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The Effects Of A Ratio-based Teaching Sequence On Performance In Fraction Equivalency For Students With Mathematics Disabilities

Jessica H. Hunt
University of Central Florida

THE EFFECTS OF A RATIO-BASED TEACHING SEQUENCE ON PERFORMANCE IN FRACTION EQUIVALENCY FOR STUDENTS WITH MATHEMATICS DISABILITIES

by

JESSICA H. HUNT

B.S., Youngstown State University, 2002
M.A., University of Central Florida, 2006

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ABSTRACT

This study examined the effects of a ratio-based supplemental teaching sequence on third grade students' equivalent fraction performance as measured by a curriculum-based measure and a standardized test. Participants included students identified as being learning disabled in mathematics (MLD), struggling (SS), or typically achieving (TA). Nineteen students were assigned to the experimental group and 19 additional students formed the control group. The difference between the two groups was that the experimental group received the ratio-based teaching sequence. Both groups continued to receive textbook based instruction in fraction equivalency concepts in their regular mathematics classroom. Qualitative interviews were employed to further investigate the thinking of each of the three types of students in the study.

Analyses of the data indicated that students in the experimental group outperformed the control group on both the curriculum-based measure and the standardized measure of fraction equivalency. All students who participated in ratio-based instruction had a higher performance in fraction equivalency than those who did not. Performance on the CBM and the standardized measure of fraction equivalency improved significantly from pre to post test for students who struggled; their performance also transferred to standardized measures. Qualitative analyses revealed that a focal student with MLD, while improving his ability to think multiplicatively, had misconceptions about fractions as ratios that persisted even after the intervention was completed. Implications for instruction, teacher preparation, and future research are provided.

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"For what does it profit a man to gain the whole world and forfeit his soul?" – Matthew 16:26

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LIST OF ACRONYMS/ABBREVIATIONS

IDEA= Individuals with Disabilities Education Improvement Act

LD = Learning Disability

MLD = Mathematics Learning Disability

NAEP = National Assessment of Educational Progress

NCTM = National Council of Teachers of Mathematics

CHAPTER ONE: INTRODUCTION

Need for the Study

The Emergence of the Field of Learning Disabilities

Throughout the past two centuries, a unique component in the field of education emerged for students with learning disabilities (LD) (Fernald & Keller, 1921; Hinshelwood, 1917; Lewandowsky & Stadelmann, 1908). Originally the field of LD was synonymous with reading and perceptual/motor disabilities (Kirk, 1933; Monroe, 1928; Orton, 1925; Strauss, 1943; Werner & Strauss, 1939, 1940, 1941). Consequently, intervention research for students with LD concentrated on motor impairment, aspects of reading, phonics, dyslexia, and sensory based impairments (Kirk, 1933; Monroe, 1928). Emerging notions of mathematics related disabilities were offered throughout the history of the field, but were largely overshadowed by the clear emphasis on reading during the field's formation (Woodward, 2004).

The ascendancy of reading, as the "unofficial" first formal definition of LD created throughout the 1960s and 1970s, influenced the foundation of knowledge related to this population (Education of All Handicapped Students Act, 1975; Kirk, 1963; United States Office of Education, 1968). The focus on students with LD predominantly having a reading disability was promoted in teacher preparation efforts (Brownell, Sindelar, Kiely, & Danielson, 2010; Gerstein, Clarke, & Mazzocco, 2006; Good, Grouws, & Ebmeier, 1983; Rosas & Campbell, 2010; Woodward, 2004) intervention studies (Bateman, 1965; Deno, 1985; Kirk, McCarthy, & Kirk, 1968) and continues to some extent in both areas today (Greer & Meyan, 2009).

Mathematics and Students with LD

Despite a strong initial focus on reading in the field of special education, students with LD have also historically underperformed in mathematics (Cawley & Miller, 1989). Authors from the National Assessment of Educational Progress (NAEP) report a great disparity between the levels of mathematics achievement for students with disabilities when compared to the results for students without disabilities (National Center for Educational Statistics, 2009). The Nation's Report Card: Mathematics 2009 reported 19% of fourth-graders with disabilities scored "at or above proficient", in comparison to 41% of fourth graders without disabilities. In eighth grade, only 9% of students with disabilities scored at or above proficient, as compared to 35% of their counterparts (National Center for Education Statistics, 2009). Furthermore, students with LD are typically two grade levels below students without disabilities in mathematics, with performance typically leveling off around the fifth grade achievement level (Cawley & Miller, 1989).

The performance in mathematics of students with LD can be disrupted for a variety of reasons pertaining to their specific strengths and weaknesses resulting from the disability. For instance, students found to have mathematics learning disabilities (MLD), in particular, experience a largely diverse array of problems related to understanding and performance, including (1) deficits in semantic memory, (2) sense of number, (3) working memory (WM), and (4) nonverbal reasoning (Geary, Hoard, Nugent, & Byrd-Craven, 2006; Lanfranchi, Lucangeli, Jerman, & Swanson, 2008; Mazzacco, 2006). The diverse types of MLD can affect students' abilities in mathematics in several ways. For instance, some students with MLD experience deficiencies in their sense of number, such as subitizing or partitioning, while others experience difficulty in making inferences from mathematical drawings (Geary et al., 2006). Many of the

difficulties experienced by students with MLD may interfere with learning mathematics when concepts are taught in ways that may play to one or several inherent weaknesses.

Fractions and MLD

Although the literature is still developing, the results of research examining performance of students with MLD in the area of fraction concepts reflect many of the inherent weaknesses noted in research. For instance, several studies have found students with MLD experience increased difficulties in acquiring fraction concepts and skills than do their peers without disabilities (Grobecker, 2000; Hecht, 1998; Hecht, Close, & Santisi, 2003, Hecht et al., 2006; Mazzacco & Devlin, 2008). Grobecker (1997, 2000) investigated students with MLD and their understanding of fraction equivalency and addition when presented through part-whole, measure, and ratio-based activities. She found students with MLD were unable to conserve the whole while imagining and reimagining the unit (e.g. reimagining $\frac{2}{8}$ as $\frac{1}{4}$) during equivalency problems in both the part-whole and measure subconstructs. These difficulties may be linked to a deficient sense of number, working memory issues, or both (Geary, 2009), yet currently specific research on deficits and resulting issues in fractions for students with MLD is lacking in the literature.

Mazzacco and Devlin (2008) discovered middle school students with MLD demonstrated statistically significant differences in identifying fraction equivalencies. When these students were presented as circular pictures and in abstract forms their performance was decreased when compared to struggling and typically achieving students. Researchers found students with MLD to have a weak sense of number related to fractions. Similar results also were found by Hecht and his colleagues in 2006. Hecht's research found deficits in conceptual knowledge and sense of number for students with MLD when fraction were based in part-whole sub constructs, a

predominant method of teaching fraction concepts and operations (Charalambous & Pitta-Pantazi, 2007).

Much of the predominance of the part-whole model in teaching fractions is rooted in historical theory. Behr, Post, and Silver (1983) proposed a hierarchy for the learning of rational number concepts and operations based on Kieran's original (1976) theoretical sub construct as shown in Figure 1.

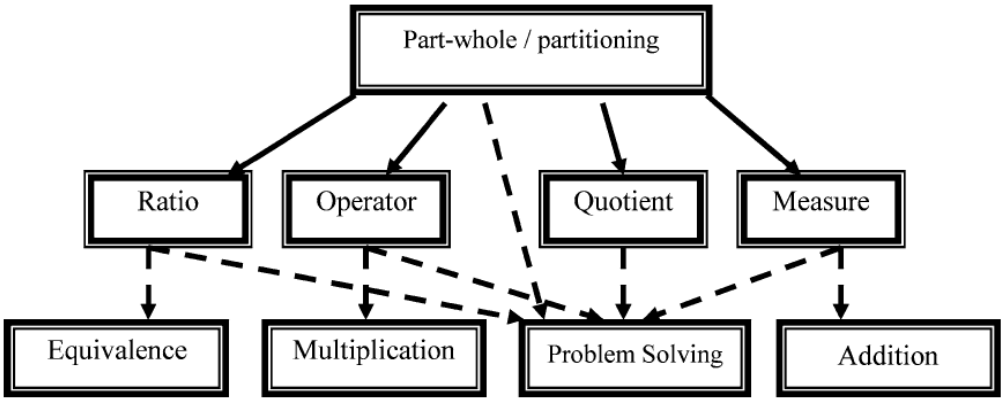


***Figure 1*. Rational number hierarchy.**

In their view, the part-whole sub construct and the act of partitioning are at the forefront of learning fractions. Although other researchers have proposed alternate relationships between the subconstructs as it relates to teaching and learning about fractions (e.g. Streefland, 1993; Lamon, 2007), the part-whole dominated hierarchy is widely accepted in most textbook and research based intervention approaches to teaching fraction concepts (e.g. Butler, Miller, Crehan, Babbit, & Pierce, 2003).

But students with MLD experience significant difficulties in acquiring fraction concepts and skills through the part-whole and even measurement based instruction (Grobecker, 1997; 2000; Hecht, 1998; Hecht et al., 2003; Hecht et al., 2006; Mazzacco & Devlin, 2008). In particular, limited or incorrect knowledge in fraction concepts involving partitioning, unitizing,

and equivalence, when largely set in part-whole and measure sub construct, have been found to interfere with students' abilities to understand and work with fraction equivalency and operations (Grobecker, 2000; Hecht et al., 2006; Mazzacco & Devlin, 2008). Although part-whole sub construct is of importance in understanding fractions, students with MLD may develop better understanding of fraction concepts through other sub constructs of rational number, such as ratios. It is possible that teaching equivalency concepts through ratios could serve as an alternative to teaching concepts to students with MLD through the part-whole sub construct if students can be taught to progress to multiplicative thinking processes (see pg. 48, 50-51) needed to understand equivalency concepts (Cortina & Zuniga, 2008; Grobecker, 1997; Lamon, 1993b). Thus, instruction that includes other fraction sub constructs may prove beneficial to those who experience difficulties in understanding fractions through activities based solely in part-whole or measure sub constructs, including students with MLD.

Statement of the Problem

Students with MLD experience significant difficulties in understanding fraction concepts centered on fraction equivalency (Hecht et al., 2006; Mazzacco & Devlin, 2008). Research suggests that the lack of understanding of fraction equivalencies may be exacerbated by a curriculum based primarily in the part-whole sub construct (Hecht et al., 2006; Lewis, 2007; Mazzacco & Devlin, 2008). Research is warranted that explores how students with MLD respond to interventions that teach equivalency and related concepts in alternate ways.

Purpose and Significance of the Study

Although many interventions in the field of special education have presented varying teaching methods to improve performance in fraction concepts (e.g. Bottge, Henrichs, Mehta, & Hung, 2002; Butler et al., 2003; Test & Ellis, 2005), few have explored presenting fractions

through a different sub construct. Researchers suggest that students with MLD have underdeveloped or limited understanding of fraction concepts based in the part-whole sub construct (Hecht et al., 2006). In the current study, the researcher investigated intervention based in the ratio sub construct for fraction understanding.

Research Questions

The current study will address the following research questions:

(1) Are there statistically significant differences in overall performance (i.e. the number of correct responses) on a curriculum-based measure of fraction and ratio equivalency and on a standardized measure for students with mathematical learning disabilities, struggling students, and typically developing students who do and do not participate in ratio-based fraction instruction?

(2) What is the multiplicative thinking and strategy usage of students when presented with ratio equivalency situations? Do strategy usage and levels of multiplicative thinking increase for students with MLD and students who struggled after participating in a ratio-based equivalency instructional sequence?

Research Design

A quasi experimental pretest-posttest mixed methods design was utilized in the study. The design examined the effects of fraction instruction based in ratios on performance of fraction equivalency between the experimental and control groups. Performance of students who struggled with MLD and students deemed as typically achieving were analyzed to identify any interactions between such factors. Patterns in performance among students with MLD, struggling students, and typically achieving students were examined qualitatively.

Instrumentation

Pre and Post Tests

To determine the effectiveness of ratio-based instruction on students' understanding of fraction and ratio equivalency, a pre and posttest of 20 items were pulled from the district curriculum, *Envision Mathematics,* Level 3, Chapter 12 (Charles, Caldwell, Crown & Fennell, 2011). Items from this chapter were used to construct a curriculum-based measure (CBM) that served as a pre and posttest. As required in the development of CBM the researchers examined each lesson within the chapter that addressed fraction equivalency. From these lessons, the researcher pulled every other problem from the text practice questions to construct the pre and posttest measure. Items in the CBM included situated problems (e.g. word problems), abstract problems, or problems that require students to judge the correctness of given equivalency statements (Deno, 1989; Foegan, Jiban, & Deno, 2007). Reliability (e.g. internal consistency reliability) and validity (e.g. convergent validity) of the CBM in subsequent text referred to as pre and post tests were confirmed (see Chapter 3).

Transfer Test

To assess whether student improvement could transfer to standardized measures, the subtest Q6 of the Brigance Comprehensive Inventory of Basic Skills- Revised (1991) was administered to students before and after instruction. The subtest measures students' ability to reduce fractions to their lowest term, increase a given fraction to a higher term equivalent, and convert improper fractions into mixed numbers.

Semi Structured Clinical Interviews

A videotaped semi-structured clinical interview pretest and posttest (see Appendix C) was administered to three strategically selected students in the experimental group. The interview covered all concepts targeted in the teaching sequence. Because the CBM and standardized measures used to evaluate research question one were not sensitive enough to uncover the strategy use and the levels of multiplicative thinking involved with understanding ratio fractions, the researcher used interviews to assess how these areas may have changed before and after instruction for the selected students. The pretest was administered the week before the commencement of teaching and the posttest was administered the week following the completion of the ratio instruction. Problems, based on the work of Battista and Borrow (1995), Lamon (1993), and Van de Walle (2004), were organized into "strands" with several variations of certain types of problem situations. Varying numerical ratio values and difficulties of the problems presented were used throughout each question. If a student could not answer two or more of the questions within a strand correctly, the remainder of the strand (i.e. questions) was not administered.

Treatment Conditions

Students

This study included students with and without MLD in the third grade. This grade level was chosen due to curriculum constraints that set the learning of fraction equivalency to a third grade maximum. Students who participated in the study were selected using several characteristics. Namely, enrolled in an inclusive third grade mathematics course, an FCAT level of 1, 2, or 3, a weakness in fraction concepts as identified by the pre-test, and the absence of limited English proficiency (LEP) or poor socio-economic status (SES) were inclusion

requirements (Murphy, Mazzacco, Hannich, and Early, 2007). The absence of LEP or poor SES status was to downgrade the chance of assigning MLD status due to confounding factors (Murphy et al.). A portion of the students selected had district confirmed exceptionalities (N=8), all of which were learning disabilities.

A total of 38 third grade students who met the selection criteria participated in the study. An a priori power analyses was utilized to compute the necessary sample size using G Power 3 statistical software (Faul, Erdfelder, Lang, & Buchner, 2007). Effect sizes from previous research in fractions for students with LD (Butler et al., 2003; Xin, Jitendra, & Deatline, 2006) ranging from 0.50 to 0.75 were considered in the analyses. Thus, using an alpha level of 0.05 and noted effect sizes, the power analyses indicated a total sample size of 38 was sufficient to produce a power of .90 for a 2 x 2 between factors MANOVA with repeated measures, with 19 students in the experimental group and 19 students in the control group (Faul et al., 2007). Sample sizes as small as 30 students are considered sufficient to detect changes in behavior (Howell, 2007). Furthermore, despite beliefs that smaller n groups sizes tend to violate assumptions of normality, thereby negating the use of parametric testing approaches, research suggests that parametric multivariate statistical analyses with sample sizes as small as eight can be conducted with a reliability of 1.00 (Ninness, Rumph, Vasquez, Bradfield, & Ninness, 2002). Thus, the sample size was deemed adequate to test research questions and to support the study design.

The sample then was split into students with MLD, struggling students, and typically achieving students. As indicated in the review of literature, an agreed upon definition of MLD currently does not exist. Thus, "in the absence of a consensus definition of MLD, it is necessary to rely on proxy definitions" (Mazzacco & Thompson, 2005, p. 146). The complexity in

defining MLD by proxies (e.g. scores on mathematics tests) has been revealed in the results of several studies (Mazzacco & Devlin, 2008; Mazzacco & Thompson, 2005; Murphy et al., 2007). However, study results provide the best guidance at the time for potential designation of MLD in research.

Tests that measured both informal (e.g. knowledge students have of mathematics that is not taught) and formal (e.g. achievement oriented mathematics knowledge) mathematics concepts were found to produce scores that stably predicted MLD over time and thus were included (Mazzacco, 2005). Additionally, test items that covered reading numerals, number constancy, magnitude judgments, and mental addition were found to be highly predictive of MLD over time (Murphy et al., 2007) and were included in the current research. Second, characteristics of MLD can change as a function of the cut off scores used to define a person as MLD (Murphy et al., 2007). The best available current research suggests the use of scores at the tenth percentile or below greatly reduced the number of false positives (e.g. students being labeled MLD despite other contributing factors to low mathematics achievement) and separated true MLD performance and characteristics from those who struggled in mathematics but did not have a MLD (Mazzacco & Devlin, 2008).

This study used several primary and secondary tests (e.g. one measuring formal knowledge and two others measuring both formal and informal knowledge of mathematics), to confirm the student as MLD, struggling, or proficient. Three subtests served as primary measures. The calculation subtest of the Woodcock Johnson III (consisting of 41 items normed for ages five through adult) (Woodcock, McGrew, & Mather, 2001) is an achievement test used to assess a person's ability to perform mathematical computations. Examples include writing single numerals and basic addition, subtraction, multiplication, and division. The Numeration

and Mental Computation subtests of the Key Math – Revised (Connolly, 1999) (consisting of 24 and 18 items) are used to assess a students' formal and informal knowledge of quantity, order, magnitude, reading numbers, counting, and mental computation of one and two digit numbers.

Cut off scores garnered from the research were used as criteria to designate a student as MLD. Students who met selection criteria were administered the calculation subtest from the WJIII Test of Achievement (Woodcock et al., 2001) along with the Numeration and Mental Computation subtests from the Key Math-R (Connolly, 1999). Students, whose performance fell in the bottom 10th percentile on two out of three measures, were considered MLD; students, whose performance fell between the 11th and 25th percentiles on two out of three measures, were considered struggling; and those whose performance was higher than the 25th percentile on two out of three measures were considered non-MLD, or typically achieving.

Students who met category criteria on only one measure (e.g. Key Math Numeration, Key math Mental Computation, or WJIII Calculation) were administered two additional (secondary) tests- the WJIII Quantitative Concepts and Applied Problems subtests (Woodcock et al., 2001). The Quantitative Concepts subtest is a test of students' ability to recognize symbols, retrieve representations, and manipulate points on a mental number line. The Applied Problems subtest measures students' ability to construct mental models and quantitative reasoning skills. To confirm the students as MLD, struggling, or typically achieving, the researcher examined results of the secondary measures. Students were confirmed as MLD, struggling, or typical if performance on *one or more* secondary measures fell within the ranges specified in the previous paragraph. The final analyses led to identification of four students as MLD, nine students as struggling in mathematics, and 26 students as typically achieving. Students were then matched on their 'student type' and randomly assigned to either a treatment or control group (Borg &

Gall, 1989). The matching was used to ensure that students were comparable across intervention conditions on relevant characteristics (Gerstein, Fuchs, Compton, Coyne, Greenwood, & Innocenti, 2005).

Intervention

The ratio-based instructional program dealt with fraction equivalency concepts using the following sequence: (1) concrete ratio-based partitioning exercises, (2) representational ratio-based unitizing exercises, (3) representational equivalency exercises utilizing additive strategies, (4) representational equivalency exercises utilizing multiplicative strategies, and (5) abstract exercises utilizing multiplicative strategies. Total instructional time was nine days. Point-by-point interrater reliability was calculated to ensure consistent delivery of instruction. The study took place in a public elementary school in central Florida in May of 2011. All students received their assigned instruction (ratio intervention or control) during school hours. During the intervention, instruction took place in a third grade classroom with five desks, three large whiteboards, and manipulatives. The instructor for the study was the researcher.

Each day of ratio-based instruction was delivered using a three part instructional sequence. In part one, a specific problem was presented to students to complete. In part two, students worked on the problem given during part one for a period of time on their own, in pairs, and then as a group. Questioning strategies were utilized by the teacher to ensure student understanding of the problem situation and solution. Finally, part three contained further questioning strategies from the teacher to the students that encouraged student reflection on the reasonableness of their solutions. The teacher utilized scripts to deliver the intervention each day. The teaching sequence was repeated for each problem in the lesson.

The first instructional session asked students to consider an amount iterated a number of times (e.g. Mauricio ordered five pieces of bacon; Nicosha ordered twice as much; Katy ordered three times as much, etc). The activity forced students to consider a double count (five to one, ten to two, fifteen to three, etc). The act of double counting set the stage for the understanding of ratios (Cortina & Zuniga, 2008).

In the second instructional session, students worked with scenarios involving relationships between cans of pancake batter and the corresponding amount of pancakes made. For instance, students could be given a scenario where one can of batter makes four pancakes. Considering the relationship, students were provided pictorial representations of a certain numbers of cans (i.e. six) and a certain amount of pancakes (i.e. 20). Next, students were asked to discuss whether the amount of pancakes shown were too few, just enough, or too many for the amount of cans (Lamon, 1993a; 2005). Students were instructed to draw pictures or use the supplied manipulatives to aid in their reasoning. The exercises and teacher questioning were designed to aid in students understanding that the relationship between cans and pancakes needed could not change when additional cans or a number of pancakes were added to a situation (Streefland, 1993).

Throughout the third and fourth instructional sessions, students worked with ten given relationships of cans and pancakes (for instance, one can makes four pancakes). From the given relation, students were asked to find missing values given certain numbers of cans or an "order" for a certain number of pancakes (i.e. given 2 batter cans make seven pancakes, how many cans of batter are needed to make 28 pancakes). As students described their thinking and illustrated through picture iterations of the unit relationship, the teacher constructed ratio tables to augment understanding (see Figure 2). In the fourth session, students were instructed to use the ratio table

exclusively to find answers to problems posed. Student understanding was aided by the use of ratio tables (see Figure 2), finding unit rates, and iterating linked quantities to comprise equivalent situations (Fosnot & Dolk, 2002; Lamon, 1993a; Streefland, 1993).

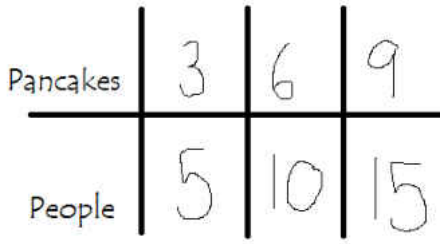


Figure 2. ***Ratio table.***

In the fifth instructional session, students were given a sample problem and several fictitious "responses" that contained drawings, ratio tables, and "shortened" ratio tables (i.e. $\frac{1}{2} = \frac{2}{4}$) displaying multiplicative between relations (e.g. between-multiplicative relations refers to the relation between the numerators and the denominators of equivalent fractions) (Van Hille & Baroody, 2002). Students were asked to determine which of the solutions were correct, why they were correct, and why the incorrect solutions were wrong (Griffin, Jitendra, & League, 2009; Grobecker, 1997; 1999; 2000; Jayanthi, Gerstein, & Baker, 2008). Moreover, students were asked to compare solution strategies exhibited as a means to attach an understanding of alternate solution strategies to an already learned strategy (Rittle-Johnson & Star, 2007).

In the final three instructional sessions, students worked with problems involving relationships between cans of pancake batter and the corresponding amount of pancakes made. The given relation changed for each problem posed. Students were directed to use long and short ratio table strategies to solve problems. Eventually, the use of pictures or tables to represent ratios was faded, and typical fraction notation was used. A solidified understanding of

equivalencies using multiplicative strategies rather than additive strategies was a goal for the last three sessions (Ni, 2001).

Fidelity of implementation. To minimize the risk of internal validity errors, fidelity of implementation of the instructional conditions were conducted by two independent observers (Gerstein et al., 2005). A checklist of the critical components of each part of instruction was created in a previous study. During observations ensuring instructional fidelity, observers used their checklists to evaluate that critical instructional components were utilized during the intervention. Percentages for agreement were calculated using point-by-point agreement. Dividing the total number of agreements by the total number on the checklist yielded a percentage of agreement.

Control

In prior months, all third grade students received textbook based instruction in fraction concepts and equivalency (e.g. Envision Mathematics, Grade 3). During the time supplemental ratio instruction took place, students in both the control group and experimental groups received instruction in equivalent fractions in their mathematics classrooms. Lessons taught by their third grade classroom teachers were taken from NCTM's *Illuminations* website. Namely, "Fun with Fractions" (lessons one through five) were used along with "Fun with Pattern Block Fractions" (lessons one through three) were utilized in students' regular mathematics classrooms during the supplemental period of ratio-based instruction. Students in the control group did not receive the ratio-based supplemental instruction.

Data Collection Procedures

Once the control and experimental groups were established, both groups were administered a pretest measuring fraction equivalency performance. Additionally, three students in the experimental group were administered a semi-structured interview to uncover their understanding of equivalency through ratio interpretations. A social validity measure of student satisfaction was administered before and after the intervention. After the pretest was completed, both groups continued to receive classroom instruction in fraction equivalency and the experimental group received the ratio-based intervention. After instruction of the intervention with the experimental group was complete, both groups were given a posttest measure of equivalency performance. A second semi-structured interview to uncover the three students' understanding of fraction equivalency through ratios was also administered.

To test the amount of change in the dependent variables as a result of the independent variable (e.g. intervention), the researcher utilized several parametric tests within a quasi-experimental pretest-intervention-posttest design. The researcher used the Statistical Package for the Social Sciences (SPSS), version 16, for statistical analyses of quantitative data. To analyze research question one, the researcher conducted a factorial MANOVA with repeated measures. Data were disaggregated by subgroups (e.g. students with MLD, struggling students, and students who were not struggling) using post hoc comparison tests to detect differences between and within groups. To analyze research question two, a pre posttest videotaped semi-structured clinical interview was administered to three students in the experimental group. A thematic analyses was conducted to determine themes relating to strategies, levels of multiplicative thinking, and representation usage conveyed by students with MLD, struggling, and typically achieving students.

Independent Variable

The independent variable for research question one was fraction instruction based in the ratio sub construct.

Dependent Variables

The dependent variables for research question one included the scores on pre and posttests of fraction equivalency performance and also on the standardized measure. The study evaluated whether the independent variable caused a change in performance as measured by the dependent variable. Moreover, a pre and post semi structured interviews were used to identify typical and atypical responses to ratio-based equivalency problems.

Limitations

Several limitations associated with this study need to be acknowledged. First, the quasi experimental part of the research design is subject to certain disadvantages- namely, the possibility of attrition of subjects as well as the possibility of fatigue, carry over effects, practice, or latency. Although counterbalancing can control for fatigue, practice, and carry over effects, the order in which treatment is delivered was not possible given the design of the intervention.

Second, the researcher provided all of the supplemental ratio-based instruction. While the instructional sessions were checked for fidelity of implementation by two independent observers, the results of the study provide no evidence of the effects of the instructional sequence implemented by other instructors.

Third, assignment of subjects, while random, is only so after students who meet certain criteria were selected. Further, selection was not truly random due to criteria for inclusion in the study. Thus, bias may be present in the selection of subjects.

Another limitation is the criteria used to deem students MLD, struggling, or typically achieving. Although care was used to employ research backed criteria to designate subgroups, the field of MLD has yet to determine a precise definition or validation process for such a designation (Mazzacco, 2006). Quantitative and qualitative differences can be found in studies that use different cutoff criterion scores to designate a group of students as MLD (Murphy et al., 2007). Thus, caution should be used in generalizing findings from this study to all those students deemed as having a MLD.

A final limitation is that the intervention was not tested against other forms of instruction outside of the one used in the textbook curriculum in the control group. The effectiveness of the instructional sequence compared to other noted effective instructional models or varying subconstructs of fractions was not evaluated.

Addressing Threats to Validity

Several possible threats to validity need to be mentioned. First, history and maturation was controlled for in the use of the control group. Second, instrumentation, scores, and observers were standardized throughout the course of the study. Third, subject selection was produced through a randomized sample of students meeting study criteria. Students were assigned to groups using a matching procedure.

Definition of Terms

Conceptual Ratio-based Fraction Intervention

An intervention sequence that teaches fraction equivalency through the following sequence: (1) concrete ratio-based partitioning exercises, (2) representational ratio-based unitizing exercises, (3) representational equivalency exercises utilizing additive strategies, (4)

representational equivalency exercises utilizing multiplicative strategies, and (5) abstract equivalency exercises.

FCAT

The FCAT began in 1998 as part of Florida's overall plan to increase student achievement by implementing higher standards. The FCAT, administered to students in Grades 3-11, consists of criterion-referenced tests (CRT) in mathematics, reading, science, and writing, which measure student progress toward meeting the state academic standards and benchmarks (FL DOE, 2009).

Inclusive

Inclusive makes reference to students who are educated primarily in general education content classrooms (IDEA, 2004).

District Confirmed Learning Disability

A disorder in one or more of the basic learning processes involved in understanding or in using language, spoken or written, that may manifest in significant difficulties affecting the ability to listen, speak, read, write, spell, or do mathematics. Associated conditions may include, but are not limited to, dyslexia, dyscalculia, dysgraphia, or developmental aphasia. A specific LD does not include learning problems that are primarily the result of a visual, hearing, motor, intellectual, or emotional/behavioral disability, limited English proficiency, or environmental, cultural, or economic factors (FLDOE, 2009).

Mathematics LD

Defines the student as falling below the tenth percentile in two out of three measures of mathematics proficiency (Murphy et al., 2007).

Part-Whole Interpretation

The understanding of a fraction as one or more equal partitions of a unit when compared to "the total number of equal portions into which the unit was divided" (Kieran, 1980; Lamon, 2005, pp. 60).

Ratio Interpretation

The understanding of a fraction as a comparison of any two quantities to one another; sets of numbers signified as a/b; where a can be but is not always part of b (Kieran, 1978; Marshall, 1993).

Struggling Student

Defines the student as falling between the 11^{th} and 25^{th} percentiles in two out of three measure of mathematics proficiency (Murphy et al., 2007).

Conclusion

Students with MLD experience significant difficulties in understanding fraction concepts centered on equivalency (Hecht et al., 2006; Mazzacco & Devlin, 2008). Results of the study add to the literature by exposing if students with MLD and struggling students performed better on tests of fraction equivalency after engaging in instruction based in the ratio sub construct. Increases in strategy use and multiplicative thinking that lead to understanding of equivalency were also assessed using qualitative analyses.

CHAPTER TWO: LITERATURE REVIEW

Introduction to the Problem

To further and more fully understand the issues surrounding the disparities in learning fractions for students with mathematics learning disabilities (MLD), a thorough review of literature is necessary. The chapter begins with a historical account of the field of study for students with LD and the limited focus on mathematics. A synopsis of notable research on students with MLD undertaken in the last two decades is presented. The next section is devoted to a discussion of the evolution of learning fraction concepts and potential issues for students with MLD. Finally, a discussion and critique of relevant studies in fraction concepts for students with MLD is presented.

The Emergence of the Field of LD

Early Growth of LD

Although much of the legislation and call for increased educational services for students with LD began in the 1960's and 1970's, the origins of the field can be traced to a much earlier time. From as early as the 1800s, European doctors worked to explore and understand people with LD through brain disorders, aphasia, and the inability to read (Broadbent, 1872; Hallahan & Mercer, 2001; Kussamaul, 1877). Physicians such as Gall, Broadbent, Kussamaul, and Ball illustrated their theories regarding the loss of reading ability through notions of varying degrees of aphasia, specifically located brain lesions, word blindness (e.g. the inability to read although a person is of normal intelligence), and eventually, spoken and written language problems resulting from stroke. The copious amounts of work done in the field in the 1800's were "among the first to make the connection between reading problems and brain dysfunction in the context of language" (Anderson & Meier-Hedde, 2001, pp. 13). As work continued in the field tying

causes of brain dysfunction to reading problems, word blindness was further distinguished and differentiated through ideas that motor or visual impairment could be connected with the disorder (Hinshelwood, 1895).

Much of the work in the field up until this point had been with adults, but over the next decades the study of LD in reading extended to students. Hinshelwood (1915) differentiated between acquired and congenital word blindness and was among the first to speculate about developmental reading problems. He proposed that interventions could be used to remediate students who could not read due to disability. Results of research efforts of the time suggested adult word blindness was caused by cerebral lesions while child word blindness was caused by underdevelopment in specific areas of the cortex (1915).

In the midst of the substantial amount of prior work established in reading and LD, the idea of a disability related to mathematics began to emerge. Two physicians, Lewandowsky and Stadelmann (1908), hypothesized about the loss of mathematics ability. They believed the ability could be impaired due to lesions in the left hemisphere of the brain from work with patients who had suffered varying traumas to the left side of the head. Since the patients did not show losses in reading facility or other abilities, the physicians concluded mathematics ability must be separate from reading and overall cognitive ability.

A decade later, Peritiz (1918) proposed the notion of a calculation center in the brain. Around the same time, Henschen (1920) proposed the term “acalculia”, or an acquired disability of mathematics due to his case study work involving students with severe aphasia in the parietal lobe of the brain with intact language abilities. It was not clear at the time whether acalculia was

caused by an inability to understand mathematics or a reading/language issue. The interest in acalculia became overshadowed by the field's emphasis on reading and language.

The Move from Cause to Treatment

In the United States, the foundational period for work in the field of LD took place beginning in the early 1900's through the 1920's and focused on language, perceptual, perceptual-motor, and reading-related disabilities (Fernald & Keller, 1921; Kirk, 1933, 1935, 1936; Monroe, 1928; Orton, 1925; Strauss, 1943; Strauss & Werner, 1943; Werner & Strauss, 1939, 1940, 1941). Remediation work began to surface in addition to causal studies.

Fernald and Keller (1921) described a multisensory approach to reading remediation involving students building from syllabic pronunciations, to full words, to sentences. Students traced words and syllables on paper and then copied the words from memory. From singular words, students eventually worked up to phrases and paragraphs, speaking on the meaning of what they wrote. Fernald and Keller found rote telling of words to students as ineffective. They note:

> at this stage of his development [referring to a student], after he had once written a word, he would almost invariably recognize it on successive presentations. Yet, on the other hand, if told a word over and over again on successive days, he failed to recognize it unless he wrote it (p. 58).

Some students learned spontaneously to associate pictures and drawings with the words. Fernald and Keller's work laid the foundation for the value of kinesthetic approaches in remediating students with LD.

Samuel Orton's (1925) report on students with LD offered new perspectives on the origin of the disability category. Orton believed that many disabilities involving reading could be overcome with special training, and suggested that psychometric testing instruments were inadequate measures of a child's intelligence. He hypothesized that a lack of dominance between the hemispheres or cortical zones of the brain was responsible for word blindness. Orton argued that when letters are learned and visually perceived by students, their concrete images are recognized as both forward and backward in recognition in the right and left hemisphere. If the dominant brain hemisphere does not agree with the associated abstraction, confusion and inability to read results. Orton calls his version of reading disability *Strephosymbolia*, or twisted symbols. Orton's work offered the word "disability" in place of "defective", and suggests phonetic training and symbol recognition for remediation.

Monroe (1928) furthered Orton's suggestions and constructed diagnostic and remediation procedures in reading. Remediation was given from results of administered tests related to LD in reading. Monroe's remediation techniques included aspects of kinesthetic tracing and phonetic methods suggested in earlier research. Her later research emphasized the phonetic aspects of remediation in reading and advocated for kinesthetic tracing only when deemed necessary (Monroe, 1932). In her approach, Monroe combined the use of visual pictures, stories, and tracing methods to promote students' ability to identify and sound out consonants, combine letter sounds in reading, and associate a kinesthetic movement with a letter (Monroe, 1932). Her work was significant in that her diagnostic methods were reminiscent of later ideas pertaining to IQ-discrepancy. She used this diagnostic information directly in her remediation efforts; others would later further the ideas Monroe implemented (e.g. Kirk).

At this point, multisensory methods were gaining support, but a large amount of research previously conducted was not validated. Kirk (1933) dealt with validating earlier kinesthetic methods introduced by earlier researchers (e.g. Monroe; Fernald). He tested two interventions in reading- the sight method and the kinesthetic method- against each other and examined the effects. The sight method consisted of rote, direct instruction where a teacher showed a student a word, told the student what the word was, and then asked the child to repeat it (look at word, hear the word, say the word). In the kinesthetic application, all of the above occurred, but students also traced the word with a dull pencil. Results showed the manual tracing methods was superior to the traditional sight method for sustained retention of reading material for students with LD.

Perceptual and motor diagnoses and remedial approaches also emerged during this period, largely from the work of Strauss and Werner (Strauss, 1943; Werner & Strauss, 1939, 1940, 1941). Based on Strauss' work with individuals with mental retardation, he contended that students with LD did not suffer from mental retardation, hearing impairment, or emotional disorder and must have some form of minimal brain damage (Werner & Strauss, 1939). He identified the "Strauss Syndrome", an identification of behaviors related to students with LD (e.g. distractibility and problems with perseverance) (Struass, 1943; Werner & Strauss, 1940). Strauss also contended that students with LD fail to differentiate the background of an image from a figure (e.g. only seeing a circle and not partitions that it is cut into) (Werner & Strauss, 1941). The problem persisted even when students were presented with concrete materials (Werner & Strauss, 1941).

In later studies, Strauss and Werner (1943) discovered that students with LD tended to make unimportant or erroneous associations between stimuli, often adding "fanciful elements

which go far beyond the content of the pictured situation" (p. 166). Interestingly, the researchers claimed that students with LD suffer from dissociation, or the inability to integrate elements into a whole or comprehensive picture. "Since he [referring to a student] cannot comprehend the pattern as a whole, the results are frequently disorganized forms [that lack] connection" (p. 169). Dissociation transcends visual, auditory, and tactile representations – students fixate on certain parts of the picture which leads them to relate objects incorrectly. From their efforts, the researchers emphasized providing a distraction free environment during learning and to remediate perceptual differences in students with LD (Struass, 1943).

Kirk and Bateman (1962) described a process in diagnosing reading and perceptual/motor LD that included determining a child's capacity for reading. Their work was an attempt to culminate previous intervention studies as well as to provide a platform linking diagnosis and remediation. The approach encompassed three steps: 1) an examination of a child's approach to reading, 2) a diagnosis of a child's disabilities to determine why he or she could not learn from instruction, and 3) a recommendation for remediation of the difficulties caused by the disability. Kirk, McCarthy, and Kirk (1968) developed the Illinois Test of Psychological Abilities (IPTA) to facilitate this approach. The IPTA is rooted in language usage and perceptual/motor issues, encompassing all areas thought to 'define' LD (Kirk et al., 1968). The test consisted of 12 subtests, each testing for a specific deficit in channels of communication, psycholinguistic processes, or levels of organization. The use of the test to guide instruction and remediating areas of deficit for students with LD was very influential throughout the 1960s, and brought attention to the nuances evident in this population (Bateman, 1965; Kirk et al., 1968).

Bateman (1965) integrated earlier notions of discrepancy between performance and intelligence in the diagnostic and remediation model. The group of students designated as having

LD is extremely diverse; Bateman claimed the only commonality within the group is the discrepancy between potential and performance. Bateman (1965) suggested a five stage plan for diagnosis and treatment for students with LD: (1) confirm an IQ-performance discrepancy [at least 1 ½ years behind for younger students; two for older students]; (2) conduct behavior analyses – a description of what performance in academics is faulty [and more importantly how the child goes about performing the academic skill- strategies used- and error analyses]; (3) use the ITPA [really consists of analyses of soft signs of brain based problems, motor awkwardness, spatial issues, etc, as well as specific educational problems]; (4) produce a summary and hypothesis; and (5) remediate by focusing on the deficiency.

Remediation and Mathematics

From over two centuries of investigation, intervention, case study, and clinical discovery, a field of study emerged for students with LD. Difficulties with reading, language, and perceptual motor disabilities became tantamount with early ideas of LD, and the field was dominated by research centered on such elements. Elegant systems of diagnosis and remediation were proposed that took into account almost all elements that were proposed as being involved with LD.

Although ideas of mathematics related disabilities were hypothesized elsewhere, the study of individuals with disabilities in learning mathematics in the United States was overshadowed by the basis of the field in reading (Woodward, 2004). The idea that much of the research in LD related to individuals whose mathematics ability was intact and reading abilities in various forms compromised (Orton, 1925), provided for a narrow focus on arithmetic (Fernald, 1928; Kirk & Bateman, 1962). Interestingly, students with MLD were virtually absent from all forms of diagnosis and remediation until after the 1960's.

Definitions and Dominance: What was Valued in LD?

As the work in the field grew, so did the need to formally define the notion of LD. From 1963 to 1968, as many as five different definitions were proposed from various stakeholders for LD. In 1963, Kirk gave the first definition of LD and is credited with starting the field with his definition. Kirk defined LD as:

> A retardation, disorder, or delayed development in one or more of the processes of speech, language, reading, writing, arithmetic, or other school subjects resulting from a psychological handicap caused by a possible cerebral dysfunction and/or emotional or behavioral disturbances. It is not the result of mental retardation, sensory deprivation, or cultural or instructional factors (p. 73).

This definition would influence subsequent definitions adopted in the field in the coming years with four additional definitions of LD emerging. The last of the four gained the greatest momentum as a national definition was adopted. In 1968, the National Advisory Committee Definition on Handicapped Students offered a definition for LD to be used for funding federal programs. The definition, rooted in comprehension of spoken and written language, states:

> Students with LD exhibit a disorder in one or more of the basic psychological processes involved in understanding or in using spoken and written language. These may be manifested in disorders of listening, thinking, talking, reading, writing, spelling or arithmetic. They include conditions which have been referred to as perceptual handicaps, brain injury, minimal brain dysfunction, dyslexia, developmental aphasia, etc. (United States Office of Education, 1968, p. 34).

Of particular importance to the field was the formal adoption of this definition contained in the Students with Specific LD Act of 1969. The Act allowed for separate classification for students with LD within special education as well as further funding at the state level to provide educational services (Hammill, 1993; Hallahan & Mercer, 2001). Therefore, the notion of LD as a reading disability was now recognized and solidified by law.

The Education of All Handicapped Students Act of 1975 "characterized an American penchant for attending to individual differences in educational settings" (Woodward, 2004, p. 19). The Act provided funding for states from the federal government to serve students with disabilities. These funds were targeted to provide services to students with disabilities, establish due process rights, and provide free and appropriate education for all students with disabilities, including LD, within the least restrictive environments (Education of All Handicapped Students Act, 1975). Definitions of LD contained in the law mirrored those proposed in 1968, and the focus of LD based in spoken and written language continued (Education of All Handicapped Students Act, 1975). Mathematics LD continued to be "a nascent concept" (Woodward, 2004 p. 35).

Definitions and Mathematics

A year earlier, however, the notion of a mathematics LD resurfaced. Kosc (1974) was the first to introduce the term "developmental dyscalculia", which defined mathematics disabilities as genetic, or due to heredity, as opposed to one acquired through trauma or due to a lack of intelligence. Kosc identified six types of dyscalculia:

- Verbal dyscalculia, or a disturbed or inability to name amounts and numbers of things, digits, numerals, operational symbols, and mathematical performances. What Kosc describes here is partly a problem with subitizing.

- Practognostic dyscalculia, or the inability to manipulate real or pictured objects. It is also said to include disturbances in estimation skills and ability to compare quantities.
- Lexical dyscalculia, or the inability to read and understanding mathematical symbols. It is often associated with problems with reading numbers horizontally, interchanging similar digits, or reversing digits.
- Graphical dyscalculia, or the inability to write numbers. It is often associated with dyslexia.
- Ideognostical dyscalculia, or the inability to understand mathematical ideas and relations; highly related to number sense. For instance, he knows that 9 = nine, but he does not know that 9 or nine is one less than 10, or 3 x 3, or one-half of 18
- Operational dyscalculia, or the inability to carry out mathematical operations (p. 167-168)

Despite the offered definition, the issue of a MLD was scantly defined in law, and continued to be scarcely acknowledged in the field outside of problems with basic arithmetic (Education for All Handicapped Act, 1975; United States Office of Education, 1977).

Understanding MLD

Through examining history, a constant pattern is evident throughout all of the changes in definitions, service delivery, and trends -- mathematics was not a central focus (Fernald & Keller, 1928; Kirk, 1933; Kirk & Bateman, 1962; Strauss, 1943; Werner & Strauss, 1939, 1940, 1941). As a result, special education teachers were prepared to know LD through reading, but had limited knowledge of LD through mathematics, a fact that largely remains the case today (Brownell et al., 2010; Maccini & Gagnon, 2006; Rosas & Campbell, 2010). For example, Maccini and Gagnon (2002) surveyed 129 general and special educators to understand their

perceptions and teaching practices that fell in line with the 2000 National Council of Teachers of Mathematics (NCTM) *Principles and Standards for School Mathematics*. Only 41% of teachers indicated they were familiar with the Standards. In 2006, Maccini and Gagnon conducted a related study and found knowledge of mathematics topics predicted quality instructional practices of special education teachers. Special educators with less mathematics preparation tended to focus on a more shallow instruction of mathematics.

Most recently, the results of Rosas and Campbell's (2010) descriptive study illustrated the limited experiences in mathematics possessed by many special education teachers. Through survey research, mathematics tests, and interviews regarding teacher knowledge of mathematics and subsequent practice in the field, researchers discovered that many special education teachers had poor preparation in mathematics content and teaching practices. Although the teachers rated their own ability as high, their preparation and practice did not correlate. The lack of awareness regarding mathematics standards, content, and teaching has implications for the quality of services students with LD receive in mathematics (National Mathematics Advisory Panel, 2008; Rosas & Campbell, 2010). Brownell and her colleagues (2010) assert:

> to assist students with disabilities, teachers should understand mathematical concepts and relationships among them and how procedural knowledge can support conceptual knowledge. Otherwise, they cannot diagnose how student understanding is breaking down and respond with the more intensive, carefully articulated math instruction that students with disabilities need (p. 368).

Greer and Meyen (2009) agreed "current preparation standards and practices may be insufficient for preparing special education teachers to effectively meet the academic needs of students with

LD in content areas and thus ensure that those students are not disadvantaged in meeting accountability mandates resulting from NCLB (2001) and IDEA (2004)" (p. 196-197). Title I of the ESEA requires each state to assess students in mathematics annually in grades 3-8, and at least once more between grades 10 – 12. The same academic assessments are given to all students, including students with LD. These assessments also must be aligned with the state's academic achievement standards, and assess higher-order thinking skills and understanding. The law holds states accountable for the continuous and substantial improvement of students with LD in mathematics. Despite the lack of focus on mathematics historically in the field of special education, teacher preparation and instructional techniques must begin to address this lost group of students – those with a MLD. Thus, the historical lack of focus on the development and remediation of a LD in the content area of mathematics has left the field with many unanswered questions today.

Fractions: An Unanswered Question

Why is a clear focus on students with MLD critical for preparation of today's teachers and in interventions for this population of students? Research indicated mathematical performance of a 17-year-old student with LD leveled off around fifth grade; with the disparities beginning around first grade and continuing to grow over time (Cawley & Miller, 1989). Simultaneously, the number of students with identified LD has grown. In fact, from 1995 through 2004, the percentage of students aged 12 through 17 receiving special education and related services increased from 6.1 percent to 6.8 percent (US Department of Education, 2009), and it is estimated that 5 to 7 percent of the school aged population experiences mathematics-related disabilities (Geary, 2009).

On standardized tests of mathematics, students with LD consistently score lower than their peers without disabilities across all grade levels. The Nation's Report Card: Mathematics 2009 reported 19% of fourth-graders with disabilities scored "at or above proficient", in comparison to 41% of fourth graders without disabilities. In eighth grade, only 9% of students with disabilities scored at or above proficient, as compared to 35% of their counterparts (National Center for Education Statistics, 2009). These numbers are largely students who have LD, as 85% of fourth graders and 75% of eighth graders with LD comprise the population of "students with disabilities" in standardized tests of mathematics achievement such as the National Assessment of Educational Progress. Further, reports suggest that their achievement levels average around 41% for fourth grade students and even lower for eighth grade students (Kitmitto & Bandeira de Mello, 2008). A disparity exists between students with LD and students without disabilities in mathematics achievement.

Beyond just a struggle in the area of mathematics in general, a target area where many students with MLD are not successful is fractions (Grobecker, 2000; Hecht, 1998; Hecht et al., 2003; Hecht et al., 2006; Mazzacco & Devlin, 2008; National Center for Educational Statistics, 2009). The 2009 U.S. National Assessment of Educational Progress (NAEP) authors reported only 50% of fourth graders could correctly identify how many fractional parts (e.g. fourths) comprised a whole; in 2005, the percentage was 53%. NAEP authors showed only 49% of eighth graders correctly identified fractions given in ascending order in 2007; in fourth grade, 64% and 41% of students were able to generate equivalent fractions and compare unit fractions to solve a problem, respectively. In 2007, less than half of the eighth grade students tested, correctly added fractions with different denominators (National Center for Educational Statistics, 2009). These data reflect the general population; typically, students with MLD would exhibit

even larger discrepancies because of their disability-related strengths and weaknesses (Kitmitto et al., 2005). The question remains - why?

Mathematics Learning Disability

What are the strengths and weaknesses of students with MLD, and how could they affect understanding of fractions for student with MLD? The evolution of knowledge regarding MLD has both similarities and differences compared to the evolution of knowledge of reading learning disability. The origins of research in the field of MLD can be found largely in the early 1900s into the mid and late 20th century, where physicians and psychologists proposed varying causes and types of MLD. As with reading LD, physicians (e.g. Lewandowsky & Stadelmann, 1908) asserted that MLD came from aphasia, or trauma to the left side of the head, although the idea was later dispelled (Gerstein, Clarke, & Mazzacco, 2006). Also like reading LD, MLD was thought to be an acquired disability (Henschen, 1919).

Presently, understanding of MLD is still in its infancy. In contrast to the field of reading LD, disagreements exist regarding factors that are valid diagnostic indicators or characteristics of MLD (Geary, 2009), because no test or operational consensus exists to identify a person as having MLD. Researchers use varying criteria to identify students as MLD in intervention research (Mazzacco, 2006). As a result, reported profiles of what constitutes MLD have been found to vary as a function of cut off scores or other definitive criteria used (Murphy et al., 2007). Additionally, much research in the field of MLD fails to differentiate between students with MLD and students who struggle in mathematics but do not have a disability (Mazzacco, 2006). However, a distinction in performance and thought processes in the two groups does exist (Mazzacco & Devlin, 2008). The summation of the above factors results in a field still immature in regards to how to best teach students with MLD mathematics content.

Nevertheless, research conducted over the past two decades suggests deficits and cognitive correlates effect outcomes for students with MLD. The deficits include semantic memory or language (Geary, 1993; Loosbroek, Dirhx, Hulstijn, & Janssen, 2008; Rouselle & Noelle, 2007), sense of number (Butterworth, 1999; Hecht & Vagi, 2010; Landerl, Bevan, & Butterworth, 2003), working memory (Geary, Bailey, & Hoard, 2009) and nonverbal or fluid reasoning (Jordon, Kaplan, & Hannich, 2003). However, "students may differ in the severity of one type of deficit or another; and students may differ in the developmental course of the deficits" (Geary, Bailey, & Hoard, 2009, p. 46). Each issue has been identified through the work of several notable researchers.

Semantic Memory

Semantic memory is a proposed deficit found amongst students with MLD. It is defined as a difficulty with retrieving basic arithmetic facts from long term semantic (language based) memory (Geary, 1993). Many researchers believe semantic memory problems to be the primary deficit of students with MLD, while others believe MLD to be based in a deficient sense of number. Baroody, Bajwa, and Eiland (2009) state:

> According to conventional wisdom (the Passive Storage View), memorizing a basic fact is a simple form of learning—merely forming and strengthening an association between an expression and its answer. The two primary reasons this simple form of learning does not occur are inadequate practice or, in cases where adequate practice has been provided, a defect in the learner (p. 74).

Geary (1993) was the first to identify semantic memory difficulties as a possible deficit of students with MLD. Much of his research at the time involved notions of deficits in the retrieval

of arithmetic facts from long-term semantic memory, in the execution of procedures for solving arithmetic problems, and in the ability to represent and interpret visuospatial representations mathematical information. Many researchers have extended his initial efforts over the decades.

A notable study that aligned with Geary's early work was conducted by Rousselle and Noelle (2007). They investigated whether students with MLD have difficulty in processing numerosities or in accessing number meaning from symbols. Forty-five students with reading and mathematics LD and MLD were compared to 41 students without disailities in tasks assessing basic numerical skills. The tests used in the study measured a student's understanding of symbolic quantity comparison and comparison problems that did not require symbolic processing. The researchers found students with MLD performed worse on the task requiring semantic memory (e.g. understanding of symbolic quantity) than on the task not requiring semantic processing. Their findings contradicted the defective number module hypothesis which implies students with MLD should be impaired in all tasks requiring them to process number magnitude. Researchers concluded students with MLD may not have issues processing quantities as a result of a deficient sense of number but because of impairment "accessing semantic information conveyed by numerical symbols" (p. 377).

Another study aligning with Geary's recommendations in 1993 was conducted by Loosbroek, Dirhx, Hulstijn, and Janssen in 2008. These researchers studied how students with and without MLD wrote numbers after hearing them as number words. The researchers timed the conversion of spoken number words in students with low and average mathematics ability to understand if the association implicated semantic or non-semantic processing. Slower processing times were recording for the low ability group, which indicated an increasing processing time. Thus, the researchers interpreted the results as evidence of the difficulties involving semantic

properties of size and order of numbers for the low ability group. Difficulties were classified as a delay for students with MLD, and researchers argued the delay could be accounted for by an incorrect linkage of words and representations of numbers on the mental number line. Other researchers argue MLD comes not from a deficient semantic memory but from a deficient sense of number.

Sense of Number

Sense of number in the early years is defined as the understanding of exact quantity of small collections of actions or objects, the symbols that represent them (e.g., '3'= * * *) and their approximate magnitude, while notions of one-to-one correspondence remain intact (Geary, 2009). Many researchers believe sense of number, and not semantic memory, to be the root of MLD deficits. Baroody and his team explain:

> According to the number sense perspective (Active Construction View), memorizing the basic combinations entails constructing a well-structured or -connected body of knowledge that involves patterns, relations, algebraic rules, and automatic reasoning processes, as well as facts. In effect, fluency with the basic number combinations begins with and grows out of number sense…. The primary cause of problems with the basic combinations, especially among students at risk for or already experiencing learning difficulties, is the lack of opportunity to develop number sense during the preschool and early school years (p.71).

Growing amounts of research are being conducted to support sense of number as a primary deficit in MLD. Shalev, Manor and Gross-Tsur (1997) studied students with co-morbid LD and reading based LD along with students with only MLD on tasks assessing subtraction and

division knowledge, number processing, and general intelligence. They found no evidence for dissociation between the two groups in language based numerical processing. The researchers claimed their findings refuted the notion that MLD stems from deficits in semantic memory processing.

Cappelletti, Butterworth and Kopelman (2001) conducted neuropsychological studies exploring the association between the ability to understand numbers and proficiency in calculation and semantic memory (language). The researchers studied calculation and language abilities of patients who experienced damage to the temporal lobe of the brain (responsible for language). These patients had intact calculation and advanced mathematics skills but severely impaired semantic memory. Results of the studies found no evidence linking the mathematics abilities to semantic memory, suggesting mathematics ability is dissociable from language. Further, results suggested the existence of a number module (an area of the brain that exclusively deals with number representations) in the brain separate from the area of the brain where language is processed. The results confirmed case study work conducted by Henschen (1920), who found severe mathematics impairment in patients with damaged parietal lobes but no disruption in language skills. The researchers argued MLD results from a dysfunction of the proposed number module (parietal lobe of the brain). Other research has confirmed this 'number module' in the brain to be responsible for enumeration and subitizing (Pinel, Dehaene, Rivie`re, & Le Bihan, 2001; Tang, Critchley, Glaser, Dolan, & Butterworth, 2006) as well as comparison of numbers (Castelli, Glaser, & Butterworth, 2006; Piazza, Mechelli, Price, & Butterworth, 2006).

Four years later, Landerl, Bevan, and Butterworth (2003) studied mental arithmetic, number comparison (both magnitude and physical size), number writing (writing spoken

numbers), verbal counting, and dot counting (subitizing) among students with MLD and reading LD. They found students with MLD demonstrated deficits in number comparison, subitizing, verbal counting and number writing and writing numbers. Researchers concluded the key deficit in MLD is the inability to represent and process numerosities, and suggested qualitative differences exist in understood meanings of numerical expressions for students with MLD. Landerl and his team explained:

> We suggest that lack of understanding of numerosity, and a poor capacity to recognize and discriminate small numerosities… prevent dyscalculics [from] developing the normal meanings for numerical expressions and lead to their difficulties in learning and retaining information regarding numbers (p. 120).

Most recently, Hecht and Vagi (2010) examined performance differences in fraction computation, estimation, and word problems among 181 elementary school students with and without MLD. They found group differences on all measures (MLD performed significantly worse). The researchers also examined which factors (working memory, classroom attentive behavior, simple arithmetic efficiency, or sense of fraction numbers) were important intermediaries of group differences in performance. Importantly, the researchers found that sense of fraction numbers and attentive behavior were consistent mediators of ability group differences in emerging fraction computation, word problems, and estimation skills. The researchers did not find any evidence that working memory or basic fact knowledge mediated group differences in fraction performance. Other researchers, however, do believe that working memory is an important source of the differences in mathematics achievement in students with MLD.

Working Memory

Working memory is a cognitive component often associated with MLD. It is defined as the coexistence of a central executive, visuospatial sketchpad, and a phonological buffer in the brain responsible for the holding of information in the mind (Geary, 1993). Working memory is often assessed in mathematics by tasks involving the verbal recall of sequences of numbers (Geary, 2004). Working memory is suggested to be a cognitive correlate involved with MLD.

Geary (2004) hypothesized that MLD would present itself as conceptual or procedural mathematics deficiencies due to deficits in working memory areas of the language or visuospatial domains of the brain. Research conducted by Geary, Hoard, Byrd-Craven, Nugent, and Numtee (2007) investigated the idea further. Students with MLD, low achievers, and typically achieving students took tests of mathematics cognition (e.g. counting, subitizing, number line estimation), working memory, and speed of processing. The MLD group showed deficits across all mathematics cognition tasks, many of which authors argued were mediated by working memory or speed of processing. For instance, students with MLD had more errors in detecting double counts during counting tasks (e.g. the students missed the fact that the research counted "2, 3, 3, 4…" instead of "2, 3, 4…"). Working memory was found to mediate the unnoticed counting errors for these students. Researchers concluded working memory deficits influence MLD, but not always in straightforward ways.

Two years later, Geary, Bailey, and Hoard (2009) sought to further define core deficits involved with MLD as well as underlying cognitive structures involved with the deficits. They administered a working memory assessment battery to 200 students as part of a longitudinal project with 200 K-9th graders. Results revealed students with MLD performed below average on working memory, number processing and representation, arithmetic procedures, and recall of

arithmetic facts. Researchers concluded that students with MLD have "broad working memory deficits and specific deficits in their sense of number that delay their learning of formal mathematics" (p.274). Geary suggested working memory issues are what separate students with MLD from students who struggle in mathematics. Other research suggests the cognitive component associated with MLD is not working memory but in fact nonverbal reasoning.

Nonverbal Reasoning

Nonverbal reasoning is defined as a fluid association between concrete number representations and abstract number representations (Gregg, 2010). It is also explained as the ability to make inferences from the act of drawing, as this skill represents two-thirds of the variance in students who possess high levels of nonverbal reasoning and those who do not (Gregg). Jordan, Kaplan, and Hannich (2003) compared students who showed poor mastery of basic facts at the end of their third grade year to third graders who had mastered their basic facts during a two year longitudinal study. Their work sought to identify if mastery deficiencies were due to weaknesses in verbal processes, nonverbal reasoning, or general intelligence. Tests measuring retrieval of number facts, calculation of addition combinations, success with word problems, and overall mathematics and reading achievement were administered four times over the two year study. The researchers' findings revealed no apparent link between deficits in fact mastery and word-level reading. They concluded that number facts are not primarily encoded in terms of their phonemic features. The conclusion was in conflict with ideas about MLD deficits originally proposed by Geary (1993). Instead, Jordan and her team proposed weaknesses in mathematics evidenced in students with MLD was a result of weaknesses in nonverbal reasoning.

A summary of the deficits and cognitive factors thought to be involved with MLD is provided in Table 1. Much research has been conducted over the past two decades concerning the deficits students with MLD experience that can disrupt mathematics learning. Links between number symbols and their meaning, the inability to subitize (a precursor to partitioning (Lamon, 1993a)), the inability to hold information in the mind, and the inability to link abstract and concrete mathematical ideas have all been offered as deficits among students with MLD. However, the ways in which named strengths and weaknesses involved with having a MLD surface as a student learns about *fractions* is not well understood, as most research in MLD has focused on primary students or basic skills (Hannich et al., 2007). Thus, the complex nature of MLD along with the infancy of knowledge relating MLD to fraction learning makes designing effective instruction for this population complex.

Table 1. ***Summary of MLD Deficits and Cognitive Components*****.**

Deficit or Cognitive Component	Disruptions in Mathematics Learning
Semantic Memory	Knowing what number symbols mean or represent. A linkage of spoken words to what numbers they represent.
Sense of Number	An inability to recognize and compare quantities. A lack of understanding numbers as quantities. The inability to subitize. Correspondence intact.
Working Memory	An inability to hold information in the mind short term. Effects on mathematical learning may not be direct.
Nonverbal Reasoning	An inability to abstract ideas presented concretely.

Designing Fraction Instruction: Students with MLD, Content, and Instructional Strategy

Adding a further layer of complexity is *how* fractions are taught to students with respect to important instructional aspects such as (1) content and (2) instructional strategy. The mathematics education literature reveals several critical elements of *content* used to promote understanding in the area of fraction concepts. Namely, four main concepts have been noted in the literature to promote depth of knowledge and understanding about fraction concepts: partitioning (Empson, 2005; Lamon, 2005; Memede & Nunes, 2008; Memede, Nunes, & Bryant, 2005; Nunes & Bryant, 1998; 2007; 2008; Streefland, 1991, 1993, 1997); unitizing (Cortina & Zuniga, 2008; Lamon, 2005; Streefland, 1991, 1993, 1997); equivalency (Fosnot & Dolk, 2002; Kamii & Clark, 1995; Lamon, 2005; Streefland, 1991, 1993, 1997); and multiplicative thinking (Lamon, 1993b; Vanhille & Baroody, 2002; Vergnaud, 1983). Yet most of the research conducted on these processes has not taken into account students with MLD and their inherent strengths and weaknesses.

Similarly, the special education literature provides several notable instructional strategies deemed as effective in *instructional strategies* used to teach mathematics content to students with MLD or students who struggle in mathematics. Namely, four main instructional strategies were noted: concrete-representational-abstract instruction (Butler et al., 2003; Jordan et al., 1999); explicit instruction in regards to instructional sequencing, concept formulation, and multiple strategy use (Jitendra et al., 2003); student use of representations to support development of mathematics knowledge (Xin et al., 2007); and verbalizing mathematics reasoning (Woodward et al., 1999). Yet most of the research conducted on these processes has not taken into account what is known about how using varying fraction sub constructs can affect teaching and learning processes of students with MLD.

Teaching Fractions: Content

How students are taught fraction *content* such as partitioning, unitizing, equivalency, and multiplicative reasoning is influenced by the *construct* involved. Fraction instruction may prove beneficial or detrimental to students with MLD if the construct used during instruction does or does not align with students' pre-instructional strengths. Many theories exist regarding fraction constructs and how fraction instruction should begin. Not all of them agree on the ideal fraction sub construct to utilize during instruction (Streefland, 1993; Lamon, 2007). Furthermore, none of the theories take into account how students with MLD may come to understand fractions through the various sub constructs. Nevertheless, an examination of fraction sub construct definitions and their implications for how fraction content is taught can be made. A comparison of the implications against what is currently known about strengths and weaknesses associated with MLD can be completed. Curricular design for students with MLD can then be made.

Fraction sub construct definitions. An examination of fraction sub construct definitions was completed to begin the process of designing fraction content for students with MLD. Kieran (1976) hypothetically identified seven rational number interpretations (e.g. fractions, decimals, ordered pairs, measures, quotients, ratios, and operators). Building on Kieran's ideas Behr, Lesh, Post, and Silver (1983) further hypothesized about the construct of rational number. Combining Kieran's work with research from the Rational Number Project, the researchers concluded that five interpretations- part-whole, ratio, measure, operator, and quotient- "stood the test of time" (p. 298), and thus gave clarity towards initial understanding of the rational number field. Behr's theory of the relationship between the constructs as a student is taught and learns about fractions defines the part-whole sub construct as the primary independent

variable influencing all other sub constructs, suggesting that fraction instruction and student understanding must be built off the part-whole meaning of fractions.

However, other theories of the relationship between rational number constructs have been hypothesized. Kieran (1980, 1988, 1993), the originator of the subconstructs, has suggested numerous times throughout his research that other sub constructs may be used to begin teaching fraction concepts, and that part-whole meanings may not be at the forefront of importance. In fact, Kieran subsumed the part-whole subconstruct in the ratio subconstruct in his earliest writings (Kieran, 1980); then in the measure and quotient subconstructs later on (Rahim & Kieran, 1988); and, in 1993, eliminated the part-whole subconstruct formally. He wrote, "To know and understand rational numbers is to know numbers that are at once quotients and ratios" (Kieran, p. 81). Kieran (1993) goes on to explain "the unit fractions along with quotients and ratio nature form a mathematical base for rational numbers" (p. 81).

Streefland (1991, 1993) also takes a different view from Behr and colleagues regarding how fraction instruction may begin. Acknowledging the importance of partitioning in the acquisition of rational number concepts as proposed by Behr, Streefland argued the context of the partitioning process appears not in the part-whole sub construct but in the ratio sub construct. Further, Streefland stated "the intertwining of ratio and fractions result in a hierarchy different from that proposed by the Rational Number Project by Behr and colleagues: ratio-equivalence-division-part-whole" (Streefland, 1993, p. 302).

Lamon (2007) also spoke of the power of beginning instruction through the ratio construct. After thoroughly investigating how students' understanding of rational number concepts and operations developed through extensive instruction in each of the five sub

constructs – measure, part-whole/unitizing, ratio, operator, and quotient – Lamon concluded students needed a strong base in one construct that played to their "pre-instructional strengths" (p. 659) to develop understanding of unit and equivalence. Like Streefland, her research suggested the ratio construct is an equally powerful starting point for fraction instruction. Moreover, students who were well versed in fractions as ratios easily transferred their knowledge to other constructs like part-whole.

Comparison of sub constructs to MLD strengths/weaknesses. An analyses of how fraction understanding typically grows through different sub constructs was then compared to available knowledge of students with MLD deficits, strengths, and weaknesses regarding instructional interventions or diagnostic interviews that were completed. A review of content literature was completed to facilitate the comparison of typical development teaching trajectories involved with fraction concepts to what is known about students with MLD. Search criteria included the following: (a) the study included students who were in elementary or middle school and included students with MLD or struggling students, (b) the study reported on student thinking and/or performance of equivalency and related concepts as the result of qualitative or quantitative analyses; and (c) the study was published between 1989 and 2010. The following web-based data sources were searched for articles: Educational Resources Information Center (ERIC), Psyc info, Web of Science, and Wilson Omni File Full Text using the key words of fractions, equivalency, LD, math disabled, at-risk, low achieving students, mathematics LD, ratio, partitioning, unitizing, fraction sub construct, quotient, intervention, mathematics instruction, and mathematical learning.

Seven studies meeting the search criteria were uncovered. Studies relating to the teaching and understanding of important fraction concepts (partitioning, unitizing, equivalency,

and multiplicative reasoning) and students with MLD were then classified (e.g. instruction used predominately a part-whole, measure, quotient, or ratio construct), reviewed, and summarized. Overall, results of the literature review indicated the ratio sub construct may be more beneficial than the part-whole sub construct for teaching fractions concepts to students with MLD. The content literature review is presented below to promote further understanding for the reader. In addition, fraction sub constructs and important content are defined and explained throughout the review.

Part-whole. In the part-whole sub construct, one or more equal partitions of a unit are compared to "the total number of equal portions into which the unit was divided" (Lamon, 2005, p. 60). The denominator *b* designates the total parts, while the numerator *a* signifies the number of equal parts taken; the notation takes the form $\frac{a}{b}$ (Behr et al., 1983). For instance, $\frac{2}{4}$ in a part-whole sub construct means that an object was partitioned into four equal parts and two of the parts are being considered. Building knowledge on the part-whole sub construct means that a student has (1) an understanding of the equality of the pieces, (2) the ability to partition a whole into equal sized parts, (3) an understanding that the parts must exhaust the whole, (4) an understanding of the inverse relationship between the number of parts and the size of each part, and (5) the notion that parts can be of equal size even if their shape is not equivalent (Charalambous & Pitta-Pantazi, 2007). Another aspect of the part-whole schema knowledge is visual. Marshall (1993) found to fully understand the part-whole sub construct, visual models for the part-whole situation need to be encoded in memory. One such model is the circular regions model, which is often used to begin instruction in partitioning.

Partitioning/unitizing, part-whole, and MLD. *Partitioning* can be defined as the process of taking an object or set of objects and dividing it equally into a number of equal parts (Empson, 2001; Lamon, 2005). Lamon (2005) believed "the process of partitioning lies at the very heart of rational number understanding" (p. 77) and is foundational for the understanding of fraction language, concepts, computation, equivalency, and multiplicative structures (e.g. multiplicative structures are defined as 'between' relations – relations between numerators and denominators of equivalent fractions – and 'within' relations – relationships between the numerator and denominator of a single fraction) (Empson et al., 2005; Vane Hille & Baroody, 2002). Partitioning is heavily associated with the part-whole sub construct of rational numbers, and it is widely held that partitioning strategies and the part-whole sub construct develop concurrently (Behr et al., 1983; Pothier & Sawada, 1983). Complementary to the notion of partitioning is unitizing. *Unitizing* is defined as "the cognitive assignment of a unit of measurement to a given quantity" (Lamon, 2005, p. 42). An example of what it means to unitize is when the fraction $\frac{3}{4}$ can be thought of as 1 unit of $\frac{3}{4}$, or 3 units of $\frac{1}{4}$. Imagining and reimagining the unit is an essential activity that promotes later understanding of fraction equivalence.

Three studies involving students with MLD or low achieving students and part-whole fractions (partitioning and unitizing) were found in the literature. First, Morris (1995) studied 31 students' who were considered to have low ability to construct meaning of fraction symbols. Her research focused on identifying variables that affected students' ability to link meaning to the fraction symbols as they worked through an instructional sequence containing work with manipulatives, pictorial representations, and abstract fraction symbols. Students were split into an experimental and a control group; the experimental group received the aforementioned

instruction in areas such as partitioning, equivalency, naming, ordering, and operating on fractions. The control group received instruction in the same areas but through abstract methods only. Morris discovered higher reasoning abilities and a greater number of problems solved correctly in the experimental group. Further, through qualitative analyses, she outlined several variables that negatively impacted student understanding, including (1) difficulties with area partitioning; although students used area models to represent problems, they experienced difficulty partitioning equal sized pieces and drawing understanding from partitioned models, and (2) difficulties transitioning from fractions as area models to fractions represented on number lines.

The second study reviewed also involved partitioning of circular and linear part-whole representations. Hecht and his colleagues (2006) studied performance patterns of typically achieving students and students with MLD on tasks involving representing fractions with pictures of partitioned circular regions, naming fractions from pictures of partitioned circular regions, and computing fractions using pictures of partitioned circular regions that follow in instruction. Significant differences between groups were found; students with MLD performed poorly. Poor performance was even more substantial when related to the part-whole sub construct of fractions (as opposed to the measure sub construct). Students with MLD were found to possess a lack of conceptual knowledge based on the understanding of part-whole pictorial representations compared to their typically achieving peers. Understandings of part -whole fractions were less developed and misunderstood in students with MLD.

Finally, Lewis (2007) conducted a case study analyses of four students' understanding of fractions as parts of wholes and the connections to understanding equivalent fractions, comparison of fractions, and beginning fraction operations. A focal student displayed atypical

understanding of shaded area models in that the student identified the line rather than the shaded quantity as the fraction (e.g. the fraction $\frac{1}{2}$ is the *partition*, not the shaded region). Further, the student understood a shaded area model representation as the amount taken away rather than a fractional quantity (e.g. student constructed $\frac{7}{12}$ - 7 pieces out of 12 shaded - but interprets as $\frac{5}{12}$ - the amount left). This misunderstanding was resistant to instruction and impeded the student's latter understanding of fraction concepts that depend on such understanding as defined by the part-whole sub construct, such as fraction equivalency.

Equivalency/multiplicative reasoning, part-whole, and students with MLD. The concept of equivalence can be viewed as the invariance of "a multiplicative relation between the numerator and the denominator, or the invariance of a quotient" (Ni, 2001, p. 400), and is a difficult concept to understand. Understanding the concept of equivalency and generalizing the concept to abstract processes involves the move from additive to multiplicative thinking (Lamon, 1993b; Battista & Borrow, 1996). Additive thinking focuses on differences between quantities, whereas multiplicative thinking focuses on a rate of change (Harel & Behr, 1990).

Equivalent fractions involve both between- and within- multiplicative relations (Van Hille & Baroody, 2002). Between relations refer to the relationship between the numerators and the denominators of two equivalent fractions. When two fractions are equivalent, the same factor is used to multiply the numerator and denominator of one fraction to achieve the other. Within relations occur between the numerator and denominator of one fraction. Effective instruction in equivalency, then, should encourage the progression of thought structures used to understand fraction equivalency from additive to multiplicative understandings.

Some educators and researchers in general and special education have sought to bring understanding to fraction equivalence through the use of manipulative models tied to the part-whole sub construct (Butler et al., 2003; Jordan et al., 1999; Morris, 1995). Although the practice is common, available research indicates possible difficulties involved with using part-whole understandings of equivalency exclusively to solve problems. For instance, Mazzacco and Devlin (2008) discovered middle school aged students with MLD have difficulty with part-whole based pictorial models of equivalency. Students with MLD demonstrated statistically significant differences in identifying fraction equivalencies in part-whole pictorial compared to their peer group of low performers and typically achieving students. The same result was found when using abstract numeric representations of fraction equivalency statements. In both situations, students with MLD identified a significant number of incorrect equivalencies in addition to their failure to identify correct equivalency statements (Mazzacco & Devlin, 2008). The researchers suggested students with MLD have "a weak rational number sense and inaccurate beliefs about rational numbers" based on the part-whole/partitioning sub construct (p. 690).

Measure. Partitioning as used to understand fraction concepts such as equivalency is also evident in the measure sub construct. Additionally, the understanding of both the magnitude and the measurement of rational numbers appears within the measure sub construct (Charalambous & Pitta-Pantazi, 2007; Lamon, 2005). In the magnitude understanding, the measure sub construct leads the student to understand how big a fraction is. Conversely, for the measurement sub construct students repetitively use a unit fraction, like $\frac{1}{b}$, to measure some interval, where the total distance can be described as $\frac{a}{b}$ (Marshall, 1993). No limit to the size of

a exists, so the measure sub construct can describe values both less than, equal to, and greater than one.

Partitioning/unitizing/equivalency, measure, and students with MLD. Lamon (1999) summarizes aspects of knowledge fundamental in understanding the measure sub construct as students being "(a)…comfortable performing partitions other than halving; (b) …able to find any number of fractions between two given fractions; and (c) …able to use a given unit interval to measure any distance from the origin" (p. 120). If students must be able to draw meaning from partitions more advanced then halving to understand fractions in the measure sub construct, is it difficult for students with MLD to grow such meaning if their understanding of partitioning is limited (Lewis, 2007)? That is, do students with MLD encounter the same difficulties regarding partitioning, unitizing, and equivalency in the measure sub construct as those evident in the part-whole subconstruct?

Grobecker (2000) researched partitioning activities that were presented within the context of dividing *number line wholes* into equal sized parts. Seven students with MLD were given a line and eight blocks. When set adjacently, the measure of the eight blocks equaled the measure of the line. Students were given various problems about the blocks and the line relationship (e.g. modeling and solving $\frac{1}{4} + \frac{1}{8}$). Twelve year old students with MLD were unable to associate the part (part) and the whole (line) to generate *equivalent relationships*, although they were found capable of understanding the relationship between the *unit* block and the line ($e.g. \frac{1}{8}\ iterated\ eight\ times\ is\ \frac{8}{8}$). The research provided some evidence students with MLD struggle with measure-based partitioning and the higher order thought structures needed to understand equivalency through the measure sub construct. But how do students with MLD

react to other fraction sub constructs that do not teach fractions with part/whole based partitioning?

Quotient. Fractions and partitioning as quotients may be an alternate way to teach concepts to students with MLD. In the quotient sub construct, rational numbers may be expressed as the result of taking *a* objects and distributing them among *b* recipients; that is, as a result of parative division (Charalambous & Pitta-Pantazi, 2007; Lamon, 1993a; Streefland, 1991). For instance, for the fraction $\frac{2}{4}$, one might consider the situation of dividing two pancakes equally among four students, or 2 ÷ 4. Quotient sub construct are unique because they provide both the problem situation and the resulting answer: $\frac{2}{4}$ represents both the sharing of two pancakes among four people as well as the part each person receives (Kieran, 1976; Mamede, Nunes, & Bryant, 2007).

Partitioning/Unitizing/Equivalency, quotients, and MLD. Quotients are distinctive in that, while involving partitioning, these numbers specifically reference the *relationship* between the sharing situation and the quantity received (Memede, Nunes, & Bryant, 2005). Students with MLD may struggle due to the pictorial element, but they may understand quotients better than part-whole sub constructs due to the emphasis on the relationship as opposed to the partitioning (Grobecker 1997). With quotients, what is necessary for understanding the fraction $\frac{1}{7}$ is not the ability to separate a circle into equal parts but to realize that $\frac{1}{7}$ is the resulting share, or name, when one object is shared between seven recipients (Empson et al., 2005). Moreover, if the emphases in the quotient sub construct is on correspondence between number of items and number of sharers as opposed to partitioning, then students with MLD may succeed in quotient

based activities if their understanding of correspondence is intact (Baroody, 1993; Geary et al., 2006).

Quotative division is also used in the quotient sub construct to understand the unit: For instance, if 2 pancakes are shared among 4 friends and each friend receives $\frac{2}{4}$ pieces, one can rebuild (use the unit) to find how many pancakes there were initially (Charalambous & Pitta-Pantazi, 2007). Lamon (1993) describes two separate situations where quotients are combined with ratio strategies to understand and use the unit to solve a problem. One method provided problems that "elicit children's counting and matching interpretations" (Lamon, 1993; p. 140). Consider the following relationship:

Figure 3. **Quotient unitizing.**

Students considered the solution by establishing the between relationship of pancakes to people in each situation (one to two and two to five, respectively) and established one of the relationships as the unit (one pancake to two people is the unit). They determined how many of these units (one pancake to two people) were contained in two pancakes for five children situation. Because one person remained after accounting for the two units, the two to five group received less. While both ideas have merit, neither have been researched concerning their effectiveness in generating fraction understanding in students with MLD.

Ratio. Fractions as ratios are another alternative to teaching concepts through part-whole and measure sub constructs. Ratios compare any two quantities to one another through one to

many correspondences, and have been described as fundamental to fraction knowledge (Lamon, 2005; Pitkethly & Hunting, 1996; Streefland, 1991). Ratios can depict both part to part and part-whole relationships, making them a distinct sub construct of rational numbers (Charalambous & Pitta-Pantazi, 2007; Lamon, 1993a). For instance, when a recipe calls for one part orange juice concentrate to three parts water, the parts are not the same but related; thus, the ratio becomes a rate in this instance (Lamon 1999). Another feature of the ratio sub construct is the relationship does not change if we wish to increase one of the parts- a person must be able to understand the unit linkage between two quantities and hold the linkage in mind to iterate the ratio (Battista & Borrow, 1996). In the orange juice example, if we use two parts concentrate, we would need six parts of water, as we need three parts water for every one part concentrate to keep the original relationship consistent. This relationship is regarded as the covariance-invariance property (Vengard, 1983) and is related to fraction ideas such as equivalence and ordering (Charalambous & Pitta-Pantazi, 2007; Streefland, 1991, 1993).

Furthermore, Battista and Borrow (1995) suggest students must move through three phases of understanding in their developing understanding of equivalency situations as ratios: (1) conceptualizing explicitly the linking action of two composite amounts; (2) understanding multiplication/division and its role in the iteration process; and (3) abstracting iterative processes and connect them to the meaning of multiplication and division. Along with multiplicative understanding, Lamon (1993b) suggests student evidence of strategy usage while developing multiplicative understandings involved with ratios: (1) Avoiding (no interaction with the problem), (2) Visual/additive (trial and error, incorrect additive linkages), (3) Pattern building (oral or written patterns without understanding number relationships), (4) Pre-proportional reasoning (pictures, charts, or manipulatives evidencing relative thinking), (5) Qualitative

proportional reasoning (ratio as unit/relative thinking/some numerical relation understandings), and (6) Quantitative Proportional reasoning (understanding of symbols, functional and scalar relationships). But can ratios be used to teach fraction concepts that underlie equivalence, such as partitioning and unitizing?

Partitioning/unitizing, ratios, and students with MLD. Cortina and Zuniga (2008) experimented with alternatives to the equi-partitioning process for supporting late elementary students' beginning notions of fraction concepts. Students were considered low achievers in mathematics, but not as students with MLD. Cortina and Zuniga's (2008) work was based on the writings of Thompson and Saldanha (2003) and also Steffe (2002), who hypothesized and produced evidence of ratio-like alternatives to evolve students' beginning knowledge of fractions Researchers argued that part-whole partitioning schemas were insufficient for such purposes.

Instead of beginning with partitioning activities, researchers asked students to consider an amount iterated a number of times (e.g. Mauricio ordered five pieces of bacon; Nicosha ordered twice as much; Katy ordered three times as much, etc). This activity forced students to consider a double count (five to one, ten to two, fifteen to three…). Next, when presented with a physical referent (e.g. a milk carton), students considered the amount of same sized cups that could be filled with the amount of milk in the carton. The rule was changed to one milk carton filling five (medium cups) and then ten cups (small cups). Results of the study showed growth in overall understanding of early fraction concepts among students who struggled. Cortina and Zuniga (2008) suggested "it is viable to engage novice learners in fraction activities such as the cups-capacity tasks, where the focus is in quantifying relationships of relative size by means different to equal partitioning" (391).

Equivalency/multiplicative reasoning, ratios, and students with MLD. Using relationships to understand fraction concepts may be a viable alternative to the equal partitioning approach for students with MLD. But can learning fractions through the ratio sub construct lead students with MLD towards the multiplicative thought structures necessary to understand important fraction concepts, such as equivalency? Grobecker (1997) investigated 84 elementary aged students with and without MLD and their ability to partition, unitize, and use multiplicative thinking over multiple age groups. Her interview task used a lion eating three grains at a time and an elephant eating two grains at a time. Students were given various scenarios requiring them to establish grouping relationships between parts and parts, wholes and wholes, and the parts and the whole (partitioning). The ability to iterate the relationship as a unit was also examined in lower levels of problem complexity.

The researcher identified four levels of understanding that encompassed all solutions of students with and without MLD: (1) The inability to manipulate grains and bundles at the same time; (2) An additive ability to count and add grains and bundles; (3) Grains and bundles represented as groups and then adding the groups, and (4) Use of mental multiplication to manipulate grains and bundles at the same time. The levels noted were similar to those found by Battista and Borrow (1996) and Lamon (1993b) with typically achieving populations. Students with and without MLD experienced difficulty advancing to higher levels of multiplicative thinking (e.g. Level 3 or Level 4). However, students with MLD were unable to advance beyond Level 2 (they used mostly additive structures to understand equivalency), while students without MLD did progress into higher levels of thought as they aged. However, it is important to note the interviews were short and were not teaching activities. It remains an empirical question whether multiplicative understandings of fraction equivalency might be

cultivated through ratios for students with MLD (Grobecker, 1997; 1999). However, through comparing the available research on deficits involved in MLD and difficulties experienced by students with MLD in understanding fraction equivalency through partitioning (see Figure 4), another method of teaching fraction equivalency that avoids partitioning and directly addresses strategies promoting multiplicative thought structures may be warranted.

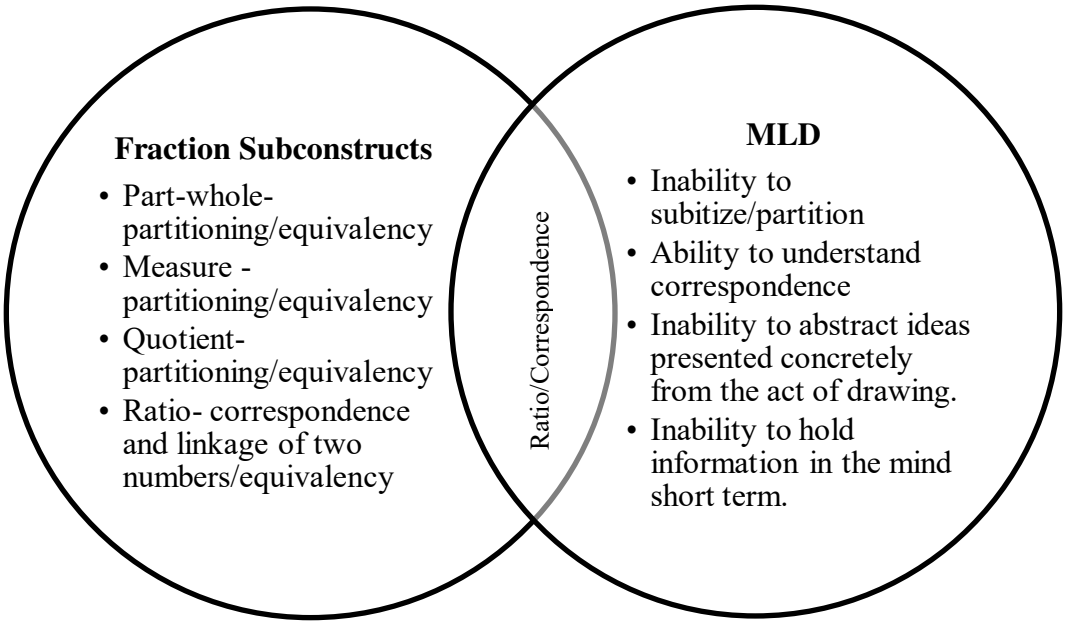


Figure 4. **Fraction sub constructs and MLD.**

Teaching Fractions: Instructional Strategy

Another piece of the empirical question relating to effective fraction instruction for students with MLD is what *instructional strategy* to employ. Uncovering possible best content approaches to teaching fractions to this unique population is important, yet content cannot be easily separated from instructional strategies. Just as fraction instruction may prove beneficial or detrimental to students with MLD if content or constructs used during instruction does or does not align with students' pre-instructional strengths, the same is true regarding the choice of instructional strategies employed. An examination of literature for teaching mathematics to

students deemed low achieving and students with MLD provided important implications of how to deliver fraction content through instructional strategies.

A review of instructional strategy literature was completed to examine effective teaching strategies for teaching mathematics to students with MLD. Search criteria included the following: (a) the study included students who were in elementary or middle school and included students with MLD or struggling students, (b) the study reported on performance of equivalency and/or related concepts as the result of qualitative or quantitative analyses; and (c) the study was published between 1989 and 2010. The following web-based data sources were searched for articles: Educational Resources Information Center (ERIC), Psyc info, Web of Science, and Wilson Omni File Full Text using the key words of fractions, equivalency, LD, math disabled, at-risk, mathematics LD, intervention, mathematics instruction, and mathematical learning. Five studies meeting the search criteria were uncovered. Studies were then classified (e.g. concrete-representational-abstract instruction, explicit instruction, instruction promoting student verbalization of mathematics thinking, or instruction promoting student use of representations), reviewed, and summarized. The instructional strategy literature review is presented below and summarized in Table 2. Results were largely in line with findings from a meta-analyses of instructional practices effective for struggling populations in mathematics conducted in 2008 by Gerstein, Chard, Jayanthi, Baker, Morphy, and Flojo.

Concrete-representational-abstract instruction. Concrete-representational-abstract (C-R-A) instruction involves the purposeful sequencing of instruction beginning with concrete manipulatives or contextualized problem situations, connecting to pictorial representations of ideas formed in the concrete, and finally connecting to abstract (numerical) representations of ideas formulated via pictorial means (Van de Walle, 2004). The strategy encompasses the doing,

seeing, and symbolic stages of understanding mathematical concepts. Two studies were uncovered relating specifically to the C-R-A approach used in teaching fraction equivalency concepts. First, Jordan, Miller, and Mercer (1999) compared two methods for teaching fraction concepts to 120 typically achieving students and five students with MLD. The experimental curriculum consisted of a graduated instructional sequence (e.g. concrete to semi-concrete to abstract). The comparison group received textbook driven instruction. Both groups' utilized instructional principles that were empirically validated. Student performance was measured using three researcher-created measures based on the Enright Diagnostic Inventory of Basic Skills and curriculum-based measures. The tests were used as repeated measures before and after instruction had concluded and again a number of weeks later. Three versions of the posttest were implemented in a rotated manner.

Table 2. ***Summary of Instructional Strategy Studies Reviewed.***

Citation	Area of Fractions	Focus	Subjects	Setting and Age	Results
Butler, Miller, Crehan, Babbit, & Pierce (2003)	Equivalent fractions	CRA	50 total; 42 students with LD	6th, 7th,and 8th grade students in resource room;	CRA outperformed RA in ratio fractions. Experimental groups outperformed control groups in word problems and improper fractions.
Jitendra, DiPipi, & Perron-Jones (2002)	Multiplicative thinking	Problem Representations	4 total; all students with LD	8th grade; one school	"Schema-based strategy was effective in substantially increasing the number of correctly solved multiplication and division word problems for all 4 students" (p. 23).
Jordan, Miller, & Mercer (1999)	Naming, equivalencies, comparison, operations	CRA	125 total; 5 students with LD	Six 4th grade classrooms; one school	Both groups improved as a result of instruction, although the CSA group improved more.
Woodward, Baxter, & Robinson (1999)	Decimal fractions; contextual and procedural	Verbalization of Thinking	44 total; 10 students with LD	8th and 9th grade; two classrooms	Conceptual instruction brought decreased errors and increased student ability to compare and order decimals.
Xin, Jitendra, & Deatline-Buchman (2006)	Conceptual and procedural; multiplicative comparison & proportions	Problem Representations	22 total; 18 students with LD	6^{th}, 7^{th}, and 8^{th} graders	Procedural knowledge increased and maintained as a result of the intervention.

A split plot ANOVA and repeated measures ANOVA were utilized to examine the effects of treatment. Significant differences were discovered for the posttests for both the treatment and control groups, with the treatment group showing significantly more gains then the control. Researchers concluded graduated instruction to be beneficial to all students involved in the study. Limitations included a small number of participation from students with LD as well as variability in control group instruction.

In a related study, Butler and colleagues (2003) compared two curricula for teaching conceptual and procedural knowledge of fraction equivalency with 42 6^{th}, 7^{th}, and 8^{th} grade students with MLD and eight students considered to be typically achieving. The experimental curricula consisted of the use of a graduated instructional sequence (e.g. concrete-representational-abstract instruction, or CRA) to teach fraction equivalency; the only difference was that one group used RA and one group used CRA. The comparison curriculum consisted of basal instruction. Both experimental curricula were scripted, employing direct instruction principles. Students were given cue cards with vocabulary and completed examples of fraction equivalency problems in the two experimental conditions. Student achievement was assessed using three tests taken from the Brigance Comprehensive Assessment of Basic Skills- Revised and included a Quantity fractions (e.g. used to measure knowledge of ratio and proportion); an Area fractions (used to measure student's ability to name pictures from geometric representations); and an Abstract fractions assessment (e.g. a test of student's ability to compute equivalent fractions). The researchers also developed a fourth measure to assess students' ability to solve word problems involving fraction equivalency.

A MANCOVA was utilized to evaluate effects of treatment between the experimental groups, with pretest scores used as covariates. CRA groups outperformed RA groups on all measures, but only one with statistical significance (e.g. Quantity fractions test). Combining the experimental groups to test for differences compared to the control group revealed statistically significant differences between groups on performance in word problem solving and improper fractions, favoring the experimental group, with large to very large effect sizes reported. Means on the abstract fractions test between the experimental and control groups were 75.1 and 83.8, favoring the control group. Researchers concluded that CRA instruction produced performance in fraction equivalency for students with LD that was comparable to student performance in general education. No maintenance results were recorded.

Verbalizing mathematics thinking. Verbalizing mathematics thinking was defined as solution format based or self-questioning based instructional strategies that resulted in increased performance in fraction concepts and equivalency (Gerstein et al., 2008). One study was found that dealt with student verbalization and fraction concepts. Woodward, Baxter, and Robinson (1999) compared the effects of conceptual versus procedural instruction of decimal concepts and operations on student achievement for 34 typically achieving students as well as 10 students with MLD and RD. The experimental curriculum consisted of lessons taken from a standards based mathematics curriculum emphasizing, among other areas, student dialoging of conceptual understanding. The control group was taught using a video disk program utilizing procedural based validated instructional practice (e.g. active teaching). No differences were found between groups prior to the start of the teaching, and the same teacher implemented all of the teaching in both groups during the study. Student achievement was assessed using three measures, a hand calculation test, a second test that was similar but allowed the use of calculators, and a third

measure that utilized interviews to assess students' ability to order decimals. Student work was also examined for error patterns.

An ANCOVA analyses was used to analyze results on all measures, with the pretests used as covariates. For the interviews, scores on the Comprehensive Test of Basic Skills was used as the covariate. Results revealed no difference on the calculator test between groups, near significant effects on the hand computation test in favor of the procedural group, and statistically significant differences on the interviews task of ordering decimals in favor of the conceptual group. Students in the conceptual group tended to make fewer errors in interpreting decimals from pictures and applying whole number concepts. Researchers stated a need to combine recursive models of instruction with models of instruction that encouraged student verbalization of thought processes in order to aid students with MLD.

Student use of representations. Student use of representations was defined as the teaching sequence requiring students to use a representation of the problem situation, solution, or both the situation and solution (Gerstein et al., 2008). The representation could have been student or teacher generated. Although four studies were found that met these criteria, only two are reported here, as the other two studies reviewed have already been reported under the C-R-A section.

Jitendra, DiPipi, and Perron-Jones (2002) evaluated the effects of schema based strategy instruction on two 8th grade students with MLD and two 8th grade students with RD's ability to solve word problems involving multiplicative reasoning in an exploratory study. Schema based instruction teaches students how to identify and represent in a picture underlying problem structures to solve problems. The experimental curriculum included vary (e.g. the size of groups

or the whole is unknown) and multiplicative comparison (e.g. the referent or what is being compared is unknown) problems that, during instruction, were used to teach students how to identify underlying problem structures and use the identification to solve the problem. Once the problem type was identified, students were taught to use broken down procedures or algorithms to arrive at the answer. Assessment measures included a word problem test and a transfer test, each including 12 items. Measures were examined for strategies utilized, and students were also given a strategy questionnaire at the conclusion of the intervention.

A multiple baseline across students design was utilized to assess student performance in response to the intervention. Experimental phases included baseline, instruction, response generalization, and maintenance. Results revealed a functional relationship between student performance and the intervention. Students' ability to generate diagrams and number sentences to solve problems increased throughout the stages of the intervention. Moreover, students maintained their ability to complete these types of problems over a period of time, and generalized their new abilities to novel problems. Results were of limited generalizability due to sample selection, low student numbers, and research design.

Xin, Jitendra, and Deatline-Buchman (2006) evaluated the effectiveness of schema instruction on 22, 8th and 9th grade, students who were either MLD or LD who were at risk for mathematics failure. Experimental instruction consisted of schema-based word problem strategy instruction where students were taught to identify problem types and apply algorithms based on the problem types. The comparison instruction included general strategy instruction (e.g. Polya's problem solving steps). "Four parallel word problem-solving test forms, each containing 16 one-step multiplication and division word problems were developed for use as the pretest, posttest, maintenance test, and follow up test" (p. 186). A pretest-posttest comparison group

design with random assignment of subjects to groups was used to examine the effects of the two instructional programs using repeated measures ANOVAs. Results of the analyses revealed significant differences on all measures between the experimental and comparison groups. Effect sizes for the experimental group were very large (Xin et al., 2006).

Explicit instruction. Explicit instruction was defined as instruction that relied on the teacher to deliver step-by-step instruction on how to solve problems (Gerstein et al., 2008). Modeling of the teacher's thinking out loud and modeling of representations used during problem solving were present throughout the studies reviewed. In each study, the sequencing of instruction (e.g. problems posed in specific, well thought out order), concept formulation, and representational strategies were systematically introduced and rehearsed using well thought out teaching trajectories. Explicit instruction, then, was found to be a staple among effective teaching in mathematics for students with MLD, although its use has not been advocated as the sole instructional strategy to be used among this population (National Mathematics Advisory Panel, 2008).

Combining Content and Instructional Strategy

In terms of content, ratios appeared to be an alternative in promoting understanding of fraction equivalency and its associated concepts. Although students with MLD exhibited lower levels of multiplicative thinking during interview tasks involving ratios, the use of ratios as representations for fraction equivalency situations during instruction may be able to build student understanding of the concept because of their reliance on *correspondence* – a concept students with MLD can understand (Grobecker, 1997). Part-whole and measure-based approaches appear to promote thinking of equivalency and fractions as actions as opposed to quantities (Lewis, 2007). Freudenthal (1983) argued against using part-whole approaches to understand

fractions exclusively, stating it to be "much too restricted not only phenomenologically but also mathematically" (p. 144). In particular, he argued that beginning and sustaining fraction instruction in the part-whole sub construct was "too narrow a start" and "one sided", and was mystified "that all attempts at innovation have disregarded this point" (p. 147). If higher level strategy use and multiplicative thinking to better understand equivalencies could be cultivated in students with MLD through ratio based instruction, this approach could prove to be a valuable innovation and access point for students with MLD regarding performance. Moreover, Lamon (2007) found that students who learned fractions, as ratios were able to transfer this knowledge to problems presented in the part-whole sub construct. Because the part-whole subconstruct is dominant in most school-based mathematics curricula (Charalambous & Pitta-Pantazi, 2007), instructional sequences that increase understanding of ratio and part-whole equivalencies simultaneously could be valuable.

In terms of instructional strategy, it seems that explicitly sequencing ratio-based equivalency concepts and teaching representations of the problem situation and solution would do well to promote understanding of equivalency among students with MLD. Teacher usage of questioning strategies that require students to verbalize their mathematical thinking should also be utilized. Moreover, the use of teacher 'think aloud', suggestions for representation use, C-R-A, and other aspects of explicit instruction seems to benefit learners with MLD in fraction instruction. Thus, an instructional strategy or plan that involves these critical aspects of instruction paired with content that allows access to fraction understanding for students with MLD could prove to be an important instructional tool to promote achievement and understanding. Student verbalization of thinking, CRA, and student use of representations can

easily be integrated with a ratio-based instructional sequence. Consider the following relationship of batter cans needed to produce a certain amount of pancakes:

***Figure 5*. Unitizing with ratios**.

Instruction can begin with such scenarios using cups and counters. The cups would represent the cans and the counters would represent the pancakes. Questions could be designed for students to develop a ratio as a unit by presenting a situation like above and several pictorial examples relating certain amounts of cans to certain amounts of pancakes. Students could be asked to model with manipulatives if there is enough batter in each situation to make the amount of pancakes pictured (Lamon, 1993a). This type of process could prove to be better instruction for students with MLD due to the absence of partitioning (Grobecker, 2000; Lewis, 2007).

To move students into pictorial representations of ideas learned in the concrete, Fosnot and Dolk (2002) suggest that students can make further sense of equivalencies through the use of ratio tables combined with unit rates. Ratio table representations may also provide crucial links between a student with MLD's understanding of equivalence through additive and then multiplicative means, aiding students in realizing the multiplicative links needed to iterate unit fractions and ratios (Streefland, 1993, 1997). Consider the following situation:

Figure 6. **Building equivalency.**

Students could use the relationship between cans and pancakes as the unit (Lamon, 1993b). To iterate the unit, students could be presented with scenarios asking them to find an unknown amount of pancakes or batter cans for a given number of cans or pancakes. When asked to consider how many cans are needed to make 36 pancakes, students may begin by pictorially iterating the unit relationship. However, teachers could explicitly model the use of a ratio table to augment student thinking using a "think aloud" strategy. Questioning strategies relating how the ratio table relates to pictorial and concrete representations could be employed by the teacher. Many studies showing student representation use as an effective instructional strategy included teachers modeling representations for student use (Gerstein et al., 2008). After an appropriate amount of practice using ratio tables, students could be prompted to shorten the tables. Between (e.g. relations between numerators and denominators of equivalent fractions) and within (e.g. relations between numerators and denominators of the same fraction) strategies to use the unit ratio to derive equivalencies could be promoted by teacher questioning and modeling. Moreover, teachers can encourage students' verbalization of the connection between long and short ratio tables and multiplication and division, enabling a further move from representational understanding toward abstract understanding (see Figure 7).

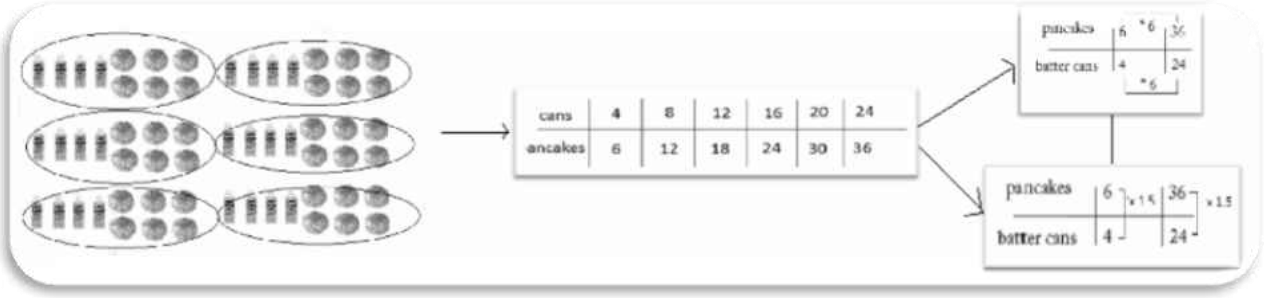


***Figure 7*. Moving from concrete to representational to abstract.**

Conclusion

Students with MLD have been found to experience difficulty understanding fraction equivalency through part-whole based fraction instruction. A need to progress to multiplicative thought structures to support performance in fraction equivalency that transfers to tests used in school curriculums is evident from review of current literature. Moreover, a need to support student learning of fraction equivalency with empirically validated instructional strategy was delineated from the review of literature. However, the benefits of combining ratio-based fraction equivalency instruction with effective instructional processes for students with MLD have not been examined empirically.

Therefore, this study investigated the impact of a ratio-based fraction teaching sequence with effective instructional strategies on performance of fraction equivalency for students identified as having a MLD. The study also evaluated struggling and non-struggling student performance and examined group differences in performance and understanding through quantitative and qualitative analyses. Research questions sought to identify if (1) performance on two tests of curriculum-based (part whole and ratio) fraction equivalency increased as a result of instruction and (2) if student understanding of equivalency situations (presented through ratios) changed with respect to (a) typical and atypical strategy usage and (b) level of multiplicative understanding uncovered in data analyses from pre to post interview.

CHAPTER THREE: METHODOLOGY

The primary purpose of this study was to investigate the impact of a ratio-based fraction teaching sequence on students identified as having a MLD or struggling in mathematics. Non-struggling student performance was also examined as well as group differences in performance and understanding through quantitative and qualitative analyses. This chapter begins with the statement of the research questions used to guide the study followed by a description of the students and settings. Next, a thorough description of the research design, instructional procedures, and data collection procedures is provided. The chapter concludes with the data analyses procedures for each of the research questions.

Research Questions

The study addressed the following research questions:

(1) Are there statistically significant differences in overall performance (i.e. the number of correct responses) on a curriculum-based measure of fraction and ratio equivalency and on a standardized measure for students with mathematical learning disabilities, struggling students, and typically developing students who do and do not participate in ratio-based fraction instruction?

(2) What are the levels of multiplicative thinking and strategy usage of students when presented with ratio equivalency situations? Do strategy use and levels of multiplicative thinking increase for students with MLD and students who struggled after participating in a ratio-based equivalency instructional sequence?

Setting and Students

Research Question One

Students in this study included students with and without MLD in the third grade. This grade level was chosen due to curriculum constraints that set the learning of fraction equivalency at this level. Students who participated in the study were selected using several characteristics. Namely, enrolled in an inclusive third grade mathematics course, an FCAT level of 1, 2, or 3, a weakness in fraction concepts as identified by the pre-test, and the absence of limited English proficiency (LEP) or poor socio-economic status (SES) were inclusion requirements (Murphy, Mazzacco, Hannich, and Early, 2007). The absence of LEP or poor SES status was to downgrade the chance of assigning MLD status due to confounding factors (Murphy et al.). A portion of the students selected had district confirmed exceptionalities (N=8), all of which were LD.

In total, 78 third grade students met the selection criteria for the study. Out of the total, 38 students returned consent forms allowing them to participate in the study and became the final student sample (N= 38). An a priori power analyses was utilized to compute the necessary sample size using G Power 3 statistical software (Faul, Erdfelder, Lang, & Buchner, 2007). Effect sizes from previous research in fractions for students with LD (Butler et al., 2003; Xin, Jitendra, & Deatline, 2006) ranging from 0.50 to 0.75 were considered in the analyses. Thus, using an alpha level of 0.05 and noted effect sizes, the power analyses indicated a total sample size of 38 was sufficient to produce a power of .90 for a 2 x 2 between factors MANOVA with repeated measures, with 19 students in the experimental group and 19 students in the control group (Faul et al., 2007). Sample sizes as small as 30 students could be considered sufficient to detect changes in behavior (Howell, 2007). Despite beliefs that smaller n groups size tends to

violate assumptions of normality, research suggests that parametric multivariate statistical analyses with sample sizes as small as eight can be conducted with a reliability of 1.00 (Ninness et al., 2002). Thus, the sample size was considered adequate to test the research questions.

The sample then was split into students with MLD, struggling students, and typically achieving students. As indicated in the review of literature, an agreed upon definition of MLD currently does not exist. Thus, "in the absence of a consensus definition of MLD, it is necessary to rely on proxy definitions" (Mazzacco & Thompson, 2005, p. 146). The complexity in defining MLD by proxies (e.g. scores on mathematics tests) has been revealed in the results of several studies (Mazzacco & Devlin, 2008; Mazzacco & Thompson, 2005; Murphy et al., 2007). However, study results provide the best guidance at the time for potential designation of MLD in research.

Tests that measured both informal (e.g. knowledge students have of mathematics that is not taught) and formal (e.g. achievement oriented mathematics knowledge) mathematics concepts were found to produce scores that stably predicted MLD over time and thus were included (Mazzacco, 2005). Additionally, test items that covered reading numerals, number constancy, magnitude judgments, and mental addition were found to be highly predictive of MLD over time (Murphy et al., 2007) and were included in the current research. Second, characteristics of MLD can change as a function of the cut off scores used to define a person as MLD (Murphy et al., 2007). The best available current research suggests the use of scores at the tenth percentile or below greatly reduced the number of false positives (e.g. students being labeled MLD despite other contributing factors to low mathematics achievement) and separated true MLD performance and characteristics from those who struggled in mathematics (Mazzacco & Devlin, 2008).

This study used several primary and secondary tests (e.g. one measuring formal knowledge and two others measuring both formal and informal knowledge of mathematics), to confirm the student as MLD, struggling, or proficient. Three subtests served as primary measures. The calculation subtest of the Woodcock Johnson-III (consisting of 41 items normed for ages five through adult) (Woodcock, McGrew, & Mather, 2001) is an achievement test used to assess a person's ability to perform mathematical computations. Examples include writing single numerals and basic addition, subtraction, multiplication, and division. The Numeration and Mental Computation subtests of the (Connolly, 1999) (consisting of 24 and 18 items) are used to assess a students' formal and informal knowledge of quantity, order, magnitude, reading numbers, counting, and mental computation of one and two digit numbers.

Cut off scores garnered from the research were used as criteria to designate a student as MLD. Students who met selection criteria were administered the calculation subtest from the WJ- III Test of Achievement (Woodcock et al., 2001) along with the Numeration and Mental Computation subtests from the Key Math-R (Connolly, 1999). Students whose performance fell in the bottom 10th percentile on two out of three measures were considered MLD; students whose performance fell between the 11th and 25th percentiles on two out of three measures were considered struggling; and those students whose performance were higher than the 25th percentile on two out of three measures were considered non-MLD, or typically achieving.

Students who met category criteria on only one measure (e.g. Key Math Numeration, Key math Mental Computation, or WJIII Calculation) were administered two additional (secondary) tests- the WJIII Quantitative Concepts and Applied Problems subtests (Woodcock et al., 2001). The Quantitative Concepts subtest is a test of students' ability to recognize symbols, retrieve representations, and manipulate points on a mental number line. The Applied Problems subtest

measures students' ability to construct mental models and quantitative reasoning skills. To confirm the students as MLD, struggling, or typically achieving, the researcher examined results of the secondary measures. Students were confirmed as MLD, struggling, or typical if performance on *one or more* secondary measures fell within the ranges specified in the previous paragraph. The final analyses led to identification of four students as MLD, nine students as struggling in mathematics, and 26 students as typically achieving. Students were then matched on their 'student type' and randomly assigned to either a treatment or control group (Borg & Gall, 1989). The matching was used to ensure that students were comparable across intervention conditions on relevant characteristics (Gerstein, Fuchs, Compton, Coyne, Greenwood, & Innocenti, 2005).

Student characteristics relating to ethnicity, gender, and grade level for the 38 students were recorded at the onset of data collection. The majority of the students were Caucasian (57%), followed by Hispanic (29%), and African American (16%). Sixty-eight percent of the students were age eight to nine and four months; 32% were age nine and five months to age ten. Twenty-one percent of students had school identified LD. All students meeting criteria for MLD status (n = 4) were already classified as LD by their school district's diagnostic criteria. Eight students were LD but did not meet MLD criteria for this study. Table 3 summarizes characteristics of the students in the experimental and control groups.

Table 3. ***Characteristics of Experimental and Control Groups***

	EG	CG
Age		
8-0 to 9-4	14	12
9-5 to 11-0	4	7
Gender		
Male	10	9
Female	11	8
Ethnicity		
Caucasian	10	12
Black	4	2
Hispanic	5	6
Disability Status		
Learning Disabilities (school defined)	4	4
MLD Status, Study Specific	2	2

The study took place in a public elementary school in central Florida. In prior months, all third grade students received textbook based instruction in fraction concepts and equivalency (e.g. Envision Mathematics, Grade 3). During the time supplemental ratio instruction took place, students in both the control group and the experimental group received instruction in equivalent fractions in their mathematics classrooms. Lessons taught by the third grade classroom teachers were taken from NCTM's *Illuminations* website. Namely, "Fun with Fractions" (lessons one through five) were used along with "Fun with Pattern Block Fractions" (lessons one through three) were utilized in students' regular mathematics classrooms during the supplemental period of ratio-based instruction. During the lessons, students worked with length and area part whole fraction models on problems involving the relative size of fractions, unitizing with fractions, naming fractions, understanding fractions as parts relative to a given whole, ordering fractions, reducing fractions, and equivalency (see http://illuminations.nctm.org/LessonDetail.aspx?id=U152 and

http://illuminations.nctm.org/LessonDetail.aspx?id=U113). Students in the control group did not receive the ratio-based supplemental instruction. The instructor for the supplemental ratio-based instruction was the researcher.

Research Question Two

For the second research question, a purposive sample of three students from the experimental group was taken. The sample was purposive because the researcher wanted to select students from the experimental group that were representative of varying mathematics performance and profiles (e.g. MLD, struggling, typically achieving). The three students chosen were most representative, on average, of the characteristics that defined their student type. In other words, their scores on the standardized tests used to define student type were in line with others also deemed MLD, struggling, or typically achieving. The three students participated in two separate semi-structured interviews to assess their levels of strategy use and multiplicative thinking before and after the ratio-based instructional sequence. Characteristics of the students who participated in the semi-structured interview sequence can be found in Table 4.

Table 4. ***Characteristics of Student Students Involved in Semi Structured Interviews***

Name*	Age	Gender	Ethnicity	Disability (School)	Student Type
Albert	9	Male	Caucasian	None	TA
Bill	9	Male	Hispanic	LD	MLD
Carl	9	Male	Hispanic	None	SS

**Names have been changed to protect student identities*

The interviews took place in a public elementary school in central Florida. Interviews were conducted in May 2011, once at the beginning and again at the end of the instructional period that lasted nine days. The interviewer was the researcher.

Instrumentation and Measurement of Variables

Research Question One

Gerstein, Fuchs, Compton, Coyne, Greenwood, and Innocenti (2005) argue that "multiple measures be used to provide an appropriate balance between measures closely aligned with [an] intervention and measures of generalized performance" (p. 151) in quality quasi-experimental and experimental research. Appropriately, the study utilized a combination of measures including a school curriculum-based pre and posttest measure of fraction equivalency and a standardized pre and post measure of transfer performance. Each of the dependent measures is discussed below.

Standardized test. The transfer test consisted of a subtest pulled from the Brigance Comprehensive Inventory of Basic Skills- Revised (1999). Specifically, subtest Q6, Converts Fractions and Mixed Numbers, were utilized as the standardized measure for this study. The Q6 subtests consists of 16 fraction items measuring fraction equivalency; the first four items asked students to scale up a fraction to produce an equivalent fraction; the following four items asked students to simplify a fraction to an equivalent fraction; the last eight items asked students to convert between mixed numbers and improper fractions. Problems were not presented in context. The Brigance has reported high levels of interrater, test-retest, and alternative forms reliability, yielding an acceptable level of internal consistency reliability. Validity information for the Brigance instrument were also at acceptable levels for reported predictive and discriminate validity in relation to other established group and individually administered achievement tests. No information was available on the subtests.

Curriculum-based measure. Because the part-whole subconstruct is dominant in most school-based mathematics curricula (Charalambous & Pitta-Pantazi, 2007), measures

documenting increases in performance of equivalencies as part-whole fractions as well as ratio fractions are necessary to ensure that increased student performance is transferrable to measures used by the student's school district. Curriculum-Based Measurement is one way to obtain this assurance. Hosp, Hosp, and Howell (2007) defined Curriculum-Based Measurement, or CBM, by several attributes:

- A focus on alterable variables
- Alignment with the curriculum that is being taught (content is the same and the questions look the same)
- Established reliability and validity based on standardized measures
- Standard procedures for implementation

To determine the effectiveness of ratio-based instruction on students' understanding of fraction and ratio equivalency, a pre and posttest of 20 items were pulled from the district curriculum, *Envision Mathematics,* Level 3, Chapter 12 (Charles, Caldwell, Crown & Fennell, 2011). Items from this chapter were used to construct a curriculum-based measure that served as a pre and posttest. As required in the development of CBM (Deno, 1989; Foegan, Jiban, & Deno, 2007), the researcher examined each lesson within the chapter that taught fraction equivalency. From these lessons, the researcher pulled every other problem from the text practice questions to construct the pre and posttest measure. Items in the CBM included situated problems (e.g. word problems), abstract problems, or problems that require students to judge the correctness of given equivalency statements (Deno, 1989; Foegan et al., 2007).

Reliability and validity of the pre and post-tests were confirmed in several ways and are reported in Chapter 4. First, with data from study students, internal consistency reliability was

calculated for the pre and posttests of non-experimental group student members by estimating how well the items that reflect the same construct yield similar results. Cronbaugh's Alpha is mathematically equivalent to the average of all possible split-half estimates and was used to examine the consistency of results for different items for the same construct within the measure. High coefficients (e.g. above 0.70) provides evidence of internal consistency reliability (Nunnally & Bernstein, 1994).

To examine the validity of the pre and posttests, the researcher used data from students' scores on the Brigance Q6 subtests as well as their scores on the pre and posttests. Validity of the pre and posttests was measured against performance on the Brigance Q6 subtest. To examine how the pre and posttests correspond with the Brigance Q6 subtest, bivariate correlation coefficients (Pearson *r*) were computed between the study students' pre and post test scores and the Brigance Q6 subtest raw scores. The calculation provided the extent to which convergent validity existed between the measures.

Research Question Two

Semi structured pre and post interviews. Because the CBM and standardized measures used to evaluate research question one were found to not be sensitive enough to uncover strategy use and levels of multiplicative thinking involved with understanding ratio fractions, a videotaped semi-structured clinical interview (see Appendix C) pretest and posttest was administered to three strategic students in each experimental group. The interview questions were used to uncover how these areas may have changed before and after instruction for these three students: One student who had a MLD, one student who struggled, and one student labeled as typically achieving. The interview covered all concepts targeted in the teaching sequence. The pretest was administered the week before the commencement of teaching and the posttest

was administered the week following the completion of the ratio instruction. Problems, which were composed by Stephen (2010) and based on the work of Battista and Borrow (1995), Lamon (1993), and Van de Walle (2004), were organized into "strands" with several variations of certain types of problem situations. Varying numerical ratio values were used throughout each question. If a student could not answer two or more of the questions within a strand correctly, the remainder of the strand (i.e. questions) was not administered. Problems included:

- Understanding a ratio as a unit, which involves the student being asked to look at a rule (e.g. one pancake can feed 3 students) and determining whether subsequent pictured situations show enough food, too much, or too little for the number of students pictured (Lamon, 1993a). Alternates for this situation are also presented to students, which involve pictures of aliens and food bars eaten. Students must determine who gets more food (Lamon, 1993b).

- Iterating linked composites, which involves the student being shown a bundle of five blue chips and a bundle of three red chips and being asked how many of the same kind of bundles would have to be there if the interviewer had 10 blue chips (Battista & Borrow, 1995; Cortina & Zuniga, 2008; Grobecker, 1997). The ability to iterate two numbers as one unit "can serve as the foundation for future meaningful dealings with ratio" (Battista & Borrow, 1995, p. 4).

- Solving situated equivalency problems by iterating linked composites, in which students are asked to determine how many of a certain quantity would be present if a certain "rule" is known (e.g. Three balloons can be bought for two dollars; how many dollars would 24 balloons cost?) (Lamon, 1993b). The purpose of these types of

questions is to uncover students' use of strategy (e.g. build up, shortened iteration, multiplication/division) and multiplicative thinking. The researcher will note the strategies used as they relate to these phases.

- Solving abstract equivalency problems in which students are asked to generate equivalent ratios and asked to determine how many equivalent ratios exist for a given unit (Grobecker, 1997; Van de Walle, 2004).

Triangulation of the data was used to improve the validity and reliability of the qualitative research (Creswell, 2007). For the purposes of this study, triangulation was achieved by the inclusion of three 3rd grade teachers (e.g. the researcher and two research assistants) in the coding and theming of data collected (Glense, 2006). Other verification strategies for the interview process including data analyses, resulting codes and themes, and guards against external threats to validity were used. First, two independent coders reviewed the code book and resulting write up at stage three data analyses (Grbich, 2007). Codes were deemed to be reliable if the coders achieve 80% agreement or greater. Coders reached a consensus on their disagreements. Second, reliability of source information was obtained through the use of verbatim translation (Grbich, 2007). Third, students interviewed were matched on their pretest scores as well as their disability status (Creswell, 2007). Finally, results of the analyses were reviewed with students as a means of member checking to ensure consistency in data reporting (Glense, 2006; Grbich 2007).

Research Design

A quasi experimental pretest-posttest design with mixed methods was utilized in the study. The design examined the effects of fraction intervention based in ratios on performance

of fraction equivalency as recorded by two measures between the experimental and control groups, between students with MLD/students who struggle and students deemed as typically achieving, and any interactions. Additionally, patterns in performance among students with MLD/struggling students and non-struggling students were examined as well as qualitative thematic analyses.

Procedures

Table 5 outlines the general procedures and research timeline used in the study.

Table 5. ***Research Procedures Timeline***

	Pretests & Interviews	Intervention	Posttests & Interviews
RG	X	Ratio Instruction	X
CG	X	----	X

The intervention group completed (1) the pretests and interviews prior to any instructional treatment and (2) the posttests and interviews immediately following the instructional treatment. The intervention and control groups also completed a social validity questionnaire immediately preceding and following the intervention instruction. The control group completed the pretests and posttests at the same time as the experimental group.

Data Collection Procedures

Once the control and experimental groups were established, both groups were administered a pretest measuring fraction equivalency performance. Additionally, three students in the experimental group were administered a semi-structured interview to uncover their understanding of equivalency through ratio interpretations. A social validity measure of student satisfaction was administered. After the pretest was completed, the experimental group entered

into the ratio-based intervention. During the experimental group completed ratio-based intervention, students in both the control group and the experimental group received instruction in equivalent fractions in their mathematics classrooms. Lessons taught by third grade classroom teachers were taken from NCTM's *Illuminations* website. Namely, "Fun with Fractions" (lessons one through five) were used along with "Fun with Pattern Block Fractions" (lessons one through three) were utilized in students' regular mathematics classrooms during the supplemental period of ratio-based instruction. Students in the control group did not receive the ratio-based supplemental instruction. After instruction in the experimental group was complete, both groups were given a posttest measure of equivalency performance. The same three students in the experimental group were given a second semi-structured interview to uncover their understanding of fraction equivalency through ratios. A social validity measure of student satisfaction was administered for a second time to both groups.

Instructional Procedures

During the study, the experimental group received supplemental conceptual fraction equivalency instruction based in ratios (RG), and the control group did not. The experimental group was taught in groups of five students. Instruction was scripted. The ratio-based instructional scripts were created and piloted in a previous study. The ratio group received instruction five times a week in 40 minute instructional sessions. Nine sessions of instruction commenced over a two week period. Two sessions focused on understanding of the unit and were delivered using concrete (e.g. manipulative-based) and representation (e.g. picture-based) instruction. Four sessions focused on the formation of conceptual understanding of additive and multiplicative means to generate equivalent fractions and were delivered using representational (e.g. pictorial or tabular) instruction. The remaining sessions focused on abstracting the fraction

symbols and the procedural processes of finding fraction equivalencies were delivered using abstract (e.g. symbol-based) instruction.

Ratio-based instructional group teaching procedure. Four instructional strategies found to promote increased outcomes in mathematics for students with MLD and students who struggle were found in the literature. These practices were all included in the ratio-based teaching sequence: concrete-representational-abstract instruction (Butler et al., 2003; Jordan et al., 1999), explicit instruction in regards to instructional sequencing, concept formulation, and multiple strategy use (Jitendra et al., 2003), student use of representations to support development of mathematics knowledge (Xin et al., 2007), and verbalizing mathematics reasoning (Woodward et al., 1999). These elements of effective mathematics instruction were incorporated into the instructional sequence used to deliver ratio based instruction. Each day of ratio-based instruction was delivered using a three part instructional sequence. Within the sequence, the previous day's work was summarized, the lesson scenario for the current day was given, and the types of problems that students would work on that day were presented. In part one, a specific problem was presented to students to complete. The teacher handed out student materials (e.g. worksheets, manipulatives, and paper) and read the scenario given at the top of the day's worksheet. The teacher then showed an example problem using a think aloud of the problem situation, solution, and representation promoted for that day's lesson as required by the teacher script. The script outlined what the teacher would say during the example problem regarding 'think aloud' used, representations employed, and the modeling of what a good explanation of a problem solution was. Then, a student was chosen to read the first problem aloud to the group.

In part two, students worked on the problem given during part one for a period of time on their own, in pairs, and then as a group. The teacher displayed a transparency with thinking questions designed to help students think about the problem and told the students to solve the problem on their own for two minutes while keeping the questions on the transparency in mind. A timer was then set for two minutes and students began to work the problem on their own using the questions on the transparency. After two minutes passed, the teacher instructed the students to share their solutions with a partner for two minutes, and a two minute timer was set again. As students shared, the teacher was instructed by the teaching script to look for a student to present their work to the class. The script also allowed for student self-selection. Observing students were asked the questions on the transparency. During each presentation of the problem solution, the teacher used questioning strategies, counterarguments, and if needed, explicit modeling and think aloud strategies to ensure student understanding of the problem situation and solution. Specifically, if the students could not agree on a correct solution (e.g. multiple answers were found that did not match) or produced an incorrect solution, the teacher utilized a counter argument (e.g. I worked this problem with a group of students last week, and they explained it like this [explanation]. What do you think of their answer?). If the counterargument did not produce a consensus on the correct answer, the teacher was directed by the script to explicitly model the problem solution utilizing a think aloud.

Finally, part three contained further questioning strategies to help students reflect on the reasonableness of their solutions. Questioning strategies such as, "Why does your answer make sense?" and "Is there an alternate solution that might be used?" were employed by the teacher. The teacher used the teaching script to know what questions to use in part three of the problem.

This sequence was repeated for each problem worked on during the lesson. The following paragraphs outline the mathematics involved in the intervention in detail (also see Table 6).

Table 6. ***Instructional Sequence - Ratios***

Lesson Number	*Lesson*	*Purpose*	*Mathematics Involved*	*Representations*	*Teaching/Questioning Sequence*
1	Order Up!	Teach	Double Counts; Multiplication	Concrete	Launch-Explore-Summarize
2	Out of Pancakes	Teach	Unitizing	Concrete/Pictorial	Launch-Explore-Summarize
3	The Kitchen	Teach	Additive Iteration	Pictorial	Launch-Explore-Summarize
4	Morning Preparation 1	Teach	Additive and Multiplicative Iteration	Pictorial/Tabular	Launch-Explore-Summarize
5	The Cook's Disagreement	Formative Assess	Additive and Multiplicative Iteration	Pictorial, Tabular, Abstract	Abbreviated Launch-Explore-Summarize
6	Morning Preparation 2	Teach	Multiplicative Iteration	Pictorial/Tabular	Launch-Explore-Summarize
7	Morning Preparation 3	Teach	Multiplicative Iteration, Between	Tabular, Abstract	Launch-Explore-Summarize
8	From the Kitchen to the Table	Teach	Multiplicative Iteration, Between and Within	Abstract	Launch-Explore-Summarize

The first instructional session (one and two) asked students to consider an amount iterated a number of times (e.g. Mauricio ordered five pieces of bacon; Nicosha ordered twice as much; Katy ordered three times as much, etc). The activity forced students to consider a double count (five to one, ten to two, fifteen to three, etc). The use of double counts set the stage for the use of correspondences found in ratio interpretations. In the next instructional session (e.g. two), students worked with scenarios involving relationships between cans of pancake batter and the

corresponding amount of pancakes made. For instance, a relationship given to students was "one can of batter makes two pancakes". Considering the relationship given, students were given pictorial representations of certain numbers of cans (i.e. six) and a certain amount of pancakes (i.e. 20) and were asked to discuss whether the amount of pancakes shown were too few, just enough, or too many for the amount of cans (Lamon, 1993a; 2005). Students were encouraged and shown how to use manipulatives and picture representations to aid in their reasoning.

The teacher used specific questioning (e.g. *How can you represent the problem situation using the manipulatives? How can you show the pancakes? How can you show the cans? How can you show the problem situation using pictures? The pancakes? The cans?*) as directed by the script to aid in student discussion of the solutions and use of representations to solve the problems. If students did not produce or agree on a correct response, the teacher used further questioning strategies (e.g. *what is a rule that you can use to determine the answers? What has to stay the same in all the situations?*) to guide the student's thinking. When the correct solution was given, the teacher further pressed the students to make sense of their thinking (e.g. *what process did you use to determine the answers to the problems? Why do your answers make sense?*). The exercises and teacher questioning used in this lesson were designed to aid in students understanding that the relationship between cans and pancakes needed cannot change when additional cans or a number of pancakes are added to a situation. This further established the recognition and importance of the concept of the unit in understanding equivalency (Streefland, 1993).

Throughout the next two instructional sessions (three and four), students worked again with several given relationships of cans and pancakes (for instance, one can makes four pancakes). From the given relation, students were asked to find missing values for a given

certain numbers of cans or an "order" for a certain number of pancakes (e.g. How many cans of batter are needed to make 28 pancakes?). Students were again prompted to use a modeled pictorial representation to aid in understanding the problem, and the teacher utilized questioning strategies if necessary to guide their thinking (e.g. *How can you show the problem situation using pictures? The pancakes? The people?*). As students described their thinking and illustrated through pictures the iteration of the unit relationship during the first two to four problems, the teacher constructed ratio tables to augment understanding, at first as a supplement to the representations used by students (Streefland, 1993). The teacher asked the students if the table he/she constructed showed the same answer as the pictures and the manipulatives used by the student. If the student said no, the teacher provided a counterargument and explicit teaching as needed.

As the instructional sessions progressed, students began to construct the tables, and were told to use ratio tables as opposed to pictures to aid in their thinking, although some students continued to draw pictures in addition to the tables. Student understanding was aided to make sense of equivalencies through the use of ratio tables, finding unit rates, and iterating linked quantities to comprise equivalent situations (Fosnot & Dolk, 2002; Lamon, 1993a; Streefland, 1993). As students constructed ratio tables, the teacher used specific questioning (e.g. *what patterns do you see in the tables that you constructed? How can you use patterns in the tables to help you find the answer?*) to aid in students' developing reasoning and use of additive processes to generate equivalent fractions. If the student responded incorrectly, the teacher utilized fictitious counter solutions, explicit teaching and think aloud strategies to aid in students' understanding. When correct answers were found, the teacher further pressed the students to make sense of their solutions through questioning (e.g. *What did you have to preserve in order to*

solve the problems? How did you use the preserved relationship to solve the problems? Why do your answers make sense? How can you prove that you are correct?).

Toward the end of the two sessions, students continued to work with ratio tables to derive solutions to the problems posed. To move students from additive to the beginnings of multiplicative strategies, the teacher used specific questioning strategies, such as "*How can you use patterns in the tables to help you find the answer?*" and "*How is multiplication and division seen in the tables you created*?" The teacher also asked students if they thought there was a way to shorten the ratio tables used to find the answer by using multiplication and division. Shortened ratio tables and the link between long and short tables were modeled by the teacher. With the last two problems involved in the sessions, students were asked to show both a longer (e.g.$\frac{3}{4} = \frac{6}{8} = \frac{9}{12} = \frac{12}{16} = \frac{15}{20}$) and a shortened (e.g.$\frac{3}{4} = \frac{15}{20}$) ratio table in their solutions if they did not supply it themselves. When correct solutions were found, the teacher further pressed the student's thinking, asking if their answers made sense and why; how they could prove they were correct; and what they had to preserve to solve each problem.

In the fifth instructional session, students were given a sample problem and several fictitious "responses" that contained drawings, ratio tables, and "shortened" ratio tables (i.e. ½ = $^2/_4$) displaying multiplicative between and within relations. After solving the problem for themselves, students were asked to determine which of the solutions were correct, why they were correct, and why the incorrect solutions were wrong (Griffin, Jitendra, & League, 2009; Jayanthi et al., 2008). Teacher facilitated student's thinking through specific questioning (e.g. *How can we build up from what we know to find the amount of needed pancakes? Which cook's method produces a correct answer? Could more than one cook be correct? How are the methods alike?*

How are they different?). Counterexamples were again used by the teacher if students' thinking is erroneous.

In the final three instructional sessions, students worked with similar problems as presented in the second and third instructional sessions that were presented both within and out of the context of the pancake scenario used previously. Many of the questions were presented in "fraction form" as opposed to pictorial (e.g. ratio tables) form. Students were encouraged to use tables to find answers to problems. As the sessions progressed, however, students were encouraged through teacher questioning (*e.g. How can we build up from what we know to find the amount of needed pancakes using multiplication? Can we use the relationship between people? Between pancakes? Between people and pancakes*?) to move toward abstract representation (e.g. traditional fraction notation) and the use of multiplicative relationships to solve the problems. By the end of the instructional sessions, students worked to shorten ratio tables used, solve traditional abstract fraction equivalency problems, determine when two fractions are equivalent, and used multiplicative thinking to derive missing equivalency values (Ni, 2001). The teacher pressed for further evaluation of student responses through specific questioning (e.g. *Was your strategy the most efficient? Can you show another way to find the missing value?).*

Control group. The control group did not receive any of the ratio-based supplemental instruction. This group continued in their normal mathematics classroom and received instruction according to the county's Order of Instruction for third grade. During the time supplemental ratio instruction took place, students in both the control group and the experimental group received instruction in equivalent fractions in their mathematics classrooms. Lessons taught by third grade classroom teachers were taken from NCTM's *Illuminations* website.

Namely, “Fun with Fractions” (lessons one through five) were used along with “Fun with Pattern Block Fractions” (lessons one through three) were utilized in students’ regular mathematics classrooms during the supplemental period of ratio-based instruction. Students in the control group did not receive the ratio-based supplemental instruction. The experimental group was separated by their exposure to supplemental instruction based in the ratio sub construct.

Fidelity Measures

Through treatment fidelity, the impact of an intervention can be concluded by: (1) determining how an intervention affects student outcomes; and (2) allowing understanding of the intervention and its potential to contribute to an outcome (Gersten et al., 2005). This study utilized a framework for fidelity of treatment developed by Bellg and his colleagues (2004). The framework consists of (a) study design, (b) training, (c) treatment delivery, (d) treatment receipt, (e) and treatment enactment. When followed correctly, internal validity of the study is increased. Thus, student outcomes due to intervention can be attributed appropriately. Table 7 shows how the framework was utilized to show treatment fidelity in this study.

Table 7. ***Treatment Fidelity***

Fidelity Area	Purpose	Evidence of Fidelity of Implementation
Study Design	Showing that design adheres to theory/practice and allows for setbacks.	Scripts to fix duration of intervention, frequency of contacts between teacher/students, frequency of student participation. Accounting for student drop out in N.
Training	Ensuring the delivery of the intervention is systematic.	Checklists, Teacher Scripts Observer training and point-by-point inner observer agreement on delivery.
Delivery of Treatment	Ensuring fidelity of treatment delivery among teachers.	Inter observer reliability with point-by-point agreement. Videotaping of sessions.
Treatment Receipt	Are students learning during treatment?	Pre and Post Measures. Interviews. Standardized measures.
Treatment Enactment	Can students employ the strategies taught?	Pre and Post Measures. Standardized measures. Interviews.

Borelli et al. (2005) defined high treatment fidelity in educational studies as those that showed evidence of as 80% or more of the following five key fidelity strategies: study design, training, treatment delivery, treatment receipt, and treatment enactment. Thus, fidelity of treatment was defined as values of at least 80% in all five areas.

Study Design, Training, and Delivery of Treatment

To guide the conceptual teaching in the ratio unit, teaching scripts were created by the researcher (see Appendix D). For each day of instruction, a checklist that contained the instructional questions and procedures was used to assess the instructor's adherence to the

assigned teaching methods (see Appendix E). To minimize the risk of internal validity errors, prevent bias, and establish reliability of instructional procedures and observation data gathered during the intervention conditions, checks were conducted by two independent observers who were undergraduate students in honors research programs (Gerstein et al., 2005). Thirty percent of conducted sessions were observed. Observers were trained on how to use the checklist.

During observations, each observer used their checklist to evaluate if critical instructional components were utilized during the intervention. Measuring the reliability of the independent variable was achieved through the measurement of rate of student participation, rate of teacher feedback, and correctness of teacher feedback as indicated by the script. For rate of student participation, observers noted if the teacher called on different students to (a) read a question out loud; (b) present their solution; (c) respond to questions posed during presented solutions; and (d) respond to the summary reflection questions. For rate of teacher feedback, observers noted (a) if teachers acknowledged correct versus incorrect explanations offered by the group during problem solutions and during reflection/summary. For correctness of teacher feedback, observers noted (a) if the teacher showed a counterargument, if students did not agree on a correct answer or agreed on an incorrect answer; and (b) if the teacher used an explicit think aloud, if the counterargument did not produce understanding of the problem from the group.

Point-by-point agreement was used to assess inter rater reliability. For each item on the checklist, observers rated if necessary instructional elements were used by the teacher (absent/present). Inter rater reliability was then calculated by dividing the number of agreements by the sum of the number of agreements and disagreements and multiplying by 100 (e.g. [#agreements/ (#agreements + #disagreements)] x 100).

Treatment Receipt and Treatment Enactment

Testing and scoring procedures. Solutions to the problems on the pre and post measures were scored as correct (1 pt) or incorrect (0 pts). Inter scorer reliability was calculated using item-by-item agreement. A level of 80% agreement was deemed an acceptable inter scorer reliability. Answers solicited from the pre post semi-structured interviews were interpreted using thematic analyses.

Inter scorer reliability. Reliability checks were conducted for each of the measures that involved scorer judgment. Inter scorer reliability was determined by having a second trained scorer independently score all of the pre and posttests completed as well as the maintenance measures used. A randomly selected sample of 25% of the tests was used to evaluate the extent of Inter scorer reliability. The scoring was compared to the researcher's scoring item-by-item to determine the number of agreements and disagreements on response types for each measure.

Social validity. Wolfe (1978) argues research-based educational interventions delivered with social validity needs to prove valuable to society on three levels: (1) the significance of the goals; (2) the acceptance of the intervention and its procedures; and (3) the satisfaction of the students with the intervention. These forms of social validity were implemented in the study in the following ways. First, students in the intervention were given a short questionnaire relating to their self-perceived ability to work with equivalent fractions and their overall satisfaction with the intervention (see Appendix F). Next, the intervention and its procedures were reviewed by an expert in mathematics education, an expert in special education, and a mathematics teacher to ensure acceptance and practical significance of the intervention were present. Finally, the

measures of fraction equivalency performance were derived from the curriculum that central Florida teachers use to educate and prepare students for such events as standardized testing.

Data Analyses

Research Question One

To test the amount of change in the dependent variables as a result of the independent variable (e.g. intervention), the researcher utilized several parametric tests within a quasi-experimental pretest-intervention-posttest design. The researcher used the Statistical Package for the Social Sciences, version 16 (SPSS. To analyze research question one, a between factors MANOVA with repeated measures (a doubly multivariate MANOVA) was utilized. Overall effects of the independent variable were analyzed over two time periods as well as the between factor effects. Follow up analyses was performed using either step down analyses (for significantly correlated dependent variables) or univariate ANOVAs (for non-statistically dependent variables). For step down analyses, an a priori decision to prioritize dependent variables was made. The standardized measure was given first priority and the CBM is given second priority.

It is important to note that, originally, "student type" (TA, SS, or MLD) was separated into three different groups. However, because only four students were identified as MLD, separating students with MLD from students who struggled would have created more cases than dependent variables in each of the MLD cells, causing a threat to statistical power (Tabachnick & Fidell, 1996). Thus, for quantitative analyses, students with MLD and students who struggled were combined into one group.

Research Question Two

The analyses of interview data involved several stages of identifying, sorting, and analyzing involved in a thematic analyses described by Grbich. First, all pre and post-interviews were videotaped and transcribed verbatim; the tapes were then destroyed. Transcripts were then entered into Microsoft Excel spreadsheet software for organization. A team made up of the researcher and two research assistants then reviewed student one's ("Albert" – TA) interview transcripts (pre and post) to identify emerging codes and themes. A quantification of various strategy usages (coding guided by previous research of Battista and Borrow (1996), Grobecker (1997), Lamon (1993b) and levels of multiplicative thinking as defined by Battista and Borrow (1996) were also undertaken. Battista and Borrow (1995) suggest students must move through three phases of understanding in their developing understanding of equivalency situations as ratios: (1) conceptualizing explicitly the linking action of two composite amounts; (2) understanding multiplication/division and its role in the iteration process; and (3) abstracting iterative processes and connect them to the meaning of multiplication and division. Along with multiplicative understanding, Lamon (1993b) suggests student evidence of strategy usage while developing multiplicative understandings involved with ratios: (1) Avoiding (no interaction with the problem), (2) Visual/additive (trial and error, incorrect additive linkages), (3) Pattern building (oral or written patterns without understanding number relationships), (4) Pre-proportional reasoning (pictures, charts, or manipulatives evidencing relative thinking), (5) Qualitative proportional reasoning (ratio as unit/relative thinking/some numerical relation understandings), and (6) Quantitative Proportional reasoning (understanding of symbols, functional and scalar relationships). But can ratios be used to teach fraction concepts that underlie equivalence, such as partitioning and unitizing? Identified codes were defined; the

codes and their definitions resulted in the first copy of the codebook for the study. Next, the research team independently examined student two's ("Carl" – SS) pre and post- interviews and met to discuss and agree on findings. Researchers added and deleted themes until they reached consensus on the information, which resulted in a revised codebook (version two). A quantification of various strategy usages occurred and levels of multiplicative thinking was undertaken. Finally, each member of the research team analyzed student three's ("Bill" – MLD) pre and post interviews first in isolation and then again as a group. The research team examined codes within and across the groups as well as quantifications for strategies and levels of multiplicative thinking. Additionally, any atypical strategies were also noted. Necessary changes were made to arrive at a consensus; final checks were performed on all codes to ensure accuracy and consensus. Related codes were condensed.

CHAPTER FOUR: RESULTS

The primary purpose of this study was to investigate the impact of a ratio-based fraction teaching sequence on students identified as having a MLD. The researcher evaluated struggling and non-struggling student performance and examined group differences in understanding through qualitative analyses. Two research questions were presented for analyses. The purpose of this chapter is to present the results related to each research question from a statistical analyses of student data collected before and after implementation of the teaching sequence.

Question One: Data Analyses and Results

Research question one addressed overall differences between experimental and control groups as measured by two tests administered before and after ratio-based instruction. Question one was as follows:

> Are there statistically significant differences in overall performance (i.e. the number of correct responses) on a curriculum-based measure of fraction and ratio equivalency and on a standardized measure for students with mathematical learning disabilities, struggling students, and typically developing students who do and do not participate in ratio-based fraction instruction?

To answer this question differences relating to time of test (pre or post intervention), student type, ethnicity, and group assignment were evaluated. Data were collected from 38 students (19 in the control group and 19 in the experimental group). Measures for the analyses were a CBM of fraction equivalency and a standardized transfer test (the Brigance Q6 subtest). The CBM measured student performance in generating equivalent fractions presented in pictorial, abstract,

and word problem format. The standardized measure contained abstract equivalency problems. Both measures were repeated measures.

A moderate to high degree of correlation was expected between the CBM and standardized measure because similar items appear on both tests. Correlations from the experimental and control groups' pretest outcomes are given in Table 8.

Table 8. ***Correlations between Pretest Means for Experimental and Control Groups***

	CBM	**Transfer Test**
CBM	1.00	0.54
Transfer Test	0.54	1.00

Outcome Analyses for Question One

To analyze the results from question one, a factorial multivariate analyses of variance (MANOVA) with repeated measures was performed on two dependent measures: A CBM and a standardized measure of fraction equivalency. The within subject factor, or repeating factor, was the time of testing (e.g. pretest or posttest). Between subject factors were specified as group (ratio intervention and control), student type (TA or SS), and student ethnicity (Caucasian, African American, or Hispanic). Originally, "student type" (TA, SS, or MLD) was separated into three different groups. However, because only four students were identified as MLD, separating students with MLD from students who struggled would have created more cases than dependent variables in each of the MLD cells, causing a threat to statistical power (Tabachnick & Fidell, 1996). Thus, for quantitative analyses, students with MLD and students who struggled were combined into one group.

Order of entry of grouping variables was group assignment, student type, and then ethnicity. Total N was 38. No univariate or multivariate within cell outliers existed at $p = 0.001$.

No data were missing. Box's Test of Equality of Covariance Matrices was not significant, $F_{(10, 1033.88)} = 0.981$, $p = 0.459$. All evaluations of assumptions of normality, linearity, and multicollinearity were satisfactory. Group means and standard deviations for intervention and control groups on the two DVs across pre and posttest time periods can be found in Table 9.

Between variable main effects. Using Wilk's criterion, the combined dependent variables were significantly affected by both group assignment, $F_{(2, 26)} = 10.597$, $p = 0.000$, and student type, $F_{(2, 26)} = 5.032$, $p = 0.014$, but not by their interaction, $F_{(2, 26)} = 0.434$, $p = 0.653$. Error degrees of freedom of over 20 confirmed the robustness of the test. The results reflected a large association between group assignment and scores on the combined dependent variables, partial $\eta^2 = 0.449$. The results reflected a moderate association between student type and scores on the combined dependent variables, partial $\eta^2 = 0.279$.

Using Wilk's criterion, the combined dependent variables were not significantly affected by ethnicity, $F_{(4, 52)} = 1.425$, $p = 0.239$, the interaction of ethnicity and student type, $F_{(4, 52)} = 0.350$, $p = 0.843$, or the interaction of ethnicity and group assignment, $F_{(4, 52)} = 0.837$, $p = 0.319$. Furthermore, the interaction of ethnicity, student type, and group assignment was not statistically significant, $F_{(2, 26)} = 0.959$, $p = 0.581$. The results reflected a small statistical association between ethnicity and scores on the combined dependent variables, partial $\eta^2 = 0.09$. The results reflected almost no association between interaction of ethnicity with student type, the interaction of ethnicity with group assignment, and the interaction of ethnicity with student type/group assignment, partial $\eta^2 = 0.026$, 0.085, and 0.041, respectively.

Within variable main effects. Using Wilk's criterion, the combined dependent variables were significantly affected by both pre-posttest time, $F_{(2, 26)} = 11.362$, $p = 0.000$, and

the interaction of pre-posttest time and group assignment, $F_{(2, 26)} = 7.892$, $p = 0.002$, but not by the interaction of student type and pre-posttest time, $F_{(2, 26)} = 0.988$, $p = 0.855$. The results reflected a large association between pre-posttest time and scores on the combined dependent variables, partial $\eta^2 = 0.466$. The results also reflected a moderately large association between the interaction of group assignment and pre-post testing time and scores on the combined dependent variables, partial $\eta^2 = 0.378$.

Results were not significant for the interaction of student type, group assignment, and pre-posttest time, $F_{(2, 26)} = 0.988$, $p = 0.855$; the interactions of pre-posttest time, student type, and ethnicity, $F_{(2, 26)} = 0.988$, $p = 0.855$; or the interaction of pre-post time, student type, ethnicity, and group assignment, $F_{(2, 26)} = 0.988$, $p = 0.855$. Error degrees of freedom of over 20 confirmed the robustness of the test. Results reflected relatively no association between the interaction of pre-post testing time and student type (partial $\eta^2 = 0.012$), pre-post testing time, ethnicity, and student type (partial $\eta^2 = 0.045$), or interaction of pre-post time, student type, ethnicity, and group assignment on the combined dependant variables (partial $\eta^2 = 0.002$).

Step-down analyses. To investigate the impact of each significant main effect on the individual dependent variables, a Roy-Bargmann step-down analyses was performed. The analyses was used because the dependent variables were found to be correlated. Thus, the sole use of univariate ANOVAs would have increased the risk of Type 1 error. (Tabachnick & Fidell, 1996). Dependent variables were prioritized in the following order: Transfer, CBM. Both dependent variables were judged to be sufficiently reliable for step-down analyses.

Table 9. ***Pre-test and Post-test Group Means***

	Experimental Group n = 19				**Control Group n = 19**			
	SS		**TA**		**SS**		**TA**	
Measure	***Pre***	***Post***	***Pre***	***Post***	***Pre***	***Post***	***Pre***	***Post***
CBM	*1.25*	*10.13*	*5.00*	*13.09*	*3.20*	*2.20*	*4.12*	*3.57*
Transfer	*1.88*	*8.50*	*4.18*	*12.55*	*2.00*	*2.40*	*4.57*	*4.64*

In step-down analyses each dependent variable was analyzed, in turn, with the higher-priority variable treated as a covariate. The highest priority dependent variable, the transfer test, was tested first via univariate factorial repeated measures ANOVA (Tabachnick & Fidell, 1996). The error rate of five percent was split between the two dependent variables, resulting in alpha levels of 0.025 for each dependent variable examined. Mauchly's Test of Sphericity was significant, so the Greenhouse-Geisser statistic was used in the analyses. A unique contribution to higher scores on the posttest standardized measure was made by the interaction of group assignment and time of testing, step-down $F_{(1, 34)} = 27.818$, $p = 0.000$. Partial eta squared was 0.45, indicating a large association between the interaction of group and test time and score on the transfer test of fraction equivalency. Students in the experimental group performed significantly better (Pretest M = 3.21; SD = 2.89; Posttest M = 10.84; SD = 4.19) from pretest to posttest then students in the control group (Pretest M = 3.89; SD = 2.601; Posttest M = 4.05; SD = 3.358). Significant contributions to the post transfer test score was also made by time of testing, step-down $F_{(2, 26)} = 31.549$, $p = 0.000$. Partial eta squared was 0.481, indicating a large association between time of testing and scores on the post transfer test for the experimental group.

Group assignment was a third unique contributor to scores on the standardized measure, step-down $F_{(1, 34)} = 16.932$, $p = 0.000$. Partial eta squared was 0.332, indicating a moderately

large association between group assignment and scores on the transfer test. Students in the experimental group (M= 10.84; SD = 4.127) performed significantly better than students in the control group (M= 4.05; SD = 3.358). Student type also contributed to increased scores on the transfer test, step-down $F_{(2, 26)} = 11.605$, $p = 0.002$. Partial eta squared was 0.254, indicating a moderate relationship between student type and scores on the post transfer test. Students deemed typically achieving (M = 8.12; SD = 3.629) performed better on the standardized measure than students who struggled (M = 4.64; SD = 5.126) in the experimental group, with similar results observed in the control group.

The CBM was tested using a factorial ANCOVA with repeated measures. The post-standardized measure served as the covariate (Tabachnick & Fidell, 1996). Alpha was set at 0.025. Mauchly's Test of Sphericity was significant, so the Greenhouse-Geisser statistic was used in the analyses. The only significant contributor to higher scores on the CBM measure was the interaction of group assignment × pre-post test time, step-down $F_{(1,33)} = 9.556$, $p = 0.004$. Partial eta squared was 0.225, indicating a moderate association between the interaction of test time and group and higher scores on the CBM measure. Students in the experimental group increased their performance significantly from pretest to posttest (Pretest M = 3.42; SD = 3.22; Posttest M = 11.84; SD = 3.96) compared to students in the control group (Pretest M = 3.37; SD 3.303; Posttest M = 3.21; SD = 2.84). All other factors and factor interactions were not significant.

Summary

Group assignment, student type, and the interaction of group assignment and testing time were significant contributors to scores on the two outcome measures of fraction equivalency taken together. There are statistically significant differences in overall performance (i.e. the

number of correct responses) on a curriculum-based measure of fraction equivalency as well as on a standardized measure for students deemed typically achieving and struggling students who do and do not participate in ratio-based fraction instruction.

Students in the experimental group outperformed students in the control group on both measures of fraction equivalence. However, successful outcomes within the experimental group did not depend on student type, as no interactions were found between the two factors.

Because significant main effects for group and student type on the two measures were found, step-down analyses was performed to understand how group assignment, type, and student type affected outcomes on each of the dependent measures examined separately. Step-down analyses confirmed the unique contributions of the interaction of group assignment and time of testing to higher performance on the transfer test, favoring the experimental group. Students labeled as typically achieving outperformed student who struggled in both groups. On the CBM, the interaction of group assignment and testing time was significant- students in the experimental group outperformed students in the control group. No significant contribution was made for student type on the CBM. That is, students who struggled/MLD and student deemed typically achieving did not differ in their performance, pretest or posttest, on the CBM in either group.

Fidelity of Implementation: Dependent Variable

Inter-scorer agreement. Reliability checks were conducted for each of the measures that involved scorer judgment. Inter scorer reliability was determined by having a second trained rater independently score all of the pre and posttests completed as well as the standardized measure used. A randomly selected sample of 50% of both the CBM and transfer pre and posttests (40 of each test) was used to evaluate the extent of inter scorer reliability. The scoring

was compared to the researcher's scoring item-by-item to determine the number of agreements and disagreements on response types for each measure. The scorers agreed on 802 items out of 820 items on the CBM tests, for an inter-scorer agreement of 98%. The scorers agreed on 638 items of 656 items transfer tests, for an inter-scorer agreement of 97%.

Fidelity of Implementation: Independent Variable

Inter-rater reliability. Teaching scripts were used to deliver instruction. Checklists were used to assess the instructor's adherence to the assigned teaching methods (overall implementation and feedback). Inter-rater reliability checks were conducted by two independent observers to ensure fidelity of implementation (Gerstein et al., 2005). Two independent observers used their checklists to ensure critical instructional components were utilized during the intervention.

Overall implementation of script. Point-by-point agreement was used to assess inter rater reliability of the teacher's overall implementation of the teaching script. For each item on the checklist, observers rated if scripted instructional elements were used by the teacher. Inter-rater reliability was then calculated by dividing the number of agreements by the sum of agreements and disagreements and multiplying by 100 (e.g. [#agreements/ (#agreements + #disagreements)] × 100) (Kazdin, 1983). A randomly selected sample of 30% of the conducted sessions was observed (Kazdin, 1983). For all observations, 89% inter-observer agreement was obtained.

Teacher feedback. For rate of teacher feedback, observers noted (a) if the teacher acknowledged correct versus incorrect explanations offered by the group during problem solving and during reflection/summary. For correctness of teacher feedback, observers noted (a) if the

teacher showed a counterargument or an explicit think aloud if the counterargument did not produce understanding of the problem from the group. Out of the 30% of instances observed and recorded on the fidelity checklists, two observers, using point-by-point agreement, noted that the frequency of teacher feedback utilized in the lesson was 85%, with correctness of teacher feedback at 100%.

Student participation. For rate of student participation, observers noted if the teacher called on different students to (a) read a question out loud; (b) present their solution; (c) ask questions during presented solutions; and (d) respond to the summary reflection questions. Out of 100% of the instances observed and recorded on the fidelity checklists, two observers, using point-by-point agreement, noted that the teacher called on students an equal number of times during the lesson 83% of the time.

Reliability of CBM

Internal consistency reliability was calculated for the pre and posttests (CBM) by estimating how well the items that reflect the same construct yield similar results. Cronbaugh's Alpha was used to examine the consistency of results for 51 non experimental group students from pre to post test. Results of the reliability analyses generated an alpha coefficient of 0.712. Because the coefficient was above 0.70, evidence of internal consistency reliability was obtained (Nunnally & Bernstein, 1994).

Validity of CBM

To examine the validity of the pre and posttests, the researcher used data from student's scores on the Brigance Q6 subtests as well as their scores on the pre and posttests. Validity of the pre and posttests was measured against performance on the Brigance Q6 subtest. To examine

how the pre and posttests corresponded with the Brigance Q6 subtest, a bi-variate correlation coefficient (Pearson *r*) was computed between a sample of 71 third grade students' pretest and posttest scores and Brigance Q6 subtest raw scores. Results of the analyses generated a coefficient of 0.773 between the pretest and the CBM. Because the coefficient was above 0.70, evidence of convergent validity was obtained (Nunnally & Bernstein, 1994). Similar results were obtained between the CBM posttest and the Brigance Q6 subtest raw scores; a coefficient of 0.782 was obtained.

Social validity. Wolfe (1978) argued research-based educational interventions with social validity needed to prove valuable to society on three levels: (1) the significance of the goals; (2) the acceptance of the intervention and its procedures; and (3) the satisfaction of the students with the intervention. These forms of social validity were implemented in the study. Students in the intervention were given a short questionnaire relating to their self-perceived ability to work with equivalent fractions and their overall satisfaction with the intervention before and after instruction. The control group was also given the questionnaire.

Responses to the questionnaire were analyzed using an ANOVA with repeated measures to detect differences between experimental and control groups. 'Pre-Post Questionnaires' was the repeated measure; group assignment, student type, and ethnicity were between subject factors. Questionnaires were scored inversely; lower scores were associated with higher levels of social validity, while higher scores were associated with lower levels of social validity. Mauchly's test of sphericity was significant, so the Greenhouse Geisser statistic was used in the analyses. Using Greenhouse Geisser, the interaction of group assignment and pre-post questionnaires was significant, $F_{(1, 27)} = 7.010$, $p = 0.013$. Partial eta squared was 0.206, indicating a small to moderate association between group assignment × time and scores on the

social validity measure. Students in the experimental group (PreM = 21.58, SD = 4.06; PostM =16.95, SD = 4.515) provided a lower score on the questionnaires then students in the control group (PreM =19.47, SD = 3.007; PostM = 21.89, SD = 3.43). No other factors or interactions of factors reached significance.

Overview of Qualitative Data Analyses

Because the CBM and standardized measures used to evaluate research question one were not sensitive enough uncover strategy use and levels of multiplicative thinking involved with understanding ratio fractions, research question two was developed to uncover how these areas may have changed before and after instruction for three students One student who had a MLD, one student who struggled, and one student labeled as typically achieving were interviewed before and after the instructional sequence. Question two was as follows:

> What is the multiplicative thinking and strategy usage of students when presented with ratio equivalency situations? Do multiplicative thinking and strategy usage increase for students with MLD and students who struggled after participating in a ratio-based equivalency instructional sequence?

Question 2: Data Analyses and Results

Table 10 provides an overview of the two categories and eight themes uncovered in data analyses. The three categories were: (a) Multiplicative Thinking and (b) Strategies. The indicator categories and nine themes include codes uncovered in both pre and post interviews. Three students, Albert (age 9, male, Caucasian, typically achieving); Bill (age 9, male, Hispanic, MLD); and Carl (age 9, male, Hispanic, Struggling) participated in two separate semi-structured interviews to assess their levels of strategy use and multiplicative thinking before and after the

ratio based instructional sequence. The interviews took place in a public elementary school in central Florida. Interviews were conducted in May 2011, once at the beginning and again at the end of the instructional period. The interviewer was the researcher.

Battista and Borrow (1995) suggest students must move through three phases of understanding in their development of understanding of equivalency situations as ratios: (1) conceptualize explicitly the linking action of two composite amounts; (2) understand multiplication/division and its role in the iteration process; and (3) abstract iterative processes and connect them to the meaning of multiplication and division. Along with multiplicative understanding, Lamon (1993b) suggests many levels of strategy usage students evidence to reach multiplicative understandings involved with ratios: (1) Avoiding (no interaction with the problem), (2) Visual/additive (trial and error, incorrect additive linkages), (3) Pattern building (oral or written patterns without understanding number relationships), (4) Pre-proportional reasoning (pictures, charts, or manipulatives evidencing relative thinking), (5) Qualitative proportional reasoning (ratio as unit/relative thinking/some numerical relation understandings), and (6) Quantitative Proportional reasoning (understanding of symbols, functional and scalar relationships). Grobecker (1997) found some evidence of these levels of multiplicative thinking and strategies amongst middle school students with MLD. It remained an empirical question as to whether ratios could be used to teach fraction concepts that underlie equivalence and grow levels of multiplicative thinking and strategy use in students who struggled and students with MLD.

Table 10. ***Multiplicative Thinking and Strategy Use***

Category	*Themes and Indicators*		
	MLD	SS	TA
Strategies	Typical, Correct *Patterning/Matching*	Typical, Correct *Patterning/Matching* *Pre Ratio/Build Up* *Abbreviated Build Up* *Iteration with Operations, Between* *Iteration with Operations, Within*	Typical, Correct *Patterning/Matching* *Pre Ratio/Build Up* *Abbreviated Build Up* *Iteration with Operations, Between* *Iteration with Operations, Within*
	Atypical, Incorrect *Incorrect Partitioning Strategy* *Misuse of Correspondence* *Misuse of Ratio Unit*		
	Typical, Incorrect *No Strategy*	Typical, Incorrect *Incorrect Additive Linking of Composites*	Typical, Incorrect *Incorrect Additive Linking of Composites*
Levels of Multiplicative Thinking	No Linking of Quantities	No Linking of Quantities	No Linking of Quantities
	Linking of Quantities as a Ratio	Linking of Quantities as a Ratio	Linking of Quantities as a Ratio
	Explicit Conceptualization of the Repeated Action of Linking Composites	Explicit Conceptualization of the Repeated Action of Linking Composites	Explicit Conceptualization of the Repeated Action of Linking Composites
		Understand Meaning of Multiplication/Division and its Role in Iterative Process	Understand Meaning of Multiplication/Division and its Role in Iterative Process
		Abstracting Iterative Process	Abstracting Iterative Process

Multiplicative thinking. The first five themes dealt with the levels of multiplicative thinking evidenced by students during the semi structured interviews. Levels were defined using the criteria set forth by Batista and Borrow (1995) as a framework for the analyses.

Theme one: No linking of quantities. *No Linking of Quantities* was defined as a student showing no evidence of the ability to coordinate two number sequences simultaneously (Batista & Borrow, 1996). This type of thinking was evident among ten of Bill's (a student with MLD) solutions, one of Carl's (a student who struggled) solutions, and three of Albert's (a student who was typically achieving) solutions:

I: "If 5 kids can be watched by one teacher, how many teachers are needed to watch 25 kids?"

Bill: "So, I have to draw twenty kids? [Draws twenty five stick figures]. There are twenty kids. I counted all of these, and then I took away five of them and I got 20."

Theme two: Linking of quantities as a ratio. Students who *Linked Quantities as a Ratio* had the ability to coordinate two numbers at the same time and may have shown ability to iterate linked composites, but did so additively and with some degree of difficulty. Students at this level of multiplicative thinking often did not show fluency with arithmetic operations of addition and subtraction and did not possess a conceptual understanding or fluent recall of multiplication or division. This type of thinking was evident in three of Carl's (SS) solutions, one of Bill's (MLD) solutions, and no solutions from Albert (TA):

I: If one plate gets four pieces of silverware, how much silverware do 14 plates get?

Bill: "The fourteen plates and the other things, 4 of them for every plate [Attempts to count silverware by four's but arrives at incorrect answer of forty]".

Carl: "[draws 14 circles each with four lines by it] 4, 8, 12, 16, 20, 24, 28, 32, 36, 40, 44, 48, 52, 56".

Theme three: Explicit conceptualization of the repeated action of linking composites. Explicit Conceptualization of the Repeated Action of Linking Composites was defined as the ability to conceptualize the necessary repeating action behind the iteration of linked composites. Students at this level of multiplicative thinking could use linked composites to iterate ratio units quite easily, but they did so without connecting the act of iteration to multiplication and division. Students at this level of multiplicative thinking often were not fluent in their understanding of multiplication and/or division and had difficulty iterating ratio units outside of familiar contexts. This type of thinking was evident among zero solutions from Bill (MLD), ten solutions from Carl (SS), and eight solutions from Albert (TA):

I: If $2.00 buys 3 balloons, how much for 24 balloons?

Carl: "...so while I was doing that I was counting by threes and kept counting by threes. I had eight groups. And then, I was done counting by threes to get to 24 so I counted by twos right where I stopped. So I got eight groups of two- 16."

I: If one food bar feeds three aliens, is there enough food pictured?"

Albert: [can 2 bars feed 9 aliens]. "Because there's three sets of aliens and every set of aliens gets one food bar and there's only two food bars"

Theme four: Understands the meaning of multiplication/division and its role in iterative process. Student thinking classified as *Understands the Meaning of Multiplication/Division and its Role in the Iterative Process* was defined as the ability to curtail longer iterative processes using multiplication and division in order to arrive at the total number

of iterations needed to solve a given equivalency situation. This type of thinking was evident in one of Carl's (SS) solutions, and one of Albert's (TA) solutions. Bill (MLD) did not use the strategy throughout the study showing perhaps a lack of understanding in this area.

I: "If three red go with five blue, how may blue go with nine red"?

Carl: "Well, with the blue…I did three times three is nine, so red adds three when blue adds five so this time it was counting like the opposite so it was 15. "

I: "If one teacher watches five kids, how many kids can 6 teachers watch?"

Albert: "[draws picture and does algorithm 5 times 6]. 30 students!"

Theme five: Abstracting iterative processes and connecting it to multiplication and division. Student thinking classified as *Abstracting Iterative Processes/Connections to Multiplication and Division* was defined as the ability to alter abstracted linked composite schemas to deal with unfamiliar problem situations. In other words, students at this level of multiplicative thought were often able to recalibrate given ratio composites to their unit values (e.g. $\frac{2}{4}$ as$\frac{1}{2}$) and then use the unit ratio to solve problems that did not make sense using whole number multiplication or division across ratios (e.g. $\frac{2}{4} = \frac{?}{10}$). This strategy was not found in any solutions of any student in the pre or post interviews.

Pretest/posttest changes. Table 11 summarizes levels of multiplicative thinking used by students before and after ratio based equivalency instruction.

Table 11. ***Pre and Post Interview Comparisons of Multiplicative Thinking***

	No Linking of Composites		***Linking of Quantities as a Ratio***		***Explicit Conceptualization of the Repeated Action of Linking Composites***		***Understand Meaning of Multiplication/Division and its Role in Iterative Process***		***Abstracting Iterative Process***	
	Pre	*Post*	*Pre*	*Post*	*Pre*	*Post*	*Pre*	*Post*	*Pre*	*Post*
Albert	3	0	0	0	8	13	1	1	0	0
Bill	10	5	1	5	0	0	0	0	0	0
Carl	1	1	3	2	10	11	1	1	0	0

Albert (TA) showed varying levels of multiplicative thinking during the pre-interview. Most of his solutions exhibited beginning levels of multiplicative thinking, and three of his solutions showed no understanding of linked composites involved in ratios. However, Albert's levels of multiplicative understanding during the post-interview showed an increased in the use of beginning multiplicative thinking. Albert did not demonstrate an understanding that ratios involved linked composites. Thus, the ratio-based instruction may have had some influence on Albert's increased tendencies to use the explicit conceptualization of repeating actions of linked composites during the post interview. Bill's move toward an increased explicit linking of quantities as a ratio can also be viewed in the pre post interview comparison of multiplicative thinking. Bill's tendency to not link composite units decreased by 50% from pre to post interview. Bill attempted to link quantities given to him in ratio equivalency problems as a unit. The results provide some evidence that Bill's multiplicative thinking increased to pre-ratio multiplicative levels after instruction. However, misconceptions in Bill's thinking processes were still very evident in the post-interview:

I: "If we have one teacher watching five kids, how many kids could six teachers watch?"

Bill: [Draws picture of a teacher with the numeral "5" beside it. Draws 30 sticks as kids and attempts to group into fives, but miscounts and leaves 3.]

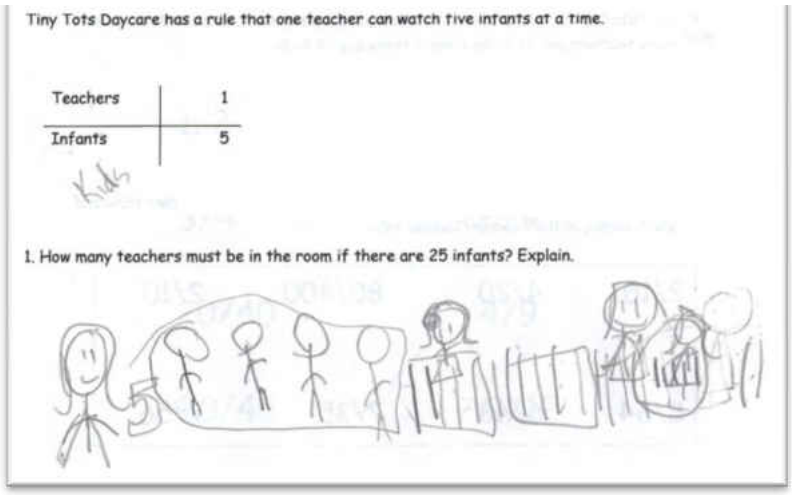


Figure 8. **Bill's teacher to "kid" representation**

"I drew 25 kids and I gave one teacher to each group. There were three kids left, so I gave one to this teacher, one to this teacher, and the last one here".

I: "Are there five kids in each group?"

Bill: [Examined his groups.] "Yes".

I: "Is it OK to split up those last three kids among the teachers like you did?"

Bill: "Yes because every teacher has to have five kids, so these three have to go to the other teachers".

Bill saw the ratio of one teacher watching five kids as meaning a teacher had to have *at least* five kids to watch. He did not keep the ratio when presented with a situation where not enough teachers were given for the amount of kids presented. Instead, Bill attempted to then "share" the

remaining kids with the teachers he had in the situation. Bill's inability to correctly recognize groups of five kids in the representation he drew contributed to his misunderstanding.

Carl's levels of multiplicative thinking remained unchanged from pre to post interview. Across all levels of multiplicative thinking evidenced through thematic analyses, Carl remained consistent in the levels of thinking he displayed throughout his solutions. Carl moved from linking quantities as a ratio to explicitly conceptualizing repeated linked composites in one solution examined.

Strategies. The next three themes defined strategies students used to solve problems presented during semi structured interviews. The analyses uncovered three themes (Batista & Borrow, 1995; Lamon, 1993; 2007): (a) typical and correct, (b) typical and incorrect, and (c) atypical and incorrect. Each theme contained several levels of strategy use (defined below). Although several similarities in strategy use were found across student groups, important differences began to emerge from Bill. He tended to have a nonuse of correct typical strategies and a use of atypical strategies that often resulted in incorrect responses to problems posed.

Theme six: Typical, correct. Strategies coded as typical, correct were defined as strategies used by most or all students interviewed that led to a correct solution. Five different forms of typical and correct strategies emerged as a result of data analyses: Patterning/Matching; Pre-Ratio Build Up; Abbreviated Build Up; Iteration with Operations, between; and Iteration with Operations, within. Names of codes were checked against and aligned with existing research on students' strategy use shown while solving ratio and proportion problems (Batista & Borrow, 1996; Lamon, 1993b). *Patterning/matching* strategies were defined as the use of two related quantities to match a specified quantity to another. The matching of quantities was

conducted either orally or pictorially (Lamon, 1993b). Students used the *patterning/matching* strategy to answer questions depicting a number of food bars needed to feed a number of aliens. Students were given situations where they needed to judge whether there was enough, too much, or too few food bars pictured for a certain number of aliens. *Patterning/matching* strategies were not used by Albert or by Bill, but were used by Carl in three solutions:

> *Carl: "One bar feeds three. So, one goes to this group, and another goes to this group, and another goes to this group. And there's an extra one that goes to…. some other guys I guess."*

Pre ratio/Build up strategies dealt with students who used a given ratio to "build up" to a given known quantity (e.g. when given "3 cans make 4 pancakes" and asked how many cans are needed for 16 pancakes, students draw the 3 to 4 ratio four times to arrive at 12 cans and 16 pancakes) (Batista & Borrow, 1996; Lamon, 1993b). Often times, students used pictures, models, or manipulatives to support their thinking. Albert used the *Pre ratio/Build up* strategy in seven separate solutions. Bill only used the *Pre ratio/Build up* strategy once during the pretest semi structured interview. The strategy was found in five solutions from Carl:

> *I: "How much money is needed to buy 24 balloons if 3 balloons cost $2.00?"*
> *Albert: "I did eight boxes and put three in it. And every three equals up to 24. And for every box of three I added two, so 2, 4, 6, 8, 10, 12, 14, 16."*
> *Carl: "[makes eight groups of three hash marks] eight threes equal 24. And I added 2, 4, 6, 8, 10, 12, 14, 16."*

Students using the strategy of *Abbreviated Build Up* were utilizing multiplicative double counting or the build up from a known ratio unit to find an unknown part of an equivalent ratio

(e.g. when given "3 cans make 4 pancakes" and asked how many cans are needed for 16 pancakes. Students counted by threes four times to arrive at 12 cans while simultaneously counting by fours four times to arrive at 16 pancakes). Albert and Bill did not use the *Abbreviated Build Up* strategy in any solutions. Carl used the *Abbreviated Build Up* strategy in three solutions:

I: "How did you figure out that 25 kids could be watched by five teachers?"

Carl: "Well, its 1, 2, 3, 4, 5 and 5, 10, 15, 20, 25 all at the same time".

Iteration with Operations, Between as a strategy was defined from the literature as an instance where students used multiplication and/or division *across ratios* to iterate or reduce a ratio quantity in order to find an unknown value in two equivalent ratios (e.g. when given a ratio that four cans makes six pancakes and asked to find how many cans for 18 pancakes, students solve by multiplying- $\frac{4 \times 3 =}{6 \times 3 =} \frac{12}{18}$) (Batista & Borrow, 1996). The *Iteration with Operations, Within* strategy was defined from the literature as an instance where students used multiplication and/or division *within a given unit ratio* to iterate or reduce a ratio quantity in order to find an unknown value in two equivalent ratios (e.g. when given a ratio that two cans makes four pancakes and asked to find how many cans for 18 pancakes, students solve by dividing 18 by 2 because they recognize that two is one-half of four). These strategies were not evident in any solution by any student in the pre or post interviews.

Pretest/posttest changes. Students' use of strategies that were typical and correct was examined pre interview and then again post interview in order to examine any changes in levels of strategies among students. Table 12 summarizes the results.

Table 12. ***Student Use of Typical, Correct Strategies***.

	Patterning or Matching		***Pre-Ratio Build Up***		***Abbreviated Build-Up***		***Iterations with Operations, Between***		***Iterations with Operations, Within***	
	Pre	*Post*	*Pre*	*Post*	*Pre*	*Post*	*Pre*	*Post*	*Pre*	*Post*
Albert	0	2	7	5	0	6	0	0	0	0
Bill	0	5	1	0	0	0	0	0	0	0
Carl	3	4	5	7	3	3	0	0	0	0

Albert progressed to using more abbreviated build-up strategies during the post interview. Additionally, Albert tended to use multiple strategies during the post-test to solve problems. He did not do so during the pre-interviews. Bill relied mostly on patterning or matching strategies during the post interview in problems solved correctly. Overall, however, Bill's strategy use could be defined mostly as atypical or typical yet incorrect. Carl did not show a change from pre to post interview in his use of typical and correct strategies to solve problems. However, Carl used multiple strategies in three problems he solved. Carl did not progress past patterning or matching strategies to solve problems in the post interviews.

Theme seven: Typical, incorrect. Strategies considered typical, incorrect were defined as strategies used by most or all students interviewed that did not lead to a correct solution but were found in the research literature as either 'common errors' made while students work with ratio equivalency problems or premature strategies utilized to attempt ratio problems. Two different forms of typical and incorrect strategies emerged as a result of data analyses: No Strategy and Incorrect Additive Linking of Composites.

No Strategy was defined as a student avoiding or using random or no apparent method or reasoning to solve the problem (Lamon, 1993b). The absence of strategy use was evident among two of Bill's solutions.

I: "How much for 24 balloons if 3 balloons cost $2.00?"

Bill: "Should I draw 24 dollars? 25 I think [after drawing 24 dollar bills]. I counted all the money. And I got 25 bucks."

Incorrect Additive Linking of Composites strategies were evident when students may have recognized that a link existed between two quantities given in a ratio, but saw the link as additive instead of multiplicative. The strategy was supported by (a) trial and error or (b) a visual judgment. The student then uses the incorrect linkage to solve the problem. For instance, the strategy was evident among two of Albert's solutions, two of Bill's solutions, and one of Carl's solutions.

I: "If five blues go with three reds, how many reds go with ten blues?"

Bill: "I think it's 7. [Draws 10 -3 and gets 7]. Seven…oh, wait … [erases and draws bundle of 3 chips and a bundle of ten chips]. I took away the three."

Albert: "Eight; because if you have ten blues. I'm thinking you just added five so I'm going to add five to the three".

Pretest/posttest changes. During Albert's (TA) post interview, he used no strategies that were classified as typical but incorrect. His tendency to use additive iterations incorrectly in certain problem solving situations (e.g. If 5 blues go with 3 reds, how many reds for 10 blues?) dissipated in the post interviews. Instead, Albert used abbreviated build up strategies to reason with the mathematics involved in the problems:

Albert, pre interview: "Eight; because if you have ten blues. I'm thinking you just added five so I'm going to add five to the three".

Albert, post-interview: "Um…six. Because you go, 3, 5; 6, 10."

Bill's use of incorrect additive iteration ceased during the post interview. However, although some incorrect typical strategies were replaced by matching (typical and correct) strategies, Bill's continued use of atypical and incorrect strategies persisted despite his participation in instruction. Conversely, Carl's use of typical but incorrect strategies did not seem to change from pre to post interview. In fact, Carl used one more incorrect additive iterative strategy in the post interview than in the pre interview.

Theme eight: Atypical, incorrect. Strategies used by students coded as atypical, incorrect displayed atypical thought patterns not witnessed among other students interviewed or found in the general education literature (Lewis, 2007). These strategies were used by Bill. The three strategies uncovered were coded as Incorrect Partitioning, Misuse of Ratio Unit and Misuse of Correspondence/Sharing strategies.

An *Incorrect Partitioning Strategy* was defined as the student using atypical ideas generated from part-whole fraction instruction to think about ratio problems. Bill used this strategy most often when the problem called for lower levels of multiplicative reasoning. Instead of viewing the given ratio as a unit quantity to be iterated, the student instead attempted to partition one of the quantities and distribute it to the other. Bill was thinking of the ratio situation as a sharing situation. However, Bill used partitioning strategies that revealed incorrect ideas about splitting into halves and a lack of counting ability:

> *Bill: [One food bar feeds three aliens…figuring if nine aliens can be fed by two food bars]. "No. There are…nine aliens and two bars. There are only two bars and if you cut them into two pieces there is only gonna be four…pieces of the bar".*
>
> *I: Oh. So are you figuring how much of one bar an alien eats?*

Bill: Well… [pause]…I think eleven. But you can't cut the bars anymore.

A *Misuse of Correspondence/Sharing strategy* was defined as the student using atypical ideas about sharing situations to think about ratio problems. Bill used this strategy most often when the problem called for the lowest and middle levels of multiplicative reasoning. Instead of viewing the given ratio as a unit quantity to be iterated, the student instead attempted to match one part of the ratio quantity to the other in a one-to-one fashion. However, the student's strategy use revealed a misapplication of one-to-one correspondence or sharing situations:

I: "If three balloons cost $2.00, how much do 24 balloons cost?"

Bill: [Draws lines between the balloons and the dollar bills shown. Upon finding he does not have enough dollar bills, he draws the relationship over again and makes a third dollar and matches it to the third balloon.] "Uh…."

***Figure 9*. Bill's misuse of correspondence**

I: [Points to the ratio of $2.00 for three balloons.] "What does this mean? Does knowing this give you a clue as to how to solve the problem?"

Bill: "It means you would get one dollar back."

Bill did not understand the given ratio as linked composites to be iterated. Instead, he thought he had to have one dollar for every balloon, or saw the scenario as a subtraction problem. This understanding of a ratio unit as a subtraction scenario leads into the final atypical strategy used by Bill.

A *Misuse of Ratio Unit* strategy was evident when Bill failed to see the linkage between composite units. Instead, Bill thought that the ratio unit relationship was telling him to subtract something:

> *I: "How many teachers are needed to watch 25 kids if the daycare rule is that one teacher can watch five kids at a time?"*
>
> *Bill: "**"**So, I have to draw twenty kids? Oh so I have to draw all of the kids? [draws twenty five stick figures]. There are twenty kids. I counted all of these and then I took away five of them and I got 20."*

Pretest/posttest changes. Many of the atypical and incorrect strategies that Bill employed to think about ratio equivalency problems during the pre-interview were used again during the post interview, despite the fact that Bill received nine days of supplemental, intervention instruction in fraction equivalency through ratios. Table 13 summarizes Bill's use (and type) of atypical and incorrect strategies before and after instruction.

Table 13. ***Bill's Atypical Strategy Use, Pre and Post Interview***

	Incorrect Partitioning Strategy		***Misuse of Correspondence or Sharing***		***Misuse of the Ratio Unit***	
	Pre	*Post*	*Pre*	*Post*	*Pre*	*Post*
Bill	3	3	1	1	2	1

Bill's use of incorrect partitioning strategies to solve problems continued from pre to post interview, indicating that the misconception is resistant to instruction. Moreover, further misunderstandings of partitioning were uncovered. For example Bill attempted to partition the four bars and divide them among the nine aliens when told, "Given one food bar feeds three aliens, is four food bars enough food for nine aliens?" Bill's misunderstanding of partitioning is evident when he takes each bar and cuts them into six pieces (using five cuts). However, when matching food to aliens, Bill matched aliens with the *partition,* not the quantity produced by the partition.

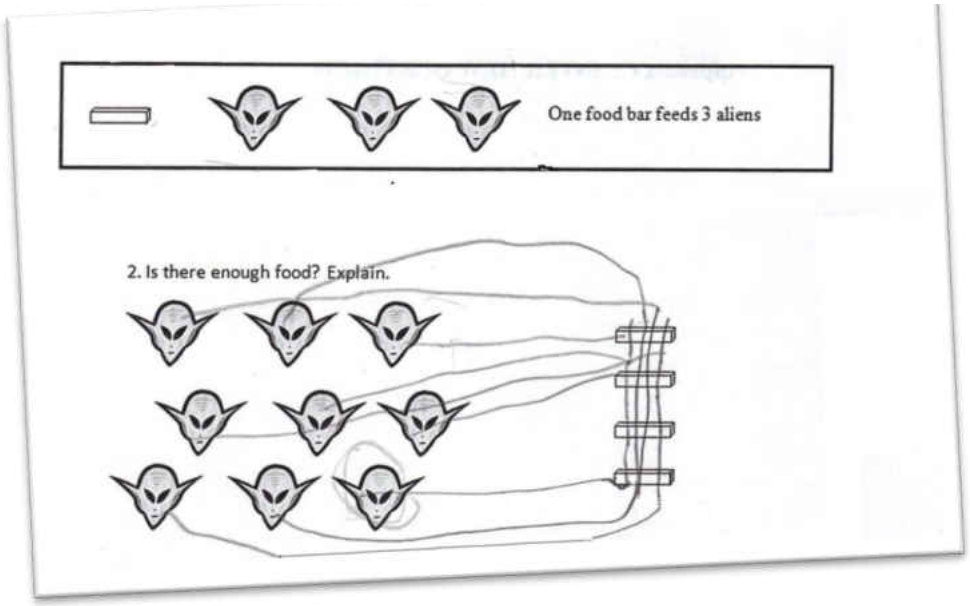


Figure 10. **Bill's partitioning strategy.**

Furthermore, Bill's difficulty with partitioning seemed to be exacerbated by situations involving more than one object that he attempted to "share". Bill continued to attempt to assign partition lines to aliens, but grew increasingly confused when presented with multiple bars; assigning multiple aliens to one cut.

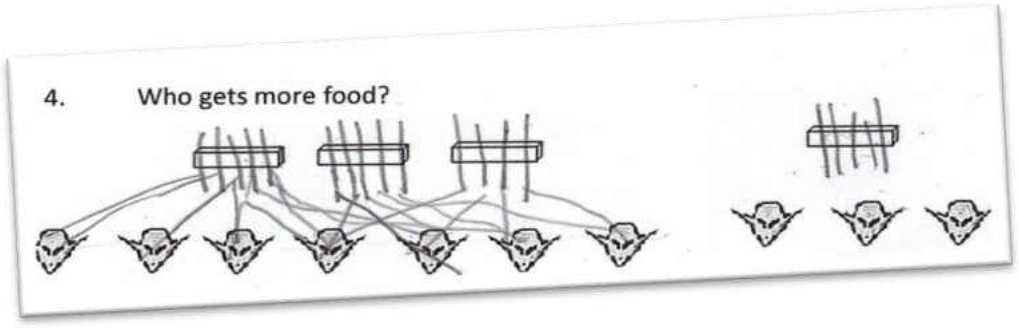


***Figure 11*. Bill's partitioning and use of correspondence.**

Bill continues to use correspondence incorrectly when considering ratio units in other problems. For example, when shown the ratio condition of one food bar feeds three aliens and asked how many food bars are needed to feed 15 aliens, Bill simply counts the aliens and states that each alien would get "one" so you would need 15 of the bars.

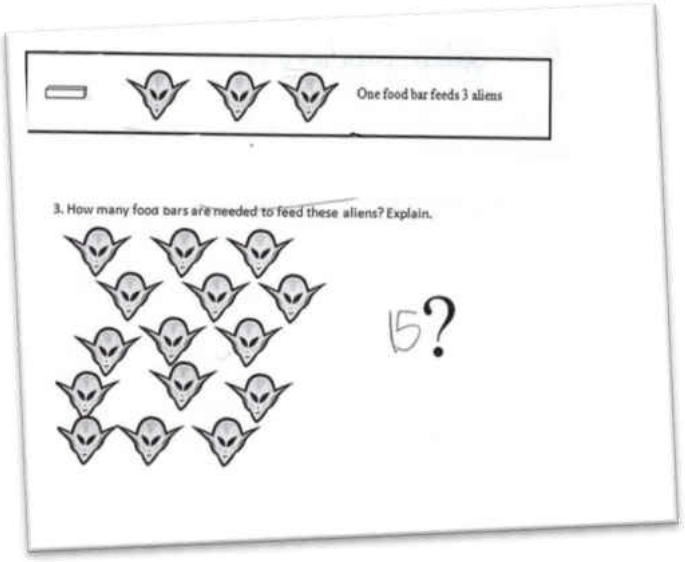


Figure 12. **Bill's application of one-to-one correspondence.**

I: "Does this [points to picture of three aliens being fed by one food bar] give us a clue on how to do this problem?"

Bill: "No."

Bill's responses to ratio equivalency situations are resistant to instruction or even teacher redirection and appear to be based in incorrect ideas regarding partitioning, correspondence, and the ratio unit. Bill was unable to use the ratio unit to solve many of the problems because he viewed the ratio as a subtraction problem or a situation of one-to-one correspondence. In other problems, Bill ignored the given ratio completely and attempted to view the problem as a sharing or partitioning situation. Unfortunately, Bill displayed incorrect ideas about halving or partitioning that did not lead to understanding the problem despite inclusion in intervention instruction on these concepts.

Reliability and Validity of Qualitative Data

Triangulation is typically a strategy for improving the validity or evaluation of findings in qualitative research (Grbich, 2007). Triangulation was achieved in two ways. First, the coding

and theming of data collected was a result of interview with three different students. Additionally, two independent coders reviewed transcripts at stage three of data analyses (Grbich, 2007). Codes were deemed to be reliable if the coders achieve 80% agreement or greater. At stage three, independent coders reached agreement on 16 out of 19 codes, resulting in 84% agreement.

Second, to verify the themes uncovered during Bill's (MLD) pre and post interviews were verified by the examination of two additional data sources (Grbich, 2007). Specifically, Bill's performance on the pre and post CBMs, which both contained measures of ratio equivalency (two problems on the pre test and one question on the post test), was examined to verify difficulties uncovered during the interviews. On the pretest, one of the questions measuring ratio equivalency was unanswered. The other question was *'At a car wash, Jim washed 8 cars per hour. David washed 6 cars per hour. How many cars did Jim wash if David washed 24 cars'?* For this question, Bill (MLD) answered 24. His answer gives additional evidence of the "Incorrect Use of Correspondence" strategy evidenced in the interviews. Furthermore, on the posttest, Bill (MLD) gave the answer of '36' for the question '*A store sells school-supply packs that contain 6 pencils and 4 pens. A customer bought enough packs to get 36 pencils. How many pens did the customer get?'*, showing again indication of a misuse of correspondence.

Patton (2002) argued in qualitative research that reliability is a consequence of the validity. Reliability of source information was obtained through the use of verbatim translation (Grbich, 2007). Student selection and matching based on their pretest scores as well as their student type was a third strategy used to ensure validity of findings (Creswell, 2007). Finally, results of the analyses were reviewed with students as a means of member checking to ensure

consistency in data reporting (Glense, 2006; Grbich 2007). Students reported overall agreement with the themes and ideas uncovered in data analyses.

Summary

Results of qualitative analyses revealed that differences in strategy usage and levels of multiplicative thinking were found before and after instruction for three students (MLD, SS, and TA) in the experimental group. Namely, levels of multiplicative thinking achieved by students from pre interview to post interview varied among the three students interviewed (see Table 11). Albert's (TA) multiplicative thinking progressed into a solidified explicit conceptualization of the iteration of linked composites, while Bill's (MLD) multiplicative thinking began to progress into pre-multiplicative levels (Batista & Borrow, 1995). While Bill's (MLD) use of the ratio as a composite unit was apparent in several of his problem solutions, he continued to show no level of multiplicative thinking in many of his solutions. Carl's (SS) thinking evidenced a slightly increased level of explicit conceptualization of iterating linked composites, but his level of thinking for the most part remained unchanged.

Strategy use among the three students varied between typical and correct, typical and incorrect, and in Bill's (MLD) case, atypical and incorrect methods (Batista & Borrow, 1995; Lamon, 1993; 2007). Tables 12 and 13 show a summary of the results. Neither Albert (TA) nor Carl (SS) surpassed using the strategy of abbreviated build up to solve problems. Albert (TA), however, showed improvement in his use of incorrect additive iteration in the post-interviews. Carl (SS) moved from pre ratio build up strategies to abbreviated build up strategies, yet continued in his use of incorrect additive iteration to solve two problems during the post interviews. Bill's (MLD) use of atypical and incorrect partitioning and correspondence

strategies, as well as the misuse of the ratio unit, were resistant to intervention instruction and were persistent in the post interviews.

Conclusion

Based upon research question one the answer was students who took part in the ratio-based instructional sequence outperformed their counterparts in the control group on two measures of fraction equivalency. Student type (SS or TA) seemed to affect results in both the control and experimental groups on the standardized measure. Students who struggled in the experimental group outperformed similar students in the control group. Likewise, students labeled typically achieving outperformed their counterparts in the control group. On the CBM, similar results were found, although student type did not have an effect on performance on the CBM in either group. Quantitative results indicated that all students in the experimental group benefitted from ratio-based fraction equivalency instruction.

For research question two, the answer was that differences in strategy usage and levels of multiplicative thinking evidenced by three students (MLD, SS, and TA) in the experimental group were found before and after instruction. Albert (TA) used strategies that were considered indicative of early to mid-ratio thinking, and overall seemed to improve from pre to post interview. Carl (SS) did not show evidence of any change in multiplicative thinking from pre to post interview. Bill (MLD), while showing increases in strategy use, did not progress past pre-multiplicative levels of thought in solving problems during the post interview. His usage of atypical and incorrect strategies persisted in spite of intervention. These findings lead to further discussion, implications and future research in the next chapter.

CHAPTER FIVE: DISCUSSION

The primary purpose of this study was to investigate the impact of a ratio-based fraction teaching sequence on students identified as having a MLD. The researcher also investigated, through quantitative and qualitative analyses, group differences of struggling and non-struggling learners in the areas of performance and understanding. The discussion is framed into three sections. First, a brief review of the study, including data analyses and results, is presented. Second, a discussion of the two research questions is provided in relation to the outcomes of this study and the literature already established in the field of MLD. The chapter concludes with a summary, including study limitations, implications for practitioners, and suggestions for future research.

Review of Rationale and Study Objectives

This study investigated the impact of a ratio-based fraction teaching sequence on students identified as having a MLD. Research questions sought to identify if (1) performance on two tests of equivalent fractions (part whole and ratio) increased as a result of instruction and (2) if understanding of ratio equivalency changed with respect to (a) strategy usage and (b) level of multiplicative thinking uncovered in data analyses from pre to post interview for students who struggle and students with MLD. On standardized tests of mathematics, students with LD consistently score lower than their peers without disabilities across all grade levels (National Center for Educational Statistics, 2009). The disparities begin around first grade and this gap continues to widen over time (Cawley & Miller, 1989). Beyond an overall struggle in mathematics in general, a target area where many students with MLD are not successful is fractions (Grobecker, 2000; Hecht, 1998; Hecht, et al., 2003; Hecht et al., 2006; Mazzacco & Devlin, 2008). This continued gap could potentially be attributed to the field's historic focus on

LD as a reading disability as opposed to a mathematics disability. Even after an exhaustive review of the literature little information was found to guide the field as to why students with LD in mathematics struggle with critical fraction concepts, such as equivalency.

The mathematics education literature indicates partitioning, unitizing, equivalency, and multiplicative thinking as critical elements of effective instruction for typically achieving students in the area of fraction concepts (Battista & Borrow, 1995; Lamon, 1993a, 1993b; Streefland, 1993). Due to limited understanding of MLD, how this type of disability affects understanding of fractions, is missing from the literature. However, some theories are emerging that lead to semantic memory (Geary, 1993) and sense of number (Landerl, et al., 2003) as being the primary deficits involved in MLD, which could affect performance and understanding of equivalent fractions. Needless to say, a divide exists among researchers regarding whether semantic memory or sense of numbers is the primary deficit among students with MLD. In addition to semantic memory and sense of number, cognitive deficits such as working memory (Geary et al., 2007) and nonverbal reasoning (Jordan et al., 2003) could work to disrupt understanding of fraction concepts. Overall though, the ways in which named strengths and weaknesses of students identified with MLD surface while learning about fractions is not well understood (Hannich et al., 2007).

Learning fraction concepts for all students could be promoted though the use of five different constructs (e.g. part-whole, measure, ratio, operator, and quotient). Despite these constructs being defined, only a small amount of research exists regarding how learning through these constructs affects students with MLD. In the part-whole sub construct, one or more equal partitions of a unit are compared to "the total number of equal portions into which the unit was divided" (Lamon, 2005, p. 60). Only a few studies involving students with MLD and their

ability to work with partitioned representations for fractions and unitize as it relates to the part-whole sub construct can be found in the literature. In one such study, researchers outlined several variables that negatively impacted student understanding, including (1) difficulties with area partitioning; although students used area models to represent problems, they experienced difficulty partitioning equal sized pieces and drawing understanding from partitioned models, and (2) difficulties transitioning from fractions as area models to fractions represented on number lines (Morris, 1995). Moreover, students with MLD were found to possess a lack of conceptual knowledge based on the understanding of part-whole representations compared to their typically achieving peers (Hecht et al., 2006).

The concept of equivalence can be viewed as the invariance of "a multiplicative relation between the numerator and the denominator, or the invariance of a quotient" (Ni, 2001, p. 400). Mazzacco and Devlin (2008) discovered middle school aged students with MLD have difficulty with part-whole based representational modes of equivalency. These researchers have suggested students with MLD have "a weak rational number sense and inaccurate beliefs about rational numbers" based on the part-whole sub construct (p. 690). Lewis (2007) reported from case study research that a focal student displayed atypical understanding of shaded area models in that the student identified the line rather than the shaded quantity as the fraction (e.g. the fraction $\frac{1}{2}$ is the *partition*, not the shaded region). Grobecker (2000) investigated students with MLD and their understanding of equivalence through the measure sub construct, which is related to part-whole in that it involves partitioning. Students were unable to associate the needed part-whole association between part of the number line and its entire length (e.g. the whole and its parts) to generate equivalent relationships.

Only one study in the research literature reported how students with MLD understand fraction equivalency through ratios (Grobecker, 1997). It is possible that teaching equivalency concepts through ratios could serve as an alternative to teaching concepts to students with MLD through the part-whole sub construct if students can be taught to progress to multiplicative thinking processes needed to understand equivalency concepts. Ratios compare any two quantities to one another, and have been described as fundamental to fraction knowledge (Lamon, 2005; Pitkethly & Hunting, 1996; Streefland, 1991). Results of research conducted with struggling students suggest that students could use ratio-like activities to begin to understand fractions (and equivalence) (Cortina & Zuniga, 2008). However, it is unknown whether students with MLD and students who struggle could understand the conceptual linkage involved in ratio units and then iterate ratio units to understand equivalency situations (Grobecker, 1997). If the linkage can be understood through instruction, then strategies and levels of multiplicative understanding may increase. Ratios may be an alternate introduction to fractions and equivalence through the linking or correspondence of two quantities. Given the evidence noted in the literature that students with MLD struggle with fractions as part-whole interpretations, another method of promoting understanding and increased performance in equivalent fractions was tested for this population.

Overview of Research Questions and Data Analyses

Research question one was constructed to address the analyses of any differences that existed between the experimental and control group as a result of ratio-based fraction instruction as well as between (e.g. time of test) and within (struggling or typical student) group effects. Data were collected from 38 students (19 in the control group and 19 in the experimental group). Measures for the analyses were a CBM of part whole and ratio fraction equivalency and a

standardized transfer test (the Brigance Q6 subtest). The CBM test measured student performance in generating part-whole and ratio equivalent fractions presented in pictorial, abstract, and word problem format. The standardized measure tested students' ability to generate equivalent fractions and was presented through abstract, symbolic notation. A factorial MANOVA with repeated measures procedures were used to evaluate pre post differences among and across students with MLD, struggling students, and typically achieving students in the experimental and control groups. Originally, "student type" (TA, SS, or MLD) was separated into three different groups. However, because only four students were identified as MLD, separating students with MLD from students who struggled would have created more cases than dependent variables in each of the MLD cells, causing a threat to statistical power (Tabachnick & Fidell, 1996). Thus, for quantitative analyses, students with MLD and students who struggled were combined into one group. Social validity questionnaires were also administered before and after instruction to both the experimental and control groups.

Research question two was evaluated by two semi structured interviews. Because the CBM and standardized measures used to evaluate research question one were not sensitive enough to uncover strategy use and levels of multiplicative thinking involved with understanding ratio fractions, interviews were used to further investigate these areas before and after instruction for three students. Data was collected through a pre and post semi-structured interview. The interviews contained questions related to student understanding of fraction equivalence through the ratio sub construct. Thematic analyses was used to uncover three categories and nine themes evident throughout the three students' solutions and responses to ratio equivalency questions.

Overall Results

The quantitative analyses of the data related to the research questions indicated that students in the experimental group scored significantly higher on the CBM and on the standardized measure than students in the control group. Students in the experimental group had better outcomes on the social validity measure relating to students' attitudes toward fraction equivalency problems than students in the control group. Experimental group students overall improved their understanding of part whole and ratio based fraction equivalency from pretest to posttest. Results indicated that group assignment, student type, and interaction of group assignment and pre and post testing time were significant contributors to scores on the two outcome measures of fraction equivalency taken together. Students in the experimental group outperformed students in the control group on measures of fraction equivalence.

In research question one, step-down analyses of effects of group assignment, student type, time of test, and ethnicity of each test individually was performed due to the main effects found by the MANOVA. The analyses confirmed the unique contributions of the interaction of group assignment and time of testing on transfer test results. Students in the experimental group did better than students in the control group on abstract problems contained in the standardized measure of fraction equivalency. Students labeled as typically achieving outperformed students who struggled in both groups. On the CBM, students in the experimental group outperformed their control group counterparts on situated problems, abstract problems, and problems asking students to judge the correctness of given equivalency statements. The interaction of group assignment and testing time was significant- students in the experimental group outperformed students in the control group. Students in both groups did not differ in their performance, pretest or posttest, on the CBM.

The analyses of research question two involved two interviews from three students in the experimental group conducted before and after intervention. From the three students whose work was further analyzed, interesting themes emerged that extended results found during quantitative analyses. For instance, a student deemed typically achieving used strategies that were considered indicative of early to mid-ratio thinking. His understanding of fractions as ratios showed overall improvement from pre to post interview. The student who struggled was able to progress into higher levels of strategy usage, but did not evidence any change in multiplicative thinking from pre to post interview. A student with MLD showed beginning increases in strategy use but did not progress past pre-multiplicative levels of thought in solving problems during the post interview. The student with MLD's overall use of atypical and incorrect strategies persisted in spite of intervention. Therefore, quantitative results showed students deemed typically achieving as well as student who struggled or who had a MLD improved their performance, but qualitative results revealed that a focal student with MLD may not have improved as much as the quantitative results indicated.

Relationship of the Findings to the Literature

The results of the present study show several connections to previous research for teaching students fraction equivalency. Namely a relationship is provided between the research and the finding of this study on (1) the differences in performance between students who struggle and students with MLD, (2) the deficits associated with MLD research, (3) the way students with MLD learn fraction content through varying sub constructs, and (4) the identified methods for teaching fractions to students with MLD.

Results and Differences in Performance

Historically, disparities have existed between students with LD and students without disabilities in mathematics achievement. Results of national tests of mathematics indicated only 19% of fourth grade students with LD and 9% of eighth grade students with LD perform at proficient levels in mathematics (National Center for Educational Statistics, 2009). Quantitative results of the present study indicated that ratio instruction improved performance for all students in the experimental group. All students in the experimental group did better on the standardized measure than students in the control group. However, students deemed typically achieving did better on the measure than students who struggled. Student type was a mediator of performance on the standardized measure. Thus, while the intervention produced better performance for students who struggled or who had MLD in the experimental group as opposed to the control group, it did not bring their performance on the standardized measure to levels of students without MLD.

Scores on the CBM, however, showed gains in performance for students who struggled or who had MLD that were comparable to students without disabilities. On the CBM, quantitative results indicated that all students in the experimental group did better than students in the control group. Unlike the standardized measure, student type did not make a difference in performance in the experimental group. Students who struggled or who had MLD in the experimental group improved their performance from pre to post test, and the fact that they struggled or had a disability made no difference in their performance compared to students without disabilities. Thus, the intervention not only produced increased levels of performance for students who struggled or who had a MLD, but their performance levels were in line with performance levels of students without disabilities.

Quantitative results showed overall increases in performance for students deemed typically achieving as well as students who struggled or who had a MLD, yet differences in performance between students who struggled and students who have MLD were yet to be uncovered. Much of the research done previously in the field of MLD fails to differentiate between students with MLD and students who struggle in mathematics but do not have a disability (Mazzacco, 2006); despite the knowledge that a distinction in performance and thought processes in the two groups exists (Mazzacco & Devlin, 2008). Quantitative results were unable to provide information relating to any differences in performance between students with MLD and students who struggled due to the sample size and statistical analyses used. Thus, it was unclear through quantitative analyses whether the ratio-based instructional sequence increased performance for students with MLD, as well as for students who struggled, but did not have a MLD.

Qualitative analyses, however, uncovered large differences in strategy use and levels of multiplicative thinking between a student with MLD and a student deemed as struggling in mathematics after instruction. The three students chosen were most representative of the characteristics that defined their student type for the comparative qualitative analyses. The qualitative results showed that the student with MLD could not give meaning to many of the ratio relations presented during the pre and post interviews. Iteration of the ratio unit necessary to understand equivalence situations was not possible for this student because he did not see the ratio as a unit quantity. Conversely, a student who struggled viewed the ratio relation as a unit quantity. His understanding of the ratio as a quantity was complete. This student may have used predominately pre ratio build up strategies, but he knew and understood that the ratio was to remain the same in situations where the ratio needed to be iterated. Thus, gains reported during

quantitative analyses may have reflected increases in performance for students who struggled yet not for students with MLD. Qualitative results of the one student selected with MLD showed he did not have the same level of understanding needed to produce higher performance on the fraction equivalency measures. This finding is one that needs to be further researched with larger populations.

Results and Types of MLD

Understanding of mathematics content and subsequent performance on testing measures can be influenced by the strengths and weaknesses produced by a student's disability (Butterworth, 1999). Namely, research conducted over the past two decades has uncovered deficits associated with mathematics performance and students with MLD. The deficits include semantic memory or language (Geary, 1993; Loosbroek et al., 2008; Rouselle & Noelle, 2007), sense of number (Butterworth, 1999; Hecht & Vagi, 2010; Landerl et al., 2003), working memory (Geary et al., 2009) and nonverbal or fluid reasoning (Jordon et al., 2003). Although it was not a goal of the current study to evaluate which "type" or definition of MLD is correct or more prevalent than others, study results can be compared and contrasted with what is currently known about MLD. Strengths and weaknesses associated with MLD surfaced in several forms throughout data analyses.

Semantic memory is defined as a difficulty with retrieving basic arithmetic facts from long term semantic (language based) memory (Geary, 1993). Researchers who believe MLD is based in semantic memory deficits contend that students with MLD may not have difficulty processing quantities as a result of a deficiency in number sense and because of impairment "accessing semantic information conveyed by numerical symbols" (Rousselle & Noelle, 2007, p. 377). In one past study, researchers found students who are MLD had decreased scores on tasks

requiring semantic memory (e.g. understanding of symbolic quantity) more often than on the task not requiring semantic processing (e.g. understanding of quantity conveyed concretely) (Rousselle & Noelle).

Unfortunately, quantitative results of the present study lend no insight into whether semantic memory had impact on the outcome of low performers and students with MLD. On the standardized measure, students who struggled or who had a MLD, even in the experimental group, performed significantly worse on standardized measure problems than typically achieving students. While items included on the standardized measure were more abstract (symbolic) in their presentation, the data analyses does not provide enough direct evidence to say that students with MLD or students who struggled have semantic memory deficits.

Qualitative results of the present study provide more insight into whether semantic memory may have been a factor in the performance of the student with MLD selected to be profiled related to his skills on equivalency items. Qualitative results showed the student with MLD could not assign meaning to ratio unit in order to iterate quantity. However, the ratio was given in non-semantic form (concrete). Therefore, results of this specific case suggest that semantic memory may not have been a cause of a student with MLD's problem understanding ratios and equivalency. Instead, the difficulties may have been rooted in number sense deficits.

This specific case leads to further discussion about number sense. Sense of number in the early years is defined as the understanding of exact quantity of small collections of objects, the symbols that represent them and their approximate magnitude, while notions of one-to-one correspondence remain intact (Geary, 2009). Researchers who believe sense of number is at the root of problems for students who are MLD argue that an inability to subitize prevents the development of meaning for numbers (Landerl et al., 2003). Two past studies indicated that

students with MLD demonstrated deficits in number comparison and subitizing (Geary, 2009; Landerl et al., 2003).

Subitizing seemed to interfere with one student with MLD's ability to assign meaning to the ratio unit. Even when he began to believe that a relationship between composite units being represented indeed could exist, the student failed to iterate the ratio unit because he could not discriminate small numerosities (e.g. could not group five students together to assign the one to five ratio unit). His lack of ability to subitize worked against any primitive understanding of ratios as a unit that might have formed. Further, the student seemed unable to detect when errors in his counting resulted in his inability to correctly iterate a ratio unit. Thus, the student with MLD began to see the relation between quantities in certain contexts, but his inability to subitize interfered with his ability to iterate the ratio through even the most immature of strategies (e.g. matching). However, the counting issues could also be explained by working memory.

Working memory is a cognitive component often associated with MLD. It is defined as the coexistence of a central executive, visuospatial sketchpad, and a phonological buffer in the brain responsible for the holding of information in the mind (Geary, 1993). Researchers who believe that working memory issues are the defining factor involved with MLD have found through their research students with MLD possess "broad working memory deficits and specific deficits in their sense of number that delay their learning of formal mathematics" (Geary et al., 2009, p.274). For instance, results of numerous studies showed deficits in counting subitizing, number line estimation, and increased errors in detecting double counts during counting tasks for students with MLD; the central executive was found to mediate performance between those with and without MLD (Geary et al., 2007, 2009).

Results of one student in the qualitative component of this study associated with research question two showed a similar trend in understanding fractions as ratios. While quantitative results showed gains in fraction equivalency performance in students who struggled and students with MLD, one student with MLD showed an inability to understand a ratio unit as a quantity compared to a student who struggled. The misunderstanding of a ratio unit may come from the inability of the student with MLD to subitize, but it may also come from errors in counting produced by working memory deficits. Unfortunately, the study design and research questions did not allow for analyses of which MLD deficit or cognitive factor lead to the misunderstanding uncovered during the quantitative or qualitative analyses of this one student. Thus, working memory, sense of number, both, or neither could have been to blame for Bill's (MLD) inability to see the ratio unit presented during the content lessons as a quantity.

Results and Content

Clearer conclusions could be made from the results of data analyses relating to the content used to teach fraction equivalency to students with MLD, students who struggled, and students deemed typically achieving. Important fraction content such as partitioning, unitizing, equivalency and the associated multiplicative reasoning can be taught through part-whole, measure, ratio, or quotient sub constructs (Behr et al., 1983; Lamon, 2005). Partitioning can be defined as the process of taking an object or set of objects and dividing it equally into a number of equal parts (Empson, 2001; Lamon, 2005). Unitizing is defined as "the cognitive assignment of a unit of measurement to a given quantity" (Lamon, 2005, p. 42). Imagining and reimagining the unit is an essential activity that promotes later understanding of fraction equivalence. The concept of equivalence can be viewed as the invariance of "a multiplicative relation between the numerator and the denominator, or the invariance of a quotient" (Ni, 2001, p. 400), and is a

difficult concept to understand. Understanding the concept of equivalency and generalizing the concept to abstract processes involves the move from additive to multiplicative thinking (Lamon, 1993b; Battista & Borrow, 1996).

Fraction instruction may prove beneficial or detrimental to students with MLD if the construct used during instruction does or does not align with students' pre-instructional strengths. It was hypothesized that students with MLD might understand fractions better through the ratio subconstruct due to their difficulties with partitioning part-whole models of fractions (Hecht et al., 2006; Lewis, 2007; Morris, 1995) and their intact understanding of correspondence (Geary, 1993). Difficulties with area partitioning, difficulties transitioning from fractions as area models to fractions represented on number lines, representing fractions with pictures of partitioned circular regions, naming fractions from pictures of partitioned circular regions, and computing fractions using pictures of partitioned circular regions were noted as difficulties experienced by students with MLD during fraction instruction based in the part-whole sub construct (Hecht et al., 2007; Morris, 1995).

More recently, Lewis (2007) described how a student with MLD displayed atypical understanding of shaded area models in that the student identified the line rather than the shaded quantity as the fraction (e.g. the fraction $\frac{1}{2}$ is the *partition*, not the shaded region). Further, the student understood a shaded area model representation as the amount taken away rather than a fractional quantity (e.g. student constructed $\frac{7}{12}$ - 7 pieces out of 12 shaded - but interprets as $\frac{5}{12}$ - the amount left).

In the current study, quantitative analyses showed growth in performance across students deemed typically achieving as well as students who struggled or who had a MLD. Their ability

to generate and identify equivalent fractions increased as a result of instruction. However, as evidenced in the interviews, one student with MLD experienced the same atypical thoughts regarding partitioning as described by Lewis. The student viewed the ratio relation as not valuable or relevant to solving the problem situations. Instead, he viewed the ratio problem scenarios as sharing situations in which the food pictured needed to be partitioned and distributed amongst the aliens. However, the student matched partition lines with the aliens, suggesting that he viewed the partitioning as an action and not associated with a fractional quantity. Also interesting was the fact that the student saw the ratio relation as a 'take away' or an 'amount left over' situation- much like the focal student in Lewis's study showed when working with part-whole shaded fraction regions. The atypical ideas regarding part-whole fractions permeated instruction in the ratio construct; the students' perceived strength relating to the use of correspondence did not lessen the students' tendencies to use part-whole based atypical strategies.

Correspondence, or linking two composites, takes place in the ratio sub construct. Ratios compare any two quantities to one another, and have been described as fundamental to fraction knowledge (Lamon, 2005; Pitkethly & Hunting, 1996; Streefland, 1991). Another feature of the ratio sub construct is the relationship does not change if we wish to increase one of the parts- a person must be able to understand the unit linkage between two quantities and hold the linkage in mind to iterate the ratio (Battista & Borrow, 1996). This relationship is regarded as the covariance-invariance property (Vengard, 1983) and is related to fraction ideas such as equivalence and ordering (Charalambous & Pitta-Pantazi, 2007; Streefland, 1991, 1993).

In contrast to expectations, correspondence seemed to produce *misunderstandings* regarding ratios as fractions (and quantities to be iterated) for one student with MLD who was

interviewed. While quantitative results of the current study suggest that fractions taught as ratios helped typical students and struggling students identify when equivalency statements were incorrect, the sequence may not have done the same for students with MLD. For instance, a student with a MLD could not make sense of ratio relations to begin to iterate the ratio unit in order to judge the correctness of equivalency situations. The student viewed given ratio situations as meaning that he had to subtract one part of the ratio from the other (e.g. $\frac{3}{4}$ means 4 – 3).

Past research indicated that students with MLD were able to make sense of the ratio relationship in order to iterate linked composites. Grobecker (1997) identified four levels of understanding that encompassed all solutions of students with and without MLD: (1) The inability to manipulate grains and bundles at the same time; (2) an additive ability to count and add grains and bundles; (3) grains and bundles represented as groups and then adding the groups, and (4) use of mental multiplication to manipulate grains and bundles at the same time. Students with and without MLD experienced difficulty advancing to higher levels of multiplicative thinking (e.g. Level 3 or Level 4). However, students with MLD were unable to advance beyond Level 2 (they used mostly additive structures to understand equivalency.

Grobecker only used interviews to uncover levels of understanding. Due to the use of correspondence involved, the researcher felt that *teaching with ratios* could cultivate understanding of fractions as ratios. Results of the current study suggest performance in generating equivalent fractions increased across all students who participated in ratio instruction. Ratios as fractions made sense to many students in the experimental group. Many of these students were able to use the ideas and strategies taught during instruction to help them solve fraction equivalency problems on the post tests. Examination of the post tests showed many

students using ratio tables, pictorial iteration of fractions as ratio units, and multiplication strategies to solve fraction equivalency problems.

However, a student with MLD could not see ratio units as iterable quantities. Differences existed in understanding ratios between one student who struggled and another who had MLD. Namely, in the case study of one struggling student, the student used mostly additive structures (as Grobecker suggested). Yet, the student with MLD could not reach even additive pre ratio strategies or 'ratio as unit' levels of multiplicative thinking on most tasks. Atypical thinking centered on viewing ratios as partitioning situations or as subtraction situations persisted in the student with MLD, despite inclusion in the ratio-based teaching intervention.

Another point can be made regarding the sub construct used to teach this student with MLD about fraction equivalency. Kieran (1993) argued, "Unit fractions along with quotients and ratio nature form a mathematical base for rational numbers" (p.81). Streefland (1993) also saw ratios as quotients – much of his instructional sequence was based on sharing (quotient) situations. Interesting to note was that during the interviews, Bill (MLD) seemed to naturally sway toward understanding fractions as quotients. He attempted to partition each food bar and match the pieces to the aliens pictured. Quotients may have been a better starting place for students with MLD to understand fractions. If the student could have been taught to understand fractional pieces as quantities to be distributed instead of the partitions themselves, their propensity to use correspondence may have produced understanding. A more explicit teaching sequence that models the partitioning and resulting fractional quantities may prove to increase understanding for students with MLD in fraction equivalency and results in a revised model of the comparison between MLD deficits and fraction sub constructs (see Figure 13). The work of

Streefland (1993) could serve as a basis for the design of future instruction in fraction equivalency for students with MLD.

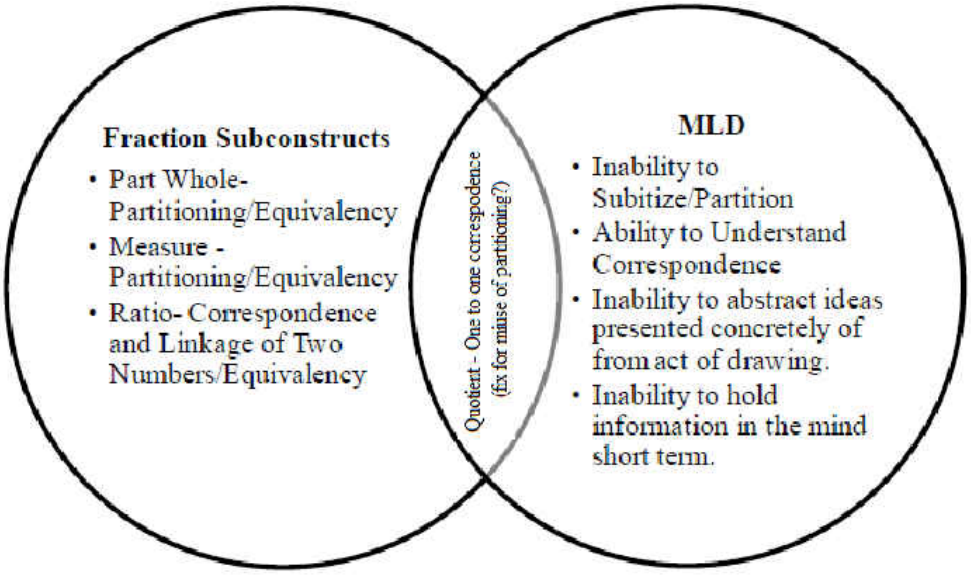


Figure 13. ***Revised comparison of fraction sub constructs and MLD.***

Results and Teaching Methods

Four instructional strategies found to promote increased outcomes in mathematics for students with MLD and students who struggle were found in the literature. These practices were all included in the ratio-based teaching sequence: concrete-representational-abstract instruction (Butler et al., 2003; Jordan et al., 1999), explicit instruction in regards to instructional sequencing, concept formulation, and multiple strategy use (Jitendra et al., 2003), student use of representations to support development of mathematics knowledge (Xin et al., 2007), and verbalizing mathematics reasoning (Woodward et al., 1999). Several implications from the current study can be made concerning the teaching methods employed during intervention.

First, verbalizing mathematical thinking was defined as student self-questioning resulting in increased performance in fraction concepts and equivalency (Gerstein et al., 2008). In prior

research, students who verbalized their mathematical thinking tended to make fewer errors in interpreting decimals from pictures and applying whole number concepts. Researchers stated a need to combine recursive models of instruction with models of instruction that encouraged student verbalization of thought processes in order to aid students with MLD.

In the current study, informally noted by the researcher was the fact that verbalization of strategies made a difference in detecting untrue equivalency statements in students who struggled and were typically achieving. Examination of the completed student post tests used in quantitative analyses revealed many students included sentences explaining their reasoning regarding why equivalency statements were true and why they were false in their answers. Furthermore, both Albert (TA) and Carl (SS) reasoned through given equivalency statements during the post interview using a ratio table to arrive at correct solutions.

Second, representation used to support development of mathematics concepts was apparent throughout the pre and post interviews. Ratio tables seemed to be helpful for struggling and typical students as they progressed in their ability to work with equivalency statements. Further, the use of the ratio table seemed to generalize to novel problems for students in the experimental group on the CBM and standardized measure used in quantitative analyses. That is, many students in the experimental group used ratio tables to assist in solving problems on the CBM and transfer posttests. The same was not true of students in the control group. But a student with MLD did not use ratio tables during the post interviews. Informal observations by the instructor found that one student with MLD did not understand the ratio table, and may not have had enough time to develop multiplicative thinking through manipulatives or pictorial representations.

Third, the use of manipulatives for understanding of the ratio as a unit and unit iteration should have been extended and explicitly modeled for students with MLD. The current ratio instructional sequence used a concrete to representational to abstract sequence, but perhaps spent too limited time in the concrete phase for students with MLD Butler's (2003) study found groups of students that began instruction with manipulatives did better in understanding fraction equivalency than students who started instruction using pictorial representations during a previous study. Manipulatives used in semi structured interviews, however, did not seem to improve student understanding past matching strategies.

Lastly, one hypothesis for the current study was that higher strategy use with ratios could be cultivated through instruction for students with MLD. The hypothesis was based on the fact that ratio-based fraction instruction could prove to be a valuable innovation and access point for students with MLD. Quantitative results showed that students deemed typically achieving and students who struggled or who had a MLD improved their performance in fraction equivalency on two tests as a result of instruction. However, qualitative results showed participation in ratio instruction cultivated the use of low level matching strategies among students with MLD but higher levels of strategy use for students who struggled. Levels of multiplicative thinking showed slight increases after instruction for the student with MLD, but the effects of atypical thought processes and strategies proved to be more dominant.

In summary, students who took part in the ratio-based instructional sequence outperformed their counterparts in the control group on two measures of fraction equivalency. Student type (SS or TA) seemed to affect results in both the control and experimental groups on the standardized measure. Students who struggled in the experimental group outperformed similar students in the control group. Likewise, students labeled typically achieving

outperformed their counterparts in the control group. Further quantitative analyses revealed that student type did not have an effect on performance on the CBM in either group. Thus, quantitative results indicated that all students in the experimental group benefitted from ratio-based fraction equivalency instruction.

Qualitatively, differences in strategy usage and levels of multiplicative thinking evidenced by three students (MLD, SS, and TA) in the experimental group were found before and after instruction. Albert (TA) used strategies that were considered indicative of early to mid-ratio thinking, and overall seemed to improve from pre to post interview. Carl (SS), while able to progress into higher levels of strategy usage, did not evidence any change in multiplicative thinking from pre to post interview. Bill, while showing increases in strategy use, did not progress past pre-multiplicative levels of thought in solving problems during the post interview. His usage of atypical and incorrect strategies persisted in spite of intervention.

Implications for Practitioners

Although numerous ideas for teachers to consider related to instructing students with MLD in fractions, four primary areas of interest supported from the findings in this study are discussed. First, mathematics content knowledge for teaching is imperative for effectively teaching and remediating students with MLD. The diminutive level of mathematics preparation that special education teachers receive before they begin teaching is well documented (Graham, Li, & Curran Buck, 2000; Maccini & Gagnon, 2006; Rosas & Campbell, 2010). Research indicates that almost 13 times as many class sessions per semester in special education methods courses are devoted to methods of teaching reading as opposed to mathematics (Parmar & Cawley, 1997). Other researchers noted that, in contrast to elementary education programs requiring 6 to 12 credits of mathematics, special education programs were found to vary in the

amount of mathematics courses necessary for certification, with some having no requirement in mathematics (Graham et al., 2000). Current researchers argue that not much has changed in recent years (Greer & Meyen, 2009; Rosas & Campbell, 2010). When two third grade teachers participated in the coding and analyses of the qualitative interviews indicated they were unaware fractions could be taught in ways that were not representative of part-whole situations. Educators who are not prepared to understand deep mathematics content as well as the multi-faceted nature of MLD will not be able to deliver methods (content based as well as instructional based) to guide this population of students to a deeper level of understanding fractions. For instance, the current study used ratio fractions as a possible access point to fraction equivalency for students with MLD due to their difficulties with partitioning. However, at the conclusion of instruction, Bill, a student with MLD, continued to exhibit misconceptions about fraction equivalency and could not use the ratio unit as an iterable quantity. If an educator is not well versed in content knowledge surrounding the teaching of fractions, he or she may not have noticed Bill's natural propensity to view fractions through sharing situations (e.g. quotients). In sharing situations, Bill could use his reliance on correspondence to match pieces to people in a one-to-one fashion. Further, educators may not be aware of Bill's continued difficulties viewing partitioning as an action, not a quantity. Thus, an educator not versed in mathematics content knowledge for teaching would have difficulty identifying alternative pathways to ensure Bills' understanding of fractions. Additionally, an educator may not know what other methods of presenting fractions are applicable and mathematically sound when one method (e.g. teaching fractions through ratios) proves not entirely beneficial.

Second, special educators and general educators need a deeper understanding of ways a learning disability can be presented in mathematics. Many special educators are not prepared to

understand the important differences between students who struggle in mathematics and those who have a true MLD. Historically, special education teachers have been prepared to respond to LD through their knowledge of reading content and disability (Kirk, 1933; Monroe, 1928; Orton, 1925; Strauss, 1943; Werner & Strauss, 1939, 1940, 1941). A deep understanding of MLD, however, is very different from reading LD and constitutes the knowledge that MLD can be comprised of one or more primary areas (Gregg, 2009). Many special education teachers are prepared to believe the primary issues involved with MLD are difficulties with word problem solving, computing, following procedures, recalling basic facts, and interpreting graphs or figures. However, these areas are often *secondary* deficits that surface in mathematics performance as a result of a breakdown in underlying mathematical understanding. As noted in the review of literature, the primary areas that most impact mathematics performance have been cited as working memory (Geary et al., 2006; Passolunghi et al., 2007), language (Rousselle & Noel, 2007), sense of number (Butterworth, 2005; Desoete et al., 2008; Geary, 2004, 2007, 2009), and fluid reasoning (Emberton, 1995; Gregg, 2009). Educators need to be prepared to identify why knowledge breaks down for a student with MLD and then use that information in preparing mathematics instruction to benefit all students.

Educators who are not prepared to understand deep mathematics content as well as the multi-faceted nature of MLD will fail to understand why this student can struggle learning important concepts. Bill (MLD) could not understand the ratio unit as a quantity. Examining the results of data analyses, Bill's difficulties with understanding ratios as units may have been due to (1) his inability or difficulty in recognizing and naming small quantities and groups, (2) his difficulty with the counting process and the implications for holding a ratio unit (and the appropriate linkage involved) constant, or some mixture of the two. Thus, Bill's problems did not stem from

an inability to follow procedures or recall facts. Instead, Bill's (MLD) performance was indicative of a breakdown in subitizing and/or a difficulty to hold information to solve a problem from the working memory. An educator who did not see the true source of Bill's difficulty may have used the teaching sequence to reiterate procedures in attempt to help Bill understand. This approach would have drilled a procedure instead of remediating the ability to subitize. The educator may never have known to address the true source of Bill's difficulty. One of the third grade teachers who assisted with qualitative data analyses was Bill's mathematics teacher. She was surprised by many of Bill's responses and indicated that she had not known to look at primary areas of mathematics, such as subitizing, for sources of misunderstanding.

A related third point to be stressed is that there is no 'magic bullet' in remediating or teaching essential mathematics content knowledge to students with MLD. In the current study, the researcher gave pre and posttests to 38 students who did and did not participate in ratio equivalency instruction. All students in the experimental group were found to improve their performance after instruction compared to those in the control group. The researcher also interviewed a student who struggled in mathematics, a student deemed typically achieving, and a student labeled as having a MLD. Important differences in thought processes and solutions strategies were found. Although the students interviewed were chosen because they were most representative of the characteristics that defined each 'student type', these findings were only representative of the three students who were interviewed. In other words, designing instruction solely based on the current study's qualitative analyses for *all* students would be faulty. Other students with similar backgrounds may have provided different patterns of results. With respect to students with MLD, "students may differ in the severity of one type of deficit or another; and students may differ in the developmental course of the deficits" (Geary et al., 2009, p. 46).

Beginning with instruction that allows many paths for understanding through instructional supports could ensure better mathematical outcomes for every student.

Finally, using assessment methods such as student interviews or probes in conjunction with paper and pencil tests could provide practitioners with clearer understanding of student thinking. If the researcher had only completed a quantitative analyses in the current study, the important differences between levels of multiplicative thought and strategy use would have gone unnoticed. Similarly, practitioners need to go beyond the standard multiple choice tests in mathematics to uncover real differences in student thinking. Then these findings could be used to inform and drive mathematics instruction and future research. For instance, error analyses is one area where educators could use the categories found in qualitative analyses to determine needed interventions. Although the categories found in this study may not be indicative of every student with MLD, teachers could use atypical categories as a reference to understand if students with MLD are evidencing potential patterns of problems and lack of or forward movement for this population. Furthermore, using multiple methods of expression are in line with the principles of Universal Design for Learning, an instructional philosophy that holds promise for creating access points to academic content for students with disabilities (Graham & Thomas, 2000).

Limitations and Future Research

As in all research, several limitations need to be acknowledged in this study. First, the quasi experimental part of the research design was subject to certain disadvantages- namely, the possibility of attrition of subjects as well as the possibility of fatigue, carry over effects, practice, or latency. Although counterbalancing can control for these effects, the order in which treatment is delivered was not possible given the design of the intervention.

Second, the researcher provided all of the instruction. While the instructional sessions were checked for fidelity of implementation by two independent observers, the results of the study provided no evidence of the effects of the instructional sequence implemented by other instructors. Future research evaluating the impact of the ratio-based instructional sequence with classroom teachers should be conducted to ensure that the results obtained herein extend to practice.

Third, assignment of subjects, while random, was only so after students who meet certain criteria were selected. Further, selection was not truly random due to criteria for inclusion in the study. Thus, bias may have been present in the selection of subjects. Another limitation was the criteria used to deem students MLD, struggling, or typically achieving. Although care was used to employ research backed criteria to designate students into subgroups, the field of mathematics learning disabilities has yet to determine a precise definition of MLD or validation processes leading to such a designation. Quantitative and qualitative differences can be found in studies that use different cutoff criterion scores to designate a group as MLD (Murphy et al., 2007). Thus, although the designation used here was defined by previous high quality research, caution should be used in generalizing findings from this study to all students labeled as having a MLD.

Fourth, the fact that students who struggled and students labeled as MLD were joined together for the quantitative analyses may have masked important differences in performance between these two types of students. Larger sample sizes may be needed in future research to better discern between performance of students with MLD, students who struggle, and students deemed typically achieving in quantitative analyses. Further, qualitative inquiry into student thinking with larger groups of students in control as well as experimental groups may do a better job of providing further validity in a study of similar design or replication.

Fifth, the CBM measure, although constructed to accurately assess the effects of the ratio-based instruction on students' understanding of part-whole and ratio fraction equivalency, did not contain many items that measured students' understanding of ratio-based fraction equivalency. Therefore, while the instrument possessed content validity with respect to the curriculum that students are expected to master, it may not have possessed an adequate content validity in measuring student knowledge of ratio fraction equivalency. A final limitation is that the intervention was not tested among other types of instruction using more traditional fraction sub constructs (e.g. part-whole) or against more (or less) explicit models to support the current ratio sequence. Many research based part-whole instructional approaches currently in existence have shown increased performance among students without disabilities (Cramer, Post, & del Mas, 2002). The effectiveness of the instructional sequence compared to other noted effective instructional models or varying subconstructs of fractions at this time was not evaluated. Future research should evaluate the extent to which ratio-based instruction in fraction equivalency increases performance when compared to instruction based in other sub constructs.

Students who participated in the ratio-based teaching sequence increased their performance on two measures of fraction equivalency. Their gains on the testing measures were significantly higher than those in the control group. However, qualitative analyses of three students showed a student with MLD had atypical misunderstandings of the ratio as a unit, and struggled to make sense of ratio equivalency situations compared to a student deemed as struggling. Further research is needed to describe and confirm the atypical misunderstandings evidenced by an interviewed student with MLD. Results indicated the need for special and general educators to clearly analyze unique learning needs of students with MLD in mathematics content.

APPENDIX A: PRE AND POST TESTS

Fractions Pre-Test

Name: _____

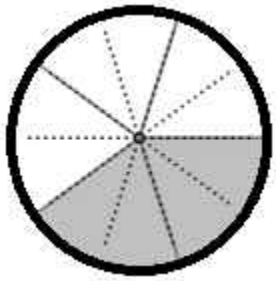
The sheets on your desk are fraction tests. All the problems are fraction equivalency problems. Look at each problem carefully before you answer it. When I say 'start,' turn them over and begin answering the problems. When you finish one side, go to the back. Are there any questions? Start.

Find an equivalent fraction for each.

1. $\frac{8}{18}$
2. $\frac{1}{3}$
3. $\frac{24}{30}$
4. $\frac{2}{15}$
5. $\frac{12}{15}$

Solve the word problems below.

6. Name ten pairs of equivalent fractions.
7. The world's largest pumpkin pie weighed 2,020 pounds. The pie was $12\frac{1}{3}$ feet across and $\frac{1}{3}$ foot thick. Write a fraction equivalent fraction to $\frac{1}{3}$.
8. Look at the model. Name three equivalent fractions for the part of the circle that is grey.



9. In the United States, $\frac{2}{5}$ of all states start with the letters M, A, or N. How can you use equivalent fractions to find out how many states this is?

10. At a car wash, Jim washed 8 cars per hour. David washed 6 cars per hour. How many cars did Jim wash if David washed 24 cars?

11. At Tara's Video Outlet, you can buy any 6 used DVDs for 48 dollars. At Sam's DVD Palace, you can buy any 4 used DVDs for 28 dollars. At which store do DVDs cost less? How much less?

12. Each day, pandas are awake for about 12 hours. They eat for 10 hours. What fraction of their time awake are pandas eating? Write your answer is simplest form.

Find the numerator that makes the fractions equivalent.

13. $\frac{1}{4} = \frac{\quad}{8}$

14. $\frac{4}{6} = \frac{\quad}{3}$

15. $\frac{1}{2} = \frac{\quad}{16}$

16. $\frac{8}{10} = \frac{\quad}{5}$

17. $\frac{3}{4} = \frac{\quad}{12}$

18. $\frac{3}{4} = \frac{\quad}{16}$

19. $\frac{1}{2} = \frac{\quad}{12}$

20. $\frac{5}{6} = \frac{\quad}{12}$

Fractions Post-Test

Name: _____

The sheets on your desk are fraction tests. All the problems are fraction equivalency problems. Look at each problem carefully before you answer it. When I say 'start,' turn them over and begin answering the problems. When you finish one side, go to the back. Are there any questions? Start.

Find an equivalent fraction for each.

1. $\frac{2}{10}$
2. $\frac{3}{5}$
3. $\frac{25}{30}$
4. $\frac{21}{28}$
5. $\frac{12}{20}$

Solve the word problems below.

6. How can you show that $\frac{3}{4}$ and $\frac{9}{12}$ are the same by multiplying and dividing?
7. James has 18 mystery books and 12 sports books. Rich has twice as many mystery books and three times as many sports books. How many books does Rich have?
8. In a school poetry contest, 15 out of the 25 students who ordered will win a small prize. Half of the remaining students receive a certificate. How many students get a certificate? Answer in simplest terms.
9. Which shows $\frac{1}{2}$ and $\frac{1}{5}$ as fractions with the same denominator?

a. $\frac{5}{10}$ and $\frac{2}{10}$

b. $\frac{1}{10}$ and $\frac{5}{10}$

c. $\frac{5}{10}$ and $\frac{1}{10}$

d. $\frac{5}{10}$ and $\frac{3}{10}$

10. Charlie had twelve cubes. He showed that $\frac{8}{12}$ is equivalent to $\frac{2}{3}$ by making three groups of 4 and drawing a circle around two of the groups. Using 12 or fewer cubes, what is another fraction that is equivalent to $\frac{2}{3}$?

11. Tyrone runs 4 miles each week. Francis runs 4 times as many miles each week. How many miles does Francis run each week?

12. A store sells school-supply packs that contain 6 pencils and 4 pens. A customer bought enough packs to get 36 pencils. How many pens did the customer get?

Find the numerator that makes the fractions equivalent.

13. $\frac{2}{5} = \frac{\quad}{10}$

14. $\frac{6}{16} = \frac{\quad}{8}$

15. $\frac{1}{3} = \frac{\quad}{12}$

16. $\frac{3}{12} = \frac{\quad}{4}$

17. $\frac{5}{8} = \frac{\quad}{16}$

18. $\frac{4}{8} = \frac{\quad}{2}$

19. $\frac{1}{2} = \frac{\quad}{10}$

20. $\frac{4}{5} = \frac{\quad}{15}$

APPENDIX B: TRANSPARENCIES

1. *How can you represent the problem situation using the objects?*
2. *How can you show the pancakes?*
3. *How can you keep track of the number of stacks?*

Transparency 1- Order Up!

How can you solve the problem situation using the given relationship?

How can you represent the problem situation using a picture?

How can you show the pancakes and the people?

Transparency 3- The Kitchen

How can you represent the problem situation using a picture?

Is there another way to represent and solve the problem instead of drawing a picture?

Transparency 4a-Morning Preparation 1

How can you represent the problem situation using a ratio table?

How can you use patterns in the tables to help you find the answer?

How is multiplication and division seen in the tables you created?

Transparency 4b- Morning Preparation 1

How are the answers alike? How are they different?

Could more than one cook be correct?

Which cook's answer produces a correct answer?

Transparency 5- The Cook's Disagreement

How can you use patterns in the tables to help you find the answer?

How is multiplication and division seen in the tables you created?

How can we use a shortened ratio table to help us find the number of pancakes?

How could you write your answers as fractions?

Transparency 6- Morning Preparation 2

How can you represent the problem situation using a ratio table (long or short) AND fractions?

How can we use multiplication and division to write shortened ratio table to help us find the number of pancakes?

What do the fractions you drew represent?

Transparency 7- Morning Preparation 3

How can we build up from what we know using multiplication and division to find the amount of needed pancakes?

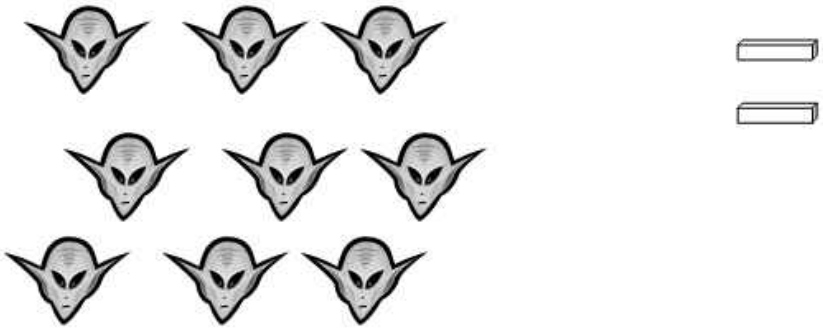
Can we use the relationship between people? Between pancakes? Between people and pancakes?

How could you write your answers as fractions?

Transparency 8- From the Kitchen to the Table

APPENDIX C: INTERVIEW QUESTIONS

1. Is there enough food? Explain.



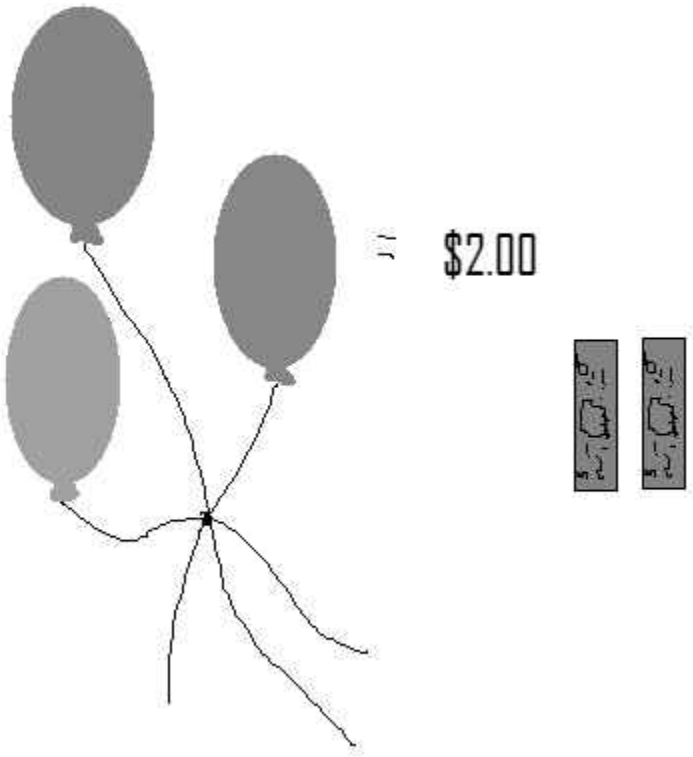
2. Is there enough food? Explain.

3. How many food bars are needed to feed these aliens? Explain.

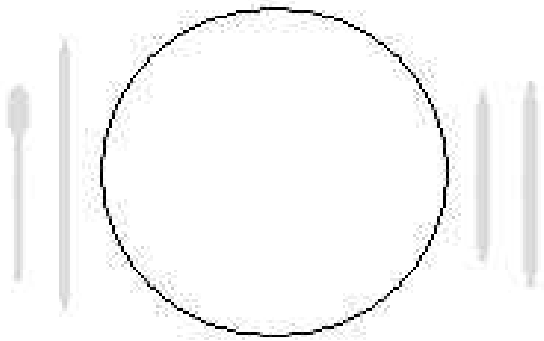
4. Who gets more food?

- Make a bundle of 5 blue and 3 red chips. Ask the student:
    - How many of the same type of bundles would be behind my back if I had 10 blue chips?
    - How many of the same type of bundles would be behind my back if I had 20 blue chips?
    - How many of the same type of bundles would be behind my back if I had 35 blue chips?
    - How many of the same type of bundles would be behind my back if I had 9 red chips?
    - How many of the same type of bundles would be behind my back if I had 15 red chips?
    - How many of the same type of bundles would be behind my back if I had 6 red chips?

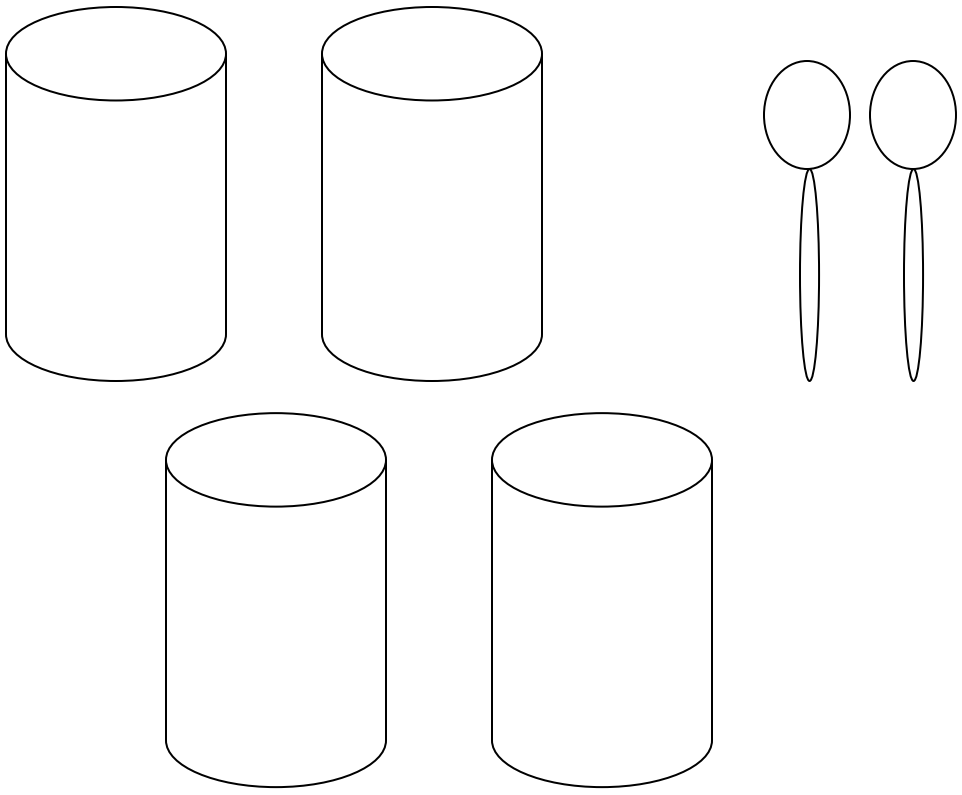
Ellen, Jim, and Steve bought 3 balloons and paid $2.00 for all three. They decided to go back and get enough balloons to give one to everyone in their class. How much did they have to pay for 24 balloons?



At a dining room table, there are 4 utensils for every 2 plates. If there are 14 plates, how many utensils are there?



2 spoons of cocoa are needed to make 4 cups of hot chocolate. How many spoons of cocoa are needed to make 10 cups of hot chocolate?



Tiny Tots Daycare has a rule that one teacher can watch five infants at a time.

Teachers	1			
Infants	5			

1. How many teachers must be in the room if there are 25 infants? Explain.

2. What is the maximum number of infants that can be in the room if there are 6 teachers? Explain.

3. How many ways could we use the fraction ratio of Teachers/Infants to determine how many infants could be watched by different numbers of teachers?

4. Which fraction ratios belong in the same groups?

1/2	2/3
3/4	6/9
20/40	4/9
20/41	9/12

5. Add two more fraction ratios to each group you created. How do you know they belong?

6. How many more fraction ratios could you add to each group?

APPENDIX D: FIDELITY CHECKLIST/TEACHER SCRIPT

TEACHER DIRECTIONS: Each numbered item in the first column lists a part of instruction that teachers need to do. Follow the script for each day of instruction.

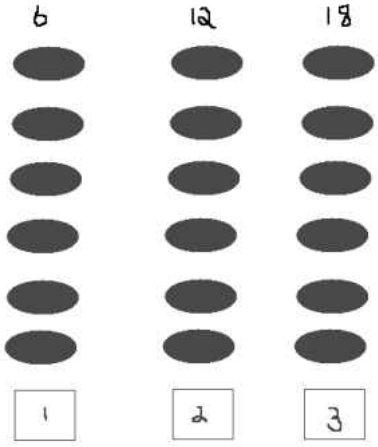
OBSERVER DIRECTIONS: For each problem, say if each numbered item that the teacher should complete in their instruction was absent ("NO") or present (checkmark).

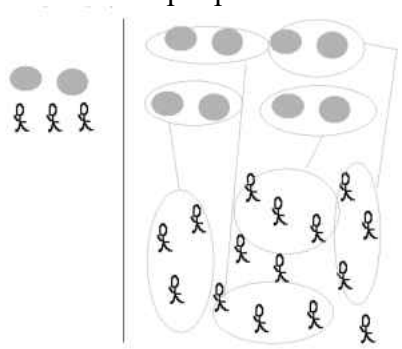
Teacher Action	Begin	P1	P2	P3	P4*	End
All materials should be handed out before students enter room. Use worksheets to assign seats. Write transparency questions on board or display on overhead. Turn on camera. Display example problem on board.	Absent/ present	Absent/ present	Absent/ present	Absent/ present	Absent/ present	Absent /present
1. Go over Example Problem (see "example problem script").						
2. Read the worksheet scenario to the students or have a student read.						
3. Ask a student to read the current problem (1, 2, 3…) aloud for the group. *Teacher: Record who read.*						
4. Say, "Solve the problem on your own for two minutes keeping in mind the questions that you see on the board."						
5. Read the questions on the board to students (first problem only).						
6. Aid any student who is confused by: • Reminding them of unit relationship having to remain the same. • Asking how they						

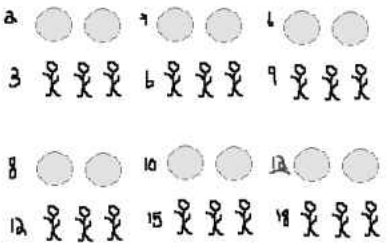
would begin the problem. • Suggesting a representation to use.						
7. When students are done with question (or two minutes), say, "Now discuss the solutions you found with a partner for the next two minutes".						
8. When students are done discussing (or two minutes), tell/select one student who will share problem solutions with the class. *Teacher: Record the student who presents each problem.*						
9. Select observing students to ask the transparency questions to student who is explaining answer. *Teacher: note who was called on to answer problems.*						
10. Say, "Does everyone agree on the answers shown?"						
a. ***If no***, explicitly model the correct answer using think aloud (see video examples).						
11. Ask students to answer reflection questions. *Teacher: record which students answered.*						
12. Was feedback for reflection questions answers? (circle one)		y/n	y/n	y/n	y/n	

REPEAT #5 – #25 for Problems 2, 3, (4)* *if it has it					
13. Tell the students to answer the last problem on their own.					
14. Collect the worksheets, turn off camera.					

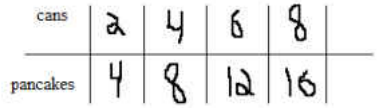
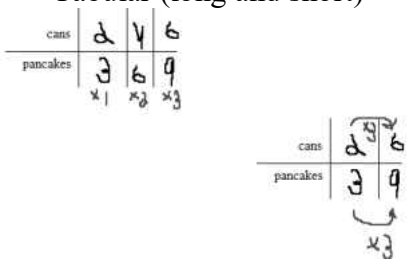
DIRECTIONS: Read example to class at beginning of each day noted.

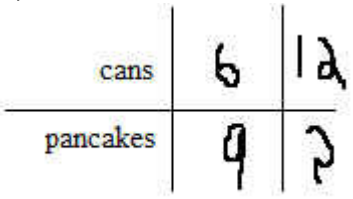
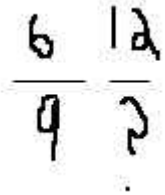
Day and Words to Say to Class	**Representation**
1. [DAY 1] Today we are going to work on problems to help out some cooks in a pancake house. They are trying to figure out how many pancakes to use if a person orders a certain number of pancake stacks. The number of pancakes in a stack is given to you on the top of the page. First, we will look at an example problem together. In our example problem ONLY, there are six pancakes in a stack. The number of pancakes in a stack in the problems you will be working on is different. Back to our example. For one order that the cooks receive, a person orders a triple stack. We need to figure out how many pancakes that is. I will use the counters and cups to model the pancakes (counters) and stacks (cups). Ready? I know that six pancakes go in one stack. So, I will start by making one stack (put six counters in a stack). Next, I know that since the customer ordered a triple stack, and triple means "three times", I need to make a total of three stacks. So I will do that (make two additional stacks of six). So, it looks like I have…(count and say) one stack (put out one cup)- that's six. Two stacks (put out another cup), that's 12…three	Use Cups and Counters to represent the problem. Also, show pictorial solution to compliment concrete representation used. 

stacks (put out a final cup), that's 18 total pancakes in three stacks. 2. If I wanted to provide a good explanation of what I did to find the answer, I might do several things. First, I would provide a drawing of how I set the problem up. I would start by drawing the number of pancakes in a stack (draw six circle to represent one stack). Next, I would show how I made three total stacks of pancakes, each with six in the stack (draw two additional "stacks" of six circles). Now, I will label my stacks…one, two, three. 3. To go with my drawing, I will write a few sentences about what I was thinking when I solved the problem (write out the following sentences and say out loud). I knew that there were six pancakes in one stack. Next, I knew that since the customer ordered a triple stack, and triple means "three times", I needed to make a total of three stacks. So, I had three stacks with six pancakes in each stack…1 stack, six pancakes; 2 stacks, 12 pancakes; three stacks, 18 pancakes. 18 pancakes.	
1. [DAY 2] Today we are going to work on problems to help the cooks again in Rusty's Pancake House. They are trying to figure out if they have enough pancakes for the number of people shown in each situation. The amount of pancakes eaten by a certain number of people is given to you at the top of the page. First, we will look at an example problem together. In our example problem ONLY, two pancakes feed three people. The number of pancakes and people you will be working on is different. Back to our example. So I am shown a situation here of eight pancakes and 15 people. I need to figure out if I	Use pictorial representation and explicitly show the linkage of pancakes to customers for example problem. 

have enough pancakes for the amount of people shown. First, I will put out 8 counters to represent the pancakes (put out eight counters). Next, I will put out 15 cups to represent the people (put out fifteen cups). So, if 2 pancakes feed 3 people, I have to take what I have and make matches of 2 pancakes for every 3 people. So I will do that (move two counters and three cups together; repeat until you cannot do any more matching). Whoops! It looks like I ran out of pancakes to match to groups of three people. So, I don't have enough pancakes to feed these 15 people. 2. If I wanted to provide a good explanation of what I did to find the answer, I might do several things. First, I would provide a drawing of how I set the problem up. But since I already have a drawing on the page, I am OK there. 3. To extend the drawing, I can show how I linked the people to pancakes. So let's connect three people to two pancakes (show on board). Last, I will write a few sentences about what I was thinking when I solved the problem (write out the following sentences and say out loud). I knew that two pancakes fed three people. Next, I matched every 2 pancakes to 3 people. I didn't have enough pancakes to go with the last group of three people. So my answer is, "too few pancakes".	
1. [DAY 3] Today we are going to work on problems to help the cooks again in Rusty's Pancake House. They are trying to figure out how many pancakes to make for a given amount of people. But in some problems, they know the amount of pancakes they have and need to know how many people those pancakes feed.	Pictorial- iterate (copy) people to pancake relationship 

The amount of pancakes eaten by a certain number of people is given to you at the top of the page. First, we will look at an example problem together. In our example problem ONLY, two pancakes feed three people. The number of pancakes and people you will be working on is different. Back to our example. So I am asked how many pancakes to make for 18 people. I think we should concentrate on our drawings today. First, I will draw out my rule of two pancakes (make two circles) and three people (draw three lines or stick figures under the two circles). I need to get to where I have 18 people, all in groups of three since that's my rule. How many groups of three go into 18? Six. So I know I have to draw six groups of three people. But every time I draw a groups of three people, I need to put two pancakes with them (model this representation). Now I will count the pancakes to see how many I have here (count pancakes like this: 1 group, 2…2 groups, 4, 3 groups, 6….). Hmm…looks like I need 12 pancakes to feed these 18 people. 2. If I wanted to provide a good explanation of what I did to find the answer, I might do several things. First, I would provide a drawing of how I set the problem up. Kind of like I have on the board here. Next, I will write a few sentences about what I was thinking when I solved the problem (write out the following sentences and say out loud). I knew that two pancakes fed three people. Next, because I knew I needed 18 people total, I figured out that I needed 6 groups of three people to do that. But every time I drew three people, I drew 2 pancakes with it. I	

did so because I know that every two pancakes fed three people. So my answer is 12 pancakes.	
1. [DAY 4] *Before first problem, use example above. After students share and report on first problem, use the following*- Could we represent this drawing another way? I was thinking, [name of student], when you were showing us how you did the problem…I was thinking of this representation (draw ratio table). So, you drew the two cans for four pancakes out until you got eight cans, right? So you saw that eight cans made only 16 pancakes…not enough. I wrote out what you were explaining to use as a table…see I put the rule here (point at two over four), and then each time you added a "two and four" can to pancake thing, I did that with numbers. So, 2, 4; 4, 8; 6, 12; 8, 16. I can represent that same thinking with this table. Try to use this table in the next problem. If you still want to draw a picture along with the table that is fine.	Pictorial and Tabular, long 
1. [DAY 5] None	None
1. [DAY 6] *Before the first problem, use example from 2nd part of Day 4. After student shows solution, use the following*- Can I shorten that table in any way? I saw that you used a table with{2,3; 4,6; 6,9} to find out that we could make 9 pancakes with 6 cans of batter, but what if I didn't want to draw out that huge table? Does anyone see a shorter way? [Elicit how multiplication can be seen in the tables – each column in a "times" column…so a shorter way would be to figure out how many times you would have to use the given pancake to can rule to get to what the questions is asking]. For example, for the 6 to 9 question, we could think, hmm…I	Tabular (long and short) 

would need 3 groups of 2 cans to get to 6 cans of batter. So, to answer the question, I would need to see if that same number of pancakes groups gets me to nine pancakes total. There are 3 in a group for pancakes. So, is 3 times 3 equal to nine? If so, than I know it works. If not, then I know it doesn't.	
1. [DAY 7] *Before problem 1, do the second part of Day 6 over again to remind students of shortened table. Then-* is this short table the same thing as a fraction? Look at the can to pancake rule you are given. I don't see any picture or table that shows me the rule. In fact, that looks like a fraction. But is there a rule there, too? [elicit understanding that the fraction given can be thought of as the can to pancake rule]. So, for an example, if the questions asks me how many pancakes can I make with twelve cans of batter, can I use the short table to  write So, is that the same thing as  [Make explicit the removal of the row labels and column line separators so that students can see that the ratio table is just showing strings of fractions that are equivalent].	Look at left column (tabular and abstract)
1. [DAY 8]. None. Just remind students of strategies and representations used thus far. Advocate for the use of short tables or fraction notation. If other representations used, try to supply	All

related fraction or short ratio representations.	
1. [DAY 9]. None. Just remind students of strategies and representations used thus far. Advocate for the use of short tables or fraction notation. If other representations used, try to supply related fraction or short ratio representations.	All

APPENDIX E: SOCIAL VALIDITY CHECKLIST

Fraction Equivalency Questionnaire

Directions: Read each sentence. Circle "1" if you agree. Circle "2" if not sure. Circle "3" if you don't agree.

START HERE

1. I know how to make one fraction the same as another by adding.	1	2	3
2. I know how to make one fraction the same as another by multiplying.	1	2	3
3. I can show that two fractions are the same two different ways.	1	2	3
4. I can tell when two fractions are the same and say why.	1	2	3
5. I can draw a picture to show why two fractions are or are not the same.	1	2	3
6. You cannot add the <u>same</u> number to the numerator and denominator to make equivalent fractions.	1	2	3
7. Equivalent fractions are different ways to show the same rule or relationship.	1	2	3
8. I feel good about how much I know about fraction equivalency.	1	2	3
9. Working with fractions does not scare me.	1	2	3
10. Fractions are not confusing.	1	2	3
11. You can multiply the same number to the numerator and denominator to make equivalent fractions.	1	2	3

APPENDIX F: IRB

University of Central Florida Institutional Review Board
Office of Research & Commercialization
12201 Research Parkway, Suite 501
Orlando, Florida 32826-3246
Telephone: 407-823-2901 or 407-882-2276
www.research.ucf.edu/compliance/irb.html

Approval of Human Research

From: **UCF Institutional Review Board #1**
FWA00000351, IRB00001138

To: **Jessica H Hunt**

Date: **February 08, 2011**

Dear Researcher:

On February 8, 2011, the IRB approved the following human participant research until 2/7/2012 inclusive:

Type of Review:	Submission Response for UCF Initial Review Submission Form Expedited Review Category #7
Project Title:	Understanding Fraction Equivalency through Conceptually Based Ratio Interpretations
Investigator:	Jessica H Hunt
IRB Number:	SBE-11-07410
Funding Agency:	None

The Continuing Review Application must be submitted 30days prior to the expiration date for studies that were previously expedited, and 60 days prior to the expiration date for research that was previously reviewed at a convened meeting. Do not make changes to the study (i.e., protocol, methodology, consent form, personnel, site, etc.) before obtaining IRB approval. A Modification Form **cannot** be used to extend the approval period of a study. All forms may be completed and submitted online at https://iris.research.ucf.edu .

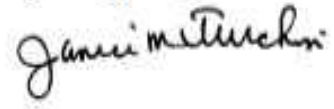
If continuing review approval is not granted before the expiration date of 2/7/2012, approval of this research expires on that date. When you have completed your research, please submit a Study Closure request in iRIS so that IRB records will be accurate.

Use of the approved, stamped consent document(s) is required. The new form supersedes all previous versions, which are now invalid for further use. Only approved investigators (or other approved key study personnel) may solicit consent for research participation. Participants or their representatives must receive a signed and dated copy of the consent form(s).

In the conduct of this research, you are responsible to follow the requirements of the Investigator Manual.

On behalf of Joseph Bielitzki, DVM, UCF IRB Chair, this letter is signed by:

Signature applied by Janice Turchin on 02/08/2011 10:55:52 AM EST



IRB Coordinator

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