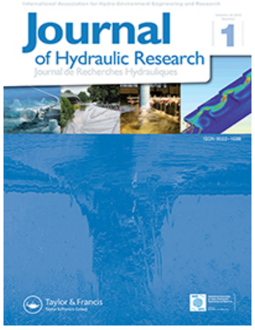

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Technical note

Advection-diffusion sediment models in a two-phase flow perspective

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ABSTRACT
Sediment profiles in open channels are usually predicted by advection-diffusion models. Most basic forms consider the terminal settling velocity of a single particle in still clear water. Alternative forms account for hindered settling at higher concentrations. It is not known, however, how these modifications relate to mass and momentum conservation of each phase. For dilute flow, it is known that the original form can be derived from a two-phase analysis, assuming a dilute suspension, neglect of inertial effects in the momentum balance and using a linear drag force formulation. Here we study how and if it is possible to understand the hindered-settling modifications for the non-dilute case, and formulate a relation between advection-diffusion models and parameters involved in the turbulent drag force. This note verifies that the transient two-phase flow solutions converge to steady state, and compares the results to experimental data.

Keywords: Fluid–particle interactions; particle-laden flows; sedimentation; suspended sediments; turbulence-sediment interactions

1 Introduction

Sediment transport mechanics is important for river morphology, mining, coastal and dredging engineering. The case of two-dimensional steady-uniform flow is considered as a basic problem to study the physics of sediment suspension (Ali & Dey, 2016), and a validation case for a wide range of modeling techniques (Dey, 2014; Jha & Bombardelli, 2014). Basic approaches consider an advection-diffusion equation of the form:

$$(1 - C)^m C V_p^\infty + \epsilon_s \frac{\mathrm{d}C}{\mathrm{d}y} = 0 \qquad (1)$$

where y is the vertical coordinate, ϵ_s is the turbulent diffusion coefficient, C is the volumetric concentration and V_p^∞ (> 0) is the terminal settling velocity of a single particle in still clear water. Different values for exponent m were considered: $m = 0$ (Rouse, 1937; Schmidt, 1925). Also $m = 1$ (Halbronn, 1949; Hunt, 1954) $m = n + 1$ (van Rijn, 1984) and $m = n$ (Winterwerp, de Groot, Mastbergen, & Verwoert, 1990), where n denotes the well-known hindered-settling exponent (Richardson & Zaki, 1954). It is not clear whether Eq. (1) is mass and momentum conserving for all considered values of m. Another question is: how does m relate to the parameters involved in drag force formulations? Finally, it is not known whether the different forms of Eq. (1) are the asymptotic states of transient two-phase flow equations. Greimann and Holly (2001) found that the basic advection-diffusion form Eq. (1) with $m = 0$ can be obtained from a two-phase model. In their derivation it is essential to neglect inertial terms in the momentum balances of both phases and to linearize the drag force formulation. In order to answer the aforementioned questions, this technical note reconsiders and extends the work of Greimann and Holly (2001).

2 Two-phase formulation

This section represents some relevant elements of the two-phase formulation of Greimann, Muste, and Holly (1999). Figure 1

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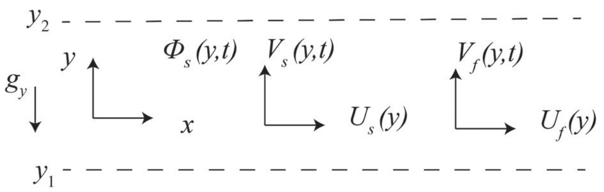

Figure 1 Definition sketch of the coordinate axes and velocity components of a fully developed particle laden flow layer

defines the coordinates and velocity components. We consider an uniform flow layer yielding $\partial_z = \partial_x = 0$.

Mass conservation requires that:

$$\frac{\partial \Phi_s}{\partial t} + \frac{\partial \Phi_s V_s}{\partial y} = 0 \tag{2}$$

where Φ_s is the solid phase fraction and V_s is the phase weighted ensemble averaged vertical velocity of the solid phase. Conservation of volume requires that: $\Phi_s + \Phi_f = 1$, where Φ_f denotes the fluid phase fraction. The momentum conservation equations for the solid and fluid phases in the vertical direction read:

$$\begin{aligned} \rho_\alpha \frac{\partial \Phi_\alpha V_\alpha}{\partial t} + \rho_\alpha \frac{\partial \Phi_\alpha V_\alpha^2}{\partial y} &= -\Phi_\alpha \frac{\partial P_\alpha}{\partial y} + \Phi_\alpha \rho_\alpha g_y \\ &+ \frac{\partial \left(\Phi_\alpha T_{\alpha yy} - \rho_\alpha \Phi_\alpha \overline{v_\alpha'^2} \right)}{\partial y} + M_{\alpha y} \end{aligned} \tag{3}$$

where $\alpha = s$ and $\alpha = f$ denote the solid and fluid phase respectively, ρ_α is the density of each phase, P_α is the ensemble averaged pressure ($P_s = P_f$), $T_{\alpha yy}$ is the ensemble averaged intraparticle stress ($\alpha = s$) or the viscous shear stress ($\alpha = f$), $\overline{v_\alpha'^2}$ represents the turbulent shear stress, g_y is the gravitational acceleration, $M_{\alpha y}$ is the coupling force between the phases ($M_{fy} = -M_{sy}$). The coupling force can be expressed as:

$$\begin{aligned} M_{sy} = \Phi_s \rho_s \tau_p^{-1} V_r + \rho_f C_A \Bigg(& \frac{\partial}{\partial t} \Phi_s V_r + \frac{\partial}{\partial y} \Phi_s (V_f + V_d)^2 \\ & - \frac{\partial \Phi_s V_s^2}{\partial y} + \frac{\partial}{\partial y} \Phi_s \overline{v_f'^2 - v_s'^2} \Bigg) \end{aligned} \tag{4}$$

where C_A is the added mass coefficient. The relative velocity is defined as: $V_r = V_f - V_s + V_d$, where V_d is the drift velocity, which relates to the correlation between the indicator function of the solid phase and velocity fluctuations of the fluid phase. It is important to distinguish V_r and the lag velocity defined as: $V_l = V_f - V_s$. The particle response time τ_p is defined as:

$$\tau_p = \frac{\rho_s d_p^2 {\Phi_f}^{\beta}}{18 \nu_f \rho_f C_f} \tag{5}$$

where d_p is the particle diameter, ν_f is the kinematic viscosity of the fluid phase, C_f is the friction coefficient of a single particle in a fluid and β is an exponent required for drag modeling in fluid–multiparticle systems (Di Felice, 1994). Both C_f and β depend on the particle Reynolds number: $\mathsf{R_p} = |u_r| d_p / \nu_f$, with velocity scale: $|u_r| = (V_r^2 + u_{ri}'^2)^{1/2}$. The friction coefficient relates to the particle Reynolds number (Wallis, 1969):

$$C_f = (1 + 0.15\, \mathsf{R_p}^{0.678}) \quad \text{for } 1 < \mathsf{R_p} < 2000 \tag{6}$$

The drift velocity is modelled as:

$$V_d = -\epsilon_{yy} \frac{1}{\Phi_s \Phi_f} \frac{\partial \Phi_s}{\partial y} = -\epsilon_{yy} \left(\frac{1}{\Phi_s} + \frac{1}{\Phi_f} \right) \frac{\partial \Phi_s}{\partial y} \tag{7}$$

The intraparticle stress T_{syy} can be expressed as a function of the solid velocity fluctuations (Greimann & Holly, 2001).

3 Generic concentration-sedimentation model for the diffusive regime

Considering local time scale τ_p, length scale $\epsilon_{yy} \overline{v_s'^2}^{-1/2}$, averaged phase velocity scale $|V_s|$ and turbulent velocity scale $\overline{v_s'^2}^{1/2}$, it follows from a scale analysis of Eqs (2) and (3) that:

$$\frac{\partial V_s}{\partial t} = -|g| - \frac{1}{\rho_s} \frac{\partial P_f}{\partial y} + \frac{1}{\rho_s \Phi_s} \left(\frac{\Phi_s \rho_s}{\tau_p} V_r + \rho_f C_A \frac{\partial \Phi_s V_r}{\partial t} \right) \tag{8}$$

and:

$$\frac{\partial V_f}{\partial t} = -|g| - \frac{1}{\rho_f} \frac{\partial P_f}{\partial y} - \frac{1}{\rho_f \Phi_f} \left(\frac{\Phi_s \rho_s}{\tau_p} V_r + \rho_f C_A \frac{\partial \Phi_s V_r}{\partial t} \right) \tag{9}$$

provided that $\overline{v_s'^2} \tau_p / \epsilon_{yy} \ll 1$ and $|V_s| \overline{v_s'^2}^{1/2} \tau_p / \epsilon_{yy} \ll 1$. From Eqs (2), (8) and (9), and volume conservation, it is possible to derive an evolution equation for the total volume flux, $J = \Phi_s V_s + \Phi_f V_f$ (zero in this study), and find an expression for the pressure gradient. Using this equation for $\partial P_f / \partial y$ and subtraction of Eq. (8) from Eq. (9) yield:

$$\begin{aligned} \left(1 + \frac{\rho_f C_A}{(\Phi_s \rho_f + \Phi_f \rho_s) \Phi_f} \right) \frac{\partial V_l}{\partial t} &= \frac{\rho_s - \rho_f}{\Phi_s \rho_f + \Phi_f \rho_s} |g| - \frac{1}{(\Phi_s \rho_f + \Phi_f \rho_s) \Phi_s \Phi_f} \\ &\times \left(\frac{\Phi_s \rho_s}{\tau_p} (V_l + V_d) + \rho_f C_A V_l \Phi_f \frac{\partial \Phi_s}{\partial t} + \rho_f C_A \frac{\partial \Phi_s V_d}{\partial t} \right) \end{aligned} \tag{10}$$

For steady state conditions $\partial / \partial t = 0$, it follows from Eqs (7) and (10) that:

$$V_l = \Phi_f \tau_p \left(1 - \frac{\rho_f}{\rho_s} \right) |g| + \epsilon_{yy} \left(\frac{1}{\Phi_s} + \frac{1}{\Phi_f} \right) \frac{\mathrm{d}\Phi_s}{\mathrm{d}y} \tag{11}$$

Since $J = 0$ for all vertical positions y, it holds that $V_s = -\Phi_f V_l$, such that it follows from Eqs (5) and (11) that:

$$-\frac{d_p^2}{18\nu_f C_f}\frac{\rho_s-\rho_f}{\rho_f}|g|{\Phi_f}^{2+\beta}\Phi_s - \epsilon_{yy}\frac{\mathrm{d}\Phi_s}{\mathrm{d}y} = V_s\Phi_s \quad (12)$$

This is the most generic form for the steady solid phase distribution in the diffusive regime.

4 Additional closures

4.1 *Drag force in fluid–multiparticle systems*

Di Felice (1994) obtained the exponent, β, as introduced in Eq. (5), from hindered-settling experiments. Eq. (12) shows that for particle setting in the absence of turbulent diffusion ($\epsilon_{yy} = 0$):

$$\frac{C_f\,(\mathsf{R}_\mathsf{p}^\infty)}{C_f\,(\mathsf{R}_\mathsf{p}^\infty{\Phi_f}^{n-1})} = {\Phi_f}^{n-2-\beta} \quad (13)$$

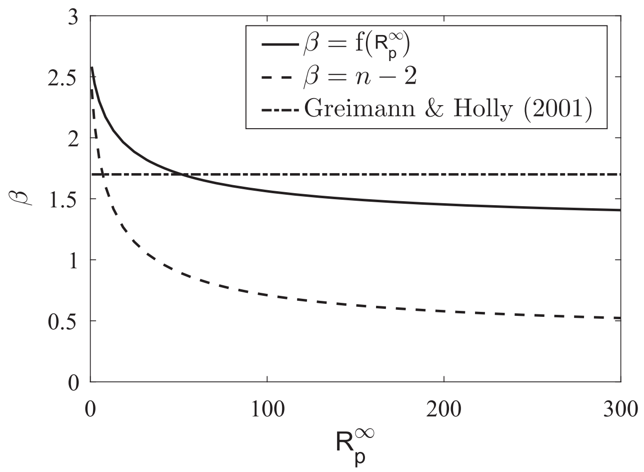

Figure 2 Exponent β as a function of the particle Reynolds number $\mathsf{R}_\mathsf{p}^\infty$

with $n(\mathsf{R}_\mathsf{p}^\infty)$. Using the approximation (Rowe, 1987):

$$n(\mathsf{R}_\mathsf{p}^\infty) = 2.35\,\frac{2+0.175(\mathsf{R}_\mathsf{p}^\infty)^{3/4}}{1+0.175(\mathsf{R}_\mathsf{p}^\infty)^{3/4}} \quad \text{for} \quad 0 < \mathsf{R}_\mathsf{p}^\infty < 10^4 \quad (14)$$

it follows for particle settling in the Stokes regime ($\mathsf{R}_\mathsf{p}^\infty \ll 1$), that $\beta = n - 2$. Figure 2 shows an optimal estimation for β as a function of $\mathsf{R}_\mathsf{p}^\infty$.

4.2 *Fluid and solid velocity fluctuations*

The particle Reynolds number, defined in Section 2, includes the velocity fluctuation of each phase. Issa and Oliviera (1997) provided appropriate estimates for the coupling: $v'_{si} = C_t v'_{fi}$. The original derivation is valid for dilute flow and particles that are smaller than a typical eddy size. For non-dilute flow, their model needs to be corrected for the mixture acceleration, yielding:

$$C_t = \frac{\frac{\rho_f}{\rho_s}(\Phi_f + C_A) + \frac{\tau_e}{\tau_p}}{(\Phi_f + C_A\frac{\rho_f}{\rho_s}) + \frac{\tau_e}{\tau_p}} \quad (15)$$

which becomes identical to the original model of Issa and Oliviera (1997) for dilute flow ($\Phi_f \approx 1$). This study estimates the eddy turn overtime, τ_e, with the ratio between the turbulent diffusivity $\epsilon_{yy} = u_*\kappa(1-y/h)$ and the velocity fluctuations of the fluid, where h denotes the channel height. The velocity fluctuations of the fluid relate to the friction velocity u_* and the measured von Kármán constant κ (Greimann & Holly, 2001).

5 Comparison with experiments

This note concerns a comparison with the experiment S11–S16 of Einstein and Chien (1955) and SF1–SF6 of Wang and Qian (1992). For these experiments the conditions for Eqs (8) and (9) apply for $0.05 < y/h < 0.95$. Figure 3 shows some transient solutions of Eq. (10) and the steady state solution Eq. (12). The phase fraction Φ_s is fixed at $y/h = 0.05$ and $\Phi_s V_s = 0$ at $y = h$.

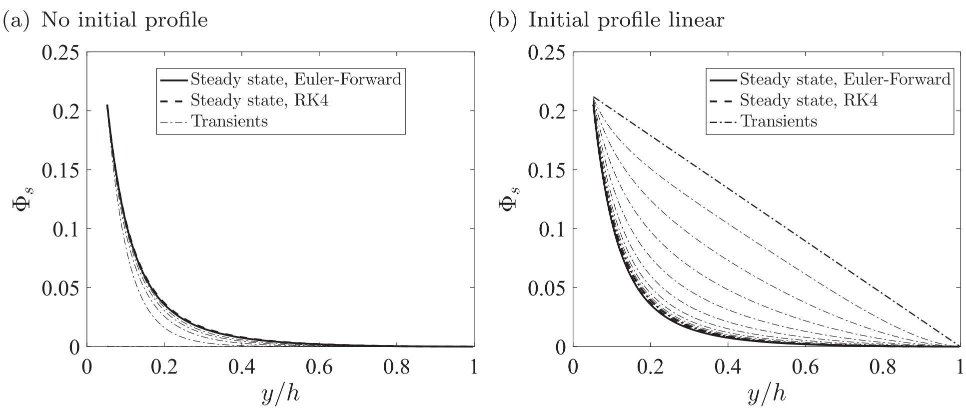

Figure 3 Steady state solutions of Eq. (12) obtained with two integration methods and transient solutions (interval 1 s) of Eqs (10), (2) with $C_A = 0$, $C_f = C_f^\infty$, $\beta = n - 2$, time step is 10^{-4} and number of grid cells is 200. Experimental conditions S16

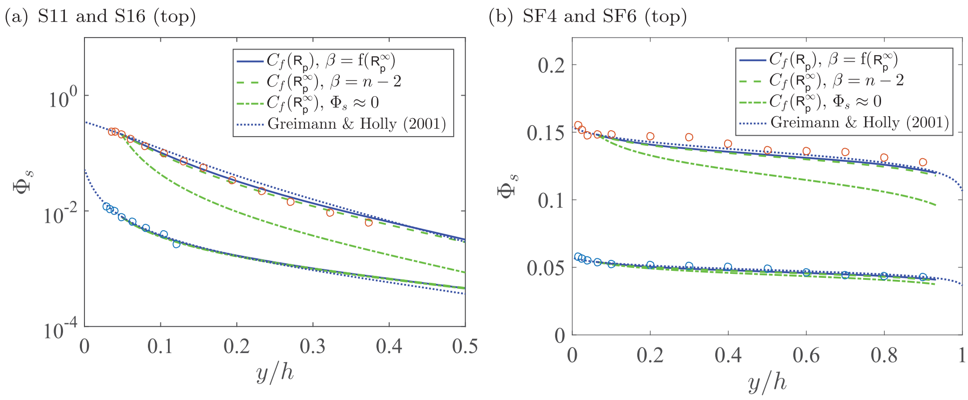

Figure 4 Comparison of different approximations in Eq. (12) and experimental data (Einstein & Chien, 1955; Wang and Qian, 1992))

The transient solution converges towards the steady state profiles. Figure 4 shows the comparison of several solutions of Eq (12) with different approximations of C_f and β. The cases $C_f\,(\mathsf{R}_p)$, $\beta = \mathbf{f}(\mathsf{R}_\mathsf{p}^\infty)$ and $C_f\,(\mathsf{R}_\mathsf{p}^\infty)$, $\beta = n-2$ yield very similar and reasonably accurate results. The approximation $C_f\,(\mathsf{R}_\mathsf{p}^\infty)$, $\Phi_s \approx 0$, which corresponds with $m = 0$ in Eq. (1) shows significant deviations from the other cases at higher concentrations. The same trends are observed in comparison to experiments S12, S13, S14, S15, SF1, SF2, SF3 and SF5 (not shown).

6 Summary and conclusion

The results can be organized using the following list of assumptions:

(i) Neglect inertial terms in the vertical momentum balance of the two phases, $\overline{v_s'^2}\tau_p/\epsilon_{yy} \ll 1$
(ii) Assume that the friction factor in two-dimensional steady-uniform flow equals the friction factor of a single settling particle in still and clear water, $C_f\,(\mathsf{R}_\mathsf{p}) \approx C_f\,(\mathsf{R}_\mathsf{p}^\infty)$
(iii) Ignore the particle crowding effect, $\beta = 0$, assume $\beta = n-2$, i.e. the value that is only valid in the Stokes regime ($C_f = 1$) or use $\beta(\mathsf{R}_\mathsf{p}^\infty)$; see Fig. 2
(iv) Assume that the water fraction is almost unity (dilute suspension).

Formulations (c) and (d), in Table 1 yield very similar and reasonably accurate predictions. The traditional form, $m = 1$, derived by Halbronn (1949) and Hunt (1954), is not found. This note verifies that the asymptotic states of the transient two-phase model correspond to solutions of Eq. (1). This remarkable observation explains why the corresponding concentration profiles are robust, and can actually develop under experimental conditions.

Notation

C	=	volumetric concentration (–)
V_p^∞	=	terminal settling velocity of a single particle in still clear water (> 0) (m s^{-1})

Table 1 Relation between the key assumption (i)–(iv) and different advection-diffusion models

Formulation	(i)	(ii)	(iii)	(iv)
(a) Eq. (1), $m = 0$	yes	yes	$\beta = 0$	yes
(b) Eq. (1), $m = 2$	yes	yes	$\beta = 0$	no
(c) Eq. (1), $m = n$	yes	yes	$\beta = n-2$	no
(d) Eq. (12)	yes	no	$\beta = \mathrm{f}(\mathsf{R}_\mathsf{p}^\infty)$	no

P_f	=	ensemble averaged fluid pressure (kg m^{-1} s^{-2})
d_p	=	particle diameter (m)
x,y,z	=	Cartesian coordinates (m)
g, g_x, g_y	=	gravity acceleration (m s^{-2})
h	=	channel height (m)
m	=	exponent in advection-diffusion equation (–)
n	=	hindered-settling exponent (–)
t	=	time (s)
$\overline{v_s'^2}$	=	ensemble averaged turbulent shear stress (kg m^{-2} s^{-1})
u_*	=	friction velocity (m s^{-1})
R_p	=	particle Reynolds number (–)
C_A	=	added mass coefficient (–)
C_f	=	friction coefficient of a single particle (–)
C_t	=	coupling factor between solid and fluid fluctuations (–)
V_f , V_s	=	phase weighted ensemble averaged vertical velocity of the fluid and solid phase (m s^{-1})
M_{sy}	=	coupling force between the solid and fluid phase (kg m^{-2} s^{-2})
T_{syy}	=	intraparticle or viscous shear stress (kg m^{-1} s^{-2})
V_r, V_d, V_l	=	relative, drift and lag velocity (m s^{-1})
Φ_f , Φ_s	=	fluid and solid phase fraction (–)
β	=	drag model exponent for fluid-multiparticle systems (–)
ϵ	=	turbulent diffusion coefficient (m^2 s^{-1})
ρ_f , ρ_s	=	density of the fluid and solid phase (kg m^{-3})
ν_f	=	kinematic viscosity of the fluid phase (m^2 s^{-1})
τ_p	=	particle response time (s)
τ_e	=	eddy turnover time (s)

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