pilot study tasks were algebra word problems that required using quadratic equations. The rest of the tasks could be solved by using linear equations. In order to obtain topical homogeneity among the tasks, I excluded those tasks and designed new tasks whose solutions involved solving a system of equations.

The difficulty of the problems was another factor I considered as I designed and selected the tasks. I designed the tasks with different levels of difficulty. I tried to include both challenging and easy tasks. The criterion for easy tasks was that they were solvable by all students but not on every student's first attempt in the pilot study. The challenging tasks took time for most of the students, but they were able to solve them after in-depth thinking.

Before discussing other factors that caused revisions in my original designs, it is important to mention the appearance and tools shown on the Geogebra screen. In the default mode of the screen view, there is an algebra window on the left and a geometry window on the right. The algebra window shows the algebraic representations of expressions; the geometry window shows the corresponding visual representations of the expressions. For example, when a person types an equation (i.e., $y=x+5$) into the input field, which is at the bottom of the page, the algebraic representation of that function appears in the algebra window, and the graph of the

function appears in the geometry window (see *Error! Reference source not found.*). In addition to the algebra window and the geometry window, there is a third window that provides a spreadsheet view. It is optional and can be activated by clicking on Spreadsheet View under the View menu at the top of the screen. This view shows the numeric representations of the expressions in the algebra and the geometry windows. Below the menu are various tools, including tools for moving objects, creating a point and creating a line.

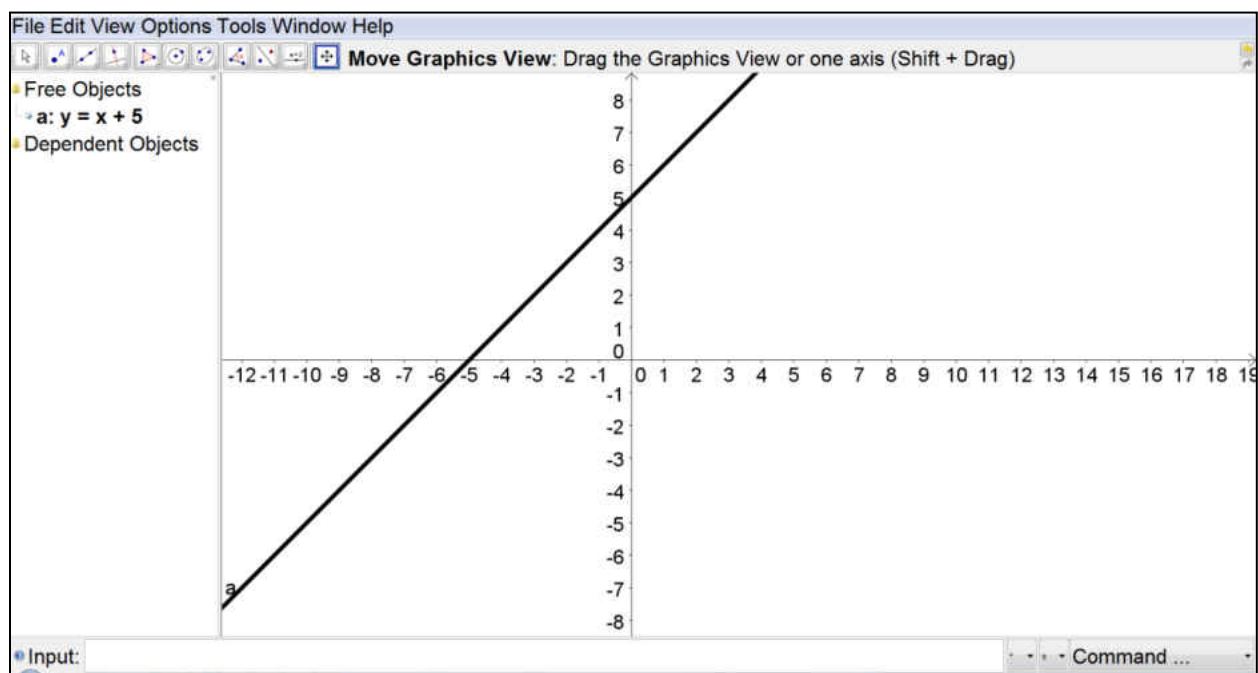


Figure 14. Screenshot from Geogebra illustrates the equation $y = x + 5$

The results of the pilot study revealed that there were two factors affecting students' solution methods for GBT-VPRs compared to their methods for PBTs. One of them was the change in media. Students were solving GBT-VPRs by using Geogebra as opposed to paper and pencil. The second factor was the effect of the representations displayed on the screen. When I compared students' solution methods for PBTs with their methods for GBT-VPRs, it was hard to determine the reason for the change. To this end, in the research study, I decided to add another type of task, called Geogebra-Based Tasks (GBTs), to my study to investigate the effect of

technology by constantly comparing them with the PBTs. GBTs included blank Geogebra files that had no preset representations of the situations described in the problems (see **Error! Reference source not found.**). Students were asked to activate their preferred windows or views according to which representations they wanted to use. For example, if students wanted to solve the task using the algebra window, they would open the algebra window. If they wanted to solve the task using a graph, they would activate axes view. If they wanted to work with a dynamic spreadsheet, they would activate spreadsheet view. Before students started to solve the GBTs, I presented them with an instruction sheet I had prepared that explained how to activate each window.

In order to investigate the effect of virtual physical representations in Geogebra, I compared their solution methods for GBT-VPRs with their methods for GBTs. With this comparison, I eliminated the effect of technology on their solutions, since both types of tasks are given and solved in Geogebra. One example each of the PBTs, GBT-VPRs and GBTs is given in **Error! Reference source not found.**, including the paper used with PBTs, the file used with GBTs and the file used with GBT-VPRs, along with the algebra word problems for each one. Thus, my pilot study revealed some of the limitations of the study, and in the light of these findings, I was able to revise the design of the tasks.

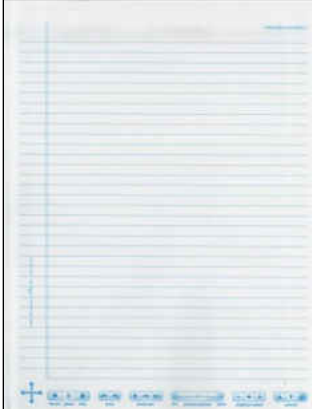

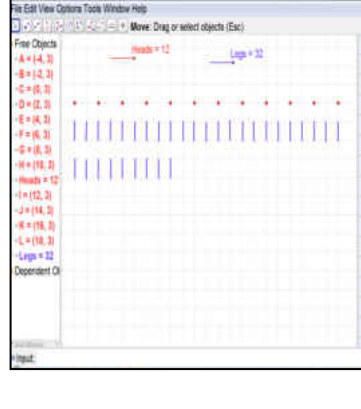
<p>PBT: One day you see 7 cycle riders and 19 wheels ride by your house. How many bikes and how many tricycles were there?</p> 	<p>GBT: The Madison Local High School marching band sold gift wrap to earn money for a band trip to Orlando, Florida. The gift wrap in solid colors sold for \$4.00 per roll, and the print gift wrap sold for \$6.00 per roll. The total number of rolls sold was 480, and the total amount of money collected was \$2340. How many rolls of each of each kind of gift wrap were sold? How many ducks and how many cows does farmer Brown have?</p> 	<p>GBT-VPR: Farmer Brown has ducks and cows. The animals have a total of 12 heads and 32 feet</p> 
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Figure 15. One example each of the PBTs, GBTs and GBT-VPRs

I classified students' solution methods for PBTs and GBT-VPRs as visual or nonvisual methods. It is important to clarify methods that are considered visual and nonvisual in Geogebra with specific examples. Even when students used representations of both visual and nonvisual methods, I attempted to determine the main solution method employed. Nonvisual methods involve algebraic, numeric, or verbal representations. For instance, if a student solves a GBT-VPR by interpreting the relations in an algebra window and inserts expressions into the input field, then this is considered a solution with algebraic representations and categorized as a nonvisual solution. If a student works primarily with spreadsheet view and solves the problem by interpreting the patterns in numbers, then this is considered a solution with numeric representations and categorized as a nonvisual solution. If a student solves the problem by using reasoning, then this method is considered a solution with verbal representation and is also categorized as a nonvisual solution.

Visual methods involve interpreting and translating graphic representations in the geometry window. That is, if a student uses graphs of expressions by interpreting the ordered pairs, manipulates the pictures by dragging and dropping on the screen, or uses the slider to observe the change in the position of pictures on the screen, then it is considered a solution with graphic representations and categorized as a visual solution.

The results revealed that access to the graphic, numeric, and algebraic representations in Geogebra affected the preferences of students while solving GBT-VPRs compared to PBTs. I did not have any difficulty determining the students' preferences regarding PBTs. However, in students' solutions for GBT-VPRs, it was difficult to categorize a solution as visual or nonvisual. Because of the ease of transition between visual and nonvisual tools in Geogebra, students' preferences in thinking were not as obvious as they were in paper-and-pencil solutions. Students who solved PBTs mainly with nonvisual methods, in particular, showed differences in their solutions for PBTs and GBT-VPRs. They tended to use more visual tools in their Geogebra solutions. This tendency might have stemmed from the availability of the representations provided in Geogebra. For example, in the pilot study I observed that the students solved some algebra word problems by creating their graphs and interpreting the information on the graphs; however, none of the students reported that they used graphs to solve the PBTs. The access to the geometry window and its scaled coordinate plane might have affected their preferences in solution methods. Different representations elicit different solution strategies in students' problem solving (Koedinger & Nathan, 2004). Therefore, the access to different tools or representations might have influenced the students' methods of solution in my study.

Another important result of the pilot study was that software facilitated both students' translations between representations and the use of multiple representations. In Geogebra, for

example, once a person had plotted a point in the geometry window, the algebraic representation of that point, which was an ordered pair, appeared in the algebra window. Moreover, if that point was recorded in a spreadsheet, then the x and y values would be shown in the spreadsheet as a numerical representation. Thus, while students were solving the problems in Geogebra, they switched back and forth between different views (algebra, geometry, and spreadsheet windows), which means different representations. The nature of the software allowed them to access different representations of the same subject at the same time. The students in the pilot study also made more translations among representations when they solved algebra word problems on the computer than when they solved problems with paper and pencil.

In my pilot study, I attempted to investigate the effects of technology on students' solution methods for algebra problems. I observed that students used different methods with and without technology. Based on the representations and the methods they used, I might conclude that their thinking patterns also changed when they switched from PBTs to GBT-VPRs. In addition to these findings, the pilot study helped me to field test GBT-VPRs. In light of students' solution methods, I revised GBT-VPRs. I eliminated and added some GBT-VPRs in addition to making changes to others. As previously mentioned, I also decided to add another type of task called GBTs. In GBTs, all views of a Geogebra file (algebra, graphic and spreadsheet views) were closed so I eliminated the effect of representations displayed in default Geogebra file. Therefore I made changes in the design of study.

Moreover, in the pilot study I used PBTs to determine students' solution preferences for visual and nonvisual solutions and to investigate how students solve algebra word problems with paper and pencil. In the research study, I administered Suwarsono's (1982) Mathematical Processing Instrument to determine students' preferences for visual and nonvisual solutions. I

used PBTs to examine students' solutions with paper and pencil. MPI has been used by many researchers (Kozhevnikov et al. 2005, Lean & Clements, 1981) to determine students' thinking preferences. After conducting the pilot study, I decided that students MPI scores would enrich the data about their solution methods.

Development of Tasks

In order to be able to compare and contrast students' solution methods I presented similar tasks in different formats. The ones that I asked students solve by using paper and pencil were called PBTs (paper-and-pencil-based tasks), and the ones that I asked students to solve by using Geogebra were called GBTs (Geogebra-based tasks) and GBT-VPRs (Geogebra-based tasks with virtual physical representations). GBT-VPRs (see Figure 16) differ from GBTs (see Figure 17) because of the presence of representations. In GBT-VPRs each problem is illustrated with virtual physical representations. In GBT the Geogebra file for the given problem is totally blank that does not display x-y coordinate, algebra view, and spreadsheet view. After selecting PBTs from different studies, I designed GBT-VPRs corresponding to PBTs. I field tested the tasks in the pilot study and revised some of the tasks and added and eliminated some problems from the tasks used in the pilot study in accordance with the results obtained from it. Moreover, in order to investigate the effect of representations I designed GBT-VPRs which includes similar algebra word problems to PBTs and GBTs.

I wrote each PBTs, GBTs and GBT-VPRs on an index card and presented them to students. For PBTs I asked students to solve the tasks by using a pen records audio and links it what you write. For GBTs and GBT-VPRs, I presented the word problem written on an index card and asked students to solve problems by using the file that I opened for the corresponding word problem.

The purposes of asking students to solve PBTs were to examine students' solution methods with paper and pencil and compare them with their solutions for GBT-VPRs and GBTs. To fulfill these purposes, I searched for types of tasks that have multiple solution strategies. I used tasks from Suwarsono's MPI (1982), problem-solving book *Crossing the River with Dogs and Other Adventures* (Herr & Johnson, 1994) and an Algebra II textbook (Holliday, Moore-Harris, Marks, Day, Cuevas, Carter, Casey & Hayek, 2004). The algebra word problems in the tasks did not require a high level of algebra knowledge. They could be solved by first degree algebraic equations or system of equations in two variables. There were no problems which require second or higher degree algebraic equations. In each category (PBT, GBT-VPR and GBT) there are ten tasks. I created a Geogebra file for each problem in GBT-VPR and a blank Geogebra file for each GBT so that, in the interviews, each word problem of GBT-VPR and GBT were given with the corresponding Geogebra file.

As previously mentioned, to compare and contrast students' solutions, I designed similar tasks and asked students to solve them with and without using technology. For example, the first algebra word problem of PBTs is similar to the problems of GBT 1 and GBT-VPR 1. Table 1, Table 2 and Table 3 show the algebra word problems that were used in those tasks.

Table 1. Interview Tasks

PBT
1. Altogether there are 8 tables in a house. Some of them have four legs, and the others have three legs. Altogether they have 27 legs. How many tables are there with four legs?
2. One day you see 7 cycle riders and 19 wheels ride by your house. How many bikes and how many tricycles were there?

PBT

3. A group of students were transported to the championship basketball game using buses and vans. When one bus and two vans unloaded, there were 55 students. A few minutes later, two more buses and one van unloaded. This time there were 89 students. In all, three buses and eight vans drove students to the game. How many students went to the game?

4. Blaise rode his bike to his friend Elroy's house, which was 18 miles away. After he had been riding for half an hour, he got a flat tire. He walked his bike the rest of the way. The total trip took him 3 hours. If his walking rate was one-fourth as fast as his riding rate, how fast did he ride?

5. Alex planned a trip from Orlando to New York. From his house to the airport he took a taxi and he flew from Orlando to New York. The total trip took 4 hours. The average speed of taxi is 60 mph and the average speed for plane is 400 mph. From her house to New York airport the distance is 1260 miles. What is the distance between her house and Orlando Airport?

6. Covell's home-mortgage payments are about \$900 per month. He is going to refinance which will cost him about \$2500 in fees, and the new payments will be \$830 per month. How long will it take him before the new loan starts saving him money?

7. Javier received a letter from his bank recently concerning his checking account. Under his current plan, each check he writes costs 15 cents, and there is a monthly fee of \$1.60. Under the proposed new plan, each check he writes will cost 12 cents, and there will be a monthly fee of \$2.75. What is the minimum number of checks Javier must write monthly in order to make new plan cost him less than the old plan?

8. The concessions manager at the Central High School football stadium offered two pay plans for people willing to sell peanuts in the stand at home football games. The first plan pays \$57.60 plus \$0.37 per bag sold. The second plan pays \$29.00 plus \$0.63 per bag sold.

- a. For what number of bags sold will these two pay plans give exactly the same pay?
- b. For what number of bags is the first plan the best choice? What about the second plan?

9. There are 9 boys to every 10 girls in a particular high school. There are 2622 students at the school. How many girls are there?

PBT
10. Cici and Amantina have lots of stickers. Cici had one third as many as Amantina had, but then Amantina gave Cici six of her stickers, and now Cici has half as many as Amantina. How many stickers did each girl start with?

Table 2. GBTs

GBT (Blank)
1. Bill has \$1.25 in nickels and dimes. He has a total of 17 coins. How many of each does he have?
2. The Madison Local High School marching band sold gift wrap to earn money for a band trip to Orlando, Florida. The gift wrap in solid colors sold for \$4.00 per roll, and the print gift wrap sold for \$6.00 per roll. The total number of rolls sold was 480, and the total amount of money collected was \$2340. How many rolls of each of each kind of gift wrap were sold?
3. Campus rentals rent 2- and 3-bedroom apartments for \$700 and \$900 per month respectively. Last month they had six vacant apartments and reported \$4600 in lost rent. How many of each type of apartment were vacant?
4. A 640-mile, 5-hour plane trip was flown at two speeds. After flying 2 hours, the tailwind picked up and the speed in the second part of the trip was two times as the speed in the first part. What was the speed in first and second part of the trip?
5. A man drove from home at an average speed of 30mph to an airport. He then boarded a helicopter and flew to the corporate office at an average speed of 60mph. The entire distance was 150 miles. The entire trip took 3 hours. Find the distance from the airport to the corporate office.
6. Cher wants to buy a refrigerator. She visits a store and finds two recommended models. The major brand is \$600 and is expected to cost \$30 per month in energy cost. The minor brand is \$400 and is expected to cost \$40 per month in energy cost. Which refrigerator would you advise Cher to buy?

GBT (Blank)
7. You contacted two local rental companies and obtained the following information for the one-day cost of renting a truck. Company 1 charges \$40.95 per day plus \$0.19 per mile, and company 2 charges \$19.95 per day plus \$0.49 per mile. How many miles do you need to make in one day to get benefit by selecting the first company?
8. A car is leaving Roseville at a constant speed of 50 km/h to go to San Francisco. Another car leaves at the same time from the next city which is 45 km closer to San Francisco at a constant speed of 40 km/h. In how many hours will the car leaving from Roseville overtake the other car?
9. There are 2 students wearing glasses to every 7 students not wearing glasses in a particular classroom. There are 63 students in that classroom. How many students are wearing glasses?
10. A and B each have a certain number of marbles. A says to B, "if you give 30 to me, I will have twice as many as left with you." B replies "if you give me 10, I will have three times as many as left with you." How many marbles does each have?

Table 3. GBT-VPRs

GBT-VPR
1. Annette has 21 coins consisting of dimes and quarters. The total amount of money she has is \$3.30. How many each coin does she have?
2. Farmer Brown has ducks and cows. The animals have a total of 12 heads and 32 feet. How many ducks and how many cows does farmer Brown have?
3. A group of exchange students from Japan went to a convalescent home to sing songs for the seniors and to demonstrate origami (Japanese paper folding). As it turned out, there was either one Japanese student at a rectangular table with three seniors or two students at a round table with four seniors. There were 11 students and 25 seniors in all. How many tables were being used to demonstrate origami?

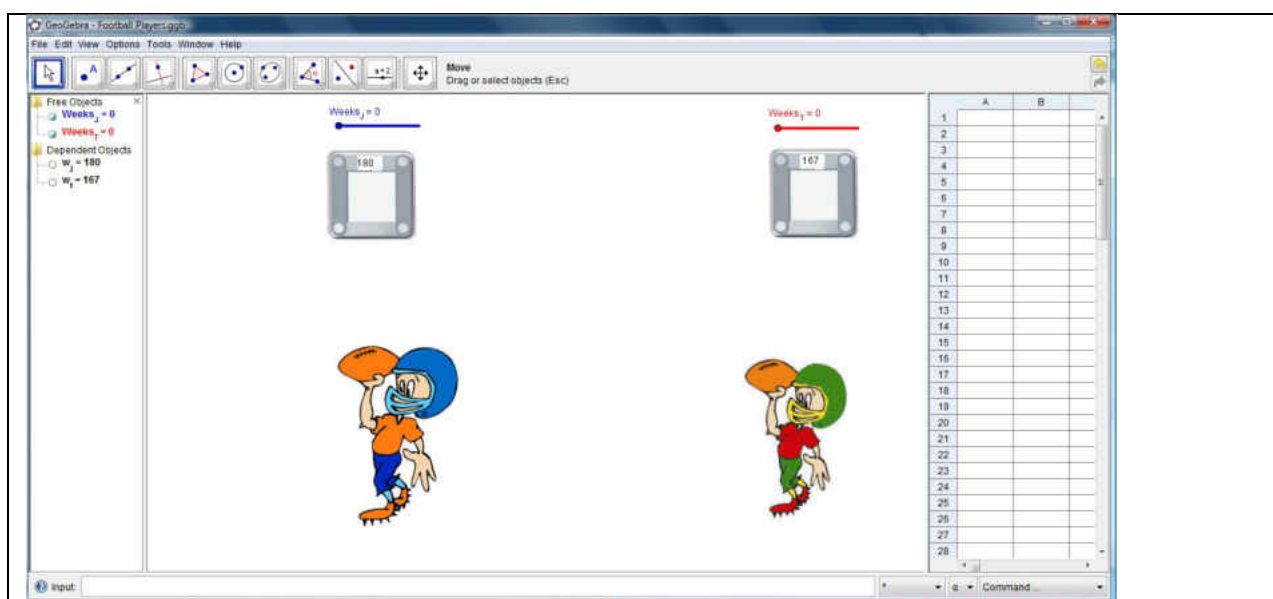
GBT-VPR
<p>4. Hillary went riding in the hills. At one point, however her horse stumbled and was hurt. Hillary decided to go back to house with her horse. Hillary figures when they were going the horse walks about twice as fast as when they were coming back. If her horse was hurt about 8 miles in, and the whole trip took 3 hours total, how fast did Horse walk when he was coming back to house?</p>
<p>5. Jason got on his bike and went for a ride. He rode at a speed of 10 km per hour from his house to his sister's house in another city. Jason and his sister then got in a car and traveled at a speed of 50 km per hour to their mother's house. The total distance from Jason's house to his mother's house is 320 km, and Jason traveled for 8 hours. How far is it from Jason's house to his sister's house?</p>
<p>6. The telephone company offers two types of service. With Plan A, you can monthly pay \$6 plus 12 cents for each minute. With Plan B, you pay \$15 monthly, plus 6 cents for each min. At least how many min would you have to use the telephone each month to make Plan B the better option?</p>
<p>7. Both Jim and Todd consider their weight a problem. Each one is trying out for the football squad and wants to weigh more. Jim is eating and working out, and has found he gains about 1 pound each week. At this point he weighs 180 pounds. Todd, on the other hand, weighs 167 and is eating, working out, and eating. He is gaining about 5 pounds every 3 weeks. How long will it take for Jim and Todd weigh the same?</p>
<p>8. Sharon and Megan have \$20 and \$26 in their piggy bank respectively. Every day, Sharon saves a quarter and Megan saves a dime. After how many days they will have same amount of money in their piggy banks?</p>
<p>9. There are 4 red stickers to every 5 yellow stickers in a pack of stickers. There are 36 stickers in one pack. How many red stickers are there?</p>
<p>10. Mike has three times as many candies as George. If he gives George six candies, he will then have twice as many as George then has. How many candies did they each have to start with?</p>

I included the problems that could be solved by visual and nonvisual methods involving representations (graphic, algebraic, numeric, and verbal). The problems were open to different solution methods and give students a chance to demonstrate their preferences with their

solutions. PBTs, including Suwarsono's problems, satisfy this condition, since they were designed by considering openness to the different solution methods and they were tested. There were problems from the Herr and Johnson's (1994) book and Algebra II textbooks. I ensured that those problems were solvable by using different methods and representations.

The content of the problem is also an important criterion in the selection process. Since Geogebra is dynamic software, it is meaningful when the dynamic construction of a problem reflects real-life situations, such as the movement of a car. Therefore, I selected the algebra word problems for GBT-VPRs that could be represented using dynamic Geogebra constructions.

While I was designing GBT-VPR, I revised and made changes to algebra word problems to create a better dynamic construct in the Geogebra version of the problems. I considered the tasks as a "conversation pieces" (Nemirovsky & Noble, 1997) to help me initiate discussions with the students about problem solving as well as allow me to observe and analyze the participants' responses and activities. At the same time, I tried to avoid possible biases that might occur in the solution process. In the Geogebra files of the problems, I did not want to lead students to focus on one representation and ignore others, so I intended to include multiple representations. All GBT-VPR files of the problems included virtual physical representations; however, they also included algebraic, graphic, and numeric data. For example, by changing the value of a slider, both the virtual physical representation of the problem and the numeric representation of the problems could be changed. The screenshot from one of the GBT-VPR's and the word problem of the GBT-VPR is presented in *Figure 16*. When the values in the sliders change, the values on the scales also change. Moreover, the size of the pictures increases when the slider values increase to have the effect of gaining weight with the time progress.



Problem: Both Jim and Todd consider their weight a problem. Each one is trying out for the football squad, and wants to weigh more. Jim is eating and working out, and has found he gains about 1 pound each week. At this point he weighs 180 pounds. Todd, on the other hand, weighs 167 and is eating, working out, and eating. He is gaining about 5 pounds every 3 weeks. How long will it take for Jim and Todd weigh the same? (Herr & Johnson, 1994, p. 482)

Figure 16. The example of GBT-VPR

Before students attempt to solve the problem, all views including algebra window and spreadsheet were open and available for their use. I told students that they were free to use a new Geogebra window if they would like. Once they opened a new window, it displayed the algebra window and geometry window by default. Therefore students could use a coordinate system whenever the geometry window of each Geogebra file was crowded by virtual physical representations of the problems. When they created a graph in a new window, their graphs would not be blocked with pictures in the geometry window. Hence, physically represented Geogebra problems allowed students to use other representations.

In GBTs, all views of a Geogebra file were closed (see Figure 17). As mentioned before the default mode of Geogebra file displayed algebra and geometry window and not spreadsheet view. The default mode might create biases for students in terms of their preferences in representations. Therefore I avoided these possible biases by providing the default of Geogebra with minor modifications (without axes and the algebra window) which allowed me to determine their initial preferences for representations.

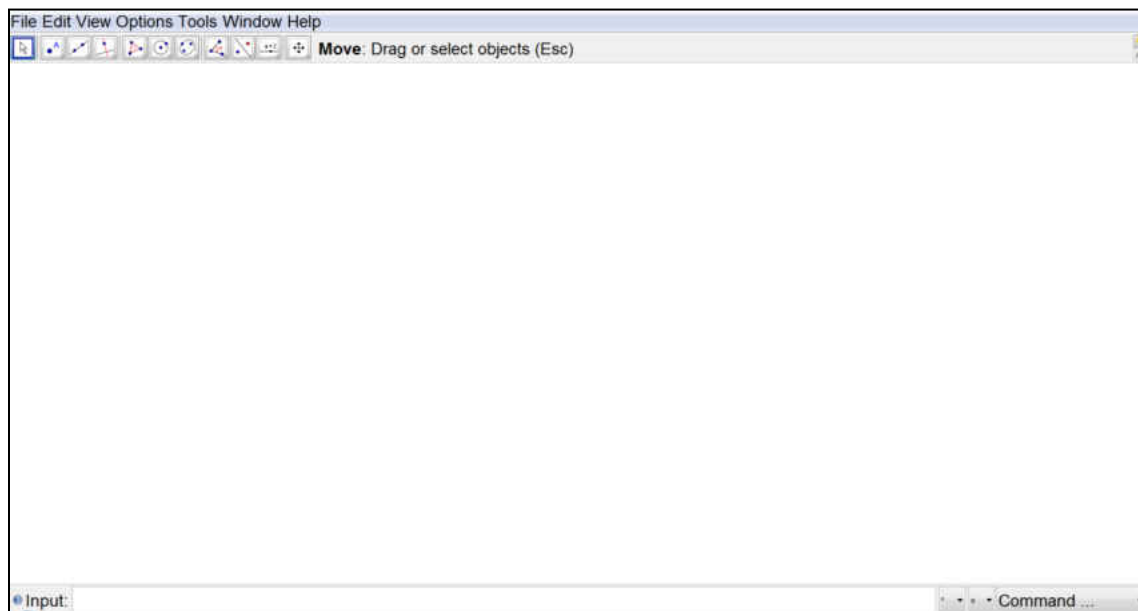


Figure 17. GBT Geogebra file

I gave an instruction sheet (see Table 4) on how to open each window or view of Geogebra in addition to algebra word problems printed on the index cards (see Table 2). According to instructions, students opened the window/view that they preferred to use. Hence I determined their initial preferences for representations by investigating their use of windows on Geogebra.

Table 4. GBT Instructions

Spreadsheet View	Geometry Window	Algebra Window
To open spreadsheet view:	To open Geometry window	To open algebra window

Spreadsheet View	Geometry Window	Algebra Window
<ul style="list-style-type: none"> • Click on the view menu • Scroll down for spreadsheet view • Click on the spreadsheet view 	<ul style="list-style-type: none"> • Click on the view menu • Scroll down for the axes • Click on the axes 	<ul style="list-style-type: none"> • Click on the view menu • Scroll down for the algebra view • Click on the algebra view

Data Collection

The data were collected by using different methods. I began with classroom observations to gather some information about their ability to verbalize their thinking and mathematical performances in class. It was important to my study work with students who did not have problems expressing their solution methods, since I made inferences about their thinking according to their external representations. Moreover, I observed their methods of solutions during their problem solving in class so that I was able to collect more information about their thinking preferences based on visual/nonvisual methods. This information helped me to have rich information about students' thinking preferences.

In this study, students' understanding was examined through clinical interviews:

If one assumes that there are a variety of ways of understanding a concept mathematically, individual differences in mathematics become the diversity in students' understandings of concepts or of mathematics itself. The clinical interview provides a means for searching for and exploring these individual understandings (Confrey, 1981, p. 400).

In clinical interview, an interviewee is required to perform carefully designed tasks in front of, and with prompting and probing from, an interviewer (Piaget, 1975). A clinical interview needs to include several series of sessions for the researcher to develop and test his or

her model of the subject's understanding concerning even the narrowest of mathematical topics. Piaget's clinical interviews supply in-depth information from which to construe an individual's thinking and cognitive processing (Ernest, 1997).

In the last twenty-five years, clinical interviews pioneered by Piaget have branched into different methods, including think-aloud problem solving protocols (Clement, 2000). In my study, the data were collected through clinical interviews with high school students and the use of a think-aloud technique. The purpose of this study was to deeply understand their problem-solving methods and their thinking patterns while solving problems with and without technology. I modeled think aloud technique with one algebra word problem. Then I mentioned that I was not interested in whether they reached correct or incorrect solutions but how they were thinking. During the interviews, when I realized that students were silent and did not verbalize their thinking, I asked them to think aloud. After one or two problems, students got used to think aloud.

After making observations, I conducted interviews to determine students' degrees of visuality on the MPI. I administered MPI in two sessions since there were 15 problems in each test. Each session lasted approximately 30 minutes. After implementing MPI, I reviewed how to use Geogebra and focused on different tools and representations of the software, such as algebra, geometry and spreadsheet views. Since students were experienced with using Geogebra, this training provided the opportunity to be certain about their knowledge of using different tools on Geogebra. I described three cases in this study. Mary, who was a nonvisual student according to her MPI score, participated in National Science Foundation granted project that included two weeks training on how to use Geogebra. Ryan and David who were visual and harmonic thinkers according to their MPI scores were the students of one of the teachers who also trained

on using Geogebra in the summer session for two weeks. In this project, the summer sessions began with the training of teachers then continued with the training of the students who were selected by teachers for the summer session training. After the summer session, teachers integrate the use of Geogebra into their Algebra II curriculum and teach Algebra II topics by using Geogebra. Therefore, Mary was one of the students who participated in the project and Ryan and David was the students of teacher who participated in the project.

The revision session was followed by the interviews. I started the interviews with PBTs and continued with GBTs and GBT-VPRs respectively. Since I had thirty tasks for interviews (see Table 1), I asked students to solve them in ten sessions; hence, I typically asked three tasks per session for each student. These were individual interview sessions and each session lasted around 30 minutes given in the order presented in *Table 1*, *Table 2*, and *Table 3*. I video-recorded students' answers and collected their work after each interview session.

For PBTs, I used the pen that audio recorded students' responses and linked their verbal responses with what they wrote on paper so I had a chance to follow students' verbal responses while they were writing. In order to observe students' gestures I set up another camera that focused on the students' face. Recording students' gestures also included clues about their solutions in addition to their written and verbal responses (Rasmussen, Stephan & Allen, 2004).

For GBT-VPRs and GBTs, I recorded students' solution methods on the computer screen by using a special software Camtesia. In addition to screen capturing, it also performs video-recording. In Camtesia files, at the bottom right corner of the screen capturing, there is a video record of the student. Therefore I had an opportunity to observe their gestures in addition to their solutions for GBT-VPR and GBTs.

Data Analysis

I analyzed the data collected through interviews, students' work and written responses, and field notes by using Krutetskii's framework (1963, 1969, 1976). In Krutetskii's (1976) study, he identifies analytic, geometric, and harmonic thinkers according to students' preferences for verbal-logical and visual-pictorial components of mathematical abilities. The analytic type is characterized by an obvious predominance of a very well developed verbal-logical component over a weak visual-pictorial one. The geometric type exhibits a very well developed visual-pictorial component, and it has predominance over the verbal-logical component. A relative equilibrium of well-developed verbal-logical and visual-pictorial components, with the former usually in the leading role, is typical of students with a modified component typified as harmonic.

In my study, I used the term "nonvisual" to describe analytic students and "visual" to describe geometric students. I also used Krutetskii's harmonic type of thinkers, as defined by him. Since I analyzed students' solution methods and made inferences about their thought processes according to their responses to the tasks, work, and representations, it was relevant to use Krutetskii's framework in my study.

I used definitions of Suwarsono (1982) and Presmeg (1986a). According to Suwarsono (1982), visual methods involve visual imagery, and nonvisual methods do not involve visual imagery. Suwarsono defined visual imagery as representations either on paper or in the mind. He defines mathematical visibility as the extent (degree) to which someone prefers to use visual methods when attempting mathematical problems that are solvable by both visual and nonvisual methods. Moreover, Presmeg (1986a) defined visualizers as individuals who prefer using visual

methods and nonvisualizers as individuals who prefer not to use visual methods when attempting to solve mathematical problems.

In order to determine these constructs, namely the students' preferences for visual or nonvisual solutions, I used Suwarsono's MPI. In order to analyze students' solutions for PBT, GBT-VPR and GBT I took MPI as a reference point and identified solutions as visual or nonvisual. To investigate the effect of technology, both solution methods for GBT-VPR and GBT were analyzed by comparing them with the solutions of PBTs. In the next section, I will illustrate some examples for the possible visual and nonvisual solutions for both PBTs, GBT-VPRs and GBTs.

Examples of solution methods for PBT

In order to exemplify solutions for a PBT, I selected a problem corresponding to one of the items in Suwarsono's instrument along with visual and nonvisual solutions. The problem and its possible visual and nonvisual solution for this task are presented below. I categorized the following two methods as nonvisual, since the first method mainly includes numeric representations, and the second method mainly includes algebraic representations. As mentioned before, methods mainly including numeric and algebraic representations are categorized as nonvisual methods. I categorized the third and fourth methods as visual methods, since they mainly include graphic representations. If the solution of the student was not one of the methods mentioned below, I analyzed the method and put in a visual/ nonvisual category according to the student's main method of solution and the representations that were used in the problem.

One day you see 7 cycle riders and 19 wheels ride by your house. How many bikes and how many tricycles were there?

Nonvisual method 1:

I solved the problem by trial and error (See Table 5):

Table 5. Numeric solution

If the number of tricycles were...	Then, the number of bicycles would be ...	So the total number of wheels would be ...
1	6	15 (NO)
2	5	16 (NO)
3	4	17 (NO)
4	3	18 (NO)
5	2	19 (YES)

Thus there are five tricycles (and two bicycles).

Nonvisual method 2:

I solved this problem using symbols and equations:

Suppose the number of tricycles = x

Then the number of wheels altogether = $3x + 2(7 - x)$

This is equal to 27; thus

$$3x + 2(7 - x) = 19$$

$$3x + 14 - 2x = 19$$

$$3x - 2x = 19 - 14$$

$$x = 5$$

Thus the number of tricycles = 5 (and the number of bicycles = 2).

Visual method 1:

To solve this problem, I drew a picture of the wheels and then categorized them into groups of three and groups of two.

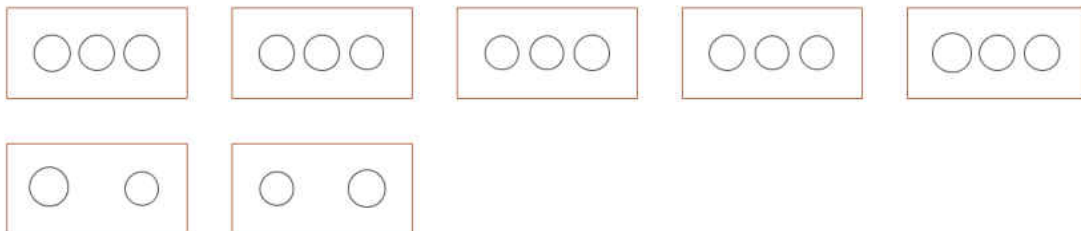


Figure 18. Graphic Solution

A group of three wheels represents a tricycle. A group of two wheels represents a bicycle. An observer can see from the picture that there are 5 groups of tricycles each and 2 groups of bicycles each. Thus, there are 5 tricycles (and 2 bicycles).

Visual method 2:

I solved this problem by drawing the cycles. First, I drew them as though all cycles are bicycles; then I kept on adding wheels to bicycles until the total number of wheels reached 19.

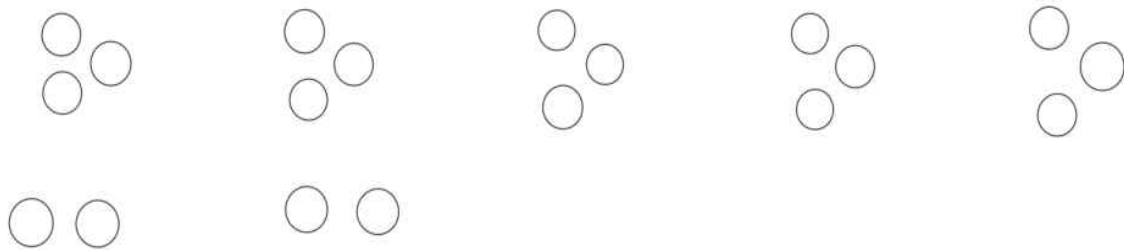


Figure 19. Graphic Solution

I found that there are 5 tricycles (and 2 bicycles).

Examples of solution methods for GBT-VPR

I selected a problem from GBT-VPRs and illustrated its possible visual and nonvisual solutions. Following the design of MPI (Suwarsono, 1982), I created visual and nonvisual solutions for GBT-VPRs. Based on the definitions for visual and nonvisual methods, I categorized the first two methods as nonvisual, since the first solution method mainly includes numeric representation and the second method includes mostly algebraic representations. I categorized the third, fourth, and fifth methods as visual methods, since they mainly include graphic representations. As mentioned before, when I categorize solutions as visual and nonvisual according to representations, I consider the main representation guiding students' solutions. For example, in the following problem, I categorize visual solution 2 as a visual solution since the main representation is graphic although it also includes algebraic representations. A student might start to solve the problem by creating two equations that are algebraic representations. However, s/he decided to shift graphic representation by creating the graphs of those equations and finding the intersection point for solution instead of solving two equations algebraically.

Problem: Farmer Brown has ducks and cows. The animals have a total of 12 heads and 32 legs. How many ducks and how many cows does farmer Brown have (Herr & Johnson, 1994, p. 470)?

The screenshot of this problem that was presented with Geogebra is given in *Figure 20*.

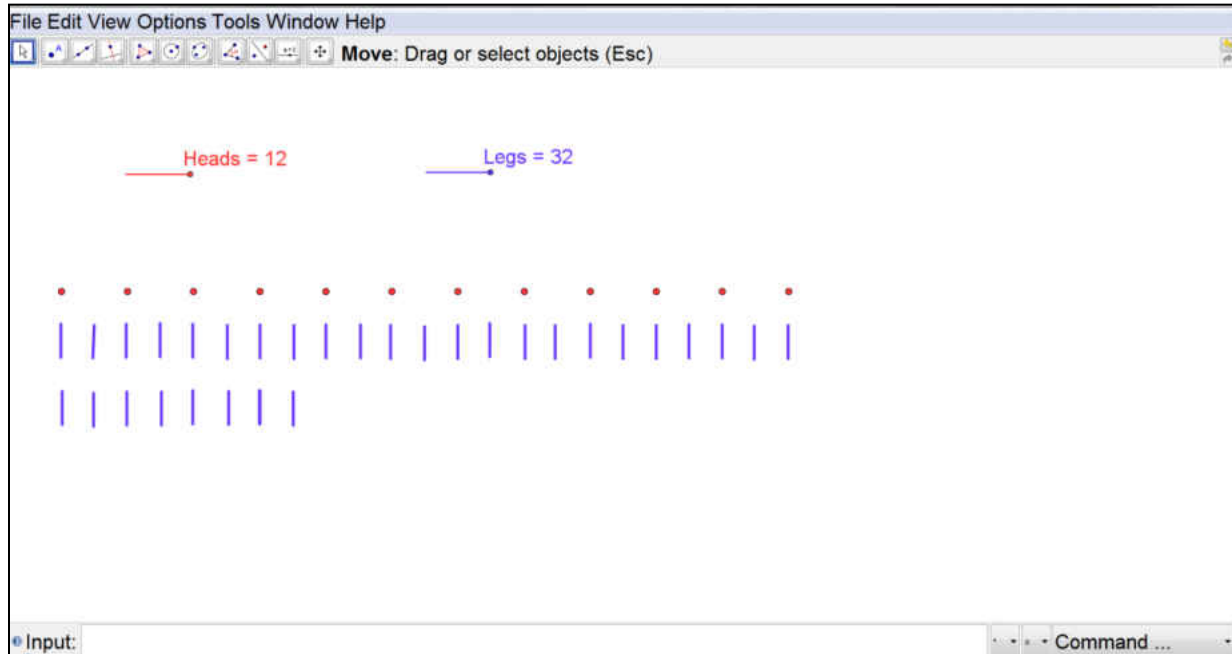


Figure 20. Geogebra version of the problem

Nonvisual solution 1:

I solved the problem by trial and error. If the number of cows was 1, then the number of ducks was 11, so the total number of legs would be 26. This is not equal to 32, so I tried 2 cows and 10 ducks, but this time it gave me 28. 28 did not work either. I checked 3 cows and 9 ducks, and the number of the legs would be 30 this time. Finally, I checked 4 cows and 8 ducks, and I got 32 legs, which gave me the number of legs that I want (See Figure 21). Therefore there are 4 cows and 8 ducks.

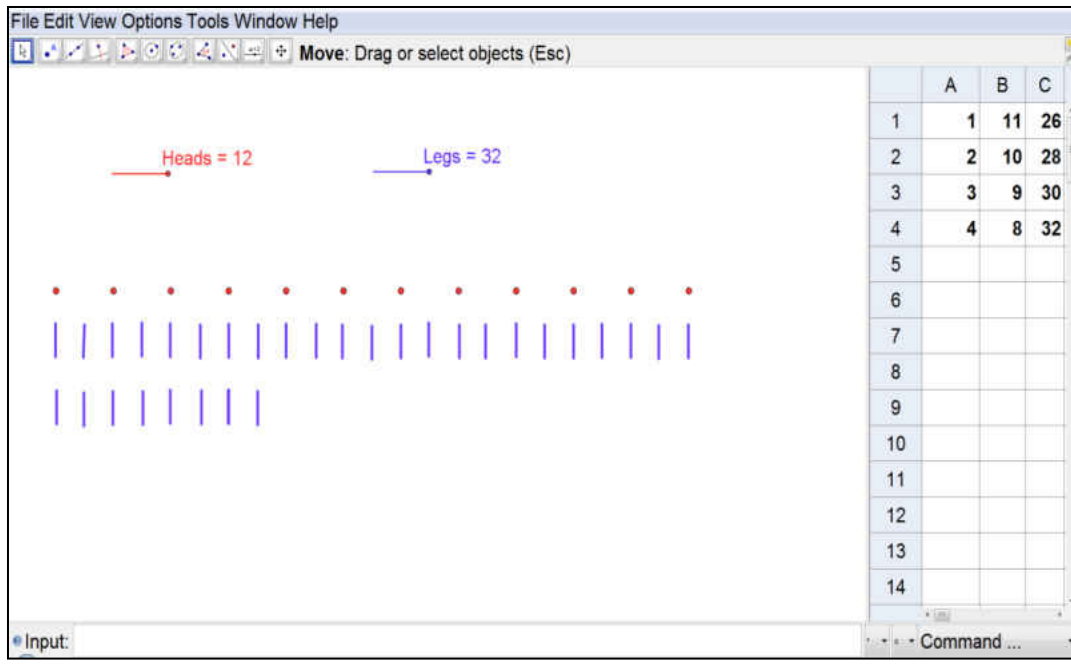


Figure 21. Nonvisual solution 1

Nonvisual solution 2

I ignored the lines and dots on the geometry window and set up the equation for the given problem. I inserted the equation, which is $32 = 4x + 2(12 - x)$, to the input field (see Figure 22). Once I entered the equation, it gave me $x = 4$ (see Figure 23) in the algebra window. Since x is the number of the cows in my equation, $12 - x$ will give me the number of ducks in this problem.

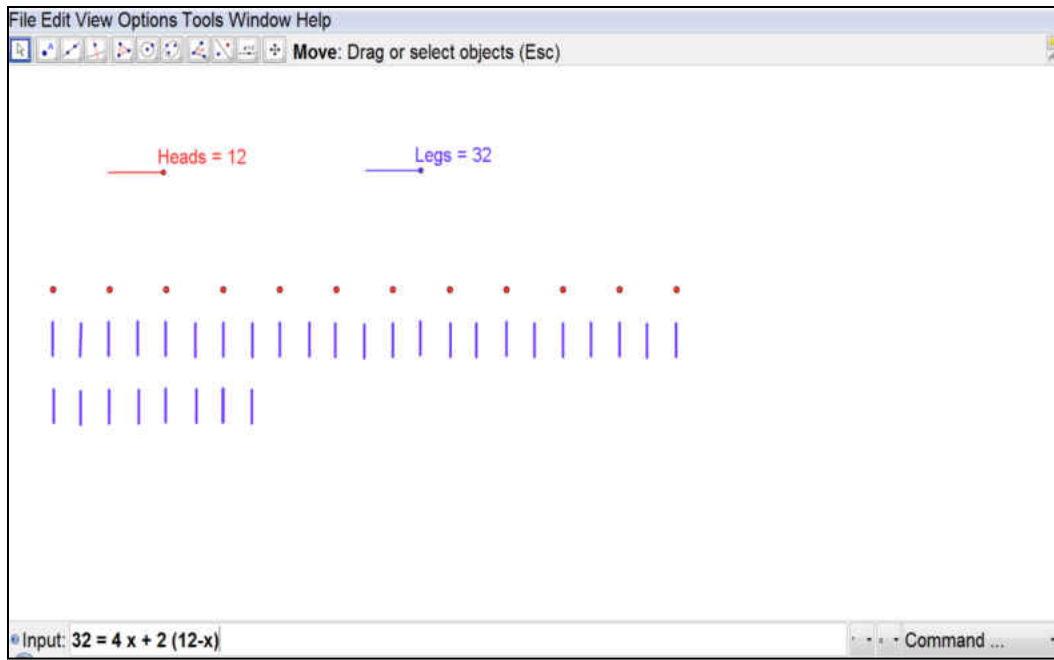


Figure 22. First step of the nonvisual solution 2

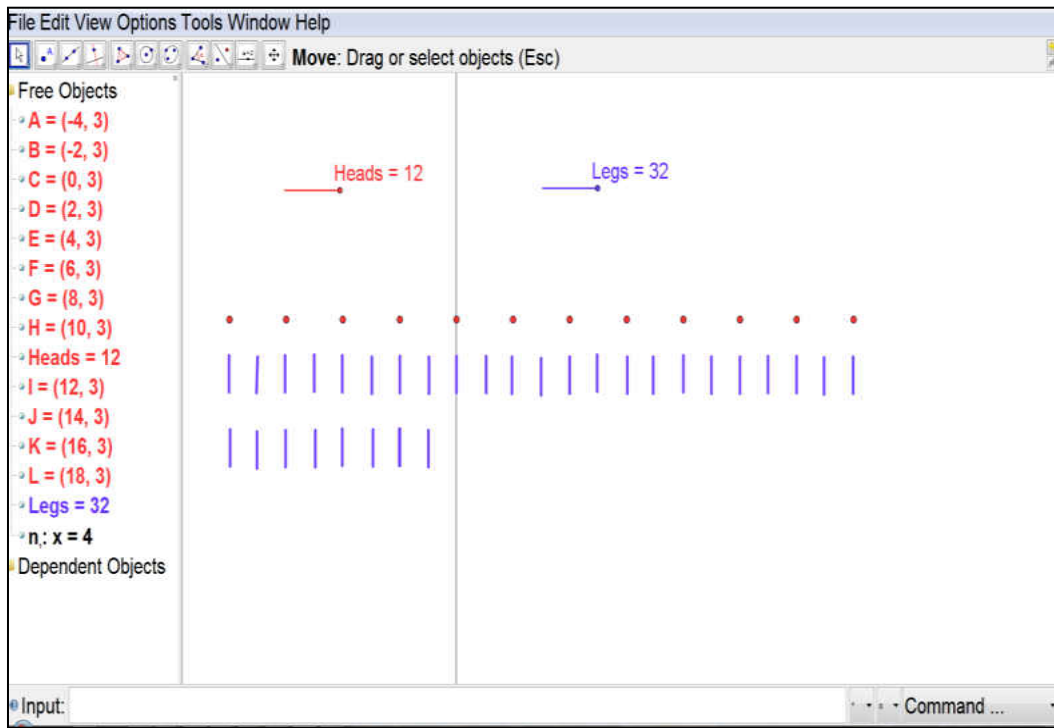


Figure 23. Second step of the nonvisual solution 2

Visual solution 1

I solved this problem by assuming that all the animals were ducks so I drew ducks with two legs (see Figure 24). Then I had 8 legs leftover. I drew cows by adding two legs to each duck until I used all of the legs (see Figure 25). I ended up with 4 cows and 8 ducks.

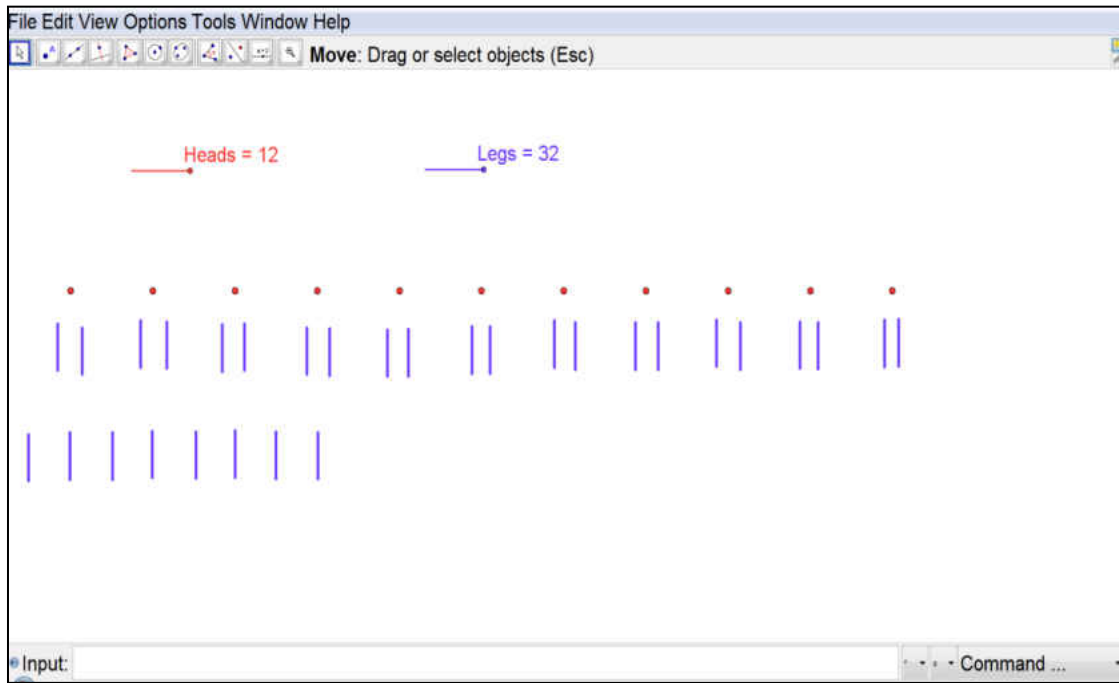


Figure 24. First step of the visual solution 1

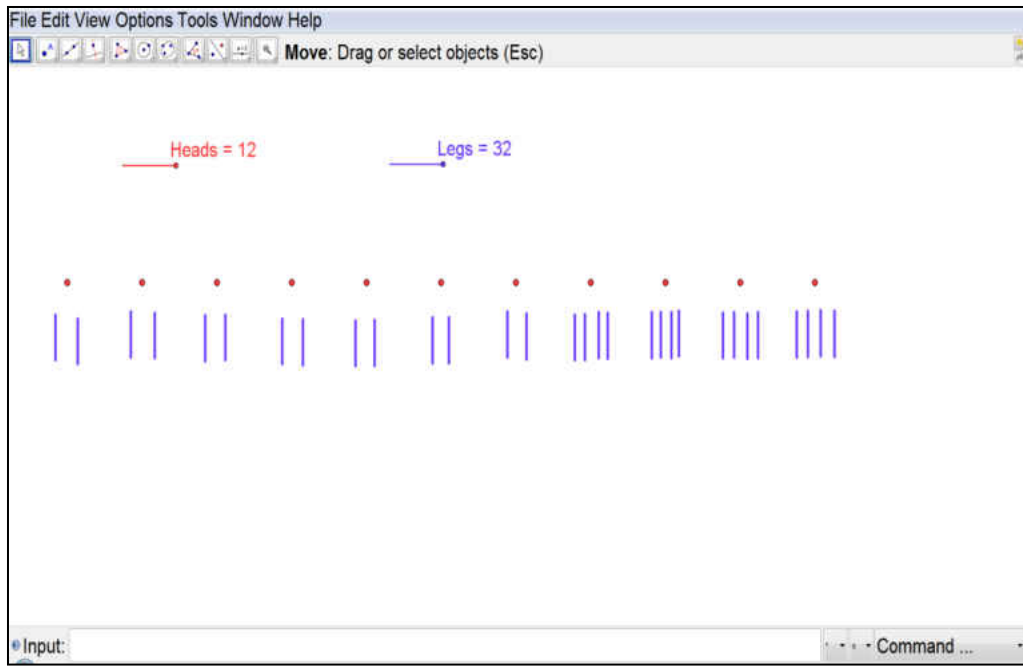


Figure 25. Second step of the visual solution 1

Visual Solution 2:

I solved this problem by creating two equations for two statements. I assumed that the number of ducks is x and that the number of cows is y . For the first statement, I inserted equation $x + y = 12$ into the input field, and the graph of the equation appeared on the geometry window (See Figure 26). Then, I created a second equation for the second statement. Since ducks have 2 legs and cows have 4 legs, I inserted the second equation, which is $2x + 4y = 32$, into the input field. The graph of the second equation also appeared on the geometry window. I observed that there was an intersection point of those two graphs. To see the coordinates of the intersection, I placed a new point which was $A = (8, 4)$ illustrated in Figure 27. Since x represents ducks and y represents cows, there are 8 ducks and 4 cows.

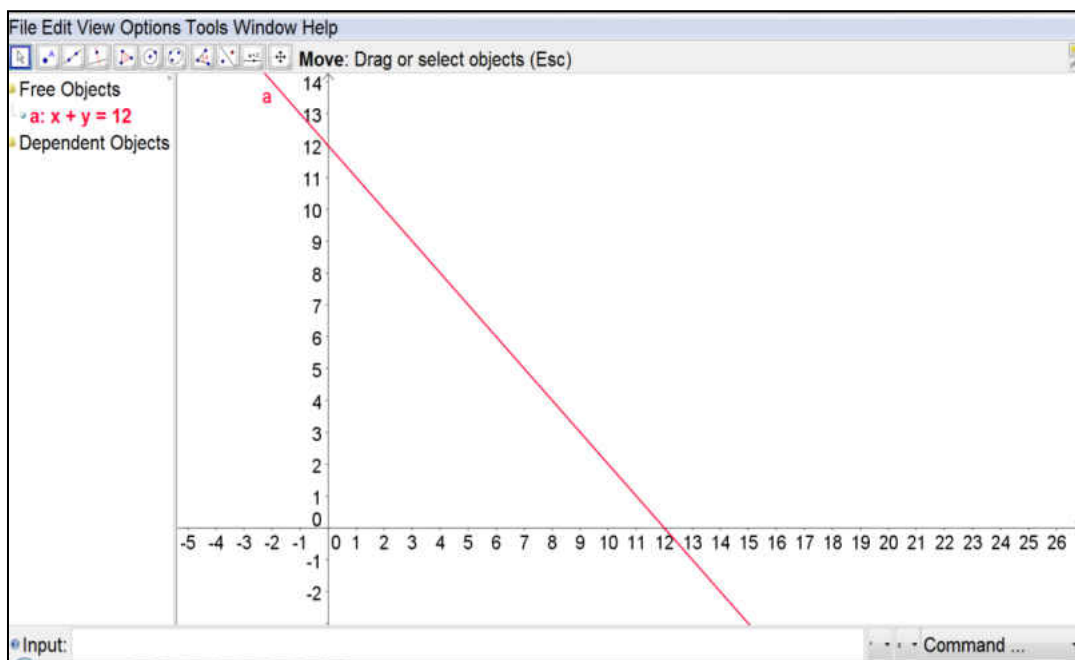


Figure 26. First step of the visual solution 2

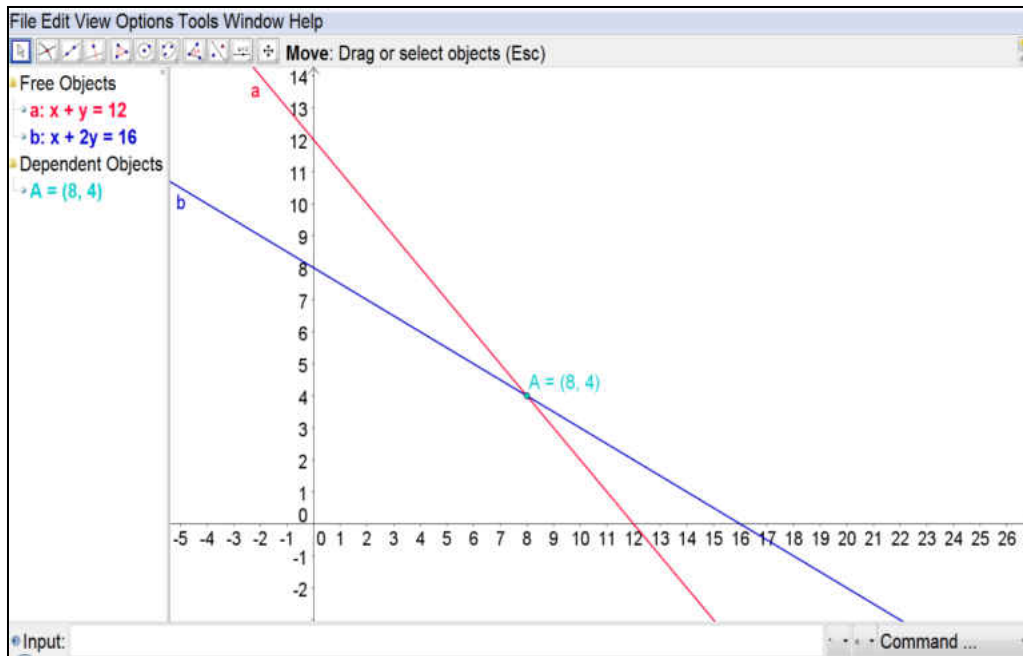


Figure 27. Second step of the visual solution 2

Visual Solution 3

I solved this problem by creating number of values for the first statement which says the total number of animals is 12. If the number of cows was 1, then the number of ducks was 11. If the number of cows was 2, then the numbers of ducks was 10. I created number of values until I had 11 cows and 1 ducks. I did the same process for the second statement which says the number of legs is 32. If the number of cows was 1, then there would be $32 - 4 = 28$ legs which mean $28 / 2 = 14$ ducks. If the number of cows were 2, then there would be $32 - 8 = 24$ legs. I applied this pattern until I had 7 cows and 2 ducks (see Figure 28).

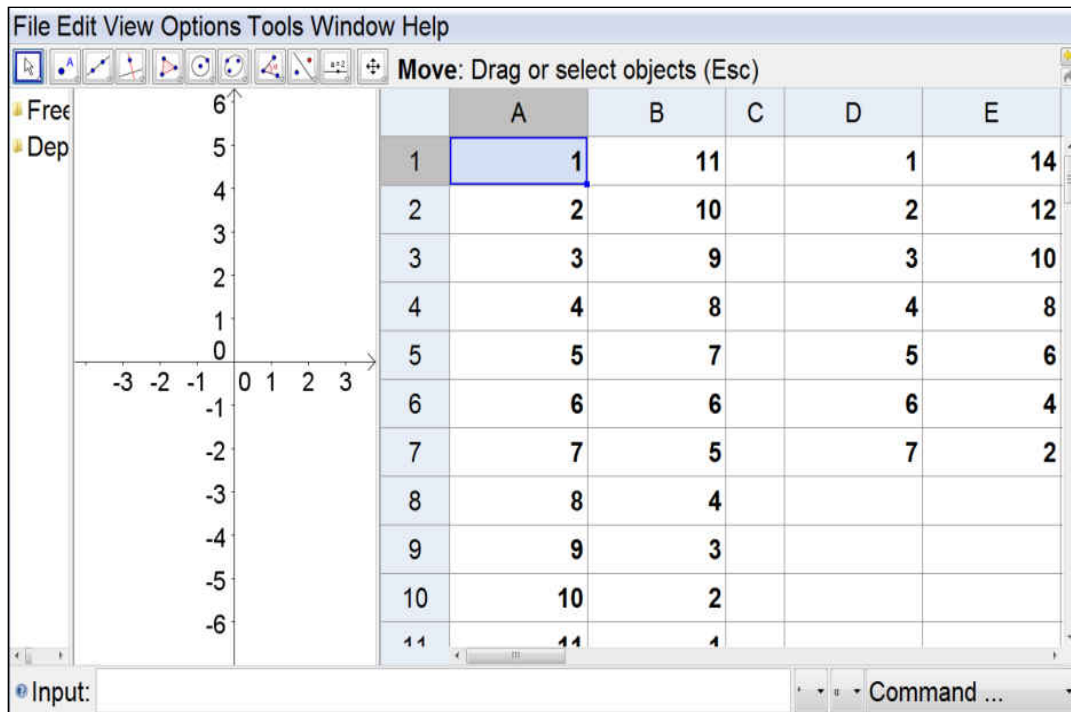


Figure 28. The first step of the visual solution 3

After creating two patterns I plotted the points belong to first statement on the graph by selecting those points and right clicking to select *create list of points*. I recognized the linear pattern on the geometry window and connected those points by clicking on *line through two points* and any two points on that list of points (see Figure 29).

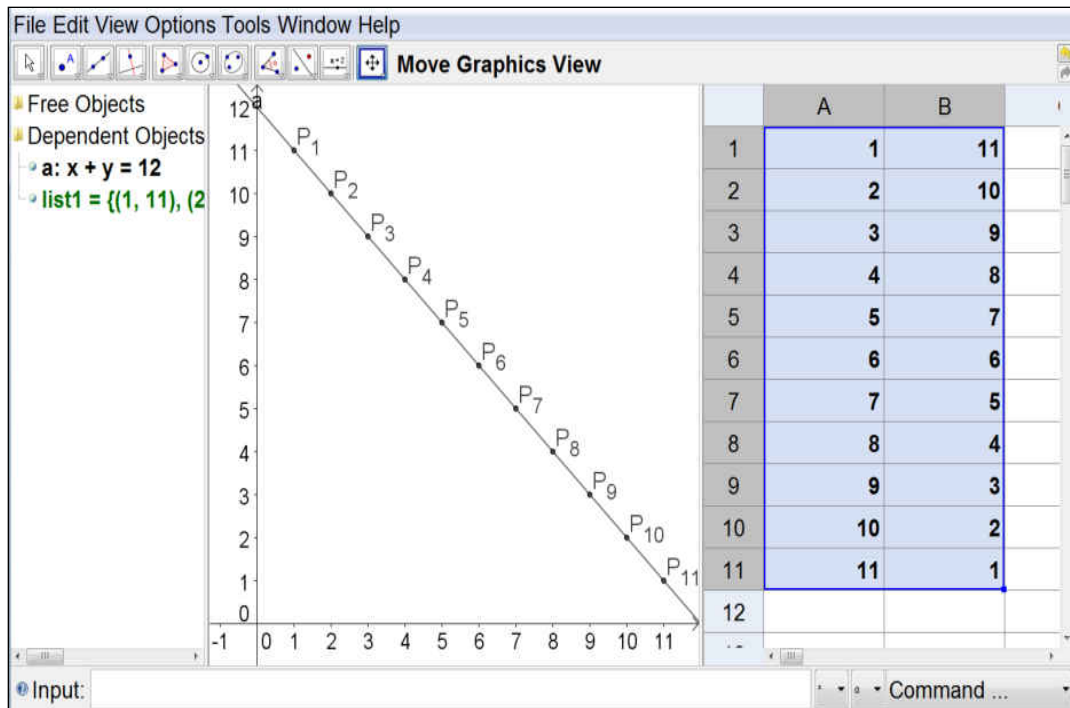


Figure 29. The second step for the visual solution 3

I implemented the same process for the second pattern to draw the graph of it. I recognized that two lines intersected at the point of $(4, 8)$. Therefore there are 4 cows and 8 ducks (see Figure 30).

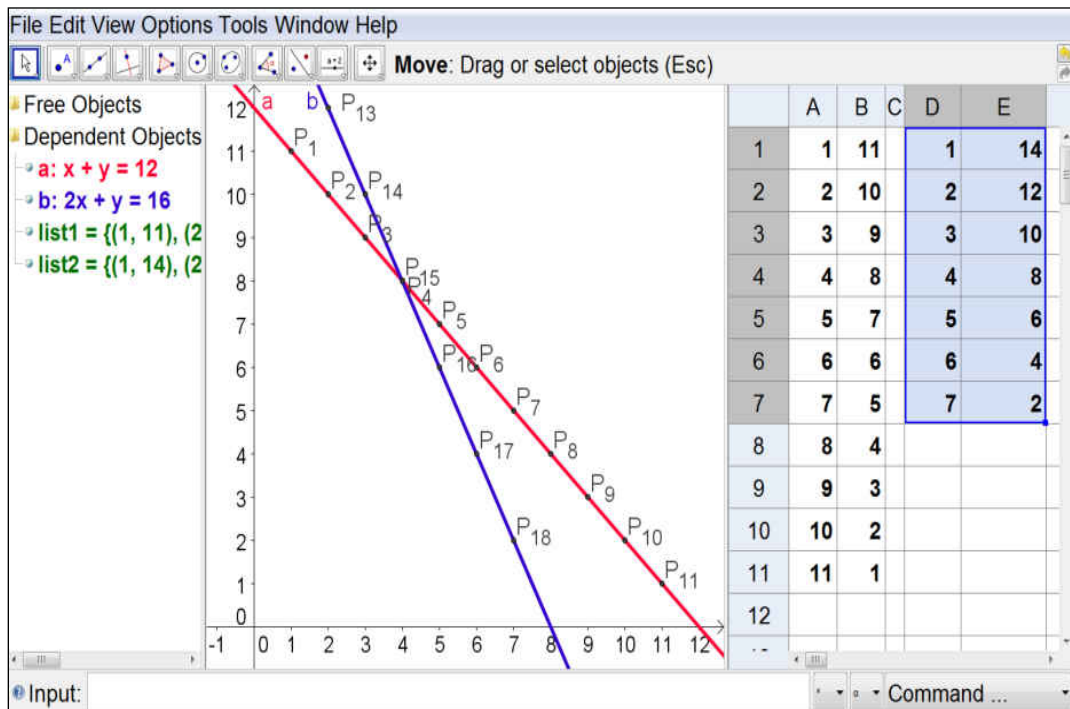


Figure 30. The third step for visual solution 3

Examples of solution methods for GBT

I selected a task from GBTs, which was the corresponding task to PBT and GBT-VPR presented above, and illustrated its possible visual and nonvisual solutions. For GBTs, I provided only blank Geogebra file for each problem. In the light of MPI (Suwarsono, 1982), I created visual and nonvisual solutions for GBTs. Based on the definitions for visual and nonvisual methods, I categorized the first two methods as nonvisual, since the first solution method mainly includes numeric representation and the second method includes mostly algebraic representations. Third and fourth solutions are categorized as visual methods although they were algebraic and numeric representations respectively. Since students used algebraic and numeric representation to draw a graph, the main method of solutions is mostly graphic representation.

Problem: The Madison Local High School marching band sold gift wrap to earn money for a band trip to Orlando, Florida. The gift wrap in solid colors sold for \$4.00 per roll, and the print gift wrap sold for \$6.00 per roll. The total number of rolls sold was 480, and the total amount of money collected was \$2340. How many rolls of each of each kind of gift wrap were sold?

Nonvisual Solution 1

I preferred to use a spreadsheet for this problem so I opened the spreadsheet view from the view menu. I started to create a table of values for the problem (see Figure 31). I thought of the problem in the following manner: If there is 1 gift wrap in solid colors sold, then the money paid for that wrap is \$4. In this case there has to be 479 print gift wraps which would cost $479 \times 6 = 2874$. Therefore the total money needed to be paid is $\$4 + \$2874 = \$2880$. \$2880 is not equal to \$2340, hence I need to check other conditions. I implemented the same process with trying 2 for the number of gift wraps in solid color. Once I acquired the numbers for 2 gift wraps in solid colors, Geogebra automatically calculated the rest of the pattern similar to excel by selecting the ten cells and dragging down until the 480th row. I checked the total amount of money which is in column E and I realized that when there are 270 solid color wraps and 210 print gift wraps, the total amount of money is equal to 2340.

File Edit View Options Tools Window Help								
Move: Drag or select objects (Esc)								
	A	B	C	D	E	F	G	
1	1	4	479	2874	2878			
2	2	8	478	2868	2876			
3	3	12	477	2862	2874			
4	4	16	476	2856	2872			
5	5	20	475	2850	2870			
6	6	24	474	2844	2868			
7	7	28	473	2838	2866			
8	8	32	472	2832	2864			
9	9	36	471	2826	2862			

Input: Command ...

Figure 31. Nonvisual solution 1

Nonvisual Solution 2

I preferred to use the algebra window for this problem so I opened algebra window. I solved the problem by creating an algebraic equation. I called x to the gift wraps sold in solid colors and $480-x$ to the sold print gift wraps. Therefore, my equation was $4x+6(480-x) = 2340$. I inserted this equation into input field and pressed enter. Geogebra calculated the x for me as 270 which was the number of gift wraps in solid colors. I found the number of print gift wraps by subtracting 270 from 480. I got 210 for the sold print gift wraps (see Figure 32).

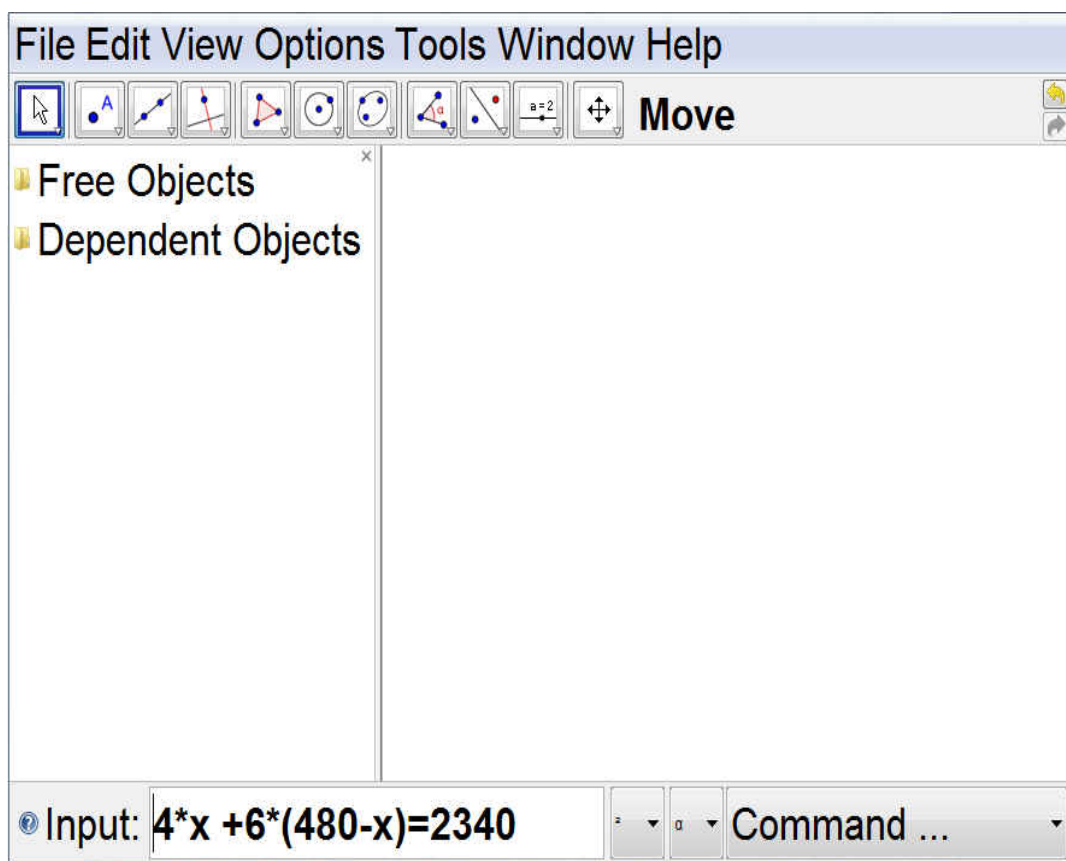


Figure 32. Nonvisual solution 2

Visual Solution 1

I preferred to use the geometry window for this problem so I clicked on the *view* menu and scrolled down to click on *axes* to display x and y coordinate on the geometry window. I created two equations to draw the graph of them. I inserted the equations $x + y = 480$ and $4x + 6y = 2340$ then pressed enter respectively. In order to find the intersection point of those two graphs, I clicked on the *intersect two objects* then the two lines and found the point $A = (270, 210)$. Therefore 270 is the number of gift wraps in solid color and 210 is the number of print gift wraps (see Figure 33).

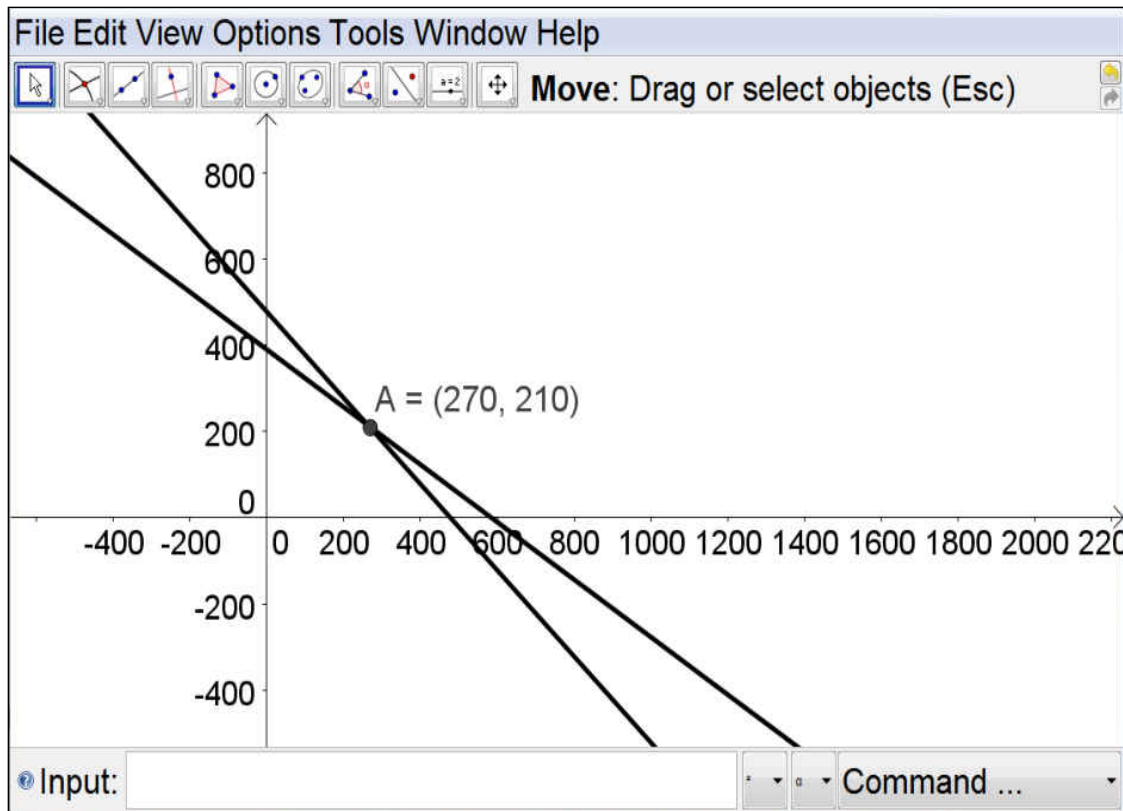


Figure 33. Visual solution 1

Visual Solution 2

I preferred to use the geometry window for this problem so I clicked on the *view* menu and scroll down to click on *axes* to display x and y coordinate on the geometry window. I wanted to use numeric information to draw my graphs. For the first graph, I used the first statement which says the total number of gift wraps is 480. I opened the *spreadsheet view* to record the points which I would plot on the coordinate system in the next step. If there was 1 gift wrap in solid colors then there would be 479 printed gift wraps. I used the same argument for the case of 2 gift wraps in solid colors. I selected those four cells and right clicked to select *create list of points*. In order to create the graph by using those two points, I clicked on *line through two points* and those two points. I implemented the same process to the second statement, which says the total amount of money collected was \$2340. If there were 3 gift wraps in solid colors, then there would be $(2340 - (3 \times 4)) / 6$ print gift wraps. I used the same argument for 6 gift wraps in solid color. As I mentioned before, I used those four cells to create two points on the coordinate system and draw their graphs. Once I completed drawing the two graphs, I found their intersection point by clicking on *intersect two objects* and their graphs. I found the intersection point as $A = (270, 210)$ which tells me 270 is the number of gift wraps in solid colors and 210 is the number of print gift wraps (see Figure 34).

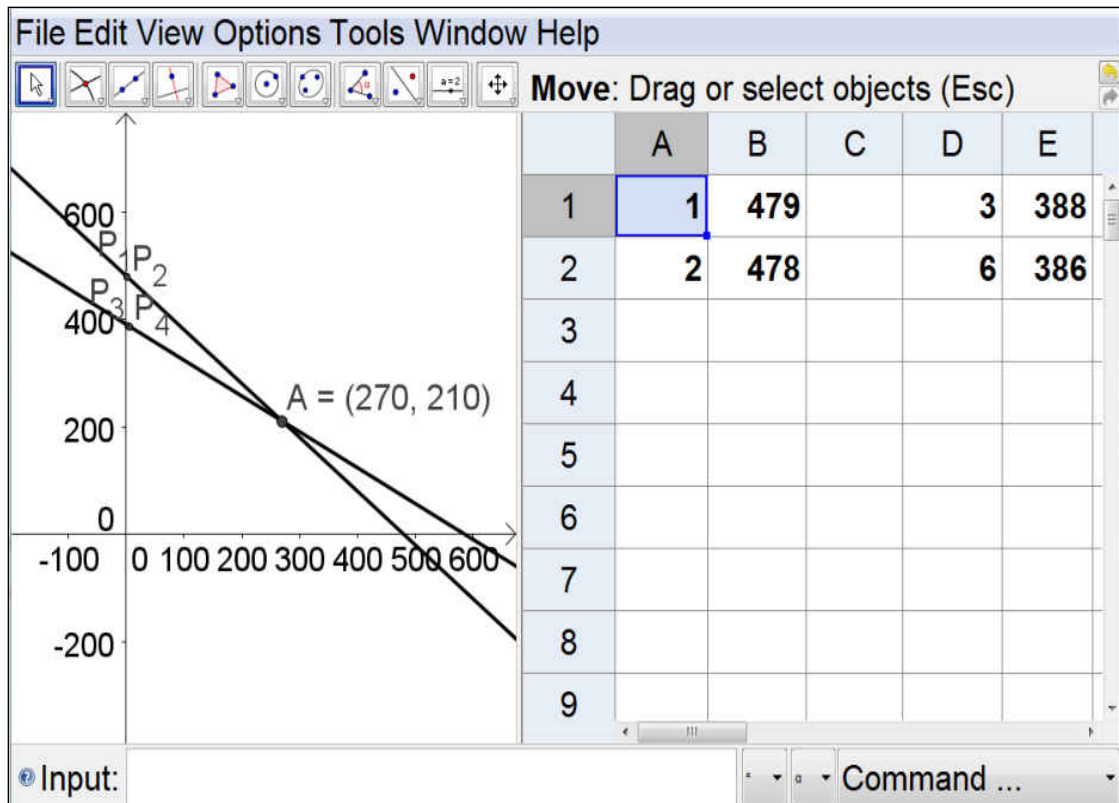


Figure 34. Visual solution 2

By comparing students' methods of solutions used in PBTs with GBTs, I attempted to investigate the effect of technology with reference to their visuality levels. Moreover, by comparing their methods of solutions between GBT-VPR and GBT, I investigated the effect of representations on students' preferences for solutions within a technology based environment. Since the software supports multiple representations and connections between them, I also observed how students used different representations and translated between those representations when solving problems with Geogebra. Students had access to Geogebra, which supports multiple representations; however, having access to software does not mean that students can use it to solve certain problems. Therefore, I examined students' use of software and attempted to describe the nature of their understanding of solving problems with Geogebra.

In order to strengthen the credibility (Lincoln & Guba, 1986) of my study by combining methods, I used three types of triangulation, which were data triangulation, investigator triangulation, and theory triangulation. For data triangulation, I used multiple data sources, such as observation field notes, students' written and verbal responses, and Geogebra screenshots for solutions of interview questions. For the investigator triangulation, I discussed my findings with committee members and doctoral students in mathematics education. For the theory triangulation, besides Krutetskii's framework, I used Suwarsono's (1982) framework to interpret my data. In order to describe and analyze students' preferences for mathematical processing, I used Krutetskii's framework. I defined and categorized visuality and preferences for visual and nonvisual solutions with respect to Suwarsono's framework.

Member checking is another method that enhanced the credibility of my study. Guba and Lincoln (1986) described member checking as verification of data and information with the participants of the study. During the study, after individual interviews, I transcribed the video records and met with students to solicit explanations of their solutions for algebra word problems to ensure that the data reflected exactly what participants intended.

CHAPTER FOUR: THE CASE OF MARY

In fall 2010, I contacted high school mathematics teachers who participated in the National Science Foundation grant-funded project on which I had been working for two years as a research assistant. I asked teachers to suggest students to participate in my study. I observed Mary in the classroom in addition to the NSF project's two-week summer session. During the two weeks of the summer course, the students learned how to use Geogebra and the teachers implemented the lesson plans that they had created with Geogebra before this two-week session. Those lessons included various topics from Algebra II. Mary had already taken Algebra II by the time she attended the project's summer session.

I conducted interviews with Mary through the end of the fall semester of 2010. At that time, she was taking Pre-Calculus. She mentioned that she had always received A grades in her math courses and that she was expecting to have an A in Pre-Calculus. She was not sure in which field she wanted to pursue a university degree, but she opined that since she loves mathematics, engineering is always an option. "I love music too but usually you don't really go very far," She said, noting that unfortunately there are not a lot of job opportunities in music. Her parents both graduated from colleges, and her mother is a business teacher. She was African-American and 16 years old.

Mary had a very quiet personality, but she did not have any difficulty in verbalizing her solutions and think-aloud techniques. During the summer session she was successful in learning both the program and the mathematics content. Although she was performing well and finding answers to the questions more quickly than her friends, she often did not raise her hand to provide answers to the questions asked during the sessions. She answered the questions, however, when teachers directly asked her to give the answer.

Before I started to interview students, I determined their preferences for visual or nonvisual thinking by using Suwarsono's Mathematical Processing Instrument (MPI). As previously mentioned, the instrument consists of two parts. In the first part, students are asked to solve mathematical word problems, and in the second part, they select their solution methods from among the given methods. If their solution methods do not correspond with the given methods, they need to explain their own methods. For each visual solution a visuality score of 2 is given and for each nonvisual solution a visuality score of 0 is given. If students do not answer the question or if there is inconsistency between the given answer and the written method a visuality score of 1 is given. There are 30 problems in the instrument. Because of time restrictions, the instrument was implemented in two sessions. The first 15 problems were given in one class and the other 15 problems were given in the second class. The highest possible score is 60 and the lowest possible score is 0.

If a student's visualization score was higher than 30, then he or she would be considered to have a preference for visual thinking in mathematics. On the other hand, if the student's visualization score was lower than 30, then he or she would be considered to have a preference for nonvisual thinking in mathematics. Those students who had visualization scores around the halfway point (30) were categorized as harmonic students. Mary scored 16 out of 60; therefore she was considered to have a preference for nonvisual thinking (nonvisual student) according to her visualization score.

Mary's Solutions of PBTs

PBT 1, 2, and 3

PBT 1: Altogether there are 8 tables in a house. Some of them have four legs, and the others have three legs. Altogether they have 27 legs. How many tables are there with four legs?

Mary attempted to solve PBT 1 in *Figure 35* by checking and guessing the numbers. She was whispering, so I asked her what she was thinking. She said, “I am putting the combinations in my head.” Once I had asked her whether she could think aloud, she started to speak audibly instead of whispering. “4-legged chair and 3-legged chair has 7 legs. 27 minus 7 equals 20 and you divide 20 by 4. It will give you 5. So six tables $[5+1=6]$ with four legs, and one table with three legs.” She realized that the total number of tables needed to be 8 instead of 7; she started to check the numbers again.

Mary: If we have 2 three legged tables that has 6 legs. Then I will be left with 21. 4 does not go in 21. Ok, 3 three legged table. It is 9 legs. 27 minus 9, 18. 4 does not go to 18. 3 times 4 is 12 and it is 15 $[27-12=15]$. 4 does not go 15. Three times five is 15, that will give you 12 $[27-15=12]$. 4 goes in 12 so. 3 tables with 4 legs and 5 tables with 3 legs. That is 8 $[5+3]$. 3 times 4 is 12 plus 15 is 27 [verifying the solution]. There you go!

Handwritten work showing Mary's solution process for PBT 1:

$$4 + 3 = 7 \quad 27 - 7 = 20$$

$$4 \overline{)20} \quad \begin{array}{l} 5 \\ 20 \\ \hline 0 \end{array} \quad \begin{array}{l} 6 \text{ tables with 4 legs} \\ 1 \text{ table with 3 legs} \end{array}$$

$$4 \times 4 = 16$$

$$3 \times 5 = 15 \text{ legs} \quad 27 - 15 = 12$$

$$4 \overline{)12} \quad \begin{array}{l} 3 \\ 12 \\ \hline 0 \end{array} \quad \begin{array}{l} 3 \text{ tables with 4 legs} \\ 5 \text{ tables with 3 legs} \end{array}$$

Figure 35. Mary's solution of PBT 1

Once Mary realized that the total number of tables was 8, she started to increase the number of tables with three legs each time by one. She multiplied the number of three-legged tables by 3 and subtracted the solution from 27 to check whether 4 is divisible with this number. She verified her solution by plugging in the numbers.

PBT 2: One day you see 7 cycle riders and 19 wheels ride by your house. How many bikes and how many tricycles were there?

Mary solved the next task in a similar way, but this time she created a table by drawing sticks for cycles (see Figure 36). The following excerpt illustrates Mary's solution of PBT 2.

Mary: 7 total and 19 wheels. So I can say 2 wheelers and 3 wheelers. One here, One here [puts one stick under two wheels and one stick under 3 wheels]. 5 wheels. 7 wheels [adds one stick to two wheelers]. 10 wheels [adds one stick to three wheelers]. 12, 15 [One stick to two wheelers and one stick to three wheelers]. And that would be 18 [adds one more stick to three wheelers]. 9 wheels plus 6, it would be 15 so I would have done another three. But that would be 18 wheels. Four two wheels is 8. If I add three wheels, lets see what happens [adds two more sticks to three wheel], that is 14. Add 3 to them, 17. If I added two that would be 19 but that is 8 wheels (total bicycle and tricycle). Himm lets see. 4 and 3 is 12. 4 two is 8. That is 20. Lets see if 5 over here (tricycles), that is 15 and then two here. There you go. So 2 and 2 is 4 and 5 and 3 is 15. 15 and 4 is 19 and 7 total. It would be 2 two wheelers (bicycles) and five tricycles.

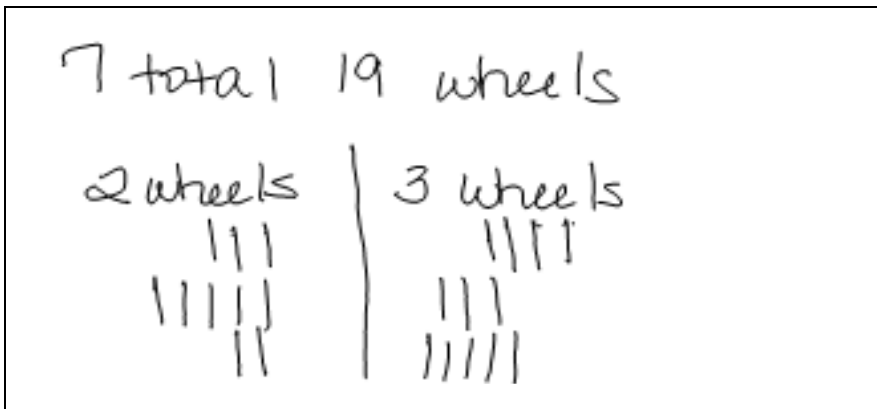


Figure 36. Mary's solution of PBT 2

Mary started by assuming 1 bicycle and 1 tricycle. She kept adding until she came close to 19 wheels. She first tried 3 bicycles and 4 tricycles, but it did not satisfy the given condition. Then she increased the number of bicycles and decreased the number of tricycles and tried 5 bicycles and 3 tricycles. Once she realized that those numbers did not satisfy, either, she

decreased the number of bicycles and increased the number of tricycles. She tried 2 bicycles and 5 tricycles, which resulted in the solution.

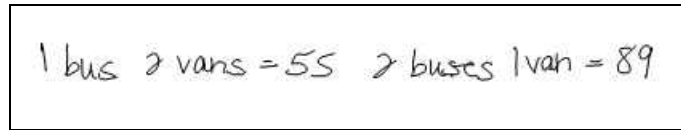
In PBT 2 Mary used sticks to represent the cycles, which was a visual representation; however, she mainly used numeric representations, constantly guessing and checking the number of cycles. Therefore her method of solution in PBT 2 was categorized as a nonvisual solution, just like that of PBT 1. Consistent with her MPI score, Mary demonstrated the characteristics of the nonvisual (analytic) type of student, whose very well-developed verbal-logical component predominates over a weak visual-pictorial one, or the abstract-harmonic type. Nonvisual (analytic) students operate easily with abstract schemes and have no need for visual supports or graphical schemes while solving mathematics problems. In order to deeply investigate her preferences for thinking and determine whether she was nonvisual or an abstract-harmonic type when solving problems with paper-and-pencil, further information about her solution methods to algebra word problems was needed.

PBT 3: A group of students were transported to the championship basketball game using buses and vans. When one bus and two vans unloaded, there were 55 students. A few minutes later, two more buses and one van unloaded. This time there were 89 students. In all, three buses and eight vans drove students to the game. How many students went to the game?

Mary attempted to solve PBT 3, but this time the numbers were larger than the numbers given in previous problems. She wrote the statement given in *Figure 37* and started to check and guess; however, she gave up with this method after trying numbers as given in the excerpt below.

Mary: I will start to try some random numbers, I do not know why. I know there is some other way to do it. Alright, let's say bus carries 20, it is too much. If it carries 30, that would be 25 (two vans) it is not equal. So bus carries 35, then this carries 20. This will make it 70 and that is 19. It is not same numbers. If it carried 34, then 21 [subtracts 34 from 55], it is not even. If I try 36, it is not even.

So I need to go odd, I guess. 37, that is 18 (subtracts 37 from 55). So bus carries 37. Wait, wait this is 1 bus and 2 vans. 37 times 2 is 74. And this [89] minus 74 would be 15. Two vans would be equal 30 but that does not add...It would take forever.



1 bus 2 vans = 55 2 buses 1 van = 89

Figure 37. Mary's solution of PBT 3

Mary abandoned guess and check method after she decided that it would take her a long time. Since the purpose of the study was to determine the students' solution methods, her attempt to solve the problem revealed that she preferred numeric methods. Therefore she dominantly used nonvisual methods when solving this task. After PBT 3, I asked PBT 4 and 5 to further investigate Mary's thinking preferences based on her responses to those tasks.

PBT 4 and 5

As with the last three tasks, Mary solved this problem by using numeric representation, which means a nonvisual solution. She solved the following two distance-speed problems (see Figure 38 and Figure 39), in a similar way. She said, "The only way I can think of is trying numbers to see what works."

PBT 4: Blaise rode his bike to his friend Elroy's house, which was 18 miles away. After he had been riding for half an hour, he got a flat tire. He walked his bike the rest of the way. The total trip took him 3 hours. If his walking rate was one-fourth as fast as his riding rate, how fast did he ride?

The following excerpt illustrates how Mary solved PBT 4.

Mary: Let's say the bike is going at a speed of 20 miles per hour. So in 30 minutes he goes 10 miles. And $\frac{1}{4}$ of 20 is 5. 2 hours that is already 10 miles. He would have gone 10 miles plus 20 miles. It is too much. So 16. I am trying to go by 4s. 30 minutes traveled, it makes 8 miles. So $\frac{1}{4}$ of 16 is 4 miles per hour. 2 hours that is

8. And 30 minutes, it is 10. Okay, so the question is, "How fast did he ride." 16 miles per hour.

Handwritten work for PBT 4:

$$\begin{array}{l} \text{total 3 hours} \quad d = 18 \text{ m} \\ \text{walking} = \frac{1}{4} \text{ bike} \\ \quad (2 \text{ hrs } 30 \text{ min}) \quad (30 \text{ min}) \\ \hline \begin{array}{l} 5 \text{ mph} = 12.5 \text{ mi} \\ 4 \text{ mph} = 10 \end{array} \quad \begin{array}{l} 20 \text{ mph} = 10 \text{ miles} \\ 16 \text{ mph} = 8 \text{ miles} \end{array} \end{array}$$

Figure 38. Mary's solutions of PBT 4

In PBT 4, Mary first checked the result if the speed of the bike was 20 *mph*. She calculated the distances traveled by bike and on foot. When she added those distances it exceeded the given distance 18. In the next step she attempted to solve problem by checking 16 *mph*. She calculated the total distance when the speed of the bike was 16 *mph*. Her result, 18 *mph*, matched the distance given in the problem. Then she read the question again and gave the answer. In order to further investigate her thinking, I asked the following task which is very similar to PBT 4. Mary's solution for this task is presented in the excerpt after the task.

PBT 5: Alex planned a trip from Orlando to New York. From his house to the airport he took a taxi and he flew from Orlando to New York. The total trip took 4 hours. The average speed of taxi is 60 *mph* and the average speed for plane is 400 *mph*. From her house to New York airport the distance is 1260 miles. What is the distance between her house and Orlando Airport?

Mary: The distance from airport to New York. That is what I am looking for. And you subtract that one from 1260 and that will give you the distance that should take with taxi. I guess I had to try different numbers. Let's see. 400 miles per hour and this one 60 miles per hour... If we do 3 hours by plane 1200, 1260 minus 1200

will give 60 and it will take 1 hour for the taxi [looks at the problem]. The distance from her house to the airport would be 60.

$$\begin{array}{l} 3 \text{ hours flying} = 1200 \text{ miles} \\ 1 \text{ hour taxi} = \underline{60 \text{ miles}} \\ 1260 \text{ miles} \end{array}$$

60 miles from her house to the airport

Figure 39. Mary's solution of PBT 5

In PBT 5, Mary's first attempt resulted in finding the answer even though she mentioned that she had to try different numbers. In order to find how much time was spent on the plane trip, she assumed 3 hours and by multiplying 3 hours with the speed of the plane she calculated the distance traveled by plane (*1200 miles*). She subtracted this distance from the total distance to calculate the distance traveled by taxi. Once she had computed the remaining distance as 60 miles, she mentioned that the time traveled by taxi was 1 hour. She read the problem again to remember what was asked in the question. She gave her solution for the distance from the house to the airport.

Mary's solutions for PBT 4 and 5 further indicated that Mary's preference is nonvisual thinking when solving mathematics problems. In all those five tasks until now, Mary did not need visual support. Mary demonstrated Krutetskii's (1976) verbal-logical type of thinking (nonvisual) when solving those problems. For PBT 4, all students in this study used graphical schemes to represent the problem except Mary. According to Krutetskii, the analytic type of students "have no need for visual supports, for visualizing objects or patterns in problem solving, even when the mathematical relations given in the problem *suggest* visual concepts." To further

investigate Mary's thinking preferences and determine the ways in which she solves problems with paper and pencil, I asked the following tasks.

PBT 6, 7, and 8

Figure 40 illustrates Mary's solutions for PBT 6. When I presented PBT 6 to Mary, the conversation given after task took place between me and her.

PBT 6: Covell's home-mortgage payments are about \$900 per month. He is going to refinance which will cost him about \$2500 in fees, and the new payments will be \$830 per month. How long will it take him before the new loan starts saving him money?

Mary: Usually I just do kind of tables. This would be 900. This will be 2500 plus 830. So after the second month, he would have paid 1800 and he would have paid plus 830 so 4160.

I: So you basically add 830.

Mary: Yeah, I added 830. This is every time plus 900. This is every time plus 830 [Creates table of values for before and after]... They will match after a while... When this starts to become more, this one was to be higher than this one. If he stayed with this, he would have been losing money.

Before	After
+900 900	3330 +830
1800	4160
2700	4990
3600	5820
4500	6650
5400	7480
6300	8310
7200	9140
8100	9970
9000	10800
9900	11630
10800	12460
11700	13290
12600	

Figure 40. Mary's solution of PBT 6

Mary created a table for two plans. For the new plan, each month she added \$830 to the refinancing fee. For the old plan she started with 0 and added \$900 each month. She mentioned that once the number of values in the same row is equal, then until that month they pay the same amount of money in both plans. She also interpreted the situation and stated that if after that month he continued to pay with the old plan, he would start to lose money compared to the new plan.

Mary used the same solution method as she had used for PBT 6 when she was solving PBT 7 and 8. She created tables and interpreted the case of what happened after the number values in the same row are equal to each other (see Figure 41). The following excerpt from Mary's solution for PBT 7 further illustrates Mary's thinking process.

PBT 7: Javier received a letter from his bank recently concerning his checking account. Under his current plan, each check he writes costs 15 cents, and there is a monthly fee of \$1.60. Under the proposed new plan, each check he writes will cost 12 cents, and there will be a monthly fee of \$2.75. What is the minimum number of checks Javier must write monthly in order to make the new plan cost him less than the old plan?

Mary: If we have only one check, this one would cost, 1.85 and this one would cost 2.87. So that is not better than this one. So if he writes 2 checks that is 2 dollars and this one would be 2.99. If he writes 3 checks this is 2.15 and 3.11. This would be 2.30 and this would be 3 dollars 23 cents. This is 2 dollars 45 cents if he writes 5 checks. And this is 3 dollars 35 cents. [She completes the table until 18th row which is \$4.40 cents for the first column and \$4.91 for the second column]... it is only 51 cents apart right here and there is 3 cents difference.... Each time this will be 3 cents less of a difference. So I guess if you divide 51 by 3 to see how many 3 cents are in 51. So 17 more and it would make 35. [Multiplies 35 by 0.12 and adds 2.75 to see if she will have the same answer when she multiplies 35 by 0.15 and adds 1.70 to that].

			5.25	4.20
			1.70	2.25
			<u>6.95</u>	<u>6.95</u>
15¢ 1.70	12¢ 2.75	3.35	4.07	
1 1.85	2.87	3.50	4.19	
2 2.00	2.99	3.65	4.31	
3 2.15	3.11	3.80	4.43	
4 2.30	3.23	3.95	4.55	
5 2.45	3.35	4.10	4.67	
6 2.60	3.47	4.25	4.79	
7 2.75	3.59	4.40	4.91	
8 2.90	3.71			
9 3.05	3.83			
10 3.20	3.95			

Figure 41. Mary's solution of PBT 7

In PBT 7, unlike PBT 6, Mary was able to tell at what point they would pay the same amount of money. Mary explored the rest of the pattern and discovered a shortcut without writing the table values after row 18. Mary used the same strategies for PBT 8 as she used for PBT 6 and 7.

PBT 8: The concessions manager at the Central High School football stadium offered two pay plans for people willing to sell peanuts in the stand at home football games. The first plan pays \$57.60 plus \$0.37 per bag sold. The second plan pays \$29.00 plus \$0.63 per bag sold.

- For what number of bags sold will these two pay plans give exactly the same pay?
- For what number of bags is the first plan the best choice? What about the second plan?

\$0.37	\$0.63	
a. 57.60	29.00	$\begin{array}{r} 43 \\ -37 \\ \hline 26 \end{array}$
57.97	29.63	
58.34	30.26	
58.71	30.89	
59.08	31.52	
59.45	32.15	
59.82	32.78	
60.19	33.41	
60.56	34.04	
3		

Figure 42. Mary's solution of PBT 8

As seen in Figure 42 Mary again created a table for the problem. Each time, she added 37 cents to the initial fee for the first plan in the first column and 63 cents to the initial fee for the second plan in the second column. She abandoned to complete the table but stated that her strategy would be adding until she reaches the same amount of money for both plans.

Mary's solution strategies for those PBTs revealed that she preferred to use numeric representations although sometimes it takes a long time for her to solve problems with this method. Mary organized tables and did not use a calculator that was built in pen and had functions similar to basic handheld calculators in most of her computations. She always confirmed her answers by substituting the value of the answer into the given problem situation. Her success in guess-and-check was not coincidence, since she knew which direction she needed to take if her guess did not satisfy the condition. For example, in PBT 4, she started her guess with 20 and continued her trials with only the numbers divisible by 4, knowing that the answer had to be less than 20. Since she recognized that the answer was not a decimal, she skipped the values 19, 18 and 17. Mary continued to use similar nonvisual strategies that were observed in PBT 1 and 2. In order to further investigate Mary's thinking preferences, I asked PBT 9 and 10.

PBT 9 and 10

PBT 9: There are 9 boys to every 10 girls in a particular high school. There are 2622 students at the school. How many girls are there?

The following excerpt presents detailed evidence of Mary's train of thought on PBT 9.

Mary: There are 9 boys to every 10 girls. I think that works. If I do that, I will do this forever. Let's say a thousand girls. If there are 1000 girls... There will be 900 boys. Lets try 200 and multiply by 200. It is 2000. So if there are 2000 girls, I am multiplying this [9] by 200. 1800, it is too much. So 2000 does not work. If I multiply by 150, that gives me 1500. And this is 1350. 1500 and 1350, it is still over. That is 2850. We need to go down...So do 140. 1400 here. This is 1260. And this 2660. It is still over. So if I go down 10, it is 38 over. If I did 136, multiply this with 136, it is 1224. That gives 2584. So this under. So lets do. I am under by 38 (subtracts 24 from 62 and gets 38). Again. First time is over now I am under. 137 [multiplies 137 by 10 and gets 1370]. 1333 [Multiplies 137 by 9]. This is 2603. Now down by 19. I think if I go one more...138 [multiplies 138 by 10]. 1242 [multiplies 138 by 9]. This would work. It will give 2622. There are 1380 girls and 1242 boys.

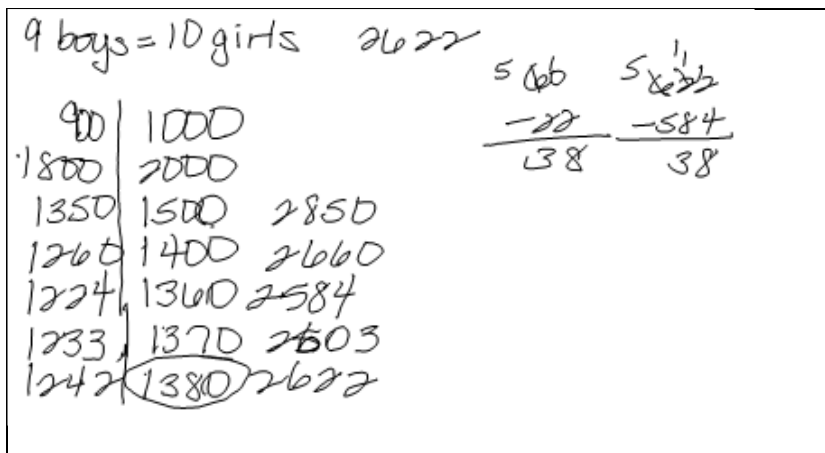


Figure 43. Mary's solution of PBT 9

Mary started to solve the task by creating a table of values (see Figure 43). She first wrote the relationship between the number of boys and girls as 9 boys equals 10 girls. Then she multiplied both sides by 100 to find the number of boys and girls and realized that the total number of students was less than 2622. Second, she multiplied by 200, however this time the

total number of students was greater than 2622. Then she computed the number of students by multiplying 150 and decided to decrease the number since her results were still above 2622. In the next step she tried 140 and compared her result with 2622. She mentioned that she had decreased the number by 10 and now her answer was 38 more than 2622. She decreased the number to 136 and her result was 38 less than 2622. She increased the number to 137 and her result was 19 less than 2622. Before she increased the number to 138, she knew that 138 was going to satisfy the condition.

As can be seen from Mary's written work and the excerpt of her dialogue with herself, her strategy was not simply check-and-guess. According to results she had reached from the previous guess, she increased or decreased the number that she tested in the next trial. Mary was consistent in her strategy while solving PBTs. Until now, she had only used numeric representations and nonvisual methods in her solutions. She did not show any evidence that she used visual support to solve problems. I asked Mary one more PBT to observe whether she would use the same method of solution as before.

PBT 10: Cici and Amantina have lots of stickers. Cici had one third as many as Amantina had, but then Amantina gave Cici six of her stickers, and now Cici has half as many as Amantina. How many stickers did each girl start with?

Mary: Let's play with numbers. If Cici had one sticker. No I do not think it is possible. If Cici has 3 stickers...If Cici had 2 then A had 6, no if she gives 6 then....And 30 minus 6 is 24, 10 plus 6 is 16. 16 is not the half of 24. If Cici had 9, A will have 27, 27 minus 6 is 21. A is only have a half of that. If Cici had 8 then A would have 24 minus 6 would be 18 and so 14 is not half of 18. So if she has 7, 21 minus 6 is 15. I am going to in wrong direction. Because either way is not working. Ok I will start with higher number. If she had 20 stickers then A would have 60 stickers. That would be 26 and 54. 21,63. 57, 21 plus 6, 27. 22,66. 66 minus 6 and 22 plus 6 is 28. 23,69. That would be 63 and 32. So my strategy would be checking and guessing again.

$$\begin{array}{l}
 \text{Cici} = \frac{1}{3} (A) \\
 10 = 30 \\
 + 6 \quad - 6 \\
 \hline
 16 \quad 24
 \end{array}$$

Figure 44. Mary's solution of PBT 10

In the last PBT, Mary started to guess and check the numbers by assuming that Cici had 1 sticker. She stated that that was impossible and then computed for 2, but she realized that it was impossible for Cici to have 2 stickers. Her next guess was 10 for Cici and 30 for Amantina (see Figure 44). She decreased her numbers by one in each of the next three trials when she realized that her approach was not going to work. She increased the number to 20 and continued to increase each time by 1 until 23. She mentioned that her strategy would be guess-and-check until the numbers satisfied the given condition.

In all ten PBTs, Mary demonstrated the characteristics of Krutetskii's (1976) verbal-logical thinking type while she was solving algebra word problems by using paper and pencil. In light of Mary's written answers and her verbal responses to the tasks, it is clear that her preferred method for solving PBTs includes mainly numeric representations, which are nonvisual methods of solutions. In the next section I investigated the effect of technology and representations on her preferences for visual or nonvisual thinking for similar tasks.

The Effect of Technology on Mary's Solutions

In this section, I attempted to compare and contrast Mary's solution methods of PBTs with GBTs in order to investigate the effect of technology on Mary's thinking preference for the

solution of problems. For PBTs and GBTs, I provided only blank paper and blank Geogebra files respectively. Hence the comparison between the solution methods used in two different media revealed the effect of technology. As was mentioned in the previous section, Mary used mainly nonvisual methods to solve PBTs. By taking her solutions as reference points, in this part I explain how Mary responded to similar tasks when using blank Geogebra files.

GBT 1, 2, and 3

GBT 1: Bill has \$1.25 in nickels and dimes. He has a total of 17 coins. How many of each does he have?

When presented GBT 1 in *Figure 45*, Mary first opened a grid, the axes and algebra window appeared in the Geogebra window. She said “x can be dimes and y can be nickels.” She entered an equation for the first statement by writing $0.10x + 0.05y = 1.25$ in the input field. Then she entered the second $x + y = 17$ for the second statement into the input field. Then she found the intersection point of those two graphs by clicking on *Intersect two objects* and the two graphs. She interpreted the intersection point and stated that there had to be 8 dimes and 9 nickels.

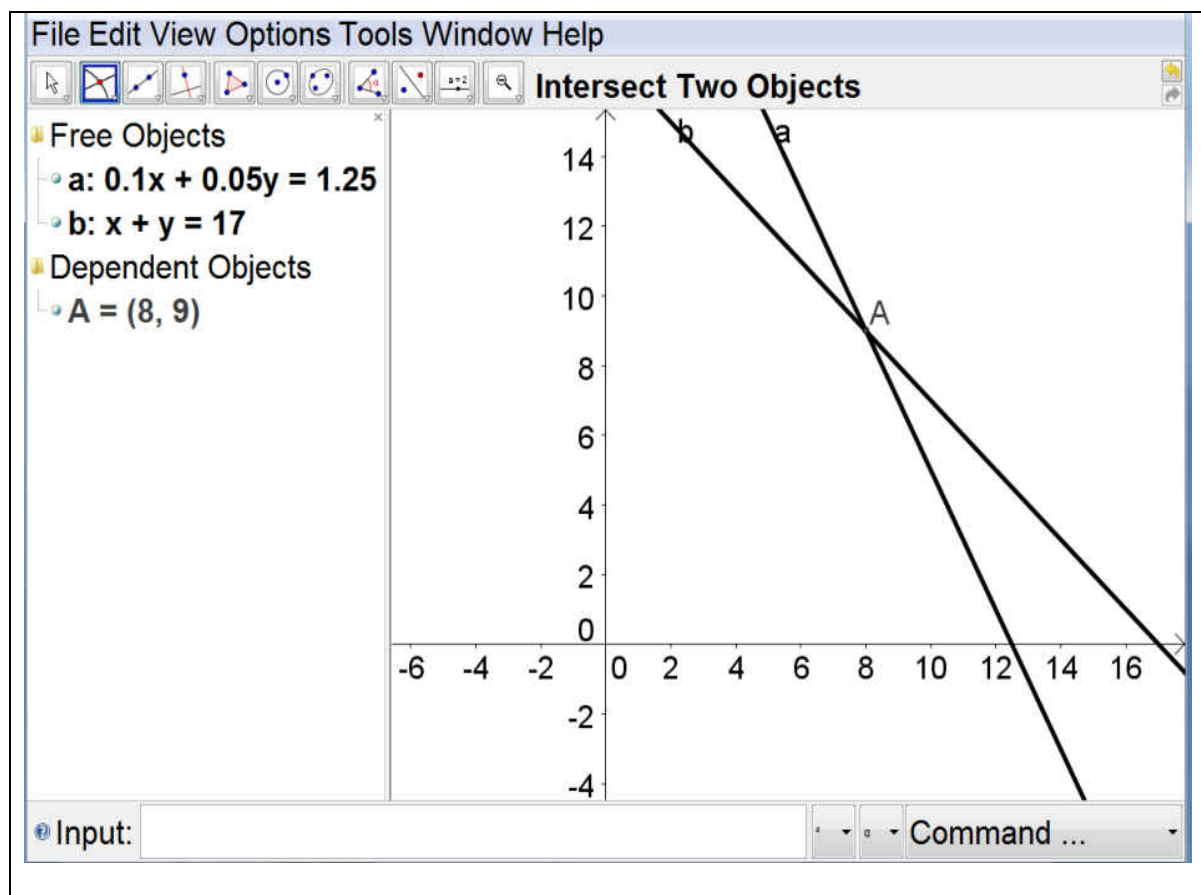


Figure 45. Mary's solution of GBT 1

I categorized Mary's solution for GBT 1 as a visual method. Although she created equations for the statements in the problem, she used those equations to draw the graph of two equations. She also mentioned at the beginning of the problem that she wanted to use a graph. Visual methods are defined as methods which include mainly graphic representations. In this problem, since she used the formula to create a graphic representation and did not continue to solve the problem in algebraic ways, her method of solution is categorized as a visual solution.

When Mary's solution for GBT 1 is compared with her solution for PBT 1, important differences are evident (see Figure 35 and Figure 45). She used mainly numeric representation in her PBT solution, but she used graphic and algebraic representations in her GBT solution. She could have solved the problem in the same way that she had solved PBT 1 by using

Spreadsheet View, but she did not prefer to do it that way when Geogebra was available. When I asked Mary why she preferred to solve the task in this way, she responded, “It is quicker.”

Although Mary showed Krutetskii’s verbal-logical thinking type in her answer to PBT 1 and was categorized as a nonvisual student according to her score on Suwarsono’s MPI (1982), her preference for solving GBT 1 was a visual method. It is important to mention that Mary verified her solution by nonvisual methods, although she used mainly visual methods to solve the problem. In order to further investigate Mary’s solutions using Geogebra, I gave Mary the following tasks.

GBT 2: The Madison Local High School marching band sold gift wrap to earn money for a band trip to Orlando, Florida. The gift wrap in solid colors sold for \$4.00 per roll, and the print gift wrap sold for \$6.00 per roll. The total number of rolls sold was 480, and the total amount of money collected was \$2340. How many rolls of each of each kind of gift wrap were sold?

The word problem given in GBT 2 was similar to the problem in GBT 1. Again Mary used the same method of solution as she used in GBT 1. For GBT 2, she activated the axes and started to solve the problem (see Figure 46).

Mary: I think it will work as same as the last problem [Clicks on axes]. $4x + 6y = 480$. Wait no, ohh, yes [She zooms out to see the graph]. Wait wait wait. I do not think that is right [deletes the formula and inserts $4x + 6y = 2340$]. $x + y = 480$ [Clicks on intersect two objects then two lines]. That is 270,210. That means 270 solid colors and 210 printed colors.

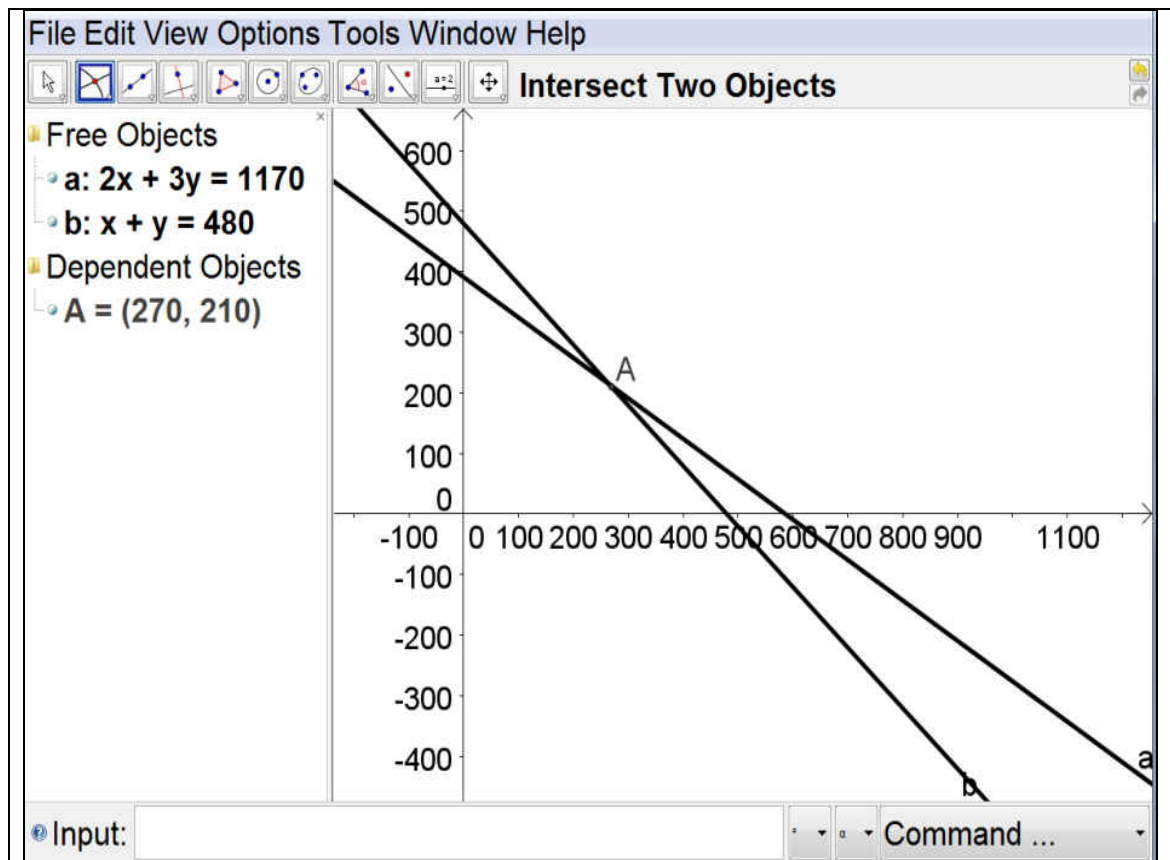


Figure 46. Mary's solution of GBT 2

She entered the first equation into the input field as $4x + 6y = 480$. She zoomed out to see the graph of the equation. Before she entered the second equation she realized her mistake and opened the algebra window, mentioning that $4x + 6y$ needed to be equal to 2340. Therefore, she deleted the equation and inserted $4x + 6y = 2340$ and $x + y = 480$ into the input field respectively. She followed the same process as before to find the intersection point. Again Mary confirmed her answer by plugging the numbers into the problem. Although Mary solved the first two GBT problems by using graphs, she continued to verify her solutions by substituting the numbers.

The accessibility of the graphical setting and automatic drawing ability of Geogebra might have affected Mary's preference to select graphic representation instead of numeric

representation while solving the problem by using Geogebra. In order to investigate in-depth how technology affected Mary's solution methods and thinking processes, I continued to ask her to solve problems using Geogebra.

GBT 3: Campus rentals rent 2- and 3-bedroom apartments for \$700 and \$900 per month respectively. Last month they had six vacant apartments and reported \$4600 in lost rent. How many of each type of apartment were vacant?

Again Mary inserted two equations ($x + y = 6$ and $700x + 900y = 4600$) to draw their graphs and find the intersection point of graphs. She chose x as the number of 2-bedroom apartments and y as the number of 3-bedroom apartments. She interpreted the intersection point according to this information. Mary solved GBT 3 (see Figure 47) just as she solved the first and second GBTs. When I compared Mary's method of solution for PBT 3 (see Figure 37) with her method for GBT 3, it was clear that she preferred to work with graphic representations. Although she demonstrated a strong preference for nonvisual thinking in this task while solving with paper and pencil, when asked to solve with Geogebra, she preferred to solve the problem using a graph. The availability of the graphing tool in Geogebra might play an important role in her preferences.

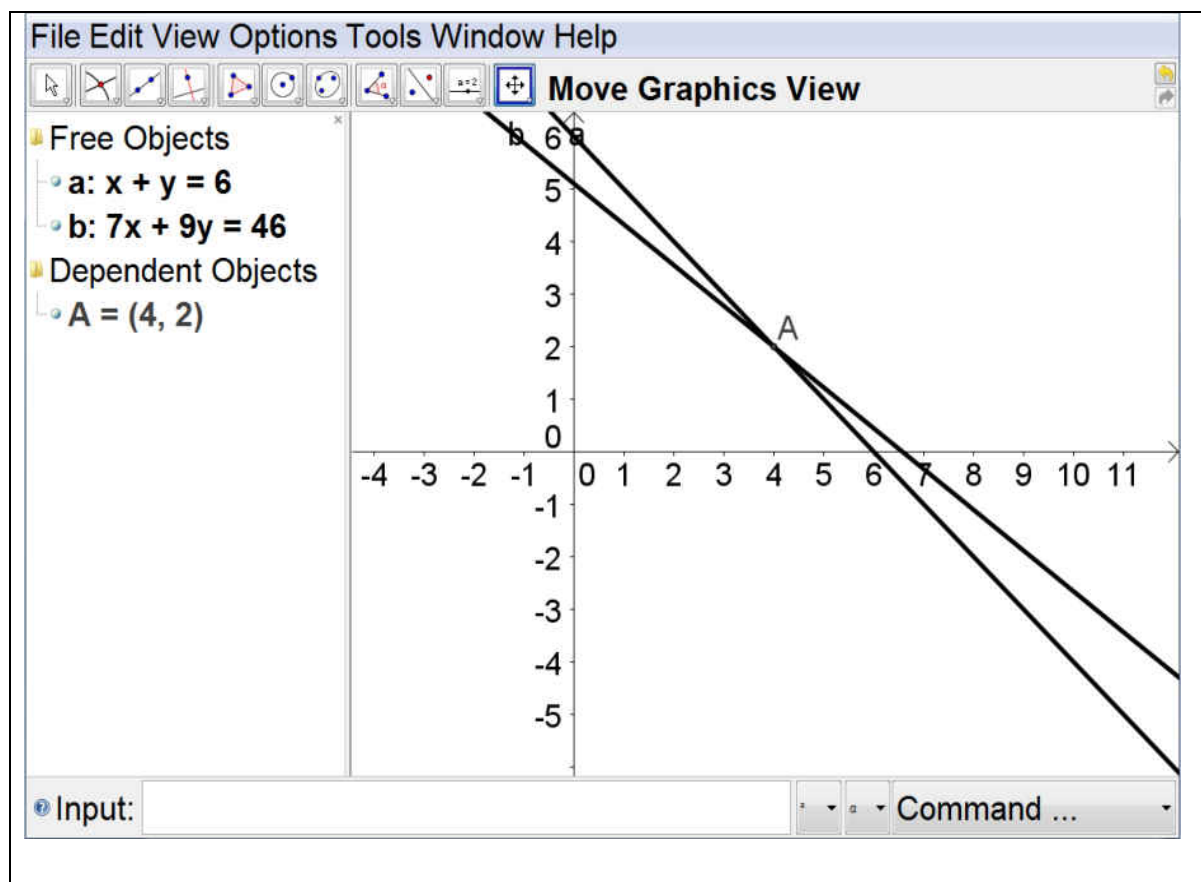


Figure 47. Mary's solution of GBT 3

GBT 4 and 5

GBT 4: A 640-mile, 5-hour plane trip was flown at two speeds. After flying 2 hours, the tailwind picked up and the speed in the second part of the trip was two times as the speed in the first part. What was the speed in first and second part of the trip?

For GBT 4 (see Figure 48), Mary only opened the spreadsheet from the given blank file, and guessed and checked the numbers. She used the formula tool of spreadsheet to facilitate her trials. She inserted 1 (hour) into cell *A1* and 2 (hours) into cell *A2*. She wrote her guesses for the distance covered in one hour for the first part of the trip (speed in the first part) into the cell *B1* and the distance covered in two hours with the same speed into *B2*. She also used that cell to represent the speed of the plane during the second part of the trip. She inserted $640 - B2$ to

represent the distance for the second part into cell *D1*. She represented the same distance by using $B2*3$, which was the speed flown during the second part times the flying time for the second part and inserted it into cell *D2*. Once she reached the same number value for *D1* and *D2*, she felt sure that her trial was right.

Mary: Lets first do 100 miles per hour. So he went to 2 hours so 200. In one hour it will be 100 miles in 2 hours it will be 200 miles if he goes 100 miles/h. $640-200$, it is 440. 3 times speed 200, that is 600. Ok it is not going to work. Change the speed to 50. 50 and the next two hours would be 100. Plus 300 is 400. This is too little. 75 and 150. 640 minus 150 is 490. 150 times 3 is 450. It is 40 short. If I try 80, that will give you 160. Ohhh. Ok. There you go. So in first two hours they are going 80 miles per hour and the last three hours would be 160 miles per hour.

	A	B	C	D	E	F	G	H	I
1	1	80		480					
2	2	160		480					
3									
4									
5									
6									
7									
8									
9									
10									
11									
12									
13									
14									

Figure 48. Mary's solution of GBT 4

Mary demonstrated a strong preference for nonvisual thinking and made no attempt in this task at any methods that are considered visual thinking. When Mary's GBT 4 was compared with her PBT 4 (see Figure 38 and Figure 48), there were no differences in terms of visual or

nonvisual solution method. She used nonvisual methods in both solutions. Thus, using Geogebra did not affect Mary's preference for this particular task. I designed another task to further examine the effect of technology on Mary's thinking preferences. The difference between the fourth and fifth task is only the context of the problem and the given information. In task 5, the speed was given and time was not given as opposed to task 4, in which time was given and speed was not given.

GBT 5: A man drove from home at an average speed of 30mph to an airport. He then boarded a helicopter and flew to the corporate office at an average speed of 60mph. The entire distance was 150 miles. The entire trip took 3 hours. Find the distance from the airport to the corporate office.

Mary's approach to PBT 5 (see Figure 39) was not different from her approach to PBT 4 and GBT 4, however, her method of solution showed differences between GBT 4 and 5. Although she used numeric representation and displayed nonvisual thinking in GBT 4, she used a visual method in GBT 5 (see Figure 49). Once Mary was presented with the task, she said that she wanted to solve it by using a graph. She opened the geometry window with axes. Then she created two equations for the given situation to draw their graph. She found the intersection of the graphs by clicking *Intersect two objects* and the two graphs. She chose x as the time that was spent driving and y as the time that was spent flying. Once she had determined x as 1 and y as 2, she calculated the distance from airport to corporate office: "If it took one hour for the first part, that means 30 miles, 2 hours for second part. This would be 120 miles. So airport was 120 miles from corporate office."

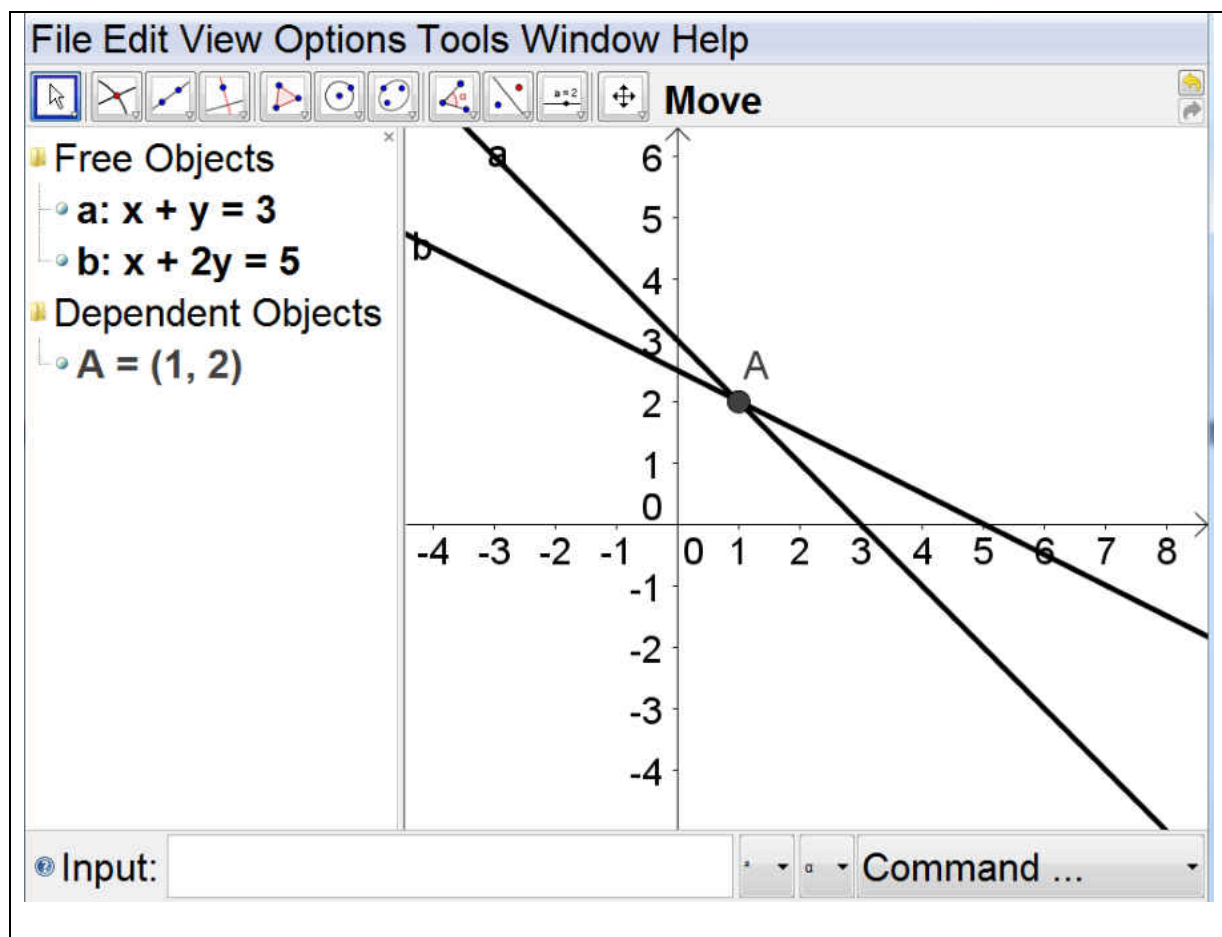


Figure 49. Mary's solution of GBT 5

Even though the fact that she mostly interpreted the PBTs by nonvisual methods, while she was attempting to solve GBT 5 she applied visual methods. The availability of the graphing tool in Geogebra and its automatic and on-scale graphing function might have affected Mary's thinking preferences. In a paper-and-pencil environment, Mary would have to create an x - y coordinate system herself and sketch graphs on scale by creating equations to find the intersection points of those graphs. The ease of doing those processes in Geogebra might have resulted in her selection of a visual solution method. The nonvisual solution of those problems using Geogebra also was an easy and short process. Her solutions of the following problems show the automation of Geogebra when creating a nonvisual solution.

GBT 6, 7, and 8

GBT 6: Cher wants to buy a refrigerator. She visits a store and finds two recommended models. The major brand is \$600 and is expected to cost \$30 per month in energy cost. The minor brand is \$400 and is expected to cost \$40 per month in energy cost. Which refrigerator would you advise Cher to buy?

Similar to PBT 6 (see Figure 40), Mary used mainly numeric representations in GBT 6 (see Figure 50). She opened only spreadsheet view and created a table just as she did when solving PBT 6. The only difference that could be observed was in the automation of creating the table when the first two patterns are given. Mary inserted *630* into cell A1 and *440* into B1 to demonstrate how much it would cost for each brand after 1 month. For the second row, Mary inserted *660* into cell A2 and *480* into cell B2 to demonstrate the cost after the second month. She selected cells A1, A2, B1 and B2 and dragged down until the 22nd row. She realized that at the 20th row the values in both columns were the same. "...So after 20 months, this one [minor brand] starts to be more." Mary moved to the next task after her solution for GBT 6.

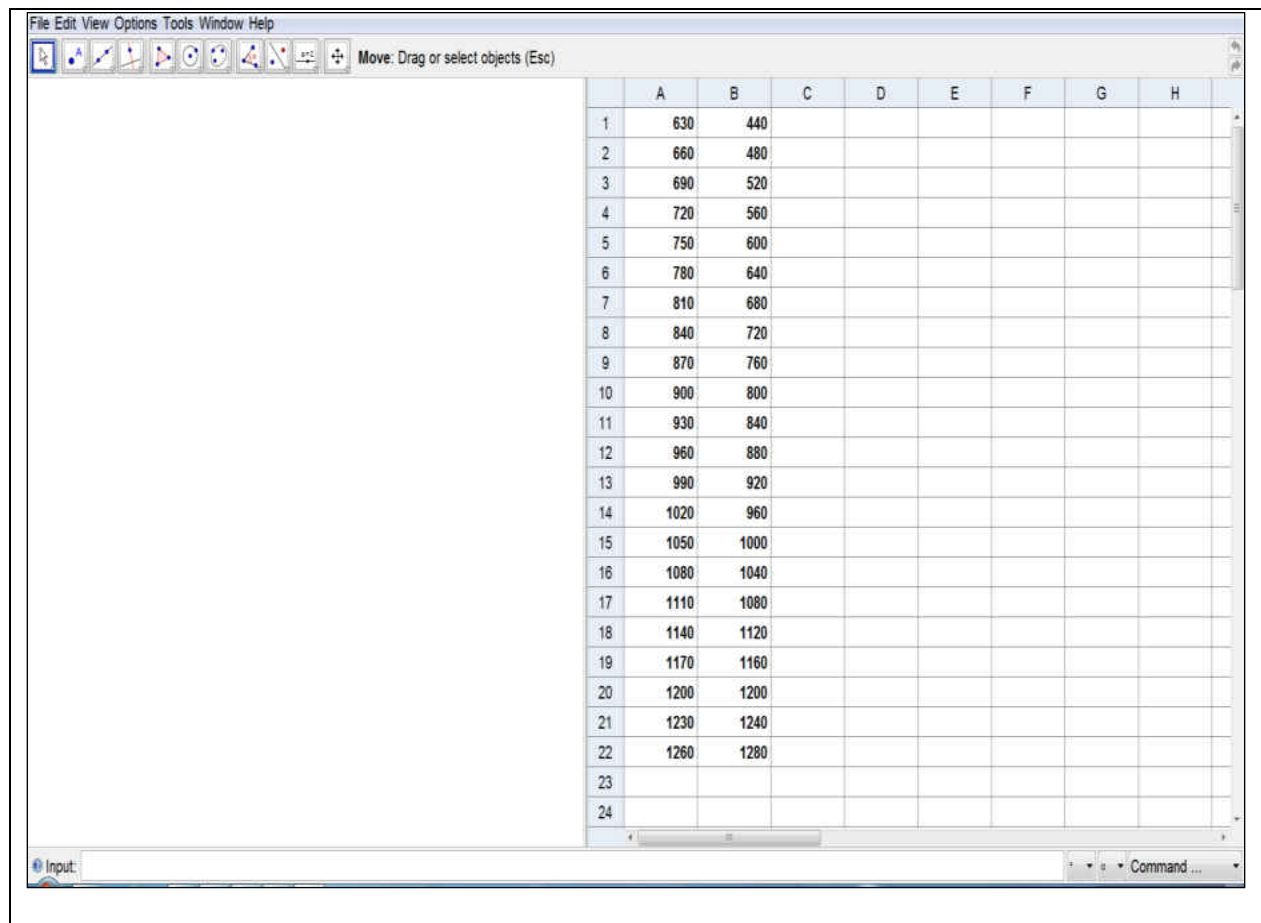
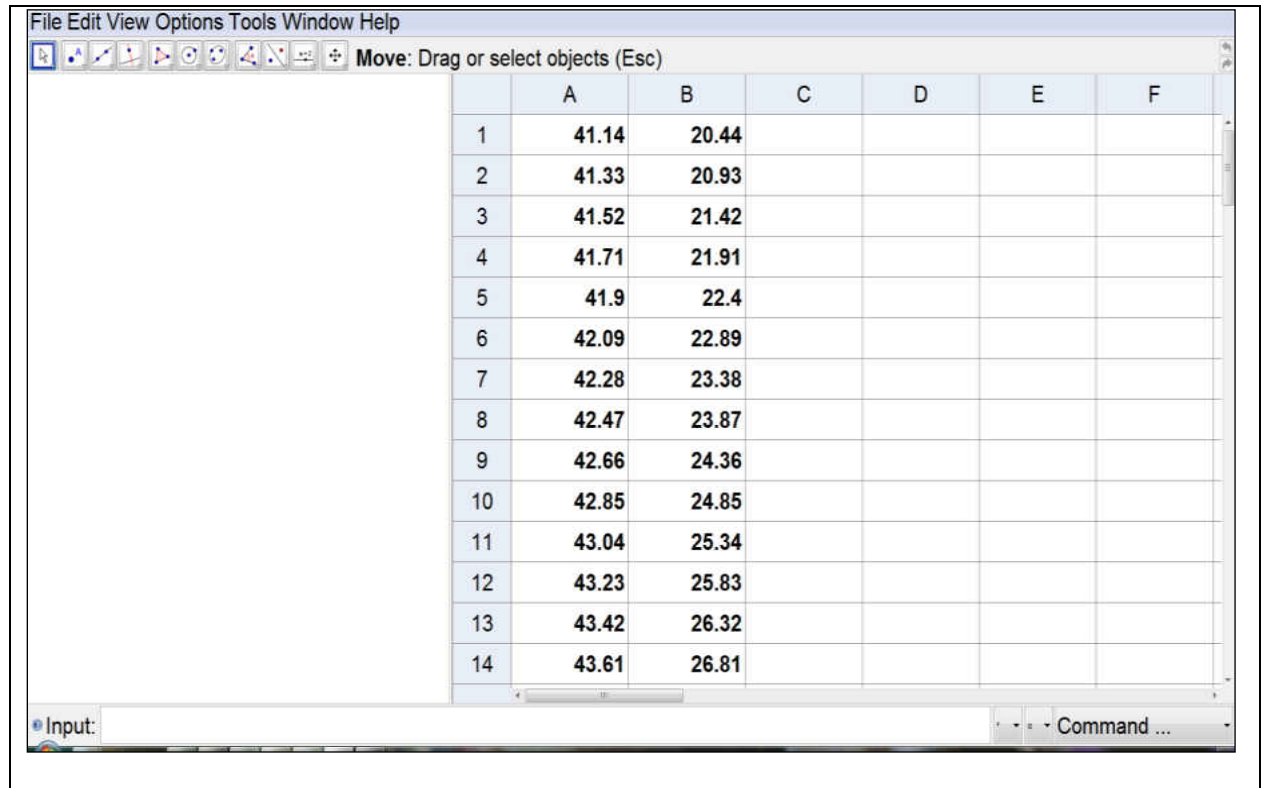


Figure 50. Mary's solution of GBT 6

GBT 7: You contacted two local rental companies and obtained the following information for the one-day cost of renting a truck. Company 1 charges \$40.95 per day plus \$0.19 per mile, and company 2 charges \$19.95 per day plus \$0.49 per mile. How many miles do you need to make in one day to get benefit by selecting the first company?

Similar to GBT 6, Mary activated the spreadsheet view and created table values for the problem. For column A and B, she entered the values for Company 1 and 2 respectively. Each time, she added \$0.19 to the \$40.95 initial fee for column A. She implemented the same procedure by adding \$0.49 to the \$19.95 initial fee for column B. By selecting A1, A2 and B1,

B2 and dragging until the 70th row, Mary was able to see where both of the companies would cost the same amount of money for the same miles driven.



	A	B	C	D	E	F
1	41.14	20.44				
2	41.33	20.93				
3	41.52	21.42				
4	41.71	21.91				
5	41.9	22.4				
6	42.09	22.89				
7	42.28	23.38				
8	42.47	23.87				
9	42.66	24.36				
10	42.85	24.85				
11	43.04	25.34				
12	43.23	25.83				
13	43.42	26.32				
14	43.61	26.81				

Figure 51. Mary's solution of GBT 8

GBT 8: A car is leaving Roseville at a constant speed of 50 km/h to go to San Francisco. Another car leaves at the same time from the next city which is 45 km closer to San Francisco at a constant speed of 40 km/h. In how many hours will the car leaving from Roseville overtake the other car?

For GBT 8, Mary implemented the same method of solution as with GBT 6 and 7. She created the table values for the distance traveled by each car (see Figure 52).

Mary : I guess I will go by hours. If he is already 45 ahead, first hour he would be at 85km. First hour for this person would be 50. Then another 40 for the second hour [inserts 125 into the next cell] and another 50 for this person [inserts 100 to the next cell]. [Selects four cell and drags it until 10th row] Ok he passed him right here [Shows 5th row]. After 5 hours he will pass the other one.

	A	B	C	D	E	F
1	85	50				
2	125	100				
3	165	150				
4	205	200				
5	245	250				
6	285	300				
7	325	350				
8	365	400				
9	405	450				
10	445	500				
11						
12						
13						
14						

Figure 52. Mary's solution of GBT 8

As seen in the excerpt Mary calculated the distances traveled by each car until 10 hours. Then Mary compared the numbers in two columns and decided at which point the car overtook the other car. Mary's solutions for GBT 6, 7, and 8 indicated that Mary demonstrated characteristics of a nonvisual thinker and technology did not affect her preferences of solution methods in those tasks. In GBT 6, I asked Mary the reason she had chosen the spreadsheet view. She reported, "That one was easier for me because the way spreadsheet works you just plug the numbers, drag it down, it will show you. You just look at the numbers where they would be best." When I asked Mary the same question about GBT 7, she said, "I do not know, I just like playing with the numbers, spreadsheet is easier way of doing that. I could have done those with graph but I preferred spreadsheet."

As seen in the excerpts, for this problem numerical methods were easier for Mary although she mentioned that she was able to solve it by graphing. The advantage of using

Geogebra—the spreadsheet function of Geogebra—might support her mathematical thinking preference for numerical methods and nonvisual thinking. Therefore, her strong preference for nonvisual thinking was clear in her answers to tasks 6, 7, and 8 of PBTs and GBTs.

GBT 9 and 10

GBT 9: There are 2 students wearing glasses to every 7 students not wearing glasses in a particular classroom. There are 63 students in that classroom. How many students are wearing glasses?

When I presented GBT 9, she expressed her intentions: “I really want to do it with graphing, but I do not see this one in graphing. I am sure you can graph it. I am going to use spreadsheet. I think this is easier for me.” She used the spreadsheet view to solve GBT 9 (see Figure 53). Her solution was similar to GBT 6, 7 and 8. She inserted 2 into cell A1 and 7 into B1 to represent the ratio between students wearing glasses and not wearing glasses. In the second row, she inserted 4 into cell A2 and 14 into cell B2. She selected those cells to create the rest of the pattern. She dragged the pattern until the 10th row.

Mary: Let’s see if they add up to 63 [Selects 4 cells and drag it until 10th row]. 20, 70 is 90. It is definitely not it. That is 54 [points 12 and 42] and this one is 63 [points 14 and 49]. So there are 14 people wearing glasses.”

Once Mary checked that the numbers in the 7th row added up to 63, she stated that there were 14 people wearing glasses. At the beginning of the problem, Mary wanted to use a graphic representation, but she stated that she would not solve it by using a graph since she could not create equations to draw the graph. Once Mary completed her numerical solution, I asked her the reason she had chosen a method that mainly employed numeric representations.

I: So do you use spreadsheet when you cannot graph it?

Mary: I do not know why I cannot graph. I am sure there is an equation but I do not find the equation. Wait wait wait. Maybe I can. 2 to 7, 63 in total. $x + y = 63$

I: What is x and y?

Mary: x is the students wearing glasses and y is the students not wearing glasses.

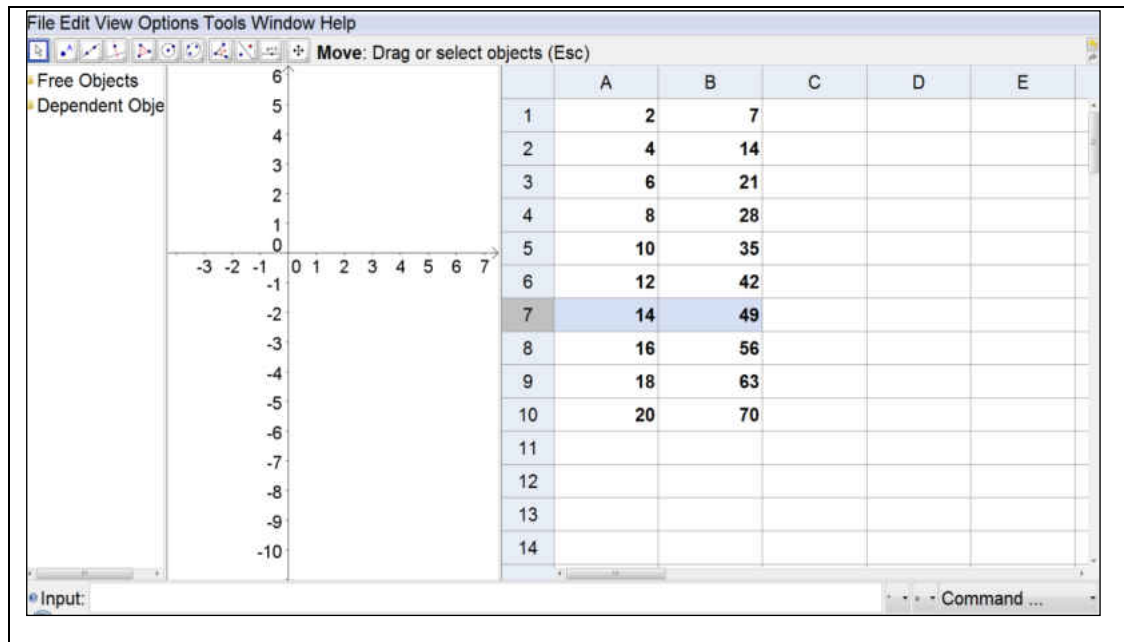


Figure 53. Mary's solution of GBT 9

As can be seen in the excerpt, when I asked Mary about graphing, her response was only focused on creating an equation and finding the graph of that equation. Mary might think that the only way of solving with graphs is creating equations and drawing their graphs, which is defined as the translation process from formulae to graph (Janvier, 1987). She might not consider a solution that includes the translation process from table to graph. Therefore, when she is not comfortable with creating equations, she might see the only alternative as solving with numeric representations.

Once Mary determined the first equation as presented in the excerpt ($x + y = 63$), she also wrote a second equation ($2y = 7x$) and drew the graph of them to find the intersection point. Even though Mary reached the solution by using a numeric representation, she had the desire to

solve the problem by creating equations and drawing the graph of those equations. The availability of the graphing tool might create a desire and preference for Mary to solve algebra problems by using visual methods. However, her reliance on and trust of nonvisual thinking might have an effect on her solution method, despite mentioning that she wanted to solve the problem by using a graph. Mary's numerical solution might facilitate her graphical solution. Once she has solved a task by using numeric representations, she might feel more comfortable using other methods.

GBT 10: A and B each have a certain number of marbles. A says to B, "if you give 30 to me, I will have twice as many as left with you." B replies "if you give me 10, I will have three times as many as left with you." How many marbles does each have?

The following excerpt illustrated Mary's method of solution for this task.

Mary: I think I am going to use graphing. Let me try to think of. So let's say A is x and B is y . Let's do B first. I am not sure about A. So the amount he has [inserts y into input field] is y plus 10 [writes $y + 10$] equals $3x$, I guess. Because x would be A and y would be B [inserts the equation $y + 10 = 3x$]. $x + 30 = 2y$. [Moves the graphing view and finds the intersection by clicking on intersect two objects and two lines] So A has 10 marbles and B has 20 marbles. Wait wait wait, if you give me 30, I will have twice as many as left with you. If he does not have 30 to give, I do not agree with that.

I: Why are you not agreed with that?

Mary: Because he has to have 30 to give... I did $y + 10 = 2x$. Because y represent the amount of marbles he has, he says give me 10 so you add 10 to it. And he will have did I say 3 times? Did I write 3? I do not remember if I wrote 3.

I: If you want you can do it again.

Mary: Ok [She deleted everything in the Geogebra file]. Would not be $x - 10$?

I: Why did you decide to write like that?

Mary: Because, if you count the marbles x will have no longer than have, it might be the problem. Because if you keep it in the equation, if you keep it with her then, technically he will give her 10. Ok so that would be the problem. This would be $y + 10 = 3(x - 10)$. $x + 30 = 2(y - 30)$ [Inserted the equation into input field]. So let's

see the line. Let me zoom out [zooms out and clicks intersect two objects and two lines]. Alright, I agree with that. So A has 34 marbles and B has 62 marbles. $62 - 30$ is 32, 30 plus 34 is 64, 2 times 32 is 64, yeap. $62 + 10$ is 72 and $34 - 10$ is 24, 24 times 3 is 72. Yeap.

For GBT 10, Mary created equations and drew the graph of those equations to find the intersection point. She called x as the amount of marbles A had and y as the amount of marbles B had. Then she wrote two equations $y + 10 = 3x$ and $x + 30 = 2y$ for the given two situations. However, she did not consider that when A or B gives some marbles away, they lose the same amount marbles while the other side gains that amount. Therefore, Mary created an incorrect equation for the given situation. Once she created the graph of equations and determined the intersection point of them, she realized that the intersection point did not satisfy the given condition. She stated that if B had 20 marbles then he would not give 30 marbles away. She realized that there was something wrong in the equations. She revised them and recognized that she did not subtract the amount of marbles from the marbles of the person who gave away. She rewrote the equations as $y + 10 = 3(x - 10)$ and $x + 30 = 2(y - 30)$. Since she did not accept without questioning and overrely on the solution that she reached by using the equation and its translation into a graph, she was able to correct herself.

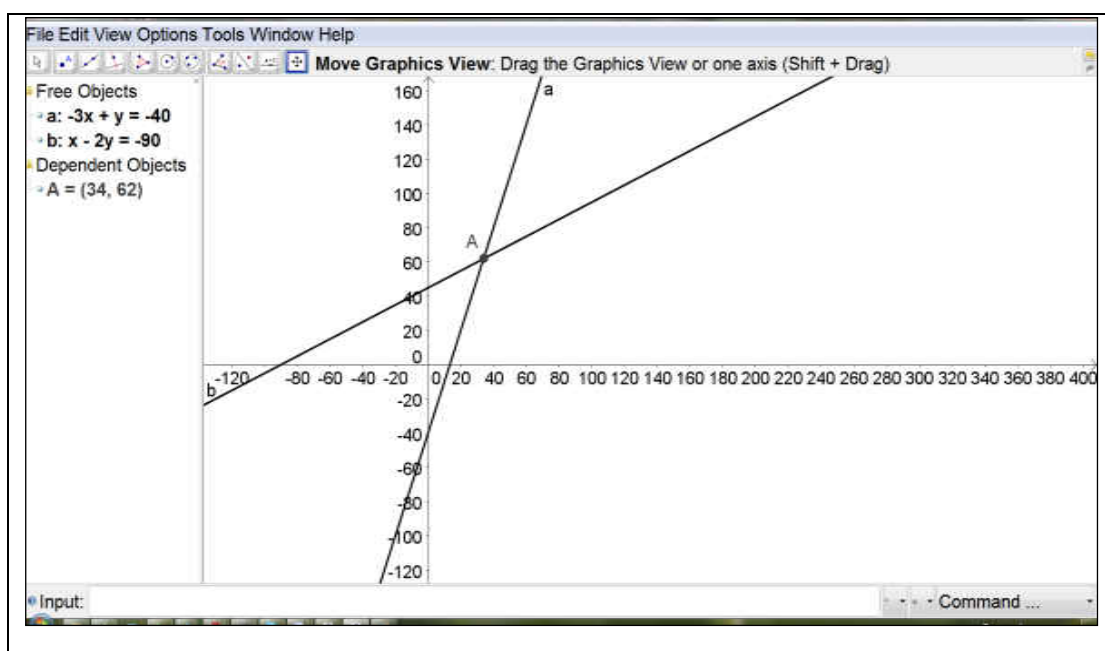


Figure 54. Mary's solution of GBT 10

As seen in the excerpt, Mary verified her solution with numerical methods even though her solution method was not numerical. She mentioned that the amount of marbles B had, had to be greater than 30 since B would give 30 marbles to A. Verifications of answers with numerical methods are also present in her other solutions, with or without Geogebra. This indicated that Mary supported her solution with the methods that she used to solve problems. It is clear, however, that the presence of the graphing tool in Geogebra affected Mary's preferences in some problems, since she did not try any visual methods in her PBTs.

After Mary completed tasks, I asked her the reason she used graphic representations in this task. The following conversation took place between her and me.

I: Why did you decide this one to do with graph?

Mary: I do not know. One reason is I want to do a graph because I rarely do that. When I read the problem, the way it is worded, you have the two numbers. You have two situations so if I see two situations I see two equations, I mean two graphs.

I: Do you see better with the graphs?

Mary: Hmm. I do not know, I think it is easier for me do it with numbers even though it is quicker with the graph.

Mary explained the occasions of using graphic representations in Geogebra. Desire to solve with a different method than she was used to was one of her reasons for changing her method. Another reason was her intention to find the way that took least time even though she was better in other solution methods; she reported that numerical methods are easier for her although graphical methods take less time. She always interpreted the solution numerically to verify its correctness. Her solution methods of PBTs did not show any diversification (visual or nonvisual solution methods) in the representations she used, but in the presence of different tools, namely those of Geogebra, her solution methods showed different representations. Especially when she was presented with a blank Geogebra file, she preferred to choose both to use numerical and graphic representations.

Geogebra offered Mary the ability to use multiple representations. For example, in GBT 9 she was not at first able to solve the problem by using a graphic representation, even though she had the desire to solve it with a graph. Once Mary solved the problem by numeric representation, she was able to solve it using graphic representations. Since Geogebra supports multiple representations, it was possible to solve problems on the same screen in different views with different representations. Solving a problem with one representation might facilitate solving the problem with another representation. Therefore, the multiple representation tools in Geogebra might have encouraged Mary to translate between representations and enabled her to solve each problem with her desired method.

By investigating only Mary's solutions of GBT problems, she could be categorized as a harmonic thinker who has a relative equilibrium of well-developed verbal-logical and visual-

pictorial components, since her solution methods included both visual and nonvisual methods. However, as previously mentioned in her solutions of PBTs, she demonstrated characteristics of nonvisual students by solving all PBTs with nonvisual methods. Therefore technology created a change in her thinking preferences while solving algebra word problems.

The Effects of Virtual Physical Representations

In this part, I investigated the effect of virtual physical representations in Geogebra on her methods of solution. In order to examine this effect, I compared Mary's solution methods for GBTs with her solutions for GBT-VPRs. As mentioned before, GBT files were totally blank files whereas GBT-VPR files included virtual physical representations of the given problem. If students did not want to use given files with representations, they would be allowed to open a default Geogebra file in which to solve the problem.

GBT-VPR 1, 2, and 3

GBT-VPR 1: Annette has 21 coins consisting of dimes and quarters. The total amount of money she has is \$3.30. How many each coin does she have?

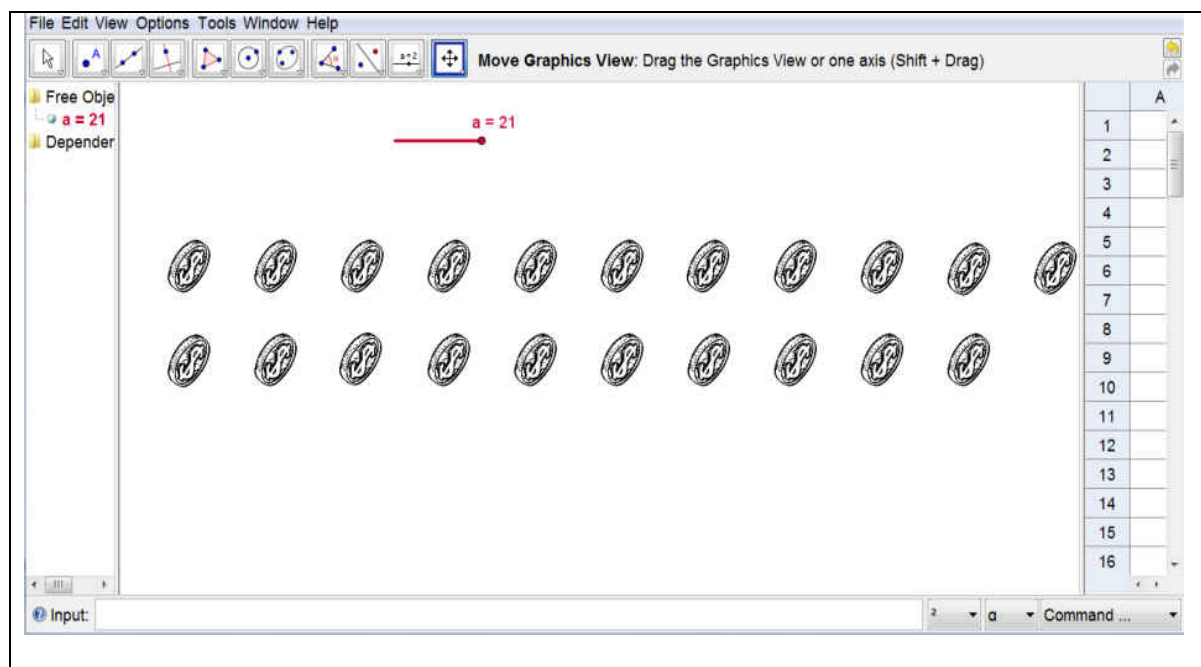


Figure 55. GBT-VPR 1

I asked Mary to solve GBT-VPR 1 (given in Figure 55) by using any representation she preferred to use. The slider a in the file controlled the number of coins by displaying them on the screen. Once Mary read the problem she said that she was going to use the graphing tool. She went to the File menu and opened a new window.

Mary: x would be the dimes, and y would be the quarters. So the number of dimes and quarters equals 21. So then we have $10x + .25y$ equals 3.30. (Clicks on *intersect two objects* and the two graphs. Zooms out to see the graphs and determines the point of intersection of them). So 13 dimes and 8 quarters. So 13 dimes is 1.30 and 8 quarters is 2 dollars. 2 dollar plus 1.30 is 3 dollars and 30 cents. 8 plus 13 is 21, yeah they add up.

Mary chose x as the number of dimes and y as the number of quarters. She created an equation for the first statement by writing $x + y = 21$ in the input field. Once she wrote the equation, she pressed Enter and Geogebra drew the graph of the equation for her. She inserted the equation $0.10x + 0.25y = 3.30$ for the second statement into the input field. Then she found the intersection point of those two graphs by clicking on *Intersect two objects* and the two

graphs. She interpreted the intersection point and stated that there had to be 13 dimes and 8 quarters (see Figure 56).

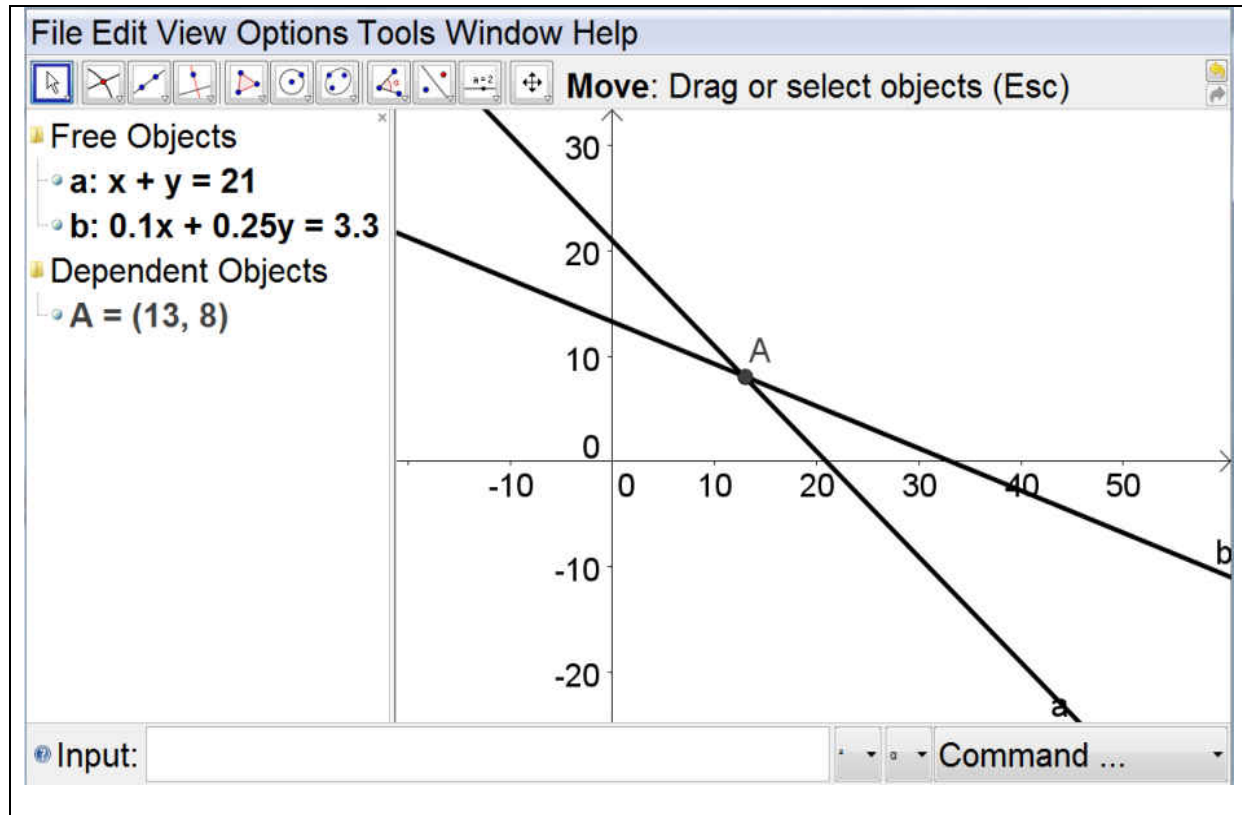


Figure 56. Mary's solution of GBT-VPR 1

Mary followed the same process as with GBT-VPR 1 when she was solving GBT 1 (see Figure 45). I categorized Mary's solution method for GBT-VPR 1 and GBT 1 as visual methods since she created equations for the statements in the problems to draw the graph of two equations. She also mentioned at the beginning of the problem that she wanted to use a graph. Moreover, she zoomed out to seek for the graph. If her intention was to solve the problem with only algebraic representations, she would not zoom out to seek for graph. She would be using only algebra window or spreadsheet to solve the algebraic equation. Visual methods are defined as the methods which include mainly graphic representations. In this problem, since she used the

formula to create a graphic representation, and she did not continue to solve the problem in an algebraic way, her method of solution is categorized as a visual solution.

GBT-VPR 2: Farmer Brown has ducks and cows. The animals have a total of 12 heads and 32 legs. How many ducks and how many cows does farmer Brown have?

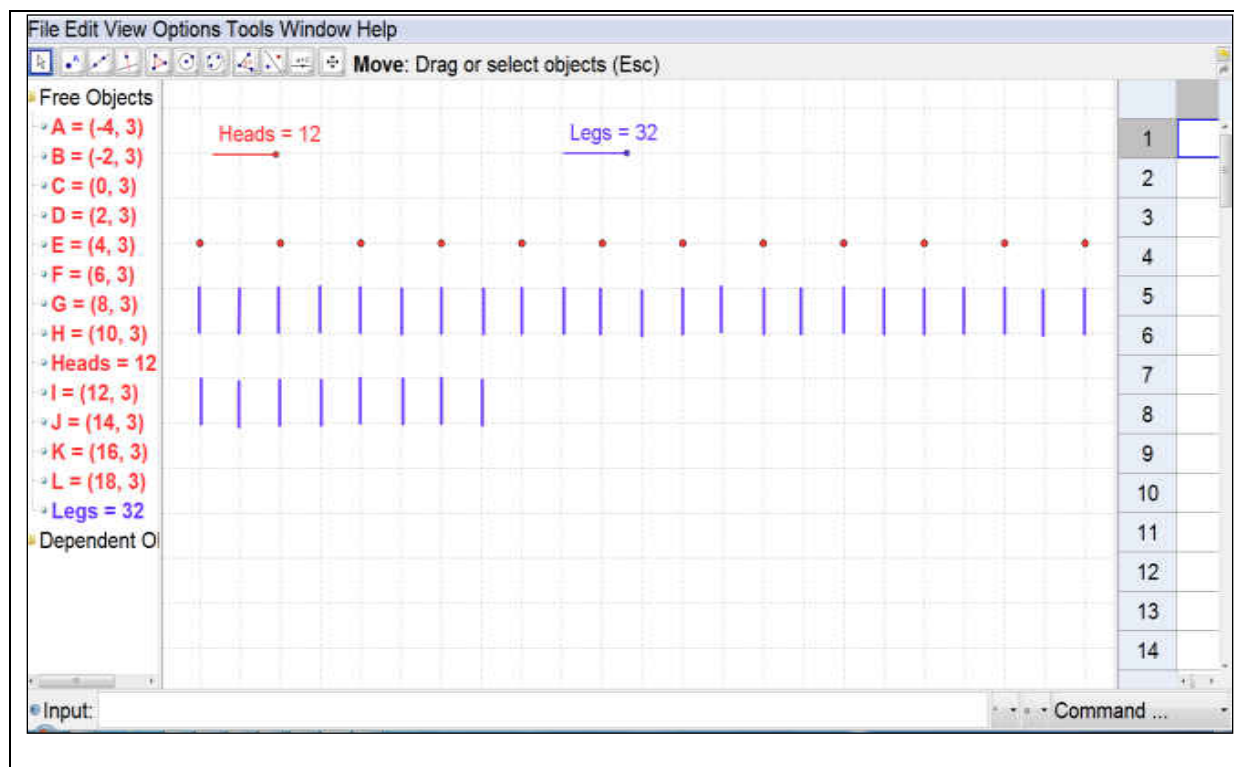


Figure 57. GBT-VPR 2

For GBT-VPR 2, Mary again did not prefer to manipulate the given virtual physical representation of the problem (see Figure 57) in which the heads of the animals were represented with the dots and their legs were represented with lines. The sliders controlled the number of the heads and legs on the screen. Mary opened a new window to draw the graph of the equations that she created for the problem as she preferred to do in GBT-VPR 1.

Mary: So total 12 heads, total 32 feet and we need to figure out how many ducks and how many cows. So ducks have 2 feet and cows have 4 feet. 12 heads, 32 feet [Opens graph view]. If I did $x + y = 12$ to represent the heads [Inserts $x + y = 12$ into input field]. Ducks has two legs so $2x$, x would be the ducks, and $4y = 32$. I am trying to figure out what this tells me (clicks on intersect two objects and the graphs to determine the

intersection point of two lines) A is 8,4 so there are 8 ducks, 4 cows. 8 times 2 is 16, and 4 times 4 is 16. 16, 16 is 32 and heads, 8, 4 is 12.

Mary chose x as the number of ducks and y as the number of cows (see Figure 58). She entered the equation $x + y = 12$ in the input field for total number of animals. Then she entered the second equation $2x + 4y = 32$ in the input field to draw its graph. She determined the intersection point of two graphs and interpreted this point by expressing that there were 8 ducks and 4 cows. The same method of solution was observed in Mary's solution for GBT 2 (see Figure 46).

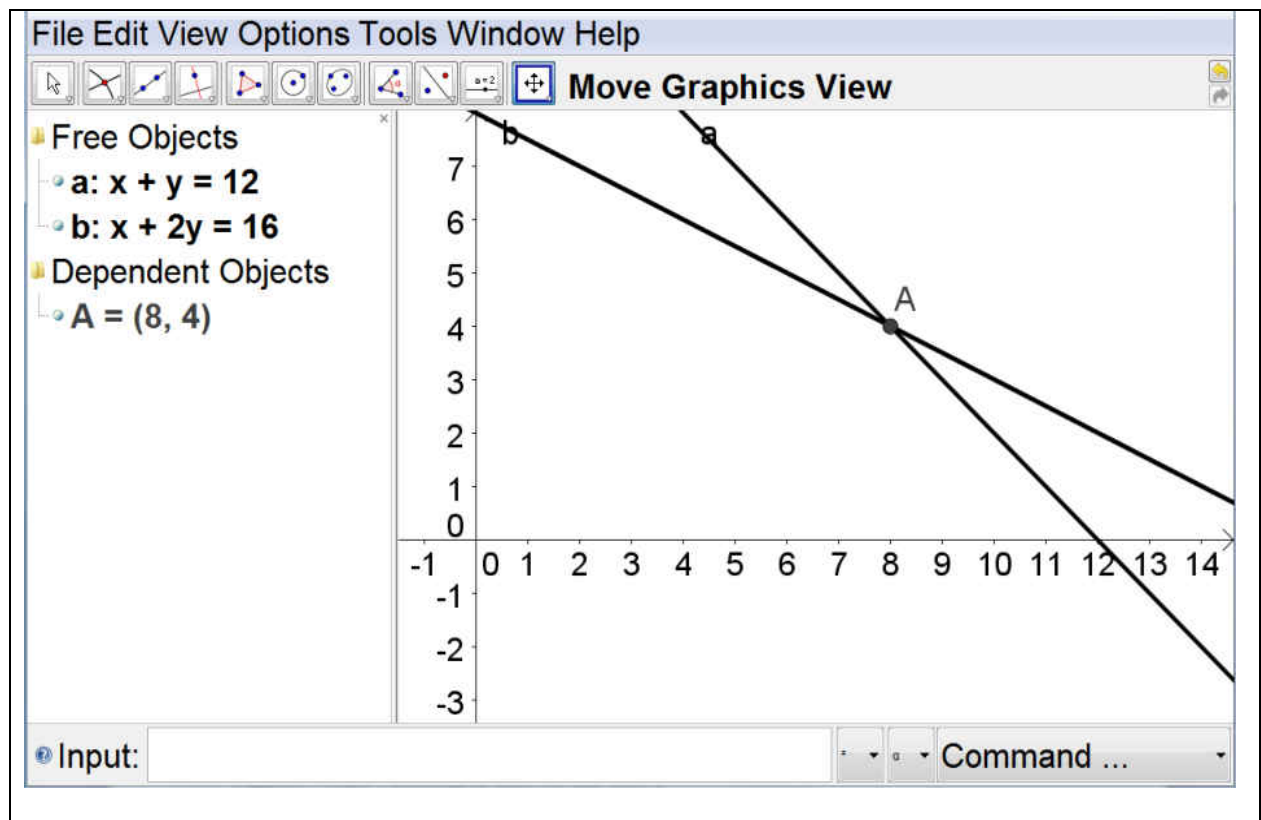


Figure 58. Mary's solution of GBT-VPR 2

When Mary's solution methods for GBT-VPRs were compared with her solutions for GBTs for the first two tasks, there was no difference between them. For those two tasks, the virtual physical representations did not affect Mary's solution method. After she read both GBT-

VPRs, she stated that she wanted to use graphic representation and opened a new file to create a graphic representation of the problems.

GBT-VPR 3: A group of exchange students from Japan went to a convalescent home to sing songs for the seniors and to demonstrate origami (Japanese paper folding). As it turned out, there was either one Japanese student at a rectangular table with three seniors or two students at a round table with four seniors. There were 11 students and 25 seniors in all. How many tables were being used to demonstrate origami?

In the GBT-VPR 3, Mary started to solve the task by focusing on the numerical aspects of the given setting. She attempted to change the values using the sliders until the number of Japanese students was equal to 11 and the number of seniors was 25 (see Figure 59). The following excerpt illustrates Mary's solution process.

Mary: [Changes the number of round table to 3 and rectangle tables to 1 by using slider] That gives me 7 students and 15 seniors. So if I put this one [changes the number of rectangle tables 2, 3, 4 and she stops at 5, the number of the students and seniors are 11 and 37 respectively], that is too many seniors. [Changes the number of round tables to 2, then 4] 4 round tables and go back [decreases the number of rectangle tables to 3]. Ooo, there it is. So 4 round tables and [checks the number to verifies the solution] and 3 rectangle tables.

Mary used guess and check as she changed the numbers of each type of tables. She used mainly numeric representations and only worked with the sliders. She neither dragged and dropped given figures nor drew the graph of the given situation. Mary made no mention of a visual interpretation of the task, thus Mary's thinking for this task is considered as nonvisual. At the end of the task, I asked Mary how she solved the problem, and she mentioned that she systematically guessed and checked the numbers.

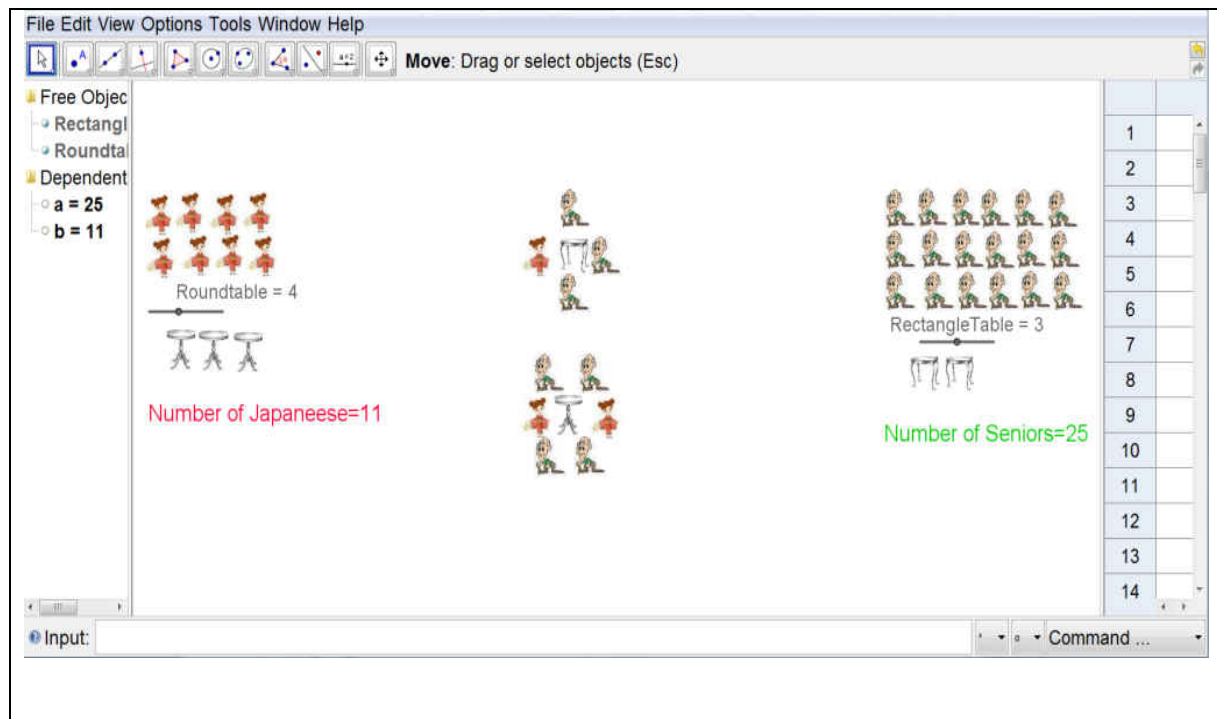


Figure 59. Mary's solution of GBT-VPR 3

Different from GBT-VPR 1 and 2, Mary's solution of GBT-VPR 3 mainly included nonvisual representations. According to Krutetskii (1976), students' focus on visual or nonvisual means while they are investigating mathematical material gives indications about their type of thinking. In this task, Mary focused on nonvisual features of the problem. The virtual physical representation of the problem affected Mary's method of solution. Therefore, there was a shift from visual solution to nonvisual solution for this task in the presence of virtual physical representations on Geogebra.

GBT-VPR 4 and 5

GBT-VPR 4: Hillary went riding in the hills. At one point, however her horse stumbled and was hurt. Hillary decided to go back to house with her horse. Hillary figures when they were going the horse walks about twice as fast as when they were coming back. If her horse was

hurt about 8 miles in, and the whole trip took 3 hours total, how fast did Horse walk when he was coming back to house?

The fourth and fifth GBT-VPR problems were speed-distance problems. Mary preferred to solve those problems by working with the given virtual physical representations. She used the sliders and similar to previous task she guessed the speed of the horse when it was going and checked the time spent for going and coming back. GBT-VPR 4 and a screen shot from Mary's solution are presented in *Figure 60*. In order to further understand her method, consider this excerpt from the interview.

Mary: This is 4 [changes the value of the slider v_1 to 4]. Then it means the speed for coming back would be 2 [changes the value of the slider v_2 to 2]. Let's see how long it will take to get 8. It will take 2 hours for the horse. So 4 miles per hour. That means there would be one hour left which probably would not be right because it would probably take her longer to get back. So let's go back [changes the value of sliders to 0]. Let's do 6 (slider $v_1=6$). This would be 3 [slider $v_2=3$]. So that is 6 miles per hour. So this is an hour and couple minutes. So lets see if it works for walking. So that means she is traveling 8 back. This would be too much. So let's go back to 0 (changes the value of slider t_1 to 0 and v_1 to 8). If this is 8 this will be 4 (the value of the slider v_2). One hour and 8 miles per hour. And this is 4 miles per hour. Let's see [changes the value of slider t until she comes 0]. It is 2 hours probably. So 2 plus 1 is 3. So that means they were going with a speed of 8 miles per hour and coming back 4 miles per hour.

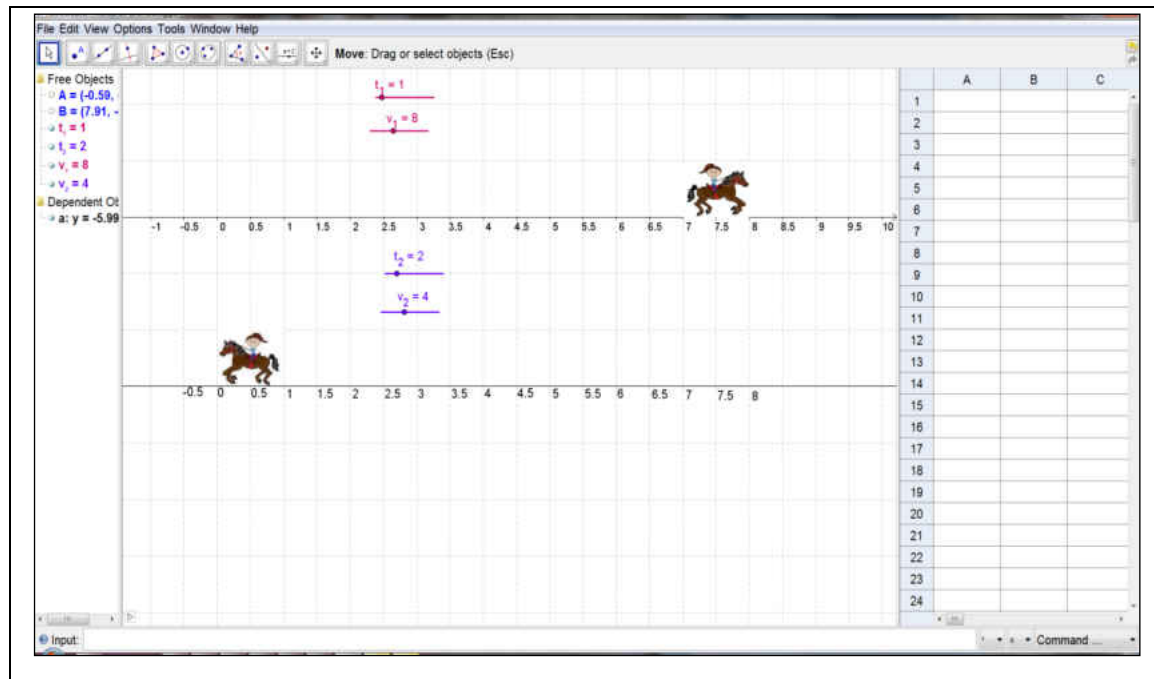


Figure 60. Mary's solution of GBT-VPR 4

First, Mary guessed the going speed as 4 mph and checked how much time should be spent to come to the point of 8 miles. Once she recognized if the speed was 4 mph then the going time would be 2 hours and coming time would be 1 hours, Mary expressed that it would not be possible since coming time would be longer than going time. Then she guessed the going time as 6 mph . When she realized that this speed did not satisfy the given condition, she tried 8 mph . She checked the going and coming time and the speed for coming back if the going speed was 8 . Once she realized the going time, 8 mph , satisfied the given conditions in the problem she expressed that they went with 8 mph and came back with 4 mph .

In this problem, Mary preferred to use sliders and virtual physical representations in the setting. Her solution includes both visual and nonvisual methods. In the setting of the problem, sliders are connected to the location of the picture. Therefore, it is difficult to differentiate between graphical and numerical methods, since they are connected. In order to determine the

primary method of her solution and which representations were dominant in her solution, I asked Mary how she solved the problem.

Mary: Basically I just plug the numbers as usual, and whatever I would put in for the horse, I would do half for her speed because she was going half as fast. Whatever, like using the sliders where she should be and how long it would take where they were going and whatever that was if it adds up to 3, then I would know what speed they were going.

According to Mary's explanation of her method, numeric representations predominated in her solution. Although she used a virtual physical representation of the problem, she mostly focused on the number value of the sliders. She constantly guessed the speed and time and checked distance. Mary's nonvisual (verbal-logical) thinking might have affected her solution method. She utilized the given setting according to her thinking preference. Similar to the previous problem, Mary used the given virtual physical representation but focused on its numerical aspect.

When Mary's solution of GBT-VPR 4 was compared with her solution of GBT 4 (see Figure 48), the main methods of solutions were not different. Her method of solution included mostly numeric representations, which indicated nonvisual thinking in those tasks. For GBT 4, Mary constantly guessed and checked the numbers by writing and deleting from cell B1 and B2 in order to determine the speeds. She implemented the same process by changing the value of sliders in GBT-VPR 4. Therefore, the use of Geogebra files with representations did not affect Mary's main method of solution.

GBT-VPR 5: Jason got on his bike and went for a ride. He rode at a speed of 10 km per hour from his house to his sister's house in another city. Jason and his sister then got in a car and traveled at a speed of 50 km per hour to their mother's house. The total distance from Jason's

house to his mother's house is 320 km, and Jason traveled for 8 hours. How far is it from Jason's house to his sister's house?

For GBT-VPR 5 Mary used the same strategy and method that she used in GBT-VPR 4.

Consider the following excerpt for insight into Mary's solution method.

Mary : They were going 50 when they get on the car. I am just going to split in the half first. If it took half the time. This one goes by 10 (speed), lets say it took (takes the value of time slider to 4) 4 hours, half of 8, that means he would have gone very little, 40 km. 320 minus 40 is 280. If this is where his sister house, so the distance from his sisters' house to her house would be 280. And it would take more than 4 hours to get to 280 miles so that does not work. Is it longer or shorter? Shorter amount of time or longer amount of time. Wait, wait, wait. 280, it would be way over. 4 hours, 5 hours. More than 5 hours to get to 280.

I: With 50 miles per hour?

Mary: Yeah, if they were in the car. I will say it took him less than 4 hours (time spent in bike). I am not sure if it is less or more. I will just try less then if it makes worse then I will go more. Ok at 2 hours, it is a long time on bike, we have very little 20 km. So that means 300. 300 divided by 50 is 6 hours. Plus 2 is 8. Oooo got it. So her house is only 20 km away from his house. Let me check this. So 10 km per hour, at 2 hours they get 20 km and then 300, 300 divided by 50 is 6. 6 hours plus 2 hours, is 8. So his house is 20 km away from her house.

As seen in the excerpt, Mary started by splitting up the time into halves. Once the distance covered by 4 hours biking and 4 hours driving did not reach the given point, she decided to decrease the biking time. She was not sure whether she was going in the right direction, but she stated that if it did not work, then she would go in the other direction (increase the biking time). Her second guess was 2 hours biking and 6 hours driving (see Figure 61). Once she plugged the numbers in, she realized that the numbers satisfied the condition. Mary again used guess-and-check. Her numerical strategies were also present in this task. She focused mostly on the numerical aspects of the solution even though she constantly guessed and checked the location of the kid given in the picture.

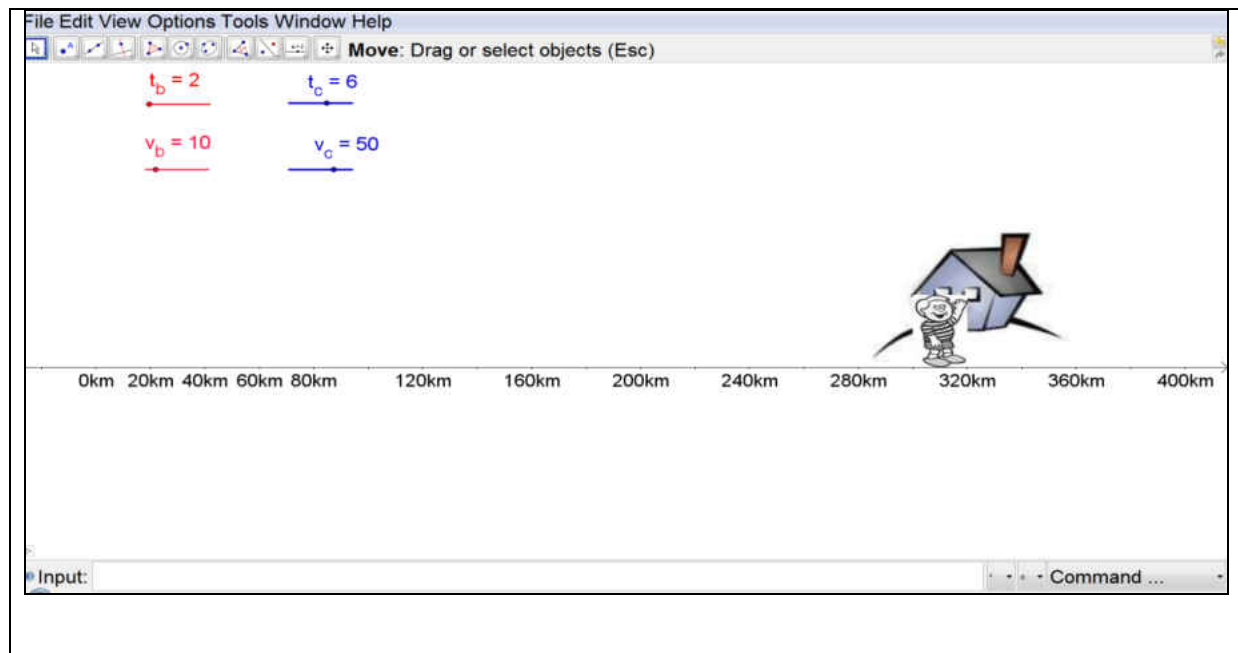


Figure 61. Mary's solution of GBT-VPR 5

According to Krutetskii (1976), analytic (nonvisual) types of students analyze the given mathematical material by nonvisual means. When Mary was solving GBT-VPR 3, 4, and 5, she focused on the nonvisual characteristics of the given setting. Therefore, the analyses of these GBT-VPRs also supported the assertion that Mary has a preference for nonvisual thinking.

Mary's strategy for GBT 5 was different than her strategy for GBT-VPR 5 (see Figure 49 and Figure 61). She created two graphs and found the intersection of them to determine the time spent on each part of the trip. Since this strategy was mainly visual, there was a shift from the visual solution to nonvisual solution when Mary was solving the problem with the virtual physical representations. Hence, Mary might reflect her preference for nonvisual thinking when she was using the virtual physical representations by focusing on the numeric representations in the given task.

GBT-VPR 6, 7, and 8

GBT-VPR 6: The telephone company offers two types of service. With Plan A, you can monthly pay \$6 plus 12 cents for each minute. With Plan B, you pay \$15 monthly, plus 6 cents for each minute. At least how many min would you have to use the telephone each month to make Plan B the better option?

Task six, seven, and eight consisted of very similar word problems. In each GBT-VPR Mary preferred to use the given virtual physical representation on the screen by focusing on its numerical aspects. In GBT-VPR 6, sliders a and b represent the minutes talked with plan A and B respectively. The number on the money bags represents how much money each bag contains depending on the minutes talked (see Figure 62). Mary constantly compared the money when the minutes were the same. She started to check from 125 and changed the value of the sliders each time.

Mary: Let's put it at 125, for this one put that 125. This (A) is still more. 135, still more. 140, still more. 145, still more. 150, they are equal. So at 150 minutes, they are equal, so I guess 151 minute, then this one (B) would be better.

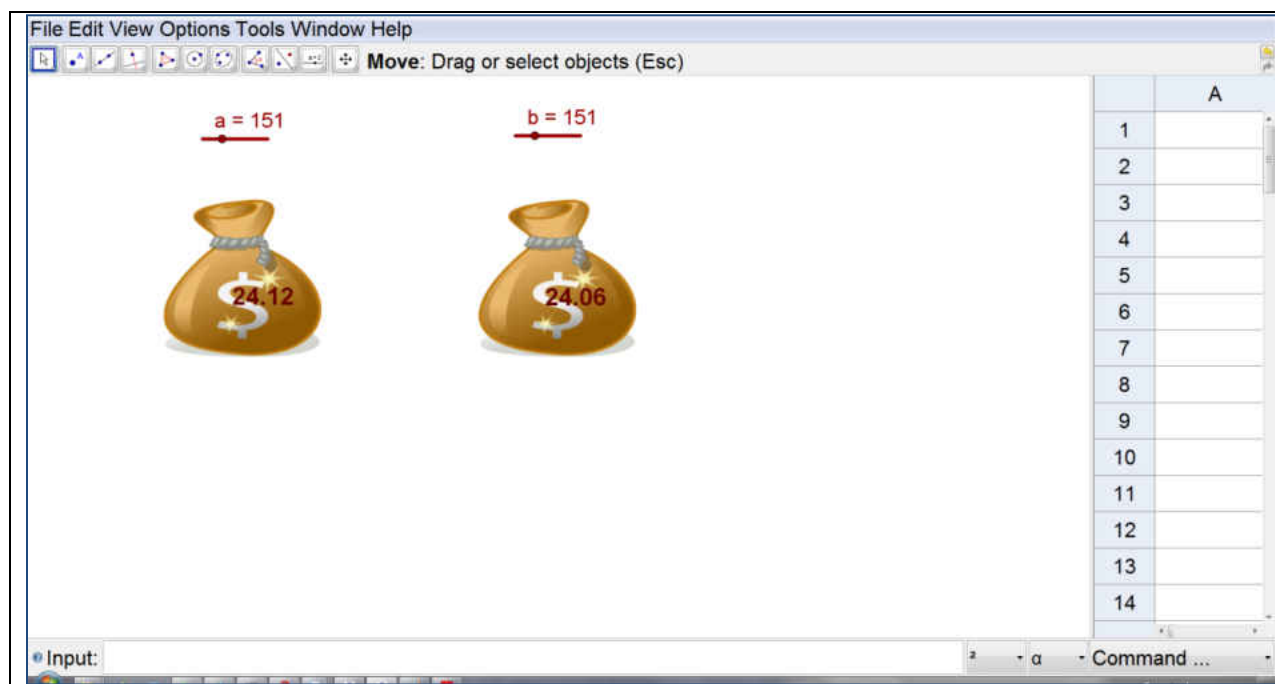


Figure 62. Mary's solution of GBT-VPR 6

The excerpt above illustrates that Mary focused on the numbers on the bags while changing the value of the sliders. Mary's nonvisual interpretation of GBT-VPR 6 was also observed in her work on GBT 6. Mary continued to use numeric representations in GBT 6. She opened only spreadsheet view and created a table. She inserted 630 into cell A1 and 440 into B1 to demonstrate how much it would cost for each brand after 1 month. For the second row, Mary inserted 660 into cell A2 and 480 into cell B2 to demonstrate the cost after the second month. She selected cells A1, A2, B1 and B2 and dragged down until the 22nd row. She realized that at the 20th row the values in both columns were the same. "...So after 20 months, this one [minor brand] starts to being more."

GBT-VPR 7: Both Jim and Todd consider their weight a problem. Each one is trying out for the football squad and wants to weigh more. Jim is eating and working out, and has found he gains about 1 pound each week. At this point he weighs 180 pounds. Todd, on the

other hand, weighs 167 and is eating, working out, and eating. He is gaining about 5 pounds every 3 weeks. How long will it take for Jim and Todd weigh the same?

Similar to GBT-VPR 7, Mary used the sliders to change the number of weeks which was connected to the weight of Jim and Todd (see Figure 63). Mary started guessing and checking while manipulating the numbers on the slider. Consider the following excerpt.

Mary : Lets' start with 5, this one is 5, so nope. 15 or maybe other 10, nope. 15 weeks, he actually gained himmm, 17 no. 20, ooo, 20. So try 19. So at 19 and half weeks they are equal. At 20 weeks he passes him.

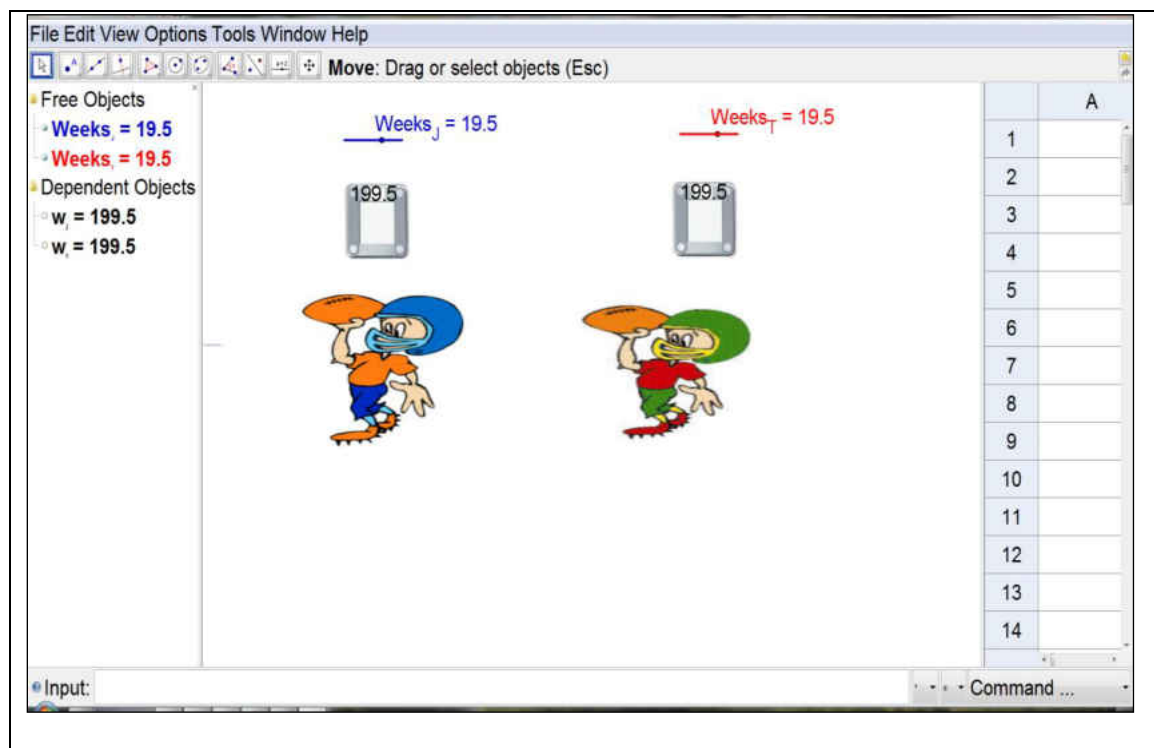


Figure 63. Mary's solution of GBT-VPR 7

Mary started her guess with 5 weeks. She increased her guess to 15, 17 and 20. In 20 weeks she realized that Todd weighs more than Jim. Then she changed the value of the number of weeks to 19.5 which resulted in the same weights for Jim and Todd. In this problem Mary again used nonvisual methods with numeric representations.

GBT-VPR 8: Sharon and Megan have \$20 and \$26 in their piggy bank respectively. Every day, Sharon saves a quarter and Megan saves a dime. After how many days they will have same amount of money in their piggy banks?

For GBT-VPR 8 Mary again used the same method with GBT-VPR 6 and 7. I designed GBT-VPR 6 and 7 using the same strategy, but GBT-VPR 8 was different in terms of automation from GBT-VPR 6 and 7. For GBT-VPR 6, the sliders showed the minutes talked for each plan, and the numbers on the money bags showed the cost of the service depending on the minutes talked. For GBT-VPR 7, the scales showed Jim and Todd's weights and the sliders represented weeks spent to gain weight. For both GBT-VPR 6 and 7, when the slider value changes (independent variable), the value on the money bag or scale (dependent variable) also changes. However, for GBT-VPR 8, the number values on the piggy banks do not change. I changed the design in GBT-VPR 8 since I wanted to investigate whether students would use this representation even though they would have to calculate the dependent variable by themselves. In other words, I wanted to test whether the reason to use the virtual physical representations in GBT-VPR 6 and 7 was choosing the solution that takes least time.

Mary preferred to use the sliders (see Figure 64) and calculated the money in each piggy bank as she was manipulating the values of slider. Consider the following excerpt.

Mary: It will take 24 quarters just to reach her. 4 times 6 is 24. That [Sharon] will be 26 dollars. 24 days 24 dimes, 2 dollars and 40 cents. She [Megan] will be at 28.40, she [Sharon] will be at 26. Ok up 2 more dollars so it will be 8 more days. So at 8 days, this will be at 28 and this will be at 29.20.

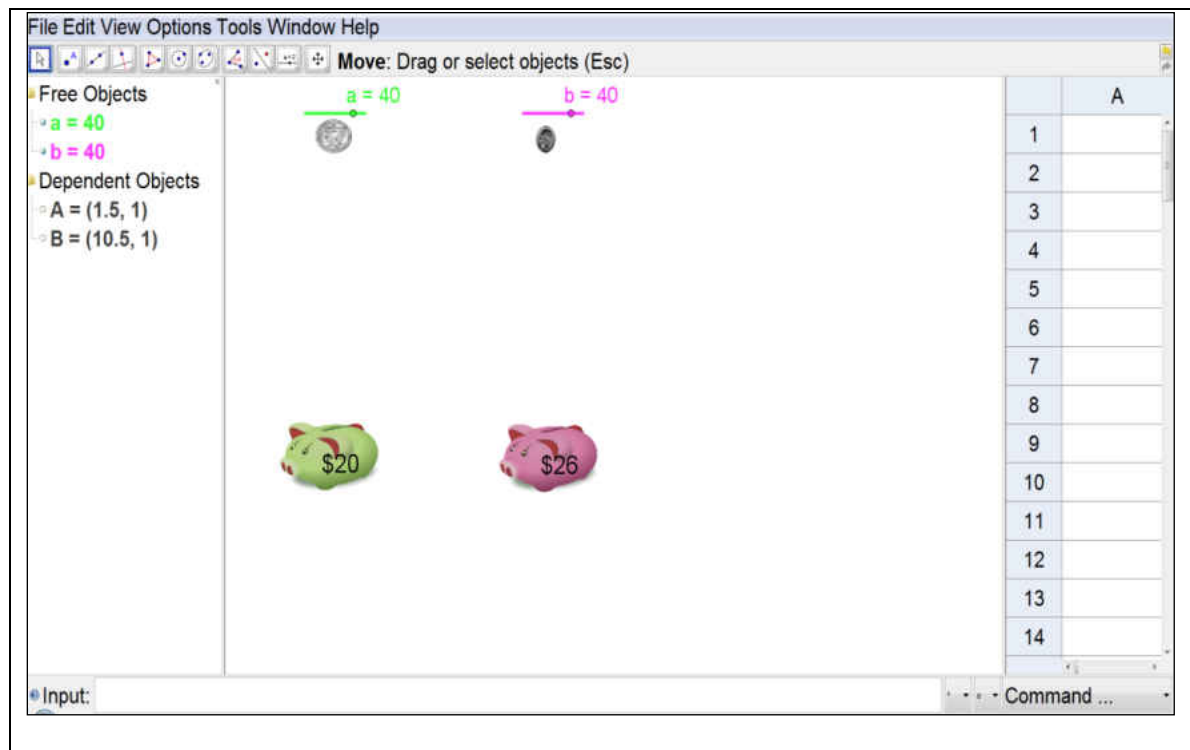


Figure 64. Mary's solution of GBT-VPR 8

Mary first calculated the number of days for Sharon to reach the money that Megan had. She determined that after 24 days Sharon would have 26 dollars. However, Megan saved a dime each day and after 24 days she had \$2.40 more money. In the rest of the problem, Mary's strategy was increasing the number of days to find when they both would have the same amount of money. Again Mary used nonvisual methods with numeric representations in this task.

In GBT-VPR 6, 7, and 8, similar to her approach to GBT-VPR 3, 4, and 5, she analyzed the virtual physical representations by focusing on nonvisual means. She used the sliders to recognize the pattern in numbers and to facilitate her guess-and-check method. Therefore there was no difference in terms of solution method between GBT-VPR and GBT for tasks six, seven, and eight. Mary's solution methods for GBT-VPR 6, 7, and 8 were aligned with her solution methods for PBTs as well as her MPI score.

GBT-VPR 9 and 10

GBT-VPR 9: There are 4 red stickers to every 5 yellow stickers in a pack of stickers.

There are 36 stickers in one pack. How many red stickers are there?

For GBT-VPR 9 (see Figure 65), Mary attempted to set up an equation but she quit and decided to solve it by using a spreadsheet.

Mary: If I did $4x + 5y = 36$ no that is not.

I: How did you set up the equation?

Mary: I tried to figure it out, I was saying 4 red stickers, 5 yellow stickers but with x and y variables would be 36 stickers totally. I do not think that works.

I: Why do you think it does not work?

Mary: I do not know, when you say 4 reds 4 to every 5, I was thinking it could not. I guess it could if x and y are the same value. Then it is 9. I think I am going to do it in spreadsheet view.

Even though Mary was successful in her attempt to reach the solution, she quit and used a spreadsheet to solve the problem. A similar tendency was also observed in Mary's solution of GBT 10.

Mary: I really want to do it with graphing but I do not see this one in graphing. I am sure you can graph it. I am going to use spreadsheet. I think this is easier for me.

Although she wanted to solve it with a graph, she chose to solve it with a spreadsheet since she was very comfortable with this method. However, after using the spreadsheet to reach the solution, she was able to solve the problem with a graph as well. In both GBT-VPR and GBT, Mary had the same intention of solving with a visual method; however, she ended up using nonvisual methods. She did not prefer to use representations given in the problem. She used the spreadsheet view, which was already open on the screen. For GBT 9, she preferred to open spreadsheet view from the given blank Geogebra file. Therefore, in this task the virtual physical representations did not affect Mary's preference of solution method.

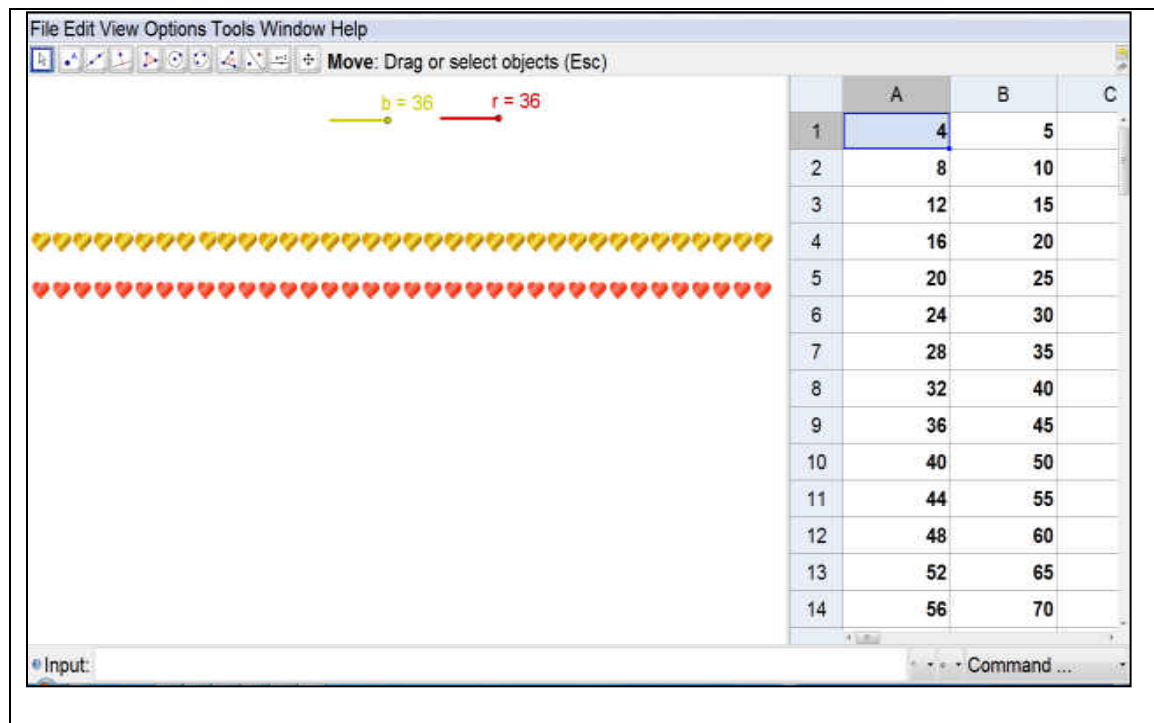


Figure 65. Mary's solution of GBT-VPR 9

GBT-VPR 10: Mike has three times as many candies as George. If he gives George six candies, he will then have twice as many as George then has. How many candies did they each have to start with?

For GBT-VPR 10 (see Figure 66), Mary used the given setting by focusing on its numerical aspects. She constantly guessed and checked the numbers by changing the value of the sliders.

Mary: If Mike has 9, George will have 3. If he gave him 6, George will have 9 and Mike will have 3. That's he will have twice as many as George [Reads the problem again]. Nope that is not it. Lets go higher, 30. George will have 10. He take away 6 from that is 24 and add 6 to George that is 16. 16 times 2 is 32. So no. That is not twice as many. [changes the slider of George to 16] Actually I will change this one first [slider of Mike]. [Changes the value of slider Mike to 60] George will have 20. If I subtract 6, that is 54. Add 6 to George will give 26. That is 52, ok. I was close to that not exactly. [Moves slider Mike to 63] That is 21. Take away 6, that is 57 but 27 is not half. That would be 54. That is more than half. Let's see something. That is [moves slider Mike to 66] 23 no 22. So that is

60, 28. [Changes the value of slider Mike to 69 and George to 23] 63 and 29, that would be 58 [$29 \times 2 = 58$]. Am I getting closer or further apart? 58 and 63. So that is 5 difference. So if I do the next to see going to right way. [Changes the value of slider of Mike to 72 and George to 24] That is 66 and 30. That is 6 away that means I should have going down then. They got further apart. This is 6 apart, 5 apart, 4 apart, 3 apart, 2 apart, 1 apart, no apart [Changes the value of slider Mike from 72 to 69, 66, 63, 60, 57 and 54 respectively]. Let's see this, 54 minus 6 is 48. That is 24. 24 is half of 48.



Figure 66. Mary's solution of GBT-VPR 10

In this task, Mary demonstrated a strong preference for nonvisual thinking and made no mention of visual interpretation of the task. The excerpt indicates that she developed a numeric strategy by recognizing the number pattern. She utilized the given virtual physical representation according to her preferred method of solution, whereas in GBT 10 (see Figure 54) she used the graphing tool to solve the task (see Figure 54). The virtual physical representation affected Mary's preference for this task by shifting her from visual representations to nonvisual representations, which indicated a shift from visual thinking to nonvisual thinking.

Table 6 shows Mary's methods of solution for GBT-VPRs and GBTs. For each task, I included the representations that she used and the categorization of her main method of solution. For example, in GBT-VPR 1 Mary solved the task by translating from algebraic to numeric representation. Her main method of solution is graphical, which is considered a visual method.

For each problem, if there were differences between the solution methods of the problems, I claimed that the virtual physical representations affected her solution method while he was using Geogebra. Mary focused on the numeric representations and used the sliders in order to determine the pattern in numbers. Consider this excerpt from the interview while she was solving GBT-VPR 10.

Mary: Am I getting closer or further apart? 58 and 63. So that is 5 difference. So if I do the next to see going to right way. [Changes the value of slider of Mike to 72 and George to 24] That is 66 and 30. That is 6 away that means I should have going down then. They got further apart. This is 6 apart, 5 apart, 4 apart, 3 apart, 2 apart, 1 apart, no apart [Changes the value of slider Mike from 72 to 69, 66, 63, 60, 57 and 54 respectively]. Let's see this, 54 minus 6 is 48. That is 24. 24 is half of 48.

The excerpt indicates that Mary used the sliders to guess and check and to create a solution with numeric representations to solve the problem. The sliders were also connected to the size of the picture, but she did not focus on any visual aspects of the virtual physical representation. The numeric manipulation with the sliders facilitated her guess and check method. The way in which representations for given GBT-VPRs are used might be a good indication of students' thinking preference. Mary demonstrated a strong preference for nonvisual thinking by using mainly numeric representations in her solutions of PBTs and GBT-VPRs.

Table 6. Mary's solution methods of GBT-VPRs and GBTs

#	GBT-VPR	GBT
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1.	Algebraic-Graphic (Visual)	Algebraic-Graphic (Visual)
2.	Algebraic-Graphic (Visual)	Algebraic-Graphic (Visual)
3.	Numeric (Nonvisual)	Algebraic-Graphic (Visual)
4.	Graphic-Numeric (Nonvisual)	Numeric (Nonvisual)
5.	Graphic-Numeric (Nonvisual)	Algebraic-Graphic (Visual)
6.	Numeric (Nonvisual)	Numeric (Nonvisual)
7.	Numeric (Nonvisual)	Numeric (Nonvisual)
8.	Numeric (Nonvisual)	Numeric (Nonvisual)
9.	Algebraic-Graphic (Visual)	Algebraic-Graphic (Visual)
10.	Numeric (Nonvisual)	Algebraic-Graphic (Visual)

When I compare Mary's solution strategies in terms of her main solution methods, differences can be observed in tasks 3, 5 and 10 between GBT-VPR and GBT. The change was from numerical methods to graphical methods. In those tasks, when Mary was presented with the virtual physical representations, she utilized the problem to solve with the numeric representations; when there was no representation on the screen, she preferred to use graphic representations.

Mary used mostly nonvisual representations in his solutions of GBT-VPRs. Mary's preferences for visual or nonvisual methods in GBT-VPRs were similar to his preferences in MPI. Her MPI score (16/60) suggested that she preferred to use nonvisual methods when solving the problems in MPI. Her solutions for GBT-VPR suggested that she also preferred nonvisual solutions when solving GBT-VPRs. In order to further investigate the effect of technology and representations on students' methods of solutions; I investigated the following cases in my study.

CHAPTER FIVE: THE CASE OF RYAN

Ryan's previous year mathematics teacher was one of our project participants from a high school in Central Florida. When I contacted his teacher, he recommended to me some students, including Ryan, who volunteered to study with me and who had earned a grade of A in their Algebra II course in the previous semester. He mentioned that he usually gets A's in his mathematics classes. Ryan was in 11th grade when the interviews were conducted in the Fall 2010 semester. He was enrolled in Pre-Calculus at that time.

Ryan was born in Central Florida. He was 16 years old. His mother is from Jamaica and his father is from United States. His father works in a field related to computer science. When I asked Ryan which direction he wanted to go for his future career, without any hesitation he mentioned that he wanted to become an aeronautical engineer. He was on the robotics team of his high school. At the time of the interviews, Ryan's team had won the regional competition and was going to compete at the state level.

When I was observing Ryan's Pre-Calculus class, I realized that Ryan was good at interpreting graphs. By the time of observation, they were sketching and interpreting the graphs of inverse trigonometric functions. Ryan was usually among the first three people who spoke up with the answers to the mathematics problems. Although he had a quiet personality, he participated in class discussions and was not shy about speaking his answer aloud when his teacher presented mathematics problems. Ryan did not like to write his answers down. Even in Pre-Calculus, his notes were very short. In the interviews I asked Ryan many times to write down the things that came to his mind and draw or explain if he had a picture or diagram in his mind.

Ryan scored 42 out of 60 in Suwarsono's MPI. This score suggests that he has a strong preference for visual thinking. Ryan mentioned that he likes to construct three-dimensional objects, which was one of the reasons for his involvement in the robotics team.

Ryan's Solutions of PBTs

PBT 1, 2, and 3

PBT 1: Altogether there are 8 tables in a house. Some of them have four legs, and the others have three legs. Altogether they have 27 legs. How many tables are there with four legs?

When I presented PBT 1, Ryan read the problem once again, then there was a long pause. When asked, he said;

Ryan: [Writes down 27 and 8 by underlining them. Under 8, he writes 3 legs and 4 legs. Under 27, he writes 4 times and 3 times]. For 4 (legged tables), I tried 2. That would be 8, and then that would be 19 [27-8=19]. 3 would not be able to work. And I tried 3 (4 legged tables) that would be 12. And then that would be 15. So then ok [writes 5 for 3 legged tables and 3 for 4 legged tables].

Figure 67. Ryan's solution of PBT 1

Ryan started to guess numbers, starting from 2 tables with four legs, and he recognized that the remaining number of legs was not divisible by 3 (see Figure 67). He increased the number of tables with four legs to 3 and multiplied it by 4 to determine if the remaining number of legs was divisible by 3. He divided the remaining numbers by 3 and confirmed the answer by writing 5 tables with three legs and 3 tables with four legs. Ryan mentioned that his method was basically guessing and checking the number of tables.

PBT 2: One day you see 7 cycle riders and 19 wheels ride by your house. How many bikes and how many tricycles were there?

For PBT 2 he said, “I will do it the same way.” When I asked him again to tell me what he was thinking, the following conversation took place.

Ryan: I am checking 2 for bicycle. If there are 2 bicycles then that would be 4 (wheels) and then that would be 15 [wheels of tricycles] so that would be 5 [number of tricycles]. So 2 bicycles and 5 tricycles.

I: Did you start to check from 2 bicycles or did you start by 1 bicycle?

Ryan: I started from 1 and went to 2.

I: So 1 did not work and you went to 2?

Ryan: Yes

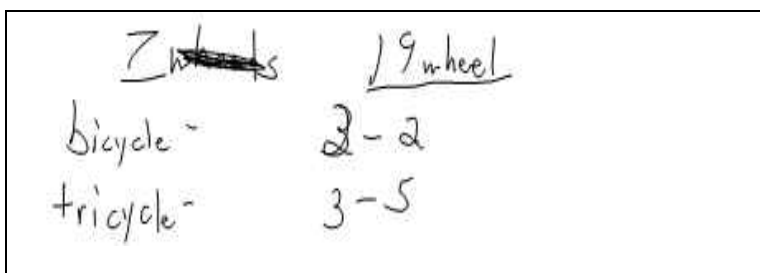


Figure 68. Ryan's solution of PBT 2

Ryan used guess-and-check method as he used in PBT 1 (see Figure 68). First he checked the number of wheels if the number of bicycles is 1 and tricycles is 6. Then he checked the number of wheels for 2 bicycles and 5 tricycles. When he realized that the total number of wheels would give 19, he said that there were 2 bicycles and 5 tricycles.

Both of the solutions for PBT 1 and 2 mainly included numeric representations. Although Ryan's visuality score was high, his solution methods for these two problems were nonvisual. It might be too early to determine Ryan's thinking type while solving problems with

paper and pencil by investigating only two tasks. In order to further probe his thinking, I continued to ask Ryan the following tasks.

PBT 3: A group of students were transported to the championship basketball game using buses and vans. When one bus and two vans unloaded, there were 55 students. A few minutes later, two more buses and one van unloaded. This time there were 89 students. In all, three buses and eight vans drove students to the game. How many students went to the game?

For PBT 3, I initiated the conversation by asking what he was thinking. I asked Ryan to tell me and to write what came to his mind first. Ryan created two equations and tried to manipulate those equations in order to generate an equation that would answer the question.

Ryan: I am just splitting it up [writes $1 \text{ bus } 2 \text{ van} = 55$ and $2 \text{ bus } 1 \text{ van} = 89$]. Long Pause [multiplies 55 by 3 and gets 165] I guess I am kind of stuck. So what I did with this I guess to make it simplifies since 1 bus and 2 vans, multiplied by 3. So there will be 3 buses and 6 vans equal 165. And then I just need to find what 2 vans equal. I guess that is where I am at.

I: You are trying to find two vans equals.

Ryan: Yes. Like how many people in two vans. I still have to go to same spot but I guess it simplifies it. Not really much but at least I can find how high the number would be. I guess my reason would be since there is only three buses, just multiplied by three so busses kind of form the vans. Then I will be still stuck back to where I was still trying to find how much one bus and one van, each one equals. Then I just add two vans to that.

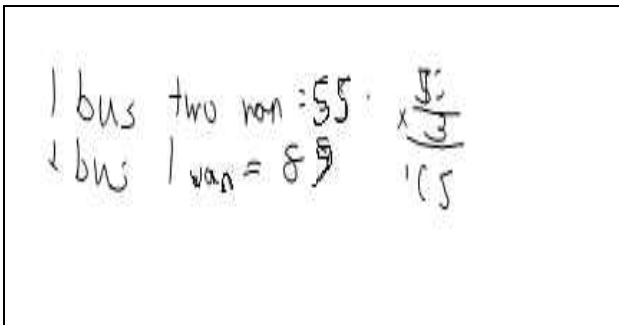

$$\begin{array}{l} 1 \text{ bus } + 2 \text{ van } = 55 \\ 2 \text{ bus } + 1 \text{ van } = 89 \end{array} \quad \begin{array}{r} 55 \\ \times 3 \\ \hline 165 \end{array}$$

Figure 69. Ryan's solution of PBT 3

Ryan multiplied the equation $1 \text{ bus} + 2 \text{ vans} = 55$ by 3 in order to have same number of busses asked in the problem. However, he mentioned that he still needed to determine the number of people in vans since the problem states that there were 3 busses and 8 vans instead of 3 busses and 6 vans. He quit the problem and wanted to continue with the next problem.

Ryan's suggested solution strategy for the task included mainly algebraic representations. In this problem, Ryan made no reference to any attempts I consider visual thinking. He did not use any visual support to solve the problem. Thus far, Ryan's solution methods indicated that he preferred to use nonvisual thinking, even though his MPI score suggests that he is a visual thinker. As previously mentioned, it is impossible to see his actual thinking process, so he might also visualize the solutions in his head but not mention this mental activity in his verbal and written responses.

PBT 4 and 5

PBT 4: Blaise rode his bike to his friend Elroy's house, which was 18 miles away. After he had been riding for half an hour, he got a flat tire. He walked his bike the rest of the way. The total trip took him 3 hours. If his walking rate was one-fourth as fast as his riding rate, how fast did he ride?

In PBT 4 (see Figure 70) after reading the problem a couple of times, there was a long pause again. In order to understand what he was thinking, I asked him to think aloud. He said he was trying to determine the miles in hours. Then the following conversation took place between him and me.

Ryan: I imagined a line and that has 3 hours and 18 miles.

I: Can you draw that for me?

Ryan: [Draws picture] 3 hours since he got a flat tire, [he draws a point on the line and writes 30 minutes and for the rest of the part writes 2 and half hours] so that would be 2 and half hours. So then [writes riding rate is $\frac{1}{4}$ of walking rate and corrects that by writing riding rate equals 4 times walking rate].

I: What did you write here?

Ryan: Riding rate is 4 times walking rate since it is 4. I guess it just visualizes better.

I: Why did you write first $\frac{1}{4}$?

Ryan: Because I was just thinking it is one fourth. I just kind a messed up. I forgot that it was four times. I: Ok.

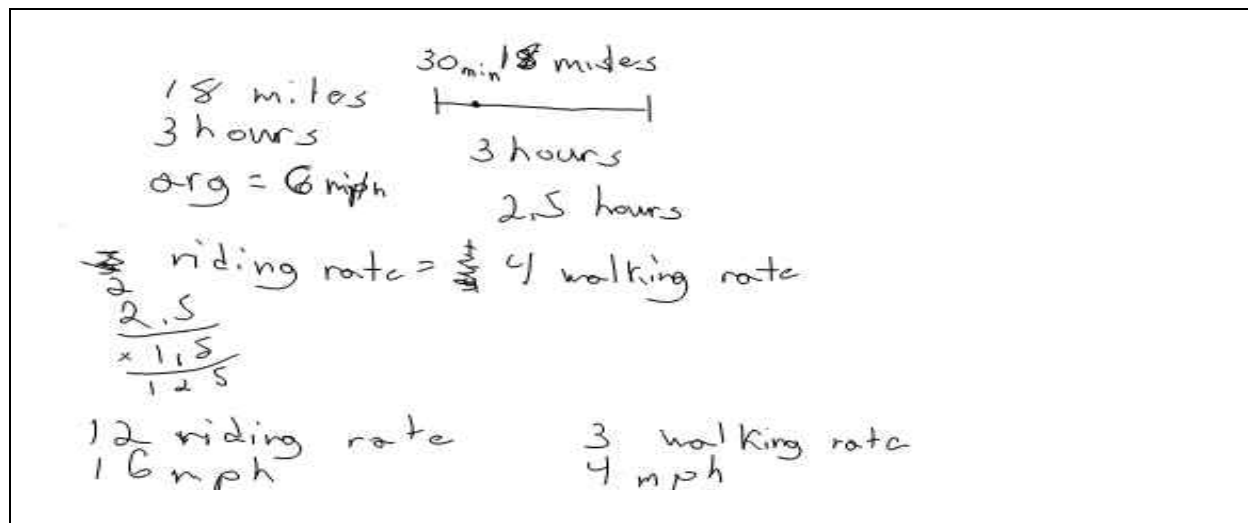


Figure 70. Ryan's solution of PBT 4

Ryan said that he imagined the picture of the trip that he drew on paper upon my request (see Figure 70). He labeled the time and the distance on the line that he created to represent the trip. Then he wrote the relationship between the riding and walking rates. He wrote “riding rate = $\frac{1}{4}$ walking rate” when he read the statement “walking rate is one fourth as fast as his riding rate.” This indicated Ryan’s difficulty when he was translating from verbal representation to algebraic representation. Janvier (1987) called this situation a contamination from verbal to formula. After a short pause, Ryan corrected his mistake by writing “*riding rate = 4 walking rate.*” The reason for the correction might be related to recognizing the meaning of the rates. He might have thought that it was meaningless to have a higher walking rate than riding rate. In the following tasks, I investigated Ryan’s difficulties translating from verbal representations to algebraic representations in more detail.

Ryan continued to solve the task. Consider the following dialogue between me and him.

Ryan: 6 miles per hour would be the average. [Long Pause then writes 1.5, 6 and 12 miles but deletes those] I am still trying to figure out, I guess just trying to figure out what the rates are, like walking rate in 2.5 hours. I just trying to figure out how many miles took him for like 30 minutes. I was first thinking 1.5 miles per hour for the walking speed and 6 miles per hour for the riding speed. But I do not think it will make any sense [Starts to multiply 1.5 by 2.5 and gets 2.75]. I did not think that it would work since it is decimals. Then I was trying to do 12 than that would be 3.

I: 12 is riding rate?

Ryan: Yeah.

I: So what is the walking rate?

Ryan: 3

I: Please write those down so we will not forget

Ryan: [Writes 12 riding rate and 3 walking rate] That will be 6 since half an hour. This is 2.5 so that would be 7.5 and the 6. It is going to add up to 13.5.

Ryan: I guess I will try 16 that would be 4. And that would be 8 and that (4) times 2.5 would equal to 10. So that would work.

After Ryan wrote the ratio between the walking and riding rate he started to use a guess and check strategy. He first tried 1.5 mph as the walking rate and 6 as the riding rate. Then he moved to his next guess which was 3 mph for the walking rate and 12 mph for the riding rate. Once he calculated the total distance was 13.5 miles with these rates, he tried 4 mph for the walking rate and 16 mph for the riding rate which resulted in a correct answer.

For this task, Ryan needed a visual support to solve the problem. Therefore his main method of solution was a visual solution. Since Ryan did not draw the picture before I asked him to think aloud, he might be a type of student who visualizes the problems in his mind but writes down only the nonvisual part of the solution. In his answers to MPI, for 21 problems out of 30 he selected the answers which start with, "I saw the diagram in my mind..." For this reason, Ryan might not need to draw diagrams or pictures in his head but prefers to imagine them.

PBT 5: Alex planned a trip from Orlando to New York. From her house to the airport he took a taxi and he flew from Orlando to New York. The total trip took 4 hours. The average speed of taxi is 60 mph and the average speed for plane is 400 mph. From her house to New York airport the distance is 1260 miles. What is the distance between her house and Orlando Airport?

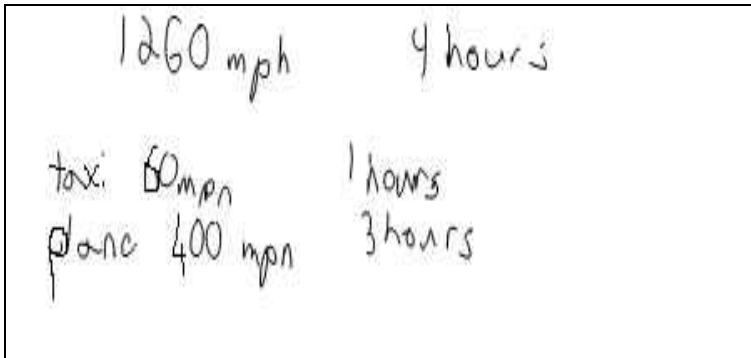


Figure 71. Ryan's solution of PBT 5

In PBT 5 (see Figure 71), Ryan guessed and checked the numbers. He said, "I tried to see how high the plane can go to figure out where as a whole number. I just tried 3 [hours] since that would make 1200 [miles]. There would be only 60 so that would make 1 hour." Ryan's first guess resulted in the correct answer. Even though PBT 5 was very similar to PBT 4, Ryan did not use visual support to solve PBT 5. Ryan's solution mainly included numeric representations, which indicate nonvisual thinking. The nature of the task might have had an effect on his solution method, I continued to ask more tasks to investigate his type of thinking.

PBT 6, 7, and 8

PBT 6: Covell's home-mortgage payments are about \$900 per month. He is going to refinance which will cost him about \$2500 in fees, and the new payments will be \$830 per month. How long will it take him before the new loan starts saving him money?

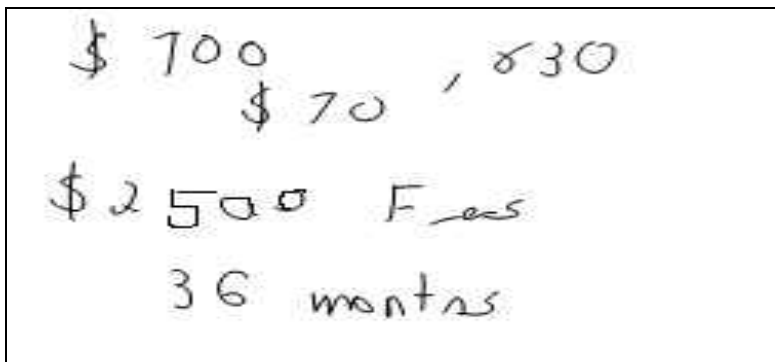
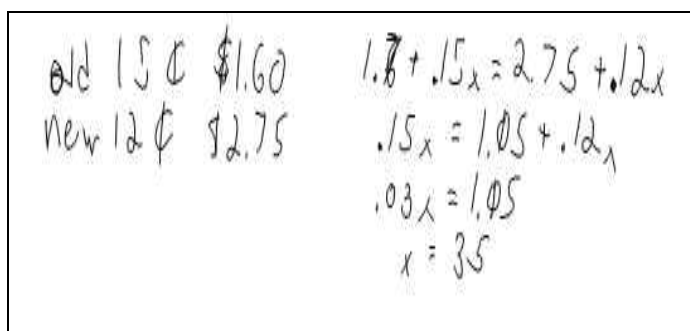


Figure 72. Ryan's solution of PBT 6

When I presented PBT 6, Ryan attempted to determine the difference between the payments (see Figure 72). He divided the refinance fee by the difference between the payments in order to calculate the number of months that the new payment costs were the same as the old payment costs. He said “I got 35.7, I guess it would be 36 months. He will start saving money. Well I guess 36 months he will have some money, I think.” Ryan’s solution method did not include any reference that can be considered as visual thinking. In order to further probe his thinking, I continued to ask Ryan tasks similar to PBT 6.

PBT 7: Javier received a letter from his bank recently concerning his checking account. Under his current plan, each check he writes costs 15 cents, and there is a monthly fee of \$1.70. Under the proposed new plan, each check he writes will cost 12 cents, and there will be a monthly fee of \$2.75. What is the minimum number of checks Javier must write monthly in order to make new plan cost him less than the old plan?



Handwritten solution for PBT 7:

old 15¢ \$1.60	$1.6 + .15x = 2.75 + .12x$
new 12¢ \$2.75	$.15x = 1.05 + .12x$
	$.03x = 1.05$
	$x = 35$

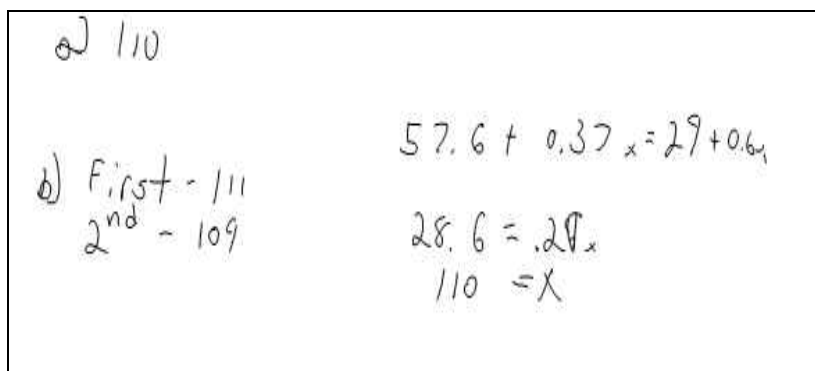
Figure 73. Ryan’s solution of PBT 7

In PBT 7 (see Figure 73) Ryan created an equation and solved that equation for x . Once Ryan determined the number of checks that costs the same amount of money, he interpreted the situation. “So after this point, the second one is a better choice. Since at 35 checks they are equal to the same number and then that would just be plus 12 instead of plus 15.” When we

compare Ryan's interpretation and his solution method, there are differences in terms of representations. Ryan used algebraic representations to find the number of checks that cost the same amount of money in both plans; however, when he was interpreting the case to find the minimum number of checks that made the new plan cost less than the old plan, he used numeric representations. He used the same method as Mary used when she was interpreting the problem. Because of the differences in representations, he translated from algebraic representation to numeric representation. Again Ryan did not show any preference for visual solutions and used algebraic and numeric representations.

PBT 8: The concessions manager at the Central High School football stadium offered two pay plans for people willing to sell peanuts in the stand at home football games. The first plan pays \$57.60 plus \$0.37 per bag sold. The second plan pays \$29.00 plus \$0.63 per bag sold.

- For what number of bags sold will these two pay plans give exactly the same pay?
- For what number of bags is the first plan the best choice? What about the second plan?



Handwritten solution for PBT 8:

$$57.6 + 0.37x = 29 + 0.63x$$

$$28.6 = .26x$$

$$110 = x$$

Additional handwritten notes on the left:

a) 110

b) First - 111
2nd - 109

Figure 74. Ryan's solution of PBT 8

For PBT 8, Ryan used the same method (see Figure 74) as he had used to solve PBT 7. He created an equation by writing the first plan to the lefthand side and the second plan to the righthand side. He solved the equation for x , which was the number of peanut bags. He stated that if 110 bags were sold, both plans would give the same pay. He continued to interpret the

case by saying, “So if it is less than 110, the second one is a best choice, and then I guess if it is higher, I think it is the first one.” Although this task was similar to PBT 7, the beneficial option in PBT 7 was the plan that pays less money among those two plans as opposed to PBT 8, whose beneficial option was the one that pays more money to the people selling peanuts. Since Ryan solved PBT 8 right after solving PBT 7, he might have interpreted both problems in the same way. Therefore, there was a problem in the translation process from verbal representation to numeric representation since Ryan interpreted the problem by using a numeric representation as he did in PBT 7.

When Ryan’s solution methods for PBT 6, 7, and 8 were examined, it was clear that he used nonvisual methods to solve them. Those solutions revealed that he preferred to use nonvisual thinking while solving problems with paper and pencil. He gave no sign that I can infer as visual thinking. In PBT 9 and 10, I further investigated his thinking preference for PBTs.

PBT 9 and 10

PBT 9: There are 9 boys to every 10 girls in a particular high school. There are 2622 students at the school. How many girls are there?

The image shows a handwritten solution for PBT 9. At the top, the ratio '9:10' is written. Below it, the number '2622' is written. To the left of '2622' is the number '138'. To the right of '2622' is the text 'boys 1242' and 'girls 1380'.

Figure 75. Ryan’s solution of PBT 9

I presented PBT 9 to further examine Ryan's thinking process. After reading the problem a couple of times, Ryan started to solve it (see Figure 75). He reported that he divided the total number of students by 19. Consider the following excerpt.

Ryan: So I guess [divides 2622 by 19], total there would be 19. So I guess divide the total number by 19. I thought it was since that is the ratio 9 per 10, there would be 19 of them in total. I just did to see how much 19 can go into 2622 and then I got 138. That is how many pairs of 9 boys and 10 girls. Multiply by 10 to get the girls and then to check my work I multiplied by 9 to get the number of boys and then add them together.

After this conversation, Ryan reported that he first imagined the groups. Ryan's solution method suggested that he used visual thinking for this task. Even though he used nonvisual methods in the previous tasks, he changed his preference for this task and used visual methods.

PBT 10: Cici and Amantina have lots of stickers. Cici had one third as many as Amantina had, but then Amantina gave Cici six of her stickers, and now Cici has half as many as Amantina. How many stickers did each girl start with?

Cici	Amantina	Check	Result
5	35	4	80/41
12	36	8	257
15	30	15	
15	45		
25	30	18	54
5	15	24	48
7	21		
13	15		
9	27		
15	21		
14	30		
20	36		
15	48		
22	42		
20	52		
26	60		
26	54		

Figure 76. Ryan's solution of PBT 10

In PBT 10, Ryan used a guess-and-check strategy to determine the number of stickers each person had. He started to solve the problem by using fractions to represent the ratio

between two people's numbers of stickers. He wrote $1/3$ for Cici and $3/3$ for Amantina. Then he wrote $2/6$ for Cici and $6/6$ for Amantina. When I asked Ryan why he preferred to write in this way, he mentioned that it was easier to understand. Then he started to create a second scenario—"Amantina gave Cici six of her stickers, and now Cici has half as many as Amantina"—by guessing and checking the number of stickers. He guessed 2 stickers for Cici and 6 stickers for Amantina. He realized that those numbers did not satisfy the condition, then he checked other numbers as seen in his work below.

Ryan's solution method for this problem included mainly numeric representations. Therefore his thinking preference for this problem might be considered nonvisual thinking since Ryan did not demonstrate any representations on methods that I can consider visual thinking. His solution strategy for PBT 10 was similar to those for PBT 1 and 2.

Ryan's solutions of PBTs included various methods. He used mainly numeric representations to solve PBT 1, 2, 5, and 10, and algebraic representations to solve PBT 3, 6, 7 and 8. In PBT 4 and 9, he used graphic representations to support his solutions. Although Ryan's MPI visuality score was high, he demonstrated visual thinking only in two PBTs. Ryan attempted to solve PBTs in mostly nonvisual ways, and most of the time he said that he did not need visual support. These findings suggest that when Ryan was solving algebra word problems with paper and pencil, he demonstrated the characteristics of Krutetskii's analytic (nonvisual) type or abstract-harmonic subtype. According to Krutetskii (1976), the difference between the abstract-harmonic and pictorial-harmonic subtype is in their use of graphic representations. The abstract-harmonic type feels no need for and does not strive to use graphic representations, whereas the pictorial-harmonic type feels a need for and often relies on graphical schemes during the solution, although both types can depict mathematical relationships equally well by pictorial

means. In the next section, I examined the effect of technology on Ryan's thinking by comparing his methods of solution for PBTs with his methods for GBTs.

The Effect of Technology on Ryan's Solutions

This part includes Ryan's detailed solution methods for GBTs in order to compare them with his solution methods for PBTs. When solving GBTs, students were presented with blank Geogebra files, similar to the blank paper with which they were presented when solving PBTs. For this part of the study, I examined the effect of technology by constantly comparing and contrasting their methods of solution.

GBT 1, 2, and 3

GBT 1: Bill has \$1.25 in nickels and dimes. He has a total of 17 coins. How many of each does he have?

In GBT 1 Ryan reported that he wanted to use a graph to solve the problems. Ryan activated the graph view of the blank Geogebra files. For all three GBTs, he created two equations and entered those equations into the input field to graph those equations. Once he finished drawing functions, he clicked *Intersect two objects* and the two graphs to find the intersection point. He interpreted that point according to the terms he set for x and y. For instance, in GBT 1 Ryan created the first equation as $0.05x + 0.10y = 1.25$ and entered that equation into the input field to draw the graph (see Figure 77). Then he created the second equation as $x + y = 17$ and entered that one into the input field as well. He called x and y the number of nickels and the number of dimes respectively. In order to find the exact intersection point of these graphs, he clicked *Intersect two objects* and then selected the graphs. He checked the intersection points by right-clicking and selecting the object properties instead of activating the algebra window. Then the coordinates of the intersection point, $A = (9, 8)$ appeared in the

geometry window. Since he called x as the number of nickels and y as the number of dimes, he said that there had to be 9 nickels and 8 dimes.

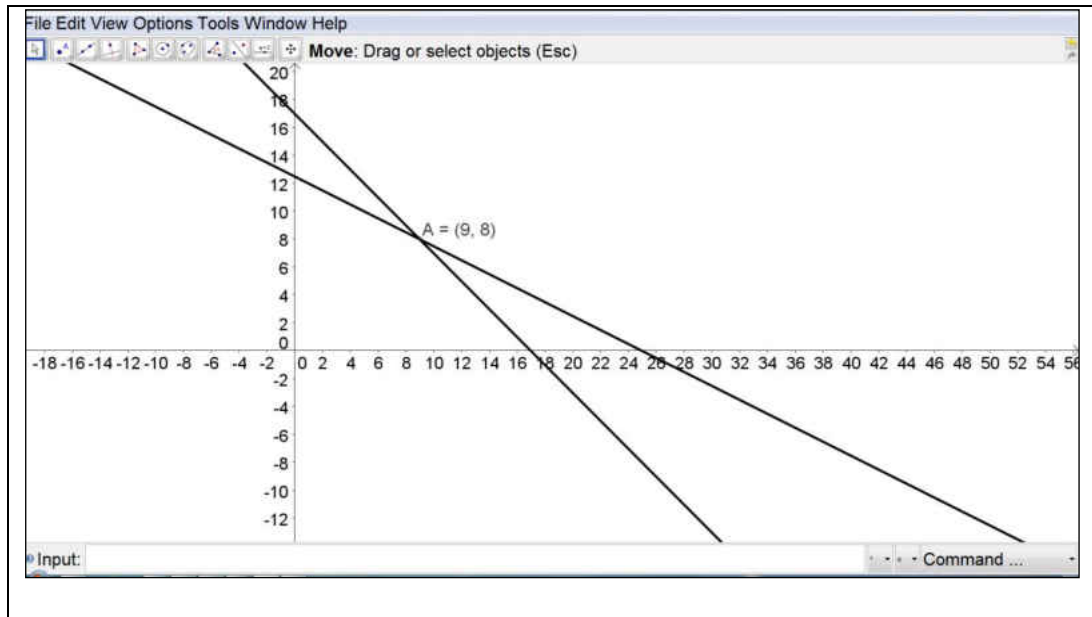


Figure 77. Ryan's solution of GBT 1

GBT 2: The Madison Local High School marching band sold gift wrap to earn money for a band trip to Orlando, Florida. The gift wrap in solid colors sold for \$4.00 per roll, and the print gift wrap sold for \$6.00 per roll. The total number of rolls sold was 480, and the total amount of money collected was \$2340. How many rolls of each of each kind of gift wrap were sold?

For GBT 2, Ryan applied the same strategy as he applied to solve GBT 1. He chose x as the number of gift wraps in solid colors and y as the number of print gift wraps. He created two equations and entered them in the input field to draw their graphs (see Figure 78). Then he determined the intersection point of those graphs and interpreted that intersection point.

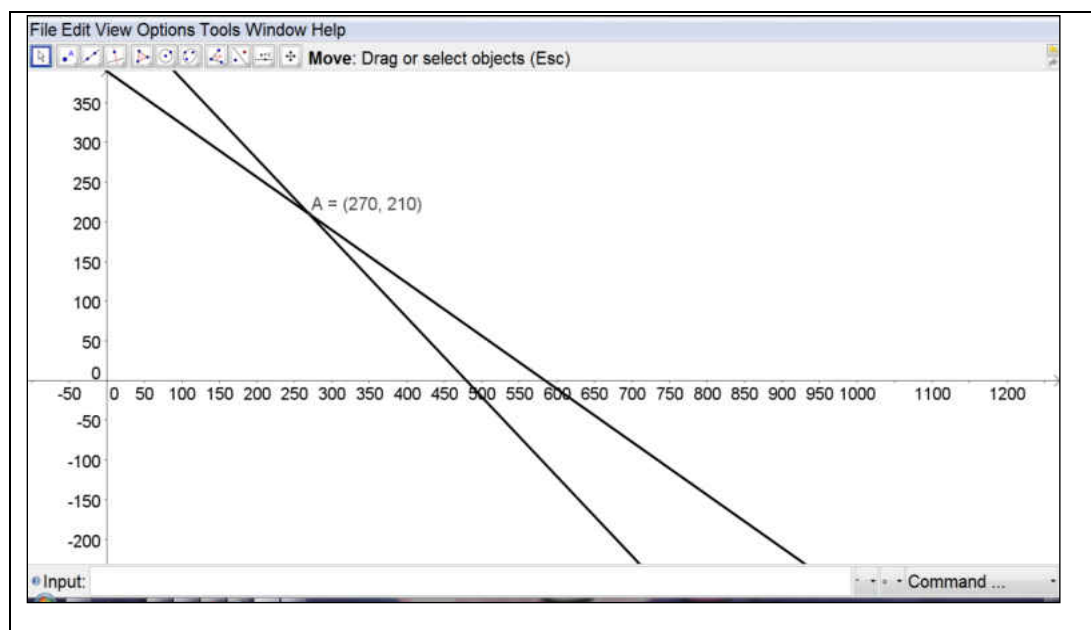


Figure 78. Ryan's solution of GBT 2

GBT 3: Campus rentals rent 2- and 3-bedroom apartments for \$700 and \$900 per month respectively. Last month they had six vacant apartments and reported \$4600 in lost rent. How many of each type of apartment were vacant?

Similar to GBT 1 and 2, Ryan again created two equations to solve GBT 3. He chose x as the number of 2-bedroom apartments and y as the number of 3-bedroom apartments. He entered the first equation $x + y = 6$ and the second equation $700x + 900y = 4600$ into the input field to draw the graph of those equations (see Figure 79). Then he determined the intersection point and interpreted that point according to the terms he set for x and y .

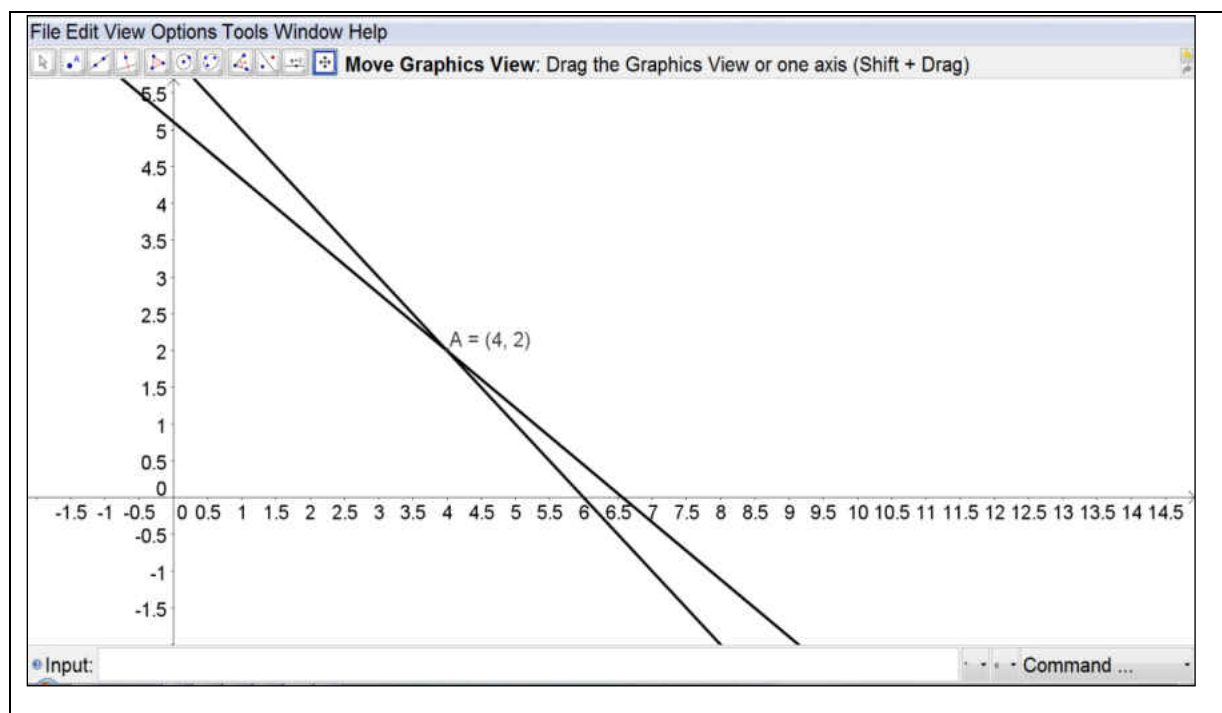


Figure 79. Ryan's solution of GBT 3

All three solutions mainly included the graphic representations which were considered as visual solutions. Even though Ryan started with creating algebraic equations, he used them to draw their graphs. Once he created the equations, he preferred to continue with their graphs instead of solving them. Hence, Ryan used visual methods in GBT 1, 2 and 3 which were an indication of a preference for visual thinking while solving those problems. When Ryan's solution method for PBT 1, 2, and 3 was compared with his method for GBT 1, 2, and 3 (see Figure 67, Figure 68 and Figure 69), there were clear differences. He preferred to solve PBT 1, 2, and 3 by mainly using numeric representations; however, he attempted to solve the GBTs by using mainly graphic representations. He demonstrated nonvisual thinking in his solutions of PBT 1, 2, and 3 as discussed in the previous section. In those three tasks, technology had an impact on his thinking preference by changing it from a nonvisual to a visual method of solution.

The availability of a coordinate system and automation in creating graphs might affect Ryan's methods of solution. In order to examine this early assertion, I continued to set tasks for Ryan.

GBT 4 and 5

GBT 4: A 640-mile, 5-hour plane trip was flown at two speeds. After flying 2 hours, the tailwind picked up and the speed in the second part of the trip was two times as the speed in the first part. What was the speed in first and second part of the trip?

When Ryan was solving GBT 4 (see Figure 80), he first started to record the given information in the spreadsheet view, then he created equations to draw the graph of those equations. He followed the same solution method that he had followed with GBT 1, 2 and 3; however, this time he used the spreadsheet view instead of the input field to insert the equations. He determined the intersection point of two graphs and the coordinates of that point. He called x as the speed in the first part and y as the speed in the second part and explained his reasoning process.

Ryan: Yes $640 = 2x + 3y$. Then I put $x = 2y$ because x is the first speed and then y would be the second part speed. It is times 2 of the first speed.

I: Is that what problem says?

Ryan: Yes because it says [reads the problem]. So I put x is the first part and y is the second part.

I: So x would be

Ryan: The speed, miles per hour and then I guess the second... but that does not make any sense.

I: Why?

Ryan: Because y should be bigger I think. [Long pause] Because I guess y would be smaller but it seems like it supposed to be bigger than first part. Well the graph makes sense I guess they are right then.

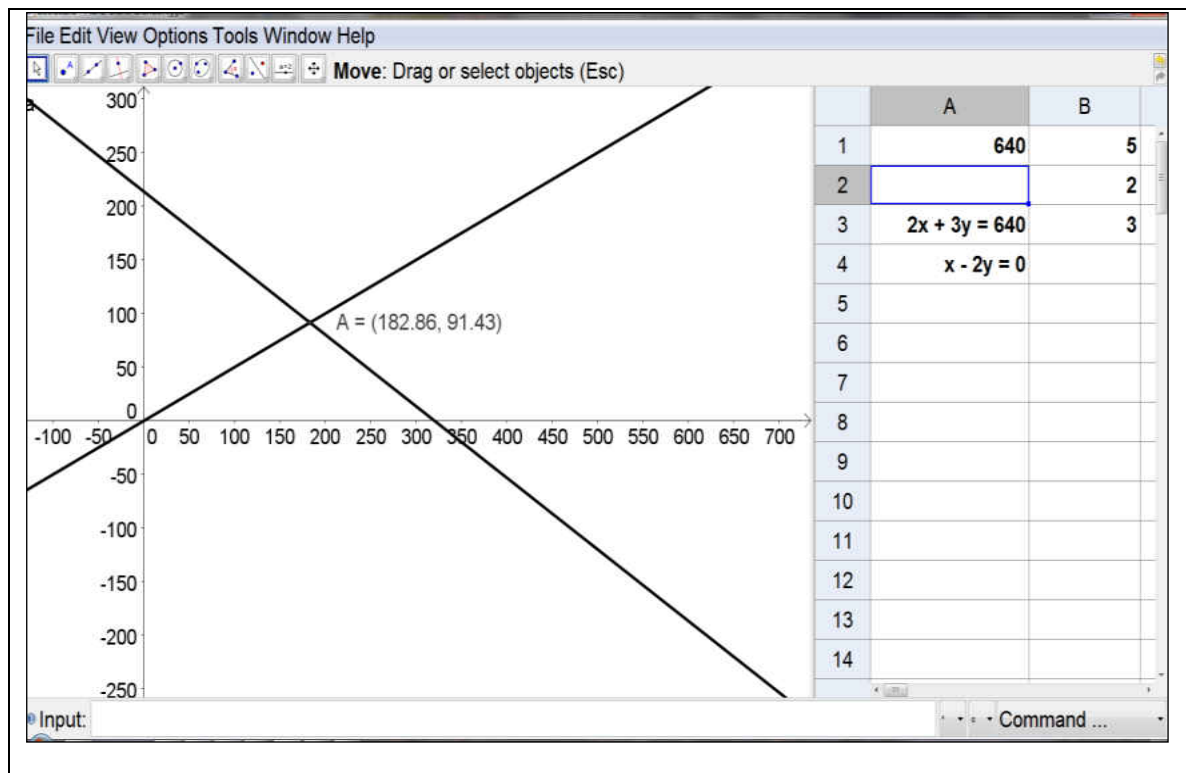


Figure 80. Ryan's solution of GBT 4

Ryan chose x as the speed in the first part and y as the speed in the second part of the trip. He wrote the first equation $2x + 3y = 640$ to represent the distance traveled in the trip. When Ryan read “the speed in the second part of the trip was two times as the speed in the first part” he wrote $x = 2y$ instead of $y = 2x$. He reported that the speed in the second part of the trip is times 2 of the speed in the first part of the trip so the equation should be $x = 2y$. Ryan's explanation of his reasoning revealed the contamination from verbal representation to algebraic representation (Janvier, 1987). This time Ryan could not correct his mistake although he read the problem aloud more than three times. His overreliance on graphic representation might have hindered him in correcting his mistake. According to Haciomeroglu (2007), one-sided thinking (only visual or nonvisual) and overreliance on a single representation create difficulties when solving mathematics problems.

When I compared Ryan's methods of solution for PBT 4 (see Figure 70) and GBT 4, both of those solutions were considered visual solutions. Ryan created a diagram for PBT 4 and graphs for GBT 4. Both of those solutions were visual solutions, however, for PBT 4 Ryan created picture of the trip, whereas for GBT 4 he created and interpreted the graphic representations of the problem.

GBT 5: A man drove from home at an average speed of 30mph to an airport. He then boarded a helicopter and flew to the corporate office at an average speed of 60mph. The entire distance was 150 miles. The entire trip took 3 hours. Find the distance from the airport to the corporate office.

In GBT 5, Ryan's response was very quick. After he read the problem one time, he said that he had determined the solution.

Ryan: Ok, alright, I think I figured out already [opens a spreadsheet], the total trip took 3 hours, the distance of 150. I guess it is only 1 hour so that would be 30 and it would be 2 hours for that would be equal 120. And then from airport to corporate office is I guess 120 miles...I just looked the paper and it made sense I guess. The numbers were easy.

	A	B	C	D	E
1		150			
2	30	30			
3	60	120			
4					
5					
6					
7					

Figure 81. Ryan’s solution of GBT 5

For both GBT 5 (see Figure 81) and PBT 5 (see Figure 71), since Ryan’s response was very quick, and he stated that the numbers made it easy to guess the solution, it was difficult to examine his train of thought. His first attempt resulted in the correct answer. He used numeric representations categorized as a nonvisual solution method; however, the nature of the problems in PBT and GBT 5 made it difficult to understand his solution methods. Therefore I asked him to solve the following tasks to further investigate the effect of technology on his thinking.

GBT 6, 7 and 8

GBT 6: Cher wants to buy a refrigerator. She visits a store and finds two recommended models. The major brand is \$600 and is expected to cost \$30 per month in energy cost. The minor brand is \$400 and is expected to cost \$40 per month in energy cost. Which refrigerator would you advise Cher to buy?

Similar to his solution methods of GBT 1, 2, 3, and 4, Ryan created two equations to draw their graphs and find the intersection point of those graphs in GBT 6. He followed exactly the same strategy in GBT 7 as well. For GBT 6 and 7, Ryan only activated the geometry window to draw graphs. In GBT 6 and 7, Ryan did not only determine the intersection point but also interpreted what happened after and before that point. Ryan interpreted the graphs by using the comparison between the positions of the graphs. As an example, consider the excerpt from Ryan's explanation of his for solution of GBT 7 below the word problem.

GBT 7: You contacted two local rental companies and obtained the following information for the one-day cost of renting a truck. Company 1 charges \$40.95 per day plus \$0.19 per mile, and company 2 charges \$19.95 per day plus \$0.49 per mile. How many miles do you need to make in one day to get benefit by selecting the first company?

Ryan: [Opens the graphic view and inserts the equation $y = 40.95 + 0.19x$ and the second equation $y = 19.95 + 0.49x$, zooms out and intersects two objects. He clicks on Show name and value for the lines and intersection points] First company, which is top one (until the intersection point), I guess it will be 71 or more.

I: Do you mean second company would be better when 71 miles driven?

Ryan: No the first company would be better. Because the second company would be this [pointing out the line of $y = 0.49x + 19.95$] so after this point the first company would be cheaper. Yeah. [Reads the problem again] and that would be 71 miles or more.

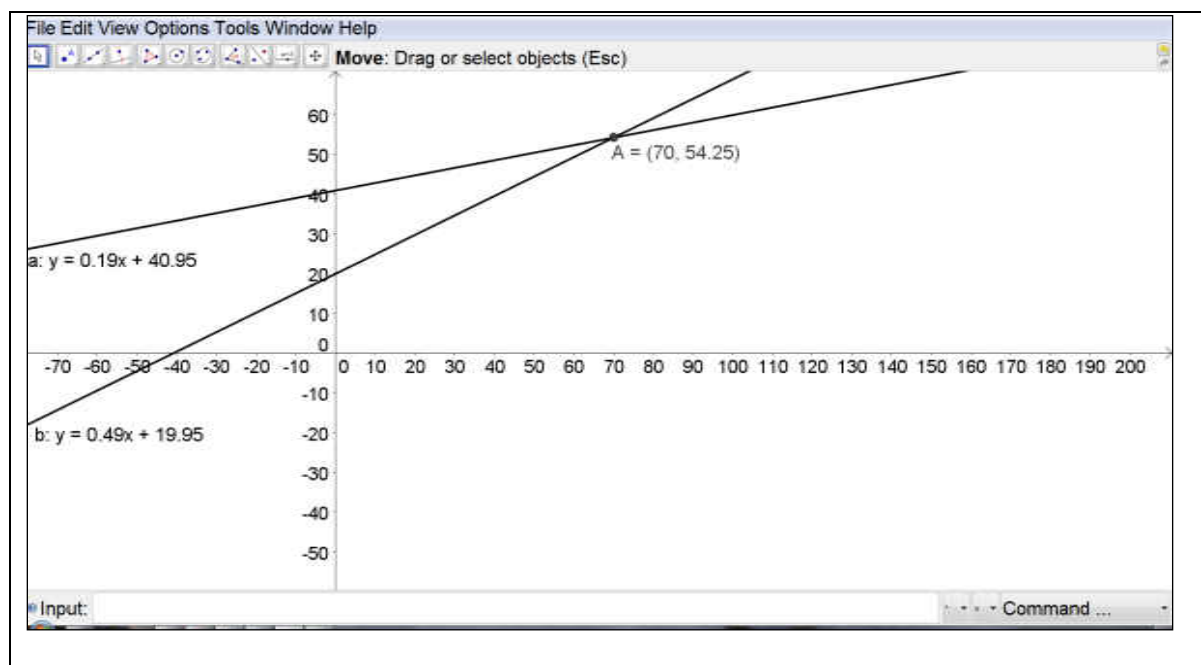


Figure 82. Ryan's solution of GBT 7

Ryan chose x as the miles driven and y as the cost of renting truck when x miles driven. He created two equations for two different rental truck companies. He entered the first equation, $y = 40.95 + 0.19x$, which represents the cost of renting truck with Company 1 into the input field. Then he entered the second equation, $y = 19.95 + 0.49x$, which represents the cost of renting truck with Company 2 into the input field. He determined the intersection of the graphs of these equations by clicking on *intersect two objects* and the graphs themselves. He interpreted that point and suggested which company should be chosen before and after that point.

Ryan's answer included his interpretation of the graphs that he drew by using equations. He made similar interpretations when solving GBT 6. These interpretations indicated that even though Ryan started to solve the problem by using equations, he translated them into a graph and interpreted the graphical information to reach the solution. His interpretations from graphs further support his visual thinking while solving those GBTs. The availability of the graphing tool in Geogebra affected Ryan's preference for solution method. Although he demonstrated

nonvisual thinking with his solutions of PBT 6, 7 and 8, his responses to GBT 6 and 7 indicate that using Geogebra affected his solution method by changing it from a nonvisual solution to a visual solution. Ryan's solution methods of GBTs (visual or nonvisual) were aligned with his MPI score which suggested that he was a visual student.

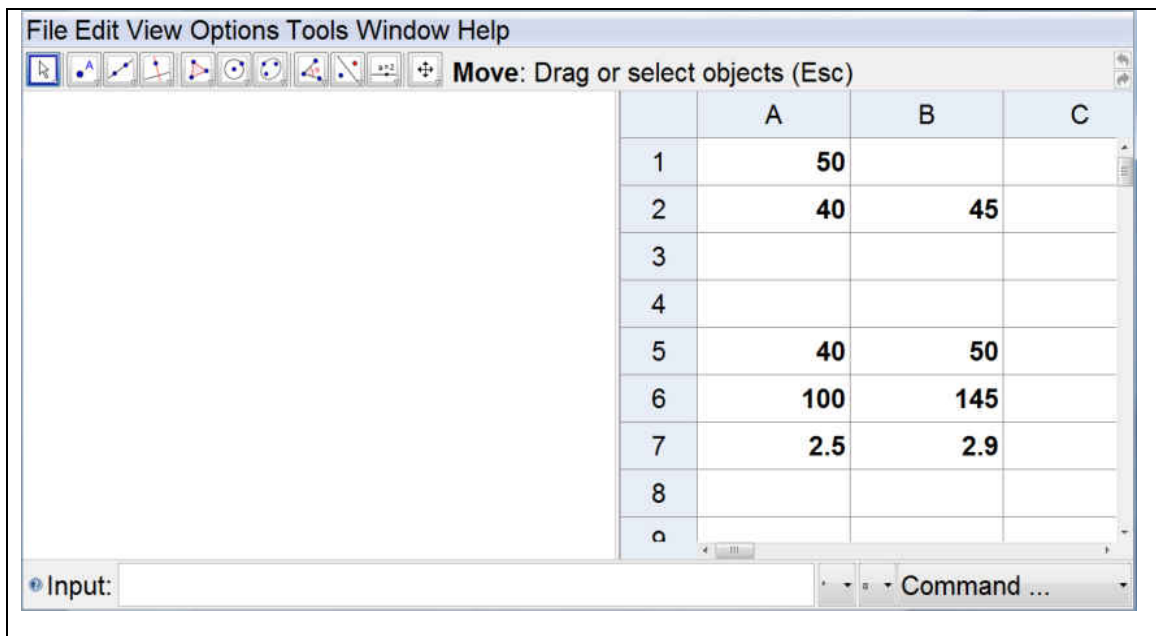
GBT 8: A car is leaving Roseville at a constant speed of 50 km/h to go to San Francisco. Another car leaves at the same time from the next city which is 45 km closer to San Francisco at a constant speed of 40 km/h. In how many hours will the car leaving from Roseville overtake the other car?

For GBT 8, Ryan started to read the problem and tried to solve it without using Geogebra. I asked him to explain what he was thinking.

Ryan: I think just since the 50 km one is going 10 km more faster and the second one is 45 closer at a speed of 40 if they both start at the same spot then it will be the second one is still needs 45. So then I would say [reads the problem again]. I would say maybe 2 hours.

I: So in 2 hours?

Ryan: It should overtake the other one. Because in 1 hour this one still be behind by a couple kilometers and in the second hour it should pass. But I am kind of lost. ...I imagine like a line graph. This one is 50. Since I do not know the total distance or total distance other than 45 km ahead, I guess I would say about 2 hours. But I am not sure.



	A	B	C
1	50		
2	40	45	
3			
4			
5	40	50	
6	100	145	
7	2.5	2.9	
8			
9			

Figure 83. Ryan's solution of GBT 8

Ryan's explanations and his difficulty indicated that he had doubts about his answer. I asked him to use Geogebra to show his work, and Ryan opened a spreadsheet to record the numbers in the spreadsheet (see Figure 83). Although Ryan mentioned that he imagined a line graph, he did not recreate his mental imagery in Geogebra. Ryan's difficulty continued until the end of the solution process. He said his guess was 2; however, he could not exactly imagine the given scenario in the problem so he was not sure.

According to Krutetskii (1976), if visual students do not succeed in creating visual supports by visualizing objects or diagrams, then they have difficulty with abstract schemes. In this problem, Ryan might not have felt comfortable because he could not visually support his answer. He attempted to solve the problem by visualizing the situation, but he did not display in the Geogebra file what he was thinking. He wrote only the numbers into the spreadsheet. His explanation "I imagined like a line graph" shows that he had a picture in his mind.

In this problem, Ryan did not use Geogebra in a way that reflected his mental imagery. Overall, Ryan's solution methods for GBT 6, 7, and 8 indicated that he used visual methods to solve those problems whereas he used nonvisual methods to solve PBT 6, 7 and 8. He interpreted the graphs to answer the questions in the given problems. When the solution methods employing Geogebra were compared with the solution methods using paper and pencil, it became clear that technology affected his solution method, which indicates a change in his mathematical thinking.

GBT 9 and 10

GBT 9: There are 2 students wearing glasses to every 7 students not wearing glasses in a particular classroom. There are 63 students in that classroom. How many students are wearing glasses?

For PBT and GBT 9, Ryan used the same strategy. As mentioned before, he solved PBT 9 (see Figure 75) by making small groups of 19 students (9 of them were boys and 10 of them were girls) and figuring out how many groups he could form with 2622 students. After he determined the number of groups by dividing the total number by 19, he calculated how many girls and boys were in the school by multiplying the number of groups by the number of girls and boys in each group. He used the same strategy in GBT 9 (see Figure 84). He followed the same reasoning, but this time he used the graph. He entered the equation $2x + 7x = 63$. Once he recognized that the graph was $x = 7$, he said, "There is 7 groups of 2 students [students wearing glasses] and 7 students [students not wearing glasses] and then times 2 to find students with glasses."

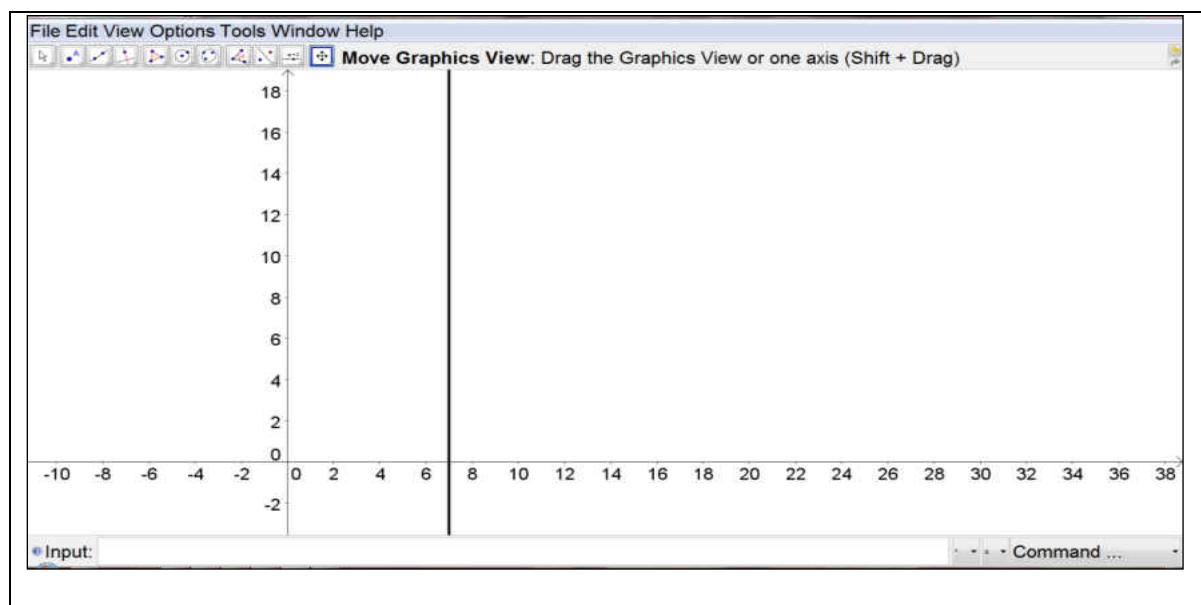


Figure 84. Ryan's solution of GBT 9

Since Ryan's solution of PBT 9 included imaging the groups, his method was categorized as a visual method, and I inferred that he had demonstrated visual thinking for PBT 9. His preference for graphic representations while solving GBT 9 indicated that he used a visual method to solve GBT 9 as well, employing mental imagery of the groups and drawing the graph.

GBT 10: A and B each have a certain number of marbles. A says to B, "If you give 30 to me, I will have twice as many as left with you." B replies, "If you give me 10, I will have three times as many as left with you." How many marbles does each have?

Ryan attempted to solve GBT 10 by creating equations and drawing their graphs (see Figure 85). Once Ryan had failed to create the correct equations, he tried numeric methods to solve this problem. One of his reasons for failure in creating the right equations stemmed from the contamination from verbal representation to algebraic representation. When he read the statement that A says to B, "If you give 30 to me, I will have twice as many as left with you," he inserted the equation $y - 30 = 2x$. He explained the reason: "Because y [B] is losing and then x [A] will have twice as much." Other than contamination from verbal expression to formula, he

also might have forgotten that while one side is losing money the other side is gaining money. When Ryan could not solve the problem by creating equations, drawing their graphs and finding the intersection points of the graphs, he started to guess and check. Then he abandoned guessing and checking and tried to set up new equations to draw graphs. He was not able to succeed in this attempt either. The focus of this study was to examine students' solution methods rather than correct or incorrect solutions, and I can assert that his methods of solution mainly included graphic representations.

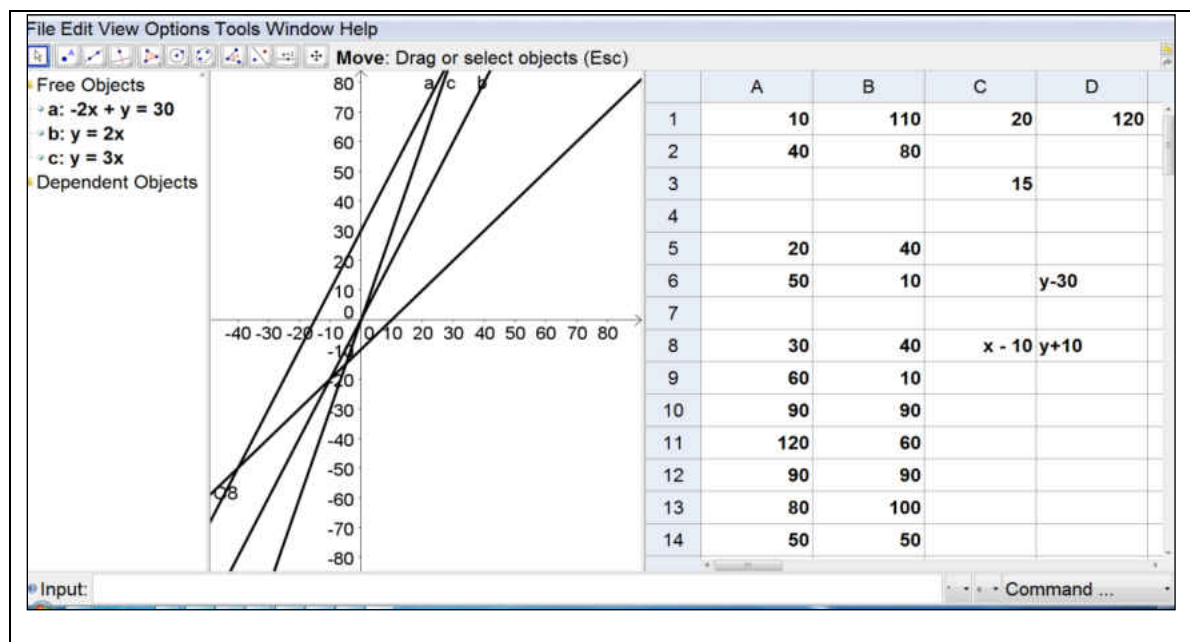


Figure 85. Ryan's solution of GBT 10

When Ryan's methods of solution for GBT 10 were compared with his methods for PBT 10, the change was from a nonvisual solution to a visual solution, which indicated that Ryan demonstrated visual thinking while solving GBT 10, whereas he demonstrated nonvisual thinking while solving PBT 10. The availability of the coordinate system and automatic graph drawing capability of Geogebra might have had an effect on Ryan's preferences while solving GBT 10.

The Effect of Virtual Physical Representations

In this session I investigated the effect of virtual physical representations with Geogebra on Ryan's methods of solution. I constantly compared and contrasted Ryan's solution methods for GBT-VPRs with his solution methods for GBTs in order to determine how Ryan used the virtual physical representations in the problem.

GBT-VPR 1, 2, and 3

GBT-VPR 1: Annette has 21 coins consisting of dimes and quarters. The total amount of money she has is \$3.30. How many each coin does she have?

For GBT-VPR 1 and 2, Ryan opened a new default Geogebra window to draw the graphs by creating equations for the given situations, and for GBT 1 and 2 (see Figure 77 and Figure 78), Ryan only activated the graph view of the blank Geogebra files. In each of the GBT-VPRs, he created two equations and entered those equations into the input field to graph them. Once he finished drawing functions, he clicked *Intersect two objects* and the graph of those equations to find the intersection point. He interpreted that point according to what meanings he assigned to x and y. For instance, in GBT-VPR 1 Ryan created the first equation as $0.10x + 0.25y = 3.30$ and entered that equation into the input field to draw the graph. Then he created the second equation as $x + y = 21$ and entered it into the input field as well. He called x and y the number of dimes and the number of quarters respectively. In order to find the exact intersection point of these two graphs, he clicked *Intersect two objects* and then selected the graphs. Then the coordinates of the intersection point, $A = (13, 8)$ appeared in the algebra window. Since he chose x as the number of dimes and y as the number of quarters, he mentioned that there had to be 13 dimes and 8 quarters.

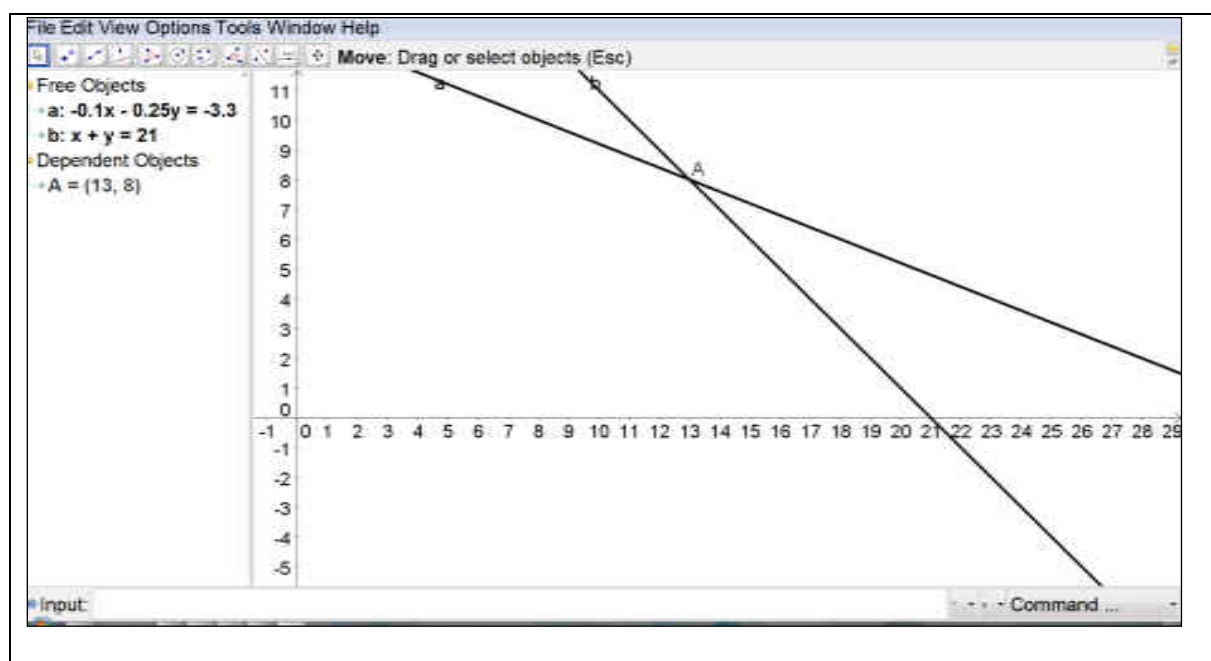


Figure 86. Ryan's solution of GBT-VPR 1

GBT-VPR 2: Farmer Brown has ducks and cows. The animals have a total of 12 heads and 32 feet. How many ducks and how many cows does farmer Brown have?

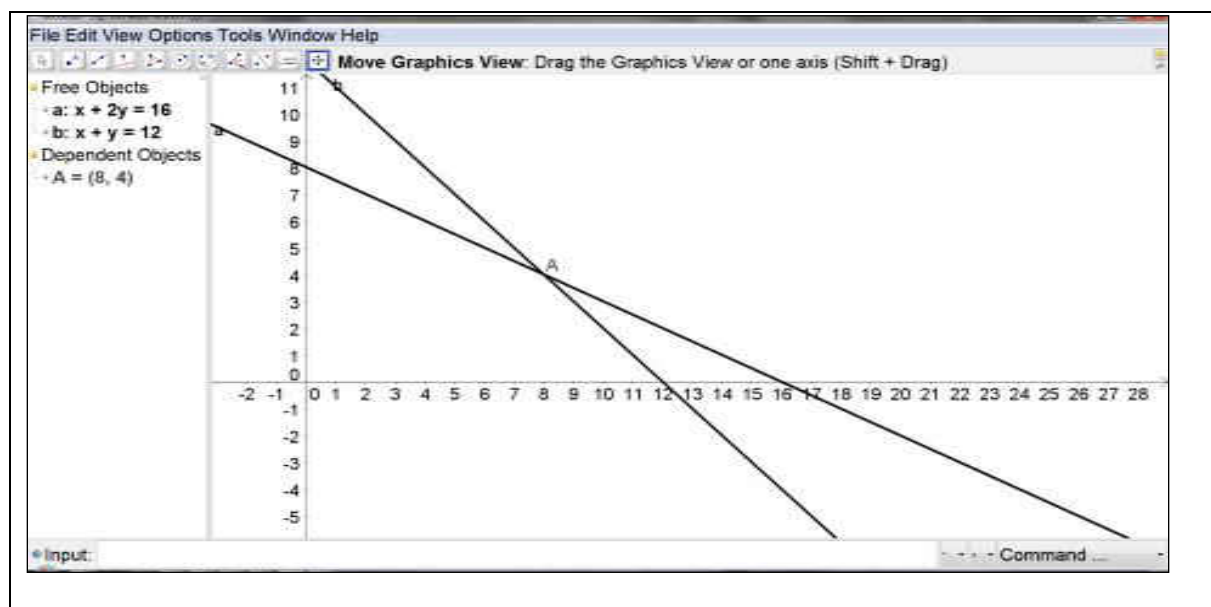


Figure 87. Ryan's solution of GBT-VPR 2

He solved GBT 1 and 2 (see Figure 77 and Figure 78) and GBT-VPR 1 and 2 (see Figure 86 and Figure 87) in a very similar way, however, Ryan only activated the geometry window to

draw the graph but not the algebra window when solving GBT 1 and 2. He followed the same process as he did in GBT-VPR 1. The only difference was in the process of checking the coordinates of the intersection point of two graphs. In GBT 1 and 2, he checked the intersection points by right-clicking and selecting the object properties instead of activating the algebra window as he did in GBT PR 1 and 2. In terms of representations and main method of solution, there were no differences between GBT-VPR 2 and GBT 2. For those tasks, the virtual physical representations did not affect Ryan's solutions.

GBT-VPR 3: A group of exchange students from Japan went to a convalescent home to sing songs for the seniors and to demonstrate origami (Japanese paper folding). As it turned out, there was either one Japanese student at a rectangular table with three seniors or two students at a round table with four seniors. There were 11 students and 25 seniors in all. How many tables were being used to demonstrate origami?

When I presented GBT-VPR 3 to Ryan, different than the previous two tasks, he wanted to work with the given setting. Similar to Mary, he observed the trends in numbers by changing the value of the sliders. He described his strategy in the following excerpt.

Ryan: I just increased the round tables to see how close I can get to 25. And it was over 25 then I lowered it and it was I think 24 then 21 where it can fit rectangle table. So I lowered the round tables and increased the rectangle tables.

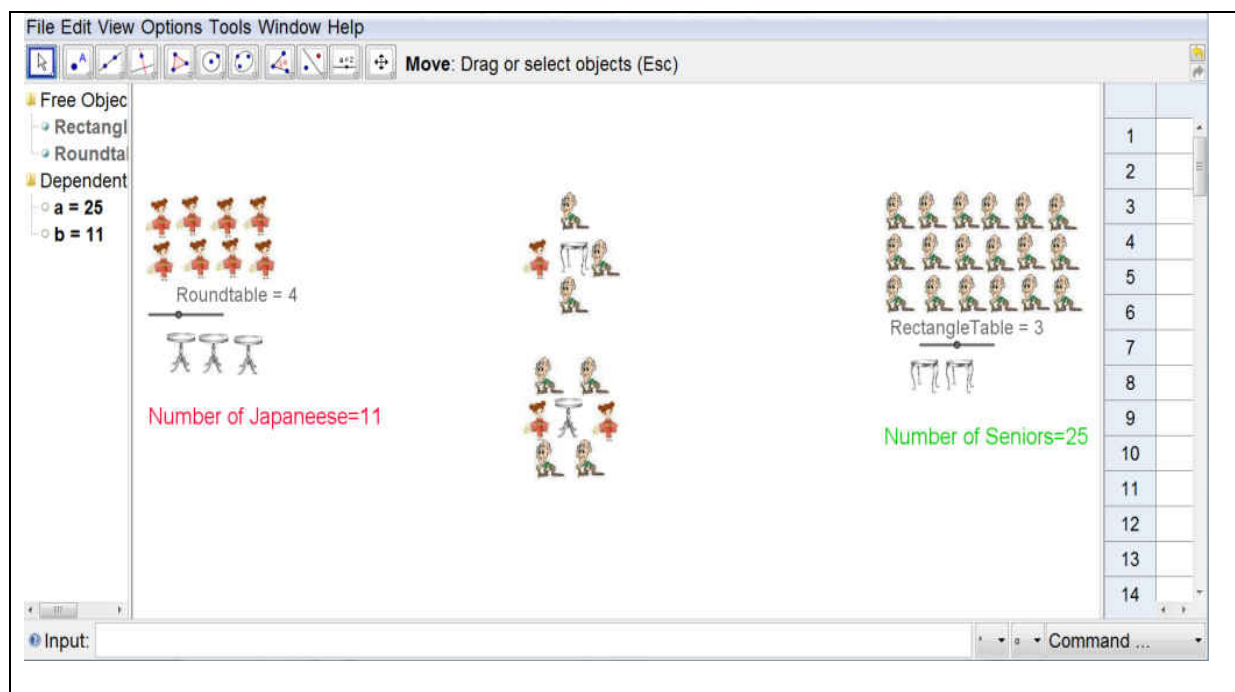


Figure 88. Ryan's solution of GBT-VPR 3

Ryan focused on the numerical aspects of the virtual physical representations and made no attempt that could be considered a sign of visual thinking. He neither manipulated the pictures given in the setting nor drew the graph. Hence, he demonstrated the nonvisual type of thinking for GBT-VPR 3, similar to his thinking for PBT 3. However, he preferred to use graphic representations when he was presented with a blank Geogebra file. He followed the same procedure in GBT 1 and 2 as he did to solve GBT 3. Ryan's solution preference for GBT 1, 2 and 3 was to use graphic representation, which was an indication of visual thinking. For task 3, virtual physical representations affected Ryan's solutions; he changed his preference from visual representations to nonvisual representations, indicating a switch to nonvisual thinking. In this task, the manipulation of numbers with sliders facilitated his guess and check method.

GBT-VPR 4 and 5

GBT-VPR 4: Hillary went riding in the hills. At one point, however her horse stumbled and was hurt. Hillary decided to go back to house with her horse. Hillary figures when they

were going the horse walks about twice as fast as when they were coming back. If her horse was hurt about 8 miles in, and the whole trip took 3 hours total, how fast did Horse walk when he was coming back to house?

For GBT-VPR 4, Ryan first changed the value of the sliders to observe how they were connected to the position of the horse. Then he checked whether the going time (slider t_1) was on 2 hours. Ryan explained his solution process as presented in the following excerpt.

Ryan: The first time is 2 [t_1] hours and then since it is twice as fast so wait wait. It [t_1] should be at 1. Because if it is the same distance covered over so it should be the other way around [changes the value of t_1 slider to 1 and t_2 slider to 2]. So it should take two hours one way. It stopped 8 miles in [shows with the mouse the point of 8]. Ooo Ok. So I guess this should be 8 [changes the value of going speed v_1 to 8] and [changes the value of the coming speed v_2 to 4]. Ok. I guess that is what it would be at then.

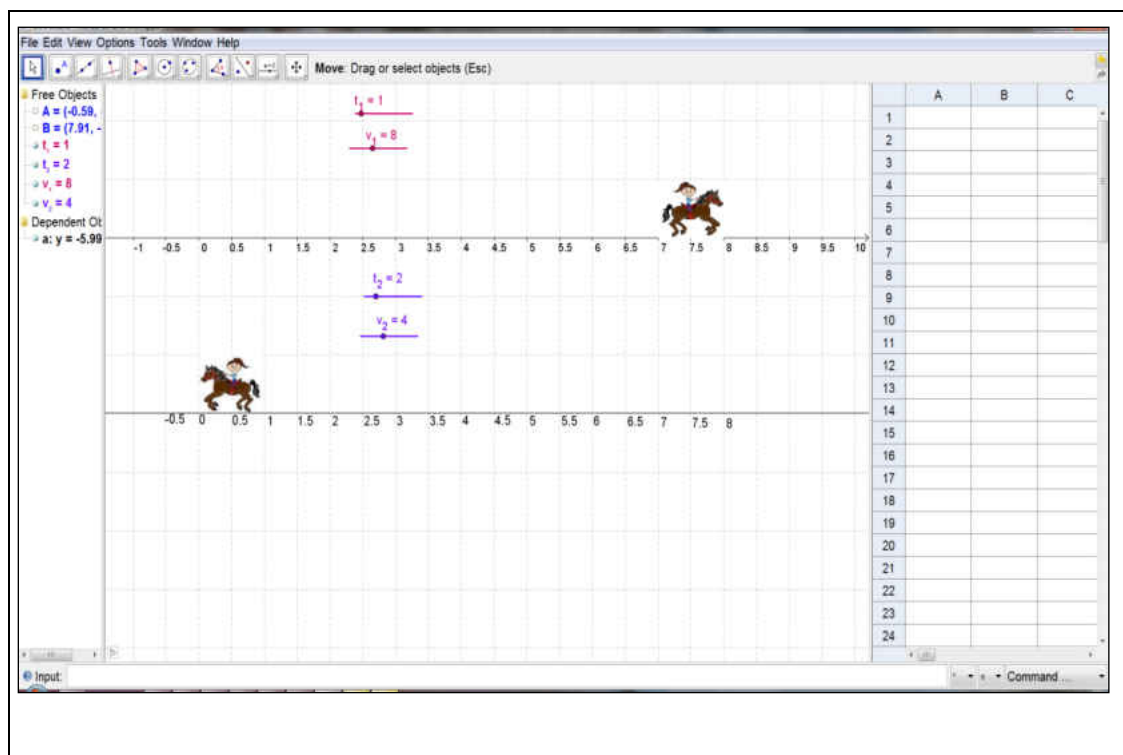


Figure 89. Ryan's solution of GBT-VPR 4

Unlike Mary, Ryan did not guess and check the numerical values of the sliders by observing the location of the car. Before Ryan attempted to solve the problem, he read the

problem over a couple of times. He used verbal representations to start the problem. He first determined the going time as 2, but upon reading the problem again he corrected himself. He demonstrated the contamination from verbal representation to numeric representation, since he split up 3 hours as 2 hours going and 1 hour coming back when the problem states “when they were going the horse walks about twice as fast as when they were coming back.” However, he corrected himself when he read the problem aloud. Once he determined the time for each part of the trip, he claimed the going speed would be 8 and the returning speed would be 4 (see Figure 89). Ryan changed the value of the speed sliders after he gave the answer, and I asked him why he did so. He reported that he first imagined the location of the horse in his head, then applied this mental image in Geogebra. The virtual physical representations helped him visualize the problem on relations. Ryan’s answer showed that he mainly used graphic representation or visual thinking to solve GBT-VPR 4.

GBT-VPR 5: Jason got on his bike and went for a ride. He rode at a speed of 10 km per hour from his house to his sister’s house in another city. Jason and his sister then got in a car and traveled at a speed of 50 km per hour to their mother’s house. The total distance from Jason’s house to his mother’s house is 320 km, and Jason traveled for 8 hours. How far is it from Jason’s house to his sister’s house?

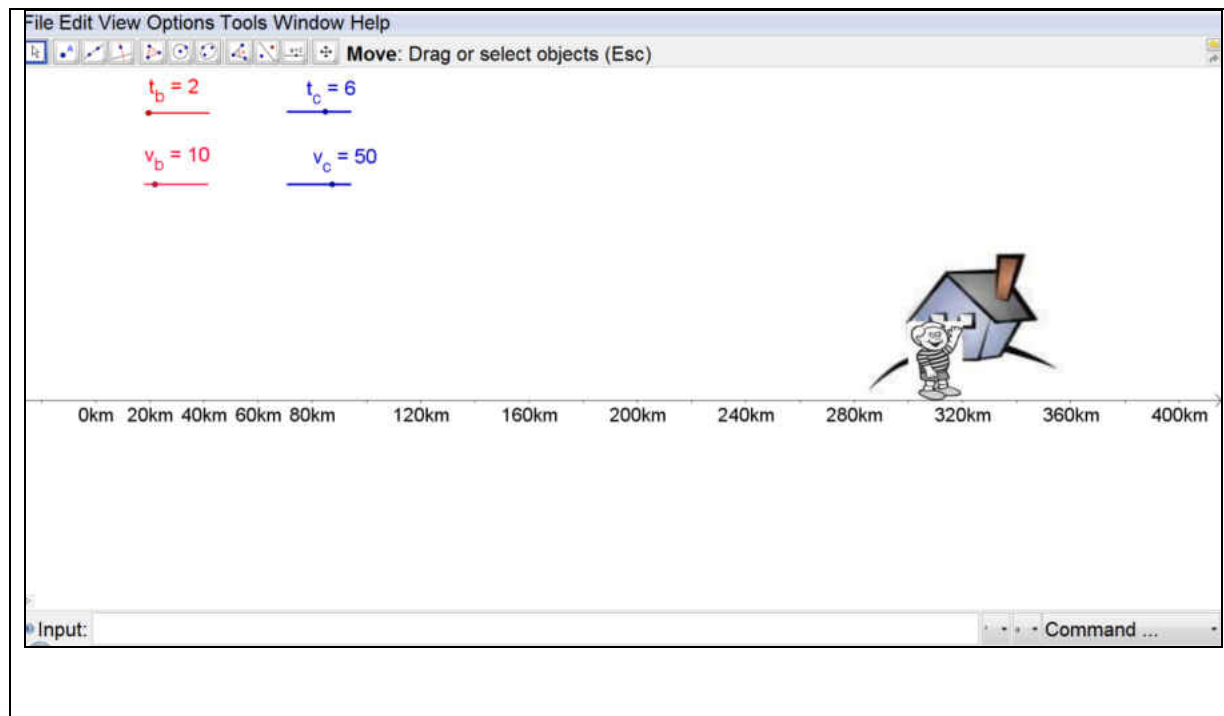


Figure 90. Ryan's solution of GBT-VPR 5

Ryan used the same method when he was solving GBT-VPR 5 (see Figure 90). Unlike GBT-VPR 4, GBT-VPR 5 included the exact speeds for each part of the trip, but the time was unknown. Ryan first set up the speed by changing the value of the speed sliders then he read the problem again and determined the travel time for each part of the trip. Ryan mentioned that he imagined a car and bicycle while they were moving. He said, "I set up the speed; I am thinking the distance. I imagined a bike moves 10 km each hour and a car moves 50 km each hour. He will be at 320 km if he rides 2 hours bike and drives 6 hours by car."

Ryan again used visual solution in this task. Unlike Mary, he did not move the time slider to observe the distance traveled by Jason. In both GBT-VPR 4 and 5, he used the sliders to show his final answer to me. Ryan's response indicated that he was using a visual support to solve this problem. Since he needed a visual support, he employed visual methods in this problem.

GBT-VPR 6, 7, and 8

GBT-VPR 6: The telephone company offers two types of service. With Plan A, you can monthly pay \$6 plus 12 cents for each minute. With Plan B, you pay \$15 monthly, plus 6 cents for each min. At least how many minute would you have to use the telephone each month to make Plan B the better option?

For GBT-VPR 6, 7 and 8, Ryan again created graphs and determined their intersection point of them which was the same strategy that he used when he was solving GBT-VPR 1 and 2. For GBT-VPR 6, 7 and 8, Ryan did not use the given virtual physical representation; instead he opened a default Geogebra window which has algebra view and the Geometry window.

Ryan chose x as the number of minutes talked and y as the cost of the monthly service when x minutes talked. He created the first equation, $y = 0.12x + 6$ to represent the cost of using Plan A for a month and the second equation, $y = 0.06x + 15$ to represent the cost of using Plan B for a month. He entered those equations into input field and determined the intersection point of their graphs. He reported that after 150 minutes Plan B would be a better option.

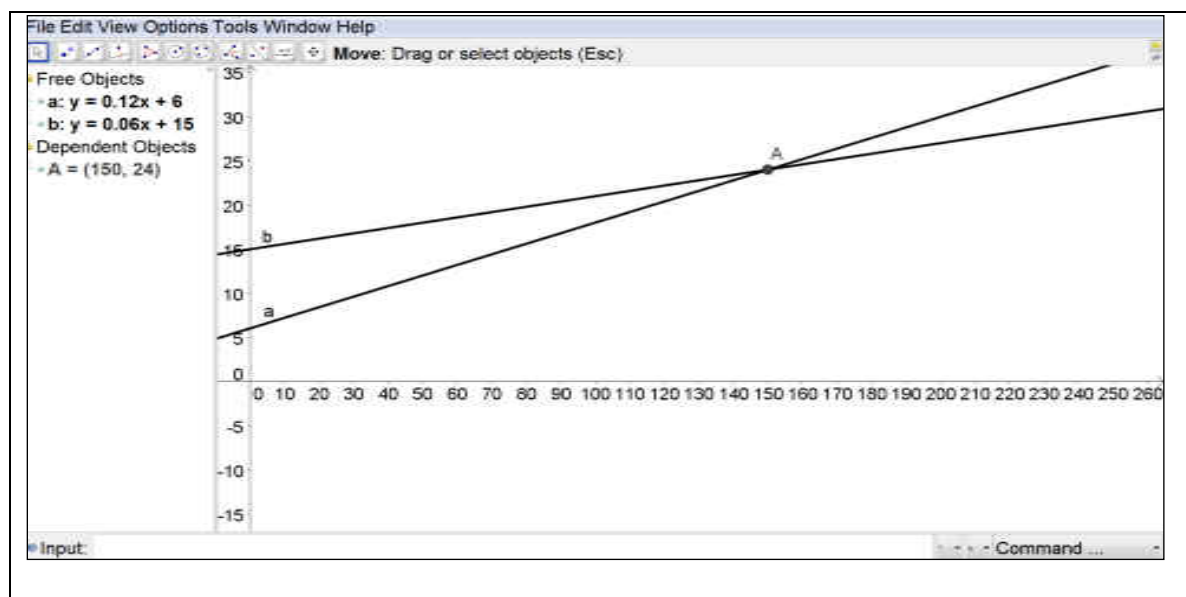


Figure 91. Ryan's solution of GBT-VPR 6

GBT-VPR 7: Both Jim and Todd consider their weight a problem. Each one is trying out for the football squad and wants to weigh more. Jim is eating and working out, and has found he gains about 1 pound each week. At this point he weighs 180 pounds. Todd, on the other hand, weighs 167 and is eating, working out, and eating. He is gaining about 5 pounds every 3 weeks. How long will it take for Jim and Todd weigh the same?

Ryan followed the same strategy as he used in GBT-VPR 6. He chose x as the number of weeks and y as the weight after x weeks (see Figure 92). He entered the first equation, $y = x + 180$, into the input field to represent the weight of Jim after x weeks and the second equation, $y = (5/3)x + 167$ to represent the weight of Todd after x weeks. Then he determined the intersection point of the equations' graphs. He interpreted the intersection point and said "after 19.5 weeks they would both weight 199.5 pounds."

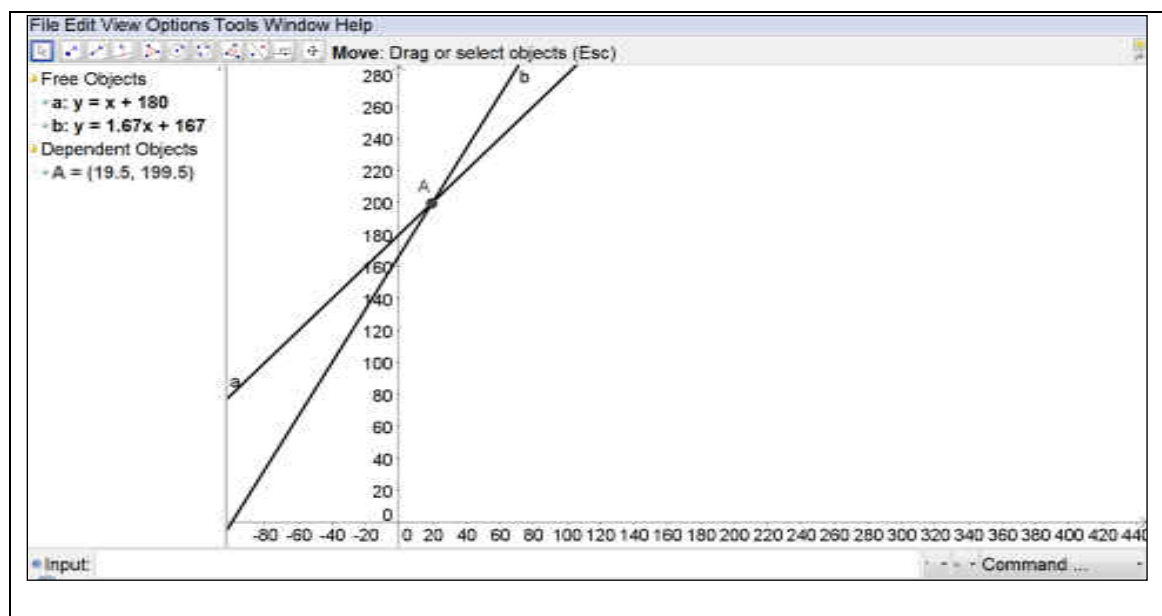


Figure 92. Ryan's solution of GBT-VPR 7

GBT-VPR 8: Sharon and Megan have \$20 and \$26 in their piggy bank respectively.

Every day, Sharon saves a quarter and Megan saves a dime. After how many days they will have same amount of money in their piggy banks?

Ryan continued to solve GBT-VPRs by using graphs. This time he chose x as the number of days and y as the money in the piggy bank after x days. He entered the first equation, $y = 0.25x + 20$ to represent the amount of money in Sharon's piggy bank after x days and the second equation, $y = 0.10x + 26$ to represent the amount of money in Megan's piggy bank after x days. Once he drew the graph of these function's equations, he determined the intersection point of graphs. He interpreted this intersection point as "after 40 days they will both have same amount of money in their piggy banks."

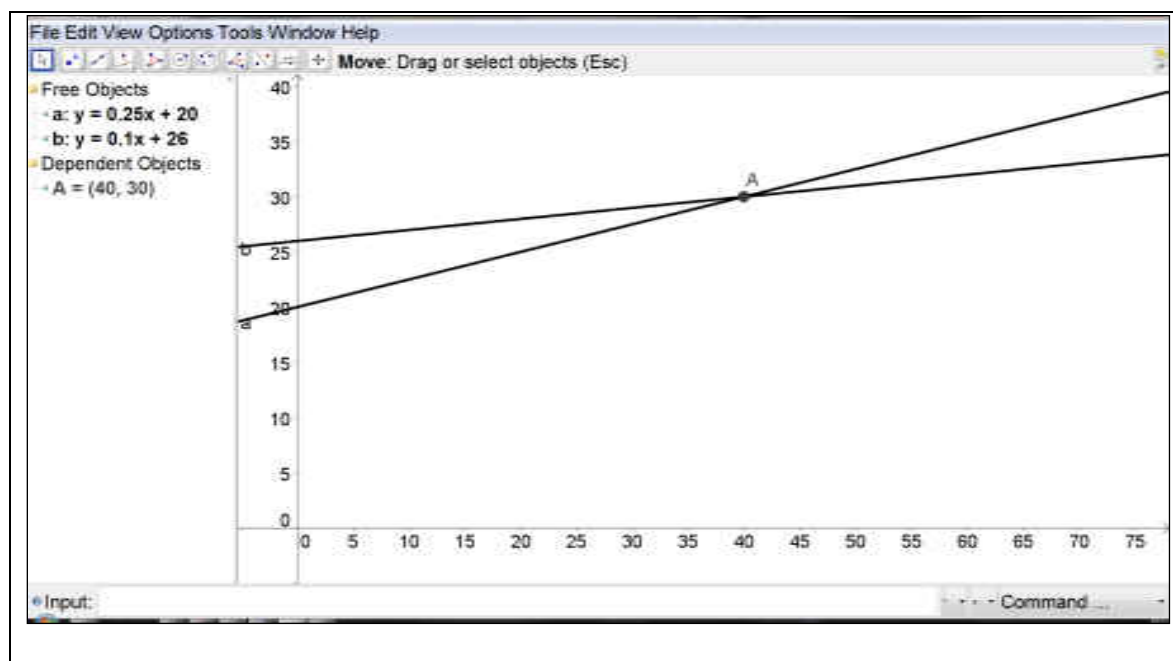


Figure 93. Ryan's solution of GBT-VPR 8

For GBT 6 and 7 (see Figure 82), Ryan only activated the Geometry window to draw graphs. Therefore, the only difference between GBT-VPR 6, 7, and 8 and GBT 6 and 7 was the algebra window. He preferred to use a visual solution in those problems. GBT-VPR He was fast in drawing the graphs and interpretation of those problems. He did not prefer to use the virtual physical representations in the task which made his solutions different from Mary in those tasks. The availability of graphs and the ease of using them in Geogebra might have an effect on his solution preference.

GBT-VPR 9 and 10

GBT-VPR 9: There are 4 red stickers to every 5 yellow stickers in a pack of stickers. There are 36 stickers in one pack. How many red stickers are there?

For GBT-VPR 9 (see Figure 94), there were yellow and red stickers on the screen whose numbers could be increased or decreased by using sliders. After Ryan read the problem, he changed the value of the yellow slider (the number of yellow stickers) to 5 and that of the red

slider to 4. Then he changed the value of the yellow slider to 20 and that of the red slider to 16. When I asked Ryan how he determined the answer, he replied, “I did the same way I did the other one with paper and pencil. This is 4 for every 5.” Ryan mentioned that he followed the same process, but this time the ratio was 4 to 5 and the total number was 36. His strategy was to find how many groups he could form by using 4 red stickers and 5 yellow stickers in a pack of 36 stickers. He said he could form 4 groups by using those 9 stickers. He stated, “I added them together and divided [36] by 9. I got 4 and then multiplied it and then checked.” As mentioned in his PBTs, his solution of PBT 9 was categorized as a visual solution since he imagined the groups to solve the problem. Since Ryan mentioned that he used the same strategy for GBT-VPR 9, this solution was also categorized as visual. In this task, Ryan used the sliders, but he used them to show the final solution. Ryan mentioned that he first imagined in his head then he changed the value of the yellow slider to 20 and that of the red slider to 16.

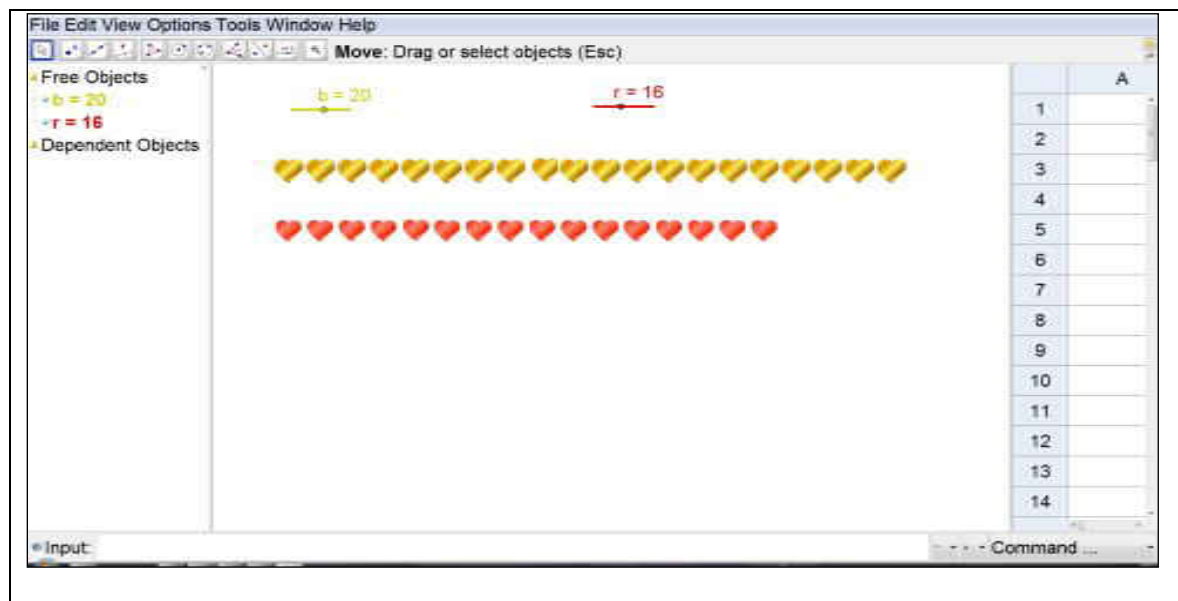


Figure 94. Ryan's solution of GBT-VPR 9

GBT-VPR 10: Mike has three times as many candies as George. If he gives George six candies, he will then have twice as many as George then has. How many candies did they each have to start with?

For the last task, Ryan again preferred to use a graphic representation for both GBT-VPR (see Figure 95) and GBT (see Figure 85). He mentioned that he wanted to use the graphing function, and he opened a new window to create graphs. He first created two equations to draw their graphs and determine the intersection point of them. He entered the first equation as $y = 3x$, with x representing the number of candies George had and y representing the number of candies Mike had. For the second equation, he entered $y = 2x + 6$ into the input field. Ryan did not realize that the second equation was not correct, and thus he thought he found intersection points of those graphs of equations. He attempted to solve GBT 10 in a similar way. He created equations and drew their graphs. Since Ryan's solutions did not differ between GBT and GBT-VPR 10 representations did not affect his method of solution.

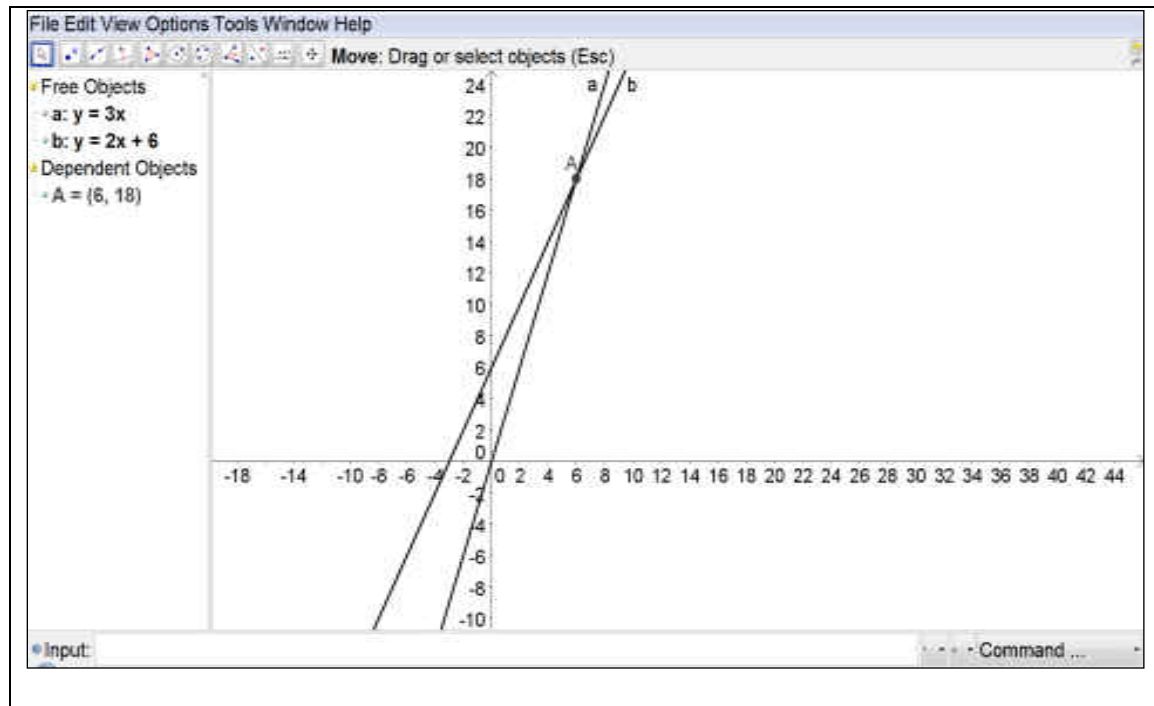


Figure 95. Ryan's solution of GBT-VPR 10

The common feature of Ryan's solutions with Geogebra was the use of graphs. Even with GBT-VPRs, in most of them he did not manipulate the virtual physical representation and preferred instead to use the geometry window to draw the graphs. Ryan's solution methods for GBT-VPRs and GBTs are illustrated in Table 7. For each task, I included the representations that he used and the categorization of his main method of solution.

Table 7. Ryan's solution methods of GBT-VPRs and GBTs

#	GBT-VPR	GBT
1.	Algebraic-Graphic (Visual)	Algebraic-Graphic (Visual)
2.	Algebraic-Graphic (Visual)	Algebraic-Graphic (Visual)
3.	Numeric (Nonvisual)	Algebraic-Graphic (Visual)
4.	Graphic-Numeric (Visual)	Algebraic-Graphic (Visual)
5.	Graphic-Numeric (Visual)	Numeric (Nonvisual)
6.	Algebraic-Graphic (Visual)	Algebraic-Graphic (Visual)
7.	Algebraic-Graphic (Visual)	Algebraic-Graphic (Visual)
8.	Algebraic-Graphic (Visual)	Graphic (Visual)
9.	Graphic (Visual)	Algebraic-Graphic (Visual)

#	GBT-VPR	GBT
10	Algebraic-Graphic (Visual)	Algebraic-Graphic (Visual)

When Ryan's solution methods for GBT-VPRs and GBTs were compared, the methods were almost identical. When he manipulated the virtual physical representations, he focused on their visual aspects (3 out of 4 tasks). There was only one task (GBT-VPR 3) in which he focused on the number patterns while using the virtual physical representations. Ryan used mostly visual representations in his solutions of GBT-VPRs. Ryan's preferences for visual or nonvisual methods in GBT-VPRs were similar to his preferences in MPI. His MPI score (42/60) suggested that he preferred to use visual methods when solving the problems in MPI. His solutions for GBT-VPR suggested that he also preferred visual solutions when solving GBT-VPRs. Overall, Ryan solved 9 out of 10 GBTs with visual methods. Therefore, Ryan's preference for visual or nonvisual solutions in GBT-VPR was similar to his preferences in MPI.

CHAPTER SIX: THE CASE OF DAVID

David's previous year mathematics teacher was also one of our project participants from a Central Florida high school mentioned in the methodology chapter. David received an A in his Algebra II class and was enrolled in Pre-Calculus in the Fall semester of 2010. David was in 11th grade at that time.

David was born in Central Florida. He was 16 years old. His parents are both from United States and they graduated from colleges. David is good at sports. He plays soccer and basketball. He is also proficient in computer use. When I asked in which field he wanted to pursue a degree, he mentioned that he had not decided yet but it might be something related with mathematics and science.

I observed David in his Pre-Calculus class. He was not a very active participant to the class but he was on task. Although he was not very talkative in the interviews, he was not shy and he had good social relationships with his friends. When I talked to his mathematics teacher, she said that David was not among the best students in the class but he was quick at understanding the mathematical topics. Even though he did not have same teacher for all his mathematics classes, his Pre-Calculus teacher included mainly nonvisual methods in her teaching even though the topic was trigonometric functions.

David scored 32 out of 60 in Suwarsono's MPI. Since David's score is close to half of the total score, it was difficult to determine his preference. This score suggests that he uses both visual and nonvisual methods in a relative equilibrium. In the interviews I further investigated his solutions and made inferences about his type of thinking.

David's solution of PBTs

PBT 1, 2, and 3

PBT 1: Altogether there are 8 tables in a house. Some of them have four legs, and the others have three legs. Altogether they have 27 legs. How many tables are there with four legs?

When I presented PBT 1, David started to guess and check the number of tables with three and four legs (see Figure 96). He started with 6 tables with four legs, then decreased the number of tables with four legs and increased the number of tables with three legs. He tried 4 tables with four legs and 4 tables with three legs, but this time the total number of legs was 28, which was greater than 27. In the last trial, he again decreased the number of tables with four legs by 1 and increased the number of tables with three legs, which resulted in the correct answer. Consider the following excerpt.

David: 4 times 6 equals 24 which leaves you 3 possible legs so that means one table with three legs. So I would say oh noo. I am checking 4 times 4 is 16 which leaves you other 4 possible tables and so 4 times 3 equals 12. 12 plus 16 is 28. Ok so 4 legs of 3 tables equal to 12. That leaves 5. 5 times 3 equal 15. That is 27 (15+12). 3 tables with 4 legs, 5 tables with 3 legs.

The image shows a box containing handwritten calculations. On the left side, there are four equations: $4 \times 6 = 24$, $4 \times 4 = 16$, $4 \times 3 = 12$, and $4 \times 3 = 12$. On the right side, there are two equations: $1 = 3 \text{ legs}$ and $4 \times 3 = 12$. Below these, there is another equation: $5 \times 3 = 15$. At the bottom center, the number 27 is written.

Figure 96. David's solution of PBT 1

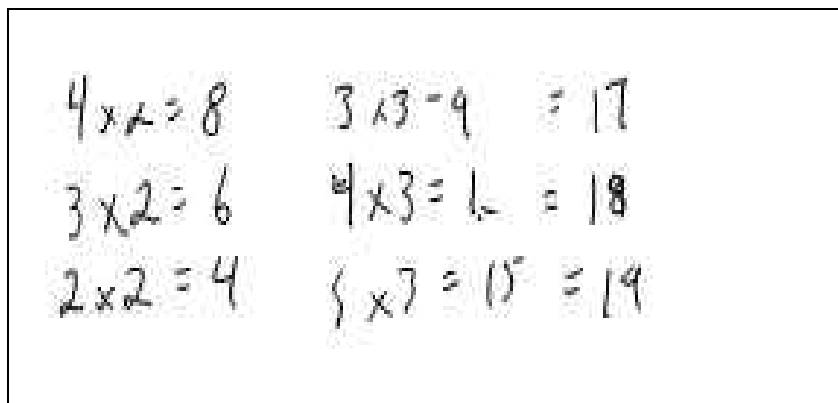
As seen in the excerpt David used numeric representations to solve PBT 1. His systematic decrease in the number of tables with four legs and increase in the number of tables

with three legs led him to the correct solution. In this problem David's thinking preference was nonvisual. I asked David to solve the following tasks in order to further understand his preferences while solving problems with paper and pencil.

PBT 2: One day you see 7 cycle riders and 19 wheels ride by your house. How many bikes and how many tricycles were there?

David approached the second task in the same way that he had approached the first one. This time there were tricycles and bicycles instead of tables with four legs and three legs. He started with 4 bicycles and 3 tricycles (see Figure 97). Each time he increased the number of tricycles by 1 and decreased the number of bicycles by 1. His third trial resulted in the correct answer.

David: Basically just like number one, I would probably do the same thing. 4 bikes with 2 wheels equals 8 and 3 bikes with 3 wheels that is 9. That equals 17. 3 bikes with 2 wheels equals 6 and we have 7 bikes in total so 4 times 3 is 12. That is 18. So 2 times 2 equals 4 and 5 times 3 equals 15 so it is 19 wheels.



$4 \times 2 = 8$	$3 \times 3 = 9$	$= 17$
$3 \times 2 = 6$	$4 \times 3 = 12$	$= 18$
$2 \times 2 = 4$	$5 \times 3 = 15$	$= 19$

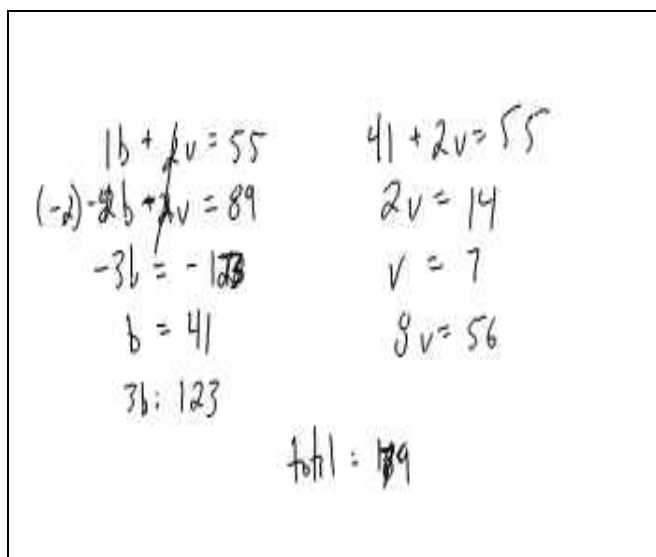
Figure 97. David's solution of PBT 2

The excerpt illustrates that David again used numeric representations to solve the problem. The same type of solution method supported his preference for the same type of problem. He made no attempt that could be described as visual thinking to obtain the solution of

these first two tasks. Therefore David's thinking preference was categorized as nonvisual for PBT 1 and 2.

PBT 3: A group of students were transported to the championship basketball game using buses and vans. When one bus and two vans unloaded, there were 55 students. A few minutes later, two more buses and one van unloaded. This time there were 89 students. In all, three buses and eight vans drove students to the game. How many students went to the game?

For PBT 3, David thought for a while and mentioned that he was trying different combinations to form 3 busses and 8 vans. Then he decided that he needed to formulate the given case. He wrote the formula given in *Figure 98*, and then he solved it by an elimination method. David said, "Basically system of equations. I am going to solve for b and v ." He multiplied the second equation by (-2) to eliminate v and find b . He calculated b as 41. When he determined b , he plugged its value into the first equation and determined v as well. Then he calculated the number of people in three busses and eight vans.



Handwritten solution for PBT 3:

$$\begin{array}{lcl} 1b + 2v = 55 & & 41 + 2v = 55 \\ (-2) \cdot 1b + 2v = 89 & & 2v = 14 \\ -3b = -123 & & v = 7 \\ b = 41 & & 8v = 56 \\ 3b = 123 & & \\ \text{Total} = 179 & & \end{array}$$

Figure 98. David's solution of PBT 3

David's solution included algebraic representations which were categorized as nonvisual methods. His nonvisual solution method indicated that he preferred nonvisual thinking while solving PBT 3. His nonvisual solutions of the first three tasks have shown that he is a representative of Ktutetskii's analytical thinking type. In order to further examine his thinking, I presented the following tasks.

PBT 4 and 5

PBT 4: Blaise rode his bike to his friend Elroy's house, which was 18 miles away. After he had been riding for half an hour, he got a flat tire. He walked his bike the rest of the way. The total trip took him 3 hours. If his walking rate was one-fourth as fast as his riding rate, how fast did he ride?

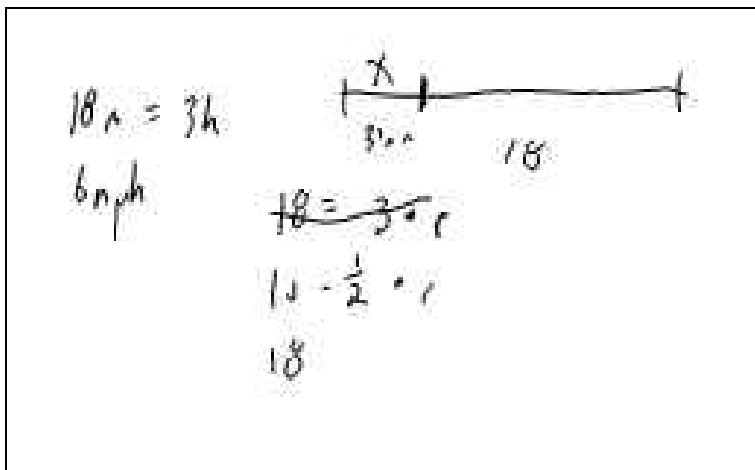


Figure 99. David's solution of PBT 4

When I presented PBT 4 (see Figure 99), David determined the average speed of the trip. Then he realized that it did not help him to reach the solution. When he could not go further with the solution, he drew a picture of the trip. He called x the speed and labeled the 30 minutes as the time for riding the bike. Then he quit solving the problem. He said, "I do not know how to split 18." In this problem, David started with algebraic representations, but when he was not

able to solve it, he needed a visual support. After creating a diagram representing the bike's route, he still did not know how to proceed. He used algebraic and graphic representations in this problem.

For this task, David demonstrated characteristics of a harmonic thinker (Krutetskii, 1976). Since he started with the algebraic representations to solve the problem, his solution suggested the abstract-harmonic subtype. Krutetskii (1976) stated that when students were presented with problems, those who belonged to the abstract-harmonic subtype preferred to start from verbal-logical formulations, while those of the pictorial-harmonic subtype preferred to start from visual-pictorial features. I continued to ask problems in order to understand David's thinking preferences while solving mathematics problems.

PBT 5: Alex planned a trip from Orlando to New York. From his house to the airport he took a taxi and he flew from Orlando to New York. The total trip took 4 hours. The average speed of taxi is 60 mph and the average speed for plane is 400 mph. From her house to New York airport the distance is 1260 miles. What is the distance between her house and Orlando Airport?

The next problem was similar to PBT 4, but this time the speeds were known while the time was unknown. This time David did not have any difficulty with solving the problem. After reading the problem, he verbally explained his solution.

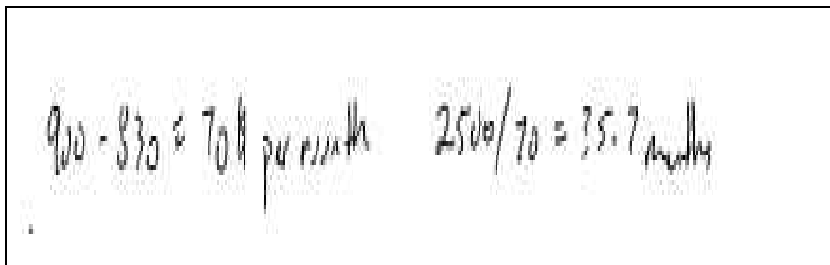
David: Since they are asking for the miles driven in taxi so if I get rid of airplane, those 1260 and it was going 400 miles per hour. In 3 hours she would have gone 1200 miles by plane. And there is 60 miles left over and she was going 60 miles per hour in the taxi. So 1 hour going with 60 miles per hour gives you 60 miles.

David did not solve this task on paper but instead mentally performed the operations. He did not draw a picture, nor mention a possible image in his mind. He started with verbal

representations, and his first numerical trial resulted in the correct answer. He used nonvisual methods to solve this task. He made no reference to anything that can be considered visual thinking. Therefore his thinking preference for this task was nonvisual thinking.

PBT 6, 7, and 8

PBT 6: Covell's home-mortgage payments are about \$900 per month. He is going to refinance which will cost him about \$2500 in fees, and the new payments will be \$830 per month. How long will it take him before the new loan starts saving him money?



The image shows a handwritten solution for PBT 6. It consists of two parts: a subtraction problem and a division problem. The first part is $900 - 830 = 70$ per month, indicating the monthly savings. The second part is $2500 / 70 = 35.7$ months, indicating the time it takes to recoup the refinancing fees.

Figure 100. David's solution of PBT 6

When presented with PBT 6 (see Figure 100), David quickly solved the problem. First he wrote the monthly payments down and calculated the difference between them. Once he determined the difference, he divided the refinance fee by that difference to find the number of months that it would take before the new loan started saving money. "Since he started with 900, he is only dropping 70 dollars. So 70 dollars per month, and you are asking how many months. If I do 2500 divided by 70, so 35.7 months." David's solution for this task included only nonvisual representations, which indicated that he demonstrated nonvisual thinking while he was solving PBT 6.

PBT 7: Javier received a letter from his bank recently concerning his checking account. Under his current plan, each check he writes costs 15 cents, and there is a monthly fee of \$1.60. Under the proposed new plan, each check he writes will cost 12 cents, and there will be a

monthly fee of \$2.75. What is the minimum number of checks Javier must write monthly in order to make new plan cost him less than the old plan?

PBT 7 was a problem very similar to PBT 6. This time David calculated the differences between the monthly fees of two plans and also the difference between the costs of each check written in two plans. Then he divided the difference between monthly fees by the difference between the costs of checks in different plans.

David: So his current plan has 15 cents for the checks he writes with a monthly fee of a dollar sixty. But the new one is 2.75 with 12 cents for each check he writes. So the difference here is a dollar five cents. Each check he writes he is only saving 3 cents. Since it is a dollar and five cent difference, I am trying to think how many checks he needs to write to get over that number. So 0.03 dollars times the check number should get over 1.05. So it would be bigger than 35 checks.

$$\begin{array}{rcl}
 0.15 & 1.60 & 1.05 \\
 0.12 & 2.75 & \\
 0.03 & & \\
 0.03x \geq 1.05 & & \\
 x = 35 & &
 \end{array}$$

Figure 101. David's solution of PBT 7

The excerpt and his written solution (see Figure 101) illustrated that David's solution mainly included algebraic representations, so his method was categorized as a nonvisual method. His similar response to PBT 6 provided further evidence his thinking preference as nonvisual for mathematical problems similar to PBT 6 and 7.

PBT 8: The concessions manager at the Central High School football stadium offered two pay plans for people willing to sell peanuts in the stand at home football games. The first plan pays \$57.60 plus \$0.37 per bag sold. The second plan pays \$29.00 plus \$0.63 per bag sold.

- a. For what number of bags sold will these two pay plans give exactly the same pay?
- b. For what number of bags is the first plan the best choice? What about the second plan?

For PBT 8, David's strategy was different than for PBT 6 and 7. He determined the difference between the initial fees of two plans. He stated, "I need to find how many bags need to be sold in the second plan to catch up \$57.60 flat fee in the first plan." Then he divided the difference between the initial fees by the cost of 1 bag in the second plan to find how many bags needed to be sold to exceed \$57.60 (see Figure 102). When he performed the calculations, he mentioned that 45 bags needed to be sold to reach that price. "While this one [second plan] increasing to 57.60, the first plan needs to also increase to something." David also determined how much the first plan paid for 45 bags. He calculated and mentioned that the first plan would increase \$16.65. He claimed that the cost of each bag in the second plan was almost two times the cost of each bag in the first plan. He divided 45 bags in half and added the result to 45. He mentioned, "67.5 bags need to be sold for the second plan catch up the first plan."

Handwritten calculations for PBT 8:

$$57.60 - 29.00 = 28.60$$

$$0.37 \quad 0.63 = 57.60$$

$$45 \text{ bags} \quad 16.65$$

$$67.5$$

Figure 102. David's solution of PBT 8

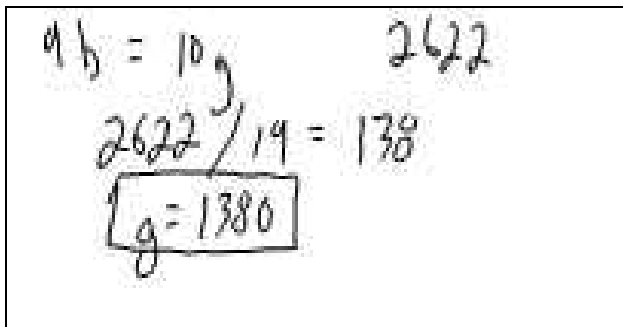
Although David made calculation errors, he used nonvisual representations in his solution similar to his approach to PBT 6 and 7. In his paper work and verbal expressions, there were no attempts which can be considered as visual thinking. Therefore I can infer that David used nonvisual thinking for PBT 6, 7, and 8.

PBT 9 and 10

PBT 9: There are 9 boys to every 10 girls in a particular high school. There are 2622 students at the school. How many girls are there?

Solving PBT 9 (see Figure 103) did not take more than 30 seconds for David. He first explained his solution strategy verbally, then he performed the required calculations. He stated:

David: So for every 9 boys there are 10 girls. My strategy would be divide the number of students in the school by 19. Because if you add these together you get 19. And that would show you how many of those groups would fit in that. So 138. From here you just multiply by 10 so 1380 girls.



The image shows a handwritten solution for PBT 9. At the top left, it says "9b = 10g". To the right of this, the number "2622" is written. Below "9b = 10g", the division "2622 / 19 = 138" is written. Below the division, the equation "1g = 1380" is written and enclosed in a rectangular box.

Figure 103. David's solution of PBT 9

David first mentioned the ratio between female and male students. Then he added the numbers of those female and male students to create a group. In each group there were 9 male and 10 female students. He divided the total number of students by the number of students in each of those groups to determine how many groups he could create. Once he calculated the number of groups, he multiplied this number by the number of female students in each group to determine the total number of girls. He mentioned that he imagined the groups and decided to determine how many groups he could create by using 2622 students. Although David did not draw the picture on the paper, his answer indicated that he had a mental imagery of groups. He did not reflect on the paper exactly what he was thinking, however, his verbal explanation of

imagining groups revealed that his solution included visual thinking. His visual support facilitated the rest of his solution.

PBT 10: Cici and Amantina have lots of stickers. Cici had one third as many as Amantina had, but then Amantina gave Cici six of her stickers, and now Cici has half as many as Amantina. How many stickers did each girl start with?

The image shows handwritten work for Problem 10. It includes several algebraic attempts and a series of numeric guesses.

Algebraic attempts:

- Top left: $C = \frac{1}{3}A$
- Below that: $C + 6 = \frac{1}{2}(A - 6)$
- Below that: $C = 18$, $A = 54$
- Bottom left: $(\frac{1}{3}A) + 6 = \frac{1}{2}(A - 6)$, followed by $\frac{A}{3} + 6 = \frac{A}{2} - 3$, $\frac{A}{3} - \frac{A}{2} = -9$, and $-\frac{A}{6} = -9$.

Numeric guesses:

- Top middle: $6 = \frac{1}{3} 18$, $12 = 12$
- Below that: $10 = \frac{1}{3}(30)$, $16 = 24$, $20 = 60$, $26 = 54$
- Top right: $8 = \frac{1}{3}(24)$, $14 = 18$
- Below that: $12 = 36$, $18 = 36$
- Bottom right: $16 = 48$, $24 = 42$
- A boxed section in the middle right contains: $18 = 54$, $24 = 48$

Figure 104. David's solution of PBT 10

In the last PBT, David attempted to create an algebraic representation for the given case. Although he created the equation, he did not prefer to solve it and instead switched to numeric representations (see Figure 104). He systematically checked and guessed the numbers. He started by assuming that Cici had 6 stickers and Amantina had 18 stickers. Once he realized that those numbers did not satisfy the second scenario, he checked 8 and 24 stickers for Cici and Amantina respectively. He continued to guess and check the number of stickers that Cici had, considering 10, 12, 16 and 20 in that order. After that, he checked 20 stickers for Cici and 60 stickers for Amantina. David said, "I should go down, it is too big." In his next trial, he checked 18 stickers for Cici and 54 stickers for Amantina. This guess resulted in the correct answer. When I asked David why he abandoned solving the system of equations and switched to guess-

and-check, he stated that he could have solved the system of equations by the substitution method. Then he solved the equations that he had created and verified the answer that he had reached by using a numerical solution.

David's solution included both algebraic and numeric representations. Since both of those representations are considered nonvisual solutions, it can be asserted that David demonstrated a nonvisual thinking preference for this problem. David made no reference to any representation or method that I can consider visual thinking for this problem.

Overall, David demonstrated the characteristics of the nonvisual thinking type while solving problems with paper and pencil. For two tasks (PBT 4 and PBT 9), David's solution method was visual and for the rest of the PBTs he used nonvisual methods.

The Effect of Technology on David's Solutions

For this section, I compared David's solution methods while solving problems with paper and pencil with his solution methods while solving problems in Geogebra. In other words, I constantly compared and contrasted his solution methods for PBTs with his methods for GBTs in order to examine the effect of technology on his preference for solving mathematics problems.

GBT 1, 2, and 3

GBT 1: Bill has \$1.25 in nickels and dimes. He has a total of 17 coins. How many of each does he have?

Once I presented GBT 1 to David, he said that he wanted to solve the problem by graphing. He opened the algebra window to enter formulas into the input field. He wrote the first equation as $x + y = 17$. Consider the following excerpt for David's method of solution.

David: x stands for nickels and y stands for dimes. 17 was in total [writes $x + y = 17$ into input field]. $x + y = 1.25$. Ohh noo, they are not going to intersect, I think the second one [equation] is not right. [Opens spreadsheet view inserts 0.05 into A1

and 0.1 into B1, 0.1 into A2 and 0.2 into B2. Selects those cells and drags until 8th row]. Since it is 17, there is no way of 17 both. There is .8 and .4 so it is 1.20 actually [drags 1 more row], this is .9 and .45 and it is 1.35. These two [points row A9=.45 and B8=.8] will give 1.25. So 9 nickels and 8 dimes.

I: Why did you give up here [graphing]?

David: I could not figure out the equation. Actually I can now. $0.05x + 0.1y = 1.25$ and intersection [he drew the graph of the equation and found the intersection point of the graphs].

After David created the first equation, $x + y = 17$ and entered into the input field, he attempted to create the second equation, $x + y = 1.25$. He realized that the graphs of those two equations were parallel to each other. Then he realized that he could not create the second equation correctly. He tried an alternative method by opening spreadsheet view. As seen in Figure 105, David created a table for increasing numbers of nickels and dimes. In column A, he inserted the amount of money into the cells with an increase in the number of nickels by 1 each time. In column B, he did the same for dimes. Since David started from the first row, the row number represented also the number of nickels and dimes. He started to create for two rows, and then he selected four cells in the first two rows and dragged them until the 8th row. Since the total number of coins was 17, he stated that it was impossible to have 17 from both nickels and dimes. He checked the 8th row and mentioned that if there were 8 nickels and 8 dimes, the amount of money would be \$1.20. Since this was not the amount given in the problem, he dragged the pattern down one more row and calculated the numbers in the 9th row. This time he checked the numbers in the 9th row and stated that there would be \$1.35 if there were 9 nickels and 9 dimes. Since the amount of money given in the problem still did not match his result, he tried numbers in different rows. Then he determined that the combination of cells A9 and B8, which represented 9 nickels and 8 dimes respectively, satisfied the given condition.

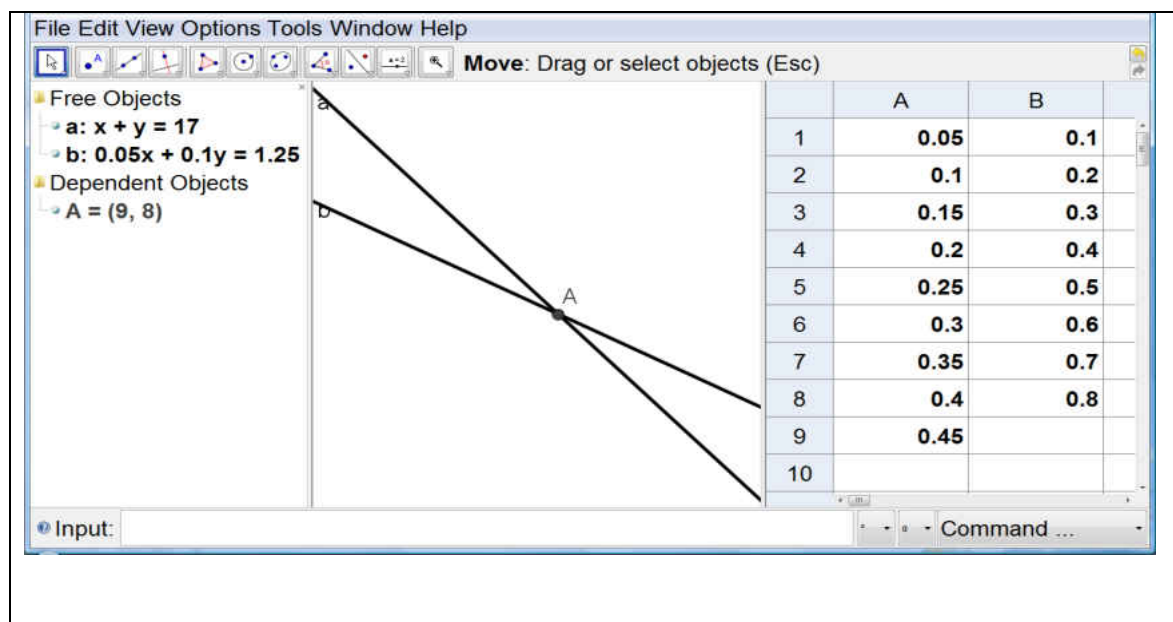


Figure 105. David's solution of GBT 1

When I asked David why he decided to switch to spreadsheet view, he replied that he was not able to create the second equation. On the other hand, after he created the table and solved it by using a spreadsheet, he said that he could solve the problem by graphing. He entered the second equation as $0.05x + 0.1y = 1.25$ into the input field and clicked *Intersect two objects* and their graphs to determine the intersection point.

David's preference was to solve the problem by using a graph. Since he could not create the second equation, he switched to the numeric representation. After solving the problem using a spreadsheet, he was able to create the second equation. Solving the problem using numeric representations might facilitate his ability to solve with graphic representations. When David's solution preferences for PBT 1 and GBT 1 were compared, it was evident that David chose a visual solution for GBT 1 as opposed to a nonvisual solution for PBT 1. In this task, technology affected David's solution method by changing it from a nonvisual to a visual method. Moreover, the availability of tools support multiple representations facilitated David's ability to translate from one representation to another.

GBT 2: The Madison Local High School marching band sold gift wrap to earn money for a band trip to Orlando, Florida. The gift wrap in solid colors sold for \$4.00 per roll ,and the print gift wrap sold for \$6.00 per roll. The total number of rolls sold was 480, and the total amount of money collected was \$2340. How many rolls of each of each kind of gift wrap were sold?

David used a graph to solve GBT 2. As with GBT 1, he inserted equations into the input field to create their graphs. The following excerpt illustrates his method of solution.

David: It is like the last problem, the solid colors represented by x and y represents the print wrap and that would be equal to 480 [inserts $x + y = 480$ into input field]. The second one is $4x + 6y = 2340$ [inserts the equation into input field and zooms out and finds intersection point]. 270 solid colors and 210 print wraps.

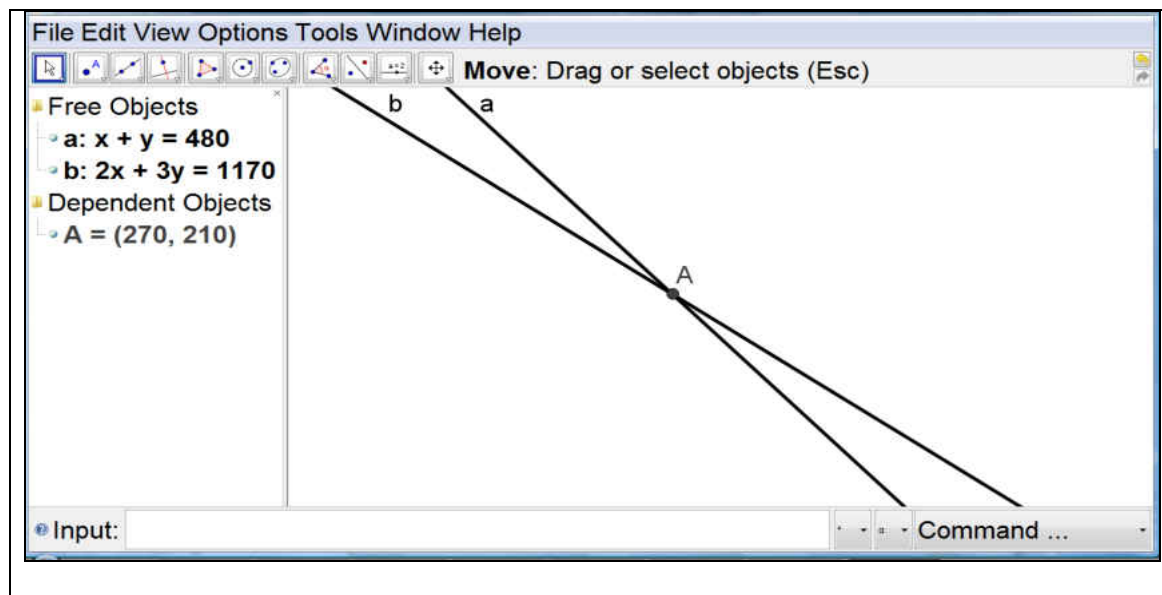


Figure 106. David's solution of GBT 2

David created two equations and drew their graphs in Geogebra. He determined their intersection point by clicking *Intersect two objects* and their graphs (see Figure 106). His verbalized thought process and his solution indicated that he used a visual representation to solve GBT 2. When I compared David's solution of GBT 2 with his solution of PBT 2 (Figure 106), the change from a nonvisual to a visual method of solution, just as in the comparison of PBT and GBT 1, was evident. Therefore, technology affected David's solution method in this task as

well. David's solution indicated visual thinking for GBT 2 as opposed to nonvisual thinking in PBT 2.

GBT 3: Campus rentals rent 2- and 3-bedroom apartments for \$700 and \$900 per month respectively. Last month they had six vacant apartments and reported \$4600 in lost rent. How many of each type of apartment were vacant?

For GBT 3, David activated the spreadsheet view (see Figure 107). First he recorded the information given in the problem in the spreadsheet. He wrote 2 into cell A1 and 3 into cell B1. Then he inserted the rents as 700 and 900 into A2 and B2 respectively. He inserted the amount of money for two- and three-bedroom apartments, starting from 1 two-bedroom and 1 three-bedroom apartments. He also wrote the amount of money for 2 two-bedroom and 2 three-bedroom apartments in the following row. Then he selected those two rows and created the rest of the pattern until the 5th row, including the amount of money for 4 two-bedroom and 4 three-bedroom apartments. He started to combine different cells to satisfy the given condition. He announced that there were 4 two-bedroom and 2 three bedroom apartments. Once he reached the solution, he deleted cells B4 and B5 to show the solution in the last cells (A5 and B3).

	A	B	C
1	2	3	
2	700	900	
3	1400	1800	
4	2100		
5	2800		
6			
7			
8			
9			

Figure 107. David's solution of GBT 3

The comparison between David's solution methods between PBT 3 (see *Figure 98*) and GBT 3 showed that there was a change from using algebraic representations to numeric representations; nevertheless both of the methods were nonvisual methods. David's thinking preference was nonvisual thinking in both of those tasks. Hence, in this task technology did not affect David's thinking preference.

GBT 4 and 5

GBT 4: A 640-mile, 5-hour plane trip was flown at two speeds. After flying 2 hours, the tailwind picked up and the speed in the second part of the trip was two times as the speed in the first part. What was the speed in first and second part of the trip?

David had a difficult time while solving GBT 4, just as he had with PBT 4. He activated the spreadsheet view to record the information given in the problem (see *Figure 108*). Once he recorded the information, he said, "If I divide 640 by 5, I will find the average speed of the trip." He began both PBT and GBT 4 by trying to find the average speed. Then he mentioned that finding the average speed would not help him solve the problem. He called x the speed in the first part of the trip and $2x$ the speed in the second part of the trip. After a long pause, he stated that he wanted to move on to the next question, similar to his reaction to PBT 4.

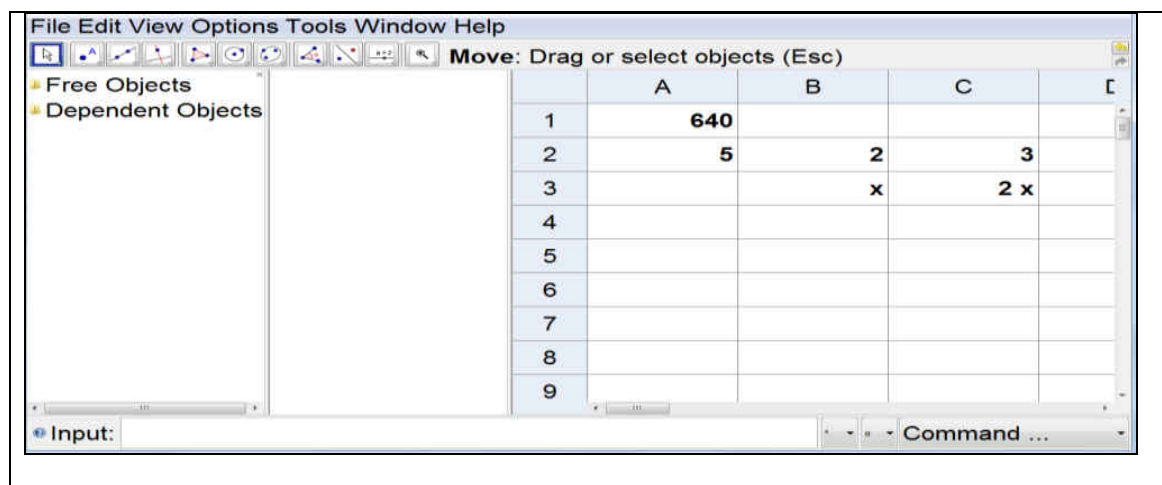


Figure 108. David's solution of GBT 4

When I compared the representations David used while he was solving PBT and GBT 4 (see Figure 99 and Figure 108), I found some differences. In PBT 4, he supported his thinking with the picture of the trip. In GBT 4, David did not give any indication of visual thinking in his verbal expressions or in his work in Geogebra. Hence his solution of GBT 4 indicated nonvisual thinking, whereas his attempted solution of PBT 4 indicated visual thinking. For task 4, technology affected his method of solution. It prompted a change from visual to nonvisual thinking for David.

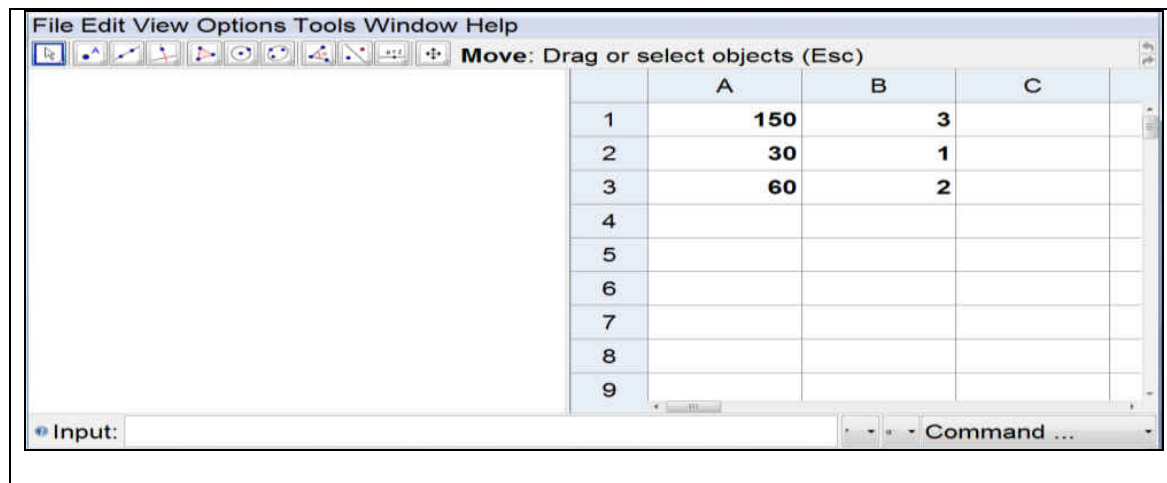
GBT 5: A man drove from home at an average speed of 30 mph to an airport. He then boarded a helicopter and flew to the corporate office at an average speed of 60mph. The entire distance was 150 miles. The entire trip took 3 hours. Find the distance from the airport to the corporate office.

For GBT 5, David again activated the spreadsheet view (see Figure 109). Then he recorded the information given in the problem. He first entered 150 miles and 3 hours. Then he inserted 30 and 60 mph. Consider the following excerpt to examine his reasoning.

David: It said the entire trip took 3 hours. One part he was going 30 miles per hour the second part he was going 60 miles per hour. If you add these together 30 and 60 is

90 plus another 60 is 150 which the entire distance was 150 miles assuming he spent 2 hours in helicopter.

David started to solve the problem by assuming 1 hour spent in each vehicle. There were 60 miles left over. Since the leftover part was 60 miles, he added that part to the helicopter ride and mentioned that the time spent in the helicopter was 2 hours. David made no reference to anything that I designate as visual thinking in either PBT 5 or GBT 5.



	A	B	C
1	150	3	
2	30	1	
3	60	2	
4			
5			
6			
7			
8			
9			

Figure 109. David's solution of GBT 5

GBT 6, 7, and 8

GBT 6: Cher wants to buy a refrigerator. She visits a store and finds two recommended models. The major brand is \$600 and is expected to cost \$30 per month in energy cost. The minor brand is \$400 and is expected to cost \$40 per month in energy cost. Which refrigerator would you advise Cher to buy?

As previously mentioned, tasks 6, 7 and 8 were highly similar word problems. When I presented GBT 6 to David, he again activated the spreadsheet view and recorded the given information (see Figure 110). He entered 600 miles and 400 miles into cells A1 and B1 respectively. Then he entered 30 under 600 and 40 under 400. He continued by entering 60 and

80 in the following row. He selected cells A2, B2, A3 and B3 and dragged them until the 12th row. David explained his solution process as follows:

David: There are two refrigerators 1 is 400 and the other one is 600 with a monthly pay of 30 and 40. There is 200 difference between them. The second month this is 80 and this is 60 [selects those 4 cells and drag it until 12th row]. We are trying to find 200 difference. [Drags until 24th row] about two years 230 [difference between the numbers in the 24th row] so it is less than that. This is 220 [difference between the numbers in the 23rd row]. This is 200 [difference between the numbers in 21st row]. The question is which one you suggest him to buy. If he will keep more than two years the expensive one [points 600] in the long run it will help him out.

In this problem, David first calculated the difference between the prices of the brands. Then he determined the total electric cost of each brand at the end of each month. He first checked whether after one year the difference between electric costs reached 200. Since the difference was less than 200 he dragged the pattern until the 24th row which he said was two years. In the 24th row the difference was 230, so he went back and pointed to the 21st row, which showed a difference of 200 between the numbers in that row. David's answer was in terms of years instead of months.

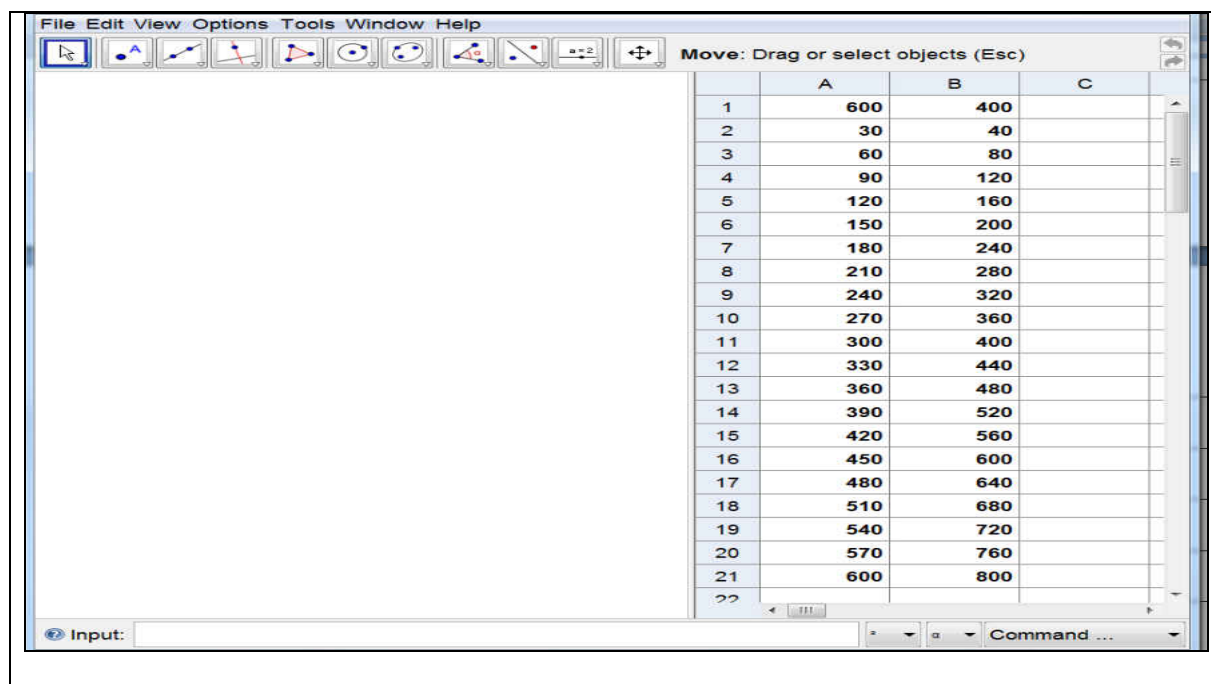


Figure 110. David's solution of GBT 6

David's solution of GBT 6 mainly included numeric representations. Unlike his solution of PBT 6, he calculated the electric cost of each month in two years. In PBT 6, David divided the initial payment by the difference between the payments. Both of those tasks mainly included nonvisual representations and were solved by nonvisual methods. Hence, technology did not affect David's thinking preference in this problem. He demonstrated a visual thinking type in both tasks.

GBT 7: You contacted two local rental companies and obtained the following information for the one-day cost of renting a truck. Company 1 charges \$40.95 per day plus \$0.19 per mile, and company 2 charges \$19.95 per day plus \$0.49 per mile. How many miles do you need to make in one day to get benefit by selecting the first company?

For GBT 7, David created equations to draw their graphs. He inserted the equations $y = 0.19x + 40.95$ and $y = 0.49x + 19.95$ into the input field and drew their graphs (see Figure 111). He zoomed out to see those two graphs and determined their intersection point by clicking

Intersect two objects and their graphs. He pointed to the graph of the first company and drew two graphs in the air. Then he said “after 70 miles, company 1 would be better.” David’s solution mainly included graphic representations. The tracing of graphs with his finger indicated that he used visual support to solve problem. Therefore his solution was considered a visual solution.

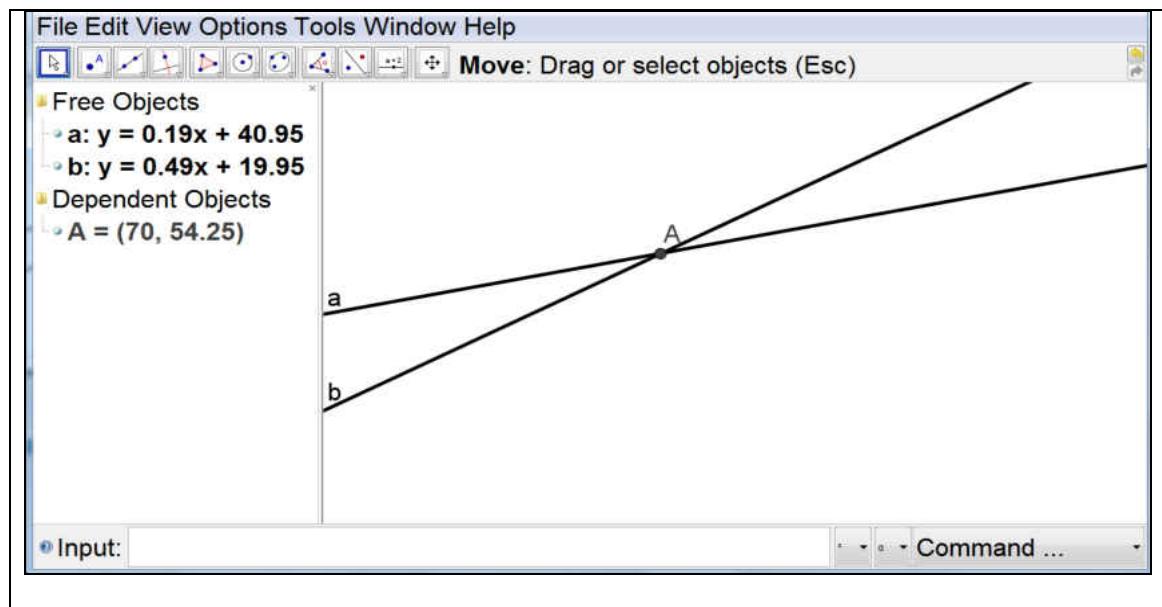


Figure 111. David’s solution of GBT 7

When I compared David’s solution strategy for GBT 7 (see Figure 111) with his strategy for PBT 7 (see Figure 101), there were clear differences in his solution preferences. David preferred to use a nonvisual method in PBT 7 as opposed to the visual method used in GBT 7. In this problem, using Geogebra affected David’s solution method and his preference for visual or nonvisual solutions.

GBT 8: A car is leaving Roseville at a constant speed of 50 km/h to go to San Francisco. Another car leaves at the same time from the next city which is 45 km closer to San Francisco at a constant speed of 40 km/h. In how many hours will the car leaving from Roseville overtake the other car?

In the next task, GBT 8, David again used the spreadsheet view (see Figure 112). He recorded the information given in the problem in the spreadsheet. The following excerpt illustrates his train of thought.

David: We have a car from San Francisco to Roseville with 50 km/h. And the other car is going 40 km/h. But that car 45 km closer [Inserts 50 to A1 and 40 to B1]. So 10 mile difference. So if we had a diagram to show distances but say 50 km one is car A and the other one is car B, car B starting at 45 and car A starting at 0. In 2 hours A would go 100 km and B would go 80 km. So there is 20 km difference. Car A still 25 behind so this will go another 50 so it is 150, and 120. This is the difference of 30 so they are 15 apart. Here [Shows the 4th row] they are 40 apart and then between 4 and 5 hours they catch each other.

I: Can you give exact time between 4 and 5?

David: Yes. Every hour he is gaining 10 km. So if you go half that will gain 5 so it would be 45. So in 4 and half hour.

When David was solving the problem, he did not create the diagram by using Geogebra; however, he traced the diagram with his finger on the computer screen. He arranged the cars' positions in the diagram. Once he determined their positions, he started to calculate the distances traveled by each car and the distances between cars. He created the pattern until the fourth row, which indicated the end of the fourth hour. He decided that the car would overtake the other car in four and half hours. Even though David did not draw the diagram by using Geogebra tools, his response to this task suggests that he employed imagery and he was demonstrating one of the characteristics of visual thinkers, who start from graphic representations and rely on visual means during a solution.

	A	B	C
1	50	40	
2	100	80	45
3	150	120	
4	200	160	
5			
6			
7			
8			
9			

Figure 112. David's solution of GBT 8

When David's solution of GBT 8 was compared with his solution of PBT 8, differences were revealed in David's verbal responses to the tasks. He verbalized his thinking process (as shown in the excerpt) and his tracing of the diagram on the computer screen with his finger suggested that he employed imagery while solving GBT 8. In PBT 8, David's solution indicated that he used nonvisual methods. Therefore, there was a change from nonvisual thinking to visual thinking in this problem.

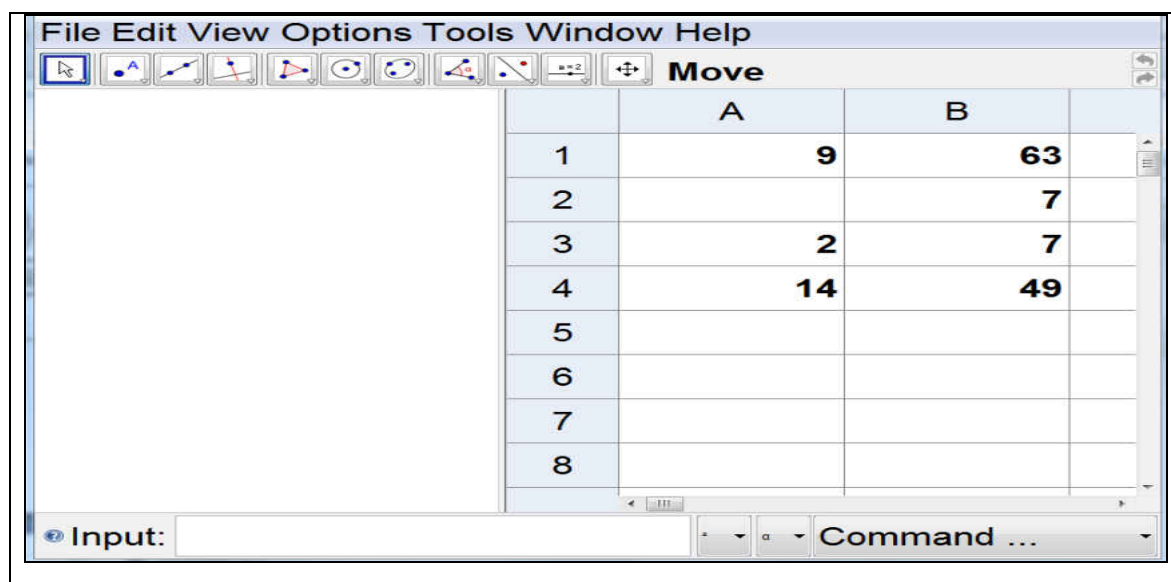
Until now, in some problems David changed his solution preference. In order to probe further his thinking on algebra word problems and understand the effect of using Geogebra, I presented him with GBT 9 and 10. The next session illustrated his solution methods for GBT 9 and 10 and compared those solutions with his solutions of PBT 9 and 10.

GBT 9 and 10

GBT 9: There are 2 students wearing glasses to every 7 students not wearing glasses in a particular classroom. There are 63 students in that classroom. How many students are wearing glasses?

When I presented the task GBT 9 to David, his response was very quick, similar to his response to PBT 9 (see Figure 103 and Figure 113). For GBT 9, David mentioned that he used the same method of solution as for PBT 9.

David: Similar to what I did before. Two students to every 7 students. That is group of 9 students. And there are 63 of them which means there is only 7 groups of those kids. Obviously 7 times 2, will give 14 students and 7 times will give you and that is 63.



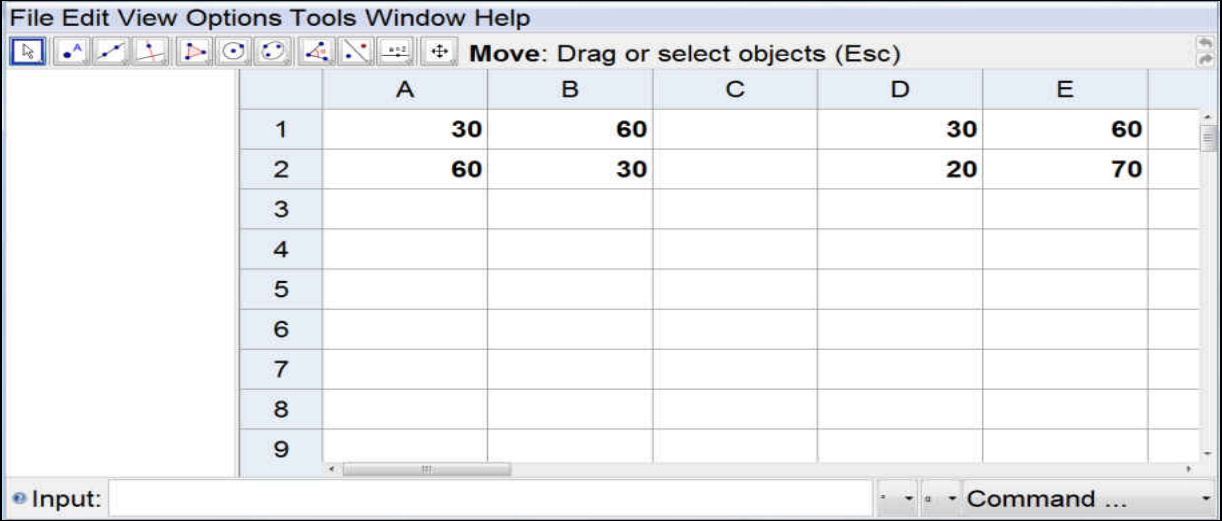
	A	B
1	9	63
2		7
3	2	7
4	14	49
5		
6		
7		
8		

Figure 113. David's solution of GBT 9

David also reported imagining the groups in both PBT 9 and GBT 9. Moreover, he used neither paper and pencil nor Geogebra tools to demonstrate the imagery in his head. David's written responses only included nonvisual representations. From his verbal responses to the tasks, I can infer that David relied on visual means during both solutions. Therefore, there was no change in his preference of solution from PBT to GBT-VPR. I can infer that technology did not affect his thinking preference in this problem.

GBT 10: A and B each have a certain number of marbles. A says to B, "if you give 30 to me, I will have twice as many as left with you." B replies "if you give me 10, I will have three times as many as left with you." How many marbles does each have?

For GBT 10, David again activated the spreadsheet and started to try different numbers. He first inserted 25 for the number of marbles A had and 65 for the number B had. Then he checked the first condition given in the problem and was not satisfied with these numbers. He deleted the numbers and inserted and tried 20 marbles for A and 60 marbles for B. Those numbers did not satisfy the first condition, either, and he deleted them. Next he tried 30 for A and 60 for B, which satisfied the first condition (see Figure 114). Then David checked the second condition for 30 marbles for A and 60 marbles for B. Once those numbers did not satisfy the second condition, David quit trying.



	A	B	C	D	E
1	30	60		30	60
2	60	30		20	70
3					
4					
5					
6					
7					
8					
9					

Figure 114. David's solution of GBT 10

David's solution method for GBT 10 included numeric representations. As previously mentioned PBT 10 also included numeric representations, however, he started the problem by creating equations in other words he used algebraic representations, then he translated to numeric representations. Since both algebraic and numeric representations were nonvisual methods, the main method of solution was nonvisual method for PBT 10. Therefore, David did not change his

thinking preference while solving GBT 10 compared to PBT 10. I can claim that using Geogebra did not affect David's preference for visual/nonvisual thinking for this task.

Overall, when I compared David's solutions of PBTs with GBTs, David changed his method of solution from nonvisual to visual in GBT 1, 2, 7 and 8. These changes were from the use of numeric representations to graphic representations. Only in GBT 4 he changed his solution from visual to nonvisual method. Geogebra evidently helped David solve the problems with graphs, since he did not create any graphs while solving problems with paper and pencil. Therefore, Geogebra supported the use of graphic representations in this problem. Moreover, the use of spreadsheet facilitated his guess and check method.

The Effects of Virtual Physical Representations

In this part, I investigated the effect of virtual physical representations on David's solution methods by comparing David's solutions of GBT-VPRs with his solutions of GBTs. The results indicated how David's thinking process was changed for similar types of problems by the virtual physical representations.

GBT-VPR 1, 2, and 3

GBT-VPR 1: Annette has 21 coins consisting of dimes and quarters. The total amount of money she has is \$3.30. How many each coin does she have?

I presented GBT-VPR 1 to David. He started to solve the problem by manipulating the virtual physical representations. He said "visual representations make it even easier." He made 2 one-dollar groups by using 4 quarters for each (see Figure 115). Then he counted the leftover coins and made another group of 1 dollar, but this time he assumed that those 13 coins were dimes. He said, "This is another 1 dollar and this is 30 cents."

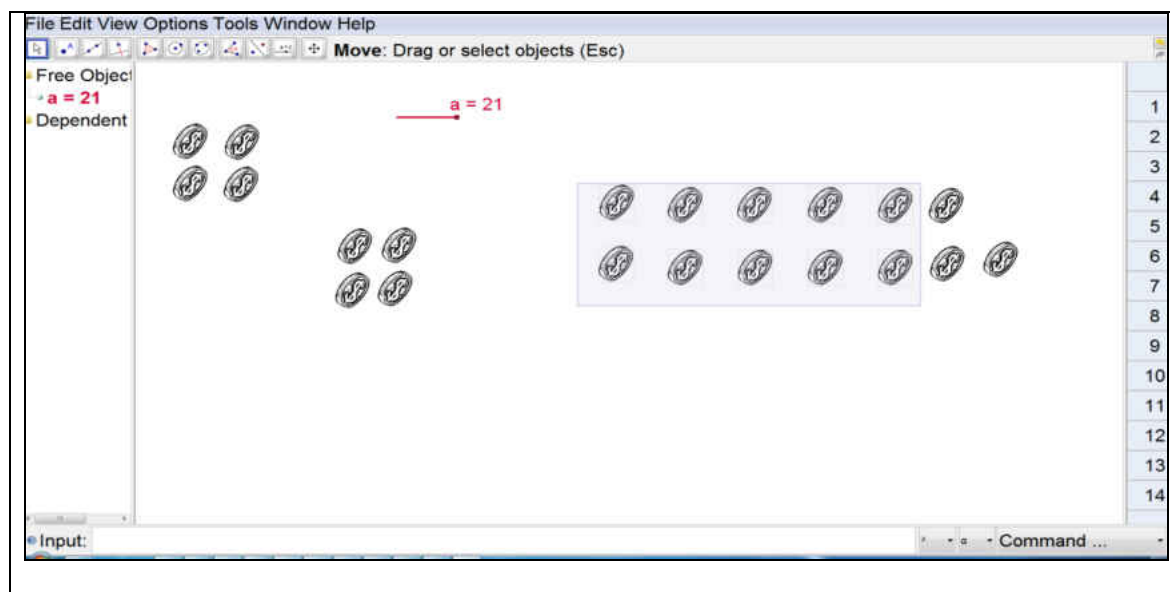


Figure 115. David's solution of GBT-VPR 1

When his solution of GBT-VPR 1 was compared with his solution of GBT 1, it was obvious that both solutions can be considered visual solutions. David performed visual thinking in both of the tasks. In GBT-VPR 1, he manipulated the given virtual physical representations; in GBT 1, he created graphs to represent the relationships given in the problem and found the intersection point to obtain the solution. Therefore, David's thinking did not change with the availability of representations in Geogebra.

Furthermore, the virtual physical representations helped David visualize the algebra word problem on relations. He stated "visual representations make it [the solution] easier." David used the virtual physical representations to create the given case in the problem. Therefore, the virtual physical representations facilitated his visual solution when using Geogebra.

GBT-VPR 2: Farmer Brown has ducks and cows. The animals have a total of 12 heads and 32 legs. How many ducks and how many cows does farmer Brown have?

For GBT-VPR 2, David again solved the given task using the virtual physical representations. When I presented him with GBT-VPR 2, he indicated the first four groups of

lines and dots with the mouse cursor and said, “This will make one cow.” When I asked him to explain in detail, he started to create the cows, which have 4 legs each, then after the 4th cow he started to create ducks, which have 2 legs. As seen in *Figure 116*, there were 8 dots representing the animals’ heads and 32 lines representing the animals’ legs. Those 32 lines were given in two rows. The first row included 23 lines and the second row included 9 lines. First, David moved the first dot between the middle of the first two lines. He used the first two lines in the 1st and 2nd row and the first dot to create an animal with 4 legs. He continued to create cows until there was only one line (leg) in the second row, then he stopped since he could not create animals with 4 legs anymore. He moved this extra leg to the first row and started to create animals with 2 legs. He moved the remaining dots (heads) to a spot between the next two lines that remained after creating cows, and continued to do that until he had used all the lines and dots. When I asked David to explain his reasoning, he showed with the mouse the picture in *Figure 116*.

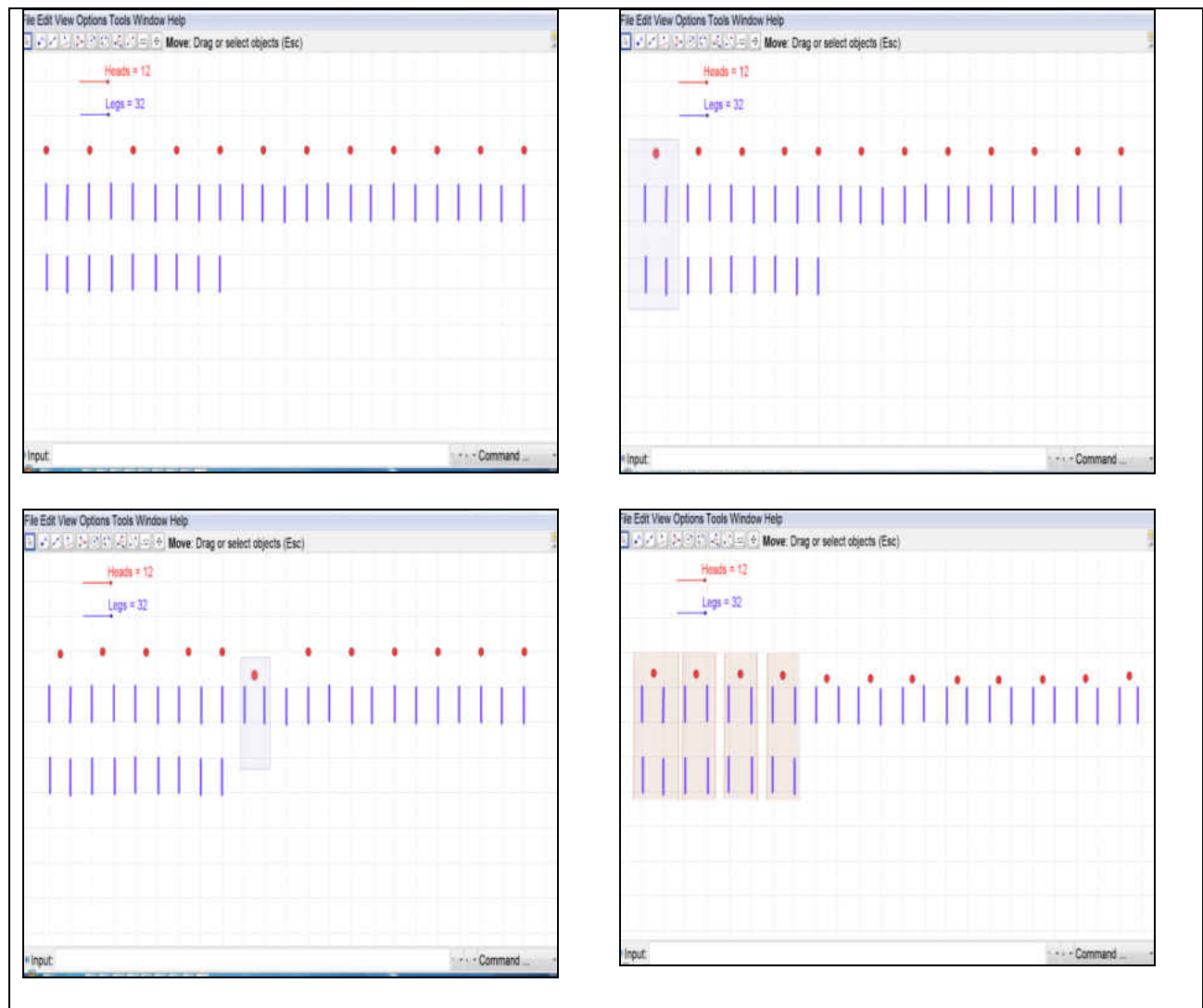


Figure 116. David's solution of GBT-VPR 2

In GBT-VPR 2, David generated a picture of cows and ducks by using dots and lines. By examining his solution of this task, I can claim that his solution demonstrated visual thinking. When David's solutions of GBT-VPR 2 and GBT 2 were compared (see Figure 106 and Figure 116), it was clear that both of them were visual solutions. GBT-VPR 2, similar to GBT-VPR 1, included the manipulation of virtual physical representations, whereas GBT 2, similar to GBT 1, included generating and interpreting the graphic representation of the problem. Similar to

previous GBT-VPR, the virtual physical representations facilitated David's visual solution and helped him visualize the algebra word problem on relations.

GBT-VPR 3: A group of exchange students from Japan went to a convalescent home to sing songs for the seniors and to demonstrate origami (Japanese paper folding). As it turned out, there was either one Japanese student at a rectangular table with three seniors or two students at a round table with four seniors. There were 11 students and 25 seniors in all. How many tables were being used to demonstrate origami?

For the next problem, David again started to manipulate the given figures. He increased the number of round tables and arranged seniors similar to the pattern given in the Geogebra file (see Figure 117). Then he said, "Ooooh, I got it" and continued to solve the problem by using only the sliders. When I asked him why he stopped creating a pattern and only used the sliders, he said, "I can imagine the rest in my head." Then David started to change the value of the sliders. After a couple of trials, he reached the solution. David attempted to solve this problem by manipulating the given figures in the problem and imagining the patterns. Since he solved this task by manipulating the virtual physical representations, his solution was considered a visual solution.

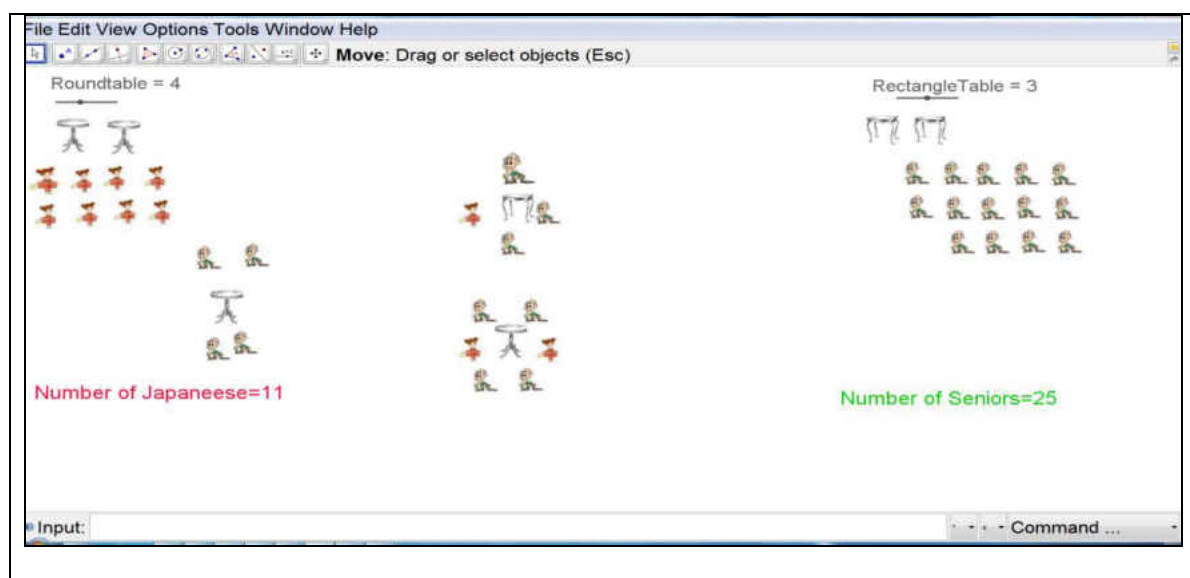


Figure 117. David's solution of GBT-VPR 3

When David's solution of GBT-VPR 3 (see Figure 117) was compared with his solution of GBT 3 (see Figure 107), it was clear that the virtual physical representations affected David's thinking preferences. He preferred to use visual representations while solving GBT-VPR 3 as opposed to nonvisual representations while solving GBT 3. The effect of the virtual physical representations on his solution method indicated that they affected his thinking preference while solving the given task. He used those virtual physical representations to create the given scenario in the problems. He manipulated the objects to determine what was asked in the problem. In the first three GBT-VPRs dynamic visual representations helped David visualize the problems on relations.

GBT-VPR 4 and 5

GBT-VPR 4: Hillary went riding in the hills. At one point, however her horse stumbled and was hurt. Hillary decided to go back to house with her horse. Hillary figures when they were going the horse walks about twice as fast as when they were coming back. If her horse was

hurt about 8 miles in, and the whole trip took 3 hours total, how fast did Horse walk when he was coming back to house?

When I presented him with GBT-VPR 4, David read the problem again. After a short pause, David explained his reasoning.

David: Since horse got hurt 8 miles up to hill, since going up is twice fast as it was when they are coming back. I said the time spent here would be at 1.5 if they were same speed. I think it took longer coming back. It would be 1 for going and 2 for coming back [changes the value of slider t_1 to 1 and slider t_2 to 2 and slowly increases the value of speed slider until 8 for going speed and until 4 for coming speed]. When it said twice as fast helped me realize that 3 hours, 2 hours is twice as 1 hour. Since it is 8 miles it takes you 1 hour to go 8 miles so speed is 8 mph. 2 times 4 is 8 so 4 mph is the speed to get back.

First David split up the 3 hours as 1 hour and 2 hours. Since the speed for going was said to be twice as fast as the speed for coming back, David mentioned that the time for going needed to be 1 hour and the time for coming back needed to be 2 hours. Then he set the slider t_1 to 1 hour and t_2 to 2 hours. For the going speed, David started to increase the value of slider v_1 from 0 to 8. When he was increasing the value of the slider, he was also checking the location of the horse. He stopped changing the value of the slider when the horse came to the point of 8 miles. David used the same process with the speed slider (v_2) for coming back. He slowly increased its value from 0 to 4 and stopped when he saw that the horse had arrived at the point of 0 (see Figure 118).

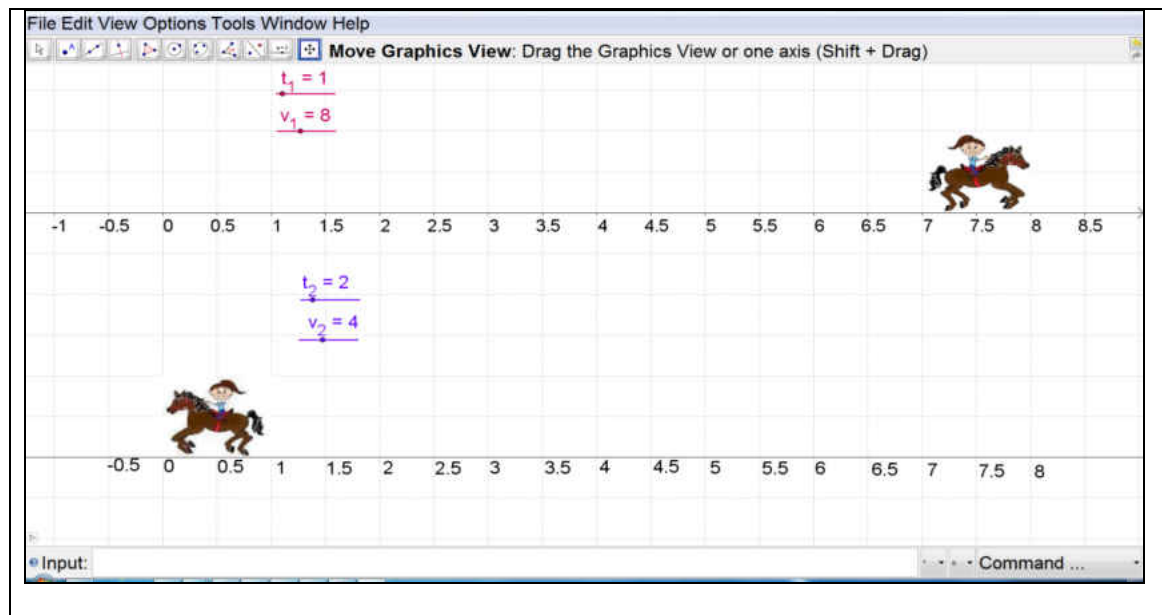


Figure 118. David's solution of GBT-VPR 4

David started by explaining the inverse relationship between speed and time, then mentioned how he split 3 hours into 2 hours and 1 hour. He used reasoning by analyzing relations described verbally in the problem. Then he started to use virtual physical representations and manipulated the horse by using sliders. When he was changing the speed of the horse, he stopped a couple of times to check the horse's location. He also stopped a couple of times when he was changing the speed of the return trip.

When his solution of GBT-VPR 4 was compared with his solution of GBT 4 (see Figure 108), important differences were evident. With the virtual physical representations, David performed visual thinking as opposed to the nonvisual thinking he performed when solving GBT 4. Thus the virtual physical representations affected his preference for solving task 4. Moreover, even though he was not able to solve GBT 4, he successfully solved GBT-VPR 4. For this task, the virtual physical representations helped him visualize the problem on relations.

GBT-VPR 5: Jason got on his bike and went for a ride. He rode at a speed of 10 km per hour from his house to his sister's house in another city. Jason and his sister then got in a car and

traveled at a speed of 50 km per hour to their mother's house. The total distance from Jason's house to his mother's house is 320 km, and Jason traveled for 8 hours. How far is it from Jason's house to his sister's house?

In the next task, GBT-VPR 5, David again manipulated the virtual physical representation (see Figure 119). Similar to GBT-VPR 4, he manipulated the dynamic picture with the sliders. He first said, "Since the bike moves slower than the car I will put more time for the bike." He started with 6 hours for the bike and 2 hours for the car. Then he set up the given speeds by changing the value of the speed slider for the bike (v_b) to 10, and that of the slider for the car (v_c) to 50. When he realized that it was not close to the destination point, he said, "So we are not quite there," and he increased the time traveled by car and decreased the biking time. His next trial was 4 hours for biking and 4 hours for driving time, and when he realized that he was still not at the destination point, he tried 2 hours for biking time and 6 hours for driving time. Then he measured the distance from Jason's house to Jason's sister house.

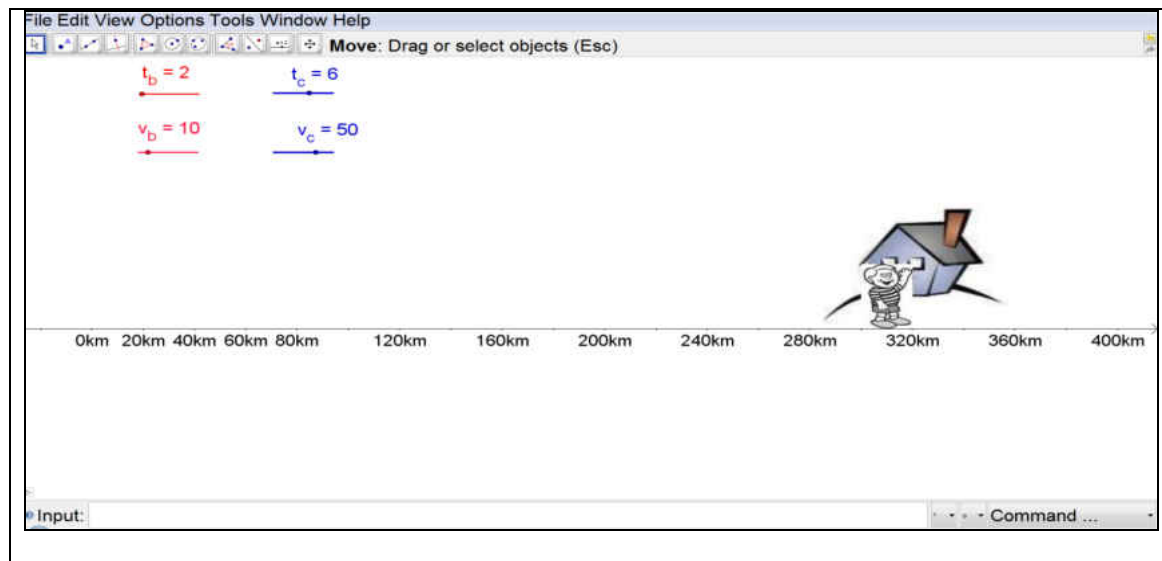


Figure 119. David's solution of GBT-VPR 5

Similar to his solution process for GBT-VPR 4, he started to solve this problem by using a verbal representation, explaining the reason to assume more time for biking and less time for driving. Then he translated from verbal representation to graphic representation by setting up the sliders using verbal reasoning, and began manipulating the dynamic picture. His main method of solution included graphic representations, thus he demonstrated visual thinking in this problem. As with the previous problem, when David was manipulating the pictures by sliders, he slowly moved and stopped a couple of times to check the distance.

When I compared David's solution methods for GBT-VPR 5 with his methods for GBT 5 (see Figure 109), there were clear differences. His solution of GBT 5 demonstrated visual thinking, whereas he made no reference to any attempts which could be considered visual thinking while solving GBT-VPR 5. . The virtual physical representations affected his solution method and helped him visualize the problem. Until now, David manipulated the virtual physical representations and solved the problems by focusing on their visual aspects.

GBT-VPR 6, 7, and 8

GBT-VPR 6: The telephone company offers two types of service. With Plan A, you can monthly pay \$6 plus 12 cents for each minute. With Plan B, you pay \$15 monthly, plus 6 cents for each min. At least how many min would you have to use the telephone each month to make Plan B the better option?

In GBT-VPR 6, 7, and 8 David focused on numeric representations. David started to solve GBT-VPR 6 and said, "Right now, I am just trying to make them equal for the same amount of minutes." He first changed the value of both sliders to *100* minutes. Then his next attempt was *150* minutes, which resulted in the same number value on both bags. David mentioned that after *150* minutes, plan B was a better option (see Figure 120).

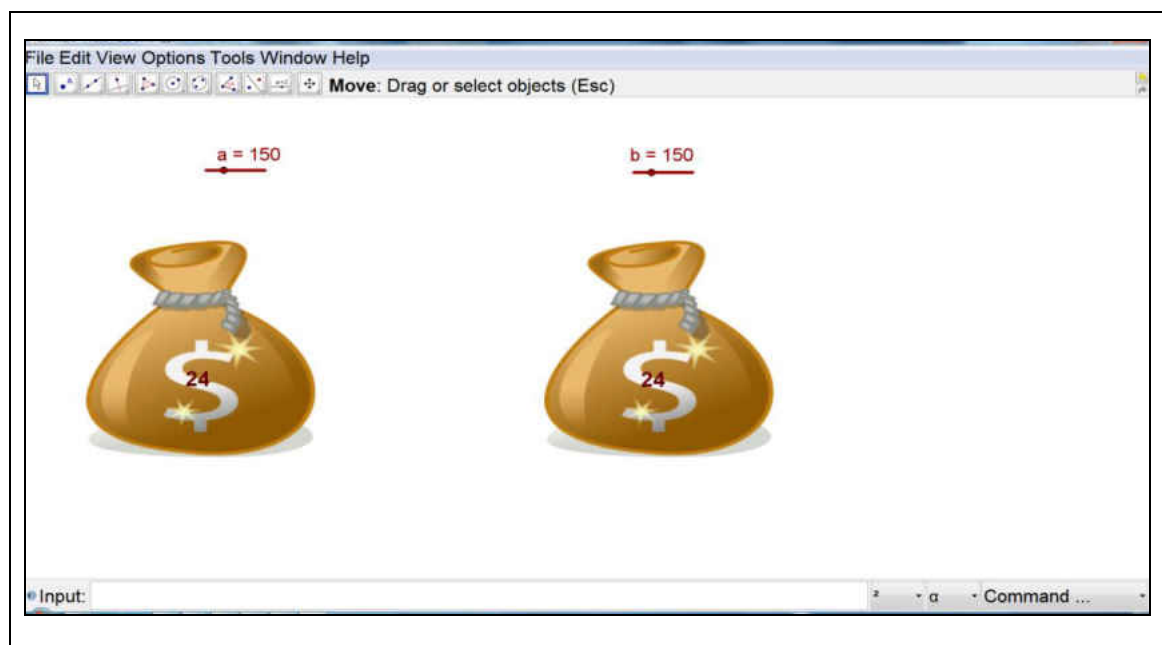


Figure 120. David's solution of GBT-VPR 6

The problems asked in GBT-VPR 6, 7, and 8 were similar problems. I designed GBT-VPR 6 and 7 using the same strategy, but GBT-VPR 8 was different in terms of automation from GBT-VPR 6 and 7. For GBT-VPR 6, the sliders showed the minutes talked for each plan, and the numbers on the money bags showed the cost of the service depending on the minutes talked. For GBT-VPR 7, the scales showed Jim and Todd's weights and the sliders represented weeks spent to gain weight. For both GBT-VPR 6 and 7, when the slider value changes (independent variable), the value on the money bag or scale (dependent variable) also changes. However, for GBT-VPR 8, the number values on the piggy banks do not change. I changed the design in GBT-VPR 8 since I wanted to investigate whether students would use this representation even though they would have to calculate the dependent variable by themselves. In other words, I wanted to test whether the reason to use the virtual physical representations in GBT-VPR 6 and 7 was choosing the method that took least time or not.

For the next GBT-VPR, David explained his reasoning as follows:

David: So this will gain 1 pound in 1 week, this will gain 5 pounds in every 3 week [changes the value of the slider to 1 and 3 respectively]. If we break it down, he gains 1 pound per week but he gains 1.66 pound per week. [Changes the value of sliders to 8 weeks] So we went from 167 to 180 which is a difference of 13 pounds. And now we are looking for a difference of only 7 pounds. So if we up to 2 weeks [checks 10 weeks], we are down to 7 pound difference. [Checks 16 then 21 then 18]. When I did 21 weeks this guy [Todd] was above 1 pound but now I am back to 18 weeks is below 1 pound. So I will check between. In 19.5 weeks they will weight same.

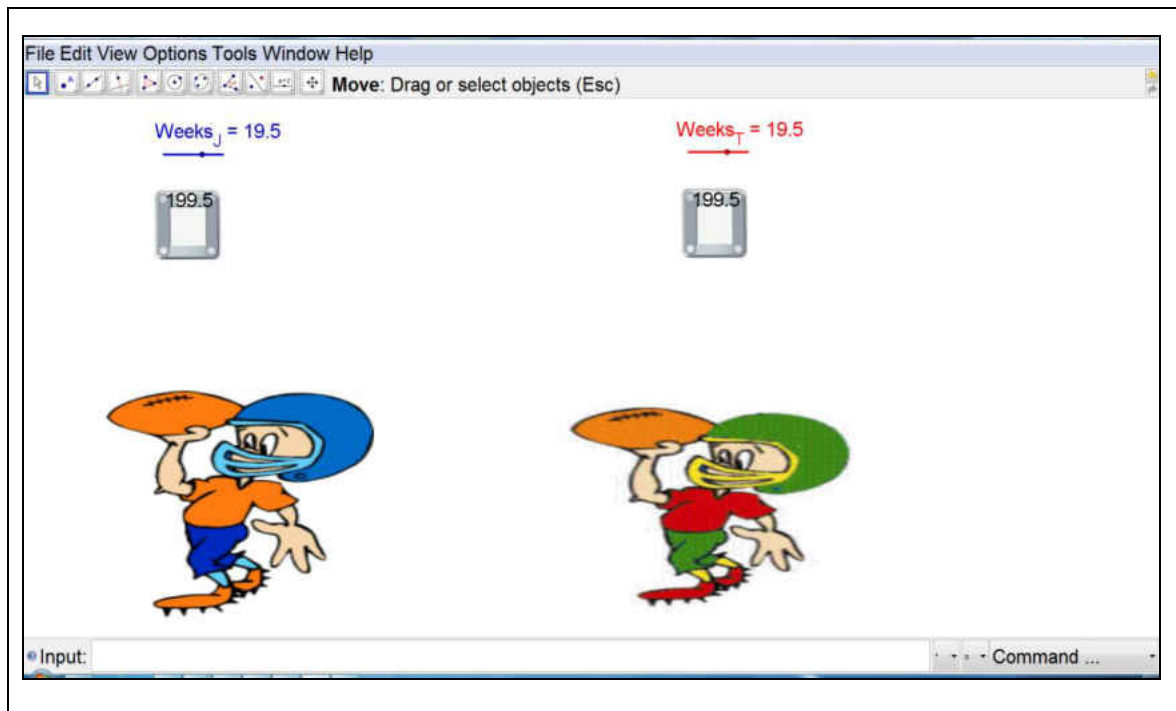


Figure 121. David's solution of GBT-VPR 7

As seen in the excerpt, David determined how many pounds each of them gained in a week. Then he checked their weights at the end of the 8th week and calculated the difference between their weights. Next, he changed the value of the sliders to 10 weeks and calculated the difference again. He tried 16, 21 and 18 weeks respectively and mentioned that Todd was 1 pound heavier than Jim after 21 weeks but lighter than Jim at 18 weeks. Then he checked the midpoint between these numbers and announced that in 19.5 weeks they would weigh the same (see Figure 121).

David used the patterns in the numbers to solve GBT-VPR 7. Therefore his solution was based on the nonvisual aspect of the problem. Even though he demonstrated nonvisual thinking for GBT-VPR 7, his solution for the corresponding blank Geogebra file was visual. Thus, for this task the virtual physical representation affected his solution, which indicated a change in his thinking process from visual thinking to nonvisual thinking. The manipulation of the numbers with sliders facilitated his guess and check method in both GBT-VPR 6 and 7.

GBT-VPR 8: Sharon and Megan have \$20 and \$26 in their piggy bank respectively. Every day, Sharon saves a quarter and Megan saves a dime. After how many days they will have same amount of money in their piggy banks?

For GBT-VPR 8, David again focused on nonvisual aspects of the given representations. GBT-VPR 8 was different than GBT-VPR 6 and 7. Dollar values on the piggy banks did not increase when the number of days changed. David started to solve the problem using verbal representations and translated the verbal representations into numeric representations.

David: Sharon was lower than Megan. Quarters are worth more than twice as much as dimes. And we know that there are 6 dollars difference. So 8 quarters will give you 2 then 3, 4, 5, 6 [increases the number of quarters each time by 4]. 24 quarters will give you 6 dollars. Without the dime these are equal 26 dollars. But this has to add 2 dollars 60 cents. So if we make it 2 dollars [Changes the value of the slider to 32] still needs 60 cents. [Changes the second slider to 32] this is 3 dollar 20 cents. [changes the first slider to 37] 37 quarters equals to 9 dollars 25 cents. This is 29.25 and this is 3 dollars and 70 cents. It is 29.70. 29.50 [increases to 38], 29.72 [increases to 39], this one is 29.90. 40 is both going to have 30 dollars.

First David calculated how many quarters Sharon needed to catch up to the amount of money that Megan had at the beginning. Once he determined that Sharon needed 24 quarters, which meant 24 days, he also calculated the amount of money that Megan would have in 24 days. David continued to get closer to the amount of money that Megan had by increasing the

number value of the slider that adjusted the number of days. Once he had calculated the amount of money that each person had at the end of 39 days, he guessed the money for 40 days, which resulted in the final solution (see Figure 122).

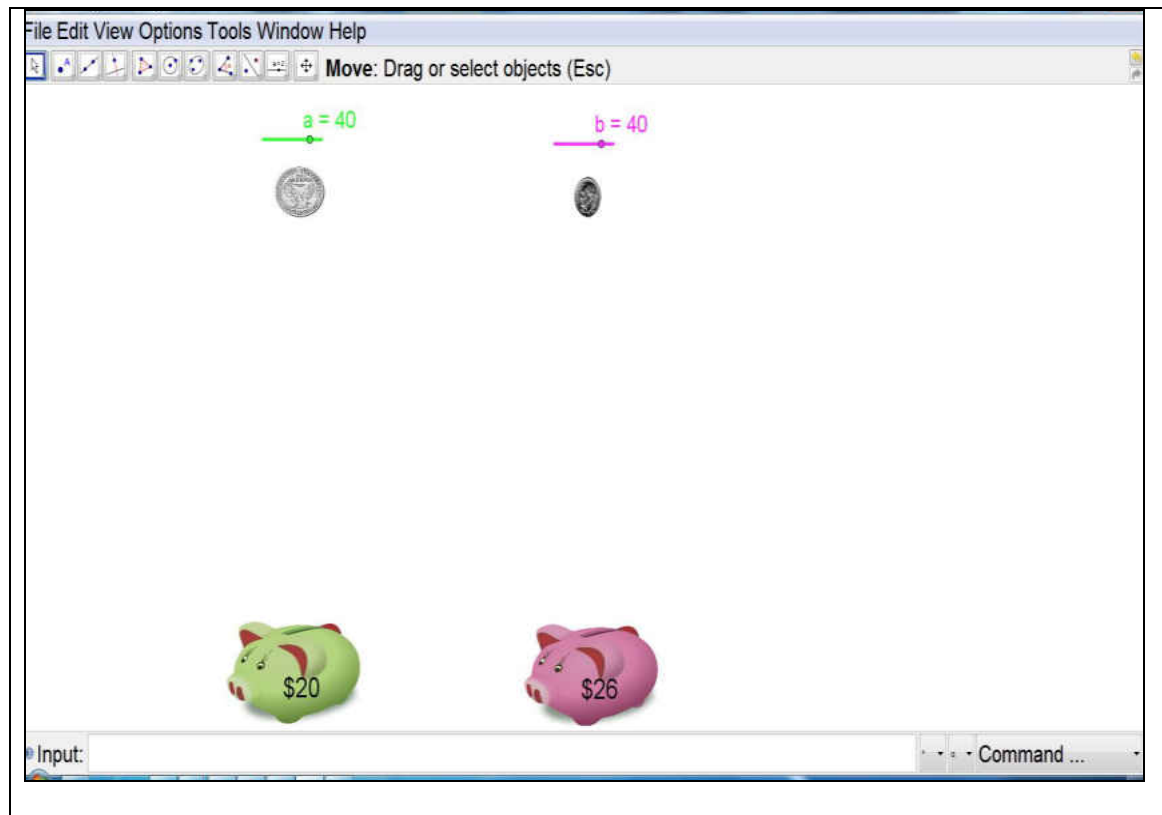


Figure 122. David's solution of GBT-VPR 8

David's method of solution for GBT-VPR 8 included both verbal and numeric representations. Since both of those representations were categorized as nonvisual methods, David demonstrated nonvisual thinking during this problem. When I compared his method of solution for GBT-VPR 8 with his method for GBT 8, a change from visual methods to nonvisual methods was evident. Despite the fact that David did not recreate the imagery in his head in the Geogebra file for GBT 8, he mentioned that his mental imagery was a diagram of the trip. David created the table by referring to his mental imagery. Therefore he demonstrated visual thinking

while he was solving GBT 8. As with the previous task, his thinking preference changed to nonvisual thinking with the virtual physical representations on Geogebra.

David did not show any difference in his thinking preference between GBT-VPR and GBT 6, however, he demonstrated visual thinking in GBT 7 and 8 and nonvisual thinking in GBT-VPR 7 and 8. The images and numbers given in those Geogebra files were dynamic, and David focused mostly on the numbers. In GBT-VPR 6 and 7, the size of the given images also increased dynamically with the given proportion, but he did not focus on those features of the virtual physical representations.

GBT-VPR 9 and 10

GBT-VPR 9: There are 4 red stickers to every 5 yellow stickers in a pack of stickers. There are 36 stickers in one pack. How many red stickers are there?

I presented GBT-VPR 9 to David. He attempted to solve GBT-VPR 9 by using the representations given in the GBT-VPR 9 Geogebra file. He mentioned that he would increase the number of groups that he could create by using 4 red and 5 yellow stickers (see Figure 123). The following excerpt shows how he proceeded to solve the given task.

David: So there are 4 red stickers to every 5 yellow stickers. We will need to get 36. If we increase red up to 8, we need 10 yellow stickers. If we move red to 12, there is going to be 15 yellow ones. That is 27 totally. If we make it 16 to 20 that will make 36.

In this problem, the sliders connected with the presence or absence of red and yellow stickers in the package of 36 stickers. David focused on the visual images displayed on the screen and increased the number of stickers by using the sliders. He also mentioned that he used the same method for this problem as he had used while solving the PBT, but that the visual representation of the stickers made the problem easier. Another important point to note is his

slow process of manipulating dynamic images. He stopped after adding each group of stickers and calculated the total number of stickers that he was using.

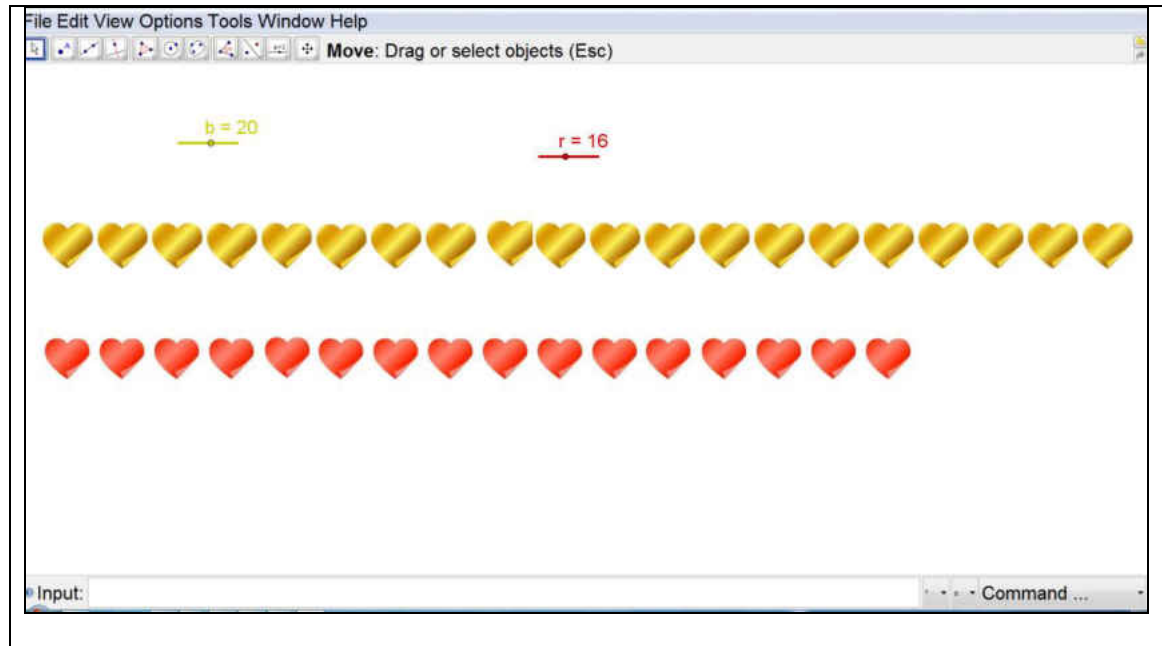


Figure 123. David's solution of GBT-VPR 9

As previously mentioned, David solved GBT 9 in a similar way by creating the mental image of groups. Even though his Geogebra file does not reflect his mental imagery, his verbal explanation indicated that he imagined the groups. Hence, the virtual physical representations did not affect his thinking, but they might have revealed his thinking process while solving the problem. Despite verbalizing his thinking process, he did not reflect his mental imagery on paper or in the blank Geogebra file. The virtual physical representations in this problem facilitated David's visualization process while solving this task.

GBT-VPR 10: Mike has three times as many candies as George. If he gives George six candies, he will then have twice as many as George then has. How many candies did they each have to start with?

For GBT-VPR 10, David used both the virtual physical representations and the spreadsheet view. He started systematically guessing and checking. He inserted the number of George's candies into column A and Mike's candies into column B. He also inserted into columns C and D the number of candies that George and Mike had after Mike gave George 6 candies.

David: If George has 4 at least to start with, Mike will have 12. If he gave over 6 then George would have 10 and Mike would have 6. That is wrong because Mike does not have twice as many as George. If George has 8 then Mike would have 24. Then if he gives 6 to George will have 14 and Mike will have 18 [selects those 4 cells and drags down until 20 candies for George].

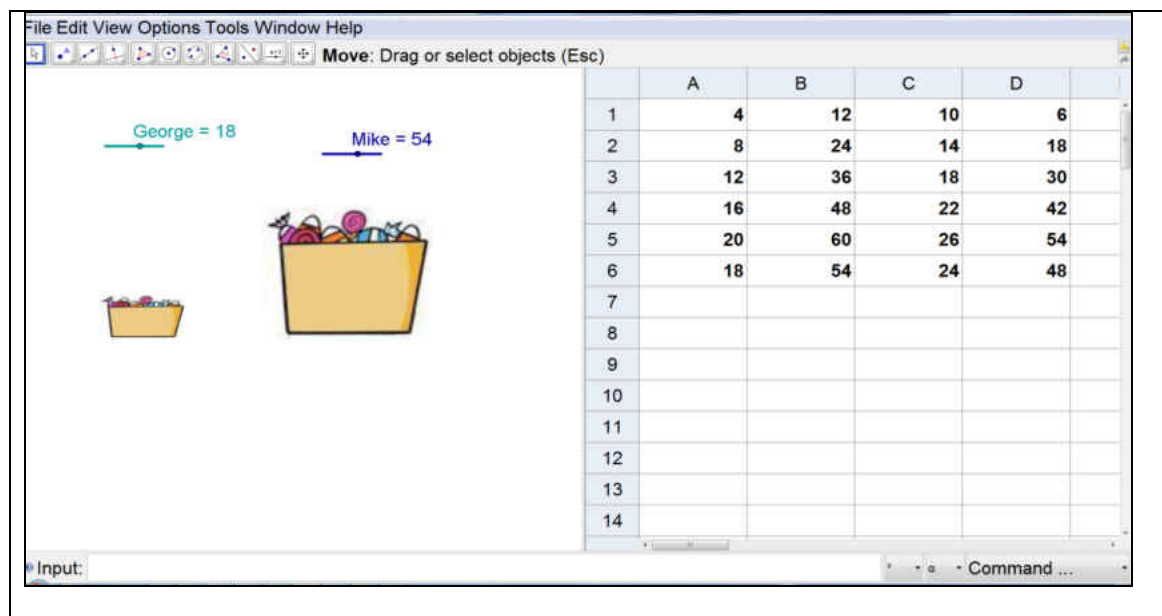


Figure 124. David's solution of GBT-VPR 10

David's guessing process started from 4 marbles for George and 12 candies for Mike (see Figure 124). He checked whether the number of Mike's candies was two times that of George's candies after Mike gave 6 candies to George. Once he realized this number did not satisfy the condition, he checked 8 candies for George and 24 candies for Mike. His trial again did not satisfy the given condition. This time he selected cells A1, B1, C1 and D1 and A2, B2, C2 and

D2 and dragged down until the 5th row. He examined the numbers in columns C and D and said, “George’s candies need to be between 16 and 20, since 22 [cell C4] times 2 is 44, which is greater than 42 [cell D1] and 26 [cell C5] times 2 is 52, which is less than 54 [cell D5].” Once he inserted 18 candies for George and 54 marbles for Mike, he realized that the second condition was satisfied. He changed the number value of the sliders to 18 for George and 54 for Mike. David demonstrated nonvisual thinking during this problem, similar to his solution of GBT 10. Hence the virtual physical representation did not influence David’s thinking preferences.

Table 8. Representations used and David’s main methods of solution

#	GBT-VPR	GBT
1.	Graphic-Numeric (Visual)	Numeric-Algebraic-Graphic (Visual)
2.	Graphic-Numeric (Visual)	Algebraic-Graphic (Visual)
3.	Graphic-Numeric (Visual)	Numeric (Nonvisual)
4.	Graphic-Numeric (Visual)	Algebraic-Numeric (Nonvisual)
5.	Graphic-Numeric (Visual)	Numeric (Nonvisual)
6.	Numeric (Nonvisual)	Numeric (Nonvisual)
7.	Numeric (Nonvisual)	Algebraic-Graphic (Visual)
8.	Numeric (Nonvisual)	Graphic-Numeric (Visual)
9.	Graphic-Numeric (Visual)	Graphic-Numeric (Visual)
10	Numeric (Nonvisual)	Numeric (Nonvisual)

As seen in *Table 8*, David used both visual methods (GBT-VPR 1, 2, 3, 4, 5, 9) and nonvisual methods (GBT-VPR 6, 7, 8, 10) while he was solving GBT-VPRs. GBT-VPRDavid’s preferences for visual or nonvisual methods in GBT-VPRs were similar to his preferences in MPI. His MPI score (32/60) suggested that he preferred to use both visual and nonvisual methods when solving the problems in MPI. His solutions for GBT-VPR suggested that he also preferred both visual and nonvisual solutions when solving GBT-VPRs.

CHAPTER SEVEN: CONCLUSION

In this study, I developed cases describing three students' solution methods for algebra word problems and investigated the effect of technology on their solution methods by making inferences about their preferences for visual or nonvisual thinking. Furthermore, I examined the students' solution methods when presented with virtual physical representations of the situations of the problems. I determined students' preferences for visual or nonvisual solution methods using Suwarsono's Mathematical Processing Instrument (MPI) prior to clinical interviews. During the interviews, students were presented with three types of tasks: paper-and-pencil-based tasks (PBTs), Geogebra-based tasks (GBTs) and Geogebra-based tasks with virtual physical representations (GBT-VPRs). Each category included 10 algebra word problems, and the problems across categories were similar (i.e., PBT 9, GBT 9 and GBT-VPR 9 were similar). By investigating students' solution methods and representations, I made inferences about their preferences for visual and nonvisual thinking regarding algebra word problems. While detailed analyses were given in the previous chapter, this chapter will reflect on the findings and discussions of the results.

The Nature of Students' Thinking When Solving Algebra Word Problems

Before describing the nature of students' thinking, it is important to mention their MPI scores. MPI included 30 word problems. For each visual solution, a score of 2 was given and for each nonvisual solution a score of 0 was given. The highest possible score on MPI was 60 and the lowest possible score was 0. Students, whose scores were lower than 30, were considered as nonvisual thinkers. Students, whose scores were higher than 30, were considered as visual thinkers. Students who scored around the cutoff point 30 were considered as harmonic

thinkers. Mary scored 16 out of 60 on the MPI, which suggested that she was a nonvisual thinker. Ryan scored 42 out of 60 on the MPI, which suggested that he was a visual thinker. David scored 32 out of 60 on the MPI, which suggested that he was a harmonic type of thinker.

Mary

When Mary was presented with PBTs, she used the guess and check method involving numeric representations. She created tables and focused on the numerical patterns. She did not show any evidence of using visual support to solve the problems. She demonstrated the characteristics of Krutetskii's (1976) nonvisual (verbal-logical thinking) type while she was solving algebra word problems.

When asked to solve GBTs, she used both visual and nonvisual methods. She solved the problems by drawing graphs as well as creating tables to recognize the patterns among numbers given in the problems. She translated between visual and nonvisual representations to solve the problems. GBTs influenced her solution method and she demonstrated characteristics of a harmonic thinker whose well-developed verbal-logical and visual-pictorial thinking exist in relative equilibrium since her solution methods included both visual and nonvisual methods. When visual methods were as accessible and easy to use as nonvisual methods, Mary preferred to use both visual and nonvisual methods. The availability of coordinate plane and the automation in graphing the equations was an important factor for Mary to solve some problems with visual methods.

On the other hand, when Mary was presented with Geogebra-based tasks with virtual physical representations (GBT-VPRs), which facilitated nonvisual problem-solving methods such as the guess-and-check strategies she had employed with PBTs, she limited the use of dynamic spreadsheets to the guess-and-check method. The ease of guess and check with the

given sliders affected Mary's choice to use guess and check strategies in her solutions for GBT-VPRs. After observing changes in variables by using sliders, it was easy for Mary to use guess and check strategy. Mary's scores on MPI suggested that she was a nonvisual student and her responses to PBTs indicated that she was using the guess and check method without any difficulty. According to Krutetskii (1976), students' solution methods and their focus on the visual or nonvisual aspects of mathematical material indicate their preferences for visual or nonvisual thinking. Mary demonstrated strong preference for nonvisual thinking in her solutions for MPI and PBTs. Her focus on the nonvisual aspects of the given representations and her preference for nonvisual solution when solving GBT-VPRs might stem from her preference for nonvisual thinking. Her responses indicated that her preference was nonvisual thinking while solving GBT-VPRs

Ryan

Ryan attempted to solve paper-and-pencil-based tasks (PBTs) in mostly nonvisual ways, and most of the time he did not need visual support. He used visual solutions only when he was solving PBT 4 and 9. These findings suggested that when Ryan was solving algebra word problems with paper and pencil, he demonstrated the characteristics of Krutetskii's nonvisual (analytic) type or abstract-harmonic subtype.

When Ryan was solving GBTs, he used graphic representations in his solution processes for nearly all of them. He created and interpreted graphs to solve the algebra word problems in Geogebra. Therefore, he employed visual thinking when he was solving GBTs. Ryan had difficulty translating from verbal to algebraic representations. Furthermore, he experienced difficulties that stemmed from overreliance on graphic representations, a tendency which was also an indication of visual thinking. According to Haciomeroglu et al. (2010), students who had

a strong preference for visual or nonvisual thinking experienced problems when synthesizing visual and nonvisual thinking because of their preferred modes of representation or thinking. Ryan's difficulties translating between different representations and his overreliance on visual thinking might stem from his strong preference for visual thinking.

During some of the GBT-VPRs, Ryan preferred to use the virtual physical representations given in Geogebra by focusing primarily on their visual aspects to solve the problems. For example, in four of the GBT-VPRs, he used the sliders to manipulate the location of the objects. On the other tasks, he continued to create graphs and interpret those graphs to solve the algebra word problems. Hence, he demonstrated the characteristics of Krutetskii's geometric (visual) type when he was solving problems using Geogebra. Again the accessibility of the coordinate system and the ease of using graphs in Geogebra affected Ryan's preferences for visual or nonvisual regarding PBTs and GBT-VPRs between paper and pencil and Geogebra solutions. Ryan's MPI score suggests that he demonstrates a preference for visual thinking. His focus on visual aspects of the virtual physical representations and solutions with visual methods might stem from his preferences for visual for visual thinking. According to Krutetskii (1976), visual (geometric) students focus on visual means of the problems when analyzing a mathematical material. Therefore, when Ryan was presented Geogebra files with virtual physical representations, he focused on visual aspects of the representations and used visual solutions to solve most of GBT-VPRs.

David

David solved PBTs primarily by using nonvisual methods. He employed visual methods only in PBT 4 and 9. Hence, for PBTs he demonstrated the characteristics of Krutetskii's analytic (nonvisual) type or abstract-harmonic subtype.

When David was presented with GBTs, he used both visual and nonvisual methods. His solution methods in Geogebra included the use of graphic and numeric representations. His solution methods changed when solving PBTs and GBTs. Again the ease of using visual methods as nonvisual methods affected David's preference. David's solution preferences for GBTs indicated that he demonstrated the characteristics of a harmonic thinker.

Again, Ryan's solutions for GBT-VPRs included both visual and nonvisual methods. In his visual solutions, he manipulated the virtual physical representations (i.e., dragged and dropped the virtual physical representations to create the given scenario in the problem), and in his nonvisual solutions, he manipulated the numbers using the guess-and-check method by using sliders. When I investigated his solutions of GBTs and GBT-VPRs, I categorized as a harmonic type (Krutetskii, 1976), which indicates that there was a relative equilibrium between his visual and nonvisual thinking.

In six of GBT-VPRs, David preferred to manipulate the virtual physical representations given in the problems and he started with and focused on their visual aspects. Moreover, in GBTs when there were no virtual physical representations, the interviews revealed that he employed visual solutions to solve the problems. For example, while solving GBT 8 he traced the line on the computer screen with his finger and arranged the locations of the cars on that diagram. For GBT 9, he mentioned that he imagined the group of people.

Apparently he felt the need for visual support in those cases. Furthermore, in GBT 1, 2, and 7, he created graphic representations of the problems and solved them. On the other hand, he was able to work with nonvisual representations as seen in his solutions of GBT-VPR 6, 7, 8, and 10. Based on his solutions of GBT-VPRs and GBTs, I inferred that he demonstrated the

characteristics of Krutetskii's (1976) pictorial-harmonic subtype while solving problems with Geogebra.

The Effect of Technology on Students' Preferences for Visual or Nonvisual thinking

Students' responses to PBTs, GBTs, and GBT-VPRs revealed important information about their thinking and their use of various representations while they were solving problems. Students' Geogebra-based solutions differed from their paper-and-pencil-based solutions. The differences in solutions indicated a change in their preferences for visual or nonvisual thinking when solving problems in two different media.

Mary was considered a nonvisual student based on her MPI score. Mary's solutions of PBTs supported this assertion since she used mostly the guess-and-check method, which are nonvisual methods with numeric representations. Her constant use of guess-and-check strategies demonstrated that either she did not know how to solve the problems by using other problem-solving methods or she was not proficient at using other methods. However, her use of both visual and nonvisual methods involving graphic, algebraic, and numeric representations while solving problems in Geogebra revealed that she knew how to solve the problems with methods other than the one she had preferred to use during the PBTs. When visual solutions of the problems were as easy as nonvisual solutions, she preferred to use both methods in her solutions of GBTs. Since Geogebra provides equal access to tools for visual solutions and for nonvisual solutions, Mary preferred to use different problem-solving methods when working on GBTs than she used when working on PBTs. The use of dynamic software revealed her ability to use both visual and nonvisual methods in her solution.

Like Mary, Ryan and David used mainly nonvisual strategies while solving PBTs. Students' solutions of PBTs differed from their solutions of MPI. PBTs included more

challenging word problems compared to the problems presented in MPI. These results supported the findings of Presmeg (1985) who claimed that students used nonvisual methods when presented with difficult problems. On the other hand, Lowrie and Kay (2001) stated that students use more visual methods when they were presented with novel and challenging tasks.

When Ryan was solving problems in Geogebra, he changed his solution methods and preferred visual thinking. On the other hand, David preferred to use both visual and nonvisual methods when he was solving GBTs. Again, when visual methods were as accessible and easy to use as nonvisual methods, students preferred to use them, demonstrating that they possessed a more complete knowledge of problem solving than their work on the PBTs suggested.

Regardless of students' preferences for visual or nonvisual methods, students used more visual methods when presented with GBTs. Students did not draw graphs on paper while solving PBTs, but they used graphs in their solutions of GBTs. These results supported the findings of Harskamp et al. (2000) who reported that the automatic graph drawing function of Geogebra and the availability of the coordinate system are important factors leading to this result. However, Geogebra does not support only visual thinking. The automatic creation of the pattern feature in spreadsheet view supports nonvisual thinking, yet students still tended to use visual solutions more often while solving GBTs than while solving PBTs. In the study of Slavit (1999), students used the graphs only when the problem required students to solve with the graph. The results of the study did not support this finding since students in this study frequently used graphs in the problems even though none of the problems were required using graphs.

In the literature, there were not sufficient numbers of studies to describe how the students' solution strategies with technology are different from their paper-pencil-based solutions. In previous studies, a commonly used research topic was investigating the effect of

technology on students' learning and success (Choi-koh, 2003; Hennesy et al., 2001). Even though there are a few studies which focus on the students' solution methods with and without technology, this study results supported the findings of Ruthven (1990), Harskamp et al. (2000) and Yerushalmy (2006) who reported that students' differed in their solutions with technology. Harskamp et al. (2000) stated that more students from experimental group used graphic strategies compare to the control group. However, there were no significant differences in the use of heuristic (guess and check) and algorithmic strategies. My study supports the first claim that there was an increase in students' solutions with graphs. However, my study also revealed that students decreased the use of guess and check methods while solving GBTs.

Technological tools provided students opportunities to use multiple representations. For instance, David started to solve GBT 1 by using graphic representations. He attempted to create equations to draw their graphs. However, he could not create the equations so he changed his strategy to use numeric representations. He solved the problem by using the spreadsheet view. He mentioned that he could create the equations once he had made the table. Numerical solutions of the problem using Geogebra facilitated his use of equations, which he used to draw their graphs. Similar methods were also observed in Mary's solutions of GBT-VPR 9 and GBT 9. Once she started to solve each problem, she mentioned that she wanted to solve with a graph. Initially, she could not create equations to draw the graphs, but her solution using spreadsheet view helped her write the equations and solve the problem using a graph. Ryan also tried to solve GBT 10 by using different representations. When students were solving PBTs, they tended to give up on solving the problem instead of using a different solution method. Hence, Geogebra provided access to alternative solution methods which enabled the students to employ their desired methods of solution.

As previously mentioned in the literature review, the National Research Council (2001) identifies the significant indicator of conceptual understanding as the ability to use different representations for mathematical situations and knowing how different representation forms can be utilized for different purposes. Hence, there is a close relationship between representational fluency and conceptual understanding, and increasing representational fluency results in the development of conceptual understanding (Lesh, Post & Behr, 1987). Multiple representations theories contend that the most important factor for understanding a mathematical idea is the ability to link representations and to gain representational fluency and interpret mathematical ideas in distinct representations (Pierce & Stacey, 2010). The results of this study revealed that the use of dynamic software facilitated students' translation processes between representations by supporting the findings of previous studies conducted by Choi-Koh (2003) and Huntley et al. (2000). Therefore, the use of dynamic software might also contribute to students' conceptual understanding by supporting their representational fluency.

GBT-VPRs facilitated students' preferred problem-solving methods, whether those methods were visual or nonvisual. For example, Mary preferred to solve GBT-VPR 6 using the guess and check method with the given sliders. The sliders controlled the minutes talked with two phone plans and the cost of the plans was displayed on the money bags. Since the cost of plans was automatically calculated when the value of the slider changed, the use of this setting facilitated Mary's guess and check strategy. On the other hand, David preferred to use both visual and nonvisual solutions when solving GBT-VPRs which facilitated the visual and nonvisual solutions. For David, dynamic virtual physical representations helped him visualize the problems on relations. For example in GBT-VPR 1 he grouped the coins as dimes and quarters to reach the amount of money given in the problem. He stated that the visual

representation made the problem easier. Therefore, the virtual physical representations facilitated his visual solution. Similar to Mary, numeric manipulation with sliders on those virtual physical representations facilitated David's use of the guess and check method which is a nonvisual method of solution. For Ryan, the virtual physical representations also helped visualize the problem on relations. For example, his solution of GBT-VPR 4 he used the sliders to observe the location of the horse. In that setting, the time and speed sliders were connected to the location of the horse. Therefore, he easily manipulated the location of the horse by using sliders. The virtual physical representations facilitated his visual solution. Therefore, although dynamic virtual physical representations facilitated both visual and nonvisual representations, students' use of virtual physical representations was consistent with their thinking preferences revealed in their MPI scores. Therefore, unlike GBTs, GBT-VPRs did not influence students' preferences for visual or nonvisual solutions. Students continued to rely on the methods that they preferred. For instance, Mary's score of 16 on MPI suggested that she was a nonvisual student and she solved GBT-VPRs mostly using nonvisual methods. Ryan's score of 42 on MPI suggested that he was a visual student and he solved GBT-VPRs mainly with visual methods. David's score of 32 on MPI suggested that he was a harmonic thinker and he solved GBT-VPRs using both visual and nonvisual methods. Therefore they preferred to use the same methods for MPI and GBT-VPRs. Krutetskii (1976) also stated that students' solution methods and their focus on the visual or nonvisual aspects of mathematical material indicate their preferences for visual or nonvisual thinking. Therefore when various solution problems equally accessible, students continued to use their preferred methods.

Teaching Implications

In this qualitative study, I investigated the effect of technology on students' preferences for visual and nonvisual solutions by comparing and contrasting their solution methods for algebra word problems with and without using technology. The results of the study suggest that the easy access to multiple representations in Geogebra influenced their preferences for visual or nonvisual thinking while solving problems. Students used visual methods more often in their solutions of Geogebra-based problems. Moreover, with the ease of using both visual and nonvisual representations in Geogebra, students revealed their knowledge of problem-solving methods other than the one each student preferred to use for paper-and-pencil-based tasks.

Since the use of technology creates a tendency to use more visual representations compared to their paper-pencil-based solutions, the use of both paper and pencil and technology might encourage the use of both visual and nonvisual representations involving visual and nonvisual thinking. As previously mentioned, researchers contend that being able to use both visual and nonvisual representations and to translate between them will result in an in-depth understanding of mathematics (de Jong & van Joolingen, 1998; Lesh, Behr, & Post, 1987). Effective use of dynamic software alongside paper-and-pencil work might create an environment in which students develop both visual and nonvisual thinking and make connections between them. This process is illustrated in *Figure 125*, where visual and nonvisual thinking together lead to an in-depth understanding of mathematics. The ability to work with mathematical subjects both visually and nonvisually results in this understanding.

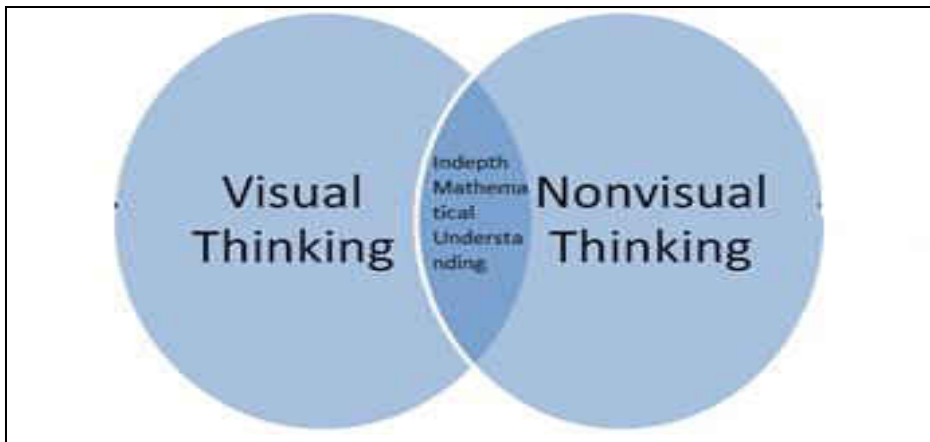


Figure 125. Mathematical thinking model

One of the important results of this study was that students' solution methods for GBT-VPRs were similar to the MPI. By using the GBT-VPRs in this study, teachers can determine students' thinking preferences by investigating which aspects of the virtual physical representations each student focuses on and how each student uses them to solve the problems. Therefore, teachers can use GBT-VPRs as a diagnostic tool similar to MPI.

Teachers should facilitate students' learning by emphasizing the strengths and weaknesses in their use of visual and nonvisual representations. For example, after assessing how students work through GBT-VPRs, teachers might recognize that some students are having difficulty in solving problems visually. Teachers should help students to overcome their difficulties by focusing on the visual aspects of the mathematical problems and connecting them with their nonvisual thinking. For example, after creating a table of values in the spreadsheet of Geogebra, students can plot these points on the graph and observe the behavior of the graph. Moreover, they can also see the algebraic representations of the graph by connecting the points. Therefore, visual and nonvisual representations would be connected and the students would have a more complete understanding of mathematics. Students should take advantage of the simple-

to-use automatic graphing ability of Geogebra. They can also benefit from Geogebra's support for nonvisual representations by using the automatic table creation and Computer Algebra Systems (CAS) features of the program. Translations between representations are also relatively easy with Geogebra compared to translations with paper and pencil. On the other hand, students should not be dependent on the technology and they should be able to solve the problems using multiple methods when solving with paper and pencil. Marchand et al.'s study (2007) also revealed that graphing software did not always show the correct graph. When students practice visual and nonvisual methods with paper and pencil, they might be able to see the pitfalls of technology. Therefore, teachers' effective use of various problem-solving tools will promote the use of visual and nonvisual thinking in their students and facilitate the translation between representations, thereby, increasing the overlap between visual and nonvisual methods shown in *Figure 125*, which will help students to possess a more in-depth understanding of mathematics.

Limitations and Future Research Implications

In this study, students used Geogebra 3.2.42.0. After conducting the study Geogebra 4 and 5 were released and they provide better tools for students. Geogebra version 5 (3-D) gives its users the ability to work in three dimensions. Kozhevnikov et al. (2005) posits an alternative model to analyze students' preferences for visual or nonvisual thinking. Kozhevnikov, et al. (2005) argue that visualizers are not a homogenous group with respect to their spatial ability. They reported that there are two types of visualizers: object and spatial visualizers. Object visualizers and spatial visualizers visualize words in different ways. One of the important differences between object and spatial visualizers lies in their ability to work with three-dimensional objects. According to Kozhevnikov et al. (2005), spatial visualizers are more accurate and faster than object visualizers in 3-D rotation tasks. On the other hand, object

visualizers are more accurate and faster in generating detailed high-resolution images. In future studies, asking students to solve GBT-VPRs, which require the generation and manipulation of 3-D objects, might provide insight into visualizers' thinking. Observing their solutions of GBT-VPRs by conducting a qualitative study, rather than presenting them with problems which measure spatial visualization ability, such as those commonly used in previous studies, might provide detailed information about their visualization process.

In this study, I selected algebra word problems that could be solved by linear equations. In future studies, this study can be extended by changing the tasks asked in the interviews. For example, the tasks, which involve algebra word problems that could be solved by using polynomials, might be used to investigate the effect of technology on their solutions. Therefore, if the findings are consistent with my study, the results will not be limited to only one topic.

Presmeg (1985) claimed that teachers' preferences for visual or nonvisual thinking are one of the important factors that affect students' preferences for visual or nonvisual thinking. For future studies, this study can be repeated by taking consideration the teachers of students' preferences for thinking. Therefore, the study results will reveal the effect of teachers' preference of thinking on students' preferences in technology environment.

In addition to future research implications, I include some limitations of my data collection tools and suggestions for the tools to be used in the data collection process of future studies. Using a pen which records students' verbal responses by linking them to their written responses when students were solving PBTs helped me record their written and verbal responses simultaneously. However, I encountered difficulty with their use of the built-in calculator. Even though the pen displayed the calculations, it did not record them. While watching the videos of students' work with the pen, it was a nuisance to find that although the pen displayed everything

students wrote in the notebook, it did not display the calculations done on the calculator.

Therefore, in future studies, it might be better to use a tool which records students' work on the calculator.

This study investigated the effect of technology on students' preferences for visual or nonvisual methods when solving algebra word problems. Study results revealed that using technology increased students' visual solutions. On the other hand, students did not change their method of solutions when presented with virtual physical representations in technology-based environment.

APPENDIX A: IRB APPROVAL



University of Central Florida Institutional Review Board
Office of Research & Commercialization
12201 Research Parkway, Suite 501
Orlando, Florida 32826-3246
Telephone: 407-823-2901 or 407-882-2276
www.research.ucf.edu/compliance/irb.html

Approval of Exempt Human Research

From: UCF Institutional Review Board #1
FWA00000351, IRB00001138

To: Sirin Cenkun

Date: March 22, 2010

Dear Researcher:

On 3/22/2010, the IRB approved the following activity as human participant research that is exempt from regulation:

Type of Review:	Exempt Determination
Project Title:	The Effects of Dynamic Mathematics Software on Students' Strategies for Solving Word Problems
Investigator:	Sirin Cenkun
IRB Number:	SBE-10-06738
Funding Agency:	
Grant Title:	
Research ID:	N/A

This determination applies only to the activities described in the IRB submission and does not apply should any changes be made. If changes are made and there are questions about whether these changes affect the exempt status of the human research, please contact the IRB. When you have completed your research, please submit a Study Closure request in iRIS so that IRB records will be accurate.

In the conduct of this research, you are responsible to follow the requirements of the Investigator Manual.

On behalf of Joseph Bisitzki, DVM, UCF IRB Chair, this letter is signed by:

Signature applied by Joanne Muratori on 03/22/2010 12:59:27 PM EST

IRB Coordinator

APPENDIX B: MATHEMATICAL PROCESSING INSTRUMENT

MATHEMATICAL PROCESSING TEST I

IMPORTANT:

1. Do not write on this problem sheet. Write your solutions on the solution sheet provided.
2. For each problem, you are required to explain your working as much as you possibly can.
3. You are required to attempt all problems, including those you find difficult.

PROBLEM 1:

John is taller than Mary. John is shorter than Jane. Who is the tallest?

PROBLEM 2:

Two years ago Mary was 8 years old. How old will she be in five years from now?

PROBLEM 3:

Two families held a party. Three members of the first family and five members of the second family attended the party. Each of the members of the first family shook hands with each of the members of the second family. How many handshakes were there altogether?

PROBLEM 4:

On one side of a scale there is a 1-kg weight and half a brick. On the other side there is one full brick. The scale is balanced. How many kg does the brick weigh?

PROBLEM 5:

Altogether there are 8 tables in a house. Some of them have four legs and the others have three legs. Altogether they have 27 legs. How many tables are there with four legs?

PROBLEM 6:

One morning a boy walked from home to school. When he got half way, he realized that he had forgotten to bring one of his books. He then walked back to get it. When he finally arrived at school, he had walked 4 km altogether. What was the distance between his home and school?

PROBLEM 7:

A girl had eleven plums. She decided to swap the plums for some apples. Her friend, who had the apples, said: 'For every 3 plums, I will give you an apple.' After the swap, how many apples and how many plums did the girl have?

PROBLEM 8:

Tim was given 79 one-cent coins by his mother. At a shop he exchanged his one-cent coins for more valuable coins, so that now he got the smallest number of coins giving the value of 79 cents. How many fifty-cent coins, twenty-cent coins, ten-cent coins, five-cent coins, and two-cent coins did he get at the shop?

PROBLEM 9:

Only four teams took part in a football competition. Each team played against each of the other teams once. How many football matches were there in the competition?

PROBLEM 10:

If the time is 8 o'clock in the morning, what was the time 9 hours ago? (Make sure you include a.m. or p.m. as part of your answer.)

PROBLEM 11:

A mother is seven times as old as her daughter. The difference between their ages is 24 years. How old are they?

PROBLEM 12:

Three quarters of a vegetable garden is occupied by potatoes. The remaining part (4 hectares) is occupied by cabbages. What is the area of the whole garden, in hectares?

PROBLEM 13:

At each of the two ends of a straight path a man planted a tree, and then every 5 meters along the path (on one side only) he also planted another tree. The length of the path is 25 meters. How many trees were planted on the path altogether?

PROBLEM 14:

A balloon first rose 200 m from the ground, then moved 100 m to the east, then dropped 100 m. It then traveled 50 m to the east, and finally dropped straight to the ground. How far was the balloon from its starting point?

PROBLEM 15:

Donny's height was 150 cm. One day he swam in a swimming pool, and when he stood upright in the water there was 28 cm of his body which was above the surface. How deep (in cm) was the water at that time?

MATHEMATICAL PROCESSING TEST II

IMPORTANT:

1. Do not write on this problem sheet. Write your solutions on the solution sheet provided.
2. For each problem, you are required to explain your working as much as you possibly can.
3. You are required to attempt all problems, including those you find difficult.

PROBLEM 1:

Dave has more money than Carol, and Mike has less money than Carol. Who has the most money?

PROBLEM 2:

In an athletics race Johnny is 10 m ahead of Peter, Tom is 4 m ahead of Jim, and Jim is 3 m ahead of Peter. How many meters is Johnny ahead of Tom?

PROBLEM 3:

A track for an athletics race is divided into three unequal sections. The length of the whole track is 450 meters. The length of the first and second sections combined is 350 m. The length of the second and third sections combined is 250 m. What is the length of each section?

PROBLEM 4:

Jack, Chris, and Karen all have birthdays on the 1st of January, but Jack is 1 year older than Chris, and Chris is 3 years younger than Karen. If Karen is 10 years old, how old is Jack?

PROBLEM 5:

One day John and Peter visit a library together. After that, John visits the library regularly every two days, at noon. Peter visits the library every three days, also at noon. If the library opens every day, how many days after the first visit will it be before they are, once again, in the library together?

PROBLEM 6:

Two children were given some money by their father. The total amount of money was twelve dollars. The first child received twice as much as the second child. How much did each of them receive?

PROBLEM 7:

One day a third of the potatoes in a storeroom were taken out of it. If 80 kg of potatoes were left in the storeroom, how many kg of potatoes were in the storeroom at first?

PROBLEM 8:

Some sparrows are sitting in two trees, with each tree having the same number of sparrows. Two sparrows then fly from the first tree to the second tree. How many sparrows does the second tree then have more than the first tree?

PROBLEM 9:

At first, the price of 1 kg of sugar was three times as much as the price of 1 kg of salt. Then the price of 1 kg of salt was increased by half its previous price, while the price of sugar was not changed. If the price of salt is now 30 cents per kilogram, what is the price of sugar per kg?

PROBLEM 10:

After a pedestrian travelled half of his journey, he still had to travel 4 km more to complete the journey. What was the length of his whole journey, in km?

PROBLEM 11:

Mr. Jones traded his horse for two cows. Next he traded the two cows, and for each cow, he got three pigs. Then, he traded the pigs, and for each pig, he got 3 sheep. Altogether, how many sheep did Mr. Jones get?

PROBLEM 12:

A saw in a sawmill saws long logs, each 16 m long, into short logs, each 2 m long. If each cut takes two minutes, how long will it take for the saw to produce eight short logs from one long log?

PROBLEM 13:

How many ways can 30 dollars be paid to a person if the money must be in 5-dollar and 2-dollar notes only, and the person must get some 5-dollar notes and some 2-dollar notes. (For each possible solution, summarize your answer by saying how many 2-dollar notes and how many 5-dollar notes the person would get.)

PROBLEM 14:

A tourist travelled some of his journey by plane, and the rest by bus. The distance that he travelled by bus was half the distance he travelled by plane. Determine the length of his entire trip if the distance that he travelled by plane was 150 km longer than the distance he travelled by bus.

PROBLEM 15:

A straight path is divided into two unequal sections. The length of the second section is half the length of the first section. What fraction of the whole path is the first section?

MATHEMATICAL PROCESSING QUESTIONNAIRE I

Name: ----- Male/Female

Date of birth: -----

School: -----

Form/Grade: -----

IMPORTANT:

On this questionnaire you are asked to consider how you did the mathematical processing problems that you were recently asked to do. Every problem is accompanied by two or more possible solutions.

1. For every problem, you are required to indicate which solution, among all the solutions presented, is the one that you used, or is very similar to the one that you used, when you first attempted the problem.

It does not matter whether you got the right or wrong answer, or whether you completed the solution or not, as long as your method of solution is very similar to any of the solutions presented on this questionnaire, you are asked to tick the box which corresponds to that solution.

2. If for any of the problems you think that none of the solutions presented is the one that you used, or is very similar to the one that you used, you are asked to explain, in the space provided, the method that you used when you first attempted the problem. Explain your solution as clearly as you possibly can.

Even if you did not get the correct answer to the problem, you are still asked to state, in writing, your method in attempting the problem.

MATHEMATICAL PROCESSING QUESTIONNAIRE I

PROBLEM 1:

John is taller than Mary. John is shorter than Jane. Who is the tallest?

☐

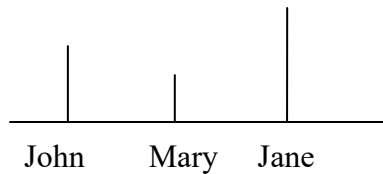
Solution 1:

To answer this question, I imagined a picture of the three children in my mind.
From this picture I could 'see' that Jane is the tallest of the three.

☐

Solution 2:

I drew a diagram representing the three children.



From the diagram it could be seen that Jane is the tallest of the three children.

☐

Solution 3:

I found the answer to this question simply by drawing conclusions from the two statements in the questions.

The two statements are: 'John is taller than Mary' and 'John is shorter than Jane'.

The second statement can be changed into another statement with the same meaning:

'John is shorter than Jane' → 'Jane is taller than John' (because 'taller' is the opposite of shorter').

Therefore, the two statements become: 'John is taller than Mary' and 'Jane is taller than John'.

Or, if the order is reversed: 'Jane is taller than John' and 'John is taller than Mary'.

Conclusion: Jane is taller than Mary. Therefore, Jane is the tallest.

☐

Solution 4:

I solved the problem by eliminating the shorter person in each statement in the problem.

First statement: 'John is taller than Mary'. In this statement 'Mary' is crossed out because Mary is the shorter person.

Second statement: 'John is shorter than Jane'. In this statement 'John' is crossed out because he is the shorter person.

Jane is the only person who is left. So she is the tallest person.

☐

I did not use any of the above methods.

I attempted the problem in this way:

PROBLEM 2:

Two years ago Mary was 8 years old. How old will she be in five years from now?

☐

Solution 1:

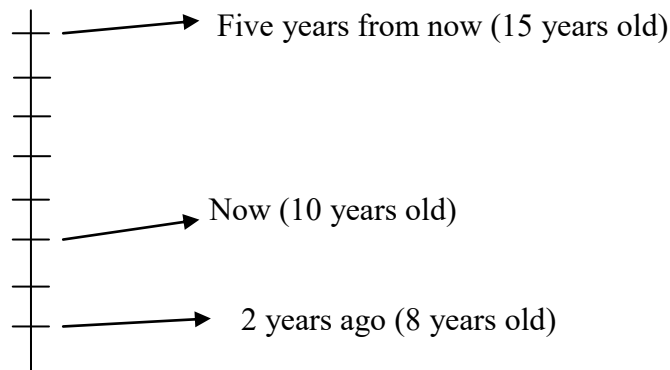
I solved this problem in this way:

Two years ago, she was 8 years old. Now, she is 10 years old. Thus, five years from now she will be 15 years old.

☐

Solution 2:

I solved this problem by drawing a diagram which represents Mary's age.



In the diagram it can be seen that five years from now Mary will be 15 years old.

☐

Solution 3:

I used the same method as for Solution 2, only I drew the diagram 'in my head' (and not on paper).

☐

I did not use any of the above methods.

I attempted the problem in this way:

PROBLEM 3:

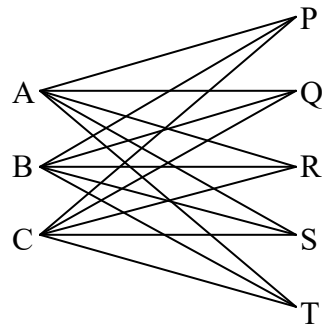
Two families held a party. Three members of the first family and five members of the second family attended the party. Each of the members of the first family shook hands with each of the members of the second family. How many handshakes were there altogether?

**Solution 1:**

I solve this problem by imagining all the handshakes and counting them in the mind. I found 15 handshakes altogether.

**Solution 2:**

I solve the problem by drawing a diagram of the handshakes and then counting them:



There were 15 handshakes altogether.

**Solution 3:**

I used a method like Solution 2, only I drew the picture ‘in my mind’ (and not on paper).

**Solution 4:**

I solved this problem by listing all the hand-shake pairs and then counting them.

First family:

A
B
C

Second family:

P
Q
R
S
T

Hand-shake pairs:

AP	BP	CP
AQ	BQ	CQ
AR	BR	CR
AS	BS	CS
AT	BT	CT

I found 15 hand-shakes pairs. Thus there were 15 hand-shakes altogether.

**Solution 5:**

I solved the problem by using the following reasoning:

Each member of the first family shook hands five times with the member of the second family. Since there were 3 members in the first family, the number of handshakes altogether = 3×5 handshakes = 15 handshakes.



I did not use any of the above methods.

I attempted the problem in this way:

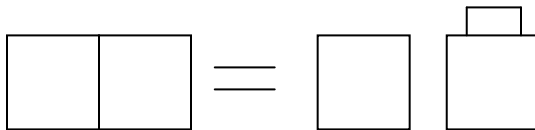
PROBLEM 4:

On one side of a scale there is a 1-kg weight and half a brick. On the other side there is one full brick. The scale is balanced. How many kg does the brick weigh?

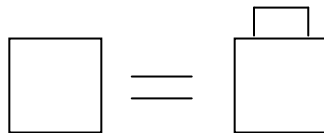


Solution 1:

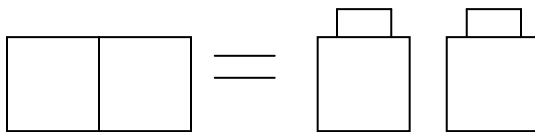
I solved the problem by drawing a diagram representing the objects.



Therefore



And



Thus the weight of one full brick is 2 kg.



Solution 2:

I solved this problem by using symbols and equations:

1 full brick = 2 halves of a brick

=

Thus = + 1 (1 = 1-kg weight)

= 1

= 2

Thus the weight of one full brick = 2 kg.

**Solution 3:**

In order to solve this problem, I imagined the scale and the objects on the two sides of it (half a brick, one 1-kg weight, one full brick).

The scale is balanced; this means that the weight of half of a brick plus one 1-kg weight equals the weight of one full brick. As one full brick equals two halves of a brick, it also means that the weight of one half of a brick equals 1-kg weight. Therefore, the weight of one full brick = 1 kg + 1 kg = 2 kg.

**I did not use any of the above methods.**

I attempted the problem in this way:

PROBLEM 5:

Altogether there are 8 tables in a house. Some of them have four legs and the others have three legs. Altogether they have 27 legs. How many tables are there with four legs?

**Solution 1:**

I solved the problem by trial and error:

If the number of tables with four legs were ...	Then, the number of tables with three legs would be ...	So the total number of legs would be ...
1	7	25 (NO)
2	6	26 (NO)
3	5	27 (YES)

Thus there are three tables which have four legs (and five tables with three legs).

**Solution 2:**

I solved this problem using symbols and equations:

Suppose the number of tables with 4 legs =

Then the number of legs altogether = $(4 \cdot \text{$

This is equal to 27; thus

$$4 \cdot \text{$$

$$4 \cdot \text{$$

$$\text{$$

$$\text{$$

Thus the number of tables with 4 legs = 3 (and the number of tables with 3 legs = 5).

**Solution 3:**

To solve this problem I drew a picture of the tables' legs, and then grouped them into groups of four and groups of three.



A group of four legs represents a table with four legs. A group of three legs represents a table with three legs. From the picture it can be seen that there are 3 groups of legs with four legs each, and 5 groups of legs with three legs each. Thus there are 3 tables with four legs (and 5 tables with three legs).

**Solution 4:**

I solved this problem by drawing the tables. First, I drew them as if all tables had three legs only, then I kept on adding a leg to tables until the total number of legs reached 27.



I found there are 3 tables with four legs (and 5 tables with three legs).

**I did not use any of the above methods.**

I attempted the problem in this way:

PROBLEM 6:

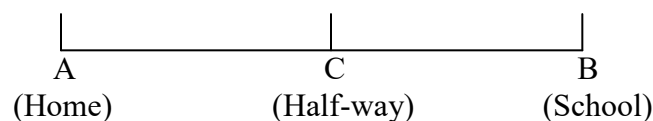
One morning a boy walked from home to school. When he got half way, he realized that he had forgotten to bring one of his books. He then walked back to get it. When he finally arrived at school, he had walked 4 km altogether. What was the distance between his home and school?

**Solution 1:**

To solve this problem, I imagined the route travelled by the boy that morning. When he finally arrived at school, he had walked twice the distance between home and school. This was equal to 4 km, so the distance between home and school was 2 km.

**Solution 2:**

I drew a diagram representing the route between his home and school.



The distance covered by the boy was AC, then CA, then AB. This means that when he finally arrived at B (school) he had walked twice the distance between his home and school. This was 4 km, so the distance between his home and school was 2 km.

☐

Solution 3:

I solved this problem by using symbols and equations.

Suppose the distance between home and school =

Then half the distance = $1/2 \cdot$

The total distance travelled that morning

$$= 1/2 \cdot \text{} + 1/2 \cdot \text{} + \text{}$$

$$= 2 \cdot \text{}$$

This was equal to 4 km. Thus = 2 km, which was the distance between his home and school.

☐

I did not use any of the above methods.

I attempted the problem in this way:

PROBLEM 7:

A girl had eleven plums. She decided to swap the plums for some apples. Her friend, who had the apples, said: 'For every 3 plums, I will give you an apple.' After the swap, how many apples and how many plums did the girl have?

☐

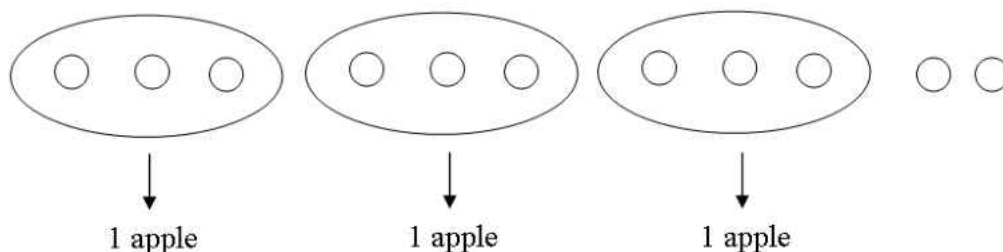
Solution 1:

The number 11 can be separated into 3, 3, 3, 2. Every three plums were swapped for an apple; so after the swap she had 3 apples and 2 plums.

☐

Solution 2:

I solved this problem by drawing the plums, and then separating the plums into groups containing 3 plums each:



In the picture it can be seen that after the swap, the girl had 3 apples and 2 plums.

☐

Solution 3:

I solved this problem by imagining the plums and the swap. I could ‘see’ in my mind that after the swap the girl had 3 apples and 2 plums.

☐

Solution 4:

I solved this problem in this way:

11 divided by 3 gives 3, remainder 2. Thus after the swap, the girl had 3 apples and 2 plums.

☐

I did not use any of the above methods.

I attempted the problem in this way:

PROBLEM 8:

Tim was given 79 one-cent coins by his mother. At a shop he exchanged his one-cent coins for more valuable coins, so that now he got the smallest number of coins giving the value of 79 cents. How many fifty-cent coins, twenty-cent coins, ten-cent coins, five-cent coins, and two-cent coins did he get at the shop?

☐

Solution 1:

I solved this problem by separating the number 79 into 50’s, 20’s, 10’s etc, so far as this is possible.

$$\begin{aligned}
\text{Thus, } 79 &= 50 + 29 \\
&= 50 + 29 + 9 \\
&= 50 + 29 + 5 + 4 \\
&= 50 + 29 + 5 + 2 + 2
\end{aligned}$$

Thus, 79 one-cent coins can be exchanged for one 50-cent coin, one 20-cent coin, one 5-cent coin, and two 2-cent coins.

☐ **Solution 2:**

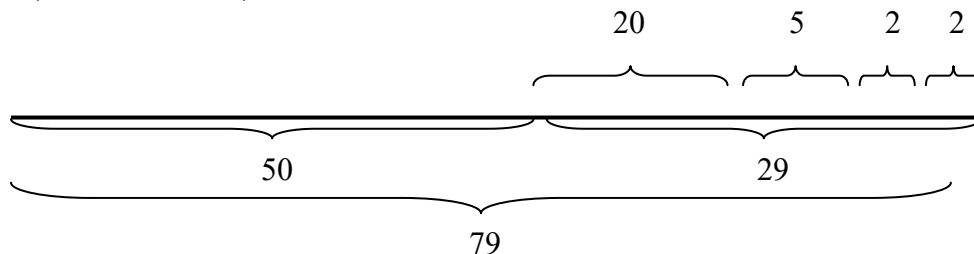
I solved this problem by imagining the 79 one-cent coins, and then trying to ‘arrange’ those coins into several groups each containing 50 one-cent coins, 20 one-cent coins, etc. I found that those coins can be arranged into:

One group containing 50 one-cent coins, one group containing 20 one-cent coins, one group containing 5 one-cent coins, and 2 groups containing 2 one-cent coins each.

Thus, 79 one-cent coins can be exchanged for one 50-cent coin, one 20-cent coin, one 5-cent coin, and two 2-cent coins.

☐ **Solution 3:**

I solved this problem by drawing a line which represents the money that Tim got from his mother. Then, I divided the line into sections, one of which containing 50 units, another 20 units, another 5 units, and two others each 2 units.



Each section represents a coin. Thus 79 one-cent coins can be exchanged for one 50-cent coin, one 20-cent coin, one 5-cent coin, and two 2-cent coins.

☐ **I did not use any of the above methods.**

I attempted the problem in this way:

PROBLEM 9:

Only four teams took part in a football competition. Each team played against each of the other teams once. How many football matches were there in the competition?



Solution 1:

I solved this problem by using the following reasoning:

Each team played against each of the three other teams once. As there were 4 teams, there would be 4×3 matches, or 12 matches, altogether.

But in that way each match had been counted twice.

So the correct answer was $= 12 \text{ matches} / 2$
 $= 6 \text{ matches}.$



Solution 2:

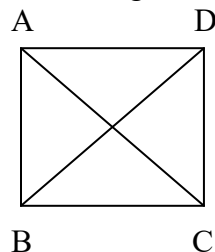
I solved this problem by listing all the match pairs and then counting them.

The teams were A, B, C, D. The match pairs were AB, AC, AD, BC, BD, and CD. There were 6 matches.



Solution 3:

I solved this problem by drawing a diagram representing the matches and then counting the matches as shown in the diagram.



There were 6 matches altogether.



Solution 4:

I did this problem like the method in solution 3, but I drew the pictures ‘in my head’ (and not on paper).



Solution 5:

I solved this problem by using the following reasoning:

As there were 4 teams, in each round there could be only 2 matches. Altogether there were 3 rounds since each team had to play each of the three other teams once — this could be done in 3 rounds.

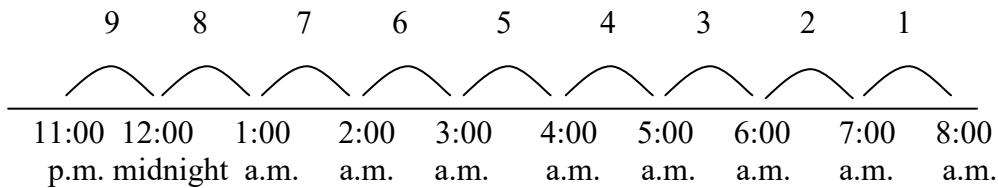
Therefore there were 3×2 matches, or 6 matches, altogether.

☐ **I did not use any of the above methods.**
I attempted the problem in this way:

PROBLEM 10:

If the time is 8 O'clock in the morning, what was the time 9 hours ago? (Make sure you include a.m. or p.m. as part of your answer.)

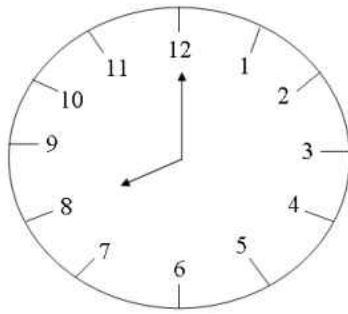
☐ **Solution 1:**
I solved this problem by drawing a line representing the time. In the diagram it can be seen that if the time now is 8 o'clock in the morning, 9 hours ago it was 11:00 o'clock at night; that is 11:00 p.m.



☐ **Solution 2:**
I used the same solution as for Solution 1, only I drew the diagram 'in my head' (and not on paper).

☐ **Solution 3:**
I did not imagine any picture, but I solved this problem by merely 'counting back' 9 hours from 8:00 a.m.

☐ **Solution 4:**
I solved this problem by drawing a clock face (or by looking at my own watch, or a clock).



Using this clock face I could work out the time 9 hours ago, which was 11:00 p.m.

☐

Solution 5:

I used the same method as for Solution 4, only I drew the clock face ‘in my mind’, and did not look at, or draw a clock face.

☐

I did not use any of the above methods.

I attempted the problem in this way:

PROBLEM 11:

A mother is seven times as old as her daughter. The difference between their ages is 24 years. How old are they?

☐

Solution 1:

I solved this problem merely by trial and error:

Daughter's age:	Mother's age:	
2 years	26 years	No
3 years	27 years	No
4 years	28 years	Yes

Thus daughter's age = 4 years, and mother's age = 28 years.

☐ **Solution 2:**

I solved the problem in this way:

Suppose daughter's age = years.

Thus, mother's age = years.

The difference between their ages = years.

= 24 years

Thus = 4.

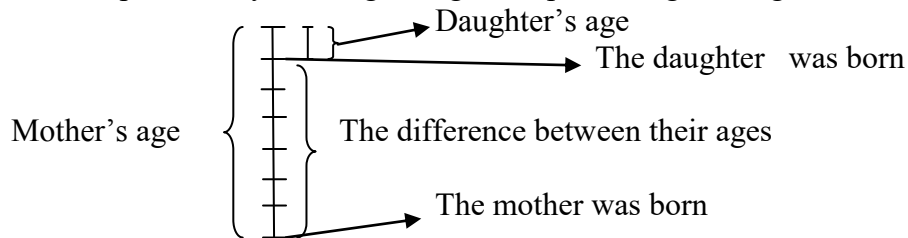
So, daughter's age = 4 years.

Mother's age = (4 + 4 + 4 + 4 + 4 + 4 + 4) years.

= 28 years.

☐ **Solution 3:**

I solved the problem by drawing a diagram representing their ages:



In the diagram it can be seen that the difference between their ages is represented by a line segment which consists of 6 equal parts. This difference = 24 years. Thus, each part represents 4 years. Daughter's age is represented by a line segment which consists of one part only. This means that daughter's age = 4 years. Mother's age is represented by a line segment which consists of 7 parts. Thus mother's age = 28 years.

☐ **Solution 4:**

I used the same method as for Solution 3, only I drew the diagram 'in my head' (and not on paper).

☐ **I did not use any of the above methods.**

I attempted the problem in this way:

PROBLEM 12:

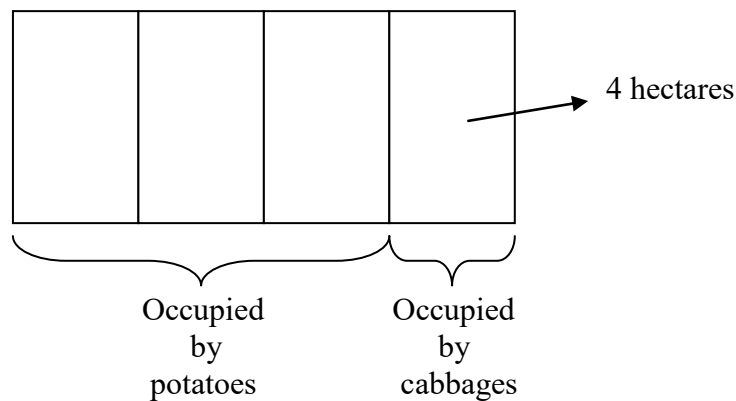
Three quarters of a vegetable garden is occupied by potatoes. The remaining part (4 hectares) is occupied by cabbages. What is the area of the whole garden, in hectares?

**Solution 1:**

I solved the problem in this way:

The part occupied by cabbages = $1 - 3/4 = 1/4$ of the whole garden. This is 4 hectares.

Therefore, the area of the whole garden is 4×4 hectares, i.e. 16 hectares.



In the picture it is clear that the area of the whole garden = $(4 + 4 + 4 + 4)$ hectares.
= 16 hectares.

**Solution 2:**

I used the same method as for Solution 2, only I drew the diagram 'in my head' (and not on paper).

**I did not use any of the above methods.**

I attempted the problem in this way:

PROBLEM 13:

At each of the two ends of a straight path a man planted a tree, and then every 5 m along the path (on one side only) he also planted another tree. The length of the path is 25 m. How many trees were planted on the path altogether?

☐**Solution 1:**

I solved the problem in this way:

Every 5 m along the path a tree was planted. This means that the path was divided into $25/5 = 5$ equal parts. Every part corresponded to one tree, but at one of the two ends of the path, the part corresponded to two trees. Therefore the number of trees was:

$$\begin{aligned} &= (4 \times 1) + (1 \times 2) \\ &= 4 + 2 \\ &= 6 \end{aligned}$$

☐**Solution 2:**

I solved the problem by imagining the path and the trees, and then counting the trees in the mind. I found there were 6 trees on the path.

☐**Solution 3:**

I solved the problem by drawing a diagram representing the path and the trees, and then counting the trees.



I found 6 trees.

☐**I did not use any of the above methods.**

I attempted the problem in this way:

PROBLEM 14:

A balloon first rose 200 m from the ground, then moved 100 m to the east, then dropped 100 m. It then traveled 50 m to the east, and finally dropped straight to the ground. How far was the balloon from its starting point?



Solution 1:

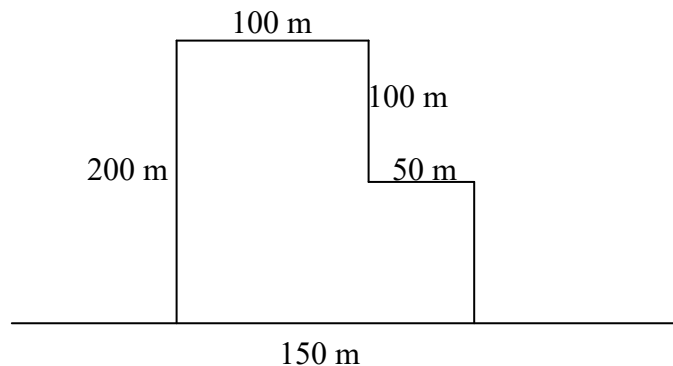
To solve this problem, I imagined the path taken by the balloon, and then worked out the distance between the starting and the finishing places, I found the distance was:

$$= 100 \text{ m} + 50 \text{ m} = 150 \text{ m}$$



Solution 2:

To solve this problem, I drew a diagram representing the path taken by the balloon, and then worked out the distance between the starting and finishing places.



The distance was $= 100 + 50 = 150\text{m}$.



Solution 3:

In order to solve this problem, I noticed only the information in the problem which was important for the solution. That is, I only noticed: 'moved 100 to the east', and 'then travelled 50 m to the east again'. Therefore, the distance between the starting and the finishing places was $100 \text{ m} + 50 \text{ m} = 150 \text{ m}$.

(I did not draw or imagine any picture at all).



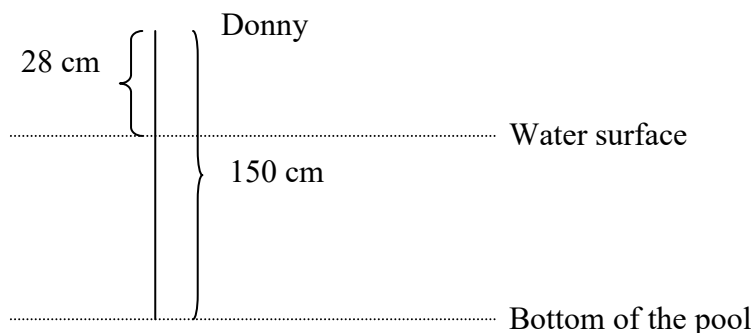
I did not use any of the above methods.

I attempted the problem in this way:

PROBLEM 15:

Donny's height was 150 cm. One day he swam in a swimming pool, and when he stood upright in the water there was 28 cm of his body which was above the surface. How deep (in cm) was the water at that time?

☐ **Solution 1:**



I found the depth of the water at that time = 122 cm.

☐ **Solution 2:**

To solve this problem, I imagined Donny and the water. I could 'see' in my mind that part of Donny's body which was below the surface was 122 cm. Thus, the depth of the water at that time was 122 cm.

☐ **Solution 3:**

To solve this problem, I only noticed the information in the problem which was important for the solution. That is, I only noticed:

'Donny's height was 150 cm' and 'there was 28 cm of his body which was above the surface'. From this information I could conclude that part of Donny's body which was below the surface was = $150 \text{ cm} - 28 \text{ cm}$
= 122 cm.

Thus the depth of the water at that time was 122 cm.

(I did not draw or imagine any picture at all).

☐ **I did not use any of the above methods.**

I attempted the problem in this way:

MATHEMATICAL PROCESSING QUESTIONNAIRE II

Name: ----- Male/Female

Date of birth: -----

School: -----

Form/Grade: -----

IMPORTANT:

On this questionnaire you are asked to consider how you did the mathematical processing problems that you were recently asked to do. Every problem is accompanied by two or more possible solutions.

1. For every problem, you are required to indicate which solution, among all the solutions presented, is the one that you used, or is very similar to the one that you used, when you first attempted the problem.

It does not matter whether you got the right or wrong answer, or whether you completed the solution or not, as long as your method of solution is very similar to any of the solutions presented on this questionnaire, you are asked to tick the box which corresponds to that solution.

2. If for any of the problems you think that none of the solutions presented is the one that you used, or is very similar to the one that you used, you are asked to explain, in the space provided, the method that you used when you first attempted the problem. Explain your solution as clearly as you possibly can.

Even if you did not get the correct answer to the problem, you are still asked to state, in writing, your method in attempting the problem.

MATHEMATICAL PROCESSING QUESTIONNAIRE II

PROBLEM 1:

Dave has more money than Carol, and Mike has less money than Carol. Who has the most money?

☐

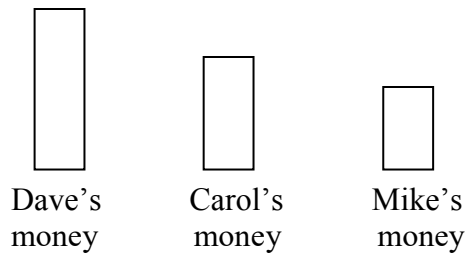
Solution 1:

I solved this problem by imagining each person's money. I could 'see' in my mind that Dave had the most money.

☐

Solution 2:

I solved this problem by drawing a diagram representing the money.



From the diagram it could be seen that Dave has the most money.

☐

Solution 3:

I solved this problem by using examples. Suppose Dave has 35 dollars; Carol has 25 dollars (as Dave has more money than Carol); and Mike has 20 dollars (as he has less money than Carol).

From these examples it can be seen that Dave has the most money.

☐

Solution 4:

I solved this problem by eliminating, in each statement in the problem, the person who has less money (as we only want the person who has more money in each statement). That is:

'Mike has less money than Carol' means that 'Mike' is crossed out since Mike is the person who has less money.

'Dave has more money than Carol' means that 'Carol' is crossed out since Carol is the person who has less money.

The only person who is left is Dave. So He has the most money.

☐

Solution 5:

I solved this problem merely by drawing conclusions from the sentences in the problem.

'Dave has more money than Carol' → 'Dave has more money than Carol'

'Mike has less money than Carol' → 'Carol has more money than Mike' (Since the opposite of 'less' is 'more').

Conclusion: Dave has more money than Mike. Thus, Dave has the most money.

☐

I did not use any of the above methods.

I attempted the problem in this way:

PROBLEM 2:

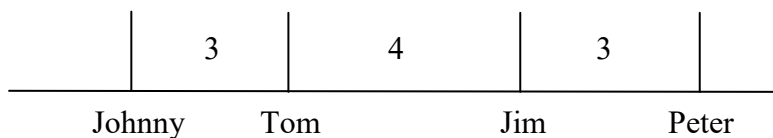
In an athletics race Johnny is 10 m ahead of Peter, Tom is 4 m ahead of Jim, and Jim is 3 m ahead of Peter. How many meters is Johnny ahead of Tom?

**Solution 1:**

To solve this problem, I imagined the four people in my mind, and then worked out the distance between Johnny and Tom. I found the distance is 3 m. So Johnny is 3 m ahead of Tom.

**Solution 2:**

I solved this problem by drawing a diagram representing the four people, and then working out the distance between Johnny and Tom.



I found the distance between Johnny and Tom is 3 m. So Johnny is 3 m ahead of Tom.

**Solution 3:**

I solved this problem merely by drawing conclusions from the sentences in the problem:

‘Tom is 4m ahead of Jim’

‘Jim is 3m ahead of Peter’

Conclusion: Tom is 7m ahead of Peter.

‘Johnny is 10 m ahead of Peter’

‘Tom is 7 m ahead of Peter’

Conclusion: Johnny is 3 m ahead of Tom.

**I did not use any of the above methods.**

I attempted the problem in this way:

PROBLEM 3:

A track for an athletics race is divided into three unequal sections. The length of the whole track is 450 meters. The length of the first and second sections combined is 350 m. The length of the second and third sections combined is 250 m. What is the length of each section?

Solution 1:

I solved this problem by imagining the track for the race and then working out the length of each section.

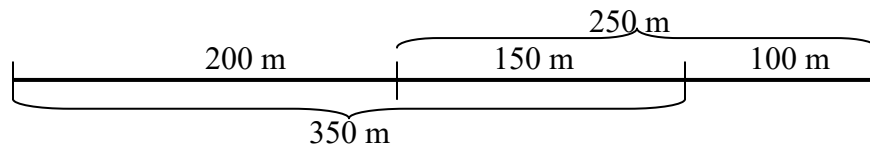
The length of the first and the second sections combined is 350 m, so the length of the third section must be 100 m (since the length of the whole track is 450 m).

The length of the second and the third sections combined is 250 m, so the the length of the first section must be 200 m.

Since the length of the first section is 200 m, and the length of the third section is 100 m, the length of the second section is 150 m.

Solution 2:

To solve this problem, I drew a diagram which represents the track and then worked out the length of each section.



The length of the first section is 200 m, the section section is 150 m, and the third section is 100 m.

Solution 3:

To solve this problem I drew conclusions from the information in the problem only, and did not imagine or draw any picture at all.

That is:

A track is divided into 3 unequal sections.

The length of whole track is 450 m.

The length of the first and the second sections combined is 350 m.

Conclusion: The length of third track = $450 - 350 = 100$ m .

The length of the second and the third sections combined is 250 m.

Conclusion: The length of first section = $450 - 250 = 200$ m.

And the length of the second section = $450 - 200 - 100 = 150$ m.

☐ **I did not use any of the above methods.**

I attempted the problem in this way:

PROBLEM 4:

Jack, Chris, and Karen all have birthdays on the 1st of January, but Jack is 1 year older than Chris, and Chris is 3 years younger than Karen. If Karen is 10 years old, how old is Jack?

☐ **Solution 1:**

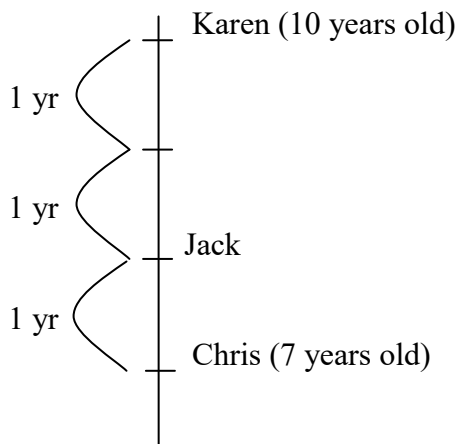
I solved the problem in this way:

Chris is 3 years younger than Karen. Karen is 10 years old.
Therefore, Chris is 7 years old.

Jack is one year older than Chris.
Therefore, Jack is 8 years old.

☐ **Solution 2:**

I solved this problem by drawing a diagram that represents their ages:



From the diagram it can be seen that Jack is 8 years old.

☐**Solution 3:**

I used the same method as for Solution 2, only I drew the line 'in my head', and not on paper.

☐**I did not use any of the above methods.**

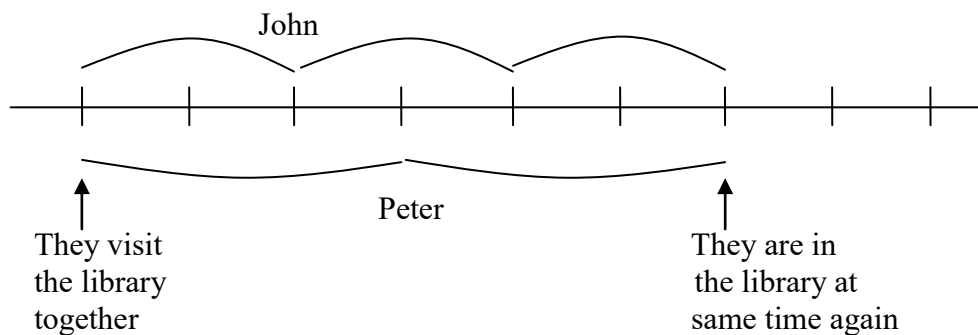
I attempted the problem in this way:

PROBLEM 5:

One day John and Peter visit a library together. After that, John visits the library regularly every two days, at noon. Peter visits the library every three days, also at noon. If the library opens every day, how many days after the first visit will it be before they are, once again, in the library together?

☐**Solution 1:**

I solved this problem by drawing a diagram representing the days after they first visit the library.



From the diagram it can be seen that, once again, they will be in the library together six days after the first visit.

☐**Solution 2:**

I used the same method as for Solution 1 only I drew the diagram 'in my head' (and not on paper).

☐ **Solution 3:**

I solved this problem by using examples. Suppose they first visit the library together on Monday. Then after that, John will visit the library on Wednesday, Friday, Sunday, Tuesday, etc., and Peter will visit the library on Thursday, Sunday, Wednesday, etc. This means that on Sunday they will be in the library at the same time again. From Monday to Sunday there are 6 days. This means that, once again, they will be in the library together six days after the first visit.

☐ **Solution 4:**

I solved this problem by saying in my mind that after the first day, John will visit the library on the third day, the fifth day, the seventh day, etc.; and Peter, after the first day, will visit the library again on the fourth day, the seventh day, etc. So on the seventh day they will be in the library at the same time again. From the first day to the seventh day there are 6 days. So, once again, they will be in the library together six days after the first visit.

☐ **I did not use any of the above methods.**

I attempted the problem in this way:

PROBLEM 6:

Two children were given some money by their father. The total amount of money was twelve dollars. The first child received twice as much as the second child. How much did each of them receive?

☐ **Solution 1:**

I solved this problem by trial and error:

First child:	Second child:	
\$6	\$6	No
\$7	\$5	No
\$8	\$4	Yes

So the first child received \$8, and the second child \$4.

☐**Solution 2:**

I solved this problem by using equations.

Suppose the second child's share =

Thus, the first child's share =

$$\text{and } \square\square + \square = 12$$

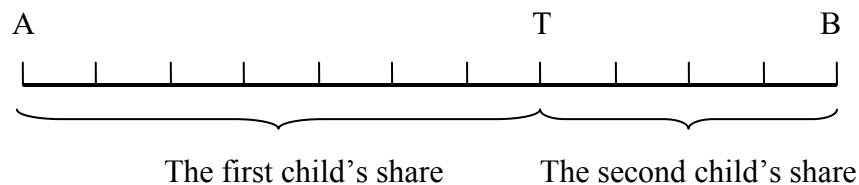
$$\square\square\square = 12$$

$$\square = 4$$

Thus, the second child received \$4, and the first child received $\$4 + \$4 = \$8$.

☐**Solution 3:**

I solved the problem by drawing a diagram which represented the money.



Then I determined the point T so that the length of AT = twice the length of TB.

In the diagram it can be seen that the first child received \$8, and the second child \$4.

☐**Solution 4:**

I used the same method as for Solution 3, only I drew the diagram 'in my mind' (and not on paper).

☐**I did not use any of the above methods.**

I attempted the problem in this way:

PROBLEM 7:

One day a third of the potatoes in a storeroom were taken out of it. If 80 kg of potatoes were left in the storeroom, how many kg of potatoes were in the storeroom at first?

**Solution 1:**

I solved the problem in this way:

One third of the potatoes were taken out, so two-thirds of the potatoes were left in the storeroom. This means that the amount of potatoes left was twice the amount taken out. It was given that the amount left was 80 kilograms, so the amount taken was 40 kg. Thus the amount of all the potatoes in the storeroom at first was

$$= 80 \text{ kg} + 40 \text{ kg} = 120 \text{ kg}.$$
**Solution 2:**

I solved the problem using symbols and equations.

Suppose the amount of potatoes at first was \square kg.

The amount taken out $= 1/3 \cdot \square$

The amount left $= 2/3 \cdot \square$

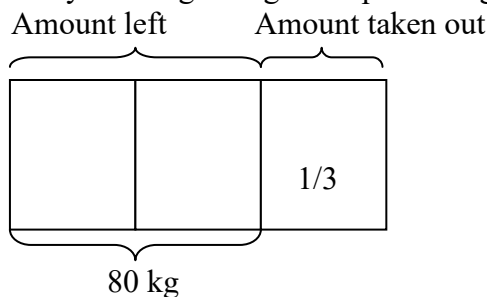
Thus, $2/3 \cdot \square = 80$

and $\square = 120$

Thus, the amount of potatoes at first = 120 kg.

**Solution 3:**

I solved this problem by drawing a diagram representing the potatoes:



From the diagram it can be seen that the amount of all the potatoes at first was

$$= 80 \text{ kg} + 40 \text{ kg} = 120 \text{ kg}.$$
**Solution 4:**

I used the same method as for Solution 3, only I drew the diagram 'in my mind' (and not on paper).

**I did not use any of the above methods.**

I attempted the problem in this way:

PROBLEM 8:

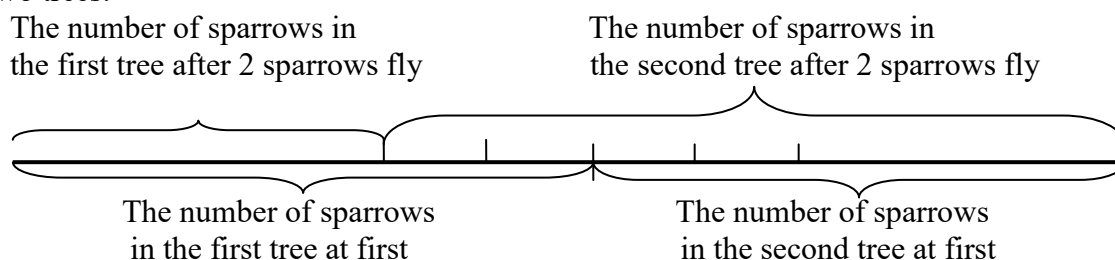
Some sparrows are sitting in two trees, with each tree having the same number of sparrows. Two sparrows then fly from the first tree to the second tree. How many sparrows does the second tree then have more than the first tree?

Solution 1:

To solve this problem, I used the following reasoning: First there is the same number of sparrows in each tree. Then 2 sparrows fly from the first tree to the second. This means that now the number of sparrows in the first tree is two less than the number before, while the number of sparrows in the second tree is two more than the number before. This means that now the second tree has 4 more sparrows than the first.

Solution 2:

In order to solve this problem, I drew a diagram representing the number of sparrows in the two trees:



From the diagram it can be seen that now the second tree has 4 more sparrows than the first.

Solution 3:

I used the same method as for Solution 2, only I drew the diagram 'in my head' (and not on paper).

Solution 4:

I solved this problem by using examples.

Suppose at first there are 8 sparrows in each tree. After the 2 sparrows fly from the first tree to the second, the number of sparrows in the first tree becomes 6, and the number of sparrows in the second tree becomes 10. So now the second tree has 4 more sparrows than the first.

Solution 5:

I solved this problem by using symbols.

Suppose the number of sparrows in each tree at first = \square . Then 2 sparrows fly from the first tree to the second.

Thus the number of sparrows in the first tree is now $(\square - 2)$, and in the second tree $(\square + 2)$.

The difference in the number of sparrows now is $(\square + 2) - (\square - 2)$.

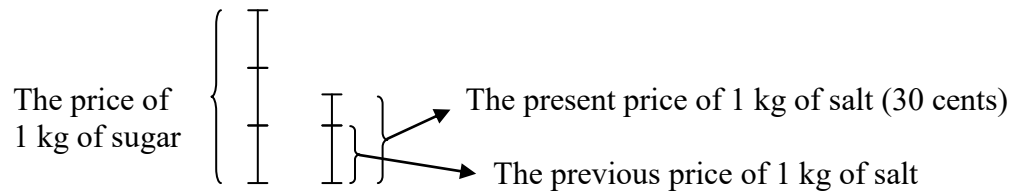
$$= \square + 2 - \square + 2 = 4$$

☐ **I did not use any of the above methods.**
 I attempted the problem in this way:

PROBLEM 9:

At first, the price of 1 kg of sugar was three times as much as the price of 1 kg of salt. Then the price of 1 kg of salt was increased by half its previous price, while the price of sugar was not changed. If the price of salt is now 30 cents per kilogram, what is the price of sugar per kg?

☐ **Solution 1:**
 I solved this problem by drawing a diagram which represents the prices of the sugar and the salt:



In the diagram it can be seen that after the price of 1 kg of salt was increased, the price of 1 kg of sugar was twice the price of 1 kg of salt.

As now the price of 1 kg of salt is 30 cents, and the price of 1 kg of sugar is 60 cents.

☐ **Solution 2:**
 I used the same method as for Solution 1, only I drew the diagram ‘in my head’ (and not on paper).

☐ **Solution 3:**
 I solved the problem in this way.
 The price of 1 kg of salt is now 30 cents. This is 1 and 1/2 times the previous price. Thus the previous price was 20 cents per kg. This means that the price of sugar is 3×20 cents, or 60 cents, per kg.

☐**Solution 4:**

I solved the problem using symbols and equations.

Suppose the price of 1 kg of salt previously = cents.

Thus the price of 1 kg of sugar = $3 \cdot$ cents.

Now, after the increase, the price of 1 kg of salt = $1 \frac{1}{2} \cdot$ cents.

This means that the price of 1 kg of sugar is twice the present price of 1 kg of salt. As the price of 1 kg of salt now = 30 cents, the price of 1 kg of sugar = 2×30 cents = 60 cents.

☐**I did not use any of the above methods.**

I attempted the problem in this way:

PROBLEM 10:

After a pedestrian travelled half of his journey, he still had to travel 4 km more to complete the journey. What was the length of his whole journey, in km?

☐**Solution 1:**

I solved this problem by using the following reasoning:

Since the pedestrian had travelled half of his journey, he still had to travel another half of the journey. This was equal to 4 km. This means that the length of the whole journey was 8 km.
(I did not draw or imagine any picture at all).

☐**Solution 2:**

I used symbols and equations to solve this problem.

Suppose the length of the whole journey = km.

He had already travelled $\frac{1}{2} \cdot$

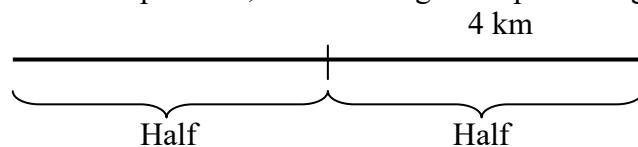
Thus, $\frac{1}{2} \cdot$ = 4

= 8

The length of the whole journey is 8 km.

**Solution 3:**

In order to solve this problem, I drew a diagram representing the journey:



From the diagram it can be seen that the length of the whole journey was 8 km.

**Solution 4:**

I used the same method as for Solution 3, only I drew the diagram ‘in my head’ (and not on paper).

**I did not use any of the above methods.**

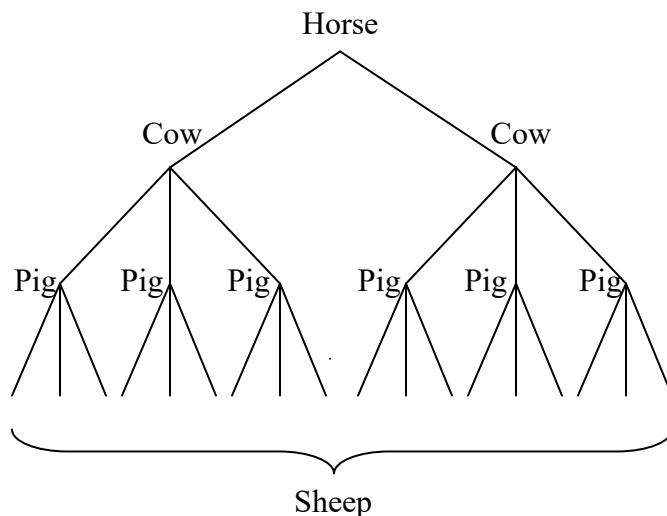
I attempted the problem in this way:

PROBLEM 11:

Mr. Jones traded his horse for two cows. Next he traded the two cows, and for each cow, he got three pigs. Then, he traded the pigs, and for each pig, he got 3 sheep. Altogether, how many sheep did Mr. Jones get?

**Solution 1:**

In order to solve this problem, I drew a diagram representing the animals:



From the diagram it can be seen that M. Jones got 18 sheep.



Solution 2:

I used the same method as for Solution 1, only I drew the diagram ‘in my head’ (and not on paper).



Solution 3:

I solved the problem using the following reasoning:

Mr. Jones traded his horse for 2 cows. Next he traded the two cows, and for each cow, he got three pigs. This means that he got 2×3 pigs, or 6 pigs. Then he traded the pigs, and for each pig, he got three sheep. This means that he got 6×3 sheep, or 18 sheep.

(I did not draw or imagine any picture at all).



Solution 4:

The number of sheep that Mr. Jones got

$$= 1 \times 2 \times 3 \times 3$$

$$= 18$$

He got 18 sheep.



I did not use any of the above methods.

I attempted the problem in this way:

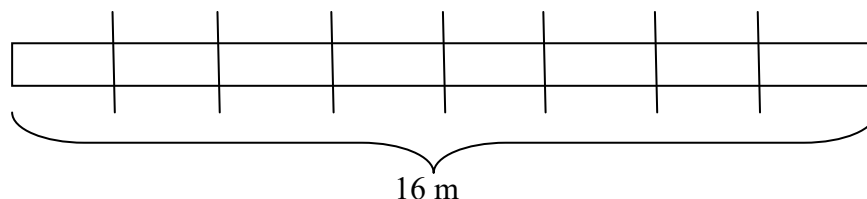
PROBLEM 12:

A saw in a sawmill saws long logs, each 16 m long, into short logs, each 2 m long. If each cut takes two minutes, how long will it take for the saw to produce eight short logs from one long log?



Solution 1:

To solve this problem, I drew a diagram showing the long log being cut into small logs.



In the diagram it can be seen that 7 cuts are needed to produce 8 short logs from one long log. Thus it will take 7×2 minutes, or 14 minutes, to produce 8 short logs from one long log.



Solution 2:

I solved this problem by imagining one long log and the cuts needed to produce the short logs. I could 'see' in my mind that 7 cuts are needed to produce 8 short logs from one long log. Thus it will take 7×2 minutes, or 14 minutes, to produce the 8 short logs.



Solution 3:

I solved the problem using the following reasoning:

If the long log were more than 16 m long, one would need 8 cuts to produce 8 short logs, each 2 m long, from that long log.

But the long log is only 16 m long, so the last cut is not needed. So one will only need $(8 - 1)$ cuts, or 7 cuts. As each cut takes 2 minutes, 7 cuts will take 7×2 minutes, or 14 minutes.



I did not use any of the above methods.

I attempted the problem in this way:

PROBLEM 13:

How many ways can 30 dollars be paid to a person if the money must be in 5-dollar and 2-dollar notes only, and the person must get some 5-dollar notes and some 2-dollar notes. (For each possible solution, summarize your answer by saying how many 2-dollar notes and how many 5-dollar notes the person would get.)



Solution 1:

I solved the problem by guessing the combinations of 5-dollar and 2-dollar notes which add up to 30 dollars?

5 5 5 5 5 2 2	= NO
5 5 5 5 2 2 2 2 2	= YES
5 5 5 2 2 2 2	= NO
5 5 2 2 2 2 2 2 2 2 2 2	= YES
5 2 2 2 2	= NO

Thus there are only two ways: 1) Four 5-dollar notes and five 2-dollar notes, and 2) Two 5-dollar notes and ten 2-dollar notes.

☐ **Solution 2:**

I solved the problem by using this reasoning: The number of 5-dollar notes should be such that the rest of the money is a multiple of 2. This means that the total amount composed by the 5-dollar notes can be 10 dollars or 20 dollars. Then the rest of the money which is composed by the 2-dollar notes can be 20 dollars or 10 dollars. Therefore, there are two ways in which the money can be paid:

1) 10 dollars composed by 5-dollar notes, and 20 dollars by 2-dollar notes. This means that there are two 5-dollar notes and ten 2-dollar notes, and 2) 20 dollars composed by 5-dollar notes, and 10 dollars by 2-dollar notes. This means that there are four 5-dollar notes and five 2-dollar notes.

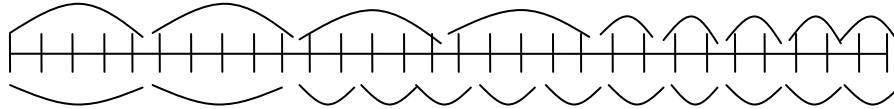
☐ **Solution 3:**

I solved the problem by using this reasoning:

The number of 2-dollar notes should be such that the rest of the money is a multiple of 5. The rest of the solution then is similar to Solution 2.

☐ **Solution 4:**

I solved the problem by drawing a diagram representing the money. The diagram is a line consisting of 30 equal parts. To solve the problem I tried to arrange the line into combinations of line segments consisting of 5 parts and 2 parts.



In the diagram it can be seen that there are two ways to make such arrangement:

- 1) Four line segments which consist of 5 parts each and five line segments which consist of 2 parts each.
2. Two line segments which consist of 5 parts each and ten line segments which consist of 2 parts each.

This means that there are 2 ways in which 30 dollars can be paid using 5-dollar and 2-dollar notes: 1) Four 5-dollar notes and five 2-dollar notes, and 2) Two 5-dollar notes and ten 2-dollar notes.

☐ **Solution 5:**

I used the same method as for Solution 4, only I drew the diagram 'in my head' (and not on paper).

☐ **I did not use any of the above methods.**

I attempted the problem in this way:

PROBLEM 14:

A tourist travelled some of his journey by plane, and the rest by bus. The distance that he travelled by bus was half the distance he travelled by plane. Determine the length of his entire trip if the distance that he travelled by plane was 150 km longer than the distance he travelled by bus.

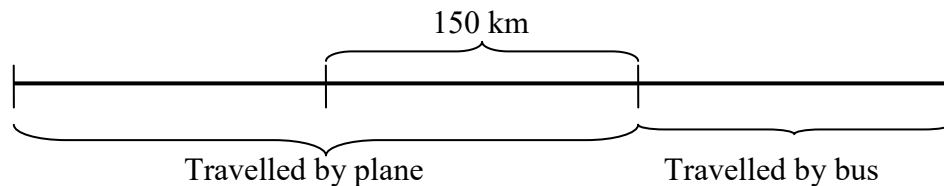
**Solution 1:**

To solve this problem, I divided the journey into three equal sections, two sections being travelled by plane, one section by bus. The difference in the distance travelled by plane and that travelled by bus was one section. This was equal to 150 km. Thus the length of the whole journey was 3×150 km, or 450 km,

(I did not draw or imagine any picture at all).

**Solution 2:**

I solved this problem by drawing a diagram of the journey.



In the diagram it can be seen that the difference between the distance travelled by plane and that travelled by bus was one section. It was equal to one section. In the diagram it is also clear that the length of the whole journey was 450 km.

**Solution 3:**

I used the same method as for Solution 2, only I drew the diagram ‘in my head’ (and not on paper).

**I did not use any of the above methods.**

I attempted the problem in this way:

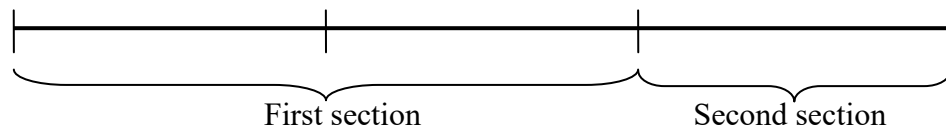
PROBLEM 15:

A straight path is divided into two unequal sections. The length of the second section is half the length of the first section. What fraction of the whole path is the first section?



Solution 1:

I solved this problem by drawing a diagram representing the path:



From the diagram it can be seen that the first section is two-thirds ($\frac{2}{3}$) of the whole path.



Solution 2:

I used the same method as for Solution 1, only I drew the diagram ‘in my head’ (and not on paper).



Solution 3:

As the length of the second section is half the length of the first section, the path can be divided into three equal parts. The first section contains two parts, and the second section one. Thus the second section is two-thirds of the whole path.

(I did not draw or imagine any picture at all.)



Solution 4:

I solved this problem by using examples. Suppose the length of the first section is 50 m, then the length of the second section is 25 m, as the length of the second section is half the length of the first. The length of the whole path then will be 75 m; This means that the first section (50 m) is two-thirds of the whole path.



I did not use any of the above methods.

I attempted the problem in this way:

TYPES OF SOLUTIONS
IN
MATHEMATICAL PROCESSING QUESTIONNAIRE I

V = Solution by a visual method

N = Solution by a nonvisual method

For the definitions of visual and nonvisual methods, see pp. 128-129

Problem 1

Solution 1 = V
Solution 2 = V
Solution 3 = N
Solution 4 = N

Problem 6

Solution 1 = V
Solution 2 = V
Solution 3 = N

Problem 11

Solution 1 = N
Solution 2 = N
Solution 3 = V
Solution 4 = V

Problem 2

Solution 1 = N
Solution 2 = V
Solution 3 = V

Problem 7

Solution 1 = N
Solution 2 = V
Solution 3 = V
Solution 4 = N

Problem 12

Solution 1 = N
Solution 2 = V
Solution 3 = V

Problem 3

Solution 1 = V
Solution 2 = V
Solution 3 = V
Solution 4 = N
Solution 5 = N

Problem 8

Solution 1 = N
Solution 2 = V
Solution 3 = V

Problem 13

Solution 1 = N
Solution 2 = V
Solution 3 = V

Problem 4

Solution 1 = V
Solution 2 = N
Solution 3 = V

Problem 9

Solution 1 = N
Solution 2 = N
Solution 3 = V
Solution 4 = V
Solution 5 = N

Problem 14

Solution 1 = V
Solution 2 = V
Solution 3 = N

Problem 5

Solution 1 = N
Solution 2 = N
Solution 3 = V
Solution 4 = V

Problem 10

Solution 1 = V
Solution 2 = V
Solution 3 = N

Problem 15

Solution 1 = V
Solution 2 = V
Solution 3 = N

TYPES OF SOLUTIONS
IN
MATHEMATICAL PROCESSING QUESTIONNAIRE II

V = Solution by a visual method

N = Solution by a nonvisual method

Problem 1

Solution 1 = V
Solution 2 = V
Solution 3 = N
Solution 4 = N
Solution 5 = N

Problem 6

Solution 1 = N
Solution 2 = N
Solution 3 = V
Solution 4 = V

Problem 11

Solution 1 = V
Solution 2 = V
Solution 3 = N
Solution 4 = N

Problem 2

Solution 1 = V
Solution 2 = V
Solution 3 = N

Problem 7

Solution 1 = N
Solution 2 = N
Solution 3 = V
Solution 4 = V

Problem 12

Solution 1 = V
Solution 2 = V
Solution 3 = N

Problem 3

Solution 1 = V
Solution 2 = V
Solution 3 = N

Problem 8

Solution 1 = N
Solution 2 = V
Solution 3 = V
Solution 4 = N
Solution 5 = N

Problem 13

Solution 1 = N
Solution 2 = N
Solution 3 = N
Solution 4 = V
Solution 5 = V

Problem 4

Solution 1 = N
Solution 2 = V
Solution 3 = V

Problem 9

Solution 1 = V
Solution 2 = V
Solution 3 = N
Solution 4 = N

Problem 14

Solution 1 = N
Solution 2 = V
Solution 3 = V

Problem 5

Solution 1 = V
Solution 2 = V
Solution 3 = N
Solution 4 = N

Problem 10

Solution 1 = N
Solution 2 = N
Solution 3 = V
Solution 4 = V

Problem 15

Solution 1 = V
Solution 2 = V
Solution 3 = N
Solution 4 = N

REFERENCES

- Abrams, J. P. (2001). Teaching mathematical modeling and the skills of representation. In A. A. Cuoco & F. R. Curcio (Eds.), *The Roles of Representation in School Mathematics* (pp. 269-282). Reston, VA: The National Council of Teachers of Mathematics.
- Ainsworth, S., Bibby, P. A., & Wood, D. J. (1997). Evaluating principles for multi representational learning environments. *Paper presented at the 7th European Conference for Research on Learning and Instruction, Athens.*
- Arcavi, A. (2003). The role of visual representations in the learning of mathematics. *Educational Studies in Mathematics*, 52, 215-241.
- Aspinwall, L., Shaw, K. L., & Presmeg, N. (1997). Uncontrollable mental imagery: Graphical connections between a function and its derivative. *Educational Studies in Mathematics*, 33, 301-317.
- Atkinson, S. (2004). A comparison of pupil learning and achievement in computer aided learning and traditionally taught situations with special reference to cognitive style and gender issues. *Educational Psychology: An International Journal of Experimental Educational Psychology*, 24(5), 659-679.
- Ben-Chaim, D., Lappan, G., & Houang R.T. (1989). Adolescents' ability to communicate spatial information: Analyzing and effecting students' performance. *Educational Studies in Mathematics*, 20, 121-146.
- Bishop, A. J. (1989). Review of the research on visualization in mathematics education. *Focus On Learning Problems in Mathematics*, 11 (1), 7-16.
- Bogdan, R. C., & Biklen, S.K. (1998). *Foundations of qualitative research in education* (3rd ed.). Boston, MA: Allyn & Bacon.

- Borba, M. C., & Villarreal, M. E. (2005). Visualization, mathematics education and computer environments. In B. S. Jones, & R. Z. Smith (Eds.), *Humans-with-Media and the Reorganization of Mathematical Thinking* (pp.89). New York: Springer.
- Choi-Koh, S. S. (2003). Effect of a graphing calculator on a 10th-grade student's study of trigonometry. *The Journal of Educational Research*, 96(6). 359-369.
- Cifarelli, V. V. (1998). The development of mental representations as a problem solving activity. *Journal of Mathematical Behavior*, 17, 239-264.
- Clement, J. (2000) Analysis of clinical interviews: Foundations and model viability. In Lesh, R. and Kelly, A., *Handbook of research methodologies for science and mathematics education* (pp. 341-385). Hillsdale, NJ: Lawrence Erlbaum.
- Clement, J., & Kaput, J. J. (1979). Letter of the editor. *The Journal of Children Mathematical Behavior*, 2 (2), 208.
- Clements, K. (1981). Visual imagery and school mathematics. *For the learning of mathematics*, 2, 2-9.
- Confrey, J. (1981). Using the clinical interview to explore students' mathematical understanding. East Lansing, MI: Institute for Research on Teaching, Science-Mathematics Teaching Center, Michigan State University.
- Davis, R. B., Young, S., & McLoughlin, P. (1982). *The roles of "understanding" in the learning of mathematics*. Urbana: University of Illinois, Curriculum laboratory.
- de Jong, T., & van Joolingen, W. R. (1998). Scientific discovery learning with computer simulations of conceptual domains. *Review of Educational Research*, 68, 179-201.
- Dixon, J. K. (1995). English language proficiency and spatial visualization in middle school students' construction of the concepts of reflection and rotation using the "The

- Geometer's Sketchpad." Ph.D. dissertation, University of Florida, United States -- Florida. Retrieved from Dissertations & Theses: Full Text. (Publication No. AAT 9607507).
- Dufour-Janvier, B., Bednarz, N., & Belanger, M. (1987). Pedagogical considerations concerning the problem of representation. In C. Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics* (pp. 109-122). Hillsdale, NJ: Erlbaum.
- Eisenberg, T., & Dreyfus, T. (1989). Spatial visualization in the mathematics curriculum. *Focus on Learning Problems in Mathematics*, 11(1), 1-5.
- Eisenberg, T., & Dreyfus, T. (1991). On the reluctance to visualize in mathematics. In W. Zimmermann & S. Cunningham (Eds.), *Visualization in Teaching and Learning Mathematics* (pp. 127 – 138). Washington, DC: MAA.
- Ferrera, F., Pratt, D., & Robutti, O. (2006). The role and uses of technologies for the teaching of algebra and calculus. In A. Gutierrez & P. Boero (Eds.), *Handbook of Research on the Psychology of Mathematics Education: Past, Present and Future* (pp. 205-235). Rotterdam, The Netherlands: Sense Publishers.
- Ferrini-Mundy, J., & Graham, K. (1991). An overview of the calculus curriculum reform effort: Issues for learning, teaching, and curriculum development. *The American Mathematical Monthly*, 98, 627-635.
- Goldin, G. (1987). Cognitive representational systems for mathematical problem solving. In C. Janvier (Eds.), *Problems of Representation in the Teaching and Learning of Mathematics* (pp. 125-146). Hillsdale, NJ: Lawrence Erlbaum Associates.
- Goldin, G. A., & Kaput, J. J. (1996). A joint perspective on the idea of representation in learning and doing mathematics. In: L. P. Steffe, P. Nesher, P. Cobb, G. A. Goldin, & B. Greer

- (Eds.), *Theories of mathematical learning* (pp. 397–430). Hillsdale, NJ: Lawrence Erlbaum.
- Goldin, G., & Shteingold, N. (2001). Systems of representations and development of mathematical concepts. In A. A. Cuoco & F. R. Curcio (Eds.), *The Roles of Representation in School Mathematics* (pp. 1-23). Reston, VA: The National Council of Teachers of Mathematics.
- Grimley, M. (2007). Learning from Multimedia Materials: The relative impact of individual differences. *Educational Psychology: An International Journal of Experimental Educational Psychology*, 27(4), 465-485.
- Guba, E.G., & Lincoln, Y.S. (1981). *Effective evaluation*. San Francisco: Jossey-Bass.
- Gutiérrez, A. (1996). Visualization in 3-dimensional geometry: In search of a framework. In L. Puig & A. Gutierrez (Eds.), *Proceedings of the 20th PME International Conference*, 1, 3-19.
- Haciomeroglu, E., Aspinwall, L., & Presmeg, N. C. (2010). Contrasting Cases of Calculus Students' Understanding of Derivative Graphs. *Mathematical Thinking & Learning*, 12(2), 152-176.
- Harskamp, E. Suhre, C. & Van Streun, A. (2000). The graphics calculator and students' solution strategies. *Mathematics Education Research Journal*, 12(1), 37-52.
- Heid, M. K., & Blume, G. W. (2008). Algebra and function development. In M. K. Heid & G. W. Blume (Eds.), *Research on technology and the teaching and learning of mathematics: Research syntheses* (Vol. 1, pp. 55–108). Charlotte, NC: Information Age.

- Hennessey, S., Fung, P. and Scanlon, E. (2001) The role of the graphic calculator in mediating graphing activity *International Journal of Mathematics for Education in Science and Technology*, 32(2), 267-290
- Hohenwarter, M. (2002) *GeoGebra - Ein Software system für dynamische Geometrie und Algebra der Ebene*. Master thesis, University of Salzburg.
- Huntley, M. A., Rasmussen, C. L., Villarubi, R. S., Sangtong, J., & Fey, J. T. (2000). Effects of Standards-based mathematics education: A study of the Core-Plus Mathematics Project algebra and functions strand. *Journal for Research in Mathematics Education*, 31(3), 328-361.
- Jackiw, N. (1995). The geometer's sketchpad [Computer software]. Berkeley, CA: Key Curriculum Press.
- Janvier, C. (1978). The interpretation of complex cartesian graphs representing situations: Studies and teaching experiments. Unpublished doctoral dissertation, University of Nottingham
- Janvier, C. (1987a). Representation and understanding: The notion of function as an example. In C. Janvier (Eds.), *Problems of Representation in the Teaching and Learning of Mathematics* (pp. 67-71). Hillsdale, NJ: Lawrence Erlbaum Associates..
- Janvier, C. (1987b). Translation processes in mathematics education. In C. Janvier (Eds.), *Problems of Representation in the Teaching and Learning of Mathematics* (pp. 27-32). Hillsdale, NJ: Lawrence Erlbaum Associates.
- Kaput, J. J. (1987a). Representation systems and mathematics. In C. Janvier (Eds.), *Problems of Representation in the Teaching and Learning of Mathematics* (pp. 19-26). Hillsdale, NJ: Lawrence Erlbaum Associates.

- Kaput, J. J. (1987b). Toward a theory of symbol use in mathematics. In C. Janvier (Eds.), *Problems of Representation in the Teaching and Learning of Mathematics* (pp. 159-195). Hillsdale, NJ: Lawrence Erlbaum Associates.
- Koedinger, K., & Nathan, M. (2004). The real story behind story problems: effects of representations on quantitative reasoning. *Journal of the Learning Sciences*, 13(2), 129-164.
- Kozhevnikov, M., Hegarty, M. & Mayer, R. E. (2002). Revising the visualizer-verbalizer dimension: evidence for two types of visualizers. *Cognition and Instruction*, 20(1), 47-77.
- Kozhevnikov, M., Kosslyn, S. M., & Shepard, J. (2005). Spatial versus object visualizers: a new characterization of visual cognitive style. *Memory and Cognition*, 33, 710–726.
- Kozhevnikov, M., Blazhenkova, O., Becker, M. (2010). Trade-Off In Object Versus Spatial Visualization Abilities: Restriction In The Development Of Visual-Processing Resources. *Psychonomic Bulletin & Review*, 17 (1), 29-35.
- Krutetskii, V. A. (1963). Some characteristics of the thinking of pupils with little capacity for mathematics. In B. Simon & J. Simon (Eds.), *Educational Psychology in the U.S.S.R.* (pp. 214-233). Stanford, CA: Stanford University Press.
- Krutetskii, V. A. (1969). *Soviet Studies in the Psychology of Learning and Teaching Mathematics, Vol. II: The Structure of Mathematical Abilities*. Chicago, IL: University of Chicago Press.
- Krutetskii, V. A. (1976). *The Psychology of Mathematical Abilities in Schoolchildren*. In J. Kilpatrick & I. Wirszup (Eds.). Chicago: The University of Chicago Press.

- Lean, G., & Clements, K. (1981). Spatial ability, visual imagery, and mathematical performance. *Educational Studies in Mathematics*, 12, 267-299.
- Lesh, R., Post, T., & Behr, M. (1987). Representations and translations among representations in mathematics learning and problem solving. In C. Janvier (Eds.), *Problems of Representation in the Teaching and Learning of Mathematics* (pp. 33-40). Hillsdale, NJ: Lawrence Erlbaum Associates.
- Lincoln, Y., & Guba, E. (1986). But is it rigorous? Trustworthiness and authenticity in naturalistic evaluation. In D. D. Williams (Ed.), *New Directions in Program Evaluation*, San Francisco, CA: Jossey-Bass
- Lin, C. Y. (1979). Imagery in mathematical thinking and learning. *International Journal of Mathematics Education in Science and Technology*, 10, 107-111.
- Lohman, D.F., (1979). *Spatial Ability: Review and Re-analysis of the Correlational Literature*. (Tech. Rep. No. 8). Stanford, CA: Stanford University School of Education Aptitude Research Project.
- Lowrie, T., & Kay, R. (2001). Relationship between visual and novisual solution methods and difficulty in elementary mathematics. *Journal of Educational Research*, 94, 248-255.
- Marchand, R. J., McDevitt, T. J., Bossé, M. J. & Nandakumar, N. R. (2007). Does Your Graphing Software Real-ly Work? *PRIMUS: Problems, Resources, and Issues in Mathematics Undergraduate Studies*, 17(2), 195-208.
- Marshall C. & Rossman G.B. (2010). *Designing Qualitative Research*. Sage Publications, London.
- Mayer, R. E. (1999). Research-based principles for the design of instructional messages: The case of multimedia explanations. *Document Design*, 2, 7-20.

- Mayer, R. E. (2001). *Multimedia learning*. New York: Cambridge University Press.
- Merriam, S.B. (1998). *Qualitative Research and Case Study Applications in Education*. San Francisco: Jossey-Bass Publishers.
- Merriweather, M. & Tharp, M. (1999). The effect of instruction with graphing calculators on how general mathematics students naturalistically solve algebraic problems. *Journal of Computers in Mathematics and Science Teaching*, 18(1), 7-22.
- Mitchelmore, M. & Cavanaugh, M. (2000). Students' difficulties in operating a graphics calculator. *Mathematics Education Research Journal*, 12 (3), 254-268.
- Moses, B.E. (1977). The nature of spatial ability and its relationship to mathematical problem solving. *Dissertation Abstracts International*, 38, 08A.
- National Council of Teachers of Mathematics. (2000). *Principles and Standards for School Mathematics*. Reston, VA: NCTM.
- National Research Council (2001). *Adding it up: Helping children learn mathematics*. J. Kilpatrick, J. Swafford & B. Findell (Eds.) Mathematics Learning Study Committee, Center for Education, Division of Behavioral and Social Sciences and Education. Washington, DC: National Academy Press.
- Nemirovsky, R., & Noble, T. (1997). On mathematical visualization and the place where we live. *Educational Studies in Mathematics*, 33, 99-131.
- Noddings, N. (1990). Constructivism in mathematics education. *Journal for Research for Mathematics Education Monograph*, 4, 7-18.
- Palmer, S. E. (1978). Fundamental aspects of cognitive representation. In E. Rosch, & B. B. Lloyd (Eds.), *Cognition and Categorization* (pp. 259-303). Hillsdale, NJ: Lawrence Erlbaum Associates.

- Parzysz, B. (1999). Visualization and modeling in problem solving: From algebra to geometry and back. In O. Zaslavsky (Ed.), *Proceedings of the 23rd PME International Conference, 1*, 212-219.
- Patton, M. Q. (2002). *Qualitative Research & Evaluation Methods*. Thousand Oaks, CA: Sage Publications, Inc.
- Pierce, R. & Stacey, K. (2010). Mapping pedagogical opportunities provided by mathematics analysis software. *International Journal of Computers for Mathematical Learning*.
- Pitta-Pantazi, D., & Christou, C. (2009). Cognitive styles, task presentation mode and mathematical performance. *Research in Mathematics Education, 11*(2), 131-148.
- Porzio, D. (1999). Effects of Differing emphases in the use of multiple representations and technology on students' understanding of calculus concepts. *Focus on Learning Problems in Mathematics, 21*(3), 1-29.
- Presmeg, N. C. (1985). The role of visually mediated processes in high school mathematics: A classroom investigation. Unpublished Ph.D. dissertation, University of Cambridge.
- Presmeg, N. C. (1986a). Visualization and Mathematical Giftedness. *Educational Studies in Mathematics, 17*, 297-311.
- Presmeg, N. C. (1986b). Visualization in high school mathematics. *For the Learning of Mathematics, 6*, 42-46.
- Rasmussen, C., Stephan, M., & Allen, K. (2004). Classroom mathematical practices and gesturing. *Journal of Mathematical Behavior, 23*, 301-323.
- Riding, R. J., & Cheema, I. (1991). Cognitive styles: an overview and integration. *Educational Psychology, 11*, 193-215.

- Riding, R. and Douglas, G. (1993). The effect of cognitive style and mode of presentation on learning performance. *British Journal of Educational Psychology*, 63, 297-307.
- Riding, R. J., & Grimley, M. (1999). Cognitive style and learning from multimedia CD-ROMs in 11-year-old children. *British Journal of Educational Technology*, 30, 43–56.
- Ruthven, K. (1990). The influence of graphic calculator use on translation from graphic to symbolic forms. *Educational Studies in Mathematics*, 21, 431-450.
- Seufert, T. (2003). Supporting coherence formation in learning from multiple representations. *Learning and Instruction*, 13, 227-237.
- Slavit, D. (1998). Three women's understandings of algebra in a Precalculus course integrated with the graphing calculator. *Journal of Mathematical Behavior*, 17(3), 355-372.
- Sweller, J. (1988). Cognitive load during problem solving: Effects on learning. *Cognitive Science*, 12, 257-285.
- Suwarsono, S. (1982). Visual imagery in the mathematical thinking of seventh-grade students. Unpublished doctoral dissertation, Monash University.
- Vergnaud, G. (1998). A comprehensive theory of representation for mathematics education. *Journal of Mathematical Behavior*, 17, 167-181.
- van Labeke, N., & Ainsworth, S. (2001). *Applying the DeFT framework to the design of multi-representational instructional simulations*. Paper presented at the AIED'2001 - 10th International Conference on Artificial Intelligence in Education, San Antonio, Texas.
- von Glaserfeld, E. (1987). Learning as a constructive activity. In C. Janvier (Eds.), *Problems of Representation in the Teaching and Learning of Mathematics* (pp. 3-17). Hillsdale, NJ: Lawrence Erlbaum Associates.

- Yerushalmy, M. (1997). Emergence for new schemes for solving algebra word problems: The impact of technology and the function approach. In E. Pehkonen (Ed.), *Proceedings of the 21st PME International Conference, 1*, 165-172.
- Yerushalmy, M., Shternberg, G., & Gilead, S. (1999). Visualization as a vehicle for meaningful problem solving in algebra. In O. Zaslavsky (Ed.), *Proceedings of the 23rd PME International Conference, 1*, 197-211.
- Yerushalmy, M. (2006). Slower algebra students meet faster tools: Solving algebra word problems with graphing software. *Journal for Research in Mathematics Education*, Vol.37, No.5: 356-387.
- Zazkis, R., Dubinsky, E., & Dautermann, J. (1996). Coordinating visual and analytic strategies: A study of students' understanding of the group D_4 . *Journal for Research in Mathematics Education*, 27, 435-457.
- Zimmermann, W., & Cunningham, S. (1991). What is mathematical visualization. In W. Zimmermann, & S. Cunningham (Eds.), *Visualization in Teaching and Learning Mathematics* (pp. 1- 8). Washington, DC: MA