2009

Analytical And Numerical Solutions Of Differentialequations Arising In Fluid Flow And Heat Transfer Problems

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STARS Citation

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ANALYTICAL AND NUMERICAL SOLUTIONS OF DIFFERENTIAL EQUATIONS ARISING IN FLUID FLOW AND HEAT TRANSFER PROBLEMS

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M.S., Minnesota State University Mankato, 2003

A dissertation submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy
in the Department of Mathematics
in the College of Sciences
at the University of Central Florida
Orlando, Florida

Fall Term, 2009

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ABSTRACT

The solutions of nonlinear ordinary or partial differential equations are important in the study of fluid flow and heat transfer. In this thesis we apply the Homotopy Analysis Method (HAM) and obtain solutions for several fluid flow and heat transfer problems. In chapter 1, a brief introduction to the history of homotopies and embeddings, along with some examples, are given. The application of homotopies and an introduction to the solutions procedure of differential equations (used in the thesis) are provided. In the chapters that follow, we apply HAM to a variety of problems to highlight its use and versatility in solving a range of nonlinear problems arising in fluid flow. In chapter 2, a viscous fluid flow problem is considered to illustrate the application of HAM. In chapter 3, we explore the solution of a non-Newtonian fluid flow and provide a proof for the existence of solutions. In addition, chapter 3 sheds light on the versatility and the ease of the application of the Homotopy Analysis Method, and its capability in handling non-linearity (of rational powers). In chapter 4, we apply HAM to the case in which the fluid is flowing along stretching surfaces by taking into the effects of "slip" and suction or injection at the surface. In chapter 5 we apply HAM to a Magneto-hydrodynamic fluid (MHD) flow in two dimensions. Here we allow for the fluid to flow between two plates which are allowed to move together or apart. Also, by considering the effects of suction or injection at the surface, we investigate the effects of

changes in the fluid density on the velocity field. Furthermore, the effect of the magnetic field is considered. Chapter 6 deals with MHD fluid flow over a sphere. This problem gave us the first opportunity to apply HAM to a coupled system of nonlinear differential equations. In chapter 7, we study the fluid flow between two infinite stretching disks. Here we solve a fourth order nonlinear ordinary differential equation. In chapter 8, we apply HAM to a nonlinear system of coupled partial differential equations known as the Drinfeld Sokolov equations and bring out the effects of the physical parameters on the traveling wave solutions. Finally, in chapter 9, we present prospects for future work.

ACKNOWLEDGMENTS

I would like to express my sincere appreciation to my advisor Dr. Kuppalapalle Vajravelu for his belief in me over the years and for his patience and guidance. I would also like to thank the other members of the committee, Dr. Ram Mohapatra, Dr. David Rollins and Dr. Alain Kassab for their contributions and encouragement through the dissertation work. I would especially like to thank Robert Van Gorder. Without Mr. Van Gorder I would most certainly not be in the position I am today.

I send my love and appreciation to my family whose support has been invaluable. I would especially like to thank my daughter Natasha for keeping me grounded and focused throughout the past few years.

TABLE OF CONTENTS

LIST OF FIGURES

xi

xiii

LIST OF TABLES

CHAPTER 1. INTRODUCTION

1.1 Definition

In topology, two continuous functions from one topological space to another are called homotopic if one can be continuously deformed into the other. Such a deformation is called a homotopy between the two functions. That is,

$$H(t,q) = qf(t) + (1-q)f(t) \tag{1.1.1}$$

would be a homotopy between the functions $f(t)$ and $g(t)$. The parameter q is called the homotopy parameter and we can see that if $q = 0$, $H(t,0) = f(t)$ and if $q = 1$, $H(t,1) = g(t)$. Thus, as q varies from 0 to 1, $f(t)$ is continuously transformed into $f(t)$.

1.2 History

In 1992, Shijun Liao introduced a method of approximating the solutions to non-linear differential equations by using a homotopy. The method is called the Homotopy Analysis Method (HAM) and can be found in detail in Liao [28]. In the years following, other techniques have been developed based on his idea. The common thread behind these techniques is the use of a homotopy to solve differential equations.

It should be mentioned that the use of homotopies in the solution process of many different types of problems was discussed in detail in E. Wasserstrom [12]. The method used by Wasserstrom was essentially the same as that given here except that an embedding, instead of a homotopy, was used. In other words, the embedding we use is given explicitly as a homotopy while the embedding used by Wasserstrom was not restricted in this way and could be changed and altered based on the type of problem needing to be solved. A simple example of the use of a homotopy to find the roots of a polynomial is given here.

Consider the application of finding the roots of a given real polynomial. The procedure is taken from Wasserstrom [12]. Here we use the simple example of finding the roots of

$$f_1(x) = 2x^2 - 7x - 1. \tag{1.2.1}$$

The method is outlined as follows.

1) Choose some initial guess polynomial of the same degree for which we know the solutions. Here we choose as our initial guess the polynomial

$$f_0(x) = x^2 - (1-i)x - i. \tag{1.2.2}$$

The solutions of which are known to be $x_0^1 = -i$ and $x_0^2 = 1$.

2) We then construct a homotopy such that at $q = 0$ we recover f_0 and at $q = 1$ we recover f_1. Here we use the following homotopy

$$H(x,q) = qf_1(x) + (1-q)f_0(x) = (1+q)x^2 - [(1-i) + q(6+i)]\, x - q(1-i) - i. \tag{1.2.3}$$

3) The next step in the process is to define the path from the solutions of the initial guess to the solutions of the given polynomial. In this case we employed a 4th order Runge-Kutta as follows:

Calculate dx/dq as:

$$\frac{dH}{dq}\frac{dq}{dq} + \frac{dH}{dx}\frac{dx}{dq} = \left[(-1+i) - (6+i)x + x^2\right] \tag{1.2.4}$$

$$+ \left[(1-q)((-1+i)+2x) + q(-7+4x)\right]\frac{dx}{dq}$$

Then

$$\frac{dH}{dq}\frac{dq}{dq} + \frac{dH}{dx}\frac{dx}{dq} = 0 \tag{1.2.5}$$

gives

$$\frac{dx}{dq} = \frac{(1-i) + (6+i)x - x^2}{(1-q)((-1+i)+2x) + q(-7+4x)}. \tag{1.2.6}$$

Using the Runge-Kutta scheme we get with a step size of 0.001 and $x_0^1 = -i$ a solution of $-0.137459 + 5.38449 \times 10^{-13}i$. With a step size of 0.01 and $x_0^2 = 1$, we get a solution of $3.63746 - 3.52729 \times 10^{-9}i$. Of course the exact solutions given by the quadratic formula are $\frac{7-\sqrt{57}}{4} \approx -0.1374586088$ and $\frac{7+\sqrt{57}}{4} \approx 3.637458609$.

The extension of this procedure is to any application in which we must solve polynomials or systems of polynomials. One such instance is that of solving eigenvalue problems $Az - \lambda z = 0$ where the matrix A can be either symmetric or non-symmetric. The method used to solve this type of problem is known as the Homotopy Continuation Method and is developed in Li, Zeng, and Cong [35]. The basic idea is to set up the following Homotopy, H : $\mathbf{C^n} \times \mathbf{C} \times [\mathbf{0, 1}] \rightarrow \mathbf{C^{n+1}}$, with the matrix A in upper Hessenberg form, given as

$$H(z, \lambda, t) = (1 - t) \begin{pmatrix} Dz - \lambda z \\ \frac{1}{2}(c_1 z_1^2 + \ldots + c_n z_n^2 - 1) \end{pmatrix} + t \begin{pmatrix} Az - \lambda z \\ \frac{1}{2}(c_1 z_1^2 + \ldots + c_n z_n^2 - 1) \end{pmatrix} \quad (1.2.7)$$

$$= \begin{pmatrix} A(t)z - \lambda z \\ \frac{1}{2}(c_1 z_1^2 + \ldots + c_n z_n^2 - 1) \end{pmatrix}$$

The claim is that if $(x_0, \lambda_0, t_0) \in H^{-1}(0) \in \mathbf{R}$, and (x_0, λ_0) is a real eigenpair of $A(t_0)$, then one can follow the path from the solution of the guess D to the the solution of A by a prediction-correction curve-following scheme via H.

In addition, to the above mentioned applications, homotopies can also be used to determine the number of solutions to a system of polynomial equations as given in Garcia and Li [5]. All applications take advantage of the property $H(x, 0) = $ *Initial Guess* and $H(x, 1) = $ *Solution of Original Problem* . The difference between the multiple applications is the choice in the path from the initial guess to solution of the original problem.

1.3 Application to Differential Equations

As given in Liao [28] for any nonlinear differential equation, say $N[f(x)] = 0$ where N represents a nonlinear differential operator, we can assume that there exists a Linear Operator L such that the solution to $L[f_0(x)] = 0$ is known. Here the Linear Operator is chosen such that the solution is known and satisfies the initial and boundary conditions given in the

original problem. The Homotopy is then defined as

$$H(\phi(x, q); q) = (1 - q)L[\phi(x, q) - f_0(x)] + qN[\phi(x, q)]. \tag{1.3.1}$$

Here, again, we use q as the homotopy parameter. We see that $q = 0$ gives $H(\phi(x, 0); 0) = L[\phi(x, 0) - f_0(x)] = 0$ which in turn gives that $\phi(x, 0) = f_0(x)$ and if $q = 1$, $H(\phi(x, 1); 1) = N[\phi(x; 1)] = 0$ gives that $\phi(x, 1)$ which is the solution to $N[\phi(x, 1)]$.

At this point Liao introduces what he calls the auxillary parameter $\hbar$ and the auxillary function $\mathcal{H}(x)$ into the homotopy so that we have:

$$H(\phi(x, q); q) = (1 - q)L[\phi(x, q) - f_0(x)] + q\hbar\mathcal{H}(x)N[\phi(x, q)]. \tag{1.3.2}$$

Setting $H = 0$ is what allows us to find the solution and is what Liao refers to as the Generalized Homotopy. The use of the auxillary parameter $\hbar$ is central to the method in that it allows us to manipulate the convergence of the solution. Liao uses the auxillary function $\mathcal{H}(x)$ to ensure that the homotopy results in solvable equations during the solution process. The path from the initial guess to the solution is defined as follows:

1) Assume a solution of the form

$$\Phi(x, 1) = f_0(x) + \sum_{m=1}^{\infty} \frac{1}{m!} \frac{\partial^m \Phi(x, q)}{\partial q^m}\bigg|_{q=0} (q^m) \tag{1.3.3}$$

In the above Liao defines

$$v_m(x) = \frac{\partial^m \Phi(x, q)}{\partial q^m}\bigg|_{q=0} \left(\frac{1}{m!}\right). \tag{1.3.4}$$

5

In essence, the solution is assumed to be the Taylor series expansion about the embedding parameter q.

2) To find an individual term in the series solution, say $v_m(x)$, we must differentiate the generalized homotopy m times with respect to q, set q to zero, and then divide by $m!$. This leads to the following which Liao refers to as the mth order deformation equation.

$$L[v_m(x) - \chi_m v_{m-1}(x)] = \hbar H(x) R_m(\overline{v}_{m-1}) : \quad \text{subject to} \quad v_m(0) = 0 \tag{1.3.5}$$

$$\text{Where:} \quad R_m(\overline{v}_{m-1}) = \frac{1}{(m-1)!} \left. \frac{\partial^{m-1} N[\Phi(x,q)]}{\partial q^{m-1}} \right|_q = 0 \tag{1.3.6}$$

$$\chi_m = \begin{cases} 0 \text{ when } m \leq 1 \\ 1 \text{ otherwise} \end{cases} \tag{1.3.7}$$

$$\overline{v}_n = \{v_0(x), v_1(x), \ldots, v_n(t)\} \tag{1.3.8}$$

$v_m(x)$ is then the solution to (1).

3) The choice of $\hbar$ that ensures the convergence of the solution $\Phi(x,1)$ is determined by taking the lth derivative at $x = 0$ for appropriate l. Liao shows that the plot of this for well chosen l will become essentially horizontal on some interval and this interval contains the value of $\hbar$ that provides the convergent solution.

1.4 Homotopy Perturbation Method: HPM

He [16] also uses a homotopy to solve nonlinear differential equations. The main difference between HAM and HPM is in the definition of the nonlinear operator N. He considers any

6

equation of the form

$$A(u) + f(\mathbf{r}) = 0, \mathbf{r} \in \Omega \qquad (1.4.1)$$

with boundary conditions

$$B(u, \partial u/\partial n) = 0, \mathbf{r} \in \Gamma. \qquad (1.4.2)$$

He assumes that A is a nonlinear differential operator that can be split as $L(u) + N(u)$ where L is linear and N is nonlinear.

He then introduces the homotopy

$$H(v, p) = (1 - p)[L(v) - L(u_0)] + p[A(v) - f(\mathbf{r})] = 0 \qquad (1.4.3)$$

$$\Rightarrow H(v, p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(\mathbf{r})] = 0 \qquad (1.4.4)$$

where $p \in [0, 1]$ is the embedding parameter and u_0 is an initial approximation of the original equation. He then assumes a solution of the form $v = u_0 + pv_1 + p^2 v_2 + \ldots$. Substituting into $H(v, p) = 0$ and equating like powers of p results in equations that can be potentially solved for the terms of the assumed perturbation solution. Like HAM, the terms v_k are the terms in the Taylor Series expansion about the parameter p.

The main difference between Liao's method and the method used by He [16] is in the choice of the non-linear operator N. The non-linear operator used by Liao may or may not be defined by the original differential equation while the non-linear operator used by He [16] may simply be the non-linear portion of the original equation. Furthermore, He [16] does not use the auxillary parameter $\hbar$. It should also be noted that in Liao [29], HPM is shown to be a special case of HAM. In addition, HAM uses the auxillary function $\mathcal{H}(x)$ to ensure

that solutions to the mth order deformations are easily solvable. HPM does not provide this assurance. In fact, in He [16], the Variational Iteration Method as found in Wang, He [31], was utilized to solve for certain terms.

1.5 Homotopy Analysis Method: Simplified

The method we utilize essentially takes advantage of the best characteristics of HAM and HPM and removes the restrictions. Unlike both methods we do not assume that the solution can be written as a Taylor Series expansion about the homotopy parameter q. Furthermore, we do not restrict ourselves in the definition of the nonlinear operator N. The solution process is as follows:

1) Choose some initial linear differential equation for which the solution is known and satisfies the given boundary conditions. This known solution is referred to as the initial guess.

2) Set up the generalized homotopy as given by HAM where the nonlinear differential operator is assumed to be the one defined by the original problem.

3) Because the generalized homotopy contains the embedding/homotopy parameter q which can take on values between 0 and 1, we assume a straight forward perturbation solution about q. That is, we assume a solution of the form

$$G(x;q) = \sum_{k=0}^{\infty} g_k(x)q^k. \tag{1.5.1}$$

Upon substituting into the generalized homotopy we simply equate like powers of q and solve the resulting differential equation. The $O(1)$ equation is held subject to the original conditions and higher order equations are held subject to homogeneous boundary conditions.

8

It is important here that the choice of the linear operator provides the initial guess as a solution to the $O(1)$ equation. In this way we ensure that our solution is a perturbation about the initial guess. The kth term of the series solution is the solution of the $O(k)$ equation and the solution to the original problem is $G(x; 1)$.

4) Once the series $G(x; 1)$ is obtained we find a suitable value of the auxillary parameter $\hbar$ by solving

$$G_m^{(l)}(x_0; 1) - G_n^{(l)}(x_0; 1) = 0 \tag{1.5.2}$$

for $\hbar$. Here, with $m < n$, $G_m^{(l)}(x_0; 1)$ is the lth derivative of the k term solutions, x_0 is some value within the solution domain and l is chosen such that the above produces values of $\hbar$. In general, this produces many values for $\hbar$. To choose between them we use two criteria. The first is that the $(l+1)^{st}$ derivative of the full solution evaluated at x_0 is well-behaved and the second is the residual generated by the solution at a particular value of $\hbar$ is "small". It should be noted that by residual we mean the result after substituting the proposed solution into the original differential equation. If the proposed solution is indeed the solution, evaluation will reveal that the result, after substitution, is essentially zero. The importance of this is that we do not require knowledge of an exact solution to determine the validity of the approximation.

One important note to make here is the use of "Simplified" in the section title. It is clear that the solution given by the traditional HAM could be of the above defined form and there would then not be a difference. The intent here is to illustrate the effectiveness of this simplified form of HAM in the solution of ODE's and PDE's arising in Fluid Flow and Heat Transfer.

CHAPTER 2. VISCOUS FLOW OVER A NONLINEARLY STRETCHING SHEET; INTRODUCTORY PROBLEM

2.1 Statement of the Problem

Here we study the problem given in Vajravelu [17] which considers the flow of a viscous fluid over a stretching sheet. This phenomena has important industrial applications, for example, in the extrusion of a polymer sheet from a dye or in the drawing of plastic films. There has been extensive studies of this situation in which the stretching of the sheet is assumed to be linear. The physical situation investigated here considers the case in which the sheet is allowed to stretch nonlinearly. Hence, we investigate the flow and heat transfer phenomenon over a nonlinearly stretching sheet.

Here we consider the flow of a viscous fluid adjacent to a wall coinciding with the plane $y = 0$, the flow being confined to $y > 0$. Two equal and opposite forces are introduced along the x-axis so that the wall is stretched keeping the origin fixed. The boundary layer equations for the steady flow and heat transfer are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (2.1.1)$$

$$u\frac{\partial u}{\partial x} + u\frac{\partial v}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \tag{2.1.2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{k}{\rho C_p}\frac{\partial^2 T}{\partial y^2}, \tag{2.1.3}$$

where u and v are the velocity components in the x and y directions, respectively, T is the temperature, ν the kinematic viscosity, ρ the density, k the thermal conductivity and C_p the specific heat at constant pressure. The boundary conditions are

$$u = Cx^n, \quad v = 0, \quad T = T_w \text{ at } y = 0 \tag{2.1.4}$$

$$u \to 0, \quad T \to T_\infty \text{ as } y \to \infty. \tag{2.1.5}$$

Upon substitution of the similarity transformations

$$\eta = y\sqrt{\frac{c(n+1)}{2\nu}}x^{(n-1)/2} \tag{2.1.6}$$

$$u = Cx^n f'(\eta) \tag{2.1.7}$$

$$v = -\sqrt{\frac{C\nu(n+1)}{2}}x^{(n-1)/2}\left[f(\eta) + \frac{n-1}{n+1}\eta f'(\eta)\right] \tag{2.1.8}$$

into the governing equations we have that

$$f''' + ff'' - \left(\frac{2n}{n+1}\right)(f')^2 = 0 \tag{2.1.9}$$

and

$$\theta'' + \sigma f\theta' = 0 \tag{2.1.10}$$

11

with the corresponding conditions

$$f' = 1, \quad f = 0 \ \text{ at } \eta = 0, \quad \text{and} \quad \lim_{\eta \to \infty} f'(\eta) = 0 \qquad (2.1.11)$$

and

$$\theta = 1 \ \text{ at } \eta = 0, \theta \to 0 \ \text{ as } \eta \to \infty. \qquad (2.1.12)$$

where $\theta(\eta) = (T - T_\infty) / (T_w - T_\infty)$ and $\sigma = \mu C_p / k$ is the Prandtl number.

Here we restrict our investigation to the Viscous Flow problem given by 2.1.9 and 2.1.11. Note that for $n = -\frac{1}{3}$, an exact solution to this is $f(\eta) = \sqrt{2} \tanh \frac{\eta}{\sqrt{2}}$. We compare the results given here to the numerical results given in Vajravelu [17] (For heat transfer results, see Afzal and Varshney [21].). It should be noted that the existence of a solution to 2.1.9 and 2.1.11 is addressed in Vajravelu and Cannon [18] in which they considered the same problem subject to $f'(R) = 0$ where R is a fixed positive number. In the following sections we illustrate the application of the solution method as well as provide an example of its versatility by providing two unique Auxillary Linear Operators that provide good approximations to the solution in a small number of iterations.

2.2 Analysis

The first step in the solution process is to choose a proper initial guess. For this problem, as in Liao and Pop [30], we used $g_0(\eta) = 1 - e^{-\eta}$. In order to find the linear operator we had to solve the third order homogeneous differential equation $u'''(\eta) + au''(\eta) + bu'(\eta) + cu(\eta) = 0$ under the assumption that one solution is similar to the initial guess. Here we assumed solutions to be $u_1(\eta) = e^{-\eta}$, $u_2(\eta) = e^{\eta}$, and $u_3(\eta) = \text{constant}$. Under these assumptions, we

12

found that $a = 0$, $b = -1$, and $c = 0$ and thus the linear operator is

$$L_1 = \frac{\partial^3}{\partial \eta^3} - \frac{\partial}{\partial \eta}. \tag{2.2.1}$$

In addition, the non-linear operator is

$$N\left[G(\eta : q)\right] = \frac{\partial^3 G(\eta; q)}{\partial \eta^3} - G(\eta; q)\frac{\partial^2 G(\eta; q)}{\partial \eta^2} - \frac{2n}{n+1}\left(\frac{\partial G(\eta; q)}{\partial \eta}\right)^2. \tag{2.2.2}$$

Putting these expressions into the generalized homotopy

$$(1 - q)\,L\left[G(\eta; q) - g_0(\eta)\right] = q\hbar N\left[G(\eta; q)\right], \tag{2.2.3}$$

equating like powers of q, and solving the resulting differential equations gives the first 3 terms of the general solution as seen in Table 2.2.1 (Note: For simplicity we set $b = \frac{2n}{n+1}$.). For $n = \frac{1}{3}$, or rather $b = -1$, we chose the parameter $\hbar$ to be -0.6329. This choice of the parameter results in the first 5 terms of the solution seen in Table 2.2.2. Again we point out that the zeroth-order differential equation was held subject to the original conditions $f(0) = 0; f'(0) = 1; \lim_{\eta \to \infty} f'(\eta) \to 0$ while the higher-order differential equations were held subject to $f(0) = 0; f'(0) = 0; \lim_{\eta \to \infty} f'(\eta) \to 0$.

As it can be seen, using the above linear operator results in the $O(1)$ solution being the initial guess. Figure 2.1 shows the plot of the exact solution for $n = -\frac{1}{3}$ versus the 20 Term HAM solution with $\hbar$ chosen to be -0.6329. From the figure we see that the obtained series solution gives good results even for a small number of terms. Also, these results agree well with the HAM results of Liao, and Pop [30].

Table 2.2.1: First 3 terms of the HAM solution of 2.1.1 and 2.1.2.

Order 0:	$1 - e^{-\eta}$
Order 1:	$-\frac{\hbar}{6} + \frac{b\hbar}{6} - \frac{1}{6}e^{-2\eta}\hbar + \frac{1}{6}be^{-2\eta}\hbar + \frac{1}{3}e^{-\eta}\hbar - \frac{1}{3}be^{-\eta}\hbar$
Order 2:	$\frac{11\hbar^2}{72} + \frac{7b\hbar^2}{72} + \frac{b^2\hbar^2}{18} - \frac{5}{144}e^{-3\eta}\hbar^2 + \frac{1}{16}be^{-3\eta}\hbar^2 - \frac{1}{36}\hbar^2 e^{-3\eta}b^2 - \frac{1}{9}be^{-2\eta}\hbar^2$ $+\frac{1}{9}b^2 e^{-2\eta}\hbar^2 + \frac{3}{16}e^{-\eta}\hbar^2 - \frac{7}{144}be^{-\eta}\hbar^2 - \frac{5}{36}b^2 e^{-\eta}\hbar^2 + \frac{1}{12}e^{-\eta}\hbar^2\eta - \frac{1}{12}be^{-\eta}\hbar^2\eta$

Table 2.2.2: First few terms in the HAM solution of 2.1.1 and 2.1.2 with $n = 1/3$ and $\hbar = -0.6329$.

Order 0:	$1 - e^{-\eta}$
Order 1:	$\frac{1}{3} + \frac{1}{3}e^{-2\eta} - \frac{2}{3}e^{-\eta}$
Order 2:	$\frac{5}{36} - \frac{1}{8}e^{-3\eta} + \frac{5}{9}e^{-2\eta} - \frac{41}{72}e^{-\eta} + \frac{1}{6}e^{-\eta}\eta$
Order 3:	$\frac{29}{720} + \frac{13}{270}e^{-4\eta} - \frac{31}{96}e^{-3\eta} + \frac{25}{36}e^{-2\eta} - \frac{1987}{4320}e^{-\eta} - \frac{1}{9}e^{-2\eta}\eta + \frac{19}{72}e^{-\eta}\eta$
Order 4:	$-\frac{527}{43200} - \frac{97}{5184}e^{-5\eta} + \frac{2711}{16200}e^{-4\eta} - \frac{623}{1152}e^{-3\eta} + \frac{4697}{6480}e^{-2\eta} - \frac{83069}{259200}e^{-\eta}$ $+\frac{1}{16}e^{-3\eta}\eta - \frac{31}{108}e^{-2\eta}\eta + \frac{1327}{4320}e^{-\eta}\eta - \frac{1}{72}e^{-\eta}\eta^2$

In Vajravelu [17], numerical solutions were given for $n = 1, 5,$ and 10 along with their first derivative. Figure 2.2 contains the plots for the solutions, and Figure 2.3 contains the plots of their derivatives along with the plot of the derivative of the solutions for $n = -\frac{1}{3}, 1, 5,$ and 10. The results obtained here agree very well with the numerical results found in Vajravelu [17]. It should be noted that the solutions for different values of n were obtained using $\hbar = -0.6329$.

2.3 Non-Uniqueness of Linear Operator

One benefit of this technique is that the choice of the linear operator is not unique. We have found that using the linear operator $L_2 = \frac{\partial^3}{\partial \eta^3} + \frac{\partial^2}{\partial \eta^2}$ also gives good results. This linear operator was found by assuming solutions to the third order homogeneous differential equation of $u_1(\eta) = e^{-\eta}$, $u_2(\eta) = \eta$, and $u_3(\eta) = \text{constant}$. We again used $g_0(\eta) = 1 - e^{-\eta}$ as our initial guess. Figure 2.4 contains the plot of the seven term HAM solution for $n = -\frac{1}{3}$ found using L_2 with $\hbar = -0.95$ versus the exact solution. The first five terms in the solution can be seen in Table 2.3.1.

We have found that the choice of the linear operator can provide a faster solution process. It was found that using $L = \frac{\partial^3}{\partial^3} - \frac{\partial}{\partial \eta}$ as opposed to $L = \frac{\partial^3}{\partial \eta^3} + \frac{\partial^2}{\partial \eta^2}$ results in less calculation time and may thus be a more desirable solution process.

The convergence for the 20-Term HAM solution can be seen in Table 2.3.2. For reference, the 20 term solution with $\hbar = -0.6329$ and $b = -1$ evaluated at $t = 5$ is 1.41181 which is the exact value.

15

Table 2.3.1: The first 5 terms of the HAM solution using L_2 with $\hbar = -0.95$.

Order 0:	$1 - e^{-\eta}$
Order 1:	$0.475 + 0.475e^{-2\eta} - 0.95e^{-\eta}$
Order 2:	$0.02375 - 0.225625e^{-3\eta} + 0.92625e^{-2\eta} - 0.724375e^{-\eta} + 0.45125e^{-\eta}\eta$
Order 3:	$-0.105985 + 0.107172e^{-4\eta} - 0.665594e^{-3\eta} + 0.948813e^{-2\eta} - 0.284406e^{-\eta}$ $-0.428687e^{-2\eta}\eta + 0.473813e^{-\eta}\eta$
Order 4:	$-0.0160164 - 0.0509066e^{-5\eta} + 0.423329e^{-4\eta} + 0.96356e^{-3\eta} + 0.542688e^{-2\eta}$ $+0.0644664e^{-\eta}0.30544e^{-\eta}\eta - 0.878809e^{-2\eta}\eta + 0.271314e^{-\eta}\eta - 0.101813e^{-\eta}\eta^2$

Table 2.3.2: Convergence of the HAM solution using L_2 with $\hbar = -0.95$ at $\eta = 5$.

20-Term Approximation to Viscous Flow evaluated at $\eta = 5$ for $\hbar = -0.6329$							
	8 Terms	10 Terms	14 Terms	16 Terms	18 Terms	19 Terms	20 Terms
$b = -1$	1.47321	1.44086	1.39208	1.39049	1.39965	1.40583	1.41181
$b = \frac{2}{3}$	1.03486	1.03423	1.03406	1.03406	1.03406	1.03407	1.03407
$b = \frac{5}{3}$	0.925017	0.925571	0.925899	0.925944	0.925965	0.925971	0.925975
$b = \frac{20}{11}$	0.911629	0.912258	0.912645	0.912701	0.912728	0.912736	0.912741

16

2.4 Choice of $\hbar$

In solving this problem with $b = -1$, using the linear operator of

$$L_1 = \frac{\partial^3}{\partial \eta^3} - \frac{\partial}{\partial \eta} \tag{2.4.1}$$

we found that $\hbar = -0.6329$ offered a satisfactory solution. We were then able to use this value of $\hbar$ to then find solutions to the problem for $n = 1, n = 5$, and, $n = 10$. However, when we used the linear operator

$$L_2 = \frac{\partial^3}{\partial \eta^3} + \frac{\partial^2}{\partial \eta^2} \tag{2.4.2}$$

we found that we needed to adjust the value for $\hbar$. As indicated earlier we used the following scheme. (Note that for illustration we consider the case for $n = 5$.)

We consider the 5-Term solution versus the 7-Term solution found using L_2. We set their second derivatives, evaluated at $t = 0$, to zero and solved the resulting equation for $\hbar$. This resulted in 6 potential values for $\hbar$. We essentially had to choose between $\hbar = -0.51363$ and $\hbar = -0.31962$. Both values resulted in the solutions third derivative evaluated at 0 being well-behaved and so we used the residual plots to make our choice. Figure 2.5 shows the residual plots for both values of the parameter $\hbar$ and it is thus apparent why we chose $\hbar = -0.51363$. Using the same method we chose $\hbar = -0.60945$ for $n = 1$, and $\hbar = -0.54188$ for $n = 10$. Figure 2.6 shows the plots of these solutions and Figure 2.7 shows the plots of the derivatives of these solutions. As can be seen, they are identical to the plots for the

17

alternate solution obtained by using

$$L_1 = \frac{\partial^3}{\partial \eta^3} - \frac{\partial}{\partial \eta}. \tag{2.4.3}$$

2.5 Results

Some results given in Vajravelu [17] are that the second derivative of the solution evaluated at 0 for $n = 5$ is -1.1945 and for $n = 10$ it is -1.2348. Using L_1 the 20-term HAM solution gives $f''(0) = -1.19449$ for $n = 5$ and $f''(0) = -1.23488$ for $n = 10$. Using L_2 the 7-term HAM solution gives $f''(0) = -1.193965$ for $n = 5$ and $f''(0) = -1.2325$ for $n = 10$.

In comparison to the solution attained using HAM as given in Liao, Pop [30] we note that implementing HAM there required the use of two homotopies to solve the problem. One homotopy was used to generate the mth order deformation equations, the other was used to provide the condition of the first derivative at 0 which in turn was used to ensure that unwanted terms in the solution were eliminated. The implementation here was far simpler. Once the generalized homotopy is set up nothing else is required except straight forward perturbation.

2.6 Figures
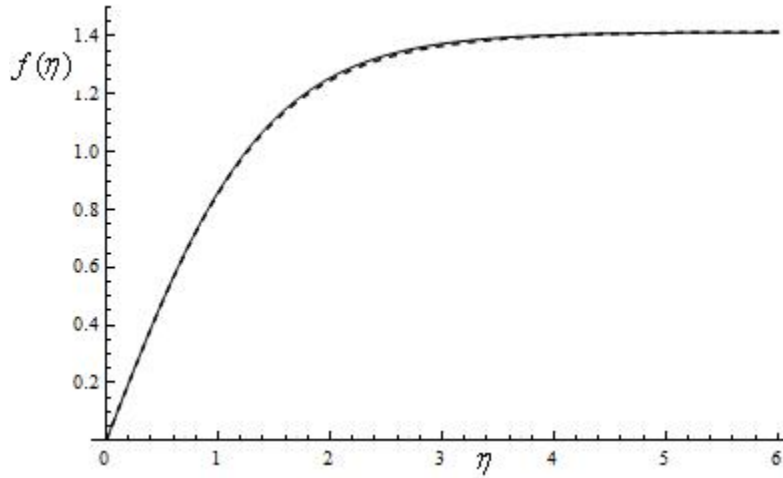
Figure 2.1: The plot of the 20 Term HAM Solution of the Viscous Flow problem vs. the exact solution $f(\eta) = \sqrt{2}\tanh\frac{\eta}{\sqrt{2}}$ for the Viscous Flow problem. The 20 Term HAM Solution is dotted and the exact solution is solid.
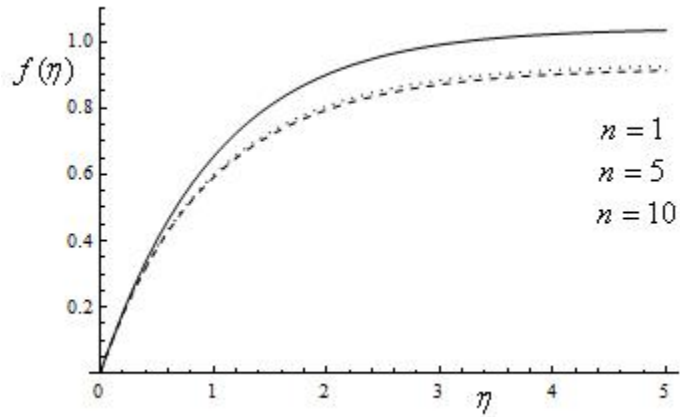
Figure 2.2: The plot of the HAM Solutions using L_1 for various values of n in the Viscous Flow problem. It can be seen that as n increases, $f(\eta)$ decreases.
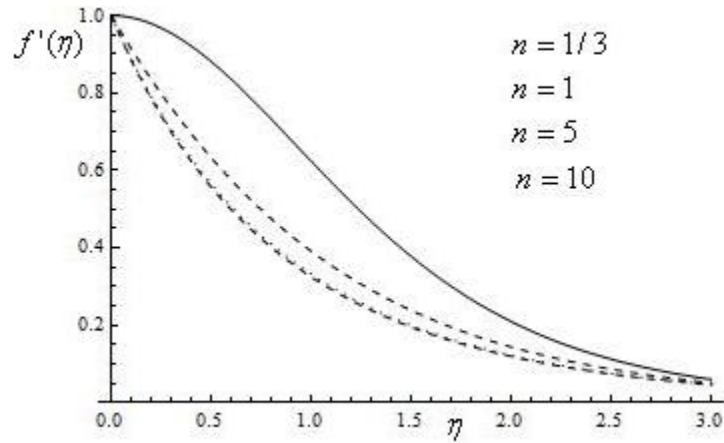
Figure 2.3: The plot of the Derivatives of the HAM Solutions found using L_1 for various values of n in the Viscous Flow problem. It can be seen that as n increases, the derivative of $f(\eta)$ decreases.
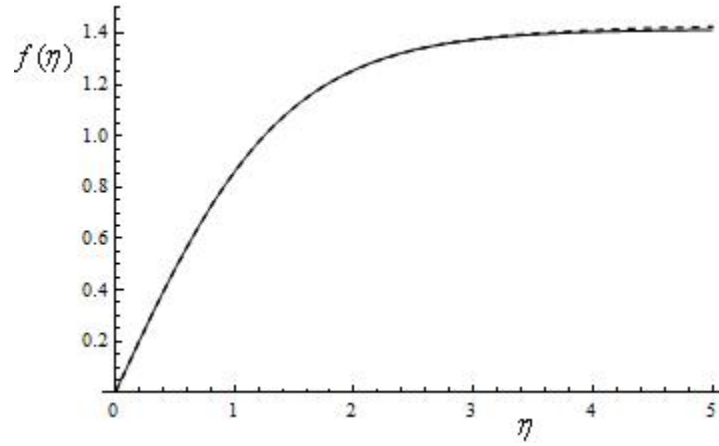
Figure 2.4: The plot of the HAM Solution for the Viscous Flow problem using L_2 versus the exact solution for $n = -\frac{1}{3}$. The HAM solution is dotted and the exact solution is solid.
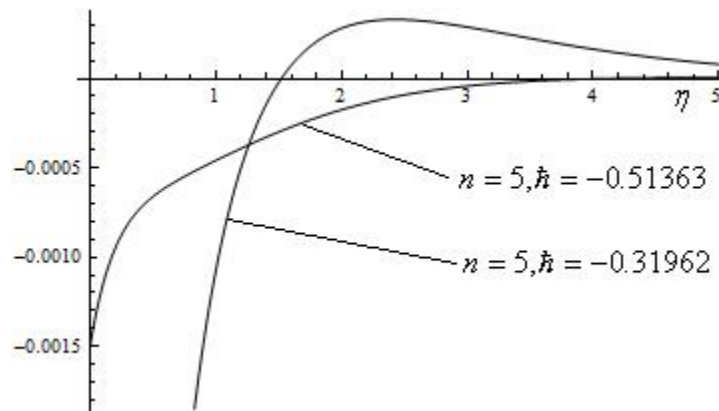
Figure 2.5: The residual plots associated with different values of $\hbar$ for $n = 5$ for the Viscous Flow problem.

21
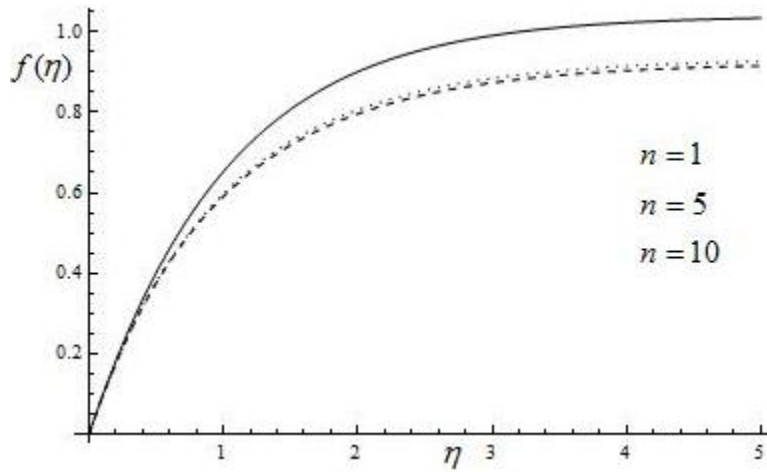
Figure 2.6: The 7-Term HAM solutions for $f(\eta)$ found using L_2 for various values of n for the Viscous Flow problem. It can be seen that as n increases, $f(\eta)$ decreases.
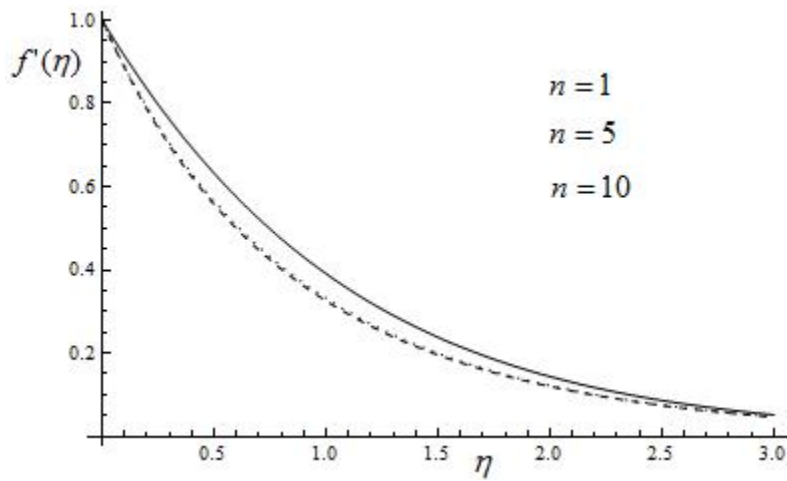
Figure 2.7: The 7-Term HAM solutions for $f'(\eta)$ found using L_2 for various values of n for the Viscous Flow problem. It can be seen that as n increases, $f'(\eta)$ decreases

22

CHAPTER 3. STEADY FLOW OF A SISKO FLUID

3.1 Statement of Problem

As given in Akyilidiz, Vajravelu, Mohapatra, Sweet and Van Gorder [14], the inadequacies of the Navier-Stokes theory to describe rheological complex fluids such as polymer solution, blood, paints, certain oils and greases have led to the development of several theories of non-Newtonian fluids. In this theory, the relation connecting shear stress and shear rate is not usually linear in that the viscosity of a non-Newtonian fluid is not constant at a given temperature and pressure but depends on the rate of shear or on the previous kinematic history of the fluid. Hence, there is no constitutive relation able to predict all non-Newtonian behaviors that can occur. Here we consider the thin film flow of a Sisko fluid on a moving belt with velocity U_0. It is noted that a survey of past work revealed that no previous attempts were made in studying this problem for non-integer values of the power index n. We will illustrate the use of HAM in obtaining approximate analytical solutions for various values of the power index n as well as provide a proof for the existence of solutions using the Implicit Function Theorem.

The thin film flow of a Sisko fluid on a moving belt with velocity U_0 is given by,

$$\frac{d^2v}{dx^2} + \frac{d}{dx}\left(b\left|\frac{dv}{dx}\right|^{n-1}\frac{dv}{dx}\right) - k = 0 \qquad (3.1.1)$$

subject to

$$v(0) = 1 \text{ and } \left.\frac{dv}{dx} + b\left(\left|\frac{dv}{dx}\right|\right)^{n-1}\frac{dv}{dx}\right|_{x=1} = 0. \qquad (3.1.2)$$

This can also be found in Tanner [25]. In what follows it is assumed that the parameter $b > 0$.

In Sajid, Hayat and Asghar [20] the problem is simplified by simply removing the absolute value. That is, the equation becomes

$$v''(x) + nb(v'(x))^{n-1}v''(x) + k_1 = 0 \qquad (3.1.3)$$

subject to

$$v(0) = 1, v'(1) = 0. \qquad (3.1.4)$$

In what follows we consider the above problem for different values of the parameter n under the assumption that the absolute value is removed and that it is retained. However, we first use the Implicit Function Theorem to show the existence of a solution.

24

3.2 Existence

3.2.1 General Implicit Function Theorem

Statement of Theorem

From Fitzpatrick [22] we have:

Let k and n be positive integers, let $\mathbf{O}$ be an open subset of $\mathbb{R}^{n+k}$ and let the mapping $\mathbf{F} : \mathbf{0} \to \mathbb{R}^k$ be continuously differentiable. Consider the equation

$$\mathbf{F}(\mathbf{u}) = \mathbf{0}, \quad \mathbf{u} \in \mathbf{O} \tag{3.2.1}$$

Note: For a point $\mathbf{u} \in \mathbb{R}^{n+k}$, separate the first n components of $\mathbf{u}$ from the last k components as

$$\mathbf{u} = (\mathbf{x}, \mathbf{y}) = (x_1, \ldots, x_n, y_1, \ldots, y_k). \tag{3.2.2}$$

Then, 3.2.1 can be written as

$$\mathbf{F}(\mathbf{x}, \mathbf{y}) = \mathbf{0}, \quad (\mathbf{x}, \mathbf{y}) \in \mathbf{O} \tag{3.2.3}$$

Suppose that $(\mathbf{x_0}, \mathbf{y_0}) \in \mathbf{O}$ such that $\mathbf{F}(\mathbf{x_0}, \mathbf{y_0}) = \mathbf{0}$ and that the $k \times k$ partial derivative matrix

$$\mathbf{D_y F}(\mathbf{x}, \mathbf{y}) \tag{3.2.4}$$

is invertible. Then there is a positive number r and a continuously differentiable mapping

$\mathbf{G} : \mathcal{N} \to \mathbb{R}^k$ where $\mathcal{N} = \mathcal{N}_r(\mathbf{x_0})$ such that

(i) $\mathbf{F}(\mathbf{x}, \mathbf{G}(\mathbf{x})) = \mathbf{0} \;\; \forall \mathbf{x} \in \mathcal{N}$

(ii) whenever $\| \mathbf{x} - \mathbf{x_0} \| < r, \| \mathbf{y} - \mathbf{y_0} \| < r,$ and $\mathbf{F}(\mathbf{x}, \mathbf{y}) = \mathbf{0},$ then $\mathbf{y} = \mathbf{G}(\mathbf{x})$

$$(3.2.5)$$

Proof

Define the auxillary mapping $\mathbf{H} : \mathbf{O} \to \mathbb{R}^{n+k}$ by

$$\mathbf{H}(\mathbf{x}, \mathbf{y}) = (\mathbf{x}, \mathbf{F}(\mathbf{x}, \mathbf{y})) \;\; \text{for} \;\; (\mathbf{x}, \mathbf{y}) \in \mathbf{O} \tag{3.2.6}$$

Make the note that $\mathbf{F}(\mathbf{x}, \mathbf{y}) = \mathbf{0}$ if and only if $\mathbf{H}(\mathbf{x}, \mathbf{y}) = (\mathbf{x}, \mathbf{0})$. Now since $\mathbf{H}$ is a continuously differentiable mapping between Euclidean spaces of the same dimension, the Inverse Function Theorem may be applied to analyze its image and more specifically to analyze the points at which $\mathbf{F}(\mathbf{x}, \mathbf{y}) = \mathbf{0}$.

The Inverse Function Theorem states:

Let $\mathbf{O}$ be an open subset of $\mathbb{R}^n$ and suppose that the mapping $\mathbf{F} : \mathbf{O} \to \mathbb{R}^n$ is continuously differentiable. Let $\mathbf{x_0}$ be a point in $\mathbf{O}$ at which the derivative matrix

$$\mathbf{DF}(\mathbf{x_0}) \tag{3.2.7}$$

is invertible. Then there is a neighborhood U of the point $\mathbf{x_0}$ and a neighborhood V of its image $\mathbf{F}(\mathbf{x_0})$ such that the mapping $\mathbf{F} : U \to V$ is one-to-one and onto. Furthermore, the inverse mapping $\mathbf{F}^{-1} : V \to U$ is also continuously differentiable, and for a point $\mathbf{y} \in V$, if

$\mathbf{x}$ *is the point in U at which* $\mathbf{F}(\mathbf{x}) = \mathbf{y}$, *then*

$$\mathbf{DF}^{-1}(\mathbf{y}) = [\mathbf{DF}(\mathbf{x})]^{-1}. \tag{3.2.8}$$

The derivative matrix $\mathbf{DH}(\mathbf{x_0}, \mathbf{y_0})$ may be partitioned as

$$\mathbf{DH}(\mathbf{x_0}, \mathbf{y_0}) = \begin{bmatrix} I_n & \mathbf{0} \\ \mathbf{D_x F}(\mathbf{x_0}, \mathbf{y_0}) & \mathbf{D_y F}(\mathbf{x_0}, \mathbf{y_0}) \end{bmatrix} \tag{3.2.9}$$

where I_n is the $n \times n$ identity matrix and $\mathbf{0}$ is the $n \times k$ matrix all of whose entries are 0. It is clear that since $\mathbf{D_y F}(\mathbf{x_0}, \mathbf{y_0})$ is invertible, $\mathbf{DH}(\mathbf{x_0}, \mathbf{y_0})$ is also invertible which implies that we can apply the Inverse Function Theorem to $\mathbf{H} : \mathbf{O} \to \mathbb{R}^{n+k}$ at $(\mathbf{x_0}, \mathbf{y_0})$ to conclude that there is a neighborhood U of $(\mathbf{x_0}, \mathbf{y_0})$ in $\mathbb{R}^{n+k}$ and a neighborhood V of its image $\mathbf{H}(\mathbf{x_0}, \mathbf{y_0}) = (\mathbf{x_0}, \mathbf{0})$ in $\mathbb{R}^{n+k}$ such that $\mathbf{H} : \mathrm{U} \to \mathrm{V}$ is one-to-one and onto. Furthermore, its inverse mapping $\mathbf{H}^{-1} : \mathrm{U} \to \mathrm{V}$ is also continuously differentiable.

Defining the inverse mapping as

$$\mathbf{H}^{-1}(\mathbf{x}, \mathbf{y}) = (\mathbf{M}(\mathbf{x}, \mathbf{y}), \mathbf{N}(\mathbf{x}, \mathbf{y})) \quad \forall (\mathbf{x}, \mathbf{y}) \in \mathrm{V} \tag{3.2.10}$$

gives

$$(\mathbf{x}, \mathbf{y}) = \left(\mathbf{H} \circ \mathbf{H}^{-1}\right)(\mathbf{x}, \mathbf{y}) = (\mathbf{M}(\mathbf{x}, \mathbf{y}), \mathbf{F}(\mathbf{M}(\mathbf{x}, \mathbf{y}), \mathbf{N}(\mathbf{x}, \mathbf{y}))) \tag{3.2.11}$$

and

$$(\mathbf{x}, \mathbf{y}) = \left(\mathbf{H}^{-1} \circ \mathbf{H}\right)(\mathbf{x}, \mathbf{y}) = (\mathbf{M}(\mathbf{x}, \mathbf{F}(\mathbf{x}, \mathbf{y})), \mathbf{N}(\mathbf{x}, \mathbf{N}(\mathbf{x}, \mathbf{y}))) \tag{3.2.12}$$

From 3.2.11 we have that $\mathbf{M}(\mathbf{x}, \mathbf{y}) = \mathbf{x}$ and $\mathbf{F}(\mathbf{x}, \mathbf{N}(\mathbf{x}, \mathbf{y})) = \mathbf{y}$.

Now since U and V are open subsets of $\mathbb{R}^{n+k}$ we choose $r > 0$ such that

$$\mathcal{N}_r(\mathbf{x_0}) \times \mathcal{N}_r(\mathbf{y_0}) \subset \mathbf{U} \text{ and } \mathcal{N}_r(\mathbf{x_0}) \times \{\mathbf{0}\} \subset V. \tag{3.2.13}$$

Define $\mathbf{G} : \mathcal{N}_r(\mathbf{x_0}) \to \mathbb{R}^k$ by

$$\mathbf{G}(\mathbf{x}) = \mathbf{N}(\mathbf{x}, \mathbf{0}) \quad \forall \mathbf{x} \in \mathcal{N}_r(\mathbf{x_0}). \tag{3.2.14}$$

Then $\mathbf{G} : \mathcal{N}_r(\mathbf{x_0}) \to \mathbb{R}^k$ is continuously differentiable and from $\mathbf{F}(\mathbf{x}, \mathbf{N}(\mathbf{x}, \mathbf{y})) = \mathbf{y}$, it follows that $\mathbf{F}(\mathbf{x}, \mathbf{G}(\mathbf{x})) = \mathbf{0} \ \forall \mathbf{x} \in \mathcal{N}_r(\mathbf{x_0})$.

To finish we note that if the point $(\mathbf{x}, \mathbf{y})$ belongs to $\mathcal{N}_r(\mathbf{x_0}) \times \mathcal{N}_r(\mathbf{y_0})$ and $\mathbf{F}(\mathbf{x}, \mathbf{y}) = \mathbf{0}$, then, since $\mathcal{N}_r(\mathbf{x_0}) \times \mathcal{N}_r(\mathbf{y_0}) \subset \mathbf{U}$, from the equality of the second components in 3.2.12 we have that

$$\mathbf{y} = \mathbf{N}(\mathbf{x}, \mathbf{F}(\mathbf{x}, \mathbf{y})) = \mathbf{N}(\mathbf{x}, \mathbf{0}) = \mathbf{G}(\mathbf{x}). \qquad \blacksquare \tag{3.2.15}$$

3.2.2 Existence of a local solution to Steady Flow of a Sisko Fluid

To apply the General Implicit Function Theorem we first note the following manipulation of the problem

$$\frac{d^2v}{dx^2} + \frac{d}{dx}\left(b \left| \frac{dv}{dx} \right|^{n-1} \frac{dv}{dx} \right) - k = 0 \tag{3.2.1}$$

subject to

$$v(0) = 1 \ \text{ and } \ \frac{dv}{dx} + b \left(\left| \frac{dv}{dx} \right| \right)^{n-1} \frac{dv}{dx} \bigg|_{x=1} = 0. \tag{3.2.2}$$

28

We note that this is equivalent to writing

$$\frac{d}{dx}\left(\frac{dv}{dx} + b\left(\left|\frac{dv}{dx}\right|\right)^{n-1}\frac{dv}{dx} - kx\right) = 0 \tag{3.2.3}$$

so that the application of the second boundary condition leads to

$$\frac{dv}{dx} + b\left(\left|\frac{dv}{dx}\right|\right)^{n-1}\frac{dv}{dx} - kx + k = 0. \tag{3.2.4}$$

To pose this problem in terms of the General Implicit Function Theorem we let $n = 2$ and $k = 1$, $\mathbf{O}$ be an open subset of $\mathbb{R}^3$ and define the mapping $\mathbf{F} : \mathbf{O} \to \mathbb{R}$ by

$$\mathbf{F}(x_1, x_2, y) = y + b|y|^{n-1}y - kx_1 + k \tag{3.2.5}$$

and consider $\mathbf{F}(x_1, x_2, y) = 0$.

We first note that as $x_1 \in [0, 1]$ we take $(\mathbf{x_0}, y_0) = (x_1, x_2, y_0) = (0.5, x_2^0, y_0)$ which gives that

$$\mathbf{F}(\mathbf{x_0}, y) = y_0 + b|y_0|^{n-1}y_0 + \frac{k}{2}. \tag{3.2.6}$$

We would then like y_0 to be a solution of the single-valued function

$$f(y) = y + b|y|^{n-1}y + \frac{k}{2}. \tag{3.2.7}$$

To verify that a solution exists consider the following cases.

29

Suppose $y \geq 0$

Under the assumption that $y \geq 0$ we have

$$f(y) = by^n + y + \frac{k}{2}. \qquad (3.2.8)$$

As this is a polynomial in y we can apply the Intermediate Value Theorem.

Letting n be odd we see that we have two cases to consider dependent on the sign and magnitude of the parameter k. We first note the value of f at -1 and 1.

$$f(-1) = -b - 1 + \frac{k}{2} \quad \text{and} \quad f(1) = b + 1 + \frac{k}{2}. \qquad (3.2.9)$$

If $k > 0$ then $f(-1) < 0$ if $0.5k < b + 1$ and $f(1) > 0$ and hence by the Intermediate Value Theorem there is a solution to $f(y) = 0$. If $k < 0$, then $f(-1) < 0$ and as long as $|0.5k| < |b + 1|$, $f(1) > 0$ and again there is a solution to $f(y) = 0$.

Letting n be even we again have two cases to consider dependent of the parameter k. Evaluating f at -1 and 0 gives

$$f(-1) = b - 1 + \frac{k}{2} \quad \text{and} \quad f(0) = \frac{k}{2}. \qquad (3.2.10)$$

If $k > 0$, then $f(0) > 0$ and $f(-1) < 0$ provided $0.5k < 1 - b$. If $k < 0$, then $f(0) < 0$ and $f(-1) > 0$ provided $b > 1 - 0.5k$. Thus, regardless of the sign of n, there is a solution to $f(y) = 0$ and we let y_0 be this solution.

30

It should also be noted that under the assumption that $y \geq 0$,

$$\frac{\partial \mathbf{F}(\mathbf{x_0}, y_0)}{\partial y} = 1 + nby_0^n \qquad (3.2.11)$$

which is not 0 and thus the assumptions of the General Implicit Function Theorem are satisfied.

Suppose $y < 0$

Under the assumption that $y < 0$ we have that

$$f(y) = -by^n + y + \frac{k}{2}. \qquad (3.2.12)$$

As this is a polynomial in y we can apply the Intermediate Value Theorem.

Letting n be odd we see that we have two cases to consider dependent on the sign and magnitude of the parameter k. Evaluating f again at -1 and 0 gives

$$f(-1) = b - 1 + \frac{k}{2} \quad \text{and} \quad f(0) = \frac{k}{2}. \qquad (3.2.13)$$

Thus if $k > 0$, $f(0) > 0$ and $f(-1) < 0$ provided $0.5k < 1 - b$. If $k < 0$ then $f(0) < 0$ and $f(-1) > 0$ provided $0.5k > 1 - b$. Moreover, since

$$\frac{\partial \mathbf{F}(\mathbf{x_0}, y_0)}{\partial y} = 1 - nby_0^n \qquad (3.2.14)$$

is not 0, the assumptions of the General Implicit Function Theorem are satisfied.

31

At this point we have that there exists a point in $\mathbb{R}^3$, $(\mathbf{x_0}, y_0)$ such that $\mathbf{F}(\mathbf{x_0}, y_0) = 0$ and $\mathbf{F_y}(\mathbf{x_0}, y_0) \neq 0$ and thus there is a positive number r and a continuously differentiable function $G : \mathcal{N} \to \mathbb{R}$, where $\mathcal{N} = \mathcal{N}_r(\mathbf{x_0})$, such that $\mathbf{F}(\mathbf{x}, \mathbf{G}(\mathbf{x})) = 0 \; \forall \mathbf{x} \in \mathcal{N}$ and $y = \mathbf{G}(\mathbf{x})$ whenever $\| \mathbf{x} - \mathbf{x_0} \| < r$, $\| y - y_0 \| < r$ and $\mathbf{F}(\mathbf{x}, y) = 0$. Or more to the point we have that for some continuously differentiable function G, that

$$v'(x) = G(x, v) \quad \text{and} \quad \mathbf{F}(x, v, G(x, v)) = 0 \tag{3.2.15}$$

Now, as $G(x, v)$ is continuously differentiable we have that $G(x, v)$ satisfies a Lipschitz condition with Lipschitz constant K on $\Omega = \{(x, v) | |x - x_0| < r, |v - v_0| < r\}$ and we choose M to be such that $|G(x, v)| \leq M$ for $(x, v) \in \Omega$. With $0 < \alpha < \min[1/K, r/M, r]$, there exists a unique solution of $v'(x) = G(x, v)$ subject to $v(x_0) = v_0$ on $|x - x_0| < r$. The proof of this statement follows from Waltman [24]

Proof

Take the complete metric space defined by $B = \{\phi | \phi \in C[x_0 - r, x_0 + r], \rho(\phi, v_0) \leq r\}$ where ρ is the metric on this space. Define the mapping T by

$$T[\phi](x) = v_0 + \int_{x_0}^{x} G(s, \phi(s)) ds.$$

Then

$$|(T\phi)(x) - v_0| \leq \left| \int_{x_0}^{x} |G(s, \phi(s)) ds \right| \leq M|x - x_0| \leq Mr \leq r \tag{3.2.16}$$

32

Since $|(T\phi)(x) - v_0|$ is continuous, it takes its maximum on $[x_0 - r, x_0 + r]$, which is then $\leq r$, implying that $\rho(T\phi, v_0) \leq r$, and so $T\phi \in B$ if $\phi \in B$ and hence, T maps B into B. Moreover, as T is a contraction mapping (this follows from the Lipschitz condition of G), it has a unique fixed point. ∎

3.3 Analysis: Analytic Solutions

3.3.1 Absolute Value Removed: Exact Solutions

Note that if we assume the condition $v'(1) = 0$ implies that $v'(x) = k_1 x - k_1$, we can rewrite

$$v''(x) + nb(v'(x))^{n-1}v''(x) + k_1 = 0 \tag{3.3.1}$$

as

$$\frac{d}{dx}\left[v'(x) + b(v'(x))^n - (k_1 x + k_1)\right] = 0. \tag{3.3.2}$$

From this we can find exact solutions for particular values of the parameters b, n, and k_1. We begin by investigating the exact solutions to the problem and then illustrate the Homotopy Analysis Method by solving the same problem and comparing the results.

- $n = 2$

With $n = 2$ we consider the problem with $b = 1$ and $k_1 = -1$. The equation reduces to

$$v'(x) + (v'(x))^2 - (x - 1) = 0 \tag{3.3.3}$$

$$\Rightarrow v'_{1,2}(x) = \frac{-1 \pm \sqrt{1 + 4(x-1)}}{2} \tag{3.3.4}$$

$$v'_{1,2}(x) = \frac{-1 \pm \sqrt{5 - 4x}}{2}. \tag{3.3.5}$$

Integrating and using the condition at 0 provides an exact solution of

$$v(x) = -\frac{1}{2}x - \frac{1}{12}(5 - 4x)^{\frac{3}{2}} + 1 + \frac{5^{\frac{3}{2}}}{12}. \tag{3.3.6}$$

Likewise, assuming that $k_1 = -2$ we get an exact solution of

$$v(x) = -\frac{1}{2} - \frac{1}{24}(9 - 8x)^{\frac{3}{2}} + \frac{17}{8}. \tag{3.3.7}$$

- $n = 3$

With $n = 3$ the equation becomes

$$v''(x) + 3b(v'(x))^2(v''(x)) - k_1 = 0. \tag{3.3.8}$$

If we again assume that the condition $v'(1) = 0$ means that $v'(x) = k_1 x - k_1$ we can rewrite the equation as

$$\left[v'(x) + b(v'(x))^3 - (k_1 x - k_1)\right]' = 0. \tag{3.3.9}$$

We then solve the equivalent equation

$$w^3 + \xi w - \xi(k_1 x - k_1) = 0, \text{ where } \xi = \frac{1}{b}. \tag{3.3.10}$$

34

Assume that w can be written as $\alpha+\beta$. Upon substitution we get that $w^3+\xi w-\xi(k_1x-k_1) = 0$ becomes

$$\alpha^3 + \beta^3 + 3\alpha\beta(\alpha + \beta) + \xi(\alpha + \beta) - \xi(k_1x - k_1) = 0 \tag{3.3.11}$$

$$\Rightarrow \alpha^3 + \beta^3 + (3\alpha\beta + \xi)(\alpha + \beta) - \xi(k_1x - k_1) = 0. \tag{3.3.12}$$

Now if $3\alpha\beta + \xi = 0$ we have

$$(\alpha^3)^2 - \xi(k_1x - k_1)\alpha^3 - \frac{\xi^3}{27} = 0 \tag{3.3.13}$$

$$\Rightarrow \alpha^3 = \frac{\xi(k_1x - k_1) \pm \sqrt{\xi^2(k_1x - k_1)^2 + \frac{4\xi^2}{27}}}{2}. \tag{3.3.14}$$

Therefore we let

$$\alpha(x) = \sqrt[3]{\frac{\xi(k_1x - k_1) + \sqrt{\xi^2(k_1x - k_1)^2 + \frac{4\xi^2}{27}}}{2}} \tag{3.3.15}$$

and

$$\beta(x) = \gamma\sqrt[3]{\frac{\xi(k_1x - k_1) - \sqrt{\xi^2(k_1x - k_1)^2 + \frac{4\xi^2}{27}}}{2}} \tag{3.3.16}$$

where γ is the cubed-root of unity that makes $\beta(x)$ real-valued for $x \in [0, 1]$. Therefore $v'(x) = \alpha(x) + \beta(x)$ and so

$$v(x) = \int \alpha(s)ds + \int \beta(s)ds + \eta \tag{3.3.17}$$

where η is chosen so that $v(0) = 1$. With this we found that the exact solutions for $k_1 = 1$ and $b = \frac{1}{4}$ give $v''(0) = 0.649791$ and for $b = \frac{1}{2}$ we get $v''(0) = 0.528689$.

- $n = 4$

Again assuming the condition $v'(1) = 0$ means that $v'(x) = k_1 x - k_1$ we have

$$(v'(x))^4 + \xi(v'(x)) - \xi(k_1 x - k_1) \text{ where } \xi = \frac{1}{b}. \tag{3.3.18}$$

In general we get a solution (using Ferari's method) of the form

$$v'(x) = \frac{\pm_a W \pm \sqrt{-(2y \pm_a \frac{2\beta}{W})}}{2} \tag{3.3.19}$$

where

$$W = \sqrt{2y},$$

$$y = -U + \frac{\xi(k_1 x - k_1)}{3U}, \tag{3.3.20}$$

$$U = \sqrt[3]{R} \quad \text{(choose the real-valued cubed root)}, \tag{3.3.21}$$

$$\text{and } R = -\frac{\xi^2}{16} + \sqrt{\frac{\xi^4}{256} + \frac{\xi^3(k_1 x - k_1)^3}{27}}. \tag{3.3.22}$$

From this we see that if $b = \frac{1}{4} \Rightarrow \xi = 4$ and $k_1 = 1$, $v'(x)$ will be real-valued if R is real which would then imply that

$$1 + \frac{64(x-1)^3}{27} \geq 0 \tag{3.3.23}$$

$$\Rightarrow x \geq 0.25 \tag{3.3.24}$$

Furthermore, to avoid division by zero in W, $x \neq 1$. We thus have an approximate interval for which $v'(x)$ will be real-valued.

Within this interval we found that the solution for $b = \frac{1}{4}$ and $k_1 = 1$ gives $v''(0.8) = 1.00811$ where

$$v'(x) = \frac{W - \sqrt{-(2y + \frac{2\beta}{W})}}{2} \tag{3.3.25}$$

and for $b = \frac{1}{6}$ and $k_1 = 1$, $v''(0.8) = 1.000538$ where

$$v'(x) = \frac{-W + \sqrt{-(2y - \frac{2\beta}{W})}}{2}. \tag{3.3.26}$$

Using the same technique we found that for $b = \frac{1}{2}$ and $k_1 = -1$, $v''(0.8) = -0.984434$ where

$$v'(x) = \frac{-W + \sqrt{-(2y - \frac{2\beta}{W})}}{2}. \tag{3.3.27}$$

3.3.2 Absolute Value Removed: HAM Solution

In applying HAM to this problem we chose the initial guess to be $v_0(t) = 1$ and the linear operator we used was the same as that used in Sajid, Hayat and Asghar [20]. That is, we used $L = \frac{\partial^2}{\partial x^2}$. To choose this linear operator we solved the second order homogeneous differential equation $v''(x) + av'(x) + bv(x) = 0$ under the assumption that $v_1(x) = \text{constant}$ and $v_2(x) = x$. Since the solution must satisfy $v(0) = 1$, we chose the constant to be 1.

We then assume a solution of the form

$$G(x; q) = \sum_{k=0}^{\infty} g_k(x) q^k \tag{3.3.28}$$

37

and with

$$N[G(x;q)] = \frac{\partial^2 G(x;q)}{\partial x^2} + nb \left(\frac{\partial G(x;q)}{\partial x} \right)^{n-1} \frac{\partial^2 G(x;q)}{\partial x^2} - k_1 \qquad (3.3.29)$$

we substituted the assumed solution into the generalized homotopy

$$(1-q) L \left[G(t;q) - g_0(t) \right] = q\hbar N \left[G(t;q) \right]. \qquad (3.3.30)$$

Equating like powers of q and solving the resulting differential equation in which the order-zero equation is held subject to $v(0) = 1$, and $v'(1) = 0$ and the higher-order differential equations are subject to $v(0) = 0$, and $v'(1) = 0$ yields the following:

- $n = 2$

With $b = 1$, the HAM solution for $k_1 = -1$, $f(x)$, yields with $\hbar = -0.075$, $f''(0) = -0.446553$ for which the exact solution, $v(x)$, gives $v''(0) = -0.447314$. For $k_1 = -2$, the HAM solution gives $f''(0) = -0.66688$ with $\hbar = -0.06562$, and the exact solution gives $v''(0) = -0.666667$. Figure 3.1 shows the plots of the HAM solutions versus the exact solutions for $k_1 = -1$ and $k_1 = -2$. Table 3.3.1 shows the first four terms of the solutions and Table 3.3.2 shows their convergence at $x = 0.1$.

- $n = 3$

For $k_1 = 1$, the 60 term HAM solution for $b = \frac{1}{2}$, yields $f''(0) = 0.528686$ (the exact solution gives 0.528689) which was attained by letting $\hbar = -0.2071701$. The 70 term HAM solution for $b = \frac{1}{4}$, yields, $f''(0) = 0.650101$ (the exact solution gives 0.649791) which was attained by letting the $\hbar = -0.213627$. Figure 3.2 shows the plots of the HAM solutions versus the

Table 3.3.1:   First four terms of the HAM solution with $n = 2$ for different values of k_1.

For $k_1 = -1$	For $k_1 = -2$
$v_0(x) = 1$	$v_0(x) = 1$
$v_1(x) = \frac{\hbar t^2}{2} - \hbar t$	$v_1(x) = \hbar t^2 - 2\hbar t$
$v_2(x) = \frac{t^2 \hbar^2}{2} - t\hbar^2 + \frac{t^2 \hbar}{2} - t\hbar$	$v_2(x) = t^2 \hbar^2 - 2t\hbar^2 + t^2 \hbar - 2t\hbar$
$v_3(x) = \frac{t^3 \hbar^3}{3} - \frac{t^2 \hbar^3}{2} + t^2 \hbar^2 - 2t\hbar^2 + \frac{t^2 \hbar}{2} - t\hbar$	$v_3(x) = -2\hbar t - 4\hbar^2 t + 2\hbar^3 t + \hbar t^2$
	$+2\hbar^2 t^2 - 3\hbar^3 t^2 + (4\hbar^3 t^3)/3$

Table 3.3.2:   Convergence of the solutions for $n = 2$ at $x = 0.1$ for different values of k_1.

$n = 2, b = 1, k_1 = -1, \hbar = -0.075$				$n = 2, b = 1, k_1 = -2, \hbar = -0.06562$			
70 terms	74 terms	78 terms	Exact	75 terms	81 terms	86 terms	Exact
1.05956	1.05954	1.05954	1.05954	1.09658	1.09662	1.09664	1.09662

exact solutions for $b = \frac{1}{2}$ and $b = \frac{1}{4}$. The tables below show the convergence for the two solutions evaluated at $x = 0.1$ and the first four terms of the solutions.

For $k_1 = -1$, the 70 term HAM solution for $b = \frac{1}{2}$ yields $f''(0) = -0.528443$ with $\hbar$ chosen to be -0.177583. The exact solution gives $v''(0) = -0.528689$. Figure 3.3 shows the plot of the HAM solution versus the exact solution for $b = \frac{1}{2}$ and $k_1 = -1$. Table 3.3.3 gives the first few terms of the HAM solution with $n = 3$ for $b = 1/4$ and $b = 1/2$ with $k_1 = 1$ and Table 3.3.4 shows their convergence at $x = 0.1$. Table 3.3.5 gives the first few terms of the HAM solution for for $n = 3$, $b = 1/2$ and $k_1 = -1$ and Table 3.3.6 gives their convergence at $x = 0.1$.

Table 3.3.3:   First four terms of the HAM solution with $n = 3$ for $b = 1/4$ and $b = 1/2$ with $k_1 = 1$.

For $b = \frac{1}{4}$	For $b = \frac{1}{2}$
$v_0(x) = 1$ $v_1(x) = \hbar t - \frac{\hbar t^2}{2}$ $v_2(x) = -\frac{1}{2}t^2\hbar^2 + t\hbar^2 - \frac{t^2\hbar}{2} + t\hbar$ $v_3(x) = -\frac{1}{2}t^2\hbar^3 + t\hbar^3 - t^2\hbar^2 + 2t\hbar^2 - \frac{t^2\hbar}{2} + t\hbar$	$v_0(x) = 1$ $v_1(x) = \hbar t - \frac{\hbar t^2}{2}$ $v_2(x) = -\frac{1}{2}t^2\hbar^2 + t\hbar^2 - \frac{t^2\hbar}{2} + t\hbar$ $v_3(x) = \hbar t + 2\hbar^2 t + \hbar^3 t$ $\quad -(\hbar t^2)/2 - \hbar^2 t^2 - (\hbar^3 t^2)/2$

Table 3.3.4:   Convergence for the HAM solutions for $n = 3$ with $k_1 = 1$ at $x = 0.1$.

$n = 3, b = \frac{1}{4}, k_1 = 1, \hbar = -0.213627$				$n = 3, b = \frac{1}{2}, k_1 = 1, \hbar = -0.2071701$			
60 terms	65 terms	70 terms	Exact	55 terms	58 terms	60 terms	Exact
0.91855	0.918532	0.918531	0.918538	0.925472	0.925382	0.92541	0.925611

Table 3.3.5:   First four terms in the HAM solution with $n = 3$ for $b = 1/2$ and $k_1 = -1$.

For $b = \frac{1}{2}$
$v_0(x) = 1$ $v_1(x) = \frac{\hbar t^2}{2} - \hbar t$ $v_2(x) = \frac{t^2\hbar^2}{2} - t\hbar^2 + \frac{t^2\hbar}{2} - t\hbar$ $v_3(x) = \frac{t^2\hbar^3}{2} - t\hbar^3 + t^2\hbar^2 - 2t\hbar^2 + \frac{t^2\hbar}{2} - t\hbar$

Table 3.3.6:   Convergence for the first four terms in the HAM solution with $n = 3$ for $b = 1/2$ and $k_1 = -1$ at $x = 0.1$.

$n = 3, b = \frac{1}{2}, k_1 = -1, \hbar = -0.177583$			
60 terms	65 terms	70 terms	Exact
1.07428	1.07451	1.074541	1.07439

Table 3.3.7: The first four terms in the HAM solution for $n = 4$, $b = 1/2$ and $k_1 = -1$.

For $b = \frac{1}{2}$
$v_0(x) = 1$ $v_1(x) = \frac{\hbar t^2}{2} - \hbar t$ $v_2(x) = \frac{t^2 \hbar^2}{2} - t\hbar^2 + \frac{t^2 \hbar}{2} - t\hbar$ $v_3(x) = \frac{t^2 \hbar^3}{2} - t\hbar^3 + t^2 \hbar^2 - 2t\hbar^2 + \frac{t^2 \hbar}{2} - t\hbar$

Table 3.3.8: Convergence for the HAM solution for $n = 4$, $b = 1/2$ and $k_1 = -1$ at $x = 0.8$.

$n = 4, b = \frac{1}{2}, k_1 = -1, \hbar = -0.108918$			
50 terms	53 terms	56 terms	Exact
0.198795	0.198959	0.199067	0.199213

- $n = 4$

With $k_1 = -1$, and $b = \frac{1}{2}$ the 56 term HAM solution with $\hbar$ chosen to be -0.108918 gives $f''(0.8) = -0.986637$. The exact solution gives $v''(0.8) = -0.984434$. Figure 3.4 shows a plot of the first derivative of the HAM solution versus the first derivative of the exact solution and Figure 3.5 shows the plot of the HAM solution. Table 3.3.7 and 3.3.8 show the first few terms of the HAM solution under the assumption that $n = 4$, $b = 1/2$ and $k_1 = -1$ with $\hbar = -0.108918$ and their convergence at $x = 0.8$.

Choice of $\hbar$

For the case of $n = 4$, $k_1 = 1$, and $b = \frac{1}{6}$, the proper choice of the auxillary parameter $\hbar$ can be explored. To ensure that the HAM solution converged, we attempted to choose the value of $\hbar$ that forced the first derivative of the 54th and 56th terms of the solution evaluated at

Table 3.3.9: First four terms of the HAM solution for $n = 4$ with $b = 1/4$ and $b = 1/6$.

For $b = \frac{1}{4}$	For $b = \frac{1}{6}$
$v_0(x) = 1$ $v_1(x) = ht - \frac{ht^2}{2}$ $v_2(x) = -\frac{1}{2}t^2h^2 + th^2 - \frac{t^2h}{2} + th$ $v_3(x) = -\frac{1}{2}t^2h^3 + th^3 - t^2h^2 + 2th^2 - \frac{t^2h}{2} + th$	$v_0(x) = 1$ $v_1(x) = ht - \frac{ht^2}{2}$ $v_2(x) = -\frac{1}{2}t^2h^2 + th^2 - \frac{t^2h}{2} + th$ $v_3(x) = -\frac{1}{2}t^2h^3 + th^3 - t^2h^2 + 2th^2$ $\qquad - \frac{t^2h}{2} + th$

$t = 0$ to be the same. This gave two potential values of -0.129611 and -0.238153. Upon further inspection, both values gave that the second derivative at $t = 0$ was well behaved and provided a convergent solution. However, when the two solutions were substituted into the original nonlinear differential equation it was found that choosing the auxillary parameter to be -0.238153 resulted in a smaller residual. The 56 term solution then gave $f''(0) = 1.00537$ whereas the exact solution gives $v''(0) = 1.00538$. Figure 3.6 shows a plot of the derivatives of the 56 term HAM solutions for the two choices of the auxillary parameter versus the derivative of the exact solution. Figure 3.7 shows the plot of the 56 term HAM solution for $b = \frac{1}{6}$ and for $b = \frac{1}{4}$. For $b = \frac{1}{4}$ the auxillary parameter was chosen to be -0.285687 which gave $f''(0) = 1.00834$. The exact solution gives $v''(0) = 1.00811$. The first four terms of the solutions for $b = 1/4$ and $b = 1/6$ are given in Table 3.3.9.

3.3.3 Absolute Value Retained: Exact Solution, Fractional Powers of n

To reiterate, we are considering the following problem:

$$\frac{d^2v}{dx^2} + \frac{d}{dx}\left(b\left|\frac{dv}{dx}\right|^{n-1}\frac{dv}{dx}\right) - k = 0 \qquad (3.3.31)$$

subject to

$$v(0) = 1 \text{ and } \frac{dv}{dx} + b\left(\left|\frac{dv}{dx}\right|\right)^{n-1}\frac{dv}{dx}\Bigg|_{x=1} = 0. \qquad (3.3.32)$$

We will consider the case when $n = 1/2$ and $b = 1$.

Assuming that $v'(x) > 0$ reduces the condition at $x = 1$ to

$v'(x) + v'(x)^{\frac{1}{2}} = 0$

$\sqrt{v'(x)}(\sqrt{v'(x)} + 1) = 0$

$\Rightarrow \sqrt{v'(1)} = 0 \text{ or } \sqrt{v'(1)} = -1$

Or simply that $\sqrt{v'(1)} = 0 \Rightarrow v'(1) = 0$

Assuming that $v'(x) < 0$ reduces the condition at $x = 1$ to

$v'(x) - v'(x)^{\frac{1}{2}} = 0$

$\sqrt{v'(x)}(\sqrt{v'(x)} - 1) = 0$

$\Rightarrow \sqrt{v'(1)} = 0 \text{ or } \sqrt{v'(1)} = 1$

$\Rightarrow v'(1) = 0 \text{ or } v'(1) = 1$

Because of this we must consider two main cases. The first is the case when $v'(x) > 0$ on $[0, 1]$ and the second is the case when $v'(x) < 0$ on $[0, 1]$. Each of these cases has subcases based on the choice of the parameter k. Here we restrict the choice of k to -1 and 1.

43

$v'(x) > 0$: *Various k*

- $k = -1$

  For $k = -1$, we have

  $$v''(x) + \frac{1}{2}\left(v'(x)\right)^{-\frac{1}{2}} v''(x) + 1 = 0 \tag{3.3.33}$$

Using the condition $v'(1) = 0$ we can write

$$v'(x) + \left(v'(x)\right)^{\frac{1}{2}} + (x - 1) = 0. \tag{3.3.34}$$

The solution to this can be found as follows:

$$\sqrt{v'(x)} = -\left[(x - 1) + v'(x)\right] \tag{3.3.35}$$

$$v'(x) = (x - 1)^2 + 2(x - 1)v'(x) + \left(v'(x)\right)^2 \tag{3.3.36}$$

$$\left(v'(x)\right)^2 + (2x - 3)v'(x) + (x - 1)^2 = 0 \tag{3.3.37}$$

$$v'(x) = \frac{-(2x - 3) \pm \sqrt{(2x - 3)^2 - 4(x - 1)^2}}{2}. \tag{3.3.38}$$

Since we need $v'(1) = 0$ we use

$$v'(x) = \frac{-(2x - 3) - \sqrt{(2x - 3)^2 - 4(x - 1)^2}}{2}. \tag{3.3.39}$$

Integrating and using $v(0) = 1$ gives

$$v(x) = -\frac{1}{2}x^2 + \frac{3}{2}x + \frac{1}{12}(5 - 4x)^{\frac{3}{2}} + 1 - \frac{5^{\frac{3}{2}}}{12}. \tag{3.3.40}$$

44

Solving

$$v''(x) + \frac{1}{2}\left(v'(x)\right)^{-\frac{1}{2}}v''(x) + 1 = 0 \tag{3.3.41}$$

with Mathematica yields

$$v_m(x) = \frac{1}{12}\left(-6x^2 - 4\sqrt{5-4x}\,x + 18x + 5\sqrt{5-4x} - 5\sqrt{5} + 12\right). \tag{3.3.42}$$

A little simplification shows this to be equivalent to the result we obtained.

- $k = 1$

   With this choice of the parameter k we have

$$v''(x) + \frac{1}{2}\left(v'(x)\right)^{-\frac{1}{2}}v''(x) - 1 = 0 \tag{3.3.43}$$

Using the condition $v'(1) = 0$ we can write

$$v'(x) + \left(v'(x)\right)^{\frac{1}{2}} - (x-1) = 0. \tag{3.3.44}$$

The solution to this can be found as follows:

$$\sqrt{v'(x)} = -\left[(x-1) - v'(x)\right] \tag{3.3.45}$$

$$v'(x) = (x-1)^2 - 2(x-1)v'(x) + \left(v'(x)\right)^2 \tag{3.3.46}$$

$$\left(v'(x)\right)^2 - (2x-1)v'(x) + (x-1)^2 = 0 \tag{3.3.47}$$

$$v'(x) = \frac{2x - 1 \pm \sqrt{4x-3}}{2}. \tag{3.3.48}$$

45

Since we need $v'(1) = 0$ we use

$$v'(x) = \frac{2x - 1 - \sqrt{4x - 3}}{2}. \tag{3.3.49}$$

Integrating and using $v(0) = 1$ gives

$$v(x) = \frac{1}{2}x^2 - \frac{1}{2}x - \frac{1}{12}(4x - 3)^{\frac{3}{2}} + 1 + \frac{3i\sqrt{3}}{12}. \tag{3.3.50}$$

$v'(x) < 0$: *Various k*

- $k = -1$ **and** $v'(1) = 1$

  With this choice of the parameters we have

  $$v''(x) + \frac{d}{dx}\left(|v'(x)|^{-\frac{1}{2}}v'(x)\right) - 1 = 0 \tag{3.3.51}$$

which we can write as

$$v''(x) + \frac{d}{dx}\left(|v'(x)|^{-\frac{1}{2}}v'(x)\right) = 1. \tag{3.3.52}$$

Integrating gives

$$|v'(x)|^{-\frac{1}{2}}v'(x) = (x + c) - v'(x). \tag{3.3.53}$$

Squaring both sides gives

$$-v'(x) = v'(x)^2 - 2(x + c)v'(x) + (x + c)^2. \tag{3.3.54}$$

Using the condition $v'(1) = 1$ and solving for the constant c gives that either $c = i$ or $c = -i$.

46

Either one results in a complex valued solution.

- $k = -1$ **and** $v'(1) = 1$

  With this choice of the parameters we have

  $$v''(x) + \frac{d}{dx}\left(|v'(x)|^{-\frac{1}{2}} v'(x)\right) - 1 = 0 \qquad (3.3.55)$$

Using the condition $v'(1) = 0$ we can assume that $v(x) = x - 1$ and write

$$v'(x) + |v'(x)|^{-\frac{1}{2}} v'(x) - (x - 1) = 0. \qquad (3.3.56)$$

The solution to this can be found as follows:

$$v'(x) + |v'(x)|^{-\frac{1}{2}} v'(x) - (x - 1) = 0 \qquad (3.3.57)$$

$$|v'(x)|^{-\frac{1}{2}} v'(x) = (x - 1) - v'(x) \qquad (3.3.58)$$

$$|v'(x)|^{-1} v'(x)^2 = v'(x)^2 - 2(x - 1)v'(x) + (x - 1)^2 \qquad (3.3.59)$$

$$-v'(x)^{-1}v'(x)^2 = v'(x)^2 - 2(x - 1)v'(x) + (x - 1)^2 \qquad (3.3.60)$$

$$v'(x)^2 - (2x - 3)v'(x) + (x - 1)^2 \qquad (3.3.61)$$

$$\Rightarrow v'(x) = \frac{2x - 3 \pm \sqrt{5 - 4x}}{2}. \qquad (3.3.62)$$

Applying $v'(1) = 0$ implies we must take

$$\Rightarrow v'(x) = \frac{2x - 3 + \sqrt{5 - 4x}}{2}. \qquad (3.3.63)$$

47

Integrating and using $v(0) = 1$ results in

$$v(x) = \frac{1}{2}x^2 - \frac{3}{2}x - \frac{1}{12}(5 - 4x)^{\frac{3}{2}} + 1 + \frac{5^{\frac{3}{2}}}{12}. \tag{3.3.64}$$

• $k = -1$ and $v'(1) = 0$

Following the same procedure as above we find that

$$v'(x) = \frac{-2x + 1 + \sqrt{4x - 3}}{2}. \tag{3.3.65}$$

Integrating and using the condition $v(0) = 1$ results in a complex valued solution.

• $k = -1$ and $v'(1) = 1$

Here we have

$$v''(x) - \frac{d}{dx}(v'(x))^{\frac{1}{2}} + 1 = 0. \tag{3.3.66}$$

Using the same technique we get that the solution is complex valued.

• **Summary, Exact Solutions:**

The two situations that give solutions are $v'(x) > 0; v'(1) = 0; k = -1$ which gives

$$v(x) = -\frac{1}{2}x^2 + \frac{3}{2}x + \frac{1}{12}(5 - 4x)^{\frac{3}{2}} + 1 - \frac{5^{\frac{3}{2}}}{12} \tag{3.3.67}$$

and $v'(x) < 0; v'(1) = 0; k = 1$ which gives

$$v(x) = \frac{1}{2}x^2 - \frac{3}{2}x - \frac{1}{12}(5 - 4x)^{\frac{3}{2}} + 1 + \frac{5^{\frac{3}{2}}}{12}. \tag{3.3.68}$$

<u>Note:</u> Monotonically Decreasing Solution: No absolute Value

With a proper choice of the parameters it is possible to obtain the same result as the monotonically decreasing solution. That is, consider the problem with $b = i$ and $k = 1$ with the absolute value removed. We get

$$v''(x) + i\frac{d}{dx}(v'(x))^{\frac{1}{2}} - 1 = 0 \tag{3.3.69}$$

subject to $v(0) = 1$ and $v'(1) = 0$. Again since we have the condition $v'(1) = 0$ we consider

$$\frac{d}{dx}\left(v'(x) + i\sqrt{v'(x)} - (x - 1)\right) = 0. \tag{3.3.70}$$

Using the same technique as before we get

$$v'^2 - 2(x - 1)v' + (x - 1)^2 = -v' \tag{3.3.71}$$

$$v'^2 - (2x - 3)v' + (x - 1)^2 = 0 \tag{3.3.72}$$

$$v' = \frac{2x - 3 \pm \sqrt{5 - 4x}}{2}. \tag{3.3.73}$$

Applying the condition at $x = 1$ implies we take

$$v' = \frac{2x - 3 + \sqrt{5 - 4x}}{2}. \tag{3.3.74}$$

Integrating and using the condition $v(0) = 1$ then gives

$$v(x) = \frac{1}{2}x^2 - \frac{3}{2}x - \frac{1}{12}(5 - 4x)^{\frac{3}{2}} + 1 + \frac{5^{\frac{3}{2}}}{12}. \tag{3.3.75}$$

3.3.4 Absolute Value Retained: HAM Solution, Fractional Powers of n

In order to apply HAM to this problem it is necessary to redefine the nonlinear operator as given in the original problem. The reason for this is that we are assuming a series solution and therefore because of the fractional power we run into computational issues in taking the square roots of large sums. Following the procedure given in Akyildiz, Vajravelu, Mohapatra, Sweet and Van Gorder [14] we do the following.

If we assume that the power $n - 1$ can be written as m/s, the original equation becomes:

$$\frac{d^2v}{dx^2} + \frac{d}{dx}\left(b\left|\frac{dv}{dx}\right|^{\frac{m}{s}}\frac{dv}{dx}\right) - k = 0 \tag{3.3.76}$$

This can then be rewritten as

$$\frac{d}{dx}\left[\frac{dv}{dx} + b\left|\frac{dv}{dx}\right|^{\frac{m}{s}}\frac{dv}{dx} - (kx - k)\right] = 0 \tag{3.3.77}$$

$$\Rightarrow \frac{dv}{dx} + b\left|\frac{dv}{dx}\right|^{\frac{m}{s}}\frac{dv}{dx} - (kx - k) = 0 \tag{3.3.78}$$

$$\Rightarrow \left|\frac{dv}{dx}\right|^{m}\left(\frac{dv}{dx}\right)^{s} - \left[\frac{1}{b}\left(kx - k - \frac{dv}{dx}\right)\right]^{s} = 0. \tag{3.3.79}$$

Using this as the non-linear operator in the generalized homotopy with the same initial guess of $v_0(t) = 1$ and the linear operator $L = \partial/\partial t$ under the monotonically increasing assumption and $L = \partial^2/\partial t^2$ for the monotonically decreasing assumption, we get, under the assumption that $b = 1$, the following.

Assuming the derivative is monotonically increasing on $[0, 1]$, the 30 term solution with

$\hbar = -0.2503$ gives $f''(0) = -0.552785$. While the exact solution of

$$v(x) = -\frac{1}{2}x^2 + \frac{3}{2}x + \frac{1}{12}(5 - 4x)^{\frac{3}{2}} + 1 - \frac{5^{\frac{3}{2}}}{12} \tag{3.3.80}$$

gives $v''(0) = -0.552786$.

Assuming the derivative is monotonically decreasing on $[0, 1]$, the 31 term solution with $\hbar = -1.17094$ gives $f''(0) = -0.554046$. While the exact solution of

$$v(x) = \frac{1}{2}x^2 \frac{3}{2}x - \frac{1}{12}(5 - 4x)^{\frac{3}{2}} + 1 + \frac{5^{\frac{3}{2}}}{12} \tag{3.3.81}$$

gives $v''(0) = -0.552786$.

Figure 3.8 shows the HAM solution versus the exact solution for the monotonically increasing assumption and Figure 3.9 shows the HAM solution versus the exact solution for the monotonically decreasing assumption.

It was shown earlier that the choice of the linear operator in the solution process is not unique in that different operators give desirable results. It is also true that the choice of the initial guess is not unique. As shown in Figure 3.10, using an initial guess of $1 - t + \frac{1}{2}t^2$ also provides accurate solutions.

3.3.5 Results

In Sajid, Hayat and Asghar [20] the authors indicate the 20 term HAM solution for $n = 3$, $b = 1/2$ and $k_1 = 1$ yields $f''(0) = 0.59928$ the HPM solution yields $f''(0) = 825872.0$. The 60 term solution provided here gives $f''(0) = 0.528686$ while the exact value is 0.528689. Therefore we see that a simplified application of HAM provides better solutions in terms of

accuracy than HAM or HPM. Moreover, the flexibility allows us to extend the process to equations involving fractional powers.

3.4 Figures
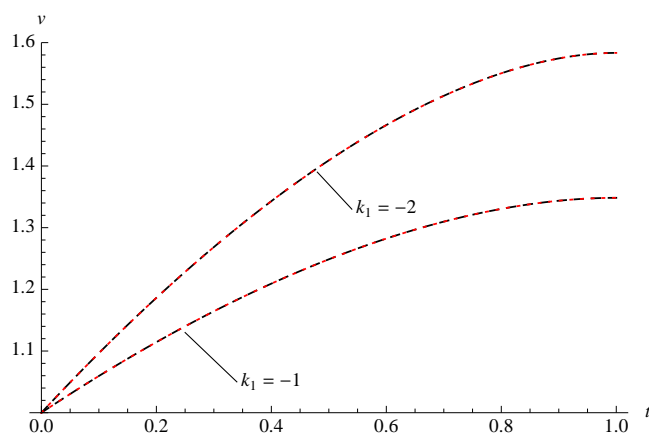
Figure 3.1: The plot of the 78 Term HAM Solution for $k_1 = -1$ and the 86 Term HAM Solution for $k_1 = -2$ vs. the corresponding exact solution for $n = 2$ in the Viscous Flow problem. The HAM Solutions are dotted and the exact solutions are dashed.
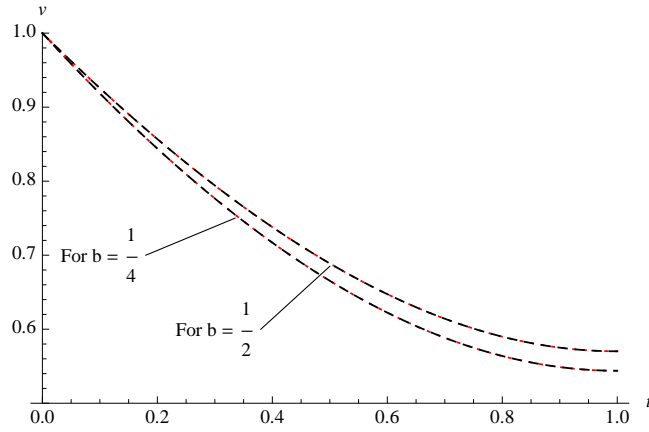
Figure 3.2: The plot of the 60 Term HAM Solution for $b = \frac{1}{2}$ and the 70 Term HAM Solution for $b = \frac{1}{4}$ vs. the corresponding exact solution for $n = 3$ and $k_1 = 1$ in the Sisko Flow problem. The HAM Solutions are dotted and the exact solutions are dashed.
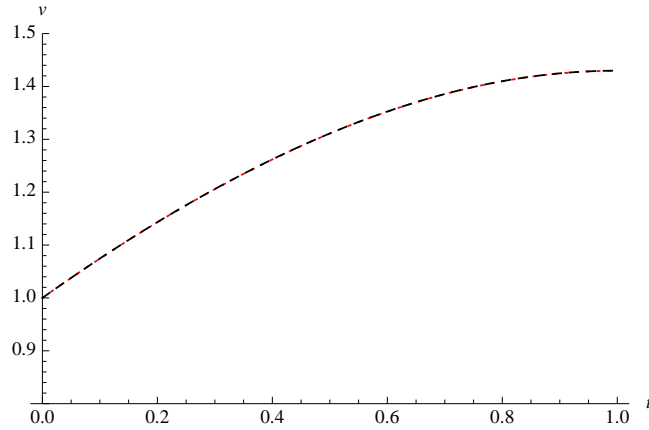
Figure 3.3: The plot of the 70 Term HAM Solution for $b = \frac{1}{2}$ vs. the corresponding exact solution for $n = 3$ and $k_1 = -1$ in the Sisko Flow problem. The HAM Solution is dotted and the exact solution is dashed.
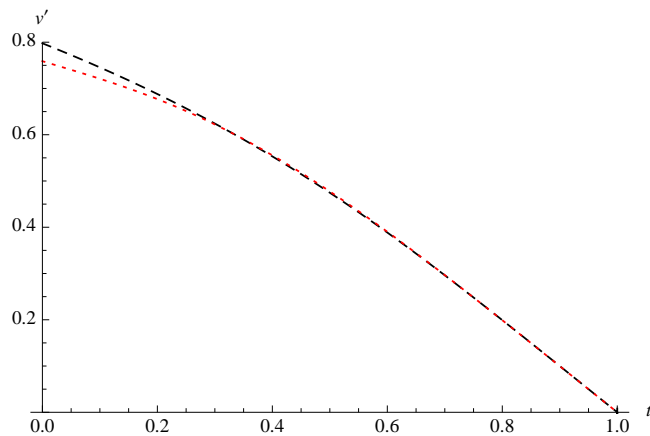
Figure 3.4: The plot of the derivative of the 56 Term HAM Solution for $b = \frac{1}{2}$ and $k_1 = -1$ vs. the first derivative of the exact solution for $n = 4$ in the Sisko Flow problem. The HAM Solution is dotted and the exact solution is dashed.
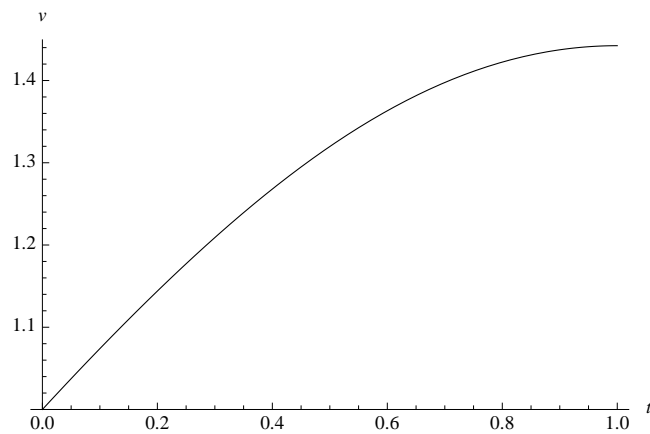
Figure 3.5: The plot of the 56 Term HAM Solution for $b = \frac{1}{2}$, $k_1 = -1$, and $n = 4$ in the Sisko Flow problem.
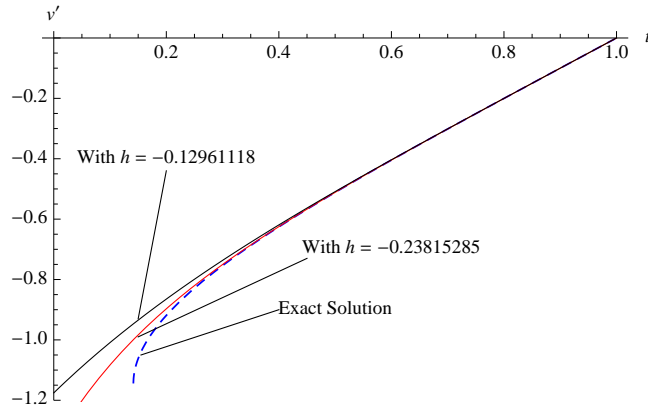
Figure 3.6: The plot of the derivative of the 56 Term HAM Solutions for the two choices of the auxillary parameter for $b = \frac{1}{6}$, $k_1 = 1$ and $n = 4$ vs. the first derivative of the exact solution in the Sisko Flow problem.
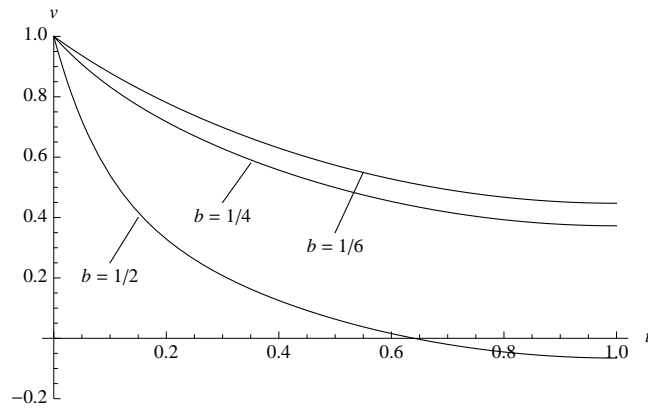
Figure 3.7: The plot of the 56 Term HAM Solutions for $b = \frac{1}{6}$, $b = \frac{1}{4}$ and $b = \frac{1}{2}$ with $k_1 = 1$ and $n = 4$ in the Sisko Flow problem.

55
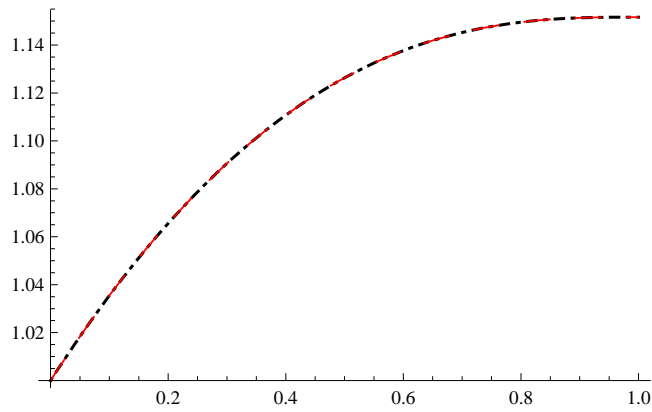
Figure 3.8: The plot of t versus $v(t)$ for the exact solution and the HAM Solution for the monotonically increasing assumption in the Sisko Flow problem.
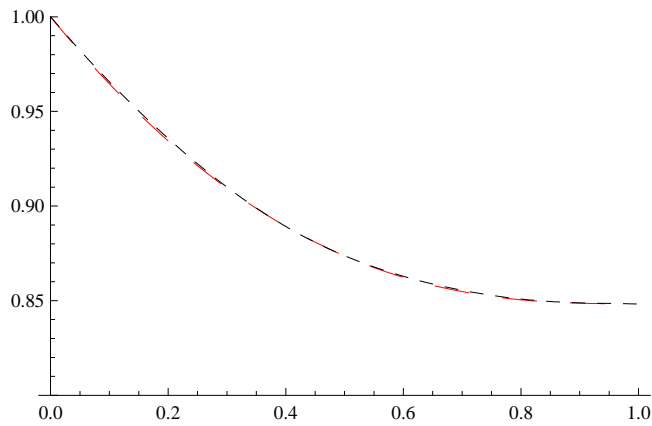
Figure 3.9: The plot of t versus $v(t)$ for the exact solution and the HAM Solution in the monotonically decreasing assumption in the Sisko Flow problem.
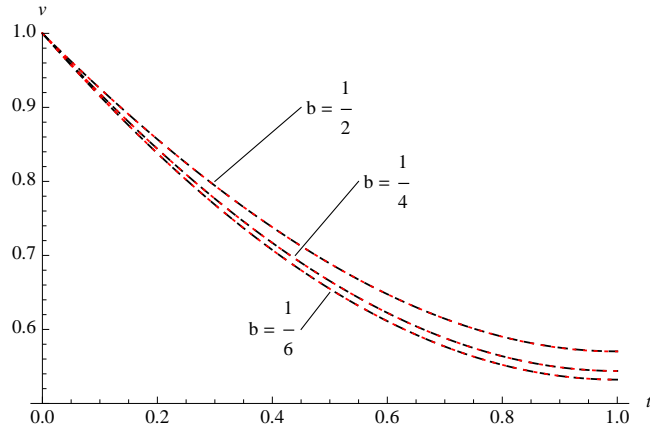
Figure 3.10: HAM solutions for $n = 3$ and $k_1 = 1$ under the assumption that the initial guess is $1 - t + \frac{1}{2}t^2$ for the Sisko Flow problem. The HAM solutions are dotted and the exact solutions are dashed.

CHAPTER 4. NANO BOUNDARY LAYERS OVER STRETCHING SURFACES

4.1 Statement of Problem

As given in Van Gorder, E. Sweet and K. Vajravelu [27], in classical boundary layer theory the condition of no-slip near solid walls is usually applied. That is, the fluid velocity component is assumed to be zero relative to the solid boundary. This is not true for fluid flows at the micro and nano scale. It is found that a certain degree of tangential slip must be allowed. We consider the model proposed by Wang [6] describing the viscous flow due to a stretching surface with both surface slip and suction (or injection). As in Wang [6], we consider two geometric situations. We allow for a two-dimensional stretching surface and an axisymmetric stretching surface. According to Wang [7], the flow due to a stretching boundary is important in extrusion processes. Examples in which partial slip between the fluid and the moving surface occur include situations when the fluid is particulate such as emulsions, suspensions, foams and polymer solutions.

Following the formulation given in Van Gorder, Sweet and Vajravelu [27] we let (u, v, w) be the velocity components in the (x, y, z) directions, respectively, and let p be the pressure.

Then the Navier-Stokes equations for the steady viscous fluid flow can be written as

$$uu_x + vu_y + wu_z = -\frac{p_z}{\rho} + \nu\left(u_{xx} + u_{yy} + u_{zz}\right) \qquad (4.1.1)$$

$$uv_x + vv_y + wu_z = -\frac{p_z}{\rho} + \nu\left(v_{xx} + v_{yy} + v_{zz}\right) \qquad (4.1.2)$$

$$uw_x + vw_y + ww_z = -\frac{p_z}{\rho} + \nu\left(w_{xx} + w_{yy} + w_{zz}\right) \qquad (4.1.3)$$

where ν is the kinematic viscosity and ρ is the density of the fluid. The continuity equation can be written as

$$u_x + u_y + u_z = 0. \qquad (4.1.4)$$

As in C.Y. Wang [6] we take the velocity on the stretching surface (on the plane $z = 0$) as

$$u = ax, \quad v = (m-1)ay, \quad w = 0 \qquad (4.1.5)$$

where $a > 0$ is the stretching rate of the sheet and m is a parameter describing the type of stretching. When $m = 1$, we have two-dimensional stretching, while for $m = 2$ we have axisymmetric stretching. To simplify the governing equations we introduce the similarity variable $\eta = \sqrt{a/\nu}\, z$ and the similarity functions

$$u = axf'(\eta), \quad v = (m-1)ayf'(\eta), \quad w = -m\sqrt{a\nu}f(\eta). \qquad (4.1.6)$$

Under these transformations the continuity equation is automatically satisfied and the Navier-

Stokes equations become

$$f'''(\eta) - (f'(\eta))^2 + mf(\eta)f''(\eta) = 0 \tag{4.1.7}$$

as there is no lateral pressure gradient at infinity. On the surface of the stretching sheet, the velocity slip is assumed to be proportional to the local sheer stress

$$u - ax = N\rho\nu\frac{\partial u}{\partial z} < 0, \quad v - (m-1)ay = N\rho\nu\frac{\partial v}{\partial z} < 0 \tag{4.1.8}$$

where N is a slip constant. By using the similarity transform described above this can be rewritten as

$$f'(0) - 1 = Kf''(0) \tag{4.1.9}$$

where $K = N\rho\sqrt{a\nu} > 0$ is a non-dimensional slip parameter. Given a suction velocity of $-W$ on the stretching surface, we have the boundary condition

$$f(0) = s \tag{4.1.10}$$

where $s = W/(m\sqrt{a\nu})$ and s is negative if injection from the surface occurs. Since there is no lateral velocity at infinity, we also have the condition

$$\lim_{\eta \to \infty} f'(\eta) = 0. \tag{4.1.11}$$

Under this formulation we solve the nonlinear boundary value problem

$$f'''(\eta) - (f'(\eta))^2 + mf(\eta)f''(\eta) = 0 \tag{4.1.12}$$

subject to

$$f(0) = s, \quad f'(0) - 1 = Kf''(0), \quad \lim_{\eta \to \infty} = 0. \tag{4.1.13}$$

4.2 Rational Approximation: Particular Case

Here we consider the version of the above problem given in Wang [7] as

$$f''' - (f')^2 + ff'' = 0 \tag{4.2.1}$$

subject to

$$\lim_{\eta \to \infty} f'(\eta) = 0, f(0) = 0, f'(0) - Kf''(0) = 1 \tag{4.2.2}$$

where K is a non-dimensional parameter indicating the relative importance of partial slip. The relevance of this problem is its solution via rational functions and the use of the Auxillary Function in the Generalized Homotopy.

The author solves the problem numerically by a change of variables by assuming $s = e^{-\eta}$ and $g(s) = f$. Upon substituting into

$$f''' - (f')^2 + ff'' = 0 \tag{4.2.3}$$

61

we get

$$s^2 g''' + s(3 - g)g'' + (sg' - g + 1)g' = 0 \qquad (4.2.4)$$

subject to

$$\lim_{s \to 0} sg'(s) = 0, g(1) = 0, Kg''(1) + (1 + K)g'(1) + 1 = 0. \qquad (4.2.5)$$

To solve this, the author guessed values of $g''(1)$ and used a fifth-order Runge-Kutta algorithm to integrate

$$s^2 g''' + s(3 - g)g'' + (sg' - g + 1)g' = 0. \qquad (4.2.6)$$

If $\lim_{s \to 0} sg'(s) = 0$, a solution is found. To guide the choice of $g''(1)$ the author used a perturbation solution in which K is assumed to be the small parameter.

To solve this with HAM, we assumed an initial guess of

$$f_0(\eta) = \left(\frac{1}{1 + 2K} \right) \left(\frac{\eta}{1 + \eta} \right). \qquad (4.2.7)$$

The linear operator L was generated by assuming solutions of

$$f_0(\eta), \quad \eta, \quad \text{and} \quad \frac{1}{1 + \eta} \qquad (4.2.8)$$

and solving the resulting system formed by substituting the assumed form of the solutions into

$$a f'''(\eta) + b f''(\eta) + c f'(\eta) + d f(\eta) = 0 \qquad (4.2.9)$$

62

for the unknowns a, b, c, and d. This resulted in defining the linear operator as

$$L = \frac{\partial^3}{\partial \eta^3} + \left(\frac{3}{1+\eta} \right) \frac{\partial^2}{\partial \eta^2}. \qquad (4.2.10)$$

The generalized homotopy we use is different than the previous ones in that we take advantage of the auxillary function $\mathcal{H}(\eta)$ in order to ensure that the resulting order k differential equation can be easily solved. Here we use

$$H[G(\eta, q); q] = (1-q)L[G(\eta, q) - f_0(\eta)] - qh \left(\frac{1}{1+\eta} \right) N[G(\eta, q)] \qquad (4.2.11)$$

where N is the operator defined by the original problem.

The results for HAM are as follows. For $K = 3/10$ and $\hbar = -0.6202876$, the 8 term solution gives $f''(0) = -0.703053$. The author gives $f''(0) = -0.701$. For $K = 1$ and $\hbar = -0.67118$, the 8 term solution gives $f''(0) = -0.428009$. The author gives $f''(0) = -0.430$. For $K = 2$ and $\hbar = -0.645073$, the 9 term solution gives $f''(0) = -0.278827$. The author gives $f''(0) = -0.284$. Figure 4.1 shows the plots for $K = 3/10$, $K = 1$ and $K = 2$

4.3 Analysis

To solve the problem in general via HAM we assume an initial guess of

$$f_0(\eta) = \left(\frac{1}{1+K} \right) \left(1 - e^{-\eta} \right) + s \qquad (4.3.1)$$

along with the linear operator

$$L = \frac{\partial^3}{\partial \eta^3} + \frac{\partial^2}{\partial \eta^2}. \tag{4.3.2}$$

The generalized homotopy is then defined to be

$$(1-q)L[G(\eta; q) - f_0(\eta)] = q\hbar \left[\frac{\partial^3}{\partial \eta^3} G(\eta; q) - \left(\frac{\partial}{\partial \eta} G(\eta; q) \right)^2 + mG(\eta; q) \frac{\partial^2}{\partial \eta^2} G(\eta; q) \right]. \tag{4.3.3}$$

This produces the mth order deformation equation

$$L[f_n(\eta) - \chi_n f_{n-1}(\eta)] = \hbar \left[f'''_{n-1}(\eta) - \sum_{k=0}^{n-1} f'_k(\eta) f'_{n-1-k}(\eta) + m \sum_{k=0}^{n-1} f_k(\eta) f''_{n-1-k}(\eta) \right] \tag{4.3.4}$$

for which χ_n is 0 for all $n \leq 1$ and 1 for all $n > 1$. For each n, the differential equation obtained from the generalized homotopy has the corresponding conditions

$$f_n(0) = 0; \lim_{\eta \to \infty} f'(\eta) = 0; f'_n(0) - K f''_n(\eta) = 0. \tag{4.3.5}$$

4.4 Results

Figures 4.2 and 4.3 show that solutions under certain assumptions on the values of the parameters m, K and s. The values of $\hbar$ that provide the solution have also been included in the plots. Under these choices of the parameters, the HAM solutions containing at least 20 terms produce an residual no greater than 10^{-7}.

64

4.5 Figures
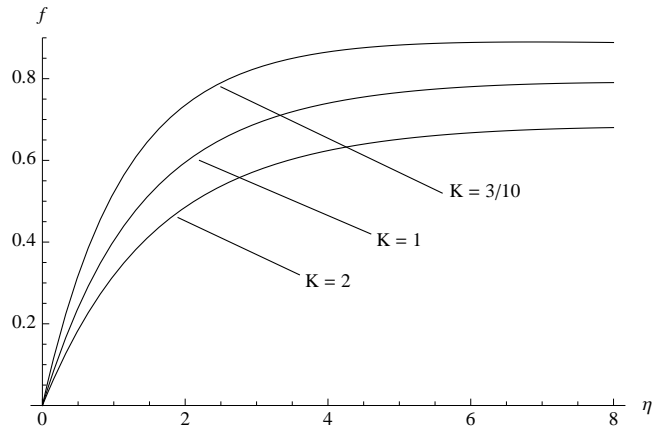
Figure 4.1: The plot of the HAM solution for the Partial Slip problem with K of 3/10, 1, and 2. It can be seen that as K increases, $f(\eta)$ decreases.
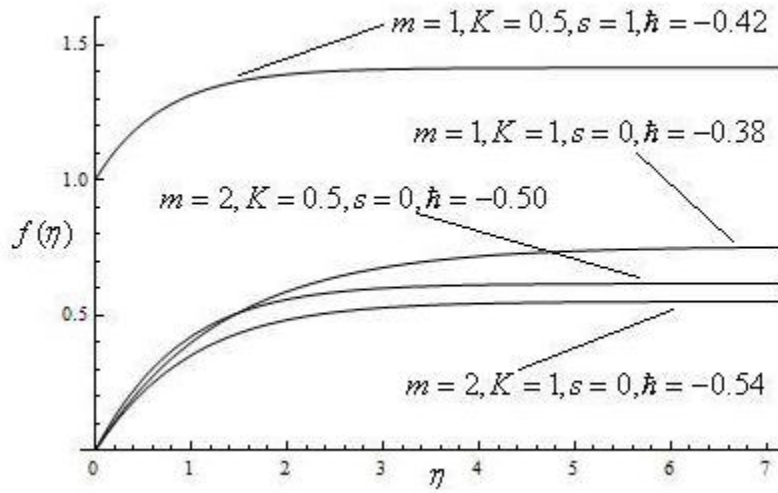
Figure 4.2: HAM solutions for $f(\eta)$ and various values of the parameters.
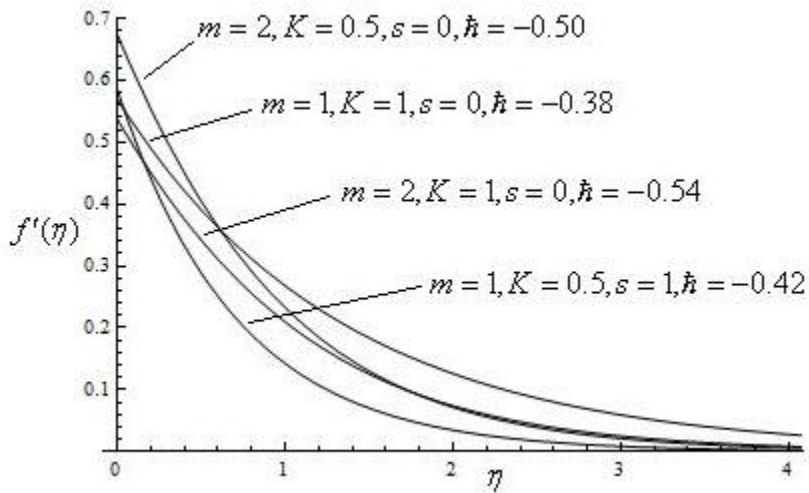
Figure 4.3: HAM solutions for $f'(\eta)$ and various values of the parameters.

CHAPTER 5. MAGNETO-HYDRODYNAMIC FLUID IN TWO DIMENSIONS; SQUEEZING FLOW

5.1 Statement of Problem

In this chapter we consider the problem given in Sweet, Van Gorder and Vajravelu [11]. The problem, originally considered in Siddiqui, Irum and Ansari [2], deals with the unsteady hydro-magnetic squeezing flow of an incompressible two-dimensional viscous fluid between two infinite plates. The flow of a fluid that is under the influence of an electromagnetic field, or an MHD fluid between moving parallel plates leads to squeezing flow. These problems lend themselves to applications in bearings with liquid-metal lubrications for example. According to [2], the use of a MHD fluid as lubricant is of interest, because it prevents the unexpected variation of lubricant viscosity with temperature under certain extreme conditions.

The unsteady mass and momentum conservation equations describing the flow are

$$u_x + v_y = 0 \tag{5.1.1}$$

$$\rho \left(u_t + u u_x + v u_y \right) = -p_x + V \left(u_{xx} + u_{yy} \right) - \sigma B_0^2 u \tag{5.1.2}$$

$$\rho \left(v_t + u v_x + v v_y \right) = -p_y + V \left(v_{xx} + v_{yy} \right) - \sigma B_0^2 u \tag{5.1.3}$$

where u and v are the velocity components along the x and y directions respectively, V denotes the kinematic viscosity, ρ denotes the density of the fluid, σ is the electrical conductivity of the fluid and B_0 is the uniform magnetic field acting along the y-axis. In this derivation the induced magnetic field is assumed to be negligible. Siddiqui, Irum and Ansari [2] defines the vorticity function ω and the generalized pressure h as

$$\omega = v_x - u_y \tag{5.1.4}$$

$$h = \frac{\rho}{2}\left(u^2 + v^2\right) + p. \tag{5.1.5}$$

From these we see that

$$h_x - p_x - \rho v \omega = \rho u u_x + \rho v v_x - \rho v v_x + \rho v u_y = \rho u u_x + \rho v u_y \tag{5.1.6}$$

and

$$h_y - p_y + \rho u \omega = \rho u u_y + \rho v v_y + \rho u v_x - \rho u u_y = \rho u v_x + \rho v v_y. \tag{5.1.7}$$

Substituting 5.1.6 and 5.1.7 into 5.1.2 and 5.1.3 respectively gives

$$h_x + \rho\left(u_t - v\omega\right) = V\left(u_{xx} + u_{yy}\right) - \sigma B_0^2 u \tag{5.1.8}$$

$$h_y + \rho\left(v_t + u\omega\right) = V\left(v_{xx} + v_{yy}\right). \tag{5.1.9}$$

Using the vorticity function and $u_x + v_y = 0$ gives

$$\omega_y = v_{xy} - u_{yy} = -u_x x - u_{yy} \Rightarrow -\omega = u_{xx} + u_{yy} \tag{5.1.10}$$

and

$$\omega_x = v_{xx} - u_{xy} = v_{xx} + v_{yy}. \tag{5.1.11}$$

Therefore 5.1.8 and 5.1.9 can be rewritten as

$$h_x + \rho \left(u_u - v\omega \right) = -V\omega_y - \sigma B_0^2 u \tag{5.1.12}$$

and

$$h_y + \rho \left(v_t + u\omega \right) = V\omega_x. \tag{5.1.13}$$

To eliminate the generalized pressure we differentiate 5.1.12 with respect to y and 5.1.13 with respect to x and subtract the two expressions to obtain the single equation

$$\rho \frac{\partial}{\partial t} \left(u_y - v_x \right) - \rho \omega \left(v_y + u_x \right) - \rho u \omega_x - \rho v \omega_y = -V\omega_{yy} - V\omega_{xx} - \sigma B_0^2 u_y. \tag{5.1.14}$$

Using 5.1.1 again allows us to write this as

$$\rho \omega_t + \rho \left(u\omega_x + v\omega_y \right) = V \, \nabla^2 \, \omega + \sigma B_0^2 u_y. \tag{5.1.15}$$

Following Siddiqui, Irum and Ansari [2] we define new variables $x^* = x$, $\eta = y/a(t)$ and $t^* = t$ so that the partial derivatives are redefined to be:

$$\frac{\partial}{\partial t^*} = \frac{\partial}{\partial \eta} \frac{\partial \eta}{\partial t} + \frac{\partial}{\partial t} = -\frac{1}{a(t)^2} y \frac{\partial}{\partial \eta} + \frac{\partial}{\partial t} = -\frac{1}{a(t)} \eta \frac{\partial}{\partial \eta} + \frac{\partial}{\partial t} \tag{5.1.16}$$

$$\frac{\partial}{\partial y} = \frac{1}{a(t)} \frac{\partial}{\partial \eta} \tag{5.1.17}$$

69

$$\frac{\partial}{\partial x^*} = \frac{\partial}{\partial x}. \tag{5.1.18}$$

Now define the dimensionless variable velocity components to be

$$u = \frac{C-x}{a(t)} v_\omega(t) f'(\eta); \quad v = v_\omega(t) f(\eta); \quad \omega = -\frac{C-x}{a(t)^2} v_\omega(t) f''(\eta) \tag{5.1.19}$$

where C is a constant related to the inlet condition and $v_\omega(t)$ is the velocity of the plates. This leads to

$$\omega_x = \frac{1}{a(t)^2} v_\omega(t) f''(\eta) \tag{5.1.20}$$

$$\omega_{xx} = 0 \tag{5.1.21}$$

$$\omega_y = -\frac{C-x}{a(t)^3} v_\omega(t) f'''(\eta) \tag{5.1.22}$$

$$\omega_{yy} = -\frac{C-x}{a(t)^4} v_\omega(t) f^{(iv)}(\eta) \tag{5.1.23}$$

$$\omega_t = \frac{C-x}{a(t)^3} \eta \left(v_\omega(t)\right)^2 f'''(\eta) + \frac{2(C-x)}{a(t)^3} \left(v_\omega(t)\right)^2 f''(\eta) - \frac{C-x}{a(t)^2} \frac{dv_\omega}{dt} f''(\eta) \tag{5.1.24}$$

$$u_y = \frac{C-x}{a(t)^2} v_\omega(t) f''(\eta) \tag{5.1.25}$$

Substituting these into 5.1.15 gives

$$\frac{\rho a(t) v_\omega(t)}{V} \left[f f''' - f' f'' - 2f'' - \eta f'''\right] + \frac{a(t)^2}{V v_\omega(t)} \frac{dv_\omega}{dt} f'' = f^{(iv)} - \frac{\sigma B_0^2 a(t)^2}{V} f''. \tag{5.1.26}$$

Defining $R = \frac{\rho a(t) v_\omega(t)}{V}$, $Q = \frac{a(t)}{\rho v_\omega(t)^2} \frac{dv_\omega(t)}{dt}$ and $M = \frac{\sigma B_0^2 a(t)^2}{V}$ we are able to write the above equation as

$$R\left[f f''' - f' f'' - 2f'' - \eta f'''\right] + RQ f'' = f^{(iv)} - M f''. \tag{5.1.27}$$

70

We make the further observations that from $R = \frac{\rho a(t) v_\omega(t)}{V}$ we have that $a(t) = \sqrt{\frac{RV}{\rho} t + k}$, where k is a constant of integration. Inserting this into the expression for Q gives that $Q = -1/\rho$. Therefore, we are left with

$$R\left(ff''' - f'f'' - \eta f''' - \left(2 + \frac{1}{\rho}\right)f''\right) = f^{(iv)} - M^2 f'' \tag{5.1.28}$$

where the $'$ denotes differentiation with respect to η. We hold 5.1.28 subject to the conditions

$$f(0) = 0, \quad f(1) = 1, \quad f'(1) = 0; \quad f''(0) = 0 \tag{5.1.29}$$

which follow from the no-slip condition and symmetry of flow. That is 5.1.29 follow from

$$v(x, 0, t) = 0 \Rightarrow v_\omega(t)f(0) = 0 \Rightarrow f(0) = 0 \tag{5.1.30}$$

$$u_\eta(x, 0, t) = 0 \Rightarrow \frac{C - x}{a(t)} v_\omega(t)f''(0) = 0 \Rightarrow f''(0) = 0 \tag{5.1.31}$$

$$u(x, 1, t) = 0 \Rightarrow \frac{C - x}{a(t)} v_\omega(t)f'(1) = 0 \Rightarrow f'(1) = 0 \tag{5.1.32}$$

$$v(x, 1, t) = v_\omega(t) \Rightarrow v_\omega(t)f(1) = v_\omega(t) \Rightarrow f(1) = 1. \tag{5.1.33}$$

It can also be seen from the definition of $a(t)$ that $R > 0$ corresponds to the case when the plates are moving away from each other and $R < 0$ corresponds to the case when the plates are moving towards each other.

With this formulation we are able to illustrate the dependence of the governing equation on the density of the fluid as well as its effect on the velocity profiles. In addition, we

71

also include solutions to the problem when the plates are allowed to move apart as well as approach as indicated by allowing the parameter R to be greater than 0 and less than 0 respectively.

5.2 Analysis

To apply HAM in this situation we let

$$f_0(\eta) = \frac{3}{2}\eta - \frac{1}{2}\eta^3 \tag{5.2.1}$$

and take as the linear operator

$$L = \frac{\partial^4}{\partial \eta^4}. \tag{5.2.2}$$

The generalized homotopy is then defined as

$$H(\eta, q, \hbar) = (1 - q)L\left[F(\eta, q) - f_0(\eta)\right] - q\hbar N\left[F(\eta, q)\right] \tag{5.2.3}$$

where $F(\eta, q) = \sum_{n=0}^{\infty} q^n f_n(\eta)$ and the nonlinear operator is

$$N\left[F(\eta, q)\right] = \frac{\partial^4}{\partial \eta^4}F(\eta, q) - M^2\frac{\partial^2}{\partial \eta^2}F(\eta, q) - RF(\eta, q)\frac{\partial^3}{\partial \eta^3}F(\eta, q) \tag{5.2.4}$$

$$+R\frac{\partial}{\partial \eta}F(\eta, q)\frac{\partial^2}{\partial \eta^2}F(\eta, q) + R\eta\frac{\partial^3}{\partial \eta^3}F(\eta, q) + \left(2 + \frac{1}{\rho}\right)R\frac{\partial^2}{\partial \eta^2}F(\eta, q).$$

72

Each $f_i(\eta)$ in the solution $F(\eta, 1)$ is then the solution of the ordinary differential equation given by

$$f_i^{(iv)}(\eta) = \hbar\left(2R - M^2\right) f_{i-1}'' + \hbar \sum_{k=0}^{i-1} \left(f_k' f_{i-1-k}' - f_k f_{i-1-k}'''\right) + \frac{\hbar R}{\rho} f_{i-1}'' \quad (5.2.5)$$

$$+\hbar R\eta f_{i-1}''' + (1+\hbar) f_{i-1}^{(iv)}.$$

The first two terms of the solution are

$$f_0(\eta) = \frac{3}{2}\eta - \frac{1}{2}\eta^3 \quad (5.2.6)$$

$$f_1(\eta) = \frac{1}{40}M^2\hbar\eta - \frac{19}{280}R\hbar\eta - \frac{1}{20}M^2\hbar\eta^3 + \frac{39}{280}M^2\hbar\eta^3 + \frac{1}{40}M^2\hbar\eta^5 \quad (5.2.7)$$

$$-\frac{3}{40}R\hbar\eta^5 + \frac{1}{280}R\hbar\eta^7 - \frac{1}{40\rho}R\hbar\eta + \frac{1}{20\rho}R\hbar\eta^3 - \frac{1}{40\rho}R\hbar\eta^5.$$

To find the values of $\hbar$ that provide convergent solutions we solved

$$\left(F_{20}^k(\eta, \hbar) - F_{18}^k(\eta, \hbar)\right)\big|_{\eta=0} = 0 \quad (5.2.8)$$

where $F_a^k(\eta, \hbar)$ represents the kth derivative with respect to η of the a-term solution. Table 5.2.1 lists the choices of k that provide the given value of $\hbar$ for the choices of the parameters M and R with $\rho = 1$ along with the solution's associated residual. For variable ρ, Table 5.2.2, with $M = 1$ and $R = 1$, gives the 20 term solutions with their corresponding values of $\hbar$ and residuals.

Table 5.2.1: Values of $\hbar$ for the HAM solution of the MHD flow with $\rho = 1$ along with corresponding residuals.

M	R	K	$\hbar$	Residual
2	-3/2	1	-0.745	10^{-9}
2	1/2	1	-0.861	10^{-9}
2	2	1	-1.224	10^{-9}
0	2	1	-1.324	10^{-7}
1	2	1	-1.319	10^{-8}
2	2	1	-1.224	10^{-9}
3	2	1	-1.224	10^{-9}
4	2	1	-0.869	10^{-9}
0	-1	5	-0.843	10^{-10}
4	-1	5	-0.611	10^{-6}
6	-1	5	-0.406	10^{-4}

Table 5.2.2: Values of $\hbar$ for the HAM solution of the MHD flow for variable ρ along with corresponding residuals.

ρ	1/6	1/4	1/2	2
$\hbar$	-1.220	-1.175	-1.072	-1.011
Residual	10^{-8}	10^{-9}	10^{-10}	10^{-10}

5.3 Results

Figures 5.1 - 5.8 show the velocity profiles for the solution under certain choices of the parameters R, M, and ρ. From the figures the dependency of the solution on the density of the fluid can be determined. Figure 5.1 shows that an increase in fluid density corresponds to a decrease in the velocity in the y-direction. Figure 5.2 shows that the initial velocity in the x-direction decreases with an increase in fluid density but increases as $\eta \to \infty$. The same type of behavior can be seen in reference to the x and y velocity profiles for fixed fluid density and decreasing R. A decrease in R could be attributed to an increase in the kinematic viscosity, a decrease in the distance between the plates or a decrease in the speed at which the plates move. In contrast, for the case when the plates are allowed to move apart, an increase in M results in velocity profiles similar to that of decreasing R. Therefore, increasing the electrical conductivity of the fluid or increasing the magnetic field results in a monotonic decrease in the velocity in the y-direction. The initial velocity in the x-direction decreases and increases as with an increase in M. Figures 5.7 and 5.8 show the same type of behavior when the plates are allowed to move together.

5.4 Figures
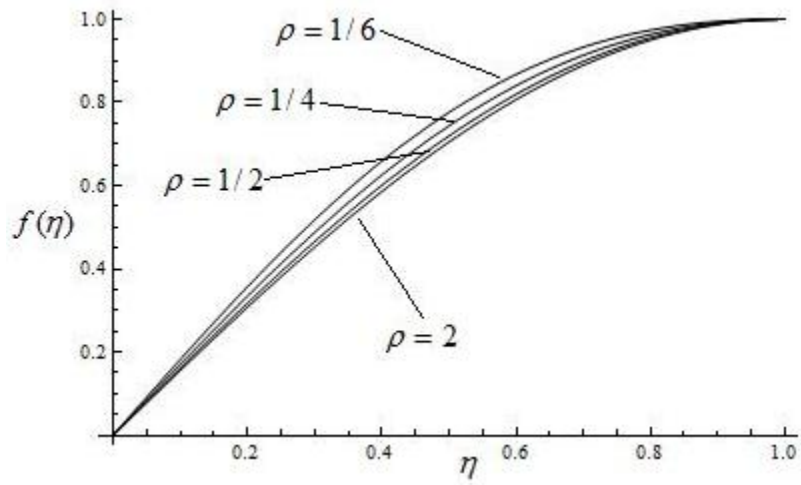
Figure 5.1: Solution showing the velocity in the y-direction for $R = 1$, $M = 1$ and variable ρ.
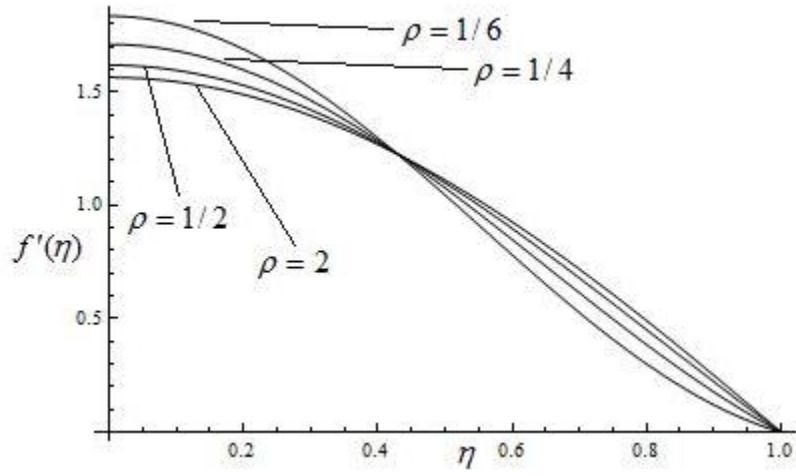
Figure 5.2: Solution showing the velocity in the x-direction for $R = 1$, $M = 1$ and variable ρ.
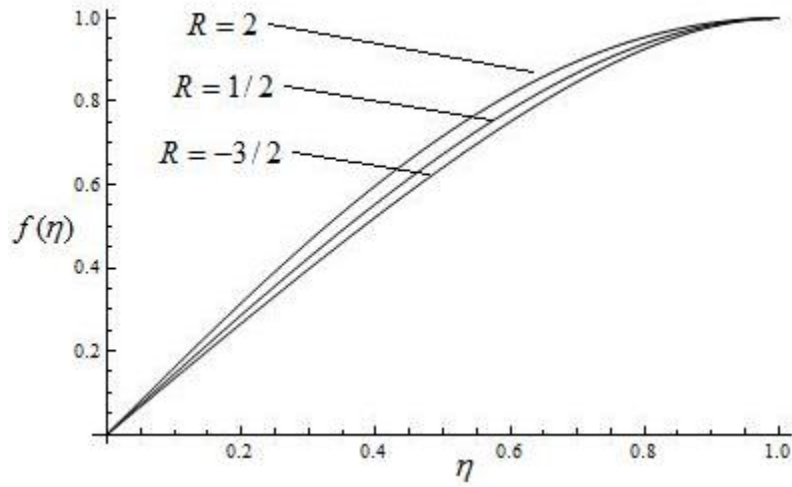
Figure 5.3: Solution showing the velocity in the y-direction for variable R, $M = 2$ and $\rho = 1$.
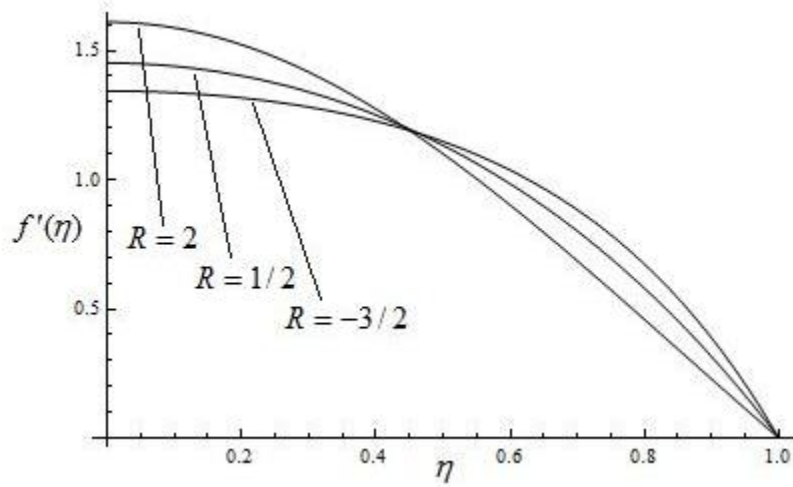
Figure 5.4: Solution showing the velocity in the x-direction for variable R, $M = 2$ and $\rho = 1$.
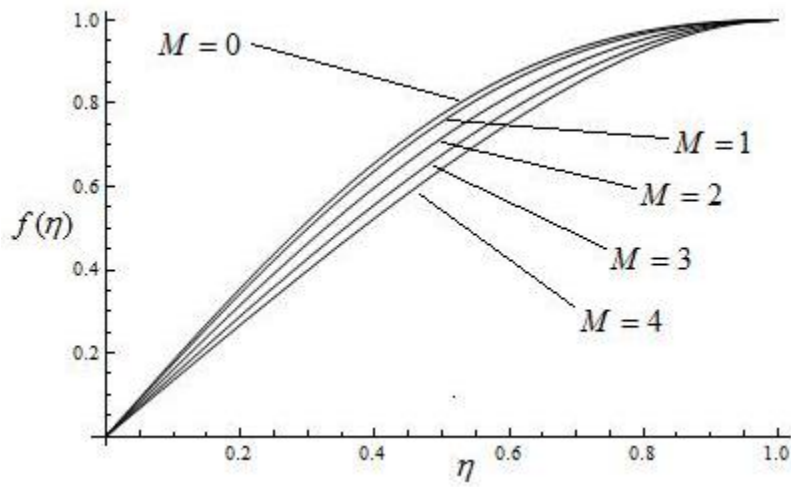
Figure 5.5: Solution showing the velocity in the y-direction for $R = 2$, variable M and $\rho = 1$.
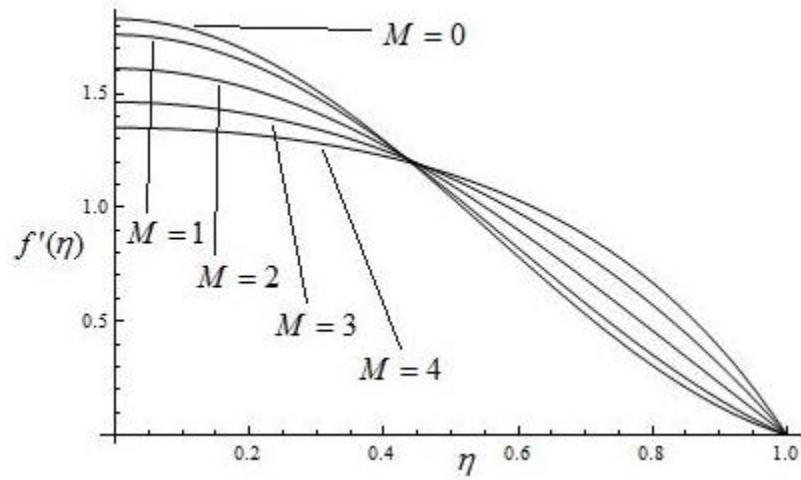
Figure 5.6: Solution showing the velocity in the x-direction for $R = 2$, variable M and $\rho = 1$.
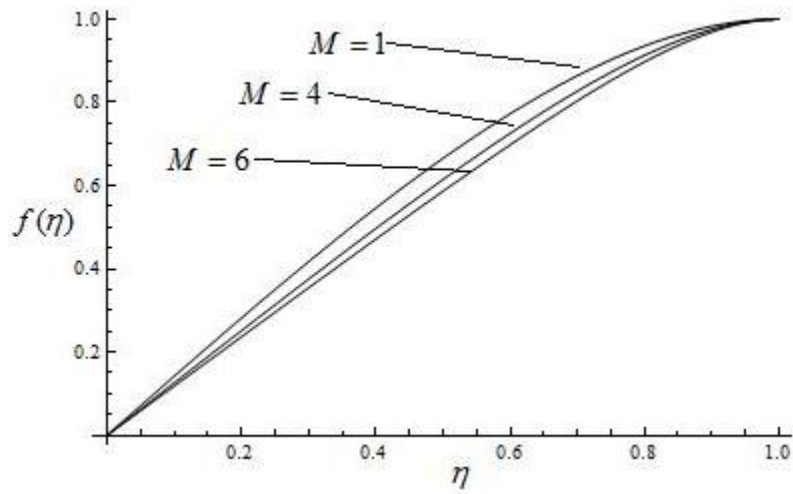
Figure 5.7: Solution showing the velocity in the y-direction for $R = -1$, variable M and $\rho = 1$.
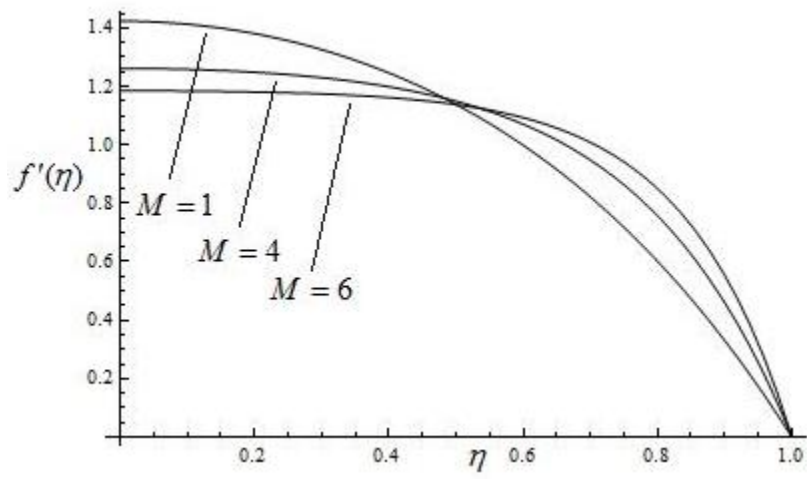
Figure 5.8: Solution showing the velocity in the x-direction for $R = -1$, variable M and $\rho = 1$.

CHAPTER 6. MAGNETO-HYDRODYNAMIC FLUID; 3D
FLOW OVER A ROTATING SPHERE

6.1 Statement of the Problem

Rotating flows over stationary or rotating bodies are known to have applications in several diverse areas such as meteorology, geophysical and cosmical fluid dynamics, gaseous and nuclear reactors and so on. Here we investigate the problem given in Sweet, Van Gorder and Vajravelu [10] which deals with a three-dimensional MHD rotating flow of a viscous fluid over a rotating sphere near the equator. An application of this would be in sun spot development or the structure of a rotating magnetic star. In what follows, the Navier-Stokes equations in spherical polar coordinates are reduced to a coupled system of nonlinear partial differential equations. Self-similar solutions are obtained for the steady state system, resulting from a coupled system of nonlinear ordinary differential equations. Analytical solutions are obtained and are used to study the effects of the magnetic field and the suction/injection parameter on the flow characteristics. The analytical solutions agree well with the numerical solutions of Chamkha, Takhar and Nath [1]. Furthermore, the obtained analytical solutions are extended so as to include the transient flow near the steady state solution.

In Chamkha, Takhar and Nath [1] the authors used spherical polar coordinates (r, θ, ϕ)

with the origin at the center of the sphere of radius a and $\theta = 0$ the axis of rotation. The motion is assumed to be axisymmetric (independent of the azimuthal angle ϕ). The magnetic field B is assumed to be applied in the r-direction and that only the applied magnetic field contributes to the Lorentz force. The components of the Lorentz force in the θ and ϕ directions are $-Mv$ and $-M(w - r\sin\theta)$ respectively where $M = (\sigma B^2)/(\rho\Omega_f)$ is the magnetic parameter that depends on the strength of the magnetic field B, the electrical conductivity of the fluid σ, the fluid density ρ and the angular velocity of the distant fluid Ω_f. The continuity and the Navier-Stokes equations governing the unsteady rotating flow over a rotation sphere are

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(v\sin\theta\right) = 0 \tag{6.1.1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial\theta} - \left(\frac{V^2 + w^2}{r}\right) = -\frac{\partial p}{\partial r} + \mathrm{Re}^{-1}\left(\nabla^2 u - \frac{2u}{r^2} - \frac{2}{r^2}\frac{\partial v}{\partial\theta} - \frac{2v\cot\theta}{r^2}\right) \tag{6.1.2}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial\theta} + \frac{uV - w^2\cot\theta}{r} = -\frac{1}{r}\frac{\partial p}{\partial\theta} + \mathrm{Re}^{-1}\left(\nabla^2 v + \frac{2}{r^2}\frac{\partial u}{\partial\theta} - \frac{v}{r^2\sin^2\theta}\right) - Mv \tag{6.1.3}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial\theta} + \frac{uw + vw\cot\theta}{r} = \mathrm{Re}^{-1}\left(\nabla^2 w - \frac{w}{r^2\sin^2\theta}\right) - M(w - r\sin\theta) \tag{6.1.4}$$

where

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right). \tag{6.1.5}$$

In the above u, v and w represent the dimensionless velocity components in the r, θ and ϕ directions respectively. The dimensionless velocity components are obtained by dividing the dimensional velocity components by $a\Omega_f$. The boundary conditions are the no-slip conditions

on the surface and the free-stream conditions far away from the surface. Therefore

$$u(1,\theta,t) = u_0, \quad v(1,\theta,t) = 0, \quad w(1,\theta,t) = \lambda(1+\varepsilon)r\sin\theta \tag{6.1.6}$$

$$u(\infty,\theta,t) = 0, \quad v(\infty,\theta,t) = 0, \quad w(\infty,\theta,t) = r\sin\theta. \tag{6.1.7}$$

Here Re= $a^2\Omega_f/v$ is the Reynolds number, v is the kinematic viscosity, $\lambda = \Omega_b/\Omega_f$ is the ratio the angular velocity of the sphere to the angular velocity of the distant fluid, $t = \Omega_f t^*$ is the dimensionless time, $r = r^*/a$ is the dimensionless radial distance, $M = \sigma B^2/\rho\Omega_f$ =Ha/Re is the magnetic parameter, Ha= $\sigma B^2 a^2/\mu$ is the Hartmann number, μ is the viscosity, a is the radius of the sphere, ε is a dimensionless constant and u_0 is the velocity at the surface of the sphere along the r-direction. $u_0 < 0$ corresponds to (radial) suction and $u_0 > 0$ corresponds to (radial) injection at the surface of the sphere.

To simplify the governing equations we transform the radial coordinate as $\eta = \sqrt{\text{Re}}(r-1)$ and redefine the velocity components and pressure as

$$u(r,\theta,t) = \frac{1}{\sqrt{\text{Re}}}U(\eta,\theta,\tau), \quad v(r,\theta,t) = V(\eta,\theta,\tau), \quad w(r,\theta,t) = W(\eta,\theta,\tau) \tag{6.1.8}$$

where $t = \tau$ and $p = P$. Under these transformations 6.1.1 - 6.1.4 become

$$U_\eta + V_\theta + V\cot\theta = 0 \tag{6.1.9}$$

$$P = \frac{1}{2}\sin^2\theta \tag{6.1.10}$$

$$V_\tau + UV_\eta + VV_\theta - W^2\cot\theta = -\sin\theta\cos\theta + V_{\eta\eta} - MV \tag{6.1.11}$$

83

$$W_\tau + UW_\eta + VW_\theta + VW \cot\theta = W_{\eta\eta} - M(W - \sin\theta). \tag{6.1.12}$$

Since we are interested in the flow near the equator $(\theta \approx \pi/2)$ we assume that

$$U(\eta, \theta, \tau) = H(\eta, \tau), \quad V(\eta, \theta, \tau) = (\theta - \pi/2)F(\eta, \tau), \quad W(\eta, \theta, \tau) = G(\eta, \tau). \tag{6.1.13}$$

Substitution into the above relations gives

$$H_\eta + F + (\theta - \pi/2)(\cot\theta)F = 0 \tag{6.1.14}$$

$$P = \frac{1}{2}\sin^2\theta \tag{6.1.15}$$

$$(\theta - \pi/2)F_\tau + (\theta - \pi/2)HF_\eta + (\theta - \pi/2)F^2 - (\cot\theta)G^2 \tag{6.1.16}$$

$$= -\sin\theta\cos\theta - (\theta - \pi/2)F_{\eta\eta} - (\theta - \pi/2)MF$$

$$G_\tau + HG_\eta - \cot\theta(\theta - \pi/2)FG = G_{\eta\eta} - M(G - \sin\theta). \tag{6.1.17}$$

Equating like powers of $(\theta - \pi/2)$ in 6.1.14 gives that either $H_\eta + F = 0$ or $F = 0$. Ignoring the trivial solution we take $F = -H_\eta = -H'$. Substitution into 6.1.16 gives

$$-(\theta - \pi/2)\frac{\partial H'}{\partial\tau} - (\theta - \pi/2)HH'' + (\theta - \pi/2)\left(H'\right)^2 - \cot\theta G^2 \tag{6.1.18}$$

$$= -\sin\theta\cos\theta - (\theta - \pi/2)H''' + (\theta - \pi/2)MH'.$$

Equating like powers of $\theta - \pi/2$ gives

$$-\frac{\partial H'}{\partial \tau} - HH'' + (H')^2 = -H''' + MH' \quad \text{or} \quad -(\cot \theta)G^2 = -\sin \theta \cos \theta. \qquad (6.1.19)$$

From the second equation $G^2 - 1 = 0$ and thus we can write the first as

$$H''' - HH'' + (H')^2 - MH' + G^2 - 1 - \frac{\partial H'}{\partial \tau} = 0. \qquad (6.1.20)$$

Again using the relation $F = -H'$ in 6.1.17 and equating like powers of $\theta - \pi/2$ gives

$$\frac{\partial G}{\partial \tau} + HG' = G'' - M(G - 1) \quad \text{or} \quad -\cot \theta H' G = 0. \qquad (6.1.21)$$

Ignoring the second trivial case we get

$$G'' - HG' - M(G - 1) - \frac{\partial G}{\partial \tau} = 0 \qquad (6.1.22)$$

The boundary conditions become

$$H(0, \tau) = A = u_0 \sqrt{\text{Re}}, \quad H'(0, \tau) = 0, \quad G(0, \tau) = \lambda(1 + \varepsilon), \qquad (6.1.23)$$

$$H'(\infty, \tau) = 0, \quad G(\infty, \tau) = 1. \qquad (6.1.24)$$

6.2 Steady Flow

Here we consider the steady state flow which comes from the assumption that $\frac{\partial G}{\partial \tau} = \frac{\partial H'}{\partial \tau} = \varepsilon = 0$. We thus study the problem

$$H''' - HH'' + (H')^2 - MH' + G^2 - 1 = 0 \tag{6.2.1}$$

$$G'' - HG' - M(G - 1) = 0 \tag{6.2.2}$$

subject to

$$H(0) = A, \quad H'(0) = 0, \quad G(0) = \lambda, \quad H'(\infty) = 0, \quad G(\infty) = 1. \tag{6.2.3}$$

As we are applying HAM to a system of coupled differential equations we have two each of initial guesses, linear operators and generalized homotopies. For 6.2.1 we take as the initial guess $h_0(\eta) = A$ while for 6.2.2 we take $g_0(\eta) = 1 + (\lambda - 1)e^{-\eta}$ in agreement with the boundary data. Assuming the analytic solutions to be of the form

$$H(\eta, q) = \sum_{n=0}^{\infty} q^n h_n(\eta) \quad \text{and} \quad G(\eta, q) = \sum_{n=0}^{\infty} q^n g_n(\eta) \tag{6.2.4}$$

we take as the linear operators for 6.2.1 and 6.2.2 to be

$$L_1 = \frac{\partial^3}{\partial \eta^3} - \frac{\partial}{\partial \eta} \quad \text{and} \quad L_2 = \frac{\partial^2}{\partial \eta^2} + \frac{\partial}{\partial \eta} \tag{6.2.5}$$

respectively. The two generalized homotopies for the coupled system are defined to be

$$H_1(\eta, q, \hbar) = (1 - q)L_1\left[H(\eta, q) - h_0(\eta)\right] - q\hbar N_1\left[H(\eta, q), G(\eta, q)\right] \qquad (6.2.6)$$

where

$$N_1\left[H(\eta, q), G(\eta, q)\right] = \frac{\partial^3}{\partial\eta^3}H(\eta, q) - H(\eta, q)\frac{\partial^2}{\partial\eta^2}H(\eta, q) + \left(\frac{\partial}{\partial\eta}H(\eta, q)\right)^2 \qquad (6.2.7)$$

$$+ (G(\eta, q))^2 - 1 - M\frac{\partial}{\partial\eta}H(\eta, q)$$

is the nonlinear operator corresponding to 6.2.1 and then take

$$H_2(\eta, q, \hbar) = (1 - q)L_2\left[G(\eta, q) - g_0(\eta)\right] - q\hbar N_2\left[H(\eta, q), G(\eta, q)\right] \qquad (6.2.8)$$

where

$$N_2\left[H(\eta, q), G(\eta, q)\right] = \frac{\partial^2}{\partial\eta^2}G(\eta, q) - H(\eta, q)\frac{\partial}{\partial\eta}G(\eta, q) - M(G(\eta, q) - 1) \qquad (6.2.9)$$

is the nonlinear operator corresponding to 6.2.2. In order to solve the coupled system we take an iterative approach. To obtain solutions we first solve the ith order deformation of 6.2.6 for the ith term in the solution of $H(\eta, q)$ and then plug it into the ith order deformation of 6.2.8 to find the ith term in the solution of $G(\eta, q)$. The ith order deformation corresponding

to 6.2.6 is given by

$$h_i''' - h_i' = \hbar \sum_{k=0}^{i-1} g_k g_{i-1-k} - (1 + \hbar M)h_{i-1}' + \hbar \sum_{k=0}^{i-1} h_k' h_{i-1-k}' - \hbar \sum_{k=0}^{i-1} h_k h_{i-1-k}'' \qquad (6.2.10)$$

$$+(1 + \hbar)h_{i-1}''' + \chi_i \hbar$$

for all $i > 0$ where

$$\chi_i = \begin{cases} 0, i = 1, \\ 1, i > 1. \end{cases} \qquad (6.2.11)$$

Similarly the ith order deformation of 6.2.8 is given by

$$g_i'' + g_i' = g_{i-1}' + (1 + \hbar)g_{i-1}'' - \hbar \sum_{k=0}^{i-1} h_k g_{i-1-k}' - \hbar M g_{i-1} + \chi_i \hbar M \qquad (6.2.12)$$

for all $i > 0$. To find the values of $\hbar$ that provide convergent solutions we solve

$$\left(\frac{\partial}{\partial} H_K(\eta, \hbar) + \frac{\partial}{\partial \eta} G_K(\eta, \hbar) \right) \Bigg|_{\eta=0} = 0 \qquad (6.2.13)$$

for $\hbar$ where $H_K(\eta, \hbar)$ and $G_K(\eta, \hbar)$ denote the Kth order HAM approximations. Table 6.2.1 provides the optimal values of $\hbar$ chosen for the solution under the given choice of the parameters along with the number of terms in the solution and the maximal residual over the domain. For brevity we include the first two terms of the solutions, which are explicitly given by:

Table 6.2.1: Values of $\hbar$, number of terms and residual in the approximate solutions of the steady state flow for the shown choices of parameters.

A	M	λ	Number of Terms	$\hbar$	Residual
1/2	1	-1/4	30	-0.241	10^{-4}
0	1	-1/4	30	-0.380	10^{-7}
-1/2	1	-1/4	30	-0.443	10^{-6}
-1	1	-1/4	30	-0.425	10^{-6}
1/2	1	-1/2	28	-0.258	10^{-4}
1/2	1	0	28	-0.271	10^{-4}
1/2	1	1/2	28	-0.292	10^{-6}
1/2	1	3/4	28	-0.305	10^{-6}
1/2	3/2	0	28	-0.389	10^{-7}
1/2	2	0	30	-0.350	10^{-6}
1/2	5/2	0	30	-0.226	10^{-5}

$$h_0(\eta) = A \tag{6.2.14}$$

$$h_1(\eta) = \frac{5}{6}\hbar - \frac{1}{6}\hbar e^{-2\eta} - \frac{2}{3}\hbar e^{-\eta} - \hbar\eta e^{-\eta} - \frac{2}{3}\hbar\lambda + \frac{1}{3}\hbar\lambda e^{-\eta} \tag{6.2.15}$$

$$+\hbar\lambda\eta e^{-\eta} - \frac{1}{6}\hbar\lambda^2 - \frac{1}{6}\hbar\lambda^2 e^{-2\eta} + \frac{1}{3}\hbar\lambda^2 e^{-\eta}$$

$$g_0(\eta) = 1 + (\lambda - 1)e^{-\eta} \tag{6.2.16}$$

$$g_1(\eta) = \hbar\eta e^{-\eta} + A\hbar\eta e^{-\eta} - M\hbar\eta e^{-\eta} - \hbar\lambda\eta e^{-\eta} - A\hbar\lambda\eta e^{-\eta} + M\hbar\lambda\eta e^{-\eta}. \tag{6.2.17}$$

6.3 Transient Flow

Let (G_*, H_*) denote the solutions for the equations governing the steady flow as were obtained in section 6.2. In order to obtain analytical solutions to the transient flow problem 6.1.20 and 6.1.22 - 6.1.24 we assume a solution $(\overline{G}, \overline{H})$ which takes the form

$$\overline{G}(\eta, \tau) = G_*(\eta) + e^{-\tau}G_1(\eta) + e^{-2\tau}G_2(\eta) + \dots \tag{6.3.1}$$

$$\overline{H}(-\eta, \tau) = H_*(\eta) + e^{-\tau}H_1(\eta) + e^{-2\tau}H_2(\eta) + \dots. \tag{6.3.2}$$

Placing these solution expressions into transient flow problem and considering the coupled linear differential equations obtained by equating powers of $e^{-\tau}$ we obtain the boundary value problems given by

$$H_*''' - H_*H_*'' + (H_*')^2 - MH_*' + G_*' - 1 = 0 \tag{6.3.3}$$

90

$$G_*''' - H_* G_*' - M(G_* - 1) = 0 \tag{6.3.4}$$

$$H_*(0) = A, \quad H_*'(0) = 0, \quad G_*(0) = \lambda, \quad H_*'(\infty) = 0, \quad G_*(\infty) = 1 \tag{6.3.5}$$

for the order 0 case,

$$H_1''' + H_* H_1'' - H_*'' H_1 + 2 H_*' H_1' - M H_1' + 2 G_* G_1 + H_1' = 0 \tag{6.3.6}$$

$$G_1'' + H_* G_1' + G_*' H_1 - M G_1 + H_1' = 0 \tag{6.3.7}$$

$$H_1(0) = 0, \quad H_1'(0) = 0, \quad G_1(0) = 0, \quad H_1'(\infty) = 0, \quad G_1(\infty) = 0 \tag{6.3.8}$$

for the order 1 case and

$$H_2''' - H_*'' H_2 - H_* H_2'' - H_1 H_1'' + 2 H_*' H_2' + (H_1')^2 - M H_2' + 2 G_* G_2 + G_1^2 + 2 H_2' = 0 \tag{6.3.9}$$

$$G_2'' + H_* G_2' + G_*' H_2 H_1 G_1' - M G_2 + 2 H_2' = 0 \tag{6.3.10}$$

$$H_2(0) = 0, \quad H_2'(0) = 0, \quad G_2(0) = 0, \quad H_2'(\infty) = 0, \quad G_2(\infty) = 0 \tag{6.3.11}$$

for the order 2 case. Continuing in this manner it can be seen that the individual coupled equations can be solved successively to obtain approximate analytical solutions of desired accuracy.

To obtain H_1 and G_1 specifically we employ HAM under the assumption that the initial approximations are $h_0^1(\eta) = 2e^{-\eta} - e^{-2\eta} - 1$ and $g_0^1(\eta) = 0$ respectively. The auxillary linear operators are as described in section 6.2 and we took as the "non-linear" operators to be those described in 6.3.6 and 6.3.7. For H_1 and G_1 it was found that taking $\hbar = -65/100$

91

provided the optimal solution. For ease of calculation we took H_* and G_* to be the four term HAM solutions with $\hbar = -53/100$.

6.4 Results

Figures 6.1 and 6.2 show that as the ratio of the angular velocity of the sphere to the angular velocity of the distant fluid decreases, the radial velocity decreases while the meridional velocity increases. This behavior is seen even in allowing the sphere and fluid to rotate in opposite directions which can be seen for negative values of λ. Figures 6.3 shows that the radial velocity decreases with decreasing injection and increasing suction. Figure 6.4 shows quite the opposite behavior in the rotational velocity. As injection and suction increase, the rotational velocity increases. Figures 6.5 - 6.7 show that an increase in the value of the parameter M causes an increase in the radial and rotational velocities and a decrease in the meridional velocity. An increase in the parameter M can be attributed to an increase in the electrical conductivity of the fluid or on increase in the magnetic field. An increase in M could also result from a decrease in the fluid density or a decrease in the angular velocity of the fluid away from the equator. Figures 6.8 - 6.10 show the behavioral dependence of the velocities on τ under the assumption that $A = 1/2$, $\lambda = 3/4$ and $M = 1$.
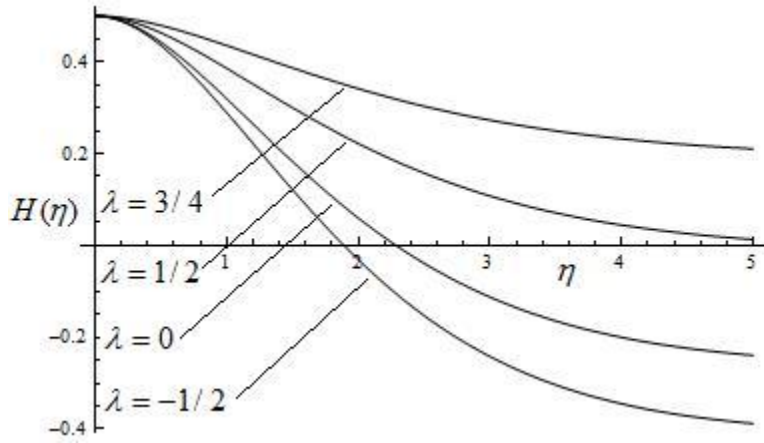
Figure 6.1: Profiles of the radial velocity $H(\eta)$ for $A = 1/2$ and $M = 1$ for variable λ.

Figure 6.2: Profiles of the meridional velocity $H'(\eta)$ for $A = 1/2$ and $M = 1$ for variable λ.
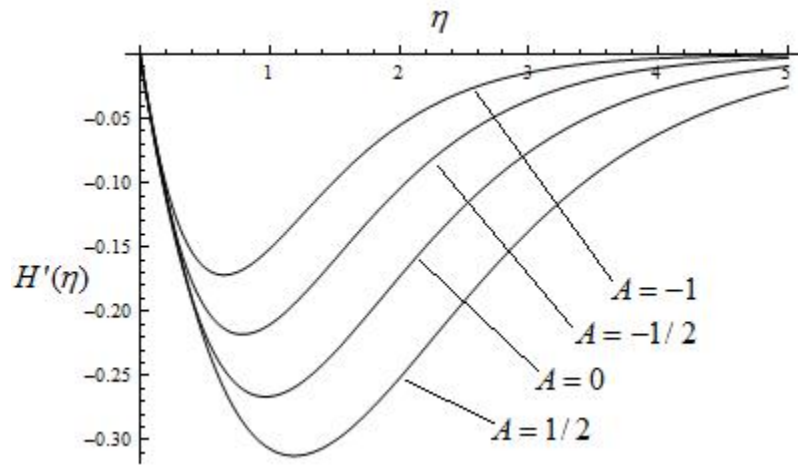
Figure 6.3: Profiles of the negative meridional velocity $H'(\eta)$ for $\lambda = -1/4$ and $M = 1$ for variable A.
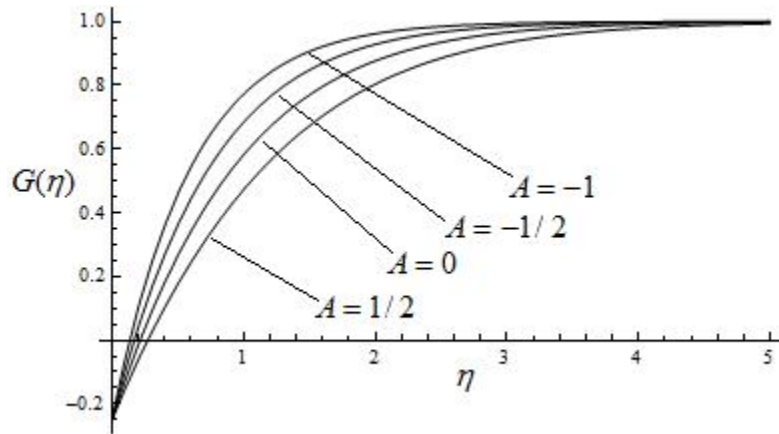
Figure 6.4: Profiles of the rotational velocity $G(\eta)$ for $\lambda = -1/4$ and $M = 1$ for variable A.
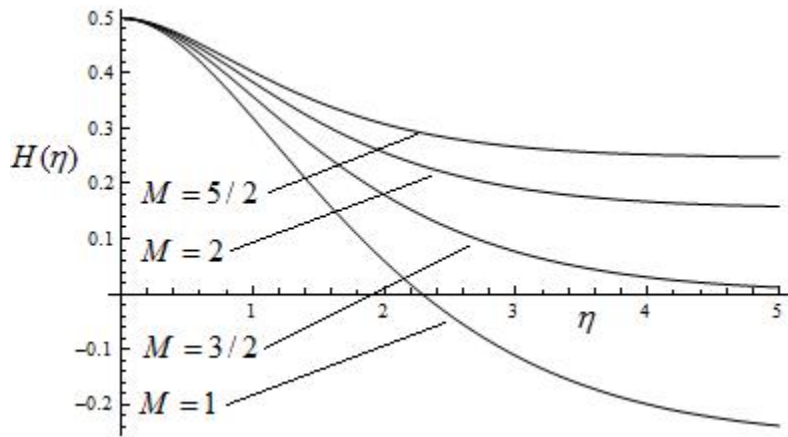
Figure 6.5: Profiles of the radial velocity $H(\eta)$ for $\lambda = 0$ and $A = 1/2$ for variable M.
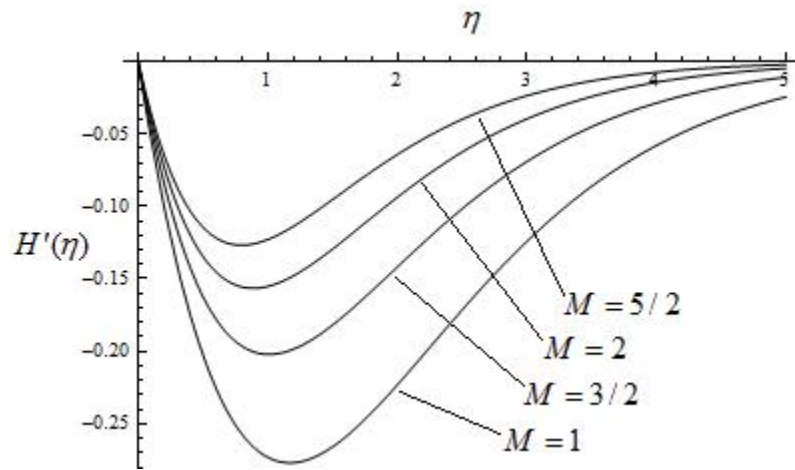
Figure 6.6: Profiles of the negative meridional velocity $H'(\eta)$ for $\lambda = 0$ and $A = 1/2$ for variable M.
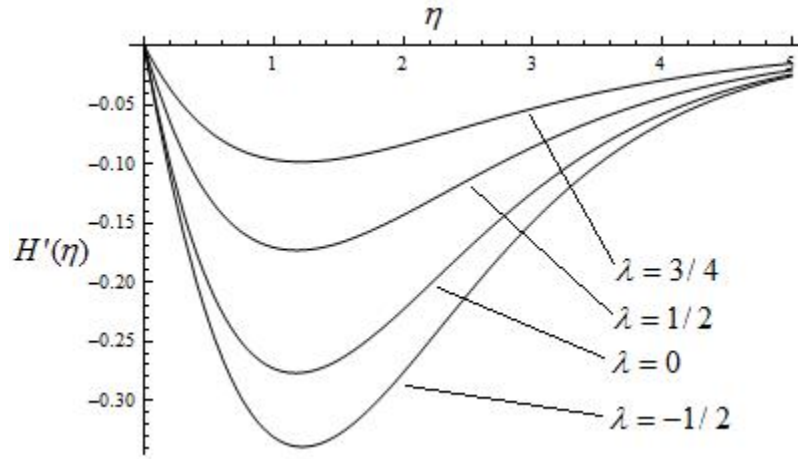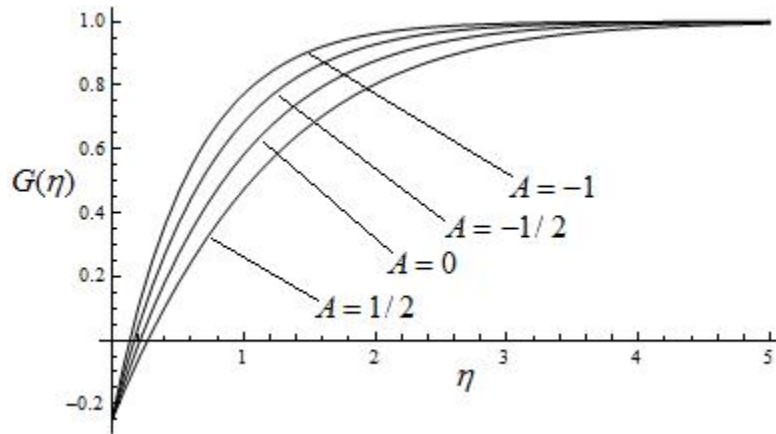

Figure 6.7: Profiles of the rotational velocity $G(\eta)$ for $\lambda = 0$ and $A = 1/2$ for variable M.
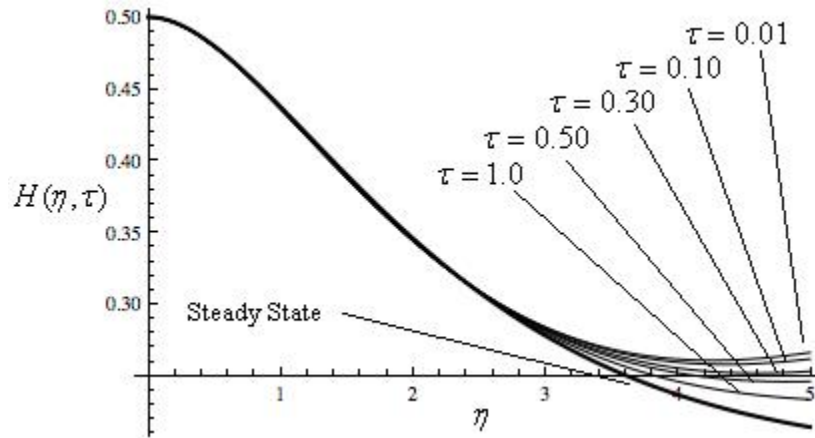
Figure 6.8: Profiles of the radial velocity $H(\eta)$ for various values of τ under the assumption that $A = 1/2$, $\lambda = 3/4$ and $M = 1$.
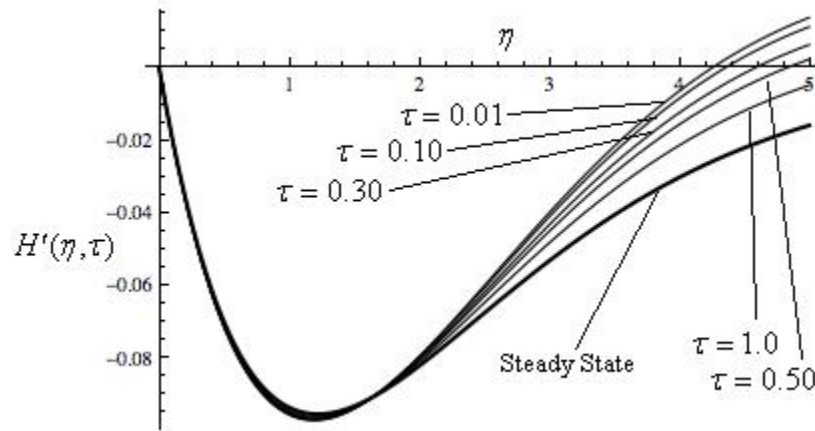
Figure 6.9: Profiles of the negative meridional velocity $H'(\eta)$ for various values of τ under the assumption that $A = 1/2$, $\lambda = 3/4$ and $M = 1$.

97
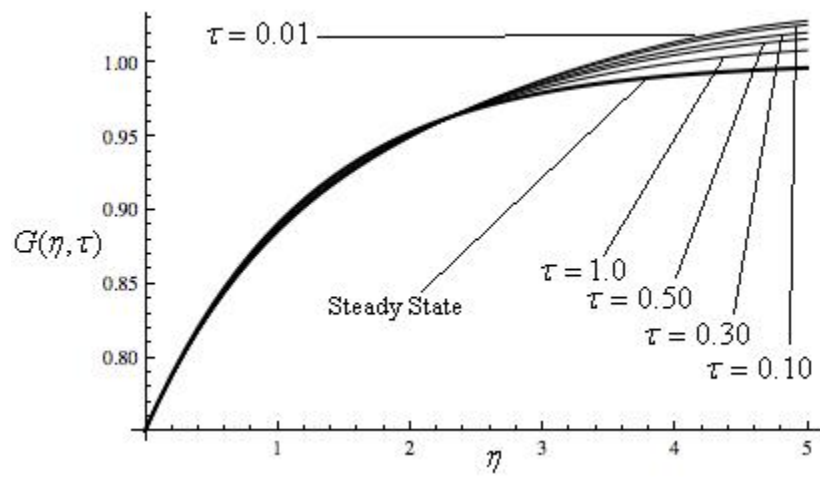
Figure 6.10: Profiles of the rotational velocity $G(\eta)$ for various values of τ under the assumption that $A = 1/2$, $\lambda = 3/4$ and $M = 1$.

CHAPTER 7. AXI-SYMMETRIC FLOW BETWEEN TWO INFINITE STRETCHING DISKS

7.1 Statement of the Problem

The study of the flow induced by a stretching boundary is of interest to those concerned with extrusion processes in the plastic and metal industries Fisher [9], Altan, Of and Gegel [33] and Tadmor and Klein [38]. The surface stretching problem was first proposed and analyzed by Sakiadis [3-4] based on boundary layer assumption, where the solution was not an exact solution of the Navier-Stokes equations. An exact solution of the two-dimensional Navier-Stokes equations for a stretching sheet where the surface stretching velocity was proportional to the distance from the slot was given in Crane [19]. The problem was extended to allow for a power-law stretching velocity by Kuiken [15] and Banks [37] and by Gupta et al. [23] to include suction and injection at the wall. The stretching boundary problem was then extended by Wang [8] to the three-dimensional case. Furthermore, by introducing a control parameter, similarity solutions for the Navier-Stokes equations were obtained numerically. It has also been shown that allowing this control parameter to vary from 0 to 1 changed the two-dimensional stretching problem into an axi-symmetric stretching problem.

Here we consider the problem given in Van Gorder, Sweet and Vajravelu [26] which is

based on the work found in Fang and Zhang [34]. That is, we consider the axi-symmetric flow between two stretchable disks, separated vertically by a distance d. Each of these disks is stretching radially with velocity proportional to the radius. the lower disk is embedded in the $z = 0$ plane, while the upper disk is embedded within the $z = d$ plane. We take the ratio of the stretching velocity of the upper disk to the lower disk to be γ. In the case of incompressible fluid without body forces and based on the axi-symmetric flow assumption, the steady Navier-Stokes equations in cylindrical coordinates (as can be found in F.M. White [13]) are given by

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{\partial}{\partial z}u_z = 0 \tag{7.1.1}$$

$$u_r\frac{\partial}{\partial r}u_r + u_z\frac{\partial}{\partial za}u_r = -\frac{1}{\rho}\frac{\partial}{\partial r}p + v\left(\frac{\partial^2}{\partial r^2}u_r + \frac{1}{r}\frac{\partial}{\partial r}u_r + \frac{\partial^2}{\partial z^2}u_r - \frac{u_r}{r^2}\right) \tag{7.1.2}$$

$$u_r\frac{\partial}{\partial r}u_z + u_z\frac{\partial}{\partial z}u_z = -\frac{1}{\rho}\frac{\partial}{\partial z}p + v\left(\frac{\partial^2}{\partial r^2}u_z + \frac{1}{r}\frac{\partial}{\partial r}u_z + \frac{\partial^2}{\partial z^2}u_z\right) \tag{7.1.3}$$

where (u_r, u_z) is the velocity field, p is the fluid pressure, ρ is the fluid density and v is the kinematic viscosity. To simplify the governing equations we follow Fang and Zhang [34] and introduce the similarity variable $\eta = z/d$ and the similarity functions

$$u_r = arF(\eta), \quad u_z = adH(\eta), \quad p = a\rho v\left(P(\eta) + \frac{\beta}{4}\frac{r^2}{d^2}\right) \tag{7.1.4}$$

where a is a parameter corresponding to the disk stretching strength. Substitution of 7.1.4 into 7.1.1 - 7.1.3 gives

$$F'' - \beta = R\left(F^2 + F'H\right) \tag{7.1.5}$$

$$H' = -2F \tag{7.1.6}$$

$$P' = 2RFH - 2F' \tag{7.1.7}$$

subject to the boundary conditions

$$F(0) = 1, \ \ H(0) = H(1) = 0, \ \ F(1) = \gamma, \ \ P(0) = 0 \tag{7.1.8}$$

where $R = ad^2/v$ is the wall stretching Reynolds number and γ is the upper disk stretching parameter giving the velocity ratio of the upper disk to the lower disk and β is an unknown parameter. In what follows we assume $0 \le \gamma \le 1$.

Using the relation given in 7.1.6 in 7.1.5 we obtain the nonlinear differential equation for the function H

$$H''' - \beta - RHH'' + \frac{R}{2}(H')^2 = 0 \tag{7.1.9}$$

subject to

$$H(0) = H(1) = 0, \ \ H'(0) = -2, \ \ H'(1) = -2\gamma. \tag{7.1.10}$$

Differentiating 7.1.9 then yields

$$H^{(iv)} - RHH''' = 0 \tag{7.1.11}$$

still subject to

$$H(0) = H(1) = 0, \ \ H'(0) = -2, \ \ H'(1) = -2\gamma. \tag{7.1.12}$$

From 7.1.9 we have the additional requirement that

$$H'''(0) = \beta - 2R. \tag{7.1.13}$$

101

As it can be seen, once we know $H(\eta)$ and $F(\eta)$, $P(\eta)$ can be obtained from 7.1.7.

7.2 Analysis

It should be noted that under the assumption that R is small or large, analytic solutions can be obtained by perturbing about R or $1/R$ respectively. Therefore, we apply HAM in this situation to find analytic approximations for intermediate values of R.

7.2.1 R between 0 and 10

Here we apply HAM to find analytic approximations for values of R between 0 and 10 over the domain $\eta \in [0, 1]$. To apply HAM we assume an initial guess of

$$H_0(\eta) = 2\eta(1 - \eta)((1 + \gamma)\eta - 1) \tag{7.2.1}$$

to satisfy the boundary conditions and we take the auxillary linear operator to be

$$L = \frac{\partial^4}{\partial \eta^4} \tag{7.2.2}$$

which agrees with the highest order term in the governing equation. Under these choices the nth order deformation is found to be

$$H_n^{(iv)}(\eta) = (\hbar + \chi_n)H_{n-1}^{(iv)}(\eta) - \hbar R \sum_{k=0}^{n-1} H_k(\eta)H_{n-1-k}'''(\eta). \tag{7.2.3}$$

7.2.2 R between 10 and 100

For values of $10 \leq R \leq 100$ we utilize the modified variable $\theta = \sqrt{R}\eta$ and a modified similarity function $G(\theta) = \sqrt{R}H(\eta)$. With these the governing equation becomes

$$G^{(iv)}(\theta) + G(\theta)G'''(\theta) = 0 \tag{7.2.4}$$

subject to

$$G(0) = G(\sqrt{R}) = G'(0) = G'(\sqrt{R}) = 0. \tag{7.2.5}$$

To apply HAM we take the initial guess to be

$$G_0(\theta) = \frac{2\theta}{R}\left(\sqrt{R} - \theta\right)((1 + \gamma)\theta - \sqrt{R}). \tag{7.2.6}$$

Using the same linear operator as 7.2.2 the nth order deformation becomes

$$G_n^{(iv)}(\theta) = (\hbar + \chi_n)G_{n-1}^{(iv)}(\theta) - \hbar\sum_{k=0}^{n-1}G_k(\theta)G_{n-1-k}'''(\theta). \tag{7.2.7}$$

After finding a sufficient number of terms in the HAM solution we transform the solution into the original independent variable $\eta = \theta/\sqrt{R}$.

7.3 Results

Figures 7.1 - 7.3 show the analytical results obtained via HAM for $F(\eta)$, $H(\eta)$ and $P(\eta)$ for various values of the parameter $\gamma \in [0, 1]$ for $R = 5$. Similar results are shown in Figures 7.4 - 7.6 for $R = 16$. For all choices of the parameters the approximate solutions are accurate

with residual on the order of 10^{-6}. It should also be noted that an analytic expression for β can be easily obtained via the relation given by 7.2.13. As shown in Figures 1 - 6, the stretching parameter of the upper wall influences the velocity distribution with a downward net flow.

7.4 Figures

Figure 7.1: HAM solutions for $F(\eta)$ for various values of the parameter γ when $R = 5$.

Figure 7.2: HAM solutions for $H(\eta)$ for various values of the parameter γ when $R = 5$.

Figure 7.3: HAM solutions for $P(\eta)$ for various values of the parameter γ when $R = 5$.
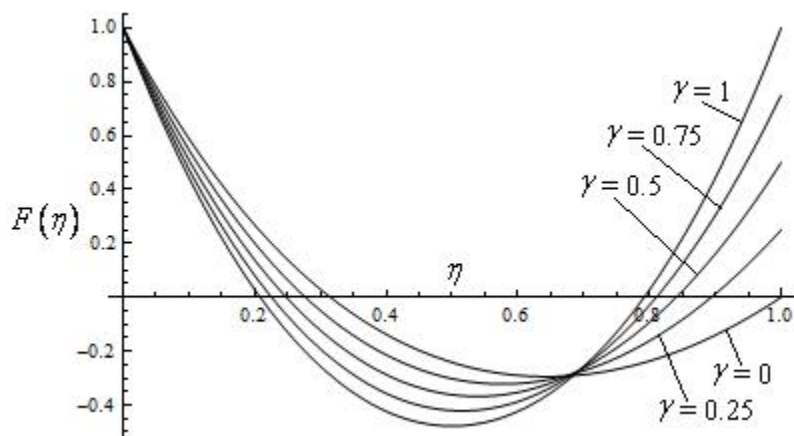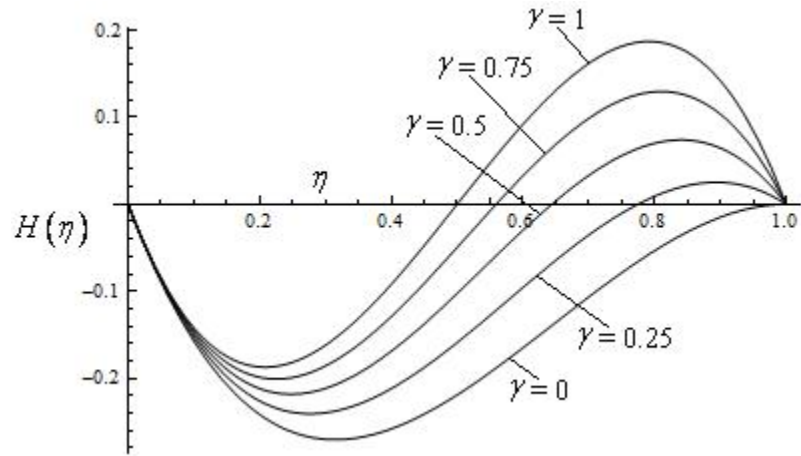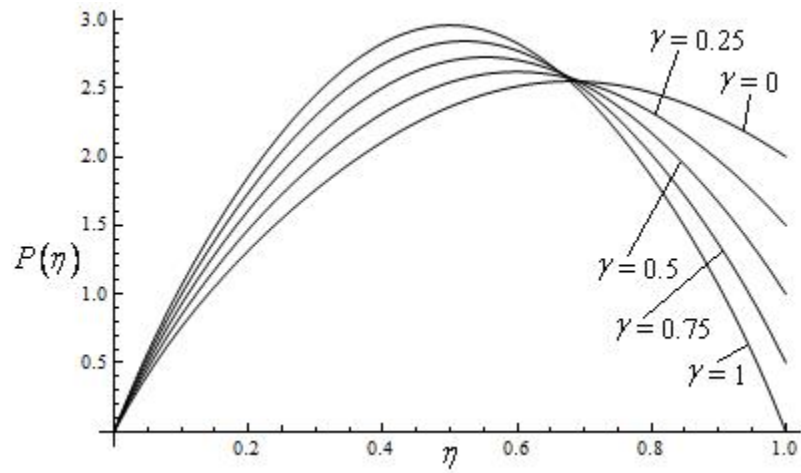
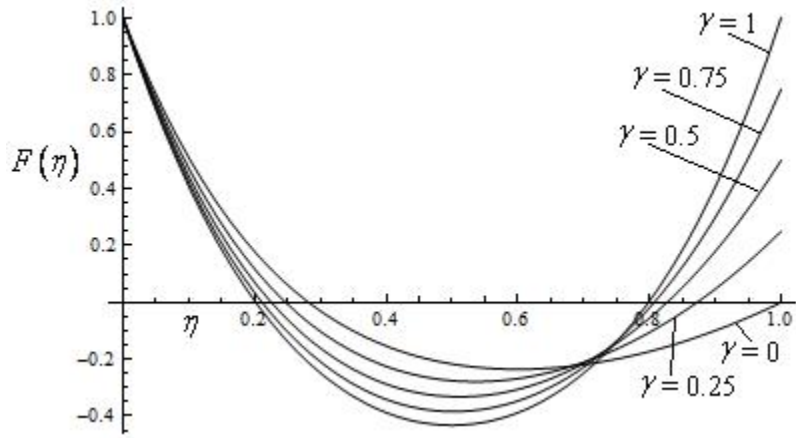Figure 7.4: HAM solutions for $F(\eta)$ for various values of the parameter γ when $R = 16$.
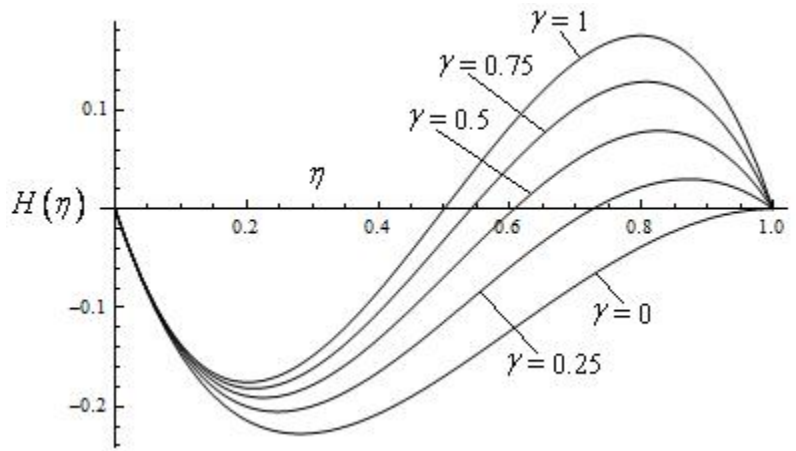

Figure 7.5: HAM solutions for $H(\eta)$ for various values of the parameter γ when $R = 16$.

106
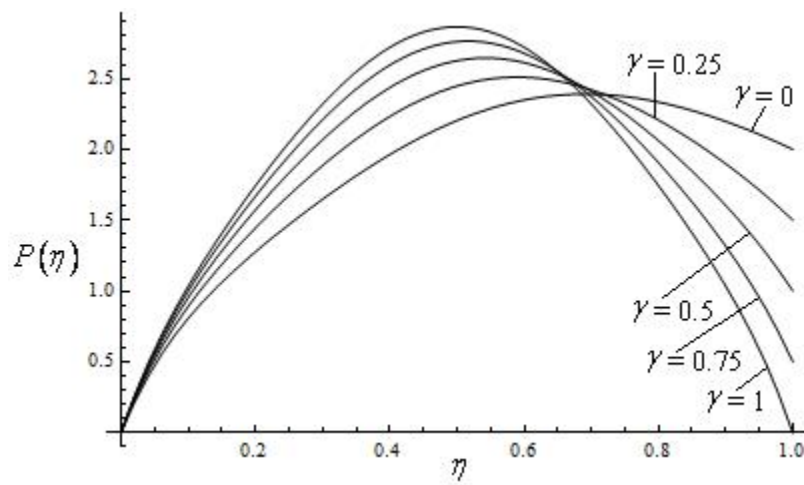
Figure 7.6: HAM solutions for $P(\eta)$ for various values of the parameter γ when $R = 16$.

CHAPTER 8. DRINFEL'D SOKOLOV EQUATION

8.1 Statement of the Problem

Here we consider the coupled nonlinear partial differential equations given in Lio, Fu and Liu [32] as

$$u_t + \alpha_1 u u_x + \beta_1 u_{xxx} + \gamma \delta(v)^{\delta-1} v_x = 0 \qquad (8.1.1)$$

$$v_t + \alpha_2 u v_x + \beta_2 v_{xxx} = 0. \qquad (8.1.2)$$

This system was introduced by Drinfel'd and Sokolov as an example of a system on nonlinear equations possessing Lax pairs of a special form as can be found in Goktas and Hereman [36]. To use HAM we must apply boundary/initial conditions. To guide the choice of the conditions we consider families of exact solutions to 8.1.1 and 8.1.2 and base our choices accordingly. The intent of this section is to not only demonstrate the use of HAM in solving systems of this type but also to illustrate the effectiveness of the method in its ability to produce local analytic approximations without the necessity of a traveling wave assumption.

8.2 Families of Exact Solutions

To find exact solutions to 8.1.1 and 8.1.2 we assume that $\alpha_2 = 0$ and $\delta = 2$.

8.2.1 Hyperbolic Solutions

Assume solutions to be of the traveling wave form $u(x,t) = a_1 + b_1 \sinh(cx + dt)$ and $v(x,t) = b_2 \cosh(cx + dt)$. Upon substitution we get that

$$\left(b_1 d + acb_1\alpha_1 + b_1 c^3 \beta_1\right) \cosh(cx + dt) + \left(b_1^2 c\alpha_1 + 2b_2^2 c\gamma\right) \cosh(cx + dt) \sinh(cx + dt) \quad (8.2.1)$$

and

$$\left(b_2 d + b_2 c^3 \beta_2\right) \sinh(cx + dt) = 0. \quad (8.2.2)$$

Setting the coefficients to zero gives the system of equations

$$d + ac\alpha_1 + c^3\beta_1 = 0 \quad (8.2.3)$$

$$b_1^2 \alpha_1 + 2b_2^2\gamma = 0 \quad (8.2.4)$$

$$d + c^3 + \beta_2 = 0. \quad (8.2.5)$$

Solving the above system gives $c = \sqrt{\frac{a\alpha_1}{\beta_2 - \beta_1}}$, $d = -c^3\beta_2$ and $b_1 = b_2\sqrt{\frac{-2\gamma}{\alpha}}$. From this we see that real-valued solutions will be obtained provided $\alpha \neq 0$, $\beta_2 - \beta_1 \neq 0$, $a\alpha_1$ and $\beta_2 - \beta_1$ are of the same sign and γ and α_1 are of opposite signs.

8.2.2 Trigonometric Solutions

Here we assume the solutions to be of the traveling wave form $u(x,t) = a + b_1 \sin(cx + dt)$ and $v(x,t) = b_2 \cos(cx + dt)$. Upon substitution into 8.1.1 and 8.1.2 and equating the proper

coefficients to zero we obtain the system of equations

$$d + ac\alpha_1 - c^3\beta_1 = 0 \tag{8.2.6}$$

$$b_1^2\alpha_1 - 2b_2^2\gamma = 0 \tag{8.2.7}$$

$$d - c^3\beta_2 = 0. \tag{8.2.8}$$

Solving this system gives $c = \sqrt{\frac{a_1\alpha_1}{\beta_1-\beta_2}}$, $d = c^3\beta_2$ and $b_1 = b_2\sqrt{\frac{2\gamma}{\alpha_1}}$. From this we see that real-valued solutions are obtained provided $\alpha_1 \neq 0$, $\beta_1 - \beta_2 \neq 0$, γ and α_1 are of the same sign and the quantities $a_1\alpha_1$ and $\beta_1 - \beta_2$ are of the same sign.

8.2.3 Exponential Solutions

If we also restrict the parameter β_2 to be -1, the exact exponential solutions are given as

$$u(x,t) = a_1 - b_2\sqrt{\frac{2|\gamma|}{\alpha}}e^{-x-t} \tag{8.2.9}$$

and

$$v(x,t) = -\frac{1 + a_1\alpha_1 + \beta_1}{\sqrt{2\alpha_1|\gamma|}} + b_2e^{-x-t}. \tag{8.2.10}$$

As it can be shown that these solutions are essentially identical to the perturbation solutions obtained by allowing $\alpha_1 = \alpha_2 = \varepsilon \ll 1$ in the governing equations they are not considered in depth here but are explored in more depth in Chapter 9.

110

8.3 Analysis

To extend the results obtained in section 8.2 we apply HAM. The initial guesses are based on the Taylor Series expansions of the exact solutions and the auxillary linear operator for both 8.1.1 and 8.1.2 is

$$L = \frac{\partial^3}{\partial x^3}. \tag{8.3.1}$$

Again assuming the solutions to be of the form $U(x,t;q) = u_0(x,t) + \sum_{n=1}^{\infty} q^n u_n(x,t)$ and $V(x,t;q) = v_0(x,t) + \sum_{n=1}^{\infty} q^n v_n(x,t)$ we find the individual terms of the solutions by solving the mth order deformations given by

$$\frac{\partial^3 u_m(x,t)}{\partial x^3} = \hbar\alpha_1 \sum_{k=0}^{m-1} u_k(x,t)\frac{\partial u_{m-1-k}(x,t)}{\partial x} + \frac{\partial^3 u_{m-1}(x,t)}{\partial x^3}(1 + \hbar\beta_1) \tag{8.3.2}$$

$$+\hbar\frac{\partial u_{m-1}(x,t)}{\partial t} + \hbar\gamma\delta\psi_m(v(x,t))$$

and

$$\frac{\partial^3 v_m(x,t)}{\partial x^3} = \hbar\alpha_2 \sum_{k=1}^{m-1} u_k(x,t)\frac{\partial v_{m-1-k}(x,t)}{\partial x} + \frac{\partial^3 v_{m-1}(x,t)}{\partial x^3}(1 + \hbar\beta_2) + \hbar\frac{\partial v_{m-1}(x,t)}{\partial t}. \tag{8.3.3}$$

$\psi_m(v(x,t))$ in 8.3.2 provides an algorithm to generate the terms in the mth order deformation as given by the nonlinear term $\gamma\delta(v)^{\delta-1}v_x$ in 8.1.1. $\psi_m(v(x,t))$ generates the terms as follows:

1) Calculate the sum $\left(\sum_{k=0}^{m-1} q^k v_k(x,t)\right)^{\delta-1}$

2) Only those term containing powers of q up to $m-1$ from (1) are retained.

3) Each term retained from (2) is then multiplied by $\frac{\partial v_{m-1-k}(x,t)}{\partial x}$.

111

As an example of the algorithm, the 4th order deformation with $\delta = 3$ would contain the terms

$$v_0(x,t)^2\frac{\partial v_3(x,t)}{\partial x} + 2v_0(x,t)v_1(x,t)\frac{\partial v_2(x,t)}{\partial x} + v_1(x,t)^2\frac{\partial v_1(x,t)}{\partial x} \qquad (8.3.4)$$

$$+2v_0(x,t)v_2(x,t)\frac{\partial v_1(x,t)}{\partial x} + 2v_1(x,t)v_2(x,t)\frac{\partial v_0(x,t)}{\partial x} + 2v_0(x,t)v_3(x,t)\frac{\partial v_0(x,t)}{\partial x}$$

as generated by $\psi_m(v(x,t))$. After a sufficient number of terms are calculated we found the value of $\hbar$ that provided the convergent solution to be found by finding the roots of the polynomial in $\hbar$ defined as

$$\left(\frac{\partial^3 U(x,t)}{\partial x^3} + \frac{\partial^3 U(x,t)}{\partial t^3} + \frac{\partial^3 V(x,t)}{\partial x^3} + \frac{\partial^3 V(x,t)}{\partial t^3}\right)\Bigg|_{x=0,t=0} = 0. \qquad (8.3.5)$$

8.3.1 Hyperbolic Approximations

To extend the results given in 8.2.1 we take our boundary/initial conditions to be

$$u(0,0) = a, \quad u_x(0,0) = b_2\sqrt{\frac{2a\gamma}{\beta_1 - \beta_2}}, \quad u_{xx}(0,0) = 0 \qquad (8.3.6)$$

and

$$v(0,0) = b_2, \quad v_x(0,0) = 0, \quad v_{xx}(0,0) = \frac{ab_2\alpha_1}{\beta_2 - \beta_1} \qquad (8.3.7)$$

and we take the initial guesses to be based on the first two terms of the Taylor Series expansions of the exact solutions. Therefore we let $u_0(x,t) = a + u_x(0,0)x + t$ and $v_0(x,t) = b_2 + \frac{1}{2}v_{xx}(0,0)x^2 + t$. The first few terms of the HAM solution under the choice of parameters

$\alpha_2 = 1$, $\alpha_1 = -2$, $\delta = 3$, $\beta_1 = 3$, $\beta_2 = 1$ and $\gamma = 1$ are

$$u_0(x,t) = a + \sqrt{a}b_2 x + t \tag{8.3.8}$$

$$v_0(x,t) = b_2 + \frac{1}{2}ab_2 x^2 + t \tag{8.3.9}$$

$$u_1(x,t) = \frac{1}{6}\hbar x^3 - \frac{1}{3}a^{3/2}b_2\hbar x^3 - \frac{1}{3}a^{1/2}b_2\hbar t x^3 - \frac{1}{12}ab_2^2\hbar x^4 + \frac{1}{8}ab_2^3\hbar x^4 + \frac{1}{4}ab_2^2\hbar t x^4 \tag{8.3.10}$$

$$+ \frac{1}{8}ab_2\hbar t^2 x^4 + \frac{1}{40}a^2 b_2^2\hbar t x^6 + \frac{1}{448}a^3 b_2^3\hbar x^8$$

$$v_1(x,t) = \frac{1}{6}\hbar x^3 + \frac{1}{24}a^2 b_2\hbar x^4 + \frac{1}{24}ab_2\hbar t x^4 + \frac{1}{60}a^{3/2}b_2^2\hbar x^5 \tag{8.3.11}$$

8.3.2 Trigonometric Approximations

Under the same solution scheme as 8.3.1 we find under the assumptions that $\alpha_2 = 1$, $\alpha_1 = 1$, $\delta = 3$, $\beta_1 = 2$, $\beta_2 = 1$ and $\gamma = 1/2$ the first few terms of the solution are

$$u_0(x,t) = a + \sqrt{a}b_2 x + t \tag{8.3.12}$$

$$v_0(x,t) = b_2 - \frac{1}{2}ab_2 x^2 + t \tag{8.3.13}$$

$$u_1(x,t) = \frac{1}{6}\hbar x^3 + \frac{1}{6}a^{3/2}b_2\hbar x^3 + \frac{1}{6}a^{1/2}b_2\hbar t x^3 + \frac{1}{24}ab_2^2\hbar x^4 - \frac{1}{16}ab_2^3\hbar x^4 - \frac{1}{8}ab_2^2\hbar t x^4 \tag{8.3.14}$$

$$- \frac{1}{16}ab_2\hbar t^2 x^4 + \frac{1}{80}a^2 b_2^3\hbar x^6 + \frac{1}{80}a^2 b_2^2\hbar t x^6 - \frac{1}{896}a^3 b_2^3\hbar x^8$$

$$v_1(x,t) = \frac{1}{6}\hbar x^3 - \frac{1}{24}a^2 b_2\hbar x^4 - \frac{1}{24}ab_2\hbar t x^4 - \frac{1}{60}a^{3/2}b_2^2\hbar x^5. \tag{8.3.15}$$

8.4 Results

Under the above mentioned choices of the parameters for $\delta = 3$, the analytic approximations based on the exact hyperbolic solution with $\hbar = -0.49$ provided 24 term solutions accurate to within 10^{-4} on the spatial interval $x \in (-1.6, 1.6)$. With $\alpha_1 = 1$, $\alpha_2 = 1$, $\delta = 2$, β_1, $\beta_2 = 1$ and $\gamma = -2$ the 31 term solutions with $\hbar = -0.6$ are accurate to 10^{-4} on the spatial interval $(-2, 2.4)$. The effect of changing the signs of α_1 and γ for $\delta = 2$ can be seen in Figures 7 and 8 and Figures 11 and 12 for $\delta = 3$. With $\alpha_1 = 1$, $\alpha_2 = 1$, $\delta = 3$, $\beta_1 = 2$, $\beta_2 = 1$ and $\gamma = 1/2$, the 25 term approximation based on the exact trigonometric solution with $\hbar = -0.5$ is accurate to 10^{-5} on the interval $(-2, 2.6)$. The 29 term solution with $\delta = 2$ and $\hbar = -0.42$ provided the same results. To extend the solutions in the spatial domain would involve the calculation of more terms.
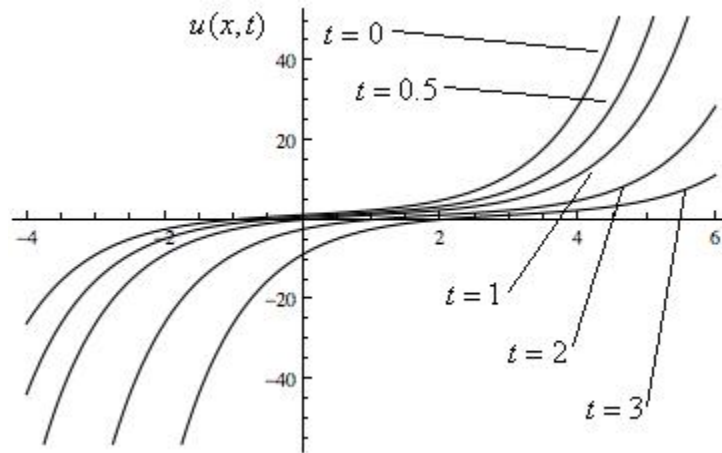
8.5 Figures

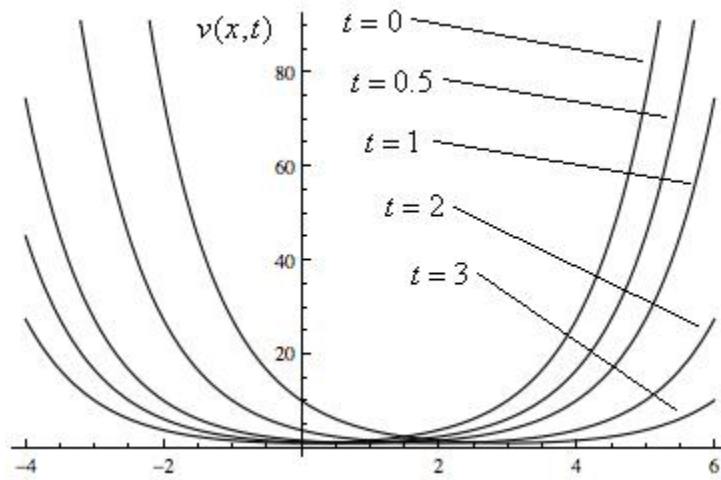Figure 8.1: The exact hyperbolic solution of $u(x,t)$ for various values of t.

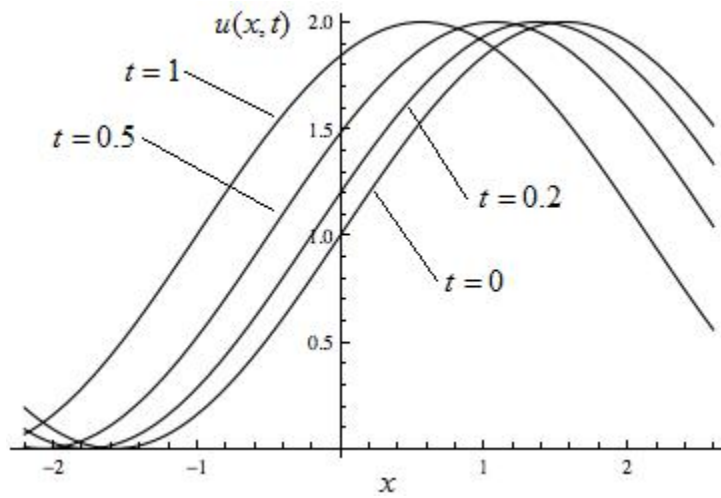Figure 8.2: The exact hyperbolic solution of $v(x,t)$ for various values of t.

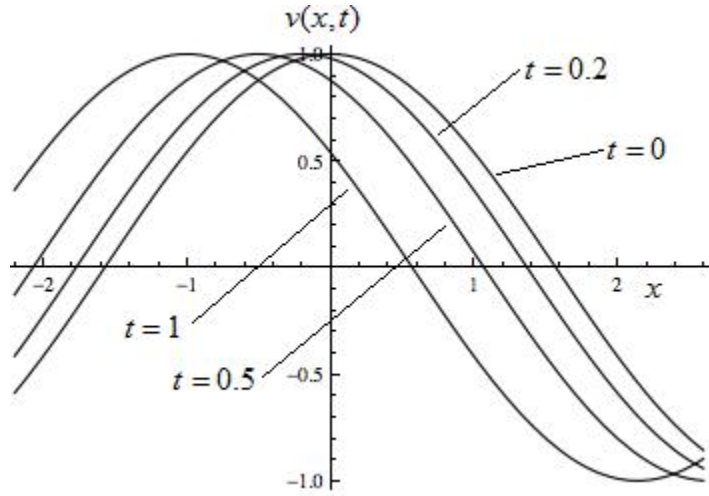Figure 8.3: The exact trigonometric solution of $u(x,t)$ for various values of t.

116

Figure 8.4: The exact trigonometric solution of $v(x, t)$ for various values of t.

Figure 8.5: 31-term HAM solution with of $u(x, t)$ with $\hbar = -0.6$ based on the exact hyperbolic solution under the assumption that $\alpha_2 = 1$, $\alpha_1 = 1$, $\delta = 2$, $\beta_1 = 1$, $\beta_2 = 1$ and $\gamma = -2$ for various values of t.

117

Figure 8.6: 31-term HAM solution with of $v(x,t)$ with $\hbar = -0.6$ based on the exact hyperbolic solution under the assumption that $\alpha_2 = 1$, $\alpha_1 = 1$, $\delta = 2$, $\beta_1 = 1$, $\beta_2 = 1$ and $\gamma = -2$ for various values of t.
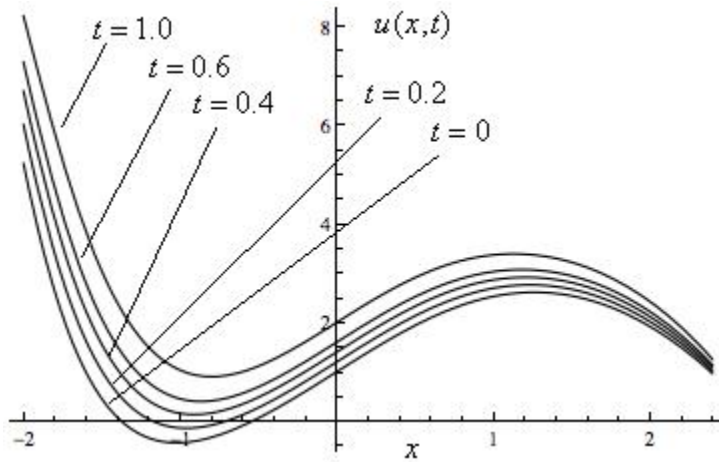

Figure 8.7: 33-term HAM solution with of $u(x,t)$ with $\hbar = -1/3$ based on the exact hyperbolic solution under the assumption that $\alpha_2 = 1$, $\alpha_1 = -2$, $\delta = 2$, $\beta_1 = 3$, $\beta_2 = 1$ and $\gamma = 1$ for various values of t.
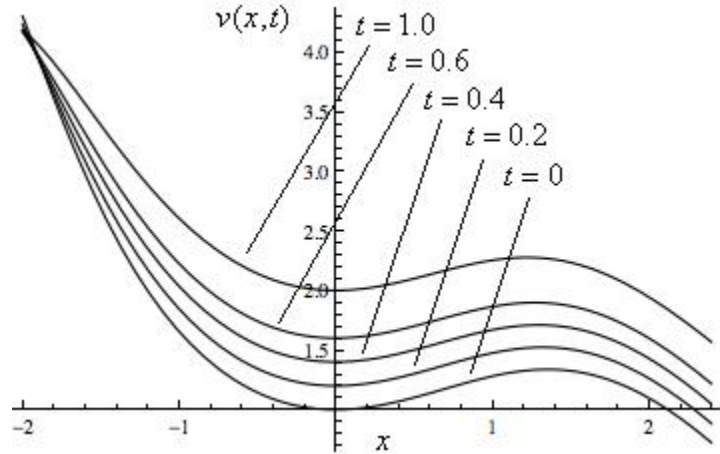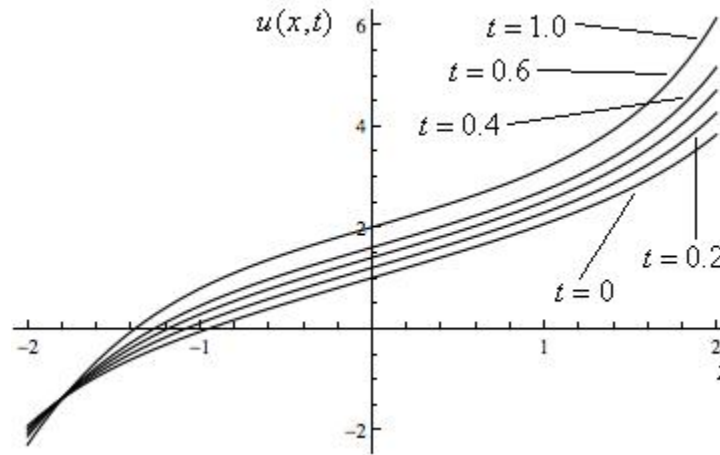
118

Figure 8.8: 33-term HAM solution with of $v(x,t)$ with $\hbar = -1/3$ based on the exact hyperbolic solution under the assumption that $\alpha_2 = 1$, $\alpha_1 = -2$, $\delta = 2$, $\beta_1 = 3$, $\beta_2 = 1$ and $\gamma = 1$ for various values of t.

Figure 8.9: 24-term HAM solution with of $u(x,t)$ with $\hbar = -0.49$ based on the exact hyperbolic solution under the assumption that $\alpha_2 = 1$, $\alpha_1 = -2$, $\delta = 3$, $\beta_1 = 3$, $\beta_2 = 1$ and $\gamma = 1$ for various values of t.

119

Figure 8.10: 24-term HAM solution with of $v(x,t)$ with $\hbar = -0.49$ based on the exact hyperbolic solution under the assumption that $\alpha_2 = 1$, $\alpha_1 = -2$, $\delta = 3$, $\beta_1 = 3$, $\beta_2 = 1$ and $\gamma = 1$ for various values of t.

Figure 8.11: 24-term HAM solution with of $u(x,t)$ with $\hbar = -0.86$ based on the exact hyperbolic solution under the assumption that $\alpha_2 = 1$, $\alpha_1 = 1$, $\delta = 3$, $\beta_1 = 1$, $\beta_2 = 1$ and $\gamma = -2$ for various values of t.
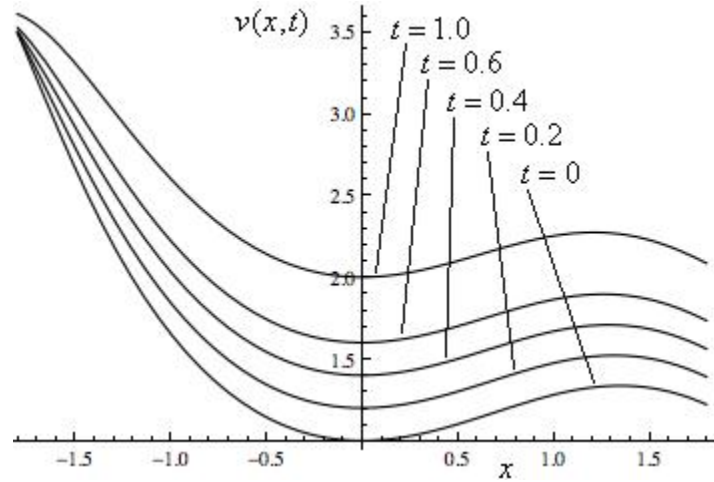
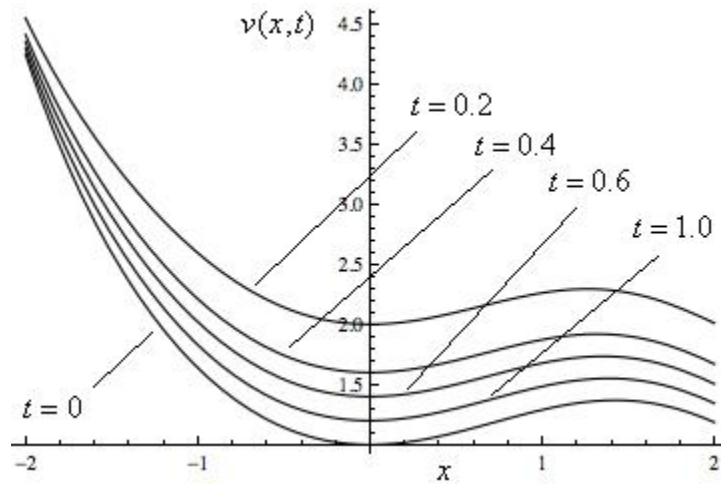Figure 8.12: 24-term HAM solution with of $v(x,t)$ with $\hbar = -0.86$ based on the exact hyperbolic solution under the assumption that $\alpha_2 = 1$, $\alpha_1 = 1$, $\delta = 3$, $\beta_1 = 1$, $\beta_2 = 1$ and $\gamma = -2$ for various values of t.
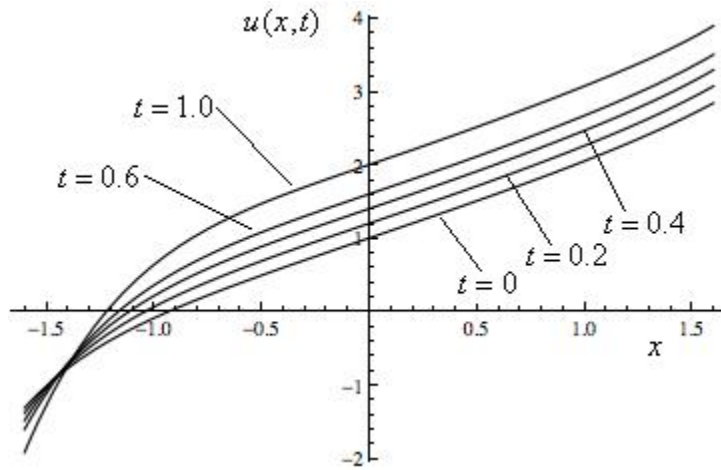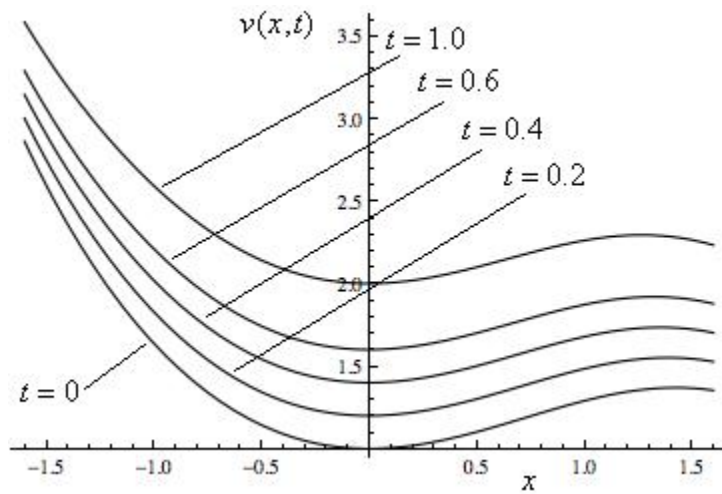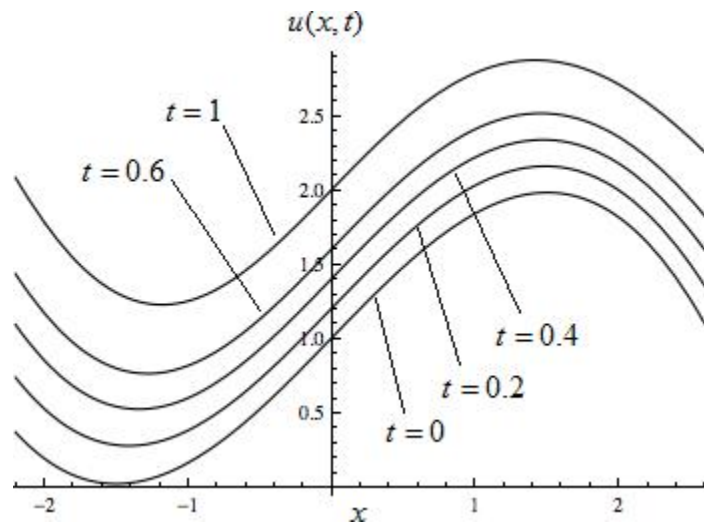

Figure 8.13: 29-term HAM solution with of $u(x,t)$ with $\hbar = -0.42$ based on the exact trigonometric solution under the assumption that $\alpha_2 = 1$, $\alpha_1 = 1$, $\delta = 2$, $\beta_1 = 2$, $\beta_2 = 1$ and $\gamma = 1/2$ for various values of t.
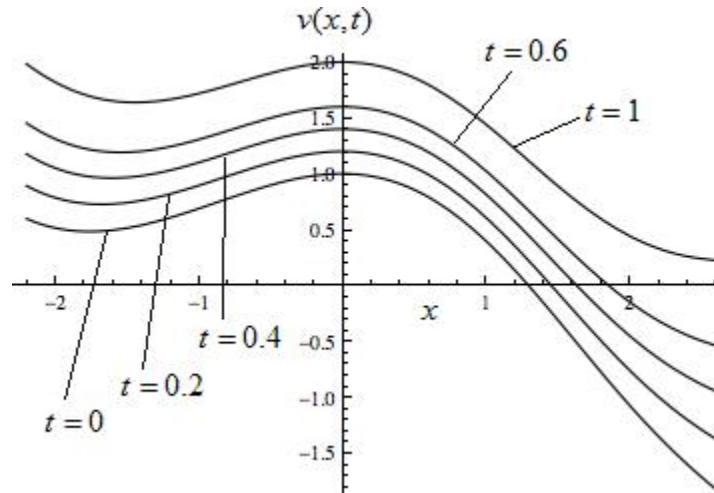
Figure 8.14: 29-term HAM solution with of $v(x, t)$ with $\hbar = -0.42$ based on the exact trigonometric solution under the assumption that $\alpha_2 = 1$, $\alpha_1 = 1$, $\delta = 2$, $\beta_1 = 2$, $\beta_2 = 1$ and $\gamma = 1/2$ for various values of t.

Figure 8.15: 25-term HAM solution with of $u(x, t)$ with $\hbar = -0.5$ based on the exact trigonometric solution under the assumption that $\alpha_2 = 1$, $\alpha_1 = 1$, $\delta = 3$, $\beta_1 = 2$, $\beta_2 = 1$ and $\gamma = 1/2$ for various values of t.
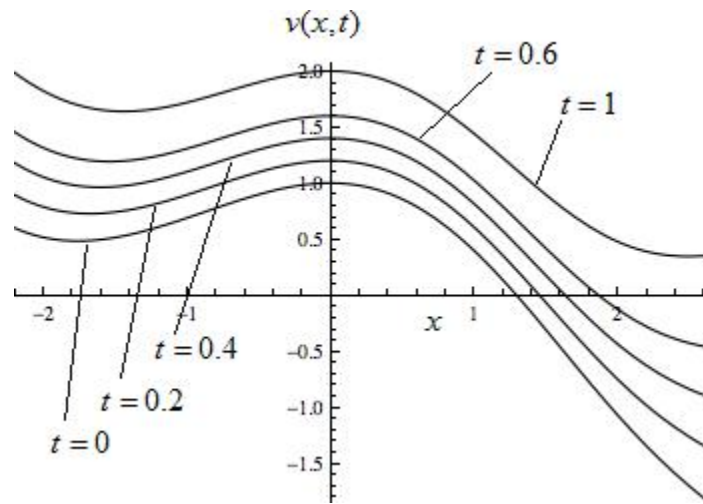
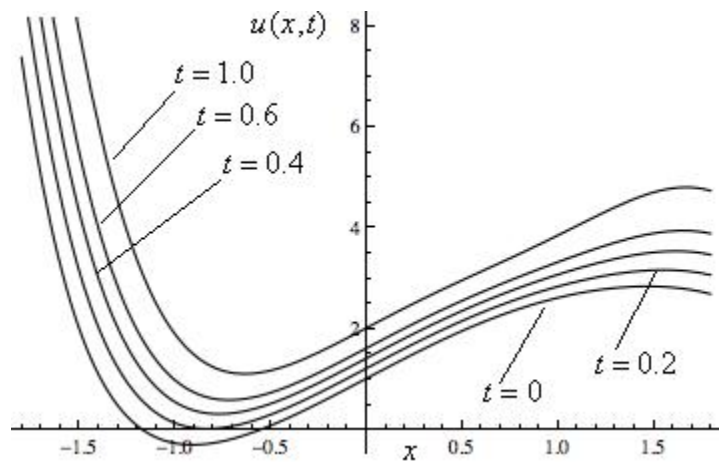Figure 8.16: 25-term HAM solution with of $u(x,t)$ with $\hbar = -0.5$ based on the exact trigonometric solution under the assumption that $\alpha_2 = 1$, $\alpha_1 = 1$, $\delta = 3$, $\beta_1 = 2$, $\beta_2 = 1$ and $\gamma = 1/2$ for various values of t.
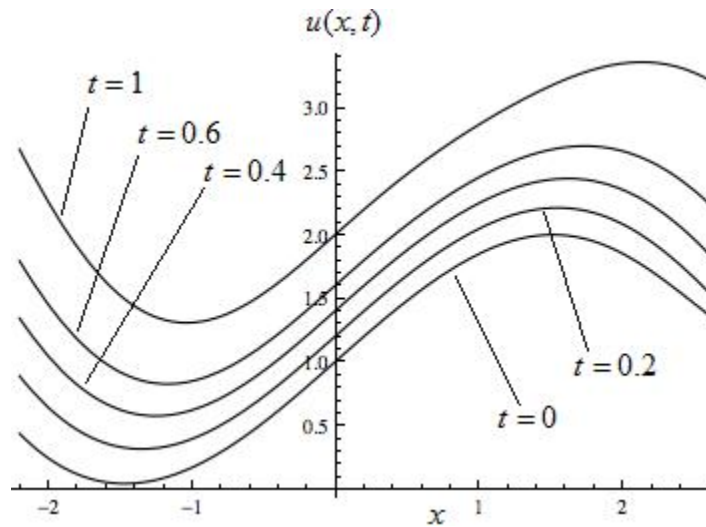
CHAPTER 9. FUTURE APPLICATIONS

As stated in the Introduction, the Homotopy Analysis Method takes advantage of homotopies or embeddings to solve nonlinear differential equations. However, it may be that the use of the generalized homotopy given by Liao is not the only tool that can be utilized in the solution process. One possible extension could be the use of an auxillary nonlinear operator instead of an auxillary linear operator to find local approximations of solutions.

That is, given the nonlinear differential equation $N[f(x)] = 0$, there may exist a nonlinear operator N^* such that a solution $f_0(x)$ is known. It may then be that $f_0(x)$ can be continuously transformed into the solution of $N[f(x)] = 0$ by means of a homotopy. This idea can be explored in more detail with the Drinfel'd Sokolov problem.

9.1 HAM extension; Nonlinear Auxillary Operator

The nonlinear coupled system of partial differential equations called the Drinfel'd Sokolov equation is

$$u_t + \alpha_1 u u_x + \beta_1 u_{xxx} + \gamma \delta(v)^{\delta-1} v_x = 0 \tag{9.1.1}$$

$$v_t + \alpha_2 u v_x + \beta_2 v_{xxx} = 0. \tag{9.1.2}$$

It can be shown that under the assumption that $\alpha_2 = 0$ and $\beta_2 = -1$ that an exact solution is

$$u(x,t) = a_1 - b_2 \sqrt{\frac{2|\gamma|}{\alpha_1}} e^{-x-t} \tag{9.1.3}$$

and

$$v(x,t) = -\frac{1 + a_1\alpha_1 + \beta_1}{\sqrt{2\alpha_1|\gamma|}} + b_2 e^{-x-t} \tag{9.1.4}$$

Define the nonlinear operators N_1^* and N_2^* by

$$N_1^*[U(\eta), V(\eta)] = \frac{dU(\eta)}{d\eta} + \alpha_1 U(\eta)\frac{dU(\eta)}{d\eta} + \beta_1\frac{d^3U(\eta)}{d\eta^3} \tag{9.1.5}$$

$$+ 2\gamma\left(V(\eta)\right)\frac{dV(\eta)}{d\eta} \tag{9.1.6}$$

and

$$N_2^*[U(\eta), V(\eta)] = \frac{dV(\eta)}{d\eta} - 2\frac{d^3V(\eta)}{d\eta^3} \tag{9.1.7}$$

where $\eta = x + t$.

We then set up the homotopies defined by

$$H_1(U(\eta), V(\eta), q, \hbar) = (1-q)N_1^*[U(\eta;q) - v_0(x,t), V(\eta;q)] \tag{9.1.8}$$

$$- q\hbar N_1[U(\eta;q), V(\eta;q)] \tag{9.1.9}$$

and

$$H_2(U(\eta;q), V(\eta;q), q, \hbar) = (1-q)N_2^*[U(\eta;q), V(\eta;q) - v_0(x,t)] \tag{9.1.10}$$

$$- q\hbar N_2[U(\eta;q), V(\eta;q)]. \tag{9.1.11}$$

125

where the nonlinear operators N_1 and N_2 are defined by the original governing equations.

As in the traditional method we assume solutions to be of the form

$$U(\eta; q) = u_0(\eta) + \sum_{n=1}^{\infty} q^m u_n(\eta) \qquad (9.1.12)$$

and

$$V(\eta; q) = v_0(\eta) + \sum_{n=1}^{\infty} q^m v_n(\eta) \qquad (9.1.13)$$

for which $u_0(\eta)$ and $v_0(\eta)$ are the exact solutions under the assumption that $\alpha_2 = 0$ and $\beta_2 = -1$.

To find the terms for the assumed solutions $U(\eta)$ and $V(\eta)$ we set $H_1(f(\eta; q), g(\eta; q), q, \hbar) = 0$ and $H_2(f(\eta; q), g(\eta; q), q, \hbar) = 0$ and solve the differential equations obtained by equating like powers of q under the assumption that the order zero case is identically satisfied by u_0 and v_0. If we allow $\alpha_2 = 0$ in the solution process then we need only be concerned with solving $H_2 = 0$ for the terms in $V(\eta)$.

9.2 Results

Figures 1 and 2 show the plots of $U(x, t)$ and $V(x, t)$ under the assumption that $\alpha_1 = 1$, $\alpha_2 = 0$, $\beta_1 = -1 = \beta_2$ and $\gamma = -2$ for $\delta = 3$. For each solution we are able to find a local analytic solution by using 6 terms for both $U(x, t)$ and $V(x, t)$. It is found that using $\hbar = -1$, the solutions are accurate to within 10^{-6}. Figures 3 and 4 show the results under the assumption that $\delta = 3$, $\alpha_1 = 1 = \alpha_2$, $\beta_1 = -1 = \beta_2$ and $\gamma = -2$. Using 6 terms for both $U(x, t)$ and $V(x, t)$ the local analytic approximation is accurate to within 10^{-4} on the spatial interval $(-1/2, 1)$. It should be noted that with $\delta = 2$ it is sufficient only to find

126

the HAM solution for $V(x,t)$ as $u_0(x,t)$ is the exact solution of 9.1.1. Figure 5 shows the 20 term HAM solution for $V(x,t)$ under the assumption that $\delta = 2$, $\alpha_1 = 1 = \alpha_2$, $\beta_1 = 1$, $\beta_2 = -1$ and $\gamma = -2$. With $\hbar = -1$ the solution was found to be accurate to within 10^{-6}. In addition, we found that as we are utilizing a nonlinear auxillary operator in the HAM solution process, the addition of more terms in the solution actually negatively affects to validity of the solution and it is thus more important to be aware of the residual associated with the solution for every additional term calculated.

Based on the success of this example we intend to investigate the use of Nonlinear auxillary operators in HAM as a solution method of nonlinear differential equations.

9.3 Figures

Figure 9.1: The 6 term solution of $U(x,t)$ with $\hbar = -1$ under the assumption that $\delta = 3$, $\alpha_1 = 1$, $\alpha_2 = 0$, $\beta_1 = -1 = \beta_2$ and $\gamma = -2$.

Figure 9.2: The 6 term solution of $V(x,t)$ with $\hbar = -1$ under the assumption that $\delta = 3$, $\alpha_1 = 1$, $\alpha_2 = 0$, $\beta_1 = -1 = \beta_2$ and $\gamma = -2$.
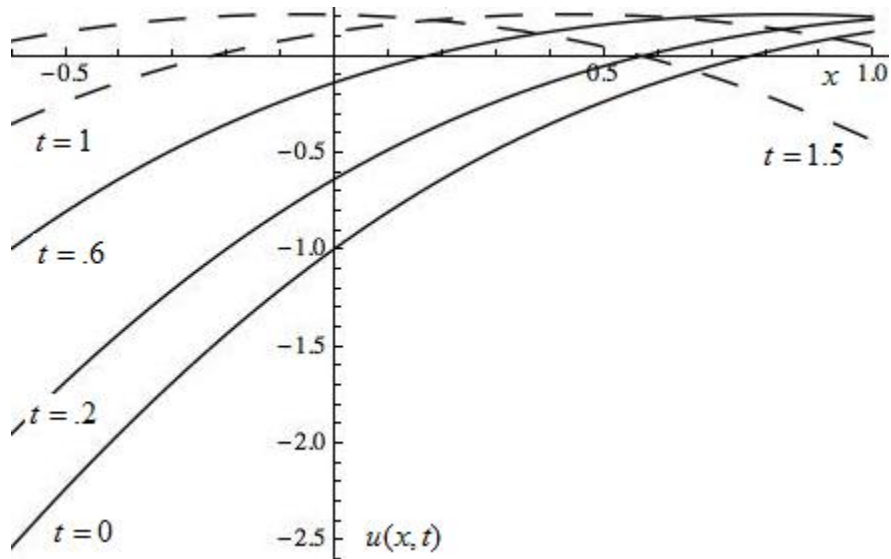
Figure 9.3: The 6 term solution of $U(x,t)$ with $\hbar = -.98$ under the assumption that $\delta = 3$, $\alpha_1 = 1$, $\alpha_2 = 1$, $\beta_1 = -1 = \beta_2$ and $\gamma = -2$. The right traveling waves are solid while the left traveling waves are dashed.

128

Figure 9.4: The 6 term solution of $V(x,t)$ with $\hbar = -.98$ under the assumption that $\delta = 3$, $\alpha_1 = 1$, $\alpha_2 = 1$, $\beta_1 = -1 = \beta_2$ and $\gamma = -2$. The right traveling waves are solid while the left traveling waves are dashed.
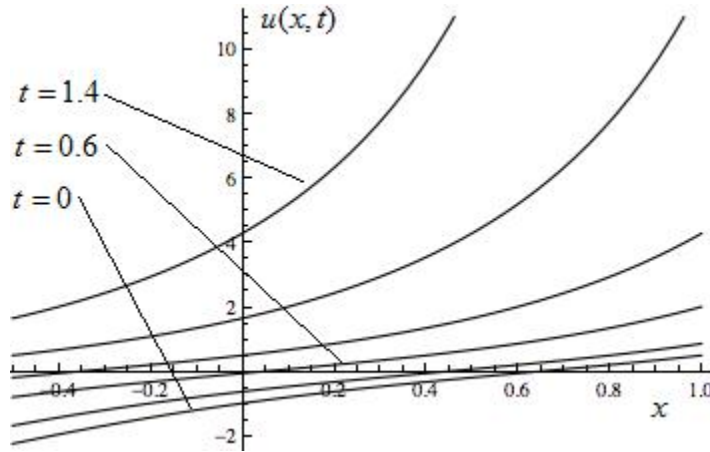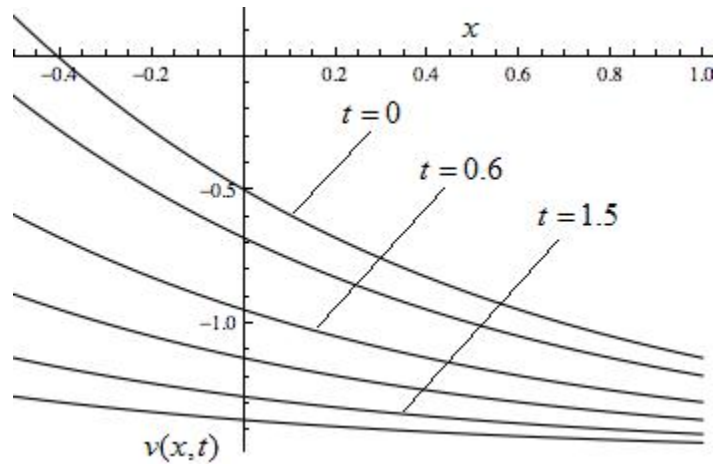

Figure 9.5: The 20 term solution of $V(x,t)$ with $\hbar = -1$ under the assumption that $\delta = 2$, $\alpha_1 = 1 = \alpha_2$, $\beta_1 = 1$, $\beta_2 = -1$ and $\gamma = -2$.
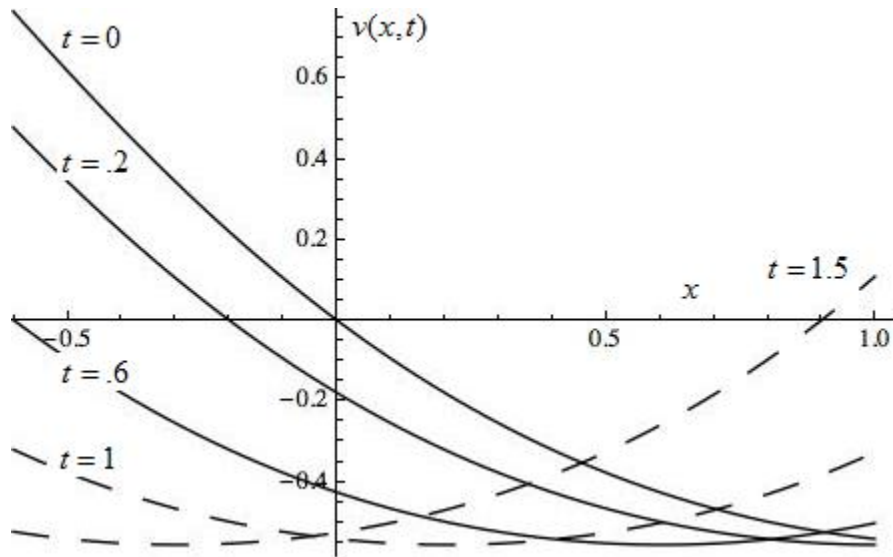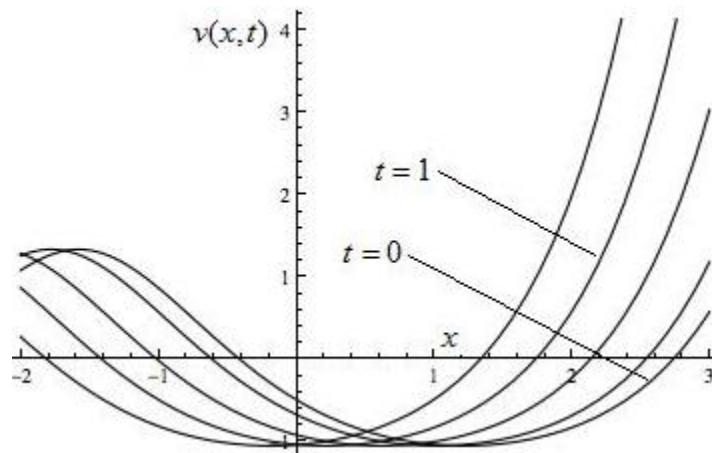
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