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Effect Of Inner Scale Atmospheric Spectrum Models On Scintillation In All Optical Turbulence Regimes

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EFFECT OF INNER SCALE ATMOSPHERIC SPECTRUM MODELS ON SCINTILLATION IN ALL OPTICAL TURBULENCE REGIMES

by

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ABSTRACT

Experimental studies have shown that a "bump" occurs in the atmospheric spectrum just prior to turbulence cell dissipation.[1,3,4] In weak optical turbulence, this bump affects calculated scintillation. The purpose of this thesis was to determine if a "non-bump" atmospheric power spectrum can be used to model scintillation for plane waves and spherical waves in moderate to strong optical turbulence regimes. Scintillation expressions were developed from an "effective" von Karman spectrum using an approach similar to that used by Andrews et al.[8,14,15] in developing expressions from an "effective" modified (bump) spectrum. The effective spectrum extends the Rytov approximation into all optical turbulence regimes using filter functions to eliminate mid-range turbulent cell size effects to the scintillation index. Filter cutoffs were established by matching to known weak and saturated scintillation results. The resulting new expressions track those derived from the effective bump spectrum fairly closely. In extremely strong turbulence, differences are minimal.

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TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

LIST OF ACRONYMS

c_1, c_2, c_3, and c_4	Asymptotic comparison constants determined from known asymptotic behavior of the scintillation index
C_n^2	Refractive index structure parameter(a measure of index of refraction fluctuation strength)
$D_{pl}(\rho)$	Wave structure function for a plane wave.
$f(\kappa l_0)$	A factor that describes basic power law for inner scale modifications
F	Phase front radius of curvature of a beam at the receiver
$G_x(\kappa)$	Large-scale turbulence cell filter
$G_y(\kappa)$	Small-scale turbulence cell filter
$h(\tau,\xi)=\begin{cases}\tau(1-\beta\xi), & \tau<\xi\\ \xi(1-\beta\tau), & \tau>\xi\end{cases}$	Non-dimensional parameter
$k=2\pi/\lambda$	Optical wave number
L	Propagation path length
l_0	Inner (micro) scale size
L_0	Outer (macro) scale cell size
$Q_l=\dfrac{L\kappa_l^2}{k}$	Non-dimensional parameter (ratio of inner scale to Fresnel zone)
$Q_m=\dfrac{L\kappa_m^2}{k}$	Non-dimensional parameter (ratio of inner scale to Fresnel zone)
W	Beam radius at the receiver
x	Associated with large-scale turbulence effects
y	Associated with small-scale turbulence effects
$\Phi_n(\kappa)$	Power spectrum model for refractive index fluctuations
$\Phi_{n,e}(\kappa)$	“Effective” atmospheric spectrum
$\eta=\dfrac{L\kappa^2}{k}$, $\eta_x=\dfrac{L\kappa_x^2}{k}$, $\eta_y=\dfrac{L\kappa_y^2}{k}$	Non-dimensional cutoff frequencies for the filter function.
κ	Scalar spatial wave number
$\kappa_l=3.3/l_0$	Inner scale wave number parameter
$\kappa_m=5.92/l_0$	Inner scale wave number parameter
$\kappa_0=2\pi/L_0$	Outer scale wave number parameter

κ_x	Spatial frequency cutoff for large-scale turbulent cells
κ_y	Spatial frequency cutoff for small-scale turbulent cells
λ	wavelength
Λ	Fresnel ratio of the beam at the receiver
Θ	Curvature parameter of a beam at the receiver
$\overline{\Theta}$	$1-\Theta$
ρ_0	Plane wave spatial coherence radius
σ_I^2	Scintillation index: normalized variance of irradiance
$\sigma_{\ln I}^2$	Log irradiance variance
$\sigma_{\ln x}^2$	Large-scale log-irradiance variance
$\sigma_{\ln y}^2$	Small-scale log-irradiance variance
σ_p^2	Rytov variance for a plane wave with inner scale
$\sigma_R^2 = 1.23C_n^2 k^{7/6} L^{11/6}$	Rytov Variance
σ_s^2	Rytov variance for a spherical wave with inner scale
σ_x^2	Large scale scintillation index
σ_y^2	Small scale scintillation index

1.0 INTRODUCTION

Classical theoretical atmospheric turbulence models,[1,2] beginning in the 1940s, assume that turbulent power is initially generated on large scale cell sizes and that dissipative forces cause the turbulent power to be transferred to smaller and smaller scales until the cells dissipate. Experimental studies have shown that at high wave numbers (small scale sizes) a "bump" occurs in the atmospheric spectrum just prior to cell dissipation.[1,3,4] In weak optical turbulence, this bump affects calculated scintillation, which is a measure of irradiance fluctuations experienced by an optical wave propagating through a random medium such as the Earth's atmosphere. Most theoretical studies on beam statistics and resulting atmospheric spectral models (including Kolmogorov, Tatarskii, and von Karman) do not reflect these observed bumps. Andrews et al.[1,5,6] developed a modified (bump) spectrum that accounts for the bump. Andrews et al.[8,14,15] also developed expressions from an "effective" modified spectrum that extends validity of the Rytov approximation into all optical turbulence regimes using filter functions to eliminate mid-range turbulent cell size effects to the scintillation index. Filter cutoffs were established by matching to known weak and saturated scintillation results.

The intent of this thesis is to determine if a simpler non-bump atmospheric power spectrum can be used to accurately calculate plane and spherical wave scintillation in moderate to strong optical turbulence regimes. Scintillation expressions for moderate to strong optical turbulence regimes were developed from an "effective" von Karman spectrum using an approach similar to that used by Andrews et al.[8,14,15]. The resulting new expressions were compared to those derived from the effective modified spectrum.

The following table, Table 1, lists existing plane and spherical wave scintillation expressions based on Kolmogorov, Modified "bump", and von Karman atmospheric power spectrums for both the weak and saturation fluctuation regimes. Table 1 also lists scintillation expressions based on effective Kolmogorov and bump spectrums that were recently calculated for moderate fluctuation regimes. An effective von Karman spectrum is developed in this thesis as well as corresponding expressions for the scintillation index. Mathematical derivations of all of these expressions are shown in the Appendix section.

Note (reference equation 15) that existing saturated turbulence scintillation expressions based on the modified atmospheric power spectra were created with the assumption that $D = D_{\chi} + D_s \cong 2D_s$ where D is the wave structure function, D_{χ} is the log-amplitude structure function, and D_s is the phase structure function. The same assumption was made in this thesis to revise existing saturated turbulence scintillation expressions based on the von Karman atmospheric power spectrum (reference equations 19 and 22). The revisions were necessary for developing new "effective" von Karman based scintillation expressions and for comparing these new expressions to the existing "effective" Modified spectrum derived expressions.

Table 1 – Existing Scintillation Expressions

Turbulence	Spectrum	Wave Type	Scintillation Expression
Weak	Kolmogorov	Plane	$\sigma_I^2 = \sigma_R^2$
		Spherical	$\sigma_I^2 = 0.4\sigma_R^2$
	Von Karman	Plane	$\sigma_I^2 = 3.86\sigma_R^2 \left\{ \left(1 + 1/Q_m^2\right)^{11/12} \sin\left[(11/6)\tan^{-1} Q_m\right] - (11/6)Q_m^{-5/6} \right\}$
		Spherical	$\sigma_I^2 = 3.86\sigma_R^2 \left[\left\{ \left(0.41 + 9/Q_m^2\right)^{11/12} \sin\left[(11/6)\tan^{-1}(Q_m/3)\right] \right\} - (11/6)Q_m^{-5/6} \right]$
	Modified	Plane	$\sigma_I^2 = 3.86\sigma_R^2 \left\{ \left(1 + \frac{1}{Q_l^2}\right)^{11/12} \left[\left(\sin \frac{11}{6} \tan^{-1} Q_l\right) + \frac{1.507}{\left(1+Q_l^2\right)^{1/4}} \sin\left(\frac{4}{3}\tan^{-1} Q_l\right) - \frac{0.273}{\left(1+Q_l^2\right)^{7/24}} \sin\left(\frac{5}{4}\tan^{-1} Q_l\right) \right] - 3.50 Q_l^{-5/6} \right\}$
		Spherical	$\sigma_I^2 = 3.86\sigma_R^2 \left\{ 0.40 \left(1 + \frac{9}{Q_l^2}\right)^{11/12} \left[\sin\left(\frac{11}{6} \tan^{-1} \frac{Q_l}{3}\right) + \frac{2.61}{\left(9+Q_l^2\right)^{1/4}} \sin\left(\frac{4}{3}\tan^{-1} \frac{Q_l}{3}\right) - \frac{0.52}{\left(9+Q_l^2\right)^{7/24}} \sin\left(\frac{5}{4}\tan^{-1} \frac{Q_l}{3}\right) \right] - 3.50 Q_l^{-5/6} \right\}$
Saturation	Kolmogorov	Plane	$\sigma_I^2 = 1 + 0.86/\left(\sigma_R^2\right)^{2/5}$
		Spherical	$\sigma_I^2 = 1 + 2.73/\left(\sigma_R^2\right)^{2/5}$
	Von Karman	Plane	$\sigma_I^2 = 1 + 3.40/\left(\sigma_R^2 Q_m^{7/6}\right)^{1/6}$
		Spherical	$\sigma_I^2 = 1 + 10.36/\left(\sigma_R^2 Q_m^{7/6}\right)^{1/6}$
	Modified	Plane	$\sigma_I^2 = 1 + 2.39/\left(\sigma_R^2 Q_l^{7/6}\right)^{1/6}$
		Spherical	$\sigma_I^2 = 1 + 7.65/\left(\sigma_R^2 Q_l^{7/6}\right)^{1/6}$
Moderate	Kolmogorov	Plane	$\sigma_I^2 = \exp\left[0.49\sigma_R^2 / \left(1 + 1.11\sigma_R^{12/5}\right)^{7/6} + 0.51\sigma_R^2 / \left(1 + 0.69\sigma_R^{12/5}\right)^{5/6}\right] - 1$
		Spherical	$\sigma_I^2 = \exp\left[0.196\sigma_R^2 / \left(1 + 0.186\sigma_R^{12/5}\right)^{7/6} + 0.204\sigma_R^2 / \left(1 + 0.230\sigma_R^{12/5}\right)^{5/6}\right] - 1$
	Modified	Plane	$\sigma_I^2 = \exp\left[\sigma_{\ln x}^2 + 0.51\sigma_\rho^2 / \left(1 + 0.69\sigma_\rho^{12/5}\right)^{5/6}\right] - 1$ where: $\sigma_{\ln x}^2 = 0.16\sigma_R^2 \left(\frac{2.61 Q_l}{2.61 + Q_l + 0.45\sigma_R^2 Q_l^{7/6}}\right)^{7/6} \left[1 + 1.75\left(\frac{2.61}{2.61 + Q_l + 0.45\sigma_R^2 Q_l^{7/6}}\right)^{1/2} - 0.25\left(\frac{2.61}{2.61 + Q_l + 0.45\sigma_R^2 Q_l^{7/6}}\right)^{7/12}\right]$ $\sigma_p^2 = 3.86\sigma_R^2 \left\{ \left(1 + \frac{1}{Q_l^2}\right)^{11/12} \left[\sin\left(\frac{11}{6} \tan^{-1} Q_l\right) + \frac{1.507}{\left(1+Q_l^2\right)^{1/4}} \sin\left(\frac{4}{3}\tan^{-1} Q_l\right) - \frac{0.273}{\left(1+Q_l^2\right)^{7/24}} \sin\left(\frac{5}{4}\tan^{-1} Q_l\right) \right] - 3.50 Q_l^{-5/6} \right\}$
		Spherical	$\sigma_I^2 = \exp\left[\sigma_{\ln x}^2 + 0.51\sigma_s^2 / \left(1 + 0.69\sigma_s^{12/5}\right)^{5/6}\right] - 1$ where: $\sigma_{\ln x}^2 = 0.016\sigma_R^2 \left(\frac{8.56 Q_l}{8.56 + Q_l + 0.08\sigma_1^2 Q_l^{7/6}}\right)^{7/6} \left[1 + 1.76\left(\frac{8.56}{8.56 + Q_l + 0.08\sigma_1^2 Q_l^{7/6}}\right)^{1/2} - 0.25\left(\frac{8.56}{8.56 + Q_l + 0.08\sigma_1^2 Q_l^{7/6}}\right)^{7/12}\right]$ $\sigma_s^2 = 3.86\sigma_R^2 \left\{ 0.40 \left(1 + \frac{9}{Q_l^2}\right)^{11/12} \left[\sin\left(\frac{11}{6} \tan^{-1} \frac{Q_l}{3}\right) + \frac{2.61}{\left(9+Q_l^2\right)^{1/4}} \sin\left(\frac{4}{3}\tan^{-1} \frac{Q_l}{3}\right) - \frac{0.52}{\left(9+Q_l^2\right)^{7/24}} \sin\left(\frac{5}{4}\tan^{-1} \frac{Q_l}{3}\right) \right] - \frac{3.50}{Q_l^{5/6}} \right\}$
	Von Karman	Plane	Thesis
		Spherical	Thesis

2.0 ATMOSPHERIC SPECTRUMS

2.1 Kolmogorov

In the early 1940s, Kolmogorov[1, 2] developed an atmospheric turbulence theory that energy is introduced into large air masses either by wind shear or convection. These air masses, under the influence of inertial forces, break up into a continuum of smaller and smaller cells from an initial outer (macro) scale cell size L_0, which is usually on the order of one to one hundred meters, to a final inner (micro) scale size l_0, on the order of millimeters. For scale sizes less than l_0, remaining energy is dissipated as heat. Each cell size has a slightly different index of refraction. Large scale sizes tend to refract (focus) optical waves, whereas small scale sizes tend to diffract the waves. The energy of these index of refraction fluctuations (optical turbulence) is described by power spectral density functions. The Kolmogorov power spectrum for refractive index fluctuations is defined by

$$\Phi_n(\kappa) = 0.033 C_n^2 \kappa^{-11/3}, \qquad \frac{1}{L_0} << \kappa << \frac{1}{l_0} \qquad (1)$$

where C_n^2 is the refractive index structure parameter (a measure of index of refraction fluctuation strength) and κ is the scalar spatial wave number.

2.2 Tatarskii

The Kolmogorov spectrum is theoretically valid only over the inertial subrange $1/L_0 << \kappa << 1/l_0$. Other spectral models have been proposed for calculations when inner scale and outer scale effects cannot be ignored. In 1971, Tatarskii[1,7] suggested an equation, first proposed by Novikov for velocity fluctuations, that uses a Gaussian function to extend the Kolmogorov Spectrum into the dissipation range $\kappa > \frac{1}{l_0}$

$$\Phi_n(\kappa) = 0.033 C_n^2 \kappa^{-11/3} \exp\left(-\kappa^2/\kappa_m^2\right), \qquad 1/L_0 << \kappa \qquad (2)$$

where $\kappa_m = 5.92/l_0$ is an inner scale wave number parameter. If $\kappa << \kappa_m$ or $l_0 \to 0$, this equation (2) reduces to the Kolmogorov spectrum (1).

2.3 Von Karman

The Tatarskii Spectrum can be extended into the range $\kappa < \frac{1}{L_0}$ using the von Karman spectrum[1]

$$\Phi_n(\kappa) = 0.033 C_n{}^2 \frac{\exp\left(-\kappa^2/\kappa_m^2\right)}{\left(\kappa^2 + \kappa_0^2\right)^{11/6}} \qquad (3)$$

where $\kappa_0 = 2\pi/L_0$ is the outer scale wave number parameter**.** If $\kappa >> \kappa_0$ or $L_0 \to \infty$, this equation (3) reduces to the Tatarskii spectrum (2).

Note that in the dissipation range, there is no physical reason to choose the von Karman spectrum over the Tatarskii spectrum. However, to allow for comparisons of scintillation expressions already developed from the von Karman spectrum, this thesis paper will refer to von Karman (as opposed to Tatarskii) with the assumption that $L_0 \to \infty$.

2.4 Modified "Bump"

Hill[1,5] performed analyses that led to a model that reflects observed bumps in experimental data. Andrews[1,6] developed the following analytic approximation to the Hill spectrum, including an outer scale parameter, called the *modified atmospheric spectrum*

$$\Phi_n(\kappa) = \frac{0.033 C_n^2 \exp\left(-\kappa^2/\kappa_l^2\right)\left[1 + 1.802(\kappa/\kappa_l) - 0.254(\kappa/\kappa_l)^{7/6}\right]}{\left(\kappa^2 + \kappa_0^2\right)^{11/6}} \qquad (4)$$

where $\kappa_l = 3.3/l_0$ is an inner scale wave number parameter.

The following two figures compare the three atmospheric spectral models[8]. Figure 1 shows that a nonzero inner scale reduces spectrum values at high wave numbers $(\kappa > 1/l_0)$ over that predicted by the Kolmogorov spectrum. Note that at low wave numbers $(\kappa < 1/L_0)$, a similar spectrum value reduction is caused by the presence of a finite outer scale.

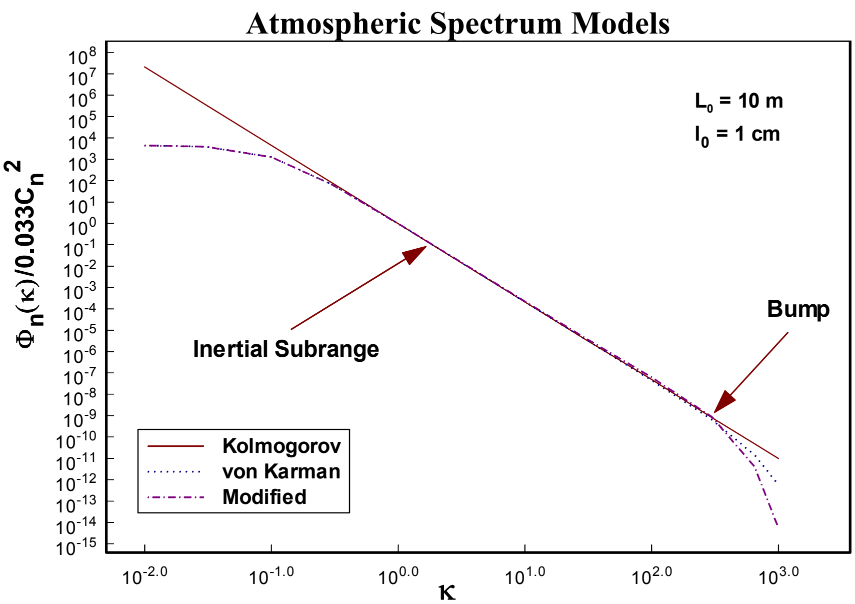


Figure 1 - Spectral models of refractive-index fluctuations

Figure 2 shows a bump for the Modified spectrum that is not reflected in the Kolmogorov or von Karman spectral models.

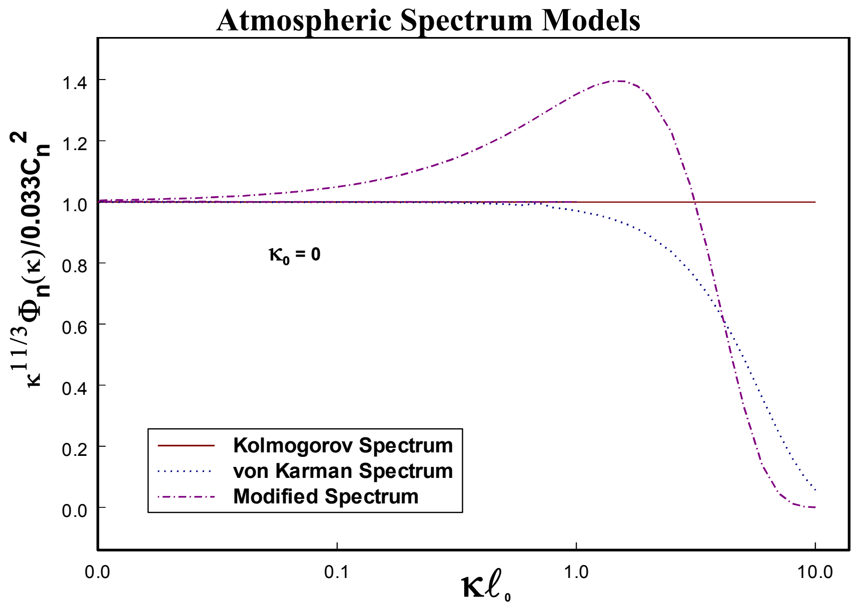


Figure 2 - Scaled spectral models of refractive-index fluctuations

3.0 SCINTILLATION IN WEAK FLUCTUATIONS

Scintillation index σ_I^2 is defined as the normalized variance of irradiance[8]

$$\sigma_I^2 = \frac{<I^2> - <I>^2}{<I>^2}$$

$$= \frac{<I^2>}{<I>^2} - 1 \qquad (5)$$

where I represents optical wave irradiance and < > denotes ensemble average.

Scintillation is often expressed in terms of the Rytov variance[1], $\sigma_R^2 = 1.23 C_n^2 k^{7/6} L^{11/6}$, where L is the propagation path length, k =optical wave number ($k = 2\pi/\lambda$), with λ being wavelength. For plane wave scintillation in weak fluctuation using the Kolmogorov spectrum, $\sigma_I^2 = \sigma_R^2$. Generally, $\sigma_R^2 << 1$ is associated with weak optical turbulence, $\sigma_R^2 >> 1$ is associated with saturated turbulence, and $\sigma_R^2 \sim 1$ is associated with moderate fluctuations.

Under weak fluctuation conditions, the on-axis scintillation component for a beam wave is defined by[1,8]

$$\sigma_{I,l}^2 = 8\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa \Phi_n(\kappa) \exp\left(-\frac{\Lambda L \kappa^2 \xi^2}{k}\right)\left\{1 - \cos\left[\frac{L\kappa^2}{k}\xi(1-\overline{\Theta}\xi)\right]\right\} d\kappa d\xi \qquad (6)$$

where $\Lambda = \frac{2L}{kW^2}$ is the Fresnel ratio of the beam at the receiver, $\Theta = 1 + \frac{L}{F}$ is a curvature parameter of a beam at the receiver, $\overline{\Theta} = 1 - \Theta$, W is the beam radius at the receiver, and F is the phase front radius of curvature at the receiver.

3.1 Plane Wave

For a plane wave, $\Lambda = 0, \Theta = 1, \overline{\Theta} = 1 - \Theta = 0$, equation 6 reduces to

$$\sigma_{I,l}^2(L) = 8\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa \Phi_n(\kappa) \left[1 - \cos\left(\frac{L\kappa^2}{k} \xi \right) \right] d\kappa d\xi \tag{7}$$

Performing the integration in equation 7, with Kolmogorov, von Karman, and modified atmospheric spectra expressions given in equations 1, 3, and 4, results in the following weak optical turbulence scintillation index expressions (see appendices A, B, and C).

Kolmogorov[1,9,10,11]

$$\sigma_I^2 = \sigma_R^2 \tag{8}$$

von Karman[1]

$$\sigma_I^2 = 3.86\sigma_R^2 \left\{ \left(1 + \frac{1}{Q_m^2} \right)^{11/12} \sin\left[\frac{11}{6} \tan^{-1} Q_m \right] - \frac{11}{6} Q_m^{-5/6} \right\} \tag{9}$$

Modified[1]

$$\sigma_I^2 = 3.86\sigma_R^2 \left\{ \left(1 + \frac{1}{Q_l^2} \right)^{11/12} [A] - 3.50 Q_l^{-5/6} \right\} \tag{10}$$

$$A = \sin\left(\frac{11}{6} \tan^{-1} Q_l \right) + \frac{1.507}{\left(1 + Q_l^2\right)^{1/4}} \sin\left(\frac{4}{3} \tan^{-1} Q_l \right) - \frac{0.273}{\left(1 + Q_l^2\right)^{7/24}} \sin\left(\frac{5}{4} \tan^{-1} Q_l \right)$$

where $Q_m = \frac{L\kappa_m^2}{k} = \frac{L(5.92/l_0)^2}{k}$ and $Q_l = \frac{L\kappa_l^2}{k} = \frac{L(3.3/l_0)^2}{k}$ are non-dimensional parameters (ratios of inner scale to Fresnel zone). Note that as $l_o \to 0$, scintillation index based on the von Karman and modified spectra reduce to that based on the Kolmogorov spectrum.

3.2 Spherical Wave

For a spherical wave, $\Lambda = 0, \Theta = 0, \overline{\Theta} = 1 - \Theta = 1$, equation 6 reduces to

$$\sigma_{I,l}^2(L) = 8\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa \Phi_n(\kappa) \left[1 - \cos\left(\frac{L\kappa^2}{k} \xi(1-\xi) \right) \right] d\kappa d\xi \tag{11}$$

Performing the integration in equation 11, with Kolmogorov, von Karman, and modified atmospheric spectra expressions given in equations 1, 3, and 4, results in the following scintillation index expressions for spherical waves in weak optical turbulence (see appendices D, E, and F).

Kolmogorov[1,10]

$$\sigma_I^2(L) = 0.4\sigma_R^2 \tag{12}$$

von Karman[1]

$$\sigma_I^2(L) = 3.86\sigma_R^2\left(\left\{0.4\left(1+\frac{9}{Q_m^2}\right)^{11/12}\sin\left[\frac{11}{6}\tan^{-1}\left(\frac{Q_m}{3}\right)\right]\right\}-\frac{11}{6}Q_m^{-5/6}\right) \tag{13}$$

Modified[1]

$$\sigma_I^2(L) = 3.86\sigma_R^2\left\{0.40\left(1+\frac{9}{Q_l^2}\right)^{11/12}[D]-3.50Q_l^{-5/6}\right\} \tag{14}$$

$$D = \sin\left(\frac{11}{6}\tan^{-1}\frac{Q_l}{3}\right)+\frac{2.61}{(9+Q_l^2)^{1/4}}\sin\left(\frac{4}{3}\tan^{-1}\frac{Q_l}{3}\right)-\frac{0.52}{(9+Q_l^2)^{7/24}}\sin\left(\frac{5}{4}\tan^{-1}\frac{Q_l}{3}\right)$$

4.0 SCINTILLATION IN SATURATION REGIME

4.1 Plane Wave

In the saturation regime, the scintillation index for an unbounded plane wave can be expressed in the form[1,8, 34]

$$\sigma_I^2(L) = 1 + 32\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa \Phi_n(\kappa) \sin^2\left[\frac{L\kappa^2}{2k} h(\xi,\xi)\right] \exp\left\{-\int_0^1 D_s\left[\frac{L\kappa}{k} h(\tau,\xi)\right] d\tau\right\} d\kappa d\xi \qquad (15)$$

where

$$h(\tau,\xi) = \begin{cases} \tau(1-\beta\xi), & \tau < \xi \\ \xi(1-\beta\tau), & \tau > \xi \end{cases}$$

$\beta = 0$ for a plane wave (note, $\beta = 1$ for a spherical wave)

D_s is the phase structure function for a plane wave

Assuming $D_{pl} = D_\chi + D_s \cong 2D_s$

where

D_{pl} is the wave structure function for a plane wave

D_χ is the log-amplitude structure function for a plane wave

Performing the integration in equation 15, with Kolmogorov, von Karman, and modified atmospheric spectra expressions given in equations 1, 3, and 4, results in the following saturated optical turbulence scintillation index expressions (see appendices G, H, and I).

Kolmogorov[1,8,12,13] $\qquad \sigma_I^2(L) = 1 + \dfrac{0.86}{\left(\sigma_R^2\right)^{2/5}}, \qquad \sigma_R^2 >> 1 \qquad (16)$

von Karman[1] $\qquad \sigma_I^2(L) = 1 + \dfrac{3.40}{\left(\sigma_R^2 Q_m^{7/6}\right)^{1/6}}, \qquad \sigma_R^2 Q_m^{7/6} >>100 \qquad (17)$

Modified[8,14] $$\sigma_I^2(L) = 1 + \frac{2.39}{\left(\sigma_R^2 Q_l^{7/6}\right)^{1/6}}, \qquad \sigma_R^2 Q_l^{7/6} >> 100 \qquad (18)$$

4.2 Spherical Wave

For β = 1, performing the integration in equation 15 with Kolmogorov, von Karman, and modified atmospheric spectra expressions results in the following spherical wave saturated optical turbulence scintillation index expressions (see appendices J, K, and L).

Kolmogorov[1,8,12,13] $$\sigma_I^2 = 1 + 2.73/\left(\sigma_R^2\right)^{2/5} \qquad (19)$$

von Karman[1] $$\sigma_I^2 = 1 + 10.36/\left(\sigma_R^2 Q_m^{7/6}\right)^{1/6} \qquad (20)$$

Modified[8,14] $$\sigma_I^2 = 1 + 7.65/\left(\sigma_R^2 Q_l^{7/6}\right)^{1/6} \qquad (21)$$

5.0 SCINTILLATION IN MODERATE TURBULENCE

When inner inner-scale effects are taken into account, the spectrum model under weak irradiance fluctuations is described by[8]

$$\Phi_n(\kappa) = 0.033C_n^2(\kappa^2 + \kappa_0^2)^{-11/6} f(\kappa l_0) \tag{22}$$

Reference equations 1, 3, and 4:

- Kolmogorov, $f(\kappa l_0) = 1$ and $\kappa_0 = 0$
- von Karman $f(\kappa l_0) = \exp(-\kappa^2/\kappa_m^2)$
- Modified $f(\kappa l_0) = \exp(-\kappa^2/\kappa_l^2)\left[1 + 1.802\kappa/\kappa_l - 0.254(\kappa/\kappa_l)^{7/6}\right]$

As optical turbulence strength increases, only very large and very small turbulence cell sizes contribute significantly to overall optical refraction and diffraction.[8] As a result, Andrews et al. [8,14,15] were able to extend validity of the Rytov approximation into all optical turbulence regimes by using filter functions to eliminate intermediate scale sizes as optical turbulence strength increases.[8] Large scale and small scale frequency "cutoffs" were established by matching to known weak and saturation results.

Andrews et al. [8,14,15] developed the following "effective" atmospheric spectrum, $\Phi_{n,e}(\kappa)$, which includes an amplitude spatial filter function for eliminating mid-range cells

$$\Phi_{n,e}(\kappa) = 0.033C_n^2\kappa^{-11/3}\left[G_x(\kappa) + G_y(\kappa)\right], \quad \frac{1}{L_0} << \kappa << \frac{1}{l_0} \tag{23}$$

where $G_x(\kappa)$ and $G_y(\kappa)$ are large-scale and small-scale turbulence cell filters, respectively

$$G_x(\kappa) = f(\kappa l_0)\exp(-\kappa^2/\kappa_x^2) \tag{24}$$

$$G_y(\kappa) = \frac{\kappa^{11/3}}{\left(\kappa^2 + \kappa_y^2\right)^{11/6}} \tag{25}$$

$f(\kappa l_0)$ is a factor that describes basic power law form inner scale modifications

κ_x is the spatial frequency cutoff for the large-scale turbulent cells

κ_y is the spatial frequency cutoff for the small-scale turbulent cells

5.1 Existing Plane Wave Equations

Using the "effective" atmospheric spectrum model and establishing large scale and small scale frequency cutoffs from the previously listed weak and saturation results, Andrews et al. [8,14,15] developed the following scintillation index expressions for plane wave propagation in all optical turbulence regimes (see appendices M and N).

Effective Kolmogorov[8]

$$\sigma_I^2(L) = \exp\left[\frac{0.49\sigma_R^2}{\left(1+1.11\sigma_R^{12/5}\right)^{7/6}} + \frac{0.51\sigma_R^2}{\left(1+0.69\sigma_R^{12/5}\right)^{5/6}}\right] - 1, \quad 0 \le \sigma_R^2 < \infty \tag{26}$$

Effective Modified[8]

$$\sigma_I^2(L) = \exp\left[\sigma_{\ln x}^2 + \frac{0.51\sigma_\rho^2}{\left(1+0.69\sigma_\rho^{12/5}\right)^{5/6}}\right] - 1 \tag{27}$$

where $\sigma_{\ln x}^2$ is the large-scale log-irradiance variance

$$\sigma_{\ln x}^2 = 0.16\sigma_R^2\left(\frac{2.61Q_l}{2.61+Q_l+0.45\sigma_R^2 Q_l^{7/6}}\right)^{7/6}[B]$$

$$[B] = \left[1+1.75\left(\frac{2.61}{2.61+Q_l+0.45\sigma_R^2 Q_l^{7/6}}\right)^{1/2} - 0.25\left(\frac{2.61}{2.61+Q_l+0.45\sigma_R^2 Q_l^{7/6}}\right)^{7/12}\right]$$

σ_p^2 is the Rytov variance for a plane wave with inner scale

$$\sigma_p^2 = 3.86\sigma_R^2\left\{\left(1+\frac{1}{Q_l^2}\right)^{11/12}[C]-3.50Q_l^{-5/6}\right\}$$

$$C = \sin\left(\frac{11}{6}\tan^{-1}Q_l\right)+\frac{1.507}{\left(1+Q_l^2\right)^{1/4}}\sin\left(\frac{4}{3}\tan^{-1}Q_l\right)-\frac{0.273}{\left(1+Q_l^2\right)^{7/24}}\sin\left(\frac{5}{4}\tan^{-1}Q_l\right)$$

Figure 3 plots the effective modified spectrum scintillation index against σ_R for various inner scale values. It shows that as inner scale increases, scintillation increases.

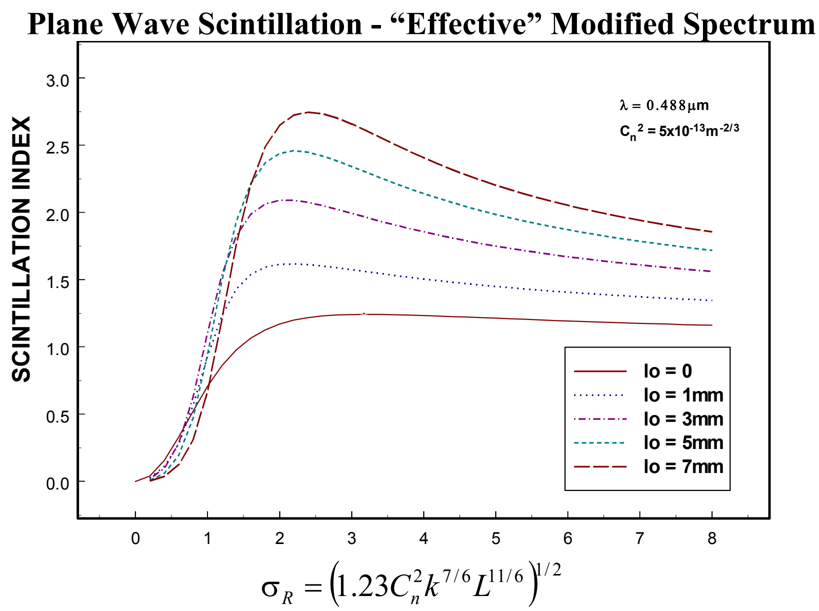


Figure 3 – Effective Modified Scintillation Index (Equation 27) vs. σ_R For Various Inner Scale Values

5.2 Existing Spherical Wave Equations

Using the "effective" atmospheric spectrum model and establishing large scale and small scale frequency cutoffs from the previously listed weak and saturation results, Andrews et al. [8,14,15] developed the following scintillation index expressions for spherical wave propagation in all optical turbulence regimes (see appendices O and P).

Kolmogorov[8,14,15]

$$\sigma_I^2(L) = \exp\left[0.196\sigma_R^2 / \left(1 + 0.186\sigma_R^{12/5}\right)^{7/6} + 0.204\sigma_R^2 / \left(1 + 0.230\sigma_R^{12/5}\right)^{5/6}\right] - 1 \tag{28}$$

Modified[8]

$$\sigma_I^2(L) = \exp\left[\sigma_{\ln x}^2(l_0) + 0.51\sigma_s^2 / \left(1 + 0.69\sigma_s^{12/5}\right)^{5/6}\right] - 1 \tag{29}$$

where

$$\sigma_{\ln x}^2 = 0.016\sigma_R^2 \left(\frac{8.56Q_l}{8.56 + Q_l + 0.08\sigma_R^2 Q_l^{7/6}}\right)^{7/6} [E]$$

$$E = 1 + 1.76\left(\frac{8.56}{8.56 + Q_l + 0.08\sigma_R^2 Q_l^{7/6}}\right)^{1/2} - 0.25\left(\frac{8.56}{8.56 + Q_l + 0.08\sigma_R^2 Q_l^{7/6}}\right)^{7/12}$$

$$\sigma_s^2 = 3.86\sigma_R^2 \left\{0.40\left(1 + \frac{9}{Q_l^2}\right)^{11/12} [F] - \frac{3.50}{Q_l^{5/6}}\right\}$$

$$F = \sin\left(\frac{11}{6}\tan^{-1}\frac{Q_l}{3}\right) + \frac{2.61}{(9 + Q_l^2)^{1/4}}\sin\left(\frac{4}{3}\tan^{-1}\frac{Q_l}{3}\right) - \frac{0.52}{(9 + Q_l^2)^{7/24}}\sin\left(\frac{5}{4}\tan^{-1}\frac{Q_l}{3}\right)$$

5.3 New Moderate Turbulence Equations

Although scintillation index expressions have been developed based on the Kolmogorov and modified spectra that are valid in all optical turbulence regimes, filter functions corresponding to the von Karman spectrum have not yet been developed. This section uses a similar approach, to that used by Andrews et al.[8,14,15], to establish these filters. The resulting scintillation index will be compared to that developed for the modified spectrum to determine the importance of using a bump spectrum as turbulence strength increases.

A more detailed breakdown of the mathematics described by this section is given in Appendices Q and R.

5.3.1 Plane Wave

5.3.1.1 Large-Scale Log-Irradiance Variance

Using the "effective" atmospheric spectrum (equation 23), the large-scale log-irradiance variance for a plane wave in the presence of a finite inner scale is[8]

$$\sigma^2_{\ln x}(L) = 8\pi^2 k^2 \int_0^L \int_0^\infty \kappa 0.033 C_n^2 \kappa^{-11/3} G_x(\kappa) \left[1 - \cos\left(\frac{\kappa^2 z}{k}\right)\right] d\kappa dz \tag{30}$$

where $G_x(\kappa) = f(\kappa l_0)\exp(-\kappa^2/\kappa_x^2) = \exp(-\kappa^2/\kappa_m^2)\exp(-\kappa^2/\kappa_x^2)$ for the von Karman spectrum.

By assuming a Taylor expansion estimate of $1 - \cos\left(\frac{\kappa^2 \xi}{k}\right) \cong \frac{1}{2}\left(\frac{\kappa^2 \xi}{k}\right)^2$, and by substituting $\xi = z/L$ and $\eta = L\kappa^2/k$, equation 30 reduces to

$$\sigma^2_{\ln x}(L) = 2\pi^2 k L^2 \int_0^1 \int_0^\infty \kappa^4 0.033 C_n^2 \kappa^{-11/3} G_x(\kappa) \xi^2 d\eta d\xi \tag{31}$$

Upon integration

$$\begin{aligned} \sigma^2_{\ln x}(L) &\cong 0.16\sigma_R^2 \left(\frac{\eta_x Q_m}{\eta_x + Q_m}\right)^{7/6} \\ &\cong 0.16\sigma_R^2 \eta_x^{7/6} \quad \text{as } l_0 \to 0 \end{aligned} \tag{32}$$

where $\eta = \dfrac{L\kappa^2}{k}$ is a non-dimensional cutoff frequency for the filter function.

5.3.1.2 Small-Scale Log-Irradiance Variance

Using the "effective" atmospheric spectrum (equation 23), the small-scale log-irradiance variance for a plane wave in the presence of a finite inner scale is[8]

$$\sigma^2_{\ln y}(L) = 8\pi^2 k^2 \int_0^L \int_0^\infty \kappa 0.033 C_n^2 \kappa^{-11/3} G_y(\kappa)\left[1-\cos\left(\frac{\kappa^2 z}{k}\right)\right] d\kappa dz$$

$$\cong 8\pi^2 k^2 \int_0^L \int_0^\infty \kappa 0.033 C_n^2 \kappa^{-11/3} G_y(\kappa) d\kappa dz \qquad (33)$$

Substituting $\xi = z/L$, and $\eta = L\kappa^2/k$, equation 33 reduces to

$$\sigma^2_{\ln y}(L) \cong 4\pi^2 k^3 \int_0^1 \int_0^\infty 0.033 C_n^2 \kappa^{-11/3} G_y(\kappa) d\eta d\xi \qquad (34)$$

where $G_y(\kappa) = \dfrac{\kappa^{11/3}}{\left(\kappa^2 + \kappa_y^2\right)^{11/6}}$

Evaluating the integral leads to

$$\sigma^2_{\ln y}(L) = 1.272\sigma_R^2 \eta_y^{-5/6} \qquad (35)$$

5.3.1.3 Asymptotic Comparisons

This thesis takes the same approach as used by Andrews, Phillips, and Hopen[14] to perform asymptotic comparisons of known scintillation behavior in weak and saturated regimes to determine frequency cutoffs for the filter functions. We assume the following functional form of η_x and η_y [see Appendix S].

$$\eta_x = \frac{1}{c_1 + c_2 L/k\rho_o^2} \cong \begin{cases} \dfrac{1}{c_1 + c_2(0)} = \dfrac{1}{c_1}, & \dfrac{L}{k\rho_0^2} << 1 \\ \dfrac{1}{0 + c_2 L/k\rho_o^2} = \dfrac{k\rho_o^2}{c_2 L}, & \dfrac{L}{k\rho_0^2} >> 1 \end{cases} \qquad (36)$$

$$\eta_y = c_3 + \frac{c_4 L}{k\rho_0^2} \cong \begin{cases} c_3 + c_4(0) = c_3, & \dfrac{L}{k\rho_0^2} << 1 \\ 0 + \dfrac{c_4 L}{k\rho_0^2} = \dfrac{c_4 L}{k\rho_0^2}, & \dfrac{L}{k\rho_0^2} >> 1 \end{cases} \qquad (37)$$

where c_1, c_2, c_3, and c_4 are constants to be determined.

5.3.1.4 Weak Turbulence

The log irradiance variance, $\sigma^2_{\ln I}$, is defined under the Rytov approximation for weak turbulence by[8]

$$\sigma^2_{\ln I} = 8\pi^2 k^2 \int_0^L \int_0^\infty \kappa \Phi_n(\kappa) \left[1 - \cos\left(\frac{\kappa^2 z}{k} \right) \right] d\kappa dz \tag{38}$$

Substituting $\xi = z/L$ and $\eta = L\kappa^2/k$, the equation reduces to

$$\sigma^2_{\ln I} = 4\pi^2 k^3 \int_0^1 \int_0^\infty \Phi_n(\kappa) [1 - \cos(\eta\xi)] d\eta d\xi \tag{39}$$

Upon evaluating the integral,

$$\sigma^2_{\ln I} = \sigma^2_R \tag{40}$$

Under weak irradiance fluctuations[8], $\sigma^2_{\ln I} << 1$,

$$\sigma^2_{\ln I} = \sigma^2_{\ln x} + \sigma^2_{\ln y} \tag{41}$$

Equating equations 40 and 41, and using equations 32 and 35, results in

$$0.16\eta_x^{7/6} + 1.272\eta_y^{-5/6} \cong 1, \quad \sigma^2_R << 1, \quad l_0 \to 0 \tag{42}$$

Assuming $\eta_x \cong \eta_y$, then $\eta_x = \eta_y = 3$ is an approximate solution.

Substituting $\eta_x = \eta_y = 3$ into equations 36 and 37

$$\eta_x = \frac{1}{c_1 + c_2 L / k\rho_o^2} \cong \frac{1}{c_1 + c_2(0)} = \frac{1}{c_1} = 3$$

$$\to c_1 \cong 1/3 \tag{43}$$

$$\eta_y = c_3 + \frac{c_4 L}{k\rho_0^2} \cong c_3 + c_4(0) = c_3 = 3$$

$$\rightarrow c_3 \cong 3 \tag{44}$$

Thus, c_1 and c_3 are now established using behavior of the scintillation index in the weak turbulence regime.

5.3.1.5 Saturated Turbulence

First, perform an asymptotic comparison for small-scale turbulent cell effects. Normalized irradiance[8] can be expressed as I = *xy* where *x* is associated with large-scale turbulent eddy affects, *y* is associated with small-scale eddy effects, *x* and *y* are statistically independent random quantities, and $<x>=<y>=1$. Given these conditions, $<I> = <x><y> = 1$.

The second moment of irradiance takes the form[8]

$$<I^2> = <x^2><y^2> = \left(1+\sigma_x^2\right)\left(1+\sigma_y^2\right) \tag{45}$$

where σ_x^2 and σ_y^2 are the large scale and small scale normalized variances, respectively. Based on equations 5 and 45, the implied scintillation index is

$$\begin{aligned} \sigma_I^2 &= \left(1+\sigma_x^2\right)\left(1+\sigma_y^2\right) - 1 \\ &= \sigma_x^2 + \sigma_y^2 + \sigma_x^2\sigma_y^2 \end{aligned} \tag{46}$$

Per equation 17, the asymptotic behavior of the scintillation index in the saturation regime is described by $\sigma_I^2 = 1 + 3.40/\left(\sigma_R^2 Q_m^{7/6}\right)^{1/6}$, $\sigma_R^2 Q_m^{7/6}$ >>100 which approaches an asymptotic limit of unity. Therefore, in saturated turbulence

$$\sigma_I^2 = \sigma_x^2 + \sigma_y^2 + \sigma_x^2\sigma_y^2 \cong 1 \tag{47}$$

In strong fluctuations, expect the large-scale scintillation terms to die out.

$$\sigma_I^2 \approx \sigma_y^2 \cong 1 \tag{48}$$

The small scale irradiance is given by[8]

$$\sigma_y^2 = \exp\left(\sigma_{\ln y}^2\right) - 1 \tag{49}$$

As a result, in strong fluctuations

$$\exp\left(\sigma_{\ln y}^2\right) - 1 \cong 1 \tag{50}$$

$$\rightarrow \sigma_{\ln y}^2 \cong \ln 2 \tag{51}$$

When $l_0 << \rho_0 << L_0$, the spatial coherence radius of a plane wave ρ_0 can be approximated by[1]

$$\rho_0 = \left(1.46 C_n^2 k^2 L\right)^{-3/5} \rightarrow \frac{L}{k\rho_0^2} = 1.22\left(\sigma_R^2\right)^{6/5} \tag{52}$$

Recall that in the presence of a finite inner scale, $\sigma_{\ln y}^2 \cong 1.272\sigma_R^2 \eta_y^{-5/6}$ (equation 35). In saturated turbulence, the small-scale cutoff is given by $\eta_y = \frac{c_4 L}{k\rho_0^2}$ (equation 37). Therefore, in saturated turbulence

$$\sigma_{\ln y}^2 \cong 1.272\sigma_R^2 \left[c_4 1.22\left(\sigma_R^2\right)^{6/5}\right]^{-5/6} \cong \ln 2 \tag{53}$$

$$\rightarrow c_4 \cong 1.7 \tag{54}$$

Now determine an asymptotic comparison for large-scale turbulent cell effects. For saturated optical turbulence[8]

$$\sigma_x^2 = \exp\left(\sigma_{\ln x}^2\right) - 1 \tag{55}$$

$$\rightarrow \exp\left(\sigma_{\ln x}^2\right) = \sigma_x^2 + 1$$

Also[8]

$$\sigma_I^2 = \exp\left(\sigma_{\ln x}^2 + \sigma_{\ln y}^2\right) - 1 \tag{56}$$

Therefore,

$$\begin{aligned}\sigma_I^2 &\cong \exp\left(\sigma_{\ln x}^2 + \ln 2\right) - 1 \\ &= 2\exp\left(\sigma_{\ln x}^2\right) - 1 \\ &= 2\left(\sigma_x^2 + 1\right) - 1 \\ &= 2\sigma_x^2 + 1 \qquad (57)\end{aligned}$$

Under weak fluctuation $\left(\sigma_x^2 << 1\right)$,

$$\sigma_x^2 = \exp\left(\sigma_{\ln x}^2\right) - 1 \cong \sigma_{\ln x}^2 \qquad (58)$$

Recall from equation 32 that $\sigma_{\ln x}^2 \cong 0.16\sigma_R^2 \eta_x{}^{7/6}$ as $l_0 \to 0$.

As a result

$$\sigma_I^2 \cong 0.32\sigma_R^2 \eta_x{}^{7/6} + 1 \qquad (59)$$

For a von Karman spectrum, the plane wave spatial coherence radius is[1] $\rho_0 = \left(1.64 C_n^2 k^2 L l_0^{-1/3}\right)^{-1/2}$.

$$\begin{aligned}L / k\rho_0^2 &= \frac{L}{k}\left[\left(1.64 C_n^2 k^2 L l_0^{-1/3}\right)^{-1/2}\right]^{-2} \\ &= 0.737\left(1.23 C_n^2 k^{7/6} L^{11/6}\right)\left(\frac{35.05L}{kl_0^2}\right)^{1/6} \\ &= 0.737\sigma_R^2 Q_m^{1/6} \qquad (60)\end{aligned}$$

Reference equation 36, assuming $\frac{L}{k\rho_0^2} >> 1$

$$\begin{aligned}\eta_x &= \frac{1}{c_1 + c_2 L / k\rho_o^2} \cong \frac{1}{c_2 L / k\rho_o^2} = \frac{1}{c_2 0.737\sigma_R^2 Q_m^{1/6}} \\ \to \sigma_I^2 &\cong 0.32\sigma_R^2\left(\frac{1}{c_2 0.737\sigma_R^2 Q_m^{1/6}}\right)^{7/6} + 1 \\ &= 0.46\left(\frac{1}{c_2}\right)^{7/6}\left(\frac{1}{\sigma_R^{1/3}}\right)\left(\frac{1}{Q_m^{7/36}}\right) + 1 \qquad (61)\end{aligned}$$

Solve for c_2 by equating 61 and equation 17.

$$1+0.46\left(\frac{1}{c_2}\right)^{7/6}\left(\frac{1}{\sigma_R^{1/3}}\right)\left(\frac{1}{Q_m^{7/36}}\right)=1+\frac{3.40}{(\sigma_R^2 Q_m^{7/6})^{1/6}} \tag{62}$$

$$\rightarrow c_2 = 0.18 \tag{63}$$

Thus, c_2 and c_4 are now established using behavior of the scintillation index in the saturated turbulence regime.

5.3.1.6 "All" Turbulence Regimes

Applying constants c_1, c_2, c_3, and c_4 to equations 32, 35, 36, 37, 52, and 60 results in the following expressions for predicting large and small scale log irradiance behavior in all turbulence regimes (including inner scale effects):

$$\sigma_{\ln x}^2 \cong 0.16\sigma_R^2\left(\frac{3Q_m}{3+Q_m+0.40\sigma_R^2 Q_m^{7/6}}\right)^{7/6} \tag{64}$$

$$\sigma_{\ln y}^2 \cong \frac{0.509\sigma_R^2}{\left(1+0.69\sigma_R^{12/5}\right)^{5/6}} \tag{65}$$

To apply inner-scale effects to the von Karman derived scintillation index,

let (ref. equation 9, plane wave in weak turbulence)

$$\sigma_I^2=\sigma_p^2=3.86\sigma_R^2\left\{\left(1+\frac{1}{Q_m^2}\right)^{11/12}\sin\left(\frac{11}{6}\tan^{-1}Q_m\right)-\frac{11}{6}Q_m^{-5/6}\right\} \tag{66}$$

impose

$$\sigma_{\ln y}^2 \cong \frac{0.509\sigma_R^2}{(1+0.69\sigma_R^{12/5})^{5/6}} \cong 0.51\sigma_R^2 = 0.51\sigma_p^2 \tag{67}$$

Recall, from equation 25, that $\sigma_{\ln y}^2(L)=1.272\sigma_R^2\eta_y^{-5/6}$. Then,

$$\eta_y \cong \left(\frac{1.272\sigma_R^2}{0.51\sigma_p^2}\right)^{6/5} \cong 3\left(\frac{\sigma_R^2}{\sigma_p^2}\right)^{6/5} \tag{68}$$

Also assume that under strong fluctuations, inner-scale effects tend to diminish such that $\eta_y \cong \frac{1.7L}{k\rho_0^2}$. Recall that $\frac{L}{k\rho_0^2} \cong 1.22\left(\sigma_R^2\right)^{6/5}$ (equation 52). Then, under strong fluctuations

$$\eta_y \cong 2.07\left(\sigma_R^2\right)^{6/5} \tag{69}$$

As a result, for all fluctuations

$$\begin{aligned} \eta_y &\cong 3\left(\frac{\sigma_R^2}{\sigma_p^2}\right)^{6/5} + 2.07\left(\sigma_R^2\right)^{6/5} \\ &= 3\left(\frac{\sigma_R^2}{\sigma_p}\right)^{12/5} (1 + 0.69\sigma_p^{12/5}) \end{aligned} \tag{70}$$

The small-scale log-irradiance variance becomes

$$\begin{aligned} \sigma_{\ln y}^2(L) &\cong 1.272\sigma_R^2\left[3\left(\frac{\sigma_R}{\sigma_p}\right)^{12/5}\left(1 + 0.69\sigma_p^{12/5}\right)\right]^{-5/6} \\ &= \frac{0.51\sigma_p^2}{\left(1 + 0.69\sigma_p^{12/5}\right)^{5/6}} \end{aligned} \tag{71}$$

Therefore, reference equation 57 $\sigma_I^2 = \exp\left(\sigma_{\ln x}^2 + \sigma_{\ln y}^2\right) - 1$ and equation 64, the resulting scintillation index for use in all optical turbulence regimes is

$$\sigma_I^2 \cong \exp\left[0.16\sigma_R^2\left(\frac{3Q_m}{3 + Q_m + 0.40\sigma_R^2 Q_m^{7/6}}\right)^{7/6} + \frac{0.51\sigma_p^2}{\left(1 + 0.69\sigma_p^{12/5}\right)^{5/6}}\right] - 1 \tag{72}$$

where $\sigma_p^2 = 3.86\sigma_R^2\left[\left(1 + \frac{1}{Q_m^2}\right)^{11/12} \sin\left(\frac{11}{6}\tan^{-1} Q_m\right) - \frac{11}{6}Q_m^{-5/6}\right]$

5.3.2 Spherical Wave

Using the “effective” atmospheric spectrum (equation 23), the large-scale log-irradiance variance for a spherical wave in the presence of a finite inner scale is given by[8]

$$\sigma_{\ln x}^2 = 8\pi^2 k^2 \int_0^L \int_0^\infty \kappa 0.033 C_n^2 \kappa^{-11/3} G_x(\kappa) \left\{1 - \cos\left[\frac{\kappa^2}{k} z\left(1 - \frac{z}{L}\right)\right]\right\} d\kappa dz \tag{73}$$

Assuming cos(x) = $1 - \frac{x^2}{2!}$, and substituting $\xi = z/L$ and $\eta = L\kappa^2/k$, results in

$$\sigma_{\ln x}^2 = 2\pi^2 k L^2 \int_0^1 \int_0^\infty 0.033 C_n^2 \kappa^{-11/3} G_x(\kappa) \kappa^4 \xi^2 (1-\xi)^2 d\eta d\xi \tag{74}$$

Integrating equation 73, results in

$$\begin{aligned}\sigma_{\ln x}^2(L) &= 0.016\sigma_R^2 \left(\frac{\eta_x Q_m}{\eta_x + Q_m}\right)^{7/6} \\ &= 0.016\sigma_R^2 \mu_x^{7/6}\end{aligned} \tag{75}$$

In the small-scale log-irradiance variance, let

$$\begin{aligned}\sigma_{\ln y}^2 &= 8\pi^2 k^2 \int_0^L \int_0^\infty \kappa 0.033 C_n^2 \kappa^{-11/3} G_y(\kappa) \left\{1 - \cos\left[\frac{\kappa^2}{k} z\left(1 - \frac{z}{L}\right)\right]\right\} d\kappa dz \\ &\cong 8\pi^2 k^2 \int_0^L \int_0^\infty \kappa 0.033 C_n^2 \kappa^{-11/3} G_y(\kappa) d\kappa dz, \qquad \kappa_y >> \sqrt{k/L}\end{aligned} \tag{76}$$

This is valid under moderate-to-strong fluctuations and leads to the same approximation derived by the plane wave solution (equation 35), $\sigma_{\ln y}^2(L) \cong 1.272\sigma_R^2 \eta_y^{-5/6}$.

Assume in equations 13 and 74 that $l_0 \to 0$, $Q_m \to \infty$

$$\sigma_I^2 = 0.4\sigma_R^2 \tag{77}$$

$$\sigma_{\ln x}^2(L) = 0.016\sigma_R^2 \eta_x^{7/6} \tag{78}$$

Under weak fluctuation, $\sigma_I^2 << 1$. As a result,

$$\sigma_I^2 = \exp\left(\sigma_{\ln I}^2\right) - 1 \cong \sigma_{\ln I}^2 \tag{79}$$

Therefore, in weak turbulence

$$\begin{aligned}\sigma_{\ln I}^2 &= \sigma_{\ln x}^2 + \sigma_{\ln y}^2 \\ &= 0.016\sigma_R^2 \eta_x^{\ 7/6} + 1.272\sigma_R^2 \eta_y^{\ -5/6} \\ &= 0.4\sigma_R^2, \quad \sigma_R^2 << 1\end{aligned} \tag{80}$$

$$\rightarrow 0.4 \cong 0.016\eta_x^{\ 7/6} + 1.272\eta_y^{\ -5/6} \tag{81}$$

Assuming $\eta_x \cong \eta_y$, then $\eta_x = \eta_y = 8$ is an approximate solution

$$c_1 \cong 1/8, \quad c_3 \cong 8 \tag{82}$$

For saturated turbulence, assuming[1] $\rho = \left(0.55 C_n^2 k^2 L\right)^{-3/5}$,

$$c_4 \cong 5.45, \quad c_2 = 0.0285 \tag{83}$$

Performing comparisons to the strong turbulence regime, using the same approach as for the plane wave calculations, results in

$$\sigma_I^2(L) = \exp\left[\sigma_{\ln x}^2\left(l_0\right) + \frac{0.51\sigma_s^2}{\left(1 + 0.69\sigma_s^{12/5}\right)^{5/6}}\right] - 1 \tag{84}$$

where

$$\sigma_{\ln x}^2\left(l_0\right) = 0.016\sigma_R^2 \left(\frac{8Q_m}{8 + Q_m + 0.0564\sigma_R^2 Q_m^{7/6}}\right)^{7/6}$$

$$\sigma_s^2 = 3.86\sigma_R^2 \left(\left\{0.4(1 + \frac{9}{Q_m^2})^{11/12} \sin\left[\left(\frac{11}{6}\right)\tan^{-1}\left(\frac{Q_m}{3}\right)\right]\right\} - \frac{11}{6} Q_m^{-5/6}\right)$$

6. 0 COMPARISONS

6.1 Comparison Of The Plane Wave Scintillation Index Models

In Figures 4 and 5, scintillation expressions based on the von Karman spectrum for weak and saturated turbulence regimes are compared to the newly derived scintillation index based on an "effective" von Karman spectrum. For assumed conditions $\lambda = 0.488\mu$m (figure 4), $\lambda = 1.55\mu$m (figure 5), $C_n^2 = 5\,\mathrm{x}\,10^{-13}\,\mathrm{m}^{-2/3}$ and l_0 = 4mm, the new plane wave expressions track previous weak and saturated turbulence curves fairly closely, providing a visual indication that calculations and assumptions used to develop it were correct.

In weak turbulence, σ_R <1, the new expressions agrees with the weak turbulence index to within 31% for $\lambda = 0.488\mu$m (figure 4) and to within 42% for $\lambda = 1.55\mu$m (figure 5). Note that in moderate turbulence, as the wavelength increases the effect of the bump decreases. Also note that in the moderate turbulence regime, of approximately 1< σ_R < 3, the new expressions provides a smooth transition between scintillation expressions predicted by existing weak and saturated turbulence scintillation expressions. For σ_R >3, figures 4 and 5 both agree with the saturated turbulence index to within 6%.

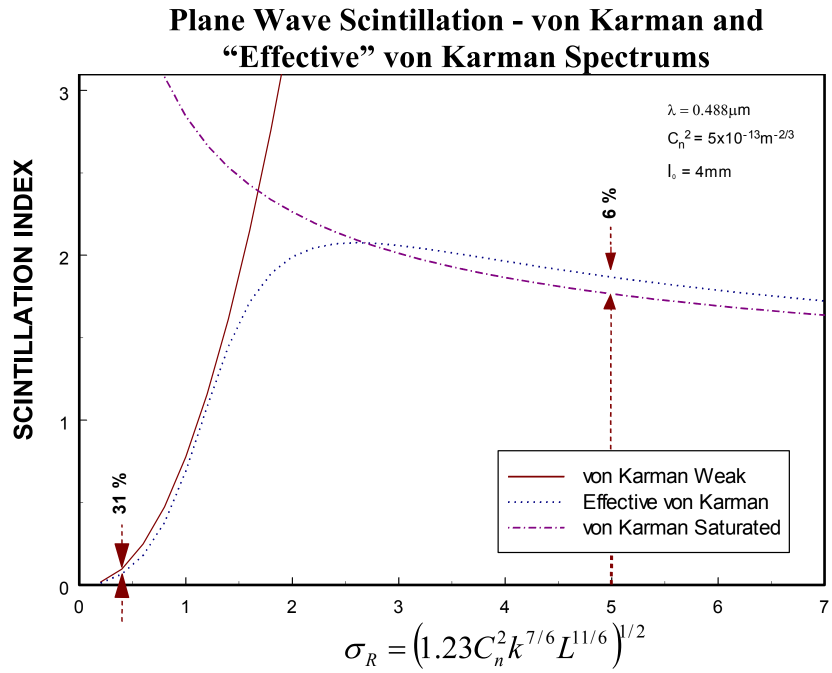


Figure 4 – Plane Wave Scintillation based on the von Karman Spectrum (weak – equation 9, medium [effective spectrum] – equation 72, saturated – equation 17)

$l_0 = 4mm$, $\lambda = 0.488\mu$m

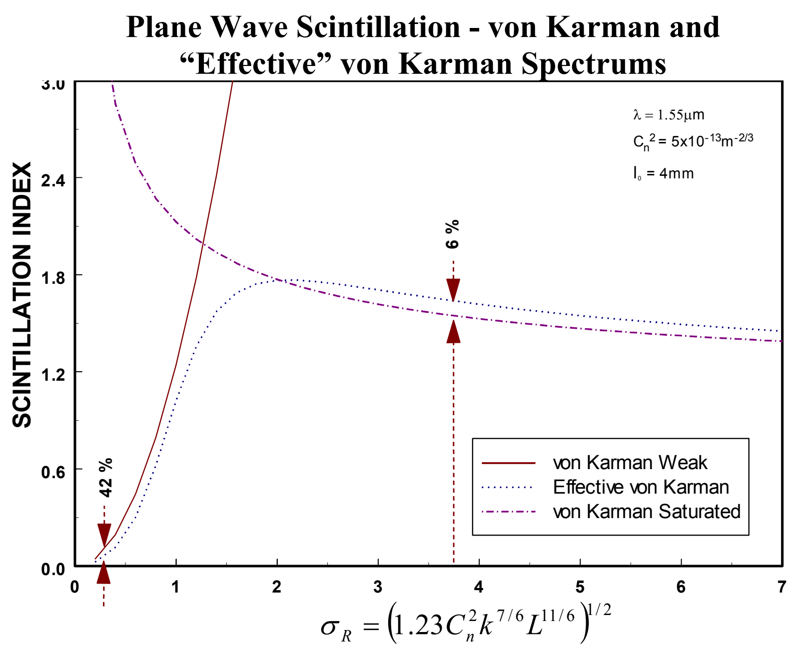


Figure 5 – Plane Wave Scintillation based on the von Karman Spectrum (weak – equation 9, medium [effective spectrum] – equation 72, saturated – equation 17)

$l_0 = 4mm$, $\lambda = 1.55\mu$m

In figures 6 and 7, scintillation expressions are compared based on three atmospheric spectra's (effective Kolmogorov, effective modified "bump", and the newly derived effective von Karman). For assumed conditions $\lambda = 0.488\mu\text{m}$, $C_n^2 = 5\,\text{x}\,10^{-13}\text{m}^{-2/3}$, l_0 = 7mm (figure 6) and l_0 = 4mm (figure 6), the new effective von Karman curves capture the effect of inner scale (which is the increase in scintillation over a non-zero inner scale expression such as Kolmogorov) but do not reach the peak of the bump spectrum in the focusing regime.

In figure 6, the effective modified curve peaks at σ_R = 2.4 with a predicted scintillation of 2.8. The corresponding effective von Karman derived scintillation is 2.3, or approximately 17% less. In figure 7, the effective modified curve peaks at σ_R = 2.2 with a predicted scintillation of 2.3. The corresponding effective von Karman derived prediction is 2.0, or approximately 11% less. Note that in figures 6 and 7, as l_0 gets smaller scintillation predicted by the newly derived effective von Karman spectrum approaches that derived by the effective von Karman spectrum. In extremely strong turbulence, the difference between scintillation derived by the new effective von Karman expression versus the effective Kolmogorov expression is minimal.

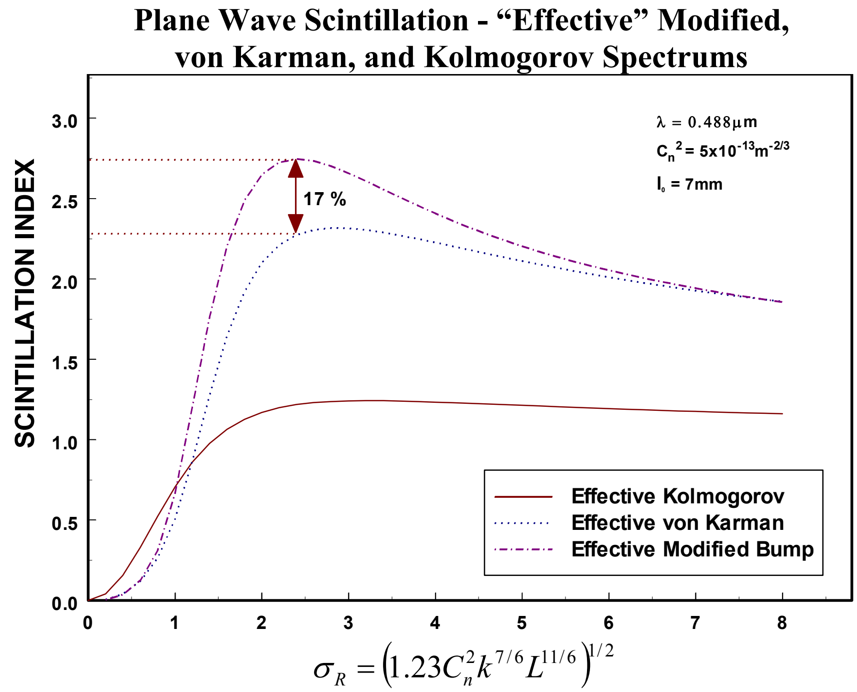


Figure 6 – Scintillation Index based on varied spectra's (equations 26, 27, and 72)
$l_0 = 7\text{mm}$, $\lambda = 0.488\mu\text{m}$

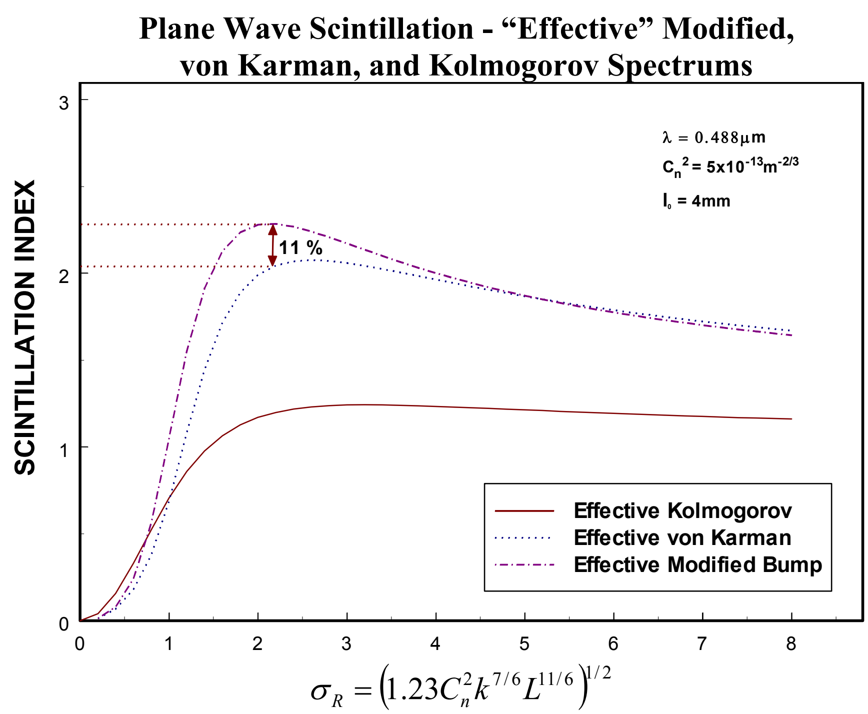
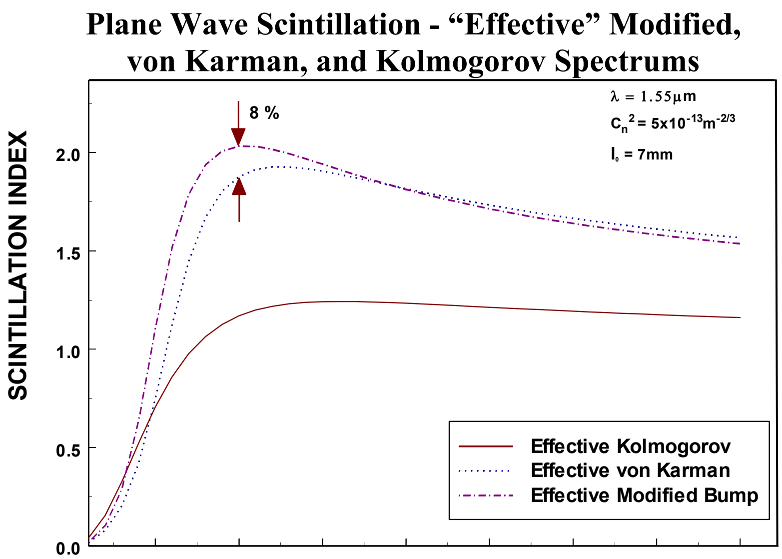


Figure 7 - Scintillation Index based on varied spectra's (equations 26, 27, and 72)
$l_0 = 4\text{mm}$, $\lambda = 0.488\mu\text{m}$

Figures 8 and 9 show the effect of a change in wavelength. In figure 8, the same conditions are assumed as in figure 6, except that the wavelength has been increased from $\lambda = 0.488\mu\text{m}$ to $\lambda = 1.55\mu\text{m}$. The resulting difference between the effective von Karman the modified derived scintillation at the peak of the curve is reduced from 17% (in figure 6) to 8%. Similarly, in figure 9, assuming the same conditions as in figure 7 except for an increase from $\lambda = 0.488\mu\text{m}$ to $\lambda = 1.55\mu\text{m}$, the resulting difference between the effective von Karman the modified derived scintillation at the peak of the curve has been reduced from 11% (in figure 7) to 2%. Note that curves in figures 8 and 9 peak at lower scintillation values than corresponding figures 6 and 7, and note that these peaks occur in the moderate turbulent regimes. Therefore, for the conditions given, in moderate turbulence increased wavelength size diminishes the bump effect.



$$\sigma_R = \left(1.23 C_n^2 k^{7/6} L^{11/6}\right)^{1/2}$$

Figure 8 - Scintillation Index based on varied spectra's (equations 26, 27, and 72)
$l_0 = 7\text{mm}$, $\lambda = 1.55\mu\text{m}$

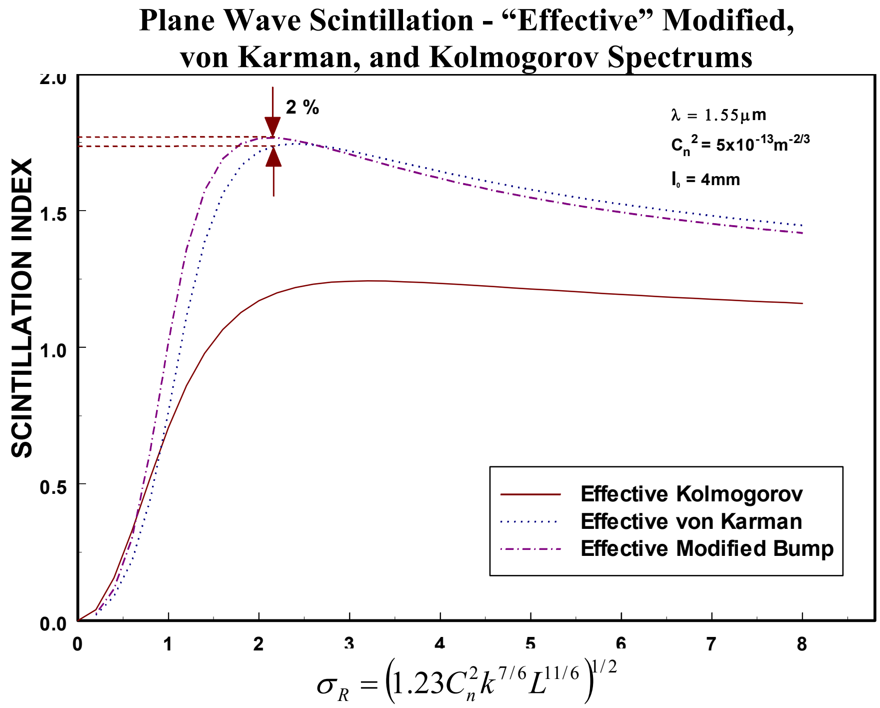
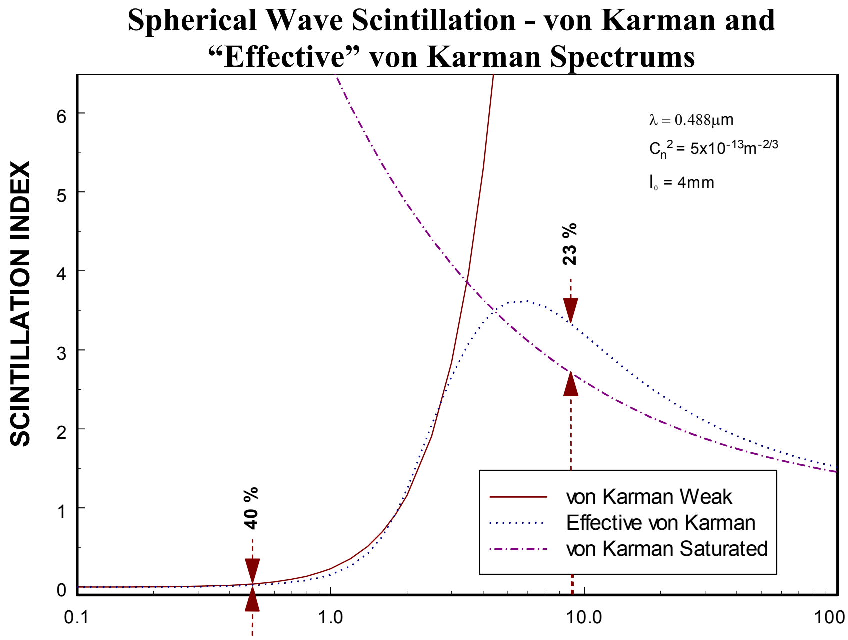


Figure 9 - Scintillation Index based on varied spectra's (equations 26, 27, and 72)
$l_0 = 4\text{mm}$, $\lambda = 1.55\mu\text{m}$

6.2 Comparison Of The Spherical Wave Scintillation Index Models

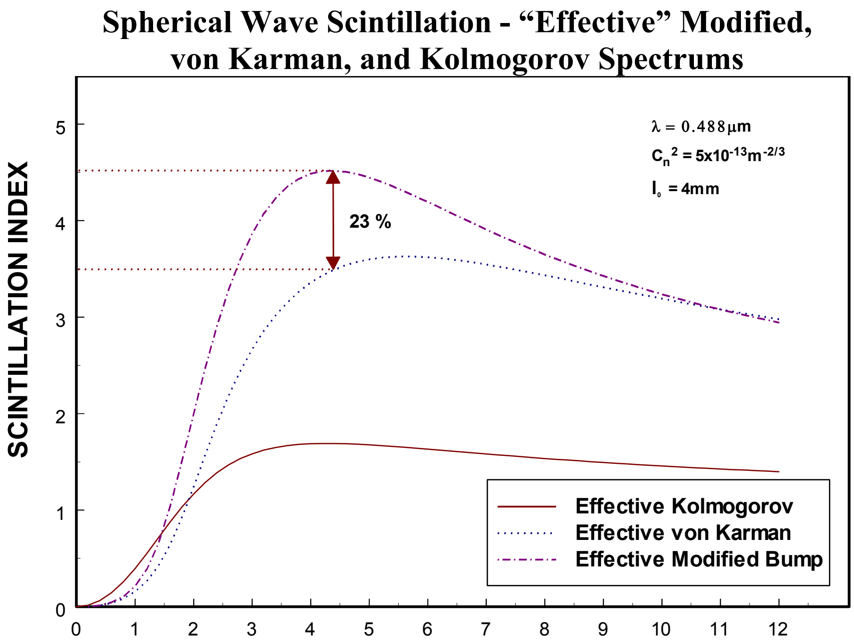
In Figure 10, spherical wave scintillation expressions based on the von Karman spectrum for weak and saturated turbulence regimes are compared to the newly derived spherical wave scintillation index. As with the new plane wave expression, the new spherical wave expression tracks previous weak and saturated turbulence curves fairly closely, providing a visual indication that calculations and assumptions used to develop it were correct. In weak turbulence, σ_R<1, the new expression agrees with the weak scintillation expression to within 40%. For σ_R>4, the new expression agrees with the saturated scintillation expression to within 23%. In the moderate turbulence regime 1< σ_R < 4, the new expression provides a smooth transition between scintillation expressions predicted by existing weak and saturated turbulence scintillation expressions.



$$\sigma_R = \left(1.23 C_n^2 k^{7/6} L^{11/6}\right)^{1/2}$$

Figure 10 – Spherical Wave Scintillation Index based on the von Karman Spectrum (weak – equation 13, medium [effective spectrum] – equation 84, saturated – equation 20)

In figures 11 and 12 scintillation differences are compared between spherical wave models based on effective Kolmogorov, effective modified (bump), and the newly derived effective von Karman spectra's. Assumed conditions are $\lambda = 0.488\mu$m (figure11), $\lambda = 1.55\mu$m (figure 12) $C_n^2 = 5x10^{-13}\text{m}^{-2/3}$, and l_0 = 4mm. In figure 11, the effective modified curve peaks at σ_R = 4.4 with a predicted scintillation of 4.5. The corresponding effective von Karman derived prediction is 3.5, or approximately 23% less. In figure 12, the effective modified curve peaks at σ_R = 3.8 with a predicted scintillation of 2.9. The corresponding effective von Karman derived prediction is 2.6, or approximately 9% less As with the new effective von Karman plane wave model, the new expression captures an increase in scintillation with inner scale, but does not reach the peak of the bump spectrum in the focusing regime. Note that curves in figures 11 peak at lower scintillation values than in figures 12, and note that these peaks occur in the moderate turbulent regimes. Therefore in moderate turbulence, for the conditions given, increased wavelength size diminishes the bump effect. In extremely strong turbulence, the difference between scintillation derived by the new effective von Karman expression versus the effective Kolmogorov expression is minimal.



$$\sigma_R = \left(1.23 C_n^2 k^{7/6} L^{11/6}\right)^{1/2}$$

Figure 11 - Scintillation Index based on varied spectra's (equations 28, 29, and 84) $l_0 = 4\text{mm}$, $\lambda = 0.488\mu\text{m}$

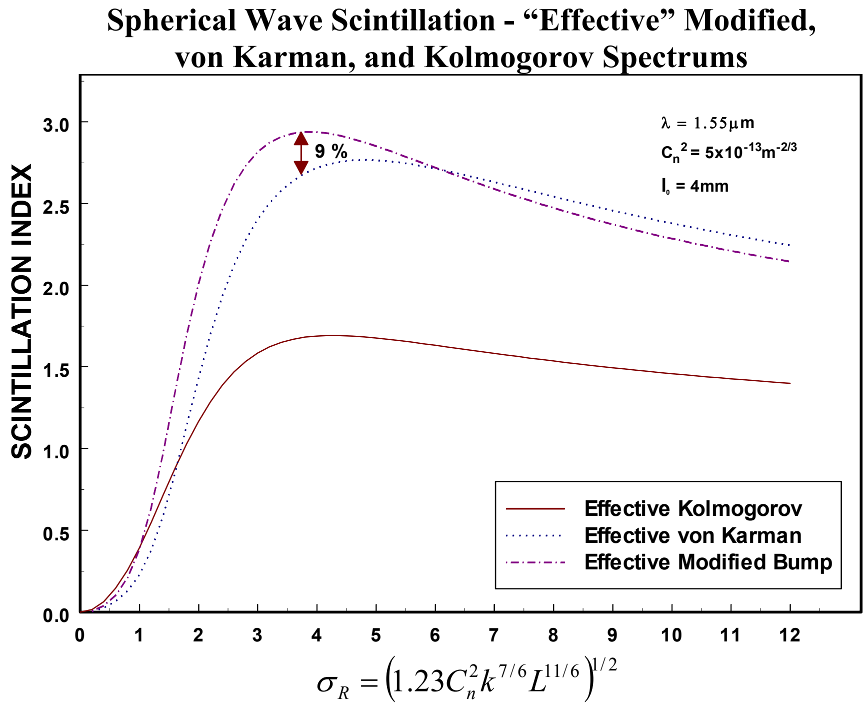


Figure 12 - Scintillation Index based on varied spectra's (equations 28, 29, and 84)
$l_0 = 4\text{mm}$, $\lambda = 1.55\mu\text{m}$

7.0 CONCLUSION

In this thesis, plane and spherical wave scintillation expressions were developed from an "effective" von Karman spectrum for use in varying fluctuation regimes. This effective spectrum was used to extend the Rytov approximation into all optical turbulence regimes by using filter functions to eliminate mid-range turbulent cell size effects to the scintillation index. Filter cutoffs were established by matching to known weak and saturated scintillation results. Although studies indicate that using a bump spectrum is important for developing accurate scintillation expressions in weak turbulence, scintillation expressions based on the effective von Karman spectrum track those based on the effective modified (bump) spectrum[8,14,15] fairly closely. In moderate turbulence, the new expressions provide a smooth transition between scintillation expressions predicted by existing weak and saturated turbulence scintillation expressions. Also in moderate turbulence, increased wavelength size diminishes the bump effect. In extremely strong turbulence, the difference between the models is minimal.

APPENDIX A: WEAK TURBULENCE, KOLMOGOROV SPECTRUM, PLANE WAVE

The Von Karman Spectrum $\phi_n(\kappa) = 0.033C_n^2 \frac{\exp(-\kappa^2/\kappa_m^2)}{(\kappa^2+\kappa_0^2)^{11/6}}$ reduces to the Kolmogorov Spectrum $\left(\phi_n(\kappa) = 0.033C_n^2\kappa^{-11/3}\right)$ if inner scale goes to zero and outer scale goes to infinity. Therefore, the scintillation index derived by the von Karman spectrum (see appendix B) reduces to that derived by the Kolmogorov spectrum if inner scale goes to zero and outer scale goes to infinity.

For a plane wave (equation 9),

$$\sigma_{I,pl}^2 = 3.86\sigma_R^2\left\{\left(1+\frac{1}{Q_m^2}\right)^{11/12}\sin\left[\frac{11}{6}\tan^{-1}(Q_m)\right] - \frac{11}{6}Q_m^{-5/6}\right\} \qquad \text{(A1)}$$

If $l_0 \to 0$, then $Q_m = L(5.92/l_0)^2/k \to \infty$

$$\begin{aligned}\sigma_{I,pl}^2 &= 0 + 3.86\sigma_R^2(1)^{11/12}\sin\left[\frac{11}{6}\tan^{-1}(\infty)\right]\\ &= 3.86\sigma_R^2\sin\left[\left(\frac{11}{6}\right)(\pi/2)\right]\\ &= \sigma_R^2\end{aligned} \qquad \text{(A2)}$$

Therefore, the scintillation index for a plane wave in weak turbulence based on the Kolmogorov spectrum is $\sigma_I^2 = \sigma_R^2$.

APPENDIX B: WEAK TURBULENCE, VON KARMAN, PLANE WAVE

Under weak fluctuation conditions, the on-axis scintillation index for a plane wave is defined by (equation 7):

$$\sigma_{I,l}^2(L) = 8\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa \Phi_n(\kappa) \left[1 - \cos\left(\frac{L\kappa^2}{k}\xi\right)\right] d\kappa d\xi \tag{B1}$$

For the von Karman spectrum,

$$\begin{aligned}
\sigma_{I,l}^2(L) &= 8\pi^2 k^2 L \int_0^1 \int_0^\infty \frac{\kappa 0.033 C_n^2 \exp\left(-\kappa^2/\kappa_m^2\right)}{\left(\kappa^2+\kappa_0^2\right)^{11/6}} \left[1 - \cos\left(\frac{L\kappa^2}{k}\xi\right)\right] d\kappa d\xi \\
&= 2.606 C_n^2 k^2 L \int_0^1 \int_0^\infty \frac{\kappa \exp\left(-\kappa^2/\kappa_m^2\right)}{\left(\kappa^2+\kappa_0^2\right)^{11/6}} \left[1 - \cos\left(\frac{L\kappa^2}{k}\xi\right)\right] d\kappa d\xi \\
&= 2.606 C_n^2 k^2 L \int_0^1 \int_0^\infty \frac{\kappa \exp\left(-\kappa^2/\kappa_m^2\right)}{\left(\kappa^2+\kappa_0^2\right)^{11/6}} \left\{1 - \mathrm{Re}\left[\exp\left(\frac{-iL\kappa^2\xi}{k}\right)\right]\right\} d\kappa d\xi \\
&= 2.606 C_n^2 k^2 L \,\mathrm{Re} \int_0^1 \int_0^\infty \frac{\kappa}{\left(\kappa^2+\kappa_0^2\right)^{11/6}} \left\{\exp\left(\frac{-\kappa^2}{\kappa_m^2}\right) - \exp\left[\left(\frac{-\kappa^2}{\kappa_m^2}\right)\left(1 + \frac{iL\kappa_m{}^2\xi}{k}\right)\right]\right\} d\kappa d\xi \\
&= 2.606 C_n^2 k^2 L \,\mathrm{Re} \int_0^1 \int_0^\infty \frac{\kappa}{\left(\kappa^2+\kappa_0^2\right)^{11/6}} \left\{\exp\left(\frac{-\kappa^2}{\kappa_m^2}\right) - \exp\left[\left(\frac{-\kappa^2}{\kappa_m^2}\right)\left(1 + iQ_m\xi\right)\right]\right\} d\kappa d\xi
\end{aligned} \tag{B2}$$

where

$$\Phi_n(\kappa) = 0.033 C_n^2 \frac{\exp\left(-\kappa^2/\kappa_m^2\right)}{\left(\kappa^2+\kappa_0^2\right)^{11/6}} \text{ (equation 3)}$$

$$\cos x = \mathrm{Re}[\exp(-ix)]$$

$$Q_m = \frac{L\kappa_m^2}{k}$$

Recognizing[1]

$$\int_0^\infty \frac{\kappa^{2\mu} \exp\left(-\kappa^2/\kappa_m^2\right)}{\left(\kappa^2+\kappa_0^2\right)^{11/6}} = \frac{1}{2}\kappa_0^{2\mu-8/3}\Gamma\left(\mu+1/2\right)u\left(\mu+1/2; \mu-1/3; \kappa_0^2/\kappa_m^2\right) \tag{B3}$$

where

$$2\mu = 1 \rightarrow \mu = 1/2,$$

$$2\mu - 8/3 = 1 - 8/3 = -5/3$$
$$\mu + 1/2 = 1$$
$$\mu - 1/3 = 1/6$$

results in

$$\sigma_{I,pl}^2 = 2.606 C_n^2 k^2 L \,\mathrm{Re}\int_0^1 \frac{1}{2}\kappa_0^{-5/3}\Gamma(1) u\left(1;1/6;\kappa_0^2/\kappa_m^2\right) - \frac{1}{2}\kappa_0^{-5/3}\Gamma(1) u\left(1;1/6;\left[\left(\kappa_0^2/\kappa_m^2\right)\left(1+iQ_m\xi\right)\right]\right) d\xi$$

$$= 1.303 C_n^2 k^2 L \kappa_0^{-5/3}\,\mathrm{Re}\int_0^1 u\left(1;1/6;\kappa_0^2/\kappa_m^2\right) - u\left(1;1/6;(\kappa_0^2/\kappa_m^2)(1+iQ_m\xi)\right) d\xi \qquad \text{(B4)}$$

Assuming[1]

$$u(a;c;z) \cong \frac{\Gamma(1-c)}{\Gamma(1+a-c)} + \frac{\Gamma(c-1) z^{1-c}}{\Gamma(a)} \qquad \text{(B5)}$$

then

$$\begin{aligned} u(1;1/6;x) &\cong \frac{\Gamma(5/6)}{\Gamma(11/6)} + \frac{\Gamma(-5/6) x^{5/6}}{\Gamma(1)} \\ &= \frac{\Gamma(5/6)}{(5/6)\Gamma(5/6)} + \left(\frac{-6}{5}\right)\Gamma\left(\frac{1}{6}\right) x^{5/6} \\ &= \frac{6}{5} + (6)\left(\frac{-6}{5}\right)\Gamma\left(\frac{7}{6}\right) x^{5/6} \\ &\cong \frac{6}{5} + (6)\left(\frac{-6}{5}\right) 0.9298 x^{5/6} \\ &= \frac{6}{5} - 6.695 x^{5/6} \qquad \text{(B6)} \end{aligned}$$

Equation B4 can be reduced to

$$\begin{aligned} \sigma_{I,pl}^2 &= 1.303 C_n^2 k^2 L \kappa_0^{-5/3}\,\mathrm{Re}\int_0^1 \left[\frac{6}{5} - 6.695\left(\kappa_0^2/\kappa_m^2\right)^{5/6}\right] - \left[\frac{6}{5} - 6.695\left(\kappa_0^2/\kappa_m^2\right)^{5/6}\left(1+iQ_m\xi\right)^{5/6}\right] d\xi \\ &= 1.303 C_n^2 k^2 L \kappa_0^{-5/3}\,\mathrm{Re}\int_0^1 \left[6.695(\kappa_0^2/\kappa_m^2)^{5/6}\right]\left[-1 + \left(1+iQ_m\xi\right)^{5/6}\right] d\xi \\ &= 8.70 C_n^2 k^2 L \kappa_m^{-5/3}\,\mathrm{Re}\left[\int_0^1 (-1) d\xi + \int_0^1 \left(1+iQ_m\xi\right)^{5/6} d\xi\right] \\ &= 8.70 C_n^2 k^2 L \kappa_m^{-5/3}\,\mathrm{Re}\left\{-1 + \frac{6}{11}\left[\frac{\left[1+iQ_m(1)\right]^{11/6}}{iQ_m} - \frac{\left[1+iQ_m(0)\right]^{11/6}}{iQ_m}\right]\right\} \end{aligned}$$

$$= 8.70 C_n^2 k^2 L \kappa_m^{-5/3} \,\mathrm{Re}\left\{ -1 + \frac{6}{11}\left[\frac{(1+iQ_m)^{11/6}}{iQ_m} - \frac{1}{iQ_m} \right] \right\}$$

$$= 8.70 C_n^2 k^2 L \kappa_m^{-5/3} \,\mathrm{Re}\frac{6}{11}\left\{ -\frac{11}{6} + \left[\frac{(1+iQ_m)^{11/6}}{iQ_m} - \frac{1}{iQ_m} \right] \right\} \qquad \text{(B7)}$$

Upon deleting the $\frac{1}{iQ_m}$ term (not real),

$$\sigma_{I,pl}^2 = 8.70 C_n^2 k^2 L \kappa_m^{-5/3} \,\mathrm{Re}\frac{6}{11}\left[-\frac{11}{6} + \frac{(1+iQ_m)^{11/6}}{iQ_m} \right]$$

$$= 4.75 C_n^2 k^2 L \kappa_m^{-5/3} \left[-\frac{11}{6} + \mathrm{Re}\frac{(1+iQ_m)^{11/6}}{iQ_m} \right] \qquad \text{(B8)}$$

Evaluate $\frac{(1+iQ_m)^{11/6}}{iQ_m}$ in polar coordinates:

$$x + iy = r\exp\left[i\tan^{-1}(y/x)\right] = \sqrt{x^2+y^2}\exp\left[i\tan^{-1}(y/x)\right]$$

$$(1+iQ_m)^{11/6} = \left\{ \sqrt{1^2+Q_m^2}\exp\left[i\tan^{-1}(Q_m/1)\right] \right\}^{11/6}$$

$$= \left(1^2+Q_m^2\right)^{11/12}\exp\left[i(11/6)\tan^{-1}(Q_m)\right] \qquad \text{(B9)}$$

Also, $x + iy = r\exp\left[i\tan^{-1}(y/x)\right] = \sqrt{x^2+y^2}\exp\left[i\tan^{-1}(y/x)\right]$

$$iQ_m = 0 + iQ_m = \sqrt{0^2+Q_m^2}\exp\left[i\tan^{-1}(\infty)\right]$$

$$= Q_m \exp(i\pi/2) \qquad \text{(B10)}$$

Then

$$\frac{1}{iQ_m} = \frac{1}{Q_m}\exp(-i\pi/2) \qquad \text{(B11)}$$

And

$$\frac{(1+iQ_m)^{11/6}}{iQ_m} = \frac{\left(1+Q_m^2\right)^{11/12}\exp\left[i(11/6)\tan^{-1}(Q_m) - i\pi/2\right]}{Q_m}$$

$$= \frac{\left(1+Q_m^2\right)^{11/12} \cos\left[(11/6)\tan^{-1}(Q_m) - \pi/2\right]}{Q_m}$$

$$= \frac{\left(1+Q_m^2\right)^{11/12} \sin\left[(11/6)\tan^{-1}(Q_m)\right]}{Q_m}$$

$$= \frac{\left[Q_m^2\left(\frac{1}{Q_m^2}+1\right)\right]^{11/12} \sin\left[(11/6)\tan^{-1}(Q_m)\right]}{Q_m}$$

$$= Q_m^{5/6}\left(\frac{1}{Q_m^2}+1\right)^{11/12} \sin\left[(11/6)\tan^{-1}(Q_m)\right] \qquad \text{(B12)}$$

where exp(ix) = cos(x), ignoring the imaginary term

Equation B8 can be rewritten as

$$\sigma_{I,pl}^2 = 4.75 C_n^2 k^2 L \kappa_m^{-5/3}\left[-\frac{11}{6} + Q_m^{5/6}\left(\frac{1}{Q_m^2}+1\right)^{11/12} \sin\left[(11/6)\tan^{-1}(Q_m)\right]\right]$$

$$= 4.75\left(0.813\sigma_R^2 Q_m^{-5/6}\right)\left\{-\frac{11}{6} + Q_m^{5/6}\left(\frac{1}{Q_m^2}+1\right)^{11/12} \sin\left[(11/6)\tan^{-1}(Q_m)\right]\right\}$$

$$= 3.86\sigma_R^2 Q_m^{-5/6}\left\{-\frac{11}{6} + Q_m^{5/6}\left(\frac{1}{Q_m^2}+1\right)^{11/12} \sin\left[(11/6)\tan^{-1}(Q_m)\right]\right\}$$

$$= 3.86\sigma_R^2 Q_m^{-5/6}\left\{Q_m^{5/6}\left(1+\frac{1}{Q_m^2}\right)^{11/12} \sin\left[(11/6)\tan^{-1}(Q_m)\right] - \frac{11}{6}\right\}$$

$$= 3.86\sigma_R^2\left\{\left(1+\frac{1}{Q_m^2}\right)^{11/12} \sin\left[\frac{11}{6}\tan^{-1}(Q_m)\right] - \frac{11}{6}Q_m^{-5/6}\right\} \qquad \text{(B13)}$$

where

$$\sigma_R^2 = 1.23 C_n^2 k^{7/6} L^{11/6}$$

$$Q_m = \frac{L\kappa_m^2}{k}$$

$$C_n^2 k^2 L \kappa_m^{-5/3} = C_n^2 k^{7/6} k^{5/6} L^{11/6} L^{-5/6} \kappa_m^{-5/3}$$

$$= C_n^2 k^{7/6} L^{11/6} \left(\frac{k}{L\kappa_m^2} \right)^{5/6}$$
$$= 0.813\sigma_R^2 Q_m^{-5/6}$$

Therefore, the plane wave expression in weak fluctuations using the von Karman spectrum is

$$\sigma_I^2 = 3.86\sigma_R^2 \left\{ (1 + \frac{1}{Q_m^2})^{11/12} \sin\left[\frac{11}{6} \tan^{-1}(Q_m) \right] - \frac{11}{6} Q_m^{-5/6} \right\}$$

APPENDIX C: WEAK TURBULENCE, MODIFIED (BUMP), PLANE WAVE

Under weak fluctuation conditions, the on-axis scintillation index for a plane wave is defined by (equation 7)

$$\sigma_{I,l}^2(L) = 8\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa \Phi_n(\kappa) \left[1 - \cos\left(\frac{L\kappa^2}{k}\right)\xi\right] d\kappa d\xi \tag{C1}$$

For the modified (bump) spectrum,

$$\sigma_{I,l}^2(L) = 8\pi^2 k^2 L \int_0^1 \int_0^\infty \frac{\kappa 0.033 C_n^2 \left[1 + 1.802\kappa/\kappa_l - 0.254(\kappa/\kappa_l)^{7/6}\right] \exp\left(-\kappa^2/\kappa_l^2\right)}{(\kappa^2 + \kappa_0^2)^{11/6}} \left[1 - \cos\left(\frac{L\kappa^2}{k}\xi\right)\right] d\kappa d\xi \tag{C2}$$

where $\Phi_n(\kappa) = \frac{0.033 C_n^2 \left[1 + 1.802\kappa/\kappa_l - 0.254(\kappa/\kappa_l)^{7/6}\right] \exp\left(-\kappa^2/\kappa_l^2\right)}{\left(\kappa^2 + \kappa_0^2\right)^{11/6}}$ (equation 4)

This equation can be split into 3 intervals. Let $\mathrm{B}(\kappa/\kappa_l)^\alpha = 1 + 1.802\kappa/\kappa_l - 0.254(\kappa/\kappa_l)^{7/6}$.

Determine a general solution for $\mathrm{B}(\kappa/\kappa_l)^\alpha$, then solve for each interval separately.

$$\begin{aligned}
\sigma_{I,l}^2(L) &= 8\pi^2 k^2 L \int_0^1 \int_0^\infty \frac{\kappa 0.033 C_n^2 \mathrm{B}(\kappa/\kappa_l)^\alpha \exp(-\kappa^2/\kappa_l^2)}{(\kappa^2 + \kappa_0^2)^{11/6}} \{1 - \cos[\frac{L\kappa^2}{k}\xi]\} d\kappa d\xi \\
&= 8\pi^2 0.033 C_n^2 k^2 L \int_0^1 \int_0^\infty \frac{\kappa \mathrm{B}(\kappa/\kappa_l)^\alpha \exp\left(-\kappa^2/\kappa_l^2\right)}{\left(\kappa^2 + \kappa_0^2\right)^{11/6}} \left[1 - \cos\left(\frac{L\kappa^2}{k}\xi\right)\right] d\kappa d\xi \\
&= 2.606 C_n^2 k^2 L \int_0^1 \int_0^\infty \frac{\kappa \mathrm{B}(\kappa/\kappa_l)^\alpha \exp\left(-\kappa^2/\kappa_l^2\right)}{\left(\kappa^2 + \kappa_0^2\right)^{11/6}} \left[1 - \cos\left(\frac{L\kappa^2}{k}\xi\right)\right] d\kappa d\xi \\
&= 2.606 C_n^2 k^2 L \int_0^1 \int_0^\infty \frac{\kappa \mathrm{B}(\kappa/\kappa_l)^\alpha \exp\left(-\kappa^2/\kappa_l^2\right)}{\left(\kappa^2 + \kappa_0^2\right)^{11/6}} \left\{1 - \mathrm{Re}\left[\exp\left(\frac{-iL\kappa^2\xi}{k}\right)\right]\right\} d\kappa d\xi \\
&= 2.606 C_n^2 k^2 L \,\mathrm{Re} \int_0^1 \int_0^\infty \frac{\kappa \mathrm{B}(\kappa/\kappa_l)^\alpha \exp\left(-\kappa^2/\kappa_l^2\right)}{\left(\kappa^2 + \kappa_0^2\right)^{11/6}} \left[1 - \exp\left(\frac{-iL\kappa^2\xi}{k}\right)\right] d\kappa d\xi \\
&= 2.606 C_n^2 k^2 L \,\mathrm{Re} \int_0^1 \int_0^\infty \frac{\kappa \mathrm{B}(\kappa/\kappa_l)^\alpha}{\left(\kappa^2 + \kappa_0^2\right)^{11/6}} \left[\exp\left(\frac{-\kappa^2}{\kappa_l^2}\right) - \exp\left(\frac{-\kappa^2}{\kappa_l^2} - \frac{iL\kappa^2\xi}{k}\right)\right] d\kappa d\xi \\
&= 2.606 C_n^2 k^2 L \,\mathrm{Re} \int_0^1 \int_0^\infty \frac{\kappa \mathrm{B}(\kappa/\kappa_l)^\alpha}{\left(\kappa^2 + \kappa_0^2\right)^{11/6}} \left\{\exp\left(\frac{-\kappa^2}{\kappa_l^2}\right) - \exp\left[\left(\frac{-\kappa^2}{\kappa_l}\right)\left(1 + \frac{iL\kappa_l{}^2\xi}{k}\right)\right]\right\} d\kappa d\xi \\
&= 2.606 C_n^2 k^2 L \,\mathrm{Re} \int_0^1 \int_0^\infty \frac{\kappa \mathrm{B}(\kappa/\kappa_l)^\alpha}{\left(\kappa^2 + \kappa_0^2\right)^{11/6}} \left\{\exp\left(\frac{-\kappa^2}{\kappa_l^2}\right) - \exp\left[\left(\frac{-\kappa^2}{\kappa_l^2}\right)(1 + iQ_l\xi)\right]\right\} d\kappa d\xi \\
&= \frac{2.606 \mathrm{B} C_n^2 k^2 L}{\kappa_l^\alpha} R \int_0^1 \int_0^\infty \frac{\kappa^{\alpha+1}}{(\left(\kappa^2 + \kappa_0^2\right)^{11/6}} \left\{\exp\left(\frac{-\kappa^2}{\kappa_l^2}\right) - \exp\left[\left(\frac{-\kappa^2}{\kappa_l^2}\right)(1 + iQ_l\xi)\right]\right\} d\kappa d\xi
\end{aligned} \tag{C3}$$

where

$$\cos x = \mathrm{Re}\left[\exp\left(-ix\right)\right]$$

$$Q_l = \frac{L\kappa_l^2}{k}$$

Recognizing[1]

$$\int_0^\infty \frac{\kappa^{2\mu}\exp\left(-\kappa^2/\kappa_m^2\right)d\kappa}{\left(\kappa^2+\kappa_0^2\right)^{11/6}} = \frac{1}{2}\mathrm{K}_0^{2\mu-8/3}\Gamma\left(\mu+1/2\right)u\left(\mu+1/2;\mu-1/3;\kappa_0^2/\kappa_m^2\right) \qquad \text{(C4)}$$

where

$$2\mu = \alpha+1 \rightarrow \mu = (\alpha+1)/2,$$
$$2\mu - 8/3 = \alpha+1-8/3 = \alpha - 5/3$$
$$\mu + 1/2 = [(\alpha+1)/2] + 1/2 = (\alpha+2)/2$$
$$\mu - 1/3 = [(\alpha+1)/2] - 1/3 = [3(\alpha+1)/6] - 2/6 = (3\alpha+3-2)/6 = (3\alpha+1)/6$$

results in

$$\sigma_{I,l}^2(L) = \frac{2.606\mathrm{B}C_n^2k^2L}{\kappa_l^\alpha}\mathrm{Re}\int_0^1\left\{\ \frac{1}{2}\kappa_0^{\alpha-5/3}\Gamma([\alpha+2]/2)u([(\alpha+2)/2];[(3\alpha+1)/6];\kappa_0^2/\kappa_l^2)\right.$$
$$\left. -\frac{1}{2}\kappa_0^{\alpha-5/3}\Gamma([\alpha+2]/2)u([(\alpha+2)/2];[(3\alpha+1)/6];\kappa_0^2/\kappa_l^2(1+iQ_l\xi)\ \right\}d\xi$$

$$= \frac{1.303\mathrm{B}C_n^2k^2L\kappa_0^{\alpha-5/3}\Gamma([\alpha+2]/2)}{\kappa_l^\alpha}\mathrm{Re}\int_0^1\left\{\ u([(\alpha+2)/2];[(3\alpha+1)/6];\kappa_0^2/\kappa_l^2)\right.$$
$$\left. -u([(\alpha+2)/2];[(3\alpha+1)/6];\kappa_0^2/\kappa_l^2(1+iQ_l\xi)\ \right\}d\xi \qquad \text{(C5)}$$

Recognizing[1]

$$u(a;c;z) \cong \frac{\Gamma(1-c)}{\Gamma(1+a-c)} + \frac{\Gamma(c-1)z^{1-c}}{\Gamma(a)} \qquad \text{(C6)}$$

then

$$u([(\alpha+2)/2];[(3\alpha+5)/6];x) \cong \frac{\Gamma\left([5-3\alpha]/6\right)}{\Gamma(11/6)} + \frac{\Gamma\left([3\alpha-5]/6\right)x^{[5-3\alpha]/6}}{\Gamma\left([\alpha+2]/2\right)} \qquad \text{(C7)}$$

where

$$\mathrm{a} = (\alpha+2)/2$$

$$\mathrm{c} = (3\alpha+1)/6$$

$$1-\mathrm{c} = 1-(3\alpha+1)/6] = [6-(3\alpha+1)]/6 = (5-3\alpha)/6$$

$$1+a-c = 1+(\alpha+2)/2-(3\alpha+1)/6 = [6+3(\alpha+2)-(3\alpha+1)]/6$$
$$= [6+3\alpha+6-3\alpha-1)]/6 = 11/6$$

Equation C5 can be reduced to

$$\sigma_{I,l}^2(L) = \frac{1.303\mathrm{B}C_n^2 k^2 L\kappa_0^{\alpha-5/3}\Gamma([\alpha+2]/2)}{\kappa_l^\alpha}\mathrm{Re}\int_0^1 \left\{ \frac{\Gamma([5-3\alpha]/6)}{\Gamma(11/6)} + \frac{\Gamma([3\alpha-5]/6)\left(\kappa_0^2/\kappa_l^2\right)^{[5-3\alpha]/6}}{\Gamma([\alpha+2]/2)} - \left[\frac{\Gamma([5-3\alpha]/6)}{\Gamma(11/6)} + \frac{\Gamma([3\alpha-5]/6)\left(\kappa_0^2/\kappa_l^2\right)^{[5-3\alpha]/6}(1+iQ_l\xi)^{[5-3\alpha]/6}}{\Gamma([\alpha+2]/2)} \right] \right\} d\xi$$

$$= \frac{1.303\mathrm{B}C_n^2 k^2 L\kappa_0^{\alpha-5/3}\Gamma([\alpha+2]/2)}{\kappa_l^\alpha}\mathrm{Re}\int_0^1 \left\{ \frac{\Gamma([3\alpha-5]/6)\left(\kappa_0^2/\kappa_l^2\right)^{[5-3\alpha]/6}}{\Gamma([\alpha+2]/2)} - \frac{\Gamma([3\alpha-5]/6)\left(\kappa_0^2/\kappa_l^2\right)^{[5-3\alpha]/6}\left(1+iQ_l\xi\right)^{[5-3\alpha]/6}}{\Gamma([\alpha+2]/2)} \right\} d\xi$$

$$= \frac{1.303\mathrm{B}C_n^2 k^2 L\kappa_0^{\alpha-5/3}\Gamma([\alpha+2]/2)\Gamma([3\alpha-5]/6)\left(\kappa_0^2/\kappa_l^2\right)^{[5-3\alpha]/6}}{\kappa_l^\alpha\Gamma([\alpha+2]/2)}\mathrm{Re}\int_0^1 \left[1-\left(1+iQ_l\xi\right)^{[5-3\alpha]/6}\right]d\xi$$

$$= \frac{1.303\mathrm{B}C_n^2 k^2 L\kappa_0^{\alpha-5/3}\kappa_0^{[5-3\alpha]/3}\Gamma([\alpha+2]/2)\Gamma([3\alpha-5]/6)}{\kappa_l^{[5-3\alpha]/3}\kappa_l^\alpha\Gamma([\alpha+2]/2)}\mathrm{Re}\int_0^1 \left[1-\left(1+iQ_l\xi\right)^{[5-3\alpha]/6}\right]d\xi$$

$$= \frac{1.303\mathrm{B}C_n^2 k^2 L\kappa_l^{-5/3}\Gamma([\alpha+2]/2)\Gamma([3\alpha-5]/6)}{\Gamma([\alpha+2]/2)}\mathrm{Re}\int_0^1 \left[1-\left(1+iQ_l\xi\right)^{[5-3\alpha]/6}\right]d\xi$$

$$= 1.059\mathrm{B}\sigma_R^2 Q_l^{-5/6}\Gamma([3\alpha-5]/6)\,\mathrm{Re}\int_0^1 \left[1-\left(1+iQ_l\xi\right)^{[5-3\alpha]/6}\right]d\xi$$

$$= \frac{1.303\mathrm{B}\left(0.813\sigma_R^2 Q_l^{-5/6}\right)\Gamma([\alpha+2]/2)\Gamma([3\alpha-5]/6)}{\Gamma([\alpha+2]/2)}\mathrm{Re}\int_0^1 \left[1-\left(1+iQ_l\xi\right)^{[5-3\alpha]/6}\right]d\xi$$

$$= 1.059\mathrm{B}\sigma_R^2 Q_l^{-5/6}\Gamma([3\alpha-5]/6)\,\mathrm{Re}\left[\int_0^1 1d\xi - \int_0^1 \left(1+iQ_l\xi\right)^{[5-3\alpha]/6}\right]d\xi$$

$$= 1.059\mathrm{B}\sigma_R^2 Q_l^{-5/6}\Gamma([3\alpha-5]/6)\,\mathrm{Re}\left\{1-\left[\frac{6\left[1+iQ_l(1)\right]^{[11-3\alpha]/6}}{(11-3\alpha)iQ_l} - \frac{6\left[1+iQ_l(0)\right]^{[11-3\alpha]/6}}{(11-3\alpha)iQ_l}\right]\right\}$$

$$= 1.059\mathrm{B}\sigma_R^2 Q_l^{-5/6}\Gamma([3\alpha-5]/6)\,\mathrm{Re}\left\{1-\left[\frac{6\left(1+iQ_l\right)^{[11-3\alpha]/6}}{(11-3\alpha)iQ_l} - \frac{6}{(11-3\alpha)iQ_l}\right]\right\}$$

$$= 1.059\mathrm{B}\sigma_R^2 Q_l^{-5/6}\Gamma([3\alpha-5]/6)\,\mathrm{Re}\frac{6}{(11-3\alpha)}\left\{\frac{(11-3\alpha)}{6} - \left[\frac{\left(1+iQ\right)_l^{[11-3\alpha]/6}}{iQ_l} - \frac{1}{iQ_l}\right]\right\} \qquad \text{(C8)}$$

where

$$\sigma_R^2 = 1.23C_n^2 k^{7/6} L^{11/6}$$

$$Q_l = \frac{L\kappa_l^2}{k}$$

$$Cn^2 k^2 L \kappa_l^{-5/3} = C_n^2 k^{7/6} k^{5/6} L^{11/6} L^{-5/6} \kappa_l^{-5/3}$$
$$= C_n^2 k^{7/6} L^{11/6} \left(\frac{k}{L\kappa_l^2} \right)^{5/6}$$
$$= 0.813 \sigma_R^2 Q_l^{-5/6}$$

Upon deleting $\frac{1}{iQ_l}$ term (not real)

$$\sigma_{I,l}^2(L) = 1.059 \mathrm{B} \sigma_R^2 Q_l^{-5/6} \Gamma([3\alpha - 5]/6) \operatorname{Re} \frac{6}{(11-3\alpha)} \left[\frac{(11-3\alpha)}{6} - \frac{(1 + iQ_l\xi)^{[11-3\alpha]/6}}{iQ_l} \right]$$
$$= \frac{6.356 \mathrm{B} \sigma_R^2 Q_l^{-5/6} \Gamma([3\alpha - 5]/6)}{(11-3\alpha)} \operatorname{Re}[\frac{(11-3\alpha)}{6} - \frac{(1 + iQ_l\xi)^{[11-3\alpha]/6}}{iQ_l}] \quad \text{(C9)}$$

Evaluate $\frac{(1 + iQ_l)^{[11-3\alpha]/6}}{iQ_l}$ in polar coordinates:

$$x + iy = r \exp[i \tan^{-1}(y/x)] = \sqrt{x^2 + y^2} \exp[i \tan^{-1}(y/x)]$$

Therefore,

$$(1 + iQ_l)^{[11-3\alpha]/6} = \left\{ \sqrt{1^2 + Q_l^2} \exp[i \tan^{-1}(Q_l/1)] \right\}^{[11-3\alpha]/6}$$
$$= (1^2 + Q_l^2)^{[11-3\alpha]/12} \exp[i([11-3\alpha]/6) \tan^{-1}(Q_l)] \quad \text{(C10)}$$

Also, $x + iy = r \exp[i \tan^{-1}(y/x)] = \sqrt{x^2 + y^2} \exp[i \tan^{-1}(y/x)]$

Therefore,

$$iQ_l = 0 + iQ_l = \sqrt{0^2 + Q_l^2} \exp[i \tan^{-1}(\infty)] = Q_l \exp(i\pi/2) \quad \text{(C11)}$$

Then

$$\frac{1}{iQ_l} = \frac{1}{Q_l} \exp(-i\pi/2) \quad \text{(C12)}$$

and

$$\frac{(1+iQ_l)^{[11-3\alpha]/6}}{iQ_l} = \frac{\left(1+Q_l^2\right)^{[11-3\alpha]/12}\exp\left[i([11-3\alpha]/6)\tan^{-1}(Q_l) - i\pi/2\right]}{Q_l}$$

$$= \frac{\left(1+Q_l^2\right)^{[11-3\alpha]/12}\cos\left[([11-3\alpha]/6)\tan^{-1}(Q_l) - \pi/2)\right]}{Q_l}$$

$$= \frac{\left(1+Q_l^2\right)^{[11-3\alpha]/12}\sin\left[([11-3\alpha]/6)\tan^{-1}(Q_l)\right]}{Q_l} \qquad \text{(C13)}$$

where exp(ix) = cos(x), ignoring the imaginary term

Therefore, the general solution is

$$\sigma_{I,l}^2(L) = \frac{6.356\mathrm{B}\,\sigma_R^2 Q_l^{-5/6}\Gamma([3\alpha-5]/6)}{(11-3\alpha)}\left\{\frac{(11-3\alpha)}{6} - \frac{\left(1+Q_l^2\right)^{[11-3\alpha]/12}\sin\left[([11-3\alpha]/6)\tan^{-1}(Q_l)\right]}{Q_l}\right\} \qquad \text{(C14)}$$

The final solution is obtained from the above by solving

$\mathrm{B}(\kappa/\kappa_l)^{\alpha} = 1 + 1.802\kappa/\kappa_l - 0.254(\kappa/\kappa_l)^{7/6}$ for each interval independently, then adding them together.

Interval 1 ($1 \rightarrow \mathrm{B} = 1, \alpha = 0$):

Interval 1 $= \frac{6.356\mathrm{B}\,\sigma_R^2 Q_l^{-5/6}\Gamma([3\alpha-5]/6)}{(11-3\alpha)}\left\{\frac{(11-3\alpha)}{6} - \frac{\left(1+Q_l^2\right)^{[11-3\alpha]/12}\sin\left[([11-3\alpha]/6)\tan^{-1}(Q_l)\right]}{Q_l}\right\}$

$$= \frac{6.356(1)\sigma_R^2 Q_l^{-5/6}\Gamma([0-5]/6)}{(11-0)}\left\{\frac{(11-3\alpha)}{6} - \frac{(1+Q_l^2)^{[11-0]/12}\sin\left[([11-0]/6)\tan^{-1}(Q_l)\right]}{Q_l}\right\}$$

$$= \frac{6.356\sigma_R^2 Q_l^{-5/6}\Gamma(-5/6)}{11}\left\{\frac{11}{6} - \frac{\left(1+Q_l^2\right)^{11/12}\sin\left[(11/6)\tan^{-1}(Q_l)\right]}{Q_l}\right\}$$

$$= \frac{6.356\sigma_R^2 Q_l^{-5/6}(-6.678)}{11}\left\{\frac{11}{6} - \frac{(1+Q_l^2)^{11/12}\sin\left[(11/6)\tan^{-1}(Q_l)\right]}{Q_l}\right\}$$

$$= 3.86\sigma_R^2 Q_l^{-5/6}\left\{\frac{\left(1+Q_l^2\right)^{11/12}\sin\left[(11/6)\tan^{-1}(Q_l)\right]}{Q_l} - \frac{11}{6}\right\}$$

$$= 3.86\sigma_R^2\left\{\frac{Q_l^{-5/6}\left(1+Q_l^2\right)^{11/12}\sin\left[(11/6)\tan^{-1}(Q_l)\right]}{Q_l} - \frac{11}{6}Q_l^{-5/6}\right\}$$

$$= 3.86\sigma_R^2 \left\{ \frac{Q_l^{-5/6}\left[\left(Q_l^2\right)\left(\frac{1}{Q_l^2}+1\right)\right]^{11/12} \sin\left[(11/6)\tan^{-1}(Q_l)\right]}{Q_l} - \frac{11}{6}Q_l^{-5/6} \right\}$$

$$= 3.86\sigma_R^2 \left\{ \frac{Q_l^{-5/6}Q_l^{11/6}\left(\frac{1}{Q_l^2}+1\right)^{11/12} \sin\left[(11/6)\tan^{-1}(Q_l)\right]}{Q_l} - \frac{11}{6}Q_l^{-5/6} \right\}$$

$$= 3.86\sigma_1^2 \left\{ \left(\frac{1}{Q_l^2}+1\right)^{11/12} \sin\left[(11/6)\tan^{-1}(Q_l)\right] - \frac{11}{6}Q_l^{-5/6} \right\} \qquad \text{(C15)}$$

<u>Interval 2</u> ($1.802\kappa/\kappa_l \rightarrow \mathrm{B} = 1.802, \alpha = 1$):

Interval 2 $= \frac{6.356(1.802)\sigma_R^2 Q_l^{-5/6}\Gamma([3(1)-5]/6)}{[11-3(1)]}\left\{\frac{[11-3(1)]}{6} - \frac{\left(1+Q_l^2\right)^{[11-3(1)]/12}\sin\left[([11-3(1)]/6)\tan^{-1}(Q_l)\right]}{Q_l}\right\}$

$$= \frac{6.356(1.802)\sigma_R^2 Q_l^{-5/6}\Gamma(-1/3)}{8}\left[\frac{4}{3} - \frac{\left(1+Q_l^2\right)^{2/3}\sin\left[(4/3)\tan^{-1}(Q_l)\right]}{Q_l}\right.$$

$$= \frac{6.356(1.802)\sigma_R^2 Q_l^{-5/6}(-4.062)}{8}\left\{\frac{4}{3} - \frac{\left(1+Q_l^2\right)^{2/3}\sin\left[(4/3)\tan^{-1}(Q_l)\right]}{Q_l}\right\}$$

$$= 5.816\sigma_R^2 Q_l^{-5/6}\left\{\frac{\left(1+Q_l^2\right)^{2/3}\sin\left[(4/3)\tan^{-1}(Q_l)\right]}{Q_l} - \frac{4}{3}\right\}$$

$$= 5.816\sigma_R^2\left\{\frac{Q_l^{-5/6}\left(1+Q_l^2\right)^{2/3}\sin\left[(4/3)\tan^{-1}(Q_l)\right]}{Q_l} - \frac{4}{3}Q_l^{-5/6}\right\}$$

$$= 5.816\sigma_R^2\left\{\frac{Q_l^{-5/6}\left(1+Q_l^2\right)^{-1/4}\left(1+Q_l^2\right)^{11/12}\sin\left[(4/3)\tan^{-1}(Q_l)\right]}{Q_l} - \frac{4}{3}Q_l^{-5/6}\right\}$$

$$= 5.816\sigma_R^2\left\{\frac{Q_l^{-5/6}Q_l^{11/6}\left(1+\frac{1}{Q_l^2}\right)^{11/12}\sin\left[(4/3)\tan^{-1}(Q_l)\right]}{\left(1+Q_l^2\right)^{1/4}Q_l} - \frac{4}{3}Q_l^{-5/6}\right\}$$

$$= 5.816\sigma_R^2\left\{\frac{\left(1+\frac{1}{Q_l^2}\right)^{11/12}\sin\left[(4/3)\tan^{-1}(Q_l)\right]}{\left(1+Q_l^2\right)^{1/4}} - \frac{4}{3}Q_l^{-5/6}\right\}$$

$$= (3.86)(1.507)\sigma_R^2 \left\{ \frac{\left(1+\frac{1}{Q_l^2}\right)^{11/12} \sin\left[(4/3)\tan^{-1}(Q_l)\right]}{\left(1+Q_l^2\right)^{1/4}} - \frac{4}{3} Q_l^{-5/6} \right\}$$

$$= 3.86\sigma_R^2 \left\{ \frac{\left(1+1/Q_l^2\right)^{11/12} (1.507)\sin\left[(4/3)\tan^{-1}(Q_l)\right]}{\left(1+Q_l^2\right)^{1/4}} - 2.01 Q_l^{-5/6} \right\} \qquad \text{(C16)}$$

<u>Interval 3</u> $(-0.254\left(\kappa/\kappa_l\right)^{7/6} \rightarrow \mathrm{B} = (-)0.254, \alpha = 7/6$

Interval 3 $= \frac{6.356(-0.254)\sigma_R^2 Q_l^{-5/6}\Gamma([3(7/6)-5]/6)}{[11-3(7/6)]} \left\{ \frac{[11-3(7/6)]}{6} - \frac{\left(1+Q_l^2\right)^{[11-3(7/6)]/12} \sin\left[([11-3(7/6)]/6)\tan^{-1}(Q_l)\right]}{Q_l} \right\}$

$$= \frac{6.356(-0.254)\sigma_R^2 Q_l^{-5/6}\Gamma(-1/4)}{(15/2)} \left\{ \frac{5}{4} - \frac{(1+Q_l^2)^{5/8} \sin\left[(5/4)\tan^{-1}(Q_l)\right]}{Q_l} \right\}$$

$$= \frac{6.356(-0.254)\sigma_R^2 Q_l^{-5/6}(-4.902)}{(15/2)} \left\{ \frac{5}{4} - \frac{\left(1+Q_l^2\right)^{5/8} \sin\left[(5/4)\tan^{-1}(Q_l\right]}{Q_l} \right\}$$

$$= 1.054\sigma_R^2 Q_l^{-5/6} \left\{ (-)\frac{\left(1+Q_l^2\right)^{5/8} \sin\left[(5/4)\tan^{-1}(Q_l)\right]}{Q_l} + \frac{5}{4} \right\}$$

$$= 1.054\sigma_R^2 \left\{ (\text{-})\frac{Q_l^{-5/6}\left(1+Q_l^2\right)^{5/8} \sin\left[(5/4)\tan^{-1}(Q_l)\right]}{Q_l} + \frac{5}{4} Q_l^{-5/6} \right\}$$

$$= 1.054\sigma_R^2 \left\{ (\text{-})\frac{Q_l^{-5/6}\left[Q_l^2\left(\frac{1}{Q_l^2}+1\right)\right]^{11/12} \sin\left[(5/4)\tan^{-1}(Q_l)\right]}{\left(1+Q_l^2\right)^{7/24} Q_l} + \frac{5}{4} Q_l^{-5/6} \right\}$$

$$= 1.054\sigma_R^2 \left\{ (\text{-})\frac{Q_l^{-5/6} Q_l^{11/6}\left(\frac{1}{Q_l^2}+1\right)^{11/12} \sin\left[(5/4)\tan^{-1}(Q_l)\right]}{\left(1+Q_l^2\right)^{7/24} Q_l} + \frac{5}{4} Q_l^{-5/6} \right\}$$

$$= 1.054\sigma_R^2 \left\{ (\text{-})\frac{\left(\frac{1}{Q_l^2}+1\right)^{11/12} \sin\left[(5/4)\tan^{-1}(Q_l)\right]}{\left(1+Q_l^2\right)^{7/24}} + \frac{5}{4} Q_l^{-5/6} \right\}$$

$$= 3.86\sigma_R^2 \left\{ \left(\frac{1}{Q_l^2}+1\right)^{11/12} (\text{-})\frac{0.273\sin\left[(5/4)\tan^{-1}(Q_l)\right]}{\left(1+Q_l^2\right)^{7/24}} + 0.341 Q_l^{-5/6} \right\} \qquad \text{(C17)}$$

Now, combine the three intervals where $\mathrm{B}(\kappa/\kappa_l)^{\alpha} = 1 + 1.802\kappa/\kappa_l - 0.254(\kappa/\kappa_l)^{7/6}$

$$
\begin{aligned}
\rightarrow \sigma_I^2(L) &= 3.86\sigma_R^2\left\{\left(\frac{1}{Q_l^2}+1\right)^{11/12}\sin\left[(11/6)\tan^{-1}(Q_l)\right] - \frac{11}{6}Q_l^{-5/6}\right\} \\
&+ 3.86\sigma_R^2\left\{\left(\frac{1}{Q_l^2}+1\right)^{11/12}\frac{(1.507)\sin\left[(4/3)\tan^{-1}(Q_l)\right]}{\left(1+Q_l^2\right)^{1/4}} - 2.01Q_l^{-5/6}\right\} \\
&+ 3.86\sigma_R^2\left\{\left(\frac{1}{Q_l^2}+1\right)^{11/12}(\text{-})\frac{0.273\sin\left[(5/4)\tan^{-1}(Q_l)\right]}{\left(1+Q_l^2\right)^{7/24}} + 0.341Q_l^{-5/6}\right\} \\
&= 3.86\sigma_R^2\left\{(1+\frac{1}{Q_l^2})^{11/12}\left[\sin\left(\frac{11}{6}\tan^{-1}Q_l\right) + \frac{1.507}{\left(1+Q_l^2\right)^{1/4}}\sin\left(\frac{4}{3}\tan^{-1}Q_l\right) - \frac{0.273}{\left(1+Q_l^2\right)^{7/24}}\sin\left(\frac{5}{4}\tan^{-1}Q_l\right)\right] - \frac{11}{6}Q_l^{-5/6} - 2.01Q_l^{-5/6} + 0.341Q_l^{-5/6}\right\} \\
&= 3.86\sigma_R^2\left\{\left(1+\frac{1}{Q_l^2}\right)^{11/12}\left[\sin\left(\frac{11}{6}\tan^{-1}Q_l\right) + \frac{1.507}{\left(1+Q_l^2\right)^{1/4}}\sin\left(\frac{4}{3}\tan^{-1}Q_l\right) - \frac{0.273}{\left(1+Q_l^2\right)^{7/24}}\sin\left(\frac{5}{4}\tan^{-1}Q_l\right)\right] - 3.50Q_l^{-5/6}\right\} \\
&= 3.86\sigma_R^2\left\{\left(1+\frac{1}{Q_l^2}\right)^{11/12}\left[\sin\left(\frac{11}{6}\tan^{-1}Q_l\right) + \frac{1.507}{\left(1+Q_l^2\right)^{1/4}}\sin\left(\frac{4}{3}\tan^{-1}Q_l\right) - \frac{0.273}{(1+Q_l^2)^{7/24}}\sin\left(\frac{5}{4}\tan^{-1}Q_l\right)\right] - 3.50Q_l^{-5/6}\right\} \qquad \text{(C18)}
\end{aligned}
$$

Therefore, the plane wave expression in weak fluctuations using the modified bump spectrum is

$$
\sigma_I^2(L) = 3.86\sigma_R^2\left\{\left(1+\frac{1}{Q_l^2}\right)^{11/12}\left[\sin\left(\frac{11}{6}\tan^{-1}Q_l\right) + \frac{1.507}{(1+Q_l^2)^{1/4}}\sin\left(\frac{4}{3}\tan^{-1}Q_l\right) - \frac{0.273}{\left(1+Q_l^2\right)^{7/24}}\sin\left(\frac{5}{4}\tan^{-1}Q_l\right)\right] - 3.50Q_l^{-5/6}\right\}
$$

APPENDIX D: WEAK TURBULENCE, KOLMOGOROV, SPHERICAL WAVE

The Von Karman Spectrum $\left(\boldsymbol{\phi_n(\kappa) = 0.033C_n^2 \exp(-\kappa^2/\kappa_m^2)/(\kappa^2+\kappa_0^2)^{11/6}}\right)$ reduces to the Kolmogorov Spectrum $\left(\phi_n(\kappa) = 0.033C_n^2\kappa^{-11/3}\right)$ if inner scale goes to zero and outer scale goes to infinity. Therefore, the scintillation index derived by the von Karman spectrum (see appendix E) reduces to that derived by the Kolmogorov spectrum if inner scale goes to zero and outer scale goes to infinity.

For a spherical wave (equation 13),

$$\sigma_{I,sp}^2 = 3.86\sigma_R^2\left\{0.4\left(1+\frac{9}{Q_m^2}\right)^{11/12}\sin\left[\frac{11}{6}\tan^{-1}\left(\frac{Q_m}{3}\right)\right]-\frac{11}{6}Q_m^{-5/6}\right\} \tag{D1}$$

If $l_0 \to 0$,

$$\begin{aligned}\sigma_{I,sp}^2 &= 3.86\sigma_R^2\left\{0.4(1+0)^{11/12}\sin\left[\frac{11}{6}\tan^{-1}(\infty)\right]-0\right\}\\ &= 3.86\sigma_R^2\left\{0.4\sin\left[\frac{11}{6}(\pi/2)\right]\right\}\\ &= 0.4\sigma_R^2\end{aligned} \tag{D2}$$

Therefore, the scintillation index for a spherical wave in weak turbulence based on the Kolmogorov spectrum is $\sigma_I^2 = 0.4\sigma_R^2$

APPENDIX E: WEAK TURBULENCE, VON KARMAN, SPHERICAL WAVE

Under weak fluctuation conditions, the on-axis scintillation index for a spherical wave is defined by (equation 11)

$$\sigma_{I,l}^2(L) = 8\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa \Phi_n(\kappa) \left[1 - \cos\left(\frac{L\kappa^2}{k}\right)\xi(1-\xi)\right] d\kappa d\xi \qquad \text{(E1)}$$

For the von Karman spectrum,

$$\begin{aligned}
\sigma_I^2 &= 8\pi^2 k^2 L \int_0^1 \int_0^\infty \frac{\kappa 0.033 C_n^2 \exp\left(-\kappa^2/\kappa_m^2\right)}{\left(\kappa^2+\kappa_0^2\right)^{11/6}} \left\{1 - \cos\left[\frac{L\kappa^2}{k}\xi(1-\xi)\right]\right\} d\kappa d\xi \\
&= 2.606 C_n^2 k^2 L \int_0^1 \int_0^\infty \frac{\kappa \exp\left(-\kappa^2/\kappa_m^2\right)}{\left(\kappa^2+\kappa_0^2\right)^{11/6}} \left\{1 - \cos\left[\frac{L\kappa^2}{k}\xi(1-\xi)\right]\right\} d\kappa d\xi \\
&= 2.606 C_n^2 k^2 L \int_0^1 \int_0^\infty \frac{\kappa \exp\left(-\kappa^2/\kappa_m^2\right)}{\left(\kappa^2+\kappa_0^2\right)^{11/6}} \left\{1 - \mathrm{Re}\left[\exp\left(\frac{-iL\kappa^2\xi(1-\xi)}{k}\right)\right]\right\} d\kappa d\xi \\
&= 2.606 C_n^2 k^2 L \,\mathrm{Re} \int_0^1 \int_0^\infty \frac{\kappa \exp\left(-\kappa^2/\kappa_m^2\right)}{\left(\kappa^2+\kappa_0^2\right)^{11/6}} \left\{1 - \exp\left[\frac{-iL\kappa^2\xi(1-\xi)}{k}\right]\right\} d\kappa d\xi \\
&= 2.606 C_n^2 k^2 L \,\mathrm{Re} \int_0^1 \int_0^\infty \frac{\kappa}{\left(\kappa^2+\kappa_0^2\right)^{11/6}} \left\{\exp\left(\frac{-\kappa^2}{\kappa_m^2}\right) - \exp\left[\frac{-\kappa^2}{\kappa_m^2} - \frac{iL\kappa^2\xi(1-\xi)}{k}\right]\right\} d\kappa d\xi \\
&= 2.606 C_n^2 k^2 L \,\mathrm{Re} \int_0^1 \int_0^\infty \frac{\kappa}{\left(\kappa^2+\kappa_0^2\right)^{11/6}} \left\{\exp\left(\frac{-\kappa^2}{\kappa_m^2}\right) - \exp\left[\left(\frac{-\kappa^2}{\kappa_m^2}\right)\left(1 + \frac{iL\kappa_m{}^2\xi(1-\xi)}{k}\right)\right]\right\} d\kappa d\xi \\
&= 2.606 C_n^2 k^2 L \,\mathrm{Re} \int_0^1 \int_0^\infty \frac{\kappa}{\left(\kappa^2+\kappa_0^2\right)^{11/6}} \left\{\exp\left(\frac{-\kappa^2}{\kappa_m^2}\right) - \exp\left[\left(\frac{-\kappa^2}{\kappa_m^2}\right)\left(1 + iQ_m\xi[1-\xi]\right)\right]\right\} d\kappa d\xi \qquad \text{(E2)}
\end{aligned}$$

where

$$\Phi_n(\kappa) = \frac{0.033 C_n^2 \exp(-\kappa^2/\kappa_m^2)}{(\kappa^2+\kappa_0^2)^{11/6}} \quad \text{(equation 3)}$$

$$\cos x = \mathrm{Re}[\exp(-ix)]$$

$$Q_m = \frac{L\kappa_m^2}{k}$$

Recognizing[1]

$$\int_0^\infty \frac{\kappa^{2\mu} \exp\left(-\kappa^2/\kappa_m^2\right)}{\left(\kappa^2+\kappa_0^2\right)^{11/6}} = \frac{1}{2} \mathrm{K}_0^{2\mu-8/3} \Gamma(\mu+1/2) u\left(\mu+1/2; \mu-1/3; \kappa_0^2/\kappa_m^2\right) \qquad \text{(E3)}$$

where

$$2\mu = 1 \rightarrow \mu = 1/2,$$

$$2\mu - 8/3 = 1 - 8/3 = -5/3$$
$$\mu + 1/2 = 1$$
$$\mu - 1/3 = 1/6$$

results in

$$\sigma_I^2 = 2.606 C_n^2 k^2 L \,\mathrm{Re}\int_0^1 \frac{1}{2}\kappa_0^{-5/3}\Gamma(1)u\left(1;1/6;\kappa_0^2/\kappa_m^2\right) - \frac{1}{2}\kappa_0^{-5/3}\Gamma(1)u\left(1;1/6;[(\kappa_0^2/\kappa_m^2)[1+iQ_m\xi(1-\xi)]\right)d\xi$$

$$= 1.303 C_n^2 k^2 L \kappa_0^{-5/3}\,\mathrm{Re}\int_0^1 u\left(1;1/6;\kappa_0^2/\kappa_m^2\right) - u\left(1;1/6;(\kappa_0^2/\kappa_m^2)[1+iQ_m\xi(1-\xi)]\right)d\xi \qquad \text{(E4)}$$

Assuming[1]

$$u(a;c;z) \cong \frac{\Gamma(1-c)}{\Gamma(1+a-c)} + \frac{\Gamma(c-1)z^{1-c}}{\Gamma(a)} \qquad \text{(E5)}$$

then

$$u(1;1/6;x) \cong \frac{\Gamma(5/6)}{\Gamma(11/6)} + \frac{\Gamma(-5/6)x^{5/6}}{\Gamma(1)}$$
$$\cong \frac{\Gamma(5/6)}{5/6\,\Gamma(5/6)} + \frac{(-6/5)\Gamma(1/6)x^{5/6}}{\Gamma(1)}$$
$$= \frac{6}{5} + 6\left(\frac{-6}{5}\right)\Gamma\left(\frac{7}{6}\right)x^{5/6}$$
$$\cong \frac{6}{5} - 6.695x^{5/6} \qquad \text{(E6)}$$

Equation E4 can be reduced to

$$\sigma_I^2 = 1.303 C_n^2 k^2 L \kappa_0^{-5/3}\,\mathrm{Re}\int_0^1 [\frac{6}{5} - 6.695\left(\kappa_0^2/\kappa_m^2\right)^{5/6}] - \left\{\frac{6}{5} - 6.695\left(\kappa_0^2/\kappa_m^2\right)^{5/6}[1+iQ_m\xi(1-\xi)]^{5/6}\right\}d\xi$$
$$= 1.303 C_n^2 k^2 L \kappa_0^{-5/3}\,\mathrm{Re}\int_0^1 \left[6.695(\kappa_0^2/\kappa_m^2)^{5/6}\right]\left\{-1 + [1+iQ_m\xi(1-\xi)]^{5/6}\right\}d\xi$$
$$= 8.70 C_n^2 k^2 L \kappa_0^{-5/3}\kappa_0^{5/3}\kappa_m^{-5/3}\,\mathrm{Re}\int_0^1 \left\{-1 + [1+iQ_m\xi(1-\xi)]^{5/6}\right\}d\xi$$
$$= 8.70 C_n^2 k^2 L \kappa_m^{-5/3}\,\mathrm{Re}\left\{-1 + \int_0^1 [1+iQ_m\xi(1-\xi)]^{5/6}d\xi\right\}$$
$$= 8.70\left(0.813\sigma_R^2 Q_m^{-5/6}\right)\mathrm{Re}\left\{-1 + \int_0^1 [1+iQ_m\xi(1-\xi)]^{5/6}d\xi\right\}$$
$$= 7.07\sigma_R^2 Q_m^{-5/6}\,\mathrm{Re}\left\{-1 + \int_0^1 [1+iQ_m\xi(1-\xi)]^{5/6}d\xi\right\} \qquad \text{(E7)}$$

where

$$\sigma_R^2 = 1.23 C_n^2 k^{7/6} L^{11/6}$$

$$Q_m = \frac{L\kappa_m^2}{k}$$

$$\begin{aligned} C_n^2 k^2 L \kappa_m^{-5/3} &= C_n^2 k^{7/6} k^{5/6} L^{11/6} L^{-5/6} \kappa_m^{-5/3} \\ &= C_n^2 k^{7/6} L^{11/6} \left(\frac{k}{L\kappa_m^2} \right)^{5/6} \\ &= 0.813 \sigma_R^2 Q_m^{-5/6} \end{aligned}$$

Evaluate $\mathrm{Re} \int_0^1 \left[1 + iQ_m \xi(1-\xi)\right]^{5/6} d\xi$

With binomial expansion, $(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n, \quad |x| < 1$

$$\mathrm{Re} \int_0^1 \left[1 + iQ_m \xi(1-\xi)\right]^{5/6} d\xi = \mathrm{Re} \int_0^1 \sum_{n=0}^{\infty} \binom{5/6}{n} \left[iQ_m \xi(1-\xi)\right]^n d\xi = \mathrm{Re} \int_0^1 \sum_{n=0}^{\infty} \binom{5/6}{n} (iQ_m)^n \left[\xi(1-\xi)\right]^n d\xi \quad \text{(E8)}$$

where $x = iQ_m \xi(1-\xi)$, $k = 5/6$

In addition,

$$\int_0^a x^m (a^n - x^n)^p \, dx = \frac{a^{m+1+np} \Gamma\left[(m+1)/n\right] \Gamma(p+1)}{n \Gamma\left[(m+1)/n + p + 1\right]} \quad \text{(E9)}$$

As a result

$$\mathrm{Re} \int_0^1 \sum_{n=0}^{\infty} \binom{5/6}{n} (iQ_m)^n \left[\xi(1-\xi)\right]^n d\xi = \mathrm{Re} \sum_{n=0}^{\infty} \binom{5/6}{n} (iQ_m)^n \frac{(1)\Gamma(n+1)\Gamma(n+1)}{\Gamma(2n+2)} \quad \text{(E10)}$$

where m=n, a=1, n=1, p=n

noting that

$$\binom{a}{n} = \frac{(-1)^n (-a)_n}{n!} \quad \text{(E11)}$$

$$\mathrm{Re}\sum_{n=0}^{\infty}\frac{(-1)^n(-5/6)_n}{n!}(iQ_m)^n\frac{\Gamma(n+1)\Gamma(n+1)}{\Gamma(2n+2)} = \mathrm{Re}\sum_{n=0}^{\infty}\frac{(-1)^n(-5/6)_n}{n!}(iQ_m)^n\frac{\Gamma(n+2)\Gamma(n+1)}{(n+1)\Gamma(2n+2)}$$

$$= \mathrm{Re}\sum_{n=0}^{\infty}\left(\frac{\Gamma(n+1)}{\Gamma(n+1)}\frac{1}{(n+1)}\right)\left(\frac{(-1)^n(-5/6)_n}{n!}\right)(iQ_m)^n\frac{\Gamma(n+2)\Gamma(n+1)}{\Gamma(2n+2)}$$

$$= \mathrm{Re}\sum_{n=0}^{\infty}\left(\frac{n!}{\Gamma(n+2)}\right)\left(\frac{(-1)^n(-5/6)_n}{n!}\right)(iQ_m)^n\frac{\Gamma(n+2)\Gamma(n+1)}{\Gamma(2n+2)}$$

$$= \mathrm{Re}\sum_{n=0}^{\infty}\left(\frac{(1)_n}{\Gamma(n+2)}\right)\left(\frac{(-1)^n(-5/6)_n}{n!}\right)(iQ_m)^n\frac{\Gamma(n+2)\Gamma(n+1)}{\Gamma(2n+2)}$$

$$= \mathrm{Re}\sum_{n=0}^{\infty}\left(\frac{(1)_n 1!}{\Gamma(n+2)}\right)\left(\frac{(-1)^n(-5/6)_n}{n!}\right)(iQ_m)^n\frac{\Gamma(n+2)\Gamma(n+1)}{\Gamma(2n+2)}$$

$$= \mathrm{Re}\sum_{n=0}^{\infty}\left(\frac{(1)_n\Gamma(2)}{\Gamma(n+2)}\right)\left(\frac{(-1)^n(-5/6)_n}{n!}\right)(iQ_m)^n\frac{\Gamma(n+2)\Gamma(n+1)}{\Gamma(2n+2)}$$

$$= \mathrm{Re}\sum_{n=0}^{\infty}\frac{(1)_n}{(2)_n}\frac{(-1)^n(-5/6)_n}{n!}(iQ_m)^n\frac{\Gamma(n+2)\Gamma(n+1)}{\Gamma(2n+2)}$$

$$= \mathrm{Re}\sum_{n=0}^{\infty}\frac{(-5/6)_n(1)_n}{(2)_n n!}(-iQ_m)^n\frac{\Gamma(n+2)\Gamma(n+1)}{\Gamma(2n+2)} \qquad \text{(E12)}$$

Equation E7 becomes

$$\sigma_{I,l}^2(L) = 7.07\sigma_R^2 Q_m^{-5/6}\left[\mathrm{Re}\sum_{n=0}^{\infty}\frac{(-5/6)_n(1)_n}{(2)_n n!}(-iQ_m)^n\frac{\Gamma(n+2)\Gamma(n+1)}{\Gamma(2n+2)} - 1\right]$$

$$= 7.07\sigma_1^2 Q_m^{-5/6}\,\mathrm{Re}\left[\sum_{n=0}^{\infty}\frac{(-5/6)_n(1)_n}{(2)_n n!}(-iQ_m)^n\,{}_2F_1(-n,n+1,n+2;1) - 1\right] \qquad \text{(E13)}$$

where

$${}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \qquad \text{(E14)}$$

Evaluate: ${}_2F_1(-n,n+1,n+2;1)$

$${}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty}\frac{(a)_k(b)_k z^k}{(c)_k k!} \cong \sum_{k=0}^{1}\frac{(a)_k(b)_k z^k}{(c)_k k!} \qquad \text{(E15)}$$

Consider $F(-n,n+1;n+2;x)$

For n = 0, $F(0,1;2;x) = \sum_{n=0}^{0}\frac{(0)_n(1)_n x^n}{(2)_n n!} = \frac{(0)_0(1)_0 x^0}{(2)_0 0!} = \frac{(1)(1)(1)}{(1)(1)} = 1$

Thus, for n=0, $F(-n,n+1;n+2;x) = F(0,1;2;x) = 1$, same as $(1-2x/3)^0$

For n = 1, $F(-1,2;3;x) = \sum_{n=1}^{1} \frac{(-1)_n (2)_n x^n}{(3)_n n!} = \frac{(-1)_1 (2)_1 x^1}{(3)_1 1!} = \frac{\left(\frac{(-1)^1 1!}{0!}\right)\left(\frac{\Gamma(2+1)}{\Gamma(2)}\right)x^1}{\frac{\Gamma(3+1)}{\Gamma(3)} 1!}$

$$= \frac{(-1)\left(\frac{2\Gamma(2)}{\Gamma(2)}\right)x}{\frac{3\Gamma(3)}{\Gamma(3)} 1!} = (-)\frac{2x}{3}$$

Thus, for n= 0:1, $F(-n, n+1; n+2; x) = 1 - \frac{2x}{3}$, same as $(1-2x/3)^1$

For n = 2, $F(-2,3;4;x) = \sum_{n=2}^{2} \frac{(-2)_n (3)_n x^n}{(4)_n n!} = \frac{(-2)_2 (3)_2 x^2}{(4)_2 2!} = \frac{\left(\frac{(-1)^2 2!}{0!}\right)\left(\frac{\Gamma(3+1)}{\Gamma(3)}\right)x^2}{\frac{\Gamma(4+1)}{\Gamma(4)} 2!}$

$$= \frac{2\left(\frac{3\Gamma(3)}{\Gamma(3)}\right)x^2}{\frac{4\Gamma(4)}{\Gamma(4)} 2!} = \frac{3x^2}{4}$$

Thus, for n= 0:2, $F(-n, n+1; n+2; x) = 1 - \frac{2x}{3} + \frac{3x^2}{4}$.

Note that $1 - \frac{2x}{3} + \frac{3x^2}{4} \cong (1-2x/3)^n$ only when n=0 & n=1. Rationale as follows:

Compare $\left(1 - \frac{2x}{3} + \frac{3x^2}{4}\right)$ to $(1-2x/3)^n$

$$n = 0 \rightarrow 1 = \left(1 - \frac{2x}{3}\right)^0 = 1$$

$$n = 1 \rightarrow 1 - \frac{2x}{3} = \left(1 - \frac{2x}{3}\right)^1 = 1 - \frac{2x}{3}$$

$$n = 2 \rightarrow 1 - \frac{2x}{3} + \frac{3x^2}{4} \cong \left(1 - \frac{2x}{3}\right)^2 = 1 - \frac{4x}{3} + \frac{4x^2}{9}, \qquad \left(\neq (1-2x/3)^n\right)$$

Therefore, when n=0 and when n=1

$$_2F_1(-n, n+1, n+2; x) \cong (1-2x/3)^n \qquad \text{(E16)}$$

$$ {}_2F_1(-n, n+1, n+2; 1) \cong (1-2/3)^n = \frac{1}{3^n} \tag{E17} $$

$$ \sum_{n=0}^{\infty} \frac{(-5/6)_n (1)_n}{(2)_n n!} \left(-iQ_m\right)^n {}_2F_1(-n, n+1, n+2; 1) = \sum_{n=0}^{\infty} \frac{(-5/6)_n (1)_n}{3^n (2)_n n!} \left(-iQ_m\right)^n $$

$$ = \sum_{n=0}^{\infty} \frac{(-5/6)_n (1)_n}{(2)_n n!} \left(\frac{-iQ_m}{3}\right)^n \tag{E18} $$

Recall from C15 that ${}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k z^k}{(c)_k k!}$.

then

$$ \rightarrow \sum_{n=0}^{\infty} \frac{(-5/6)_n (1)_n}{(2)_n n!} \left(\frac{-iQ_m}{3}\right)^n = {}_2F_1\left(-5/6, 1; 2; \frac{-iQ_m}{3}\right) $$

$$ = \frac{\left(1 + \frac{iQ_m}{3}\right)^{11/6} - 1}{\frac{11}{6}\left(\frac{iQ_m}{3}\right)} $$

$$ = \left(\frac{6}{11}\right) \frac{\left(1 + \frac{iQ_m}{3}\right)^{11/6} - 1}{\frac{iQ_m}{3}} \tag{E19} $$

where

$$ {}_2F_1(1-a, 1; 2; -x) = \frac{(1+x)^a - 1}{ax} $$

$$ 1 - a = 5/6 \rightarrow a = 11/6 $$

Equation E13 can be re-written as

$$ \sigma_{I,l}^2(L) = 7.07\sigma_R^2 Q_m^{-5/6} \operatorname{Re}\left[\sum_{n=0}^{\infty} \frac{(-5/6)_n (1)_n}{(2)_n n!} \left(-iQ_m\right)^n {}_2F_1(-n, n+1, n+2; 1) - 1\right] $$

$$ = 7.07\sigma_R^2 Q_m^{-5/6} \operatorname{Re}\left[\left(\frac{6}{11}\right)\left(\frac{\left(1 + \frac{iQ_m}{3}\right)^{11/6} - 1}{\frac{iQ_m}{3}}\right) - 1\right] $$

$$= 7.07\left(\frac{6}{11}\right)\sigma_R^2 Q_m^{-5/6}\,\mathrm{Re}\left[\left(\frac{\left(1+\frac{iQ_m}{3}\right)^{11/6}-1}{\frac{iQ_m}{3}}\right)-\frac{11}{6}\right]$$

$$= 3.86\sigma_R^2 Q_m^{-5/6}\,\mathrm{Re}\left[\left(\frac{(1+\frac{iQ_m}{3})^{11/6}}{\frac{iQ_m}{3}}-\frac{1}{\frac{iQ_m}{3}}\right)-\frac{11}{6}\right] \tag{E20}$$

Getting rid of the imaginary term,

$$\sigma_{I,l}^2(L) = 3.86\sigma_R^2 Q_m^{-5/6}\,\mathrm{Re}\left[\left(\frac{\left(1+\frac{iQ_m}{3}\right)^{11/6}}{\frac{iQ_m}{3}}\right)-\frac{11}{6}\right] \tag{E21}$$

Evaluate $\dfrac{\left(1+\frac{iQ_m}{3}\right)^{11/6}}{iQ_m/3}$ in polar coordinates:

$$x+iy = r\exp\left[i\tan^{-1}(y/x)\right] = \sqrt{x^2+y^2}\exp\left[i\tan^{-1}(y/x)\right]$$

Therefore,

$$\left(1+\frac{iQ_m}{3}\right)^{11/6} = \left\{\sqrt{1^2+\left(\frac{Q_m}{3}\right)^2}\exp\left[i\tan^{-1}\left(\frac{Q_m}{3}\right)\right]\right\}^{11/6}$$

$$= \left\{\left(1+\frac{Q_m^2}{9}\right)^{11/12}\exp\left[i\tan^{-1}\left(\frac{Q_m}{3}\right)\right]\right\}^{11/6}$$

$$= \left(1+\frac{Q_m^2}{9}\right)^{11/12}\exp\left[i\left(\frac{11}{6}\right)\tan^{-1}\left(\frac{Q_m}{3}\right)\right] \tag{E22}$$

Also, $x+iy = r\exp\left[i\tan^{-1}(y/x)\right] = \sqrt{x^2+y^2}\exp\left[i\tan^{-1}(y/x)\right]$.

Therefore,

$$i\frac{Q_m}{3} = 0 - i\frac{Q_m}{3} = \sqrt{0^2+\frac{Q_m^2}{9}}\exp\left[i\tan^{-1}(\infty)\right] = \frac{Q_m}{3}\exp(i\pi/2) \tag{E23}$$

$$\frac{1}{iQ_m/3} = \frac{3}{Q_m}\exp(-i\pi/2) = \frac{\exp(-i\pi/2)}{Q_m/3} \qquad \text{(E24)}$$

$$\frac{\left(1+\frac{iQ_m}{3}\right)^{11/6}}{iQ_m/3} = \frac{\left(1+\frac{Q_m^2}{9}\right)^{11/12}\exp\left[i(\frac{11}{6})\tan^{-1}\left(\frac{Q_m}{3}\right)-i\pi/2\right]}{Q_m/3}$$

$$= \frac{\left(1+\frac{Q_m^2}{9}\right)^{11/12}\sin\left[\left(\frac{11}{6}\right)\tan^{-1}\left(\frac{Q_m}{3}\right)\right]}{Q_m/3}$$

$$= \frac{\left[\frac{Q_m^2}{9}\left(\frac{9}{Q_m^2}+1\right)\right]^{11/12}\sin\left[\left(\frac{11}{6}\right)\tan^{-1}\left(\frac{Q_m}{3}\right)\right]}{Q_m/3}$$

$$= \frac{\frac{Q_m^{11/6}}{7.49}\left(\frac{9}{Q_m^2}+1\right)^{11/12}\sin\left[\left(\frac{11}{6}\right)\tan^{-1}\left(\frac{Q_m}{3}\right)\right]}{Q_m/3}$$

$$= 0.4Q_m^{5/6}\left(\frac{9}{Q_m^2}+1\right)^{11/12}\sin\left[\left(\frac{11}{6}\right)\tan^{-1}\left(\frac{Q_m}{3}\right)\right] \qquad \text{(E25)}$$

where exp(ix) = cos(x), ignoring the imaginary term

Thus, for a Spherical Wave Scintillation in Weak Turbulence using von Karman Spectrum, equation E21 reduces to

$$\sigma_{I,l}^2(L) = 3.86\sigma_R^2\left[\left\{0.4\left(1+\frac{9}{Q_m^2}\right)^{11/12}\sin\left[\left(\frac{11}{6}\right)\tan^{-1}\left(\frac{Q_m}{3}\right)\right]\right\}-\frac{11}{6}Q_m^{-5/6}\right] \qquad \text{(E26)}$$

APPENDIX F: WEAK TURBULENCE, MODIFIED (BUMP), SPHERICAL WAVE

Under weak fluctuation conditions, the on-axis scintillation index for a spherical wave is defined by (equation 11):

$$\sigma_{I,l}^2(L) = 8\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa \Phi_n(\kappa) \left[1 - \cos\left(\frac{L\kappa^2}{k}\right)\xi(1-\xi)\right] d\kappa d\xi \qquad \text{(F1)}$$

For the modified spectrum

$$\sigma_{I,l}^2(L) = 8\pi^2 k^2 L \int_0^1 \int_0^\infty \frac{\kappa 0.033 C_n^2 \left[1 + 1.802\kappa/\kappa_l - 0.254(\kappa/\kappa_l)^{7/6}\right] \exp\left(-\kappa^2/\kappa_l^2\right)}{\left(\kappa^2 + \kappa_0^2\right)^{11/6}} \left\{1 - \cos\left[\frac{L\kappa^2}{k}\xi(1-\xi)\right]\right\} d\kappa d\xi \qquad \text{(F2)}$$

where

$$\Phi_n(\kappa) = \frac{0.033 C_n^2 \left[1 + 1.802\kappa/\kappa_l - 0.254(\kappa/\kappa_l)^{7/6}\right] \exp\left(-\kappa^2/\kappa_l^2\right)}{\left(\kappa^2 + \kappa_0^2\right)^{11/6}} \quad \text{(equation 4)}$$

This equation can be split into 3 intervals. Let $\mathrm{B}(\kappa/\kappa_l)^\alpha = 1 + 1.802\kappa/\kappa_l - 0.254(\kappa/\kappa_l)^{7/6}$.

Determine a general solution for $\mathrm{B}(\kappa/\kappa_l)^\alpha$, then solve for each interval separately.

$$\sigma_{I,sp}^2 = 8\pi^2 k^2 L \int_0^1 \int_0^\infty \frac{\kappa 0.033 C_n^2 \mathrm{B}(\kappa/\kappa_l)^\alpha \exp\left(-\kappa^2/\kappa_l^2\right)}{(\kappa^2 + \kappa_0^2)^{11/6}} \left\{1 - \cos\left[\frac{L\kappa^2}{k}\xi(1-\xi)\right]\right\} d\kappa d\xi$$

$$= 2.606 C_n^2 k^2 L \int_0^1 \int_0^\infty \frac{\kappa \mathrm{B}(\kappa/\kappa_l)^\alpha \exp\left(-\kappa^2/\kappa_l^2\right)}{\left(\kappa^2 + \kappa_0^2\right)^{11/6}} \left\{1 - \cos\left[\frac{L\kappa^2}{k}\xi(1-\xi)\right]\right\} d\kappa d\xi$$

$$= 2.606 C_n^2 k^2 L \int_0^1 \int_0^\infty \frac{\kappa \mathrm{B}(\kappa/\kappa_l)^\alpha \exp\left(-\kappa^2/\kappa_l^2\right)}{\left(\kappa^2 + \kappa_0^2\right)^{11/6}} \left\{1 - \mathrm{Re}\left[\exp\left(\frac{-iL\kappa^2\xi(1-\xi)}{k}\right)\right]\right\} d\kappa d\xi$$

$$= 2.606 C_n^2 k^2 L \,\mathrm{Re} \int_0^1 \int_0^\infty \frac{\kappa \mathrm{B}(\kappa/\kappa_l)^\alpha \exp\left(-\kappa^2/\kappa_l^2\right)}{\left(\kappa^2 + \kappa_0^2\right)^{11/6}} \left\{1 - \exp\left[\frac{-iL\kappa^2\xi(1-\xi)}{k}\right]\right\} d\kappa d\xi$$

$$= 2.606 C_n^2 k^2 L \,\mathrm{Re} \int_0^1 \int_0^\infty \frac{\kappa \mathrm{B}(\kappa/\kappa_l)^\alpha}{\left(\kappa^2 + \kappa_0^2\right)^{11/6}} \left\{\exp\left(\frac{-\kappa^2}{\kappa_l^2}\right) - \exp\left[\frac{-\kappa^2}{\kappa_l^2} - \frac{iL\kappa^2\xi(1-\xi)}{k}\right]\right\} d\kappa d\xi$$

$$= 2.606 C_n^2 k^2 L \,\mathrm{Re} \int_0^1 \int_0^\infty \frac{\kappa \mathrm{B}(\kappa/\kappa_l)^\alpha}{\left(\kappa^2 + \kappa_0^2\right)^{11/6}} \left\{\exp\left(\frac{-\kappa^2}{\kappa_l^2}\right) - \exp\left[\left(\frac{-\kappa^2}{\kappa_l}\right)\left(1 + \frac{iL\kappa_l{}^2\xi(1-\xi)}{k}\right)\right]\right\}] d\kappa d\xi$$

$$= 2.606 C_n^2 k^2 L \,\mathrm{Re} \int_0^1 \int_0^\infty \frac{\kappa \mathrm{B}(\kappa/\kappa_l)^\alpha}{\left(\kappa^2 + \kappa_0^2\right)^{11/6}} \left\{\exp\left(\frac{-\kappa^2}{\kappa_l^2}\right) - \exp\left[\left(\frac{-\kappa^2}{\kappa_l^2}\right)\left(1 + iQ_l\xi(1-\xi)\right)\right]\right\} d\kappa d\xi$$

$$= \frac{2.606 \mathrm{B} C_n^2 k^2 L}{\kappa_l^\alpha} \,\mathrm{Re} \int_0^1 \int_0^\infty \frac{\kappa^{\alpha+1}}{\left(\kappa^2 + \kappa_0^2\right)^{11/6}} \left\{\exp\left(\frac{-\kappa^2}{\kappa_l^2}\right) - \exp\left[\left(\frac{-\kappa^2}{\kappa_l^2}\right)\left(1 + iQ_l\xi(1-\xi)\right)\right]\right\} d\kappa d\xi \qquad \text{(F3)}$$

where

$$\cos x = \mathrm{Re}[\exp(-ix)]$$

$$Q_l = \frac{L\kappa_l^2}{k}$$

Recognizing[1]

$$\int_0^\infty \frac{\kappa^{2\mu}\exp\left(-\kappa^2/\kappa_m^2\right)d\kappa}{\left(\kappa^2+\kappa_0^2\right)^{11/6}} = \frac{1}{2}\mathrm{K}_0^{2\mu-8/3}\Gamma(\mu+1/2)u(\mu+1/2;\mu-1/3;\kappa_0^2/\kappa_m^2) \qquad \text{(F4)}$$

where

$$2\mu = \alpha+1 \rightarrow \mu = (\alpha+1)/2\,,$$
$$2\mu - 8/3 = \alpha+1-8/3 = \alpha-5/3$$
$$\mu+1/2 = [(\alpha+1)/2]+1/2 = (\alpha+2)/2$$
$$\mu-1/3 = [(\alpha+1)/2]-1/3 = [3(\alpha+1)/6]-2/6 = (3\alpha+3-2)/6 = (3\alpha+1)/6$$

results in

$$\sigma_{I,sp}^2 = \frac{2.606\mathrm{B}C_n^2k^2L}{\kappa_l^\alpha}\mathrm{Re}\int_0^1\left\{\ \frac{1}{2}\kappa_0^{\alpha-5/3}\Gamma([\alpha+2]/2)u([(\alpha+2)/2];[(3\alpha+1)/6];\kappa_0^2/\kappa_l^2)\right.$$
$$\left.-\frac{1}{2}\kappa_0^{\alpha-5/3}\Gamma([\alpha+2]/2)u([(\alpha+2)/2];[(3\alpha+1)/6];\kappa_0^2/\kappa_l^2(1+iQ_l\xi(1-\xi)\ \right\}d\xi$$

$$= \frac{1.303\mathrm{B}C_n^2k^2L\kappa_0^{\alpha-5/3}\Gamma([\alpha+2]/2)}{\kappa_l^\alpha}\mathrm{Re}\int_0^1\left\{\ u([(\alpha+2)/2];[(3\alpha+1)/6];\kappa_0^2/\kappa_l^2)\right.$$
$$\left.-u([(\alpha+2)/2];[(3\alpha+1)/6];\kappa_0^2/\kappa_l^2\left[1+iQ_l\xi(1-\xi)\right]\ \right\}d\xi \qquad \text{(F5)}$$

Applying[1]

$$u(a;c;z) \cong \frac{\Gamma(1-c)}{\Gamma(1+a-c)} + \frac{\Gamma(c-1)z^{1-c}}{\Gamma(a)} \qquad \text{(F6)}$$

then

$$u([(\alpha+2)/2];[(3\alpha+5)/6];x) \cong \frac{\Gamma([5-3\alpha]/6)}{\Gamma(11/6)} + \frac{\Gamma([3\alpha-5]/6)x^{[5-3\alpha]/6}}{\Gamma([\alpha+2]/2)} \qquad \text{(F7)}$$

where

a = $(\alpha+2)/2$
c = $(3\alpha+1)/6$
1 – c= $1-(3\alpha+1)/6] = [6-(3\alpha+1)]/6 = (5-3\alpha)/6$
1 + a – c = $1+(\alpha+2)/2-(3\alpha+1)/6 = [6+3(\alpha+2)-(3\alpha+1)]/6 = [6+3\alpha+6-3\alpha-1)]/6 = 11/6$

equation F5 can be reduced to

$$\sigma^2_{I,sp} = \frac{1.303\mathrm{B}C_n^2 k^2 L\kappa_0^{\alpha-5/3}\Gamma([\alpha+2]/2)}{\kappa_l^{\alpha}} \mathrm{Re}\int_0^1 \left\{ \frac{\Gamma([5-3\alpha]/6)}{\Gamma(11/6)} + \frac{\Gamma([3\alpha-5]/6)(\kappa_0^2/\kappa_l^2)^{[5-3\alpha]/6}}{\Gamma([\alpha+2]/2)} \right.$$
$$\left. -\left[\frac{\Gamma([5-3\alpha]/6)}{\Gamma(11/6)} + \frac{\Gamma([3\alpha-5]/6)(\kappa_0^2/\kappa_l^2)^{[5-3\alpha]/6}\left[1+iQ_l\xi(1-\xi)\right]^{[5-3\alpha]/6}}{\Gamma([\alpha+2]/2)}\right]\right\} d\xi$$

$$= \frac{1.303\mathrm{B}C_n^2 k^2 L\kappa_0^{\alpha-5/3}\Gamma([\alpha+2]/2)}{\kappa_l^{\alpha}} \mathrm{Re}\int_0^1 \left\{ \frac{\Gamma([3\alpha-5]/6)(\kappa_0^2/\kappa_l^2)^{[5-3\alpha]/6}}{\Gamma([\alpha+2]/2)} \right.$$
$$\left. - \frac{\Gamma([3\alpha-5]/6)(\kappa_0^2/\kappa_l^2)^{[5-3\alpha]/6}\left[1+iQ_l\xi(1-\xi)\right]^{[5-3\alpha]/6}}{\Gamma([\alpha+2]/2)} \right\} d\xi$$

$$= \frac{1.303\mathrm{B}C_n^2 k^2 L\kappa_0^{\alpha-5/3}\Gamma([\alpha+2]/2)\Gamma([3\alpha-5]/6)(\kappa_0^2/\kappa_l^2)^{[5-3\alpha]/6}}{\kappa_l^{\alpha}\Gamma([\alpha+2]/2)} \mathrm{Re}\int_0^1 \left\{1-\left[1+iQ_l\xi(1-\xi)\right]^{[5-3\alpha]/6}\right\} d\xi$$

$$= \frac{1.303\mathrm{B}C_n^2 k^2 L\kappa_0^{\alpha-5/3}\kappa_0^{[5-3\alpha]/3}\Gamma([\alpha+2]/2)\Gamma([3\alpha-5]/6)}{\kappa_l^{[5-3\alpha]/3}\kappa_l^{\alpha}\Gamma([\alpha+2]/2)} \mathrm{Re}\int_0^1 \left\{1-\left[1+iQ_l\xi(1-\xi)\right]^{[5-3\alpha]/6}\right\} d\xi$$

$$= \frac{1.303\mathrm{B}C_n^2 k^2 L\kappa_l^{-5/3}\Gamma([\alpha+2]/2)\Gamma([3\alpha-5]/6)}{\Gamma([\alpha+2]/2)} \mathrm{Re}\int_0^1 \left\{1-\left[1+iQ_l\xi(1-\xi)\right]^{[5-3\alpha]/6}\right\} d\xi$$

$$= \frac{1.303\mathrm{B}(0.813\sigma_R^2 Q_l^{-5/6})\Gamma([\alpha+2]/2)\Gamma([3\alpha-5]/6)}{\Gamma([\alpha+2]/2)} \mathrm{Re}\int_0^1 \left\{1-\left[1+iQ_l\xi(1-\xi)\right]^{[5-3\alpha]/6}\right\} d\xi$$

$$= 1.059 \mathrm{B} \sigma_R^2 Q_l^{-5/6} \Gamma([3\alpha - 5]/6) \,\mathrm{Re} \int_0^1 \left\{ 1 - \left[1 + iQ_l \xi (1-\xi) \right]^{[5-3\alpha]/6} \right\} d\xi$$

$$= 1.059 \mathrm{B} \sigma_R^2 Q_l^{-5/6} \Gamma([3\alpha - 5]/6) \,\mathrm{Re} \int_0^1 \left\{ 1 - \left[1 + iQ_l \xi (1-\xi) \right]^{[5-3\alpha]/6} \right\} d\xi$$

$$= 1.059 \mathrm{B} \sigma_R^2 Q_l^{-5/6} \Gamma([3\alpha - 5]/6) \,\mathrm{Re} \left\{ \int_0^1 1 d\xi - \int_0^1 \left[1 + iQ_l \xi (1-\xi) \right]^{[5-3\alpha]/6} d\xi \right\}$$

$$= 1.059 \mathrm{B} \sigma_R^2 Q_l^{-5/6} \Gamma([3\alpha - 5]/6) \left\{ 1 - \mathrm{Re} \int_0^1 (1 + iQ_l \xi (1-\xi))^{[5-3\alpha]/6} d\xi \right\} \qquad \text{(F8)}$$

where

$$\sigma_R^2 = 1.23 C_n^2 k^{7/6} L^{11/6}$$

$$Q_l = \frac{L\kappa_l^2}{k}$$

$$\begin{aligned} C_n^2 k^2 L \kappa_l^{-5/3} &= C_n^2 k^{7/6} k^{5/6} L^{11/6} L^{-5/6} \kappa_l^{-5/3} \\ &= C_n^2 k^{7/6} L^{11/6} \left(\frac{k}{L\kappa_l^2} \right)^{5/6} \\ &= 0.813 \sigma_R^2 Q_l^{-5/6} \end{aligned}$$

Evaluate $\mathrm{Re} \int_0^1 \left[1 + iQ_m \xi (1-\xi) \right]^{[5-3\alpha]/6} d\xi$:

With binomial expansion, $(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$, $|x| < 1$

$$\mathrm{Re} \int_0^1 \left[1 + iQ_m \xi (1-\xi) \right]^{[5-3\alpha]/6} d\xi = \mathrm{Re} \int_0^1 \sum_{n=0}^{\infty} \binom{[5-3\alpha]/6}{n} \left[iQ_m \xi (1-\xi) \right]^n d\xi = \mathrm{Re} \int_0^1 \sum_{n=0}^{\infty} \binom{[5-3\alpha]/6}{n} (iQ_m)^n \left[\xi (1-\xi) \right]^n d\xi \qquad \text{(F9)}$$

where $x = iQ_m \xi (1-\xi)$, $k = [5-3\alpha]/6$

In addition

$$\int_0^a x^m (a^n - x^n)^p \, dx = \frac{a^{m+1+np} \Gamma[(m+1)/n] \Gamma(p+1)}{n \Gamma[(m+1)/n + p + 1]} \qquad \text{(F10)}$$

As a result

$$\mathrm{Re}\sum_{n=0}^{\infty}\binom{[5-3\alpha]/6}{n}(iQ_m)^n\ \frac{(1)\Gamma[(n+1)]\Gamma(n+1)}{\Gamma[(2n+2]} \tag{F11}$$

where m=n, a=1, n=1, p=n

note that

$$\binom{a}{n}=\frac{(-1)^n(-a)_n}{n!} \tag{F12}$$

Therefore,

$$\begin{aligned}
\mathrm{Re}\int_0^1\left[1+iQ_m\xi(1-\xi)\right]^{[5-3\alpha]/6}d\xi &= \mathrm{Re}\sum_{n=0}^{\infty}\frac{(-1)^n(-[5-3\alpha]/6)_n}{n!}(iQ_m)^n\ \frac{\Gamma[(n+1)]\Gamma(n+1)}{\Gamma[(2n+2]}\\
&= \mathrm{Re}\sum_{n=0}^{\infty}\frac{(-1)^n(-[5-3\alpha]/6)_n}{n!}(iQ_m)^n\ \frac{\Gamma[(n+2)]\Gamma(n+1)}{(n+1)\Gamma[(2n+2]}\\
&= \mathrm{Re}\sum_{n=0}^{\infty}\left(\frac{1}{(n+1)}\right)\frac{(-1)^n(-[5-3\alpha]/6)_n}{n!}(iQ_m)^n\ \frac{\Gamma[(n+2)]\Gamma(n+1)}{\Gamma[(2n+2]}\\
&= \mathrm{Re}\sum_{n=0}^{\infty}\left[\frac{\Gamma(n+1)}{\Gamma(n+1)}\left(\frac{1}{(n+1)}\right)\right]\frac{(-1)^n(-[5-3\alpha]/6)_n}{n!}(iQ_m)^n\ \frac{\Gamma[(n+2)]\Gamma(n+1)}{\Gamma[(2n+2]}\\
&= \mathrm{Re}\sum_{n=0}^{\infty}\left[\frac{n!}{\Gamma(n+2)}\right]\frac{(-1)^n(-[5-3\alpha]/6)_n}{n!}(iQ_m)^n\ \frac{\Gamma[(n+2)]\Gamma(n+1)}{\Gamma[(2n+2]}\\
&= \mathrm{Re}\sum_{n=0}^{\infty}\left[\frac{(1)_n}{\Gamma(n+2)}\right]\frac{(-1)^n(-[5-3\alpha]/6)_n}{n!}(iQ_m)^n\ \frac{\Gamma[(n+2)]\Gamma(n+1)}{\Gamma[(2n+2]}\\
&= \mathrm{Re}\sum_{n=0}^{\infty}\left[\frac{(1)_n 1!}{\Gamma(n+2)}\right]\frac{(-1)^n(-[5-3\alpha]/6)_n}{n!}(iQ_m)^n\ \frac{\Gamma[(n+2)]\Gamma(n+1)}{\Gamma[(2n+2]}\\
&= \mathrm{Re}\sum_{n=0}^{\infty}\left[\frac{(1)_n\Gamma(2)}{\Gamma(n+2)}\right]\frac{(-1)^n(-[5-3\alpha]/6)_n}{n!}(iQ_m)^n\ \frac{\Gamma[(n+2)]\Gamma(n+1)}{\Gamma[(2n+2]}\\
&= \mathrm{Re}\sum_{n=0}^{\infty}\frac{(1)_n}{(2)_n}\ \frac{(-1)^n(-[5-3\alpha]/6)_n}{n!}(iQ_m)^n\ \frac{\Gamma[(n+2)]\Gamma(n+1)}{\Gamma[(2n+2]}\\
&= \mathrm{Re}\sum_{n=0}^{\infty}\frac{(-[5-3\alpha]/6)_n(1)_n}{(2)_n n!}(-iQ_m)^n\ \frac{\Gamma[(n+2)]\Gamma(n+1)}{\Gamma[(2n+2]}
\end{aligned} \tag{F13}$$

Equation F8 becomes

$$\sigma_{I,sp}^2=1.059\mathrm{B}\sigma_R^2 Q_l^{-5/6}\Gamma([3\alpha-5]/6)\left\{1-\mathrm{Re}\sum_{n=0}^{\infty}\frac{(-[5-3\alpha]/6)_n(1)_n}{(2)_n n!}(-iQ_m)^n\ \frac{\Gamma[(n+2)]\Gamma(n+1)}{\Gamma[(2n+2]}\right\} \tag{F14}$$

Evaluate $\sum_{n=0}^{\infty} \frac{(-[5-3\alpha]/6)_n (1)_n}{(2)_n n!} (-iQ_m)^n \frac{\Gamma(n+2)\Gamma(n+1)}{\Gamma(2n+2)}$:

Note that

$$ {}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \qquad \text{(F15)} $$

Compare to $\frac{\Gamma(n+2)\Gamma(n+1)}{\Gamma(2n+2)}$

c=n+2
c-a-b=n+1
c-a=2n+2
c-b=1 or 2 (reason: $\Gamma(2)=\Gamma(1)=1$)

As a result:

c: c = n+2 (from above)
a: $c-a=2n+2 \rightarrow n+2-a=2n+2 \rightarrow a=-n$
b: c – a - b= n +1 $\rightarrow n+2-(-n)-b=n+1 \rightarrow b=n+1$

Incorporating this into equation F14,

$$ \begin{aligned} \sigma_{I,sp}^2 &= 1.059\mathrm{B}\sigma_R^2 Q_l^{-5/6}\Gamma([3\alpha-5]/6)\{1-\mathrm{Re}\sum_{n=0}^{\infty}\frac{(-[5-3\alpha]/6)_n(1)_n}{(2)_n n!}(-iQ_m)^n\,{}_2F_1(-n,n+1,n+2;1)\} \\ &= (-)1.059\mathrm{B}\sigma_R^2 Q_l^{-5/6}\Gamma([3\alpha-5]/6)\{\mathrm{Re}\sum_{n=0}^{\infty}\frac{(-[5-3\alpha]/6)_n(1)_n}{(2)_n n!}(-iQ_m)^n\,{}_2F_1(-n,n+1,n+2;1)-1\} \end{aligned} \qquad \text{(F16)} $$

Evaluate: ${}_2F_1(-n,n+1,n+2;1)$

$$ {}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty}\frac{(a)_k(b)_k z^k}{(c)_k k!} \cong \sum_{k=0}^{1}\frac{(a)_k(b)_k z^k}{(c)_k k!} \qquad \text{(F17)} $$

Consider $F(-n,n+1;n+2;x)$

For n = 0, $F(0,1;2;x) = \sum_{n=0}^{0}\frac{(0)_n(1)_n x^n}{(2)_n n!} = \frac{(0)_0(1)_0 x^0}{(2)_0 0!} = \frac{(1)(1)(1)}{(1)(1)} = 1$

Thus, for n=0, $F(-n,n+1;n+2;x) = F(0,1;2;x) = 1$, same as $(1-2x/3)^0$

For n = 1, $F(-1,2;3;x)=\sum_{n=1}^{1}\frac{(-1)_n(2)_n x^n}{(3)_n n!} = \frac{(-1)_1(2)_1 x^1}{(3)_1 1!} = \frac{\left(\frac{(-1)^1 1!}{0!}\right)\left(\frac{\Gamma(2+1)}{\Gamma(2)}\right)x^1}{\frac{\Gamma(3+1)}{\Gamma(3)}1!}$

$$= \frac{(-1)\left(\frac{2\Gamma(2)}{\Gamma(2)}\right)x}{\frac{3\Gamma(3)}{\Gamma(3)}1!} = (-)\frac{2x}{3}$$

Thus, for n= 0:1, $F(-n,n+1;n+2;x)=1-\frac{2x}{3}$, same as $(1-2x/3)^1$

For n = 2, $F(-2,3;4;x)=\sum_{n=2}^{2}\frac{(-2)_n(3)_n x^n}{(4)_n n!} = \frac{(-2)_2(3)_2 x^2}{(4)_2 2!} = \frac{\left(\frac{(-1)^2 2!}{0!}\right)\left(\frac{\Gamma(3+1)}{\Gamma(3)}\right)x^2}{\frac{\Gamma(4+1)}{\Gamma(4)}2!}$

$$= \frac{2\left(\frac{3\Gamma(3)}{\Gamma(3)}\right)x^2}{\frac{4\Gamma(4)}{\Gamma(4)}2!} = \frac{3x^2}{4}$$

Thus, for n= 0:2, $F(-n,n+1;n+2;x)=1-\frac{2x}{3}+\frac{3x^2}{4}$.

Note that $1-\frac{2x}{3}+\frac{3x^2}{4}\cong(1-2x/3)^n$ only when n=0 & n=1. Rationale as follows:

Compare $1-\frac{2x}{3}+\frac{3x^2}{4}$ to $(1-2x/3)^n$

$$n=0 \rightarrow 1=\left(1-\frac{2x}{3}\right)^0=1$$

$$n=1 \rightarrow 1-\frac{2x}{3}=\left(1-\frac{2x}{3}\right)^1=1-\frac{2x}{3}$$

$$n=2 \rightarrow 1-\frac{2x}{3}+\frac{3x^2}{4}\cong\left(1-\frac{2x}{3}\right)^2=1-\frac{4x}{3}+\frac{4x^2}{9}, \qquad \left(\neq(1-2x/3)^n\right)$$

Therefore, when n=0 and when n=1,

$$ {}_2F_1(-n,n+1,n+2;x)\cong(1-2x/3)^n \qquad \text{(F18)}$$

$$ {}_2F_1(-n,n+1,n+2;1) \cong (1-2/3)^n = \frac{1}{3^n} \qquad \text{(F19)} $$

$$ \begin{aligned} \sum_{n=0}^{\infty}\frac{(-[5-3\alpha]/6)_n(1)_n}{(2)_n n!}(-iQ_l)^n\,{}_2F_1(-n,n+1,n+2;1) &= \sum_{n=0}^{\infty}\frac{(-(5-3\alpha)/6)_n(1)_n}{3^n(2)_n n!}\left(-iQ_l\right)^n \\ &= \sum_{n=0}^{\infty}\frac{(-[5-3\alpha]/6)_n(1)_n}{(2)_n n!}\left(\frac{-iQ_l}{3}\right)^n \qquad \text{(F20)} \end{aligned} $$

Recalling from equation F15 that ${}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty}\frac{(a)_k(b)_k z^k}{(c)_k k!}$,

$$ \begin{aligned} \sum_{n=0}^{\infty}\frac{(-[5-3\alpha]/6)_n(1)_n}{(2)_n n!}\left(\frac{-iQ_l}{3}\right)^n &= {}_2F_1(-[5-3\alpha]/6,1;2;\frac{-iQ_l}{3}) \\ &= \frac{\left(1+\frac{iQ_l}{3}\right)^{[11-3\alpha]/6}-1}{\frac{[11-3\alpha]}{6}\left(\frac{iQ_l}{3}\right)} \qquad \text{(F21)} \end{aligned} $$

where

$$ {}_2F_1(1-a,1;2;-x) = \frac{(1+x)^a-1}{ax} $$

$$ 1-a = -[5-3\alpha]/6 $$

$$ \begin{aligned} a &= 1+[5-3\alpha]/6 \\ &= [6+5-3\alpha]/6 \\ &= [11-3\alpha]/6 \end{aligned} $$

Equation F16 can be re-written as

$$ \begin{aligned} \sigma_{I,sp}^2 &= 1.059\mathrm{B}\sigma_R^2 Q_l^{-5/6}\Gamma([3\alpha-5]/6)\{1-\mathrm{Re}\sum_{n=0}^{\infty}\frac{(-[5-3\alpha]/6)_n(1)_n}{(2)_n n!}(-iQ_m)^n\,{}_2F_1(-n,n+1,n+2;1)\} \\ &= (-)1.059B\sigma_R^2 Q_l^{-5/6}\Gamma([3\alpha-5]/6)\,\mathrm{Re}[\sum_{n=0}^{\infty}\frac{(-[5-3\alpha]/6)_n(1)_n}{(2)_n n!}(-iQ_l)^n\,{}_2F_1(-n,n+1,n+2;1)-1] \\ &= (-)1.059B\sigma_R^2 Q_l^{-5/6}\Gamma([3\alpha-5]/6)\,\mathrm{Re}\left\{\frac{\left[\left(1+\frac{iQ_l}{3}\right)^{[11-3\alpha]/6}-1\right]}{\left(\frac{11-3\alpha}{6}\right)\left(\frac{iQ_l}{3}\right)}-1\right\} \end{aligned} $$

$$= (-)1.059B\left(\frac{6}{11-3\alpha}\right)\sigma_R^2 Q_l^{-5/6}\Gamma([3\alpha-5]/6)\,\mathrm{Re}\left\{\frac{\left[\left(1+\frac{iQ_l}{3}\right)^{1+[5-3\alpha]/6}-1\right]}{\frac{iQ_l}{3}}-\frac{11-3\alpha}{6}\right\}$$

$$= (-)1.059B\left(\frac{6}{11-3\alpha}\right)\sigma_R^2 Q_l^{-5/6}\Gamma([3\alpha-5]/6)\,\mathrm{Re}\left[\frac{\left(1+\frac{iQ_l}{3}\right)^{[11-3\alpha]/6}}{\frac{iQ_l}{3}}-\frac{1}{\frac{iQ_l}{3}}-\frac{11-3\alpha}{6}\right] \qquad \text{(F22)}$$

Get rid of the imaginary term:

$$\sigma_{I,sp}^2(L) = (-)1.059B\left(\frac{6}{11-3\alpha}\right)\sigma_R^2 Q_l^{-5/6}\Gamma\left([3\alpha-5]/6\right)\mathrm{Re}\left[\frac{\left(1+\frac{iQ_l}{3}\right)^{[11-3\alpha]/6}}{\frac{iQ_l}{3}}-\frac{11-3\alpha}{6}\right] \qquad \text{(F23)}$$

Evaluate $\dfrac{\left(1+\frac{iQ_l}{3}\right)^{[11-3\alpha]/6}}{iQ_l/3}$ in polar coordinates:

$$x+iy = r\exp[i\tan^{-1}(y/x)] = \sqrt{x^2+y^2}\exp[i\tan^{-1}(y/x)]$$

Therefore

$$\left(1+\frac{iQ_l}{3}\right)^{[11-3\alpha]/6} = \left\{\sqrt{1^2+\left(\frac{Q_l}{3}\right)^2}\exp\left[i\tan^{-1}\left(\frac{Q_l}{3}\right)\right]\right\}^{[11-3\alpha]/6}$$

$$= \left\{\left(1+\frac{Q_l^2}{9}\right)^{[11-3\alpha]/12}\exp\left[i\tan^{-1}\left(\frac{Q_l}{3}\right)\right]\right\}^{[11-3\alpha]/6}$$

$$= \left(1+\frac{Q_l^2}{9}\right)^{[11-3\alpha]/12}\exp\left[i\frac{(11-3\alpha)}{6}\tan^{-1}\left(\frac{Q_l}{3}\right)\right] \qquad \text{(F24)}$$

Also, $x+iy = r\exp[i\tan^{-1}(y/x)] = \sqrt{x^2+y^2}\exp[i\tan^{-1}(y/x)]$.

Therefore,

$$i\frac{Q_l}{3} = 0 - i\frac{Q_l}{3} = \sqrt{0^2+\frac{Q_l^2}{9}}\exp\left[i\tan^{-1}(\infty)\right] = \frac{Q_l}{3}\exp\left(i\pi/2\right) \qquad \text{(F25)}$$

$$\frac{1}{iQ_l/3} = \frac{3}{Q_l}\exp(-i\pi/2) = \frac{\exp(-i\pi/2)}{Q_l/3} \tag{F26}$$

$$\begin{aligned}
\frac{\left(1+\frac{iQ_l}{3}\right)^{[11-3\alpha]/6}}{iQ_l/3} &= \frac{\left(1+\frac{Q_l^2}{9}\right)^{[11-3\alpha]/12} \exp\left[i\left(\frac{11-3\alpha}{6}\right)\tan^{-1}\left(\frac{Q_l}{3}\right) - i\pi/2\right]}{Q_l/3} \\
&= \frac{\left(1+\frac{Q_l^2}{9}\right)^{[11-3\alpha]/12} \cos\left[\left(\frac{11-3\alpha}{6}\right)\tan^{-1}\left(\frac{Q_l}{3}\right) - \pi/2\right]}{Q_l/3} \\
&= \frac{\left(1+\frac{Q_l^2}{9}\right)^{[11-3\alpha]/12} \sin\left[\left(\frac{11-3\alpha}{6}\right)\tan^{-1}\left(\frac{Q_l}{3}\right)\right]}{Q_l/3} \\
&= \frac{\left[\frac{Q_l^2}{9}\left(\frac{9}{Q_l^2}+1\right)\right]^{[11-3\alpha]/12} \sin\left[\left(\frac{11-3\alpha}{6}\right)\tan^{-1}\left(\frac{Q_l}{3}\right)\right]}{Q_l/3} \\
&= \frac{\left(\frac{Q_l^2}{9}\right)^{[11-3\alpha]/12}\left(\frac{9}{Q_l^2}+1\right)^{[11-3\alpha]/12} \sin\left[\left(\frac{11-3\alpha}{6}\right)\tan^{-1}\left(\frac{Q_l}{3}\right)\right]}{Q_l/3}
\end{aligned} \tag{F27}$$

where exp(ix) = cos(x), ignoring the imaginary term.

Thus, the general solution is

$$\sigma_{I,sp}^2(L) = (-)1.059B\left(\frac{6}{11-3\alpha}\right)\sigma_r^2 Q_l^{-5/6}\Gamma([3\alpha-5]/6)\left\{\frac{\left(\frac{Q_l^2}{9}\right)^{[11-3\alpha]/12}\left(\frac{9}{Q_l^2}+1\right)^{[11-3\alpha]/12}\sin\left[\left(\frac{11-3\alpha}{6}\right)\tan^{-1}\left(\frac{Q_l}{3}\right)\right]}{Q_l/3} - \frac{11-3\alpha}{6}\right\} \tag{F28}$$

The final solution is obtained from the above by solving

$B(\kappa/\kappa_l)^{\alpha} = 1 + 1.802\kappa/\kappa_l - 0.254(\kappa/\kappa_l)^{7/6}$ for each interval independently, then adding them together:

<u>Interval 1</u> $B = 1, \alpha = 0$, $\Gamma(-5/6) = (-6.680)$

Interval 1 =

$$(-)1.059B\left(\frac{6}{11-3\alpha}\right)\sigma_R^2 Q_l^{-5/6}\Gamma([3\alpha-5]/6)\left\{\frac{\left(\frac{Q_l^2}{9}\right)^{[11-3\alpha]/12}\left(\frac{9}{Q_l^2}+1\right)^{[11-3\alpha]/12}\sin\left[\left(\frac{11-3\alpha}{6}\right)\tan^{-1}\left(\frac{Q_l}{3}\right)\right]}{Q_l/3}-\frac{11-3\alpha}{6}\right\}$$

$$=(-)1.059\left(\frac{6}{11}\right)\sigma_R^2 Q_l^{-5/6}\Gamma(-5/6)\left\{\frac{\left(\frac{Q_l^2}{9}\right)^{11/12}\left(\frac{9}{Q_l^2}+1\right)^{11/12}\sin\left[\left(\frac{11-3\alpha}{6}\right)\tan^{-1}\left(\frac{Q_l}{3}\right)\right]}{Q_l/3}-\frac{11}{6}\right\}$$

$$=(-)1.059\left(\frac{6}{11}\right)\left(\frac{1}{9}\right)^{11/12}\sigma_R^2 Q_l^{-5/6}\Gamma(-5/6)\left\{\frac{3Q_l^{11/6}\left(\frac{9}{Q_l^2}+1\right)^{11/12}\sin\left[\left(\frac{11-3\alpha}{6}\right)\tan^{-1}\left(\frac{Q_l}{3}\right)\right]}{Q_l}-\frac{11}{6}(9)^{11/12}\right\}$$

$$=(-)1.059\left(\frac{6}{11}\right)\left(\frac{1}{9}\right)^{11/12}(3)\left(\frac{0.4}{0.4}\right)\sigma_R^2 Q_l^{-5/6}\Gamma(-5/6)\left\{Q_l^{5/6}\left(\frac{9}{Q_l^2}+1\right)^{11/12}\sin\left[\left(\frac{11-3\alpha}{6}\right)\tan^{-1}\left(\frac{Q_l}{3}\right)\right]-\frac{11}{6}(9)^{11/12}\left(\frac{1}{3}\right)\right\}$$

$$=(-)1.059\left(\frac{6}{11}\right)\left(\frac{1}{9}\right)^{11/12}(3)\left(\frac{1}{0.4}\right)0.4\sigma_R^2 Q_l^{-5/6}\Gamma(-5/6)\left\{Q_l^{5/6}\left(\frac{9}{Q_l^2}+1\right)^{11/12}\sin\left[\frac{11}{6}\tan^{-1}\left(\frac{Q_l}{3}\right)\right]-\frac{11}{6}(9^{11/12})(\frac{1}{3})(0.4)\right\}$$

$$=(-)1.059\left(\frac{6}{11}\right)\left(\frac{1}{9}\right)^{11/12}(3)\left(\frac{1}{0.4}\right)\sigma_R^2\Gamma(-5/6)\left\{0.4\left(\frac{9}{Q_l^2}+1\right)^{11/12}\sin\left[\frac{11}{6}\tan^{-1}\left(\frac{Q_l}{3}\right)\right]-\frac{11}{6}(9)^{11/12}\left(\frac{1}{3}\right)(0.4)Q_l^{-5/6}\right\}$$

$$=3.86\sigma_R^2\left\{0.4\left(\frac{9}{Q_l^2}+1\right)^{11/12}\sin\left[\frac{11}{6}\tan^{-1}\left(\frac{Q_l}{3}\right)\right]-1.83Q_l^{-5/6}\right\} \qquad \text{(F29)}$$

Interval 2 ($B=1.802, \alpha=1$):

Interval 2 =

$$(-)1.059B\left(\frac{6}{11-3\alpha}\right)\sigma_R^2 Q_l^{-5/6}\Gamma([3\alpha-5]/6)\left\{\frac{\left(\frac{Q_l^2}{9}\right)^{[11-3\alpha]/12}\left(\frac{9}{Q_l^2}+1\right)^{[11-3\alpha]/12}\sin\left[\left(\frac{11-3\alpha}{6}\right)\tan^{-1}\left(\frac{Q_l}{3}\right)\right]}{Q_l/3}-\frac{11-3\alpha}{6}\right\}$$

$$=(-)1.059(1.802)\left(\frac{6}{11-3}\right)\sigma_R^2 Q_l^{-5/6}\Gamma(-1/3)\left\{\frac{\left(\frac{Q_l^2}{9}\right)^{[11-3]/12}\left(\frac{9}{Q_l^2}+1\right)^{[11-3]/12}\sin\left[\left(\frac{8}{6}\right)\tan^{-1}\left(\frac{Q_l}{3}\right)\right]}{Q_l/3}-\frac{11-3}{6}\right\}$$

$$=(-)1.059(1.802)\left(\frac{6}{8}\right)\sigma_R^2 Q_l^{-5/6}\Gamma(-1/3)\left\{\frac{\left(\frac{Q_l^2}{9}\right)^{8/12}\left(\frac{9}{Q_l^2}+1\right)^{8/12}\sin\left[\frac{4}{3}\tan^{-1}\left(\frac{Q_l}{3}\right)\right]}{Q_l/3}-\frac{4}{3}\right\}$$

$$=(-)1.431\sigma_R^2 Q_l^{-5/6}\Gamma(-1/3)\left\{\frac{\left(\frac{Q_l^2}{9}\right)^{2/3}\left(\frac{9}{Q_l^2}+1\right)^{2/3}\sin\left[\frac{4}{3}\tan^{-1}\left(\frac{Q_l}{3}\right)\right]}{Q_l/3}-\frac{4}{3}\right\}$$

$$= (-)1.431\sigma_R^2 Q_l^{-5/6}\Gamma(-1/3)\left\{\frac{3\left(\frac{Q_l^2}{9}\right)^{2/3}\left(\frac{9}{Q_l^2}+1\right)^{2/3}\sin\left[\frac{4}{3}\tan^{-1}\left(\frac{Q_l}{3}\right)\right]}{Q_l}-\frac{4}{3}\right\}$$

$$= (-)1.431\sigma_R^2 Q_l^{-5/6}\Gamma(-1/3)(3)\left(\frac{1}{9}\right)^{2/3}[\frac{Q_l^{4/3}\left(\frac{9}{Q_l^2}+1\right)^{2/3}\sin\left[\frac{4}{3}\tan^{-1}\left(\frac{Q_l}{3}\right)\right]}{Q_l}-\frac{4}{3}(\frac{1}{3})(9)^{2/3}]$$

$$= (-)1.431\sigma_R^2 Q_l^{-5/6}\Gamma(-1/3)(3)\left(\frac{1}{9}\right)^{2/3}\left\{\frac{Q_l^{4/3}\left(\frac{9}{Q_l^2}+1\right)^{2/3}\sin\left[\frac{4}{3}\tan^{-1}\left(\frac{Q_l}{3}\right)\right]}{Q_l}-\frac{4}{3}\left(\frac{1}{3}\right)(9)^{2/3}\right\}$$

$$= (-)0.992\sigma_R^2\Gamma(-1/3)\left\{\frac{Q_l^{-5/6}Q_l^{4/3}\left(\frac{9}{Q_l^2}+1\right)^{2/3}\sin\left[\frac{4}{3}\tan^{-1}\left(\frac{Q_l}{3}\right)\right]}{Q_l}-1.92Q_l^{-5/6}\right\}$$

$$= (-)0.992\sigma_R^2\Gamma(-1/3)\left\{\frac{Q_l^{-5/6}Q_l^{4/3}\left(\frac{9}{Q_l^2}+1\right)^{11/12}\left(\frac{9}{Q_l^2}+1\right)^{-1/4}\sin\left[\frac{4}{3}\tan^{-1}\left(\frac{Q_l}{3}\right)\right]}{Q_l}-1.92Q_l^{-5/6}\right\}$$

$$= (-)0.992\sigma_R^2\Gamma(-1/3)\left\{\frac{Q_l^{-5/6}Q_l^{4/3}\left(\frac{9}{Q_l^2}+1\right)^{11/12}\left[\frac{1}{Q_l^2}\left(9+Q_l^2\right)\right]^{-1/4}\sin\left[\frac{4}{3}\tan^{-1}\left(\frac{Q_l}{3}\right)\right]}{Q_l}-1.92Q_l^{-5/6}\right\}$$

$$= (-)0.992\sigma_R^2\left\{\frac{Q_l^{-5/6}Q_l^{4/3}\left(\frac{9}{Q_l^2}+1\right)^{11/12}\left(\frac{1}{Q_l^2}\right)^{-1/4}\left(9+Q_l^2\right)^{-1/4}\sin\left[\frac{4}{3}\tan^{-1}\left(\frac{Q_l}{3}\right)\right]}{Q_l}-1.92Q_l^{-5/6}\right\}$$

$$= (-)0.992\sigma_R^2\Gamma(-1/3)\left\{\frac{Q_l^{-5/6}Q_l^{4/3}Q_1^{-1}\left(\frac{1}{Q_l^2}\right)^{-1/4}\left(\frac{9}{Q_l^2}+1\right)^{11/12}\sin\left[\frac{4}{3}\tan^{-1}\left(\frac{Q_l}{3}\right)\right]}{\left(9+Q_l^2\right)^{-1/4}}-1.92Q_l^{-5/6}\right\}$$

$$= (-)0.992\sigma_R^2\Gamma(-1/3)\left\{\frac{Q_l^{-10/12}Q_l^{16/12}Q_1^{-12/12}Q_l^{6/12}\left(\frac{9}{Q_l^2}+1\right)^{11/12}\sin\left[\frac{4}{3}\tan^{-1}\left(\frac{Q_l}{3}\right)\right]}{\left(9+Q_l^2\right)^{1/4}}-1.92Q_l^{-5/6}\right\}$$

$$= (-)0.992\sigma_R^2\Gamma(-1/3)[\left\{\frac{\left(\frac{9}{Q_l^2}+1\right)^{11/12}\sin\left[\frac{4}{3}\tan^{-1}\left(\frac{Q_l}{3}\right)\right]}{\left(9+Q_l^2\right)^{1/4}} - 1.92Q_l^{-5/6}\right\}]$$

$$= (-)0.992(\frac{1}{0.4})(\frac{1}{2.610})\sigma_R^2(-4.062)\left\{\frac{(0.4)\left(\frac{9}{Q_l^2}+1\right)^{11/12}(2.610)\sin\left[\frac{4}{3}\tan^{-1}\left(\frac{Q_l}{3}\right)\right]}{\left(9+Q_l^2\right)^{1/4}} - 1.92(0.4)(2.610)Q_l^{-5/6}\right\}$$

$$= 3.86\sigma_R^2\left\{\frac{(0.4)\left(\frac{9}{Q_l^2}+1\right)^{11/12}(2.610)\sin\left[\frac{4}{3}\tan^{-1}\left(\frac{Q_l}{3}\right)\right]}{\left(9+Q_l^2\right)^{1/4}} - 2.00Q_l^{-5/6}\right\} \quad \text{(F30)}$$

<u>Interval 3</u> ($B = -0.254, \alpha = 7/6$):

Interval 3

$$= (-)1.059B\left(\frac{6}{11-3\alpha}\right)\sigma_R^2 Q_l^{-5/6}\Gamma([3\alpha-5]/6)\left\{\frac{\left(\frac{Q_l^2}{9}\right)^{[11-3\alpha]/12}\left(\frac{9}{Q_l^2}+1\right)^{[11-3\alpha]/12}\sin\left[\frac{(11-3\alpha)}{6}\tan^{-1}\left(\frac{Q_l}{3}\right)\right]}{Q_l/3} - \frac{11-3\alpha}{6}\right\}$$

$$= (-)1.059(-0.254)(\frac{4}{5})\sigma_R^2 Q_l^{-5/6}\Gamma\left(-\frac{1}{4}\right)\left\{\frac{\left(\frac{Q_l^2}{9}\right)^{5/8}\left(\frac{9}{Q_l^2}+1\right)^{5/8}\sin\left[\frac{5}{4}\tan^{-1}\left(\frac{Q_l}{3}\right)\right]}{Q_l/3} - \frac{5}{4}\right\}$$

$$= (0.215)\sigma_R^2 Q_l^{-5/6}\Gamma\left(-\frac{1}{4}\right)\left\{\frac{\left(\frac{Q_l^2}{9}\right)^{5/8}\left(\frac{9}{Q_l^2}+1\right)^{5/8}\sin\left[\frac{5}{4}\tan^{-1}\left(\frac{Q_l}{3}\right)\right]}{Q_l/3} - \frac{5}{4}\right\}$$

$$= (0.215)(3)\left(\frac{1}{9}\right)^{5/8}\sigma_R^2 Q_l^{-5/6}\Gamma\left(-\frac{1}{4}\right)\left\{\frac{Q_l^{5/4}\left(\frac{9}{Q_l^2}+1\right)^{5/8}\sin\left[\frac{5}{4}\tan^{-1}\left(\frac{Q_l}{3}\right)\right]}{Q_l} - \frac{5}{4}\left(\frac{1}{3}\right)(9)^{5/8}\right\}$$

$$= (0.164)\sigma_R^2 Q_l^{-5/6}\Gamma\left(-\frac{1}{4}\right)\left\{\frac{Q_l^{5/4}\left(\frac{9}{Q_l^2}+1\right)^{5/8}\sin\left[\frac{5}{4}\tan^{-1}\left(\frac{Q_l}{3}\right)\right]}{Q_l} - 1.645\right\}$$

$$= (0.133)\sigma_R^2\Gamma\left(-\frac{1}{4}\right)\left\{Q_l^{-5/6}Q_l^{5/4}Q_1^{-1}\left(\frac{9}{Q_l^2}+1\right)^{5/8}\sin\left[\frac{5}{4}\tan^{-1}\left(\frac{Q_l}{3}\right)\right] - 1.645Q_l^{-5/6}\right\}$$

$$= (0.164)\sigma_R^2\Gamma\left(-\frac{1}{4}\right)\left\{Q_l^{-5/6}Q_l^{5/4}Q_1^{-1}\left(\frac{9}{Q_l^2}+1\right)^{11/12}\left(\frac{9}{Q_l^2}+1\right)^{-7/24}\sin\left[\frac{5}{4}\tan^{-1}\left(\frac{Q_l}{3}\right)\right]-1.645Q_l^{-5/6}\right\}$$

$$= (0.164)\sigma_1^2\Gamma\left(-\frac{1}{4}\right)\left\{\frac{Q_l^{-5/6}Q_l^{5/4}Q_1^{-1}\left(\frac{9}{Q_l^2}+1\right)^{11/12}\sin\left[\frac{5}{4}\tan^{-1}\left(\frac{Q_l}{3}\right)\right]}{(\frac{9}{Q_l^2}+1)^{-7/24}}-1.645Q_l^{-5/6}\right\}$$

$$= (0.164)\sigma_R^2\Gamma\left(-\frac{1}{4}\right)\left\{\frac{Q_l^{-5/6}Q_l^{5/4}Q_1^{-1}\left(\frac{9}{Q_l^2}+1\right)^{11/12}\sin\left[\frac{5}{4}\tan^{-1}\left(\frac{Q_l}{3}\right)\right]}{\left[\frac{1}{Q_l^2}\left(9+Q_l^2\right)\right]^{-7/24}}-1.645Q_l^{-5/6}\right\}$$

$$= (0.164)\sigma_R^2\Gamma\left(-\frac{1}{4}\right)\left\{\frac{Q_l^{-5/6}Q_l^{5/4}Q_1^{-1}Q_l^{-7/12}\left(\frac{9}{Q_l^2}+1\right)^{11/12}\sin\left[\frac{5}{4}\tan^{-1}\left(\frac{Q_l}{3}\right)\right]}{\left(9+Q_l^2\right)^{7/24}}-1.645Q_l^{-5/6}\right\}$$

$$= (0.164)\sigma_R^2\Gamma\left(-\frac{1}{4}\right)\left\{\frac{Q_l^{-10/12}Q_l^{15/12}Q_1^{-12/12}Q_l^{7/12}\left(\frac{9}{Q_l^2}+1\right)^{11/12}\sin\left[\frac{5}{4}\tan^{-1}\left(\frac{Q_l}{3}\right)\right]}{\left(9+Q_l^2\right)^{7/24}}-1.645Q_l^{-5/6}\right\}$$

$$= (0.164)\left(\frac{1}{0.4}\right)\left(\frac{1}{0.518}\right)\sigma_R^2(-4.902)\left\{\frac{0.4\left(\frac{9}{Q_l^2}+1\right)^{11/12}(0.518)\sin\left[\frac{5}{4}\tan^{-1}\left(\frac{Q_l}{3}\right)\right]}{\left(9+Q_l^2\right)^{7/24}}-1.645(0.4)(0.518)Q_l^{-5/6}\right\}$$

$$= (-)3.86\sigma_R^2\left\{\frac{0.4\left(1+\frac{9}{Q_l^2}\right)^{11/12}(0.518)\sin\left[\frac{5}{4}\tan^{-1}\left(\frac{Q_l}{3}\right)\right]}{\left(9+Q_l^2\right)^{7/24}}-0.341Q_l^{-5/6}\right\} \tag{F31}$$

$\sigma_{I,sp}^2(L)$ = Interval 1 + Interval 2 + Interval 3

$$\sigma_{I,sp}^2(L) = 3.86\sigma_R^2\left\{0.4\left(\frac{9}{Q_l^2}+1\right)^{11/12}\sin\left[\frac{11}{6}\tan^{-1}\left(\frac{Q_l}{3}\right)\right]-1.83Q_l^{-5/6}\right\}$$

$$+3.86\sigma_R^2\left\{\frac{0.4\left(\frac{9}{Q_l^2}+1\right)^{11/12}(2.610)\sin\left[\frac{4}{3}\tan^{-1}\left(\frac{Q_l}{3}\right)\right]}{(9+Q_l^2)^{1/4}}-2.00Q_l^{-5/6}\right\}$$

$$-3.86\sigma_R^2\left\{\frac{0.4\left(1+\frac{9}{Q_l^2}\right)^{11/12}(0.518)\sin\left[\frac{5}{4}\tan^{-1}\left(\frac{Q_l}{3}\right)\right]}{\left(9+Q_l^2\right)^{7/24}}-0.341Q_l^{-5/6}\right\}$$

$$
= 3.86\sigma_1^2 \left\{ \begin{array}{l} \left\{ 0.4\left(\frac{9}{Q_l^2}+1\right)^{11/12} \sin\left[\frac{11}{6}\tan^{-1}(\frac{Q_l}{3})\right] - 1.83Q_l^{-5/6} \right\} + \left\{ \frac{(0.4)\left(\frac{9}{Q_l^2}+1\right)^{11/12}(2.610)\sin\left[\frac{4}{3}\tan^{-1}\left(\frac{Q_l}{3}\right)\right]}{\left(9+Q_l^2\right)^{1/4}} - 2.00Q_l^{-5/6} \right\} \\ -\left\{ \frac{0.4\left(1+\frac{9}{Q_l^2}\right)^{11/12}(0.518)\sin\left[\frac{5}{4}\tan^{-1}(\frac{Q_l}{3})\right]}{\left(9+Q_l^2\right)^{7/24}} - 0.341Q_l^{-5/6} \right\} \end{array} \right\}
$$

$$
= 3.86\sigma_R^2 \left\{ 0.4\left(\frac{9}{Q_l^2}+1\right)^{11/12} \sin\left[\frac{11}{6}\tan^{-1}\left(\frac{Q_l}{3}\right) + \frac{(0.4)\left(\frac{9}{Q_l^2}+1\right)^{11/12}(2.610)\sin[\frac{4}{3}\tan^{-1}(\frac{Q_l}{3})]}{(9+Q_l^2)^{1/4}} - \frac{0.4(1+\frac{9}{Q_l^2})^{11/12}(0.518)\sin[\frac{5}{4}\tan^{-1}(\frac{Q_l}{3})]}{(9+Q_l^2)^{7/24}}\right] - 3.5Q_l^{-5/6}] \right\}
$$

$$
= 3.86\sigma_R^2 \left\{ 0.4\left(1+\frac{9}{Q_l^2}\right)^{11/12} \left[\sin\left[\frac{11}{6}\tan^{-1}\left(\frac{Q_l}{3}\right)\right] + \frac{2.610\sin\left[\frac{4}{3}\tan^{-1}\left(\frac{Q_l}{3}\right)\right]}{\left(9+Q_l^2\right)^{1/4}} - \frac{0.518\sin\left[\frac{5}{4}\tan^{-1}\left(\frac{Q_l}{3}\right)\right]}{\left(9+Q_l^2\right)^{7/24}} \right] - 3.5Q_l^{-5/6}] \right\} \qquad \text{(F32)}
$$

Thus, for a spherical wave scintillation in weak turbulence using modified spectrum

$$
\sigma_{I,sp}^2(L) = 3.86\sigma_R^2 \left\{ 0.4\left(1+\frac{9}{Q_l^2}\right)^{11/12} \left[\sin\left[\frac{11}{6}\tan^{-1}\left(\frac{Q_l}{3}\right)\right] + \frac{2.610\sin\left[\frac{4}{3}\tan^{-1}\left(\frac{Q_l}{3}\right)\right]}{\left(9+Q_l^2\right)^{1/4}} - \frac{0.518\sin\left[\frac{5}{4}\tan^{-1}\left(\frac{Q_l}{3}\right)\right]}{\left(9+Q_l^2\right)^{7/24}} \right] - 3.5Q_l^{-5/6}] \right\}
$$

APPENDIX G: SATURATED TURBULENCE, KOLMOGOROV, PLANE WAVE

Reference equation 15, in the saturation regime, the scintillation index for an unbounded plane wave can be expressed in the form

$$\sigma_I^2(L) = 1 + 32\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa \Phi_n(\kappa) \sin^2\left[\frac{L\kappa^2}{2k} h(\xi,\xi)\right] \exp\left\{-\int_0^1 D_s\left[\frac{L\kappa}{k} h(\tau,\xi)\right] d\tau\right\} d\kappa d\xi \qquad \text{(G1)}$$

For the Kolmogorov spectrum (inner scale not included) $D_s = D_{pl}$. As a result,

$$\sigma_I^2(L) = 1 + 32\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa Q_n(\kappa) \sin^2\left[\frac{L\kappa^2}{2k} h(\xi,\xi)\right] \exp\left\{-\int_0^1 D_{pl}\left[\frac{L\kappa}{k} h(\tau,\xi)\right] d\tau\right\} d\kappa d\xi \qquad \text{(G2)}$$

Assume the following approximation for the plane wave structure function based on the Kolmogorov spectrum[1]

$$D_{pl}(\rho) = 2.914 C_n^2 k^2 L \rho^{5/3}, \quad l_0 << \rho << L_0 \qquad \text{(G3)}$$

where $\rho = \left[\frac{L\kappa}{k} h(\tau,\xi)\right]$

Then,

$$\begin{aligned}
\sigma_I^2(L) &= 1 + 32\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa Q_n(\kappa) \sin^2\left[\frac{L\kappa^2}{2k} h(\xi,\xi)\right] \exp\left\{-\int_0^1 2.914 C_n^2 k^2 L \left[\frac{L\kappa}{k} h(\tau,\xi)\right]^{5/3} d\tau\right\} d\kappa d\xi \\
&= 1 + 32\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa Q_n(\kappa) \sin^2\left[\frac{L\kappa^2}{2k} h(\xi,\xi)\right] \exp\left\{-\int_0^1 (2.914) 0.813 \sigma_R^2 \left(\frac{k}{L}\right)^{5/6} \left[\frac{L\kappa}{k} h(\tau,\xi)\right]^{5/3} d\tau\right\} d\kappa d\xi \\
&= 1 + 32\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa Q_n(\kappa) \sin^2\left[\frac{L\kappa^2}{2k} h(\xi,\xi)\right] \exp\left\{-2.369 \sigma_R^2 \left(\frac{L}{k}\right)^{5/6} \kappa^{5/3} \int_0^1 [h(\tau,\xi)]^{5/3} d\tau\right\} d\kappa d\xi \\
&= 1 + 32\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa Q_n(\kappa) \sin^2\left[\frac{L\kappa^2}{2k} h(\xi,\xi)\right] \exp\left\{-2.369 \sigma_R^2 \left(\frac{L}{k}\right)^{5/6} \kappa^{5/3} \left[\int_0^\xi h(\tau,\xi)^{5/3} d\tau + \int_\xi^1 h(\tau,\xi)^{5/3} d\tau\right]\right\} d\kappa d\xi \\
&= 1 + 32\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa Q_n(\kappa) \sin^2\left[\frac{L\kappa^2}{2k} h(\xi,\xi)\right] \exp\left\{-2.369 \sigma_R^2 \left(\frac{L}{k}\right)^{5/6} \kappa^{5/3} \left[\int_0^\xi [\tau(1-\beta\xi)]^{5/3} d\tau + \int_\xi^1 [\xi(1-\beta\tau)]^{5/3} d\tau\right]\right\} d\kappa d\xi \\
&= 1 + 32\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa Q_n(\kappa) \sin^2\left[\frac{L\kappa^2}{2k} h(\xi,\xi)\right] \exp\left\{-2.369 \sigma_R^2 \left(\frac{L}{k}\right)^{5/6} \kappa^{5/3} \left[\int_0^\xi \tau^{5/3} d\tau + \int_\xi^1 \xi^{5/3} d\tau\right]\right\} d\kappa d\xi \\
&= 1 + 32\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa Q_n(\kappa) \sin^2\left[\frac{L\kappa^2}{2k} h(\xi,\xi)\right] \exp\left\{-2.369 \sigma_R^2 \left(\frac{L}{k}\right)^{5/6} \kappa^{5/3} \xi^{5/3} \left(1 - \frac{5}{8}\xi\right)\right\} d\kappa d\xi
\end{aligned} \qquad \text{(G4)}$$

where

$$C_n^2 k^2 L = C_n^2 k^{7/6} k^{5/6} L^{11/6} L^{-5/6}$$

$$= (C_n^2 k^{7/6} L^{11/6}) \frac{k^{5/6}}{L^{5/6}}$$

$$= 0.813 \sigma_R^2 \left(\frac{k}{L} \right)^{5/6}$$

$$h(\tau, \xi) = \tau(1 - \beta\xi), \qquad \tau < \xi$$

$$h(\tau, \xi) = \xi(1 - \beta\tau), \qquad \tau > \xi$$

$$\beta = 0$$

Assume, for x<<1, that

$$\sin(\mathrm{x}) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \cong x$$

$$\sin^2(x) \cong (x)^2$$

$$\sin^2\left[\frac{L\kappa^2}{2k} h(\xi, \xi) \right] \cong \left[\frac{L\kappa^2}{2k} h(\xi, \xi) \right]^2$$

$$= \left[\frac{L\kappa^2}{2k} \xi(1 - \beta\xi) \right]^2$$

$$= \left[\frac{L^2\kappa^4}{4k^2} \xi^2 (1 - \beta\xi)^2 \right]$$

$$= \left[\frac{L^2\kappa^4}{4k^2} \xi^2 \right]$$

Then, equation G4 can be written as

$$\sigma_{I,pl}^2(L) = 1 + 32\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa Q_n(\kappa) \frac{L^2\kappa^4}{4k^2} \xi^2 \exp\left\{ -2.369 \sigma_R^2 \left(\frac{L}{k} \right)^{5/6} \kappa^{5/3} \xi^{5/3} \left(1 - \frac{5}{8}\xi \right) \right\} d\kappa d\xi$$

$$= 1 + 8\pi^2 L^3 \int_0^1 \int_0^\infty \kappa^5 Q_n(\kappa) \xi^2 \exp\left\{ -2.369 \sigma_R^2 \left(\frac{L}{k} \right)^{5/6} \kappa^{5/3} \xi^{5/3} \left(1 - \frac{5}{8}\xi \right) \right\} d\kappa d\xi$$

$$= 1 + (0.033) 8\pi^2 (0.813) \sigma_R^2 \left(\frac{k}{L} \right)^{5/6} \frac{L^2}{k^2} - \int_0^1 \int_0^\infty \kappa^{4/3} \xi^2 \exp\left\{ -2.369 \sigma_R^2 \left(\frac{L}{k} \right)^{5/6} \kappa^{5/3} \xi^{5/3} \left(1 - \frac{5}{8}\xi \right) \right\} d\kappa d\xi$$

$$= 1 + 2.118 \sigma_R^2 \left(\frac{L}{k} \right)^{7/6} \int_0^1 \xi^2 \left[\int_0^\infty \kappa^{4/3} \exp\left\{ -2.369 \sigma_R^2 \left(\frac{L}{k} \right)^{5/6} \xi^{5/3} \left(1 - \frac{5}{8}\xi \right) \kappa^{5/3} \right\} d\kappa \right] d\xi \qquad \text{(G5)}$$

where

$Q_n(\kappa) = 0.033 C_n^2 \kappa^{-11/3}$ (equation 1)

$Cn^2 k^2 L = 0.813 \sigma_R^2 \left(\frac{k}{L}\right)^{5/6}$

Evaluate $\int_0^\infty \kappa^{4/3} \exp\{-s\kappa^{5/3}\} d\kappa$, where s = $2.369\sigma_R^2 (\frac{L}{k})^{5/6} \xi^{5/3} \left(1 - \frac{5}{8}\xi\right)$

Substituting $t = \kappa^{5/3} \quad \rightarrow \kappa = t^{3/5},\ d\kappa = \frac{3}{5} t^{-2/5} dt$

$$\int_0^\infty \kappa^{4/3} \exp\{-s\kappa^{5/3}\} d\kappa = \int_0^\infty \left(t^{3/5}\right)^{4/3} \exp\{-st\} \frac{3}{5} t^{-2/5} dt = \frac{3}{5} \int_0^\infty t^{2/5} \exp\{-st\} dt \quad \text{(G6)}$$

Note that[1]

$$\int_0^\infty t^{x-1} \exp\{-st\} dt = \frac{\Gamma(x)}{s^x} \quad \text{(G7)}$$

Therefore

$$\int_0^\infty \kappa^{4/3} \exp\{-s\kappa^{5/3}\} d\kappa = \frac{3}{5} \frac{\Gamma(7/5)}{\left[2.369\sigma_R^2 \left(\frac{L}{k}\right)^{5/6} \xi^{5/3} \left(1 - \frac{5}{8}\xi\right)\right]^{7/5}}$$
$$= \frac{0.159}{(\sigma_R^2)^{7/5} \left(\frac{L}{k}\right)^{7/6} \xi^{7/3} \left(1 - \frac{5}{8}\xi\right)^{7/5}} \quad \text{(G8)}$$

Then

$$\sigma_{I,pl}^2(L) = 1 + 2.118\sigma_R^2 \left(\frac{L}{k}\right)^{7/6} \int_0^1 \xi^2 \left[\frac{0.159}{(\sigma_R^2)^{7/5} \left(\frac{L}{k}\right)^{7/6} \xi^{7/3} \left(1 - \frac{5}{8}\xi\right)^{7/5}}\right] d\xi$$
$$= 1 + \frac{0.337}{\left(\sigma_R^2\right)^{2/5}} \int_0^1 \xi^{-1/3} \left(1 - \frac{5}{8}\xi\right)^{-7/5} d\xi \quad \text{(G9)}$$

Applying[35]

$$\int_0^u \frac{x^{\mu-1}dx}{(1+\beta x)^{\nu}} = \frac{u^{\mu}}{\mu}[{}_2F_1(\nu,\mu;1+\mu,-\beta u)] \tag{G10}$$

Then

$$\begin{aligned}
\sigma_{I,pl}^2(L) &= 1+\frac{0.337}{\left(\sigma_R^2\right)^{2/5}}\frac{u^{\mu}}{\mu}[{}_2F_1(\nu,\mu;1+\mu,-\beta u)] \\
&= 1+\frac{0.337}{\left(\sigma_R^2\right)^{2/5}}\frac{1}{2/3}[{}_2F_1(\frac{7}{5},\frac{2}{3};\frac{5}{3},\frac{5}{8})] \\
&= 1+\frac{0.506}{\left(\sigma_R^2\right)^{2/5}}[1.704] \\
&= 1+\frac{0.86}{\left(\sigma_R^2\right)^{2/5}}
\end{aligned} \tag{G11}$$

Therefore, for a plane wave in saturated turbulence using the Kolmogorov spectrum,

$$\sigma_{I,pl}^2(L) = 1+\frac{0.86}{\left(\sigma_R^2\right)^{2/5}}, \qquad \sigma_1^2 >> 1$$

APPENDIX H: SATURATED TURBULENCE, VON KARMAN, PLANE WAVE

Reference equation 15, in the saturation regime, the scintillation index for an unbounded plane wave can be expressed in the form

$$\sigma_I^2(L) = 1 + 32\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa \Phi_n(\kappa) \sin^2\left[\frac{L\kappa^2}{2k} h(\xi,\xi)\right] \exp\left\{-\int_0^1 D_s\left[\frac{L\kappa}{k} h(\tau,\xi)\right] d\tau\right\} d\kappa d\xi \qquad \text{(H1)}$$

For the von Karman spectrum (inner scale included) assume $D_s(\rho, L) \approx \frac{1}{2} D(\rho, l)$. As a result,

$$\sigma_I^2(L) \cong 1 + 32\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa Q_n(\kappa) \sin^2\left[\frac{L\kappa^2}{2k} h(\xi,\xi)\right] \exp\left\{-\int_0^1 \frac{1}{2} D_{pl}\left[\frac{L\kappa}{k} h(\tau,\xi)\right] d\tau\right\} d\kappa d\xi \qquad \text{(H2)}$$

Assume the following approximation for the plane wave structure functions based on the von Karman spectrum[1]

$$D_{pl} = 3.280 C_n^2 k^2 L l_0^{-1/3} \rho^2 \qquad \text{(H3)}$$

where $\rho = \left[\frac{L\kappa}{k} h(\tau,\xi)\right]$

Then

$$\sigma_I^2(L) \cong 1 + 32\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa \Phi_n(\kappa) \sin^2\left[\frac{L\kappa^2}{2k} h(\xi,\xi)\right] \exp\left\{-\int_0^1 \left(\frac{1}{2}\right)\left(3.280 C_n^2 k^2 L l_0^{-1/3} \left[\frac{L\kappa}{k} h(\tau,\xi)\right]^2\right) d\tau\right\} d\kappa d\xi$$

$$= 1 + 32\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa \Phi_n(\kappa) \sin^2\left[\frac{L\kappa^2}{2k} h(\xi,\xi)\right] \exp\left\{-\int_0^1 1.640 C_n^2 k^2 L l_0^{-1/3} \left[\frac{L\kappa}{k} h(\tau,\xi)\right]^2 d\tau\right\} d\kappa d\xi$$

$$= 1 + 32\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa \Phi_n(\kappa) \sin^2\left[\frac{L\kappa^2}{2k} h(\xi,\xi)\right] \exp\left\{-\int_0^1 1.640\left[0.813 \sigma_R^2 \left(\frac{k}{L}\right)^{5/6}\right] l_0^{-1/3} \left[\frac{L\kappa}{k} h(\tau,\xi)\right]^2 d\tau\right\} d\kappa d\xi$$

$$= 1 + 32\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa \Phi_n(\kappa) \sin^2\left[\frac{L\kappa^2}{2k} h(\xi,\xi)\right] \exp\left\{-\int_0^1 1.333 \sigma_R^2 \frac{L^{7/6}\kappa^2}{k^{7/6}} l_0^{-1/3} [h(\tau,\xi)]^2 d\tau\right\} d\kappa d\xi$$

$$= 1 + 32\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa \Phi_n(\kappa) \sin^2\left[\frac{L\kappa^2}{2k} h(\xi,\xi)\right] \exp\left\{-\int_0^1 \frac{1.333}{(35.05)^{1/6}} \sigma_R^2 \kappa^2 \left(\frac{L}{k}\right)\left(\frac{35.05 L}{k l_0^2}\right)^{1/6} [h(\tau,\xi)]^2 d\tau\right\} d\kappa d\xi$$

$$= 1 + 32\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa \Phi_n(\kappa) \sin^2\left[\frac{L\kappa^2}{2k} h(\xi,\xi)\right] \exp\left\{-\int_0^1 0.737 \sigma_R^2 \kappa^2 \left(\frac{L}{k}\right) Q_m^{1/6} [h(\tau,\xi)]^2 d\tau\right\} d\kappa d\xi$$

$$= 1 + 32\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa \Phi_n(\kappa) \sin^2\left[\frac{L\kappa^2}{2k} h(\xi,\xi)\right] \exp\left\{-0.737 \sigma_R^2 \kappa^2 \left(\frac{L}{k}\right) Q_m^{1/6} \int_0^1 [h(\tau,\xi)]^2 d\tau\right\} d\kappa d\xi$$

$$= 1 + 32\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa \Phi_n(\kappa) \sin^2\left[\frac{L\kappa^2}{2k} h(\xi,\xi)\right] \exp\left\{-0.737 \sigma_R^2 \kappa^2 \left(\frac{L}{k}\right) Q_m^{1/6} \left[\int_0^\xi h(\tau,\xi)^2 d\tau + \int_\xi^1 h(\tau,\xi)^2 d\tau\right]\right\} d\kappa d\xi$$

$$= 1 + 32\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa \Phi_n(\kappa) \sin^2\left[\frac{L\kappa^2}{2k} h(\xi,\xi)\right] \exp\left\{-0.737 \sigma_R^2 \kappa^2 \left(\frac{L}{k}\right) Q_m^{1/6} \left[\int_0^\xi [\tau(1-\beta\xi)]^2 d\tau + \int_\xi^1 [\xi(1-\beta\tau)]^2 d\tau\right]\right\} d\kappa d\xi$$

$$
\begin{aligned}
&= 1+32\pi^2k^2L\int_0^1\int_0^\infty \kappa\Phi_n(\kappa)\sin^2\left[\frac{L\kappa^2}{2k}h(\xi,\xi)\right]\exp\left\{-0.737\sigma_R^2\kappa^2\left(\frac{L}{k}\right)Q_m^{1/6}\left[\int_0^\xi \tau^2 d\tau+\int_\xi^1 \xi^2 d\tau\right]\right\}d\kappa d\xi \\
&= 1+32\pi^2k^2L\int_0^1\int_0^\infty \kappa\Phi_n(\kappa)\sin^2\left[\frac{L\kappa^2}{2k}h(\xi,\xi)\right]\exp\left\{-0.737\sigma_R^2\kappa^2\left(\frac{L}{k}\right)Q_m^{1/6}\left[\frac{\xi^3}{3}+\xi^2(1-\xi)\right]\right\}d\kappa d\xi \\
&= 1+32\pi^2k^2L\int_0^1\int_0^\infty \kappa\Phi_n(\kappa)\sin^2\left[\frac{L\kappa^2}{2k}h(\xi,\xi)\right]\exp\left\{-0.737\sigma_R^2\kappa^2\left(\frac{L}{k}\right)Q_m^{1/6}\left[\xi^2\left(\frac{\xi}{3}+(1-\xi)\right)\right]\right\}d\kappa d\xi \\
&= 1+32\pi^2k^2L\int_0^1\int_0^\infty \kappa\Phi_n(\kappa)\sin^2\left[\frac{L\kappa^2}{2k}h(\xi,\xi)\right]\exp\left\{-0.737\sigma_R^2\kappa^2\left(\frac{L}{k}\right)Q_m^{1/6}\left[\xi^2\left(\frac{\xi}{3}+(1-\xi)\right)\right]\right\}d\kappa d\xi \\
&= 1+32\pi^2k^2L\int_0^1\int_0^\infty \kappa\Phi_n(\kappa)\sin^2\left[\frac{L\kappa^2}{2k}h(\xi,\xi)\right]\exp\left\{-0.737Q_m^{1/6}\sigma_R^2\left(\frac{L\kappa^2}{k}\right)\left[\xi^2\left(1-\frac{2}{3}\xi\right)\right]\right\}d\kappa d\xi
\end{aligned}
\tag{H4}
$$

where

$$\sigma_R^2 = 1.23C_n^2k^{7/6}L^{11/6}$$

$Q_m = \dfrac{L\kappa_m^2}{k} = \dfrac{L(5.92)^2}{kl_0^2} = \dfrac{35.05L}{kl_0^2}$ is a non-dimensional inner scale parameter

$$
\begin{aligned}
C_n^2k^2L\kappa_m^{-5/3} &= C_n^2k^{7/6}k^{5/6}L^{11/6}L^{-5/6}\kappa_m^{-5/3} \\
&= C_n^2k^{7/6}L^{11/6}\left(\frac{k}{L\kappa_m^2}\right)^{5/6} \\
&= 0.813\sigma_R^2Q_m^{-5/6}
\end{aligned}
$$

$h(\tau,\xi) = \tau(1-\beta\xi)\,, \quad \tau < \xi$

$h(\tau,\xi) = \xi(1-\beta\tau)\,, \quad \tau > \xi$

$\beta = 0$ for a plane wave

Assume, for x<<1, that

$$\sin(\mathrm{x}) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \cong x$$

$$\sin^2(x) \cong (x)^2$$

$$\sin^2\left[\frac{L\kappa^2}{2k}h(\xi,\xi)\right] \cong \left[\frac{L\kappa^2}{2k}h(\xi,\xi)\right]^2$$

$$=\left[\frac{L\kappa^2}{2k}\xi(1-\beta\xi)\right]^2$$

$$=\left[\frac{L^2\kappa^4}{4k^2}\xi^2(1-\beta\xi)^2\right]$$

$$=\left[\frac{L^2\kappa^4}{4k^2}\xi^2\right]$$

Then,

$$\sigma_I^2(L)=1+32\pi^2k^2L\int_0^1\int_0^\infty \kappa\Phi_n(\kappa)\left[\frac{L^2\kappa^4}{4k^2}\xi^2\right]\exp\left[-0.737Q_m^{1/6}\sigma_R^2\left(\frac{L\kappa^2}{k}\right)\xi^2\left(1-\frac{2}{3}\xi\right)\right]d\kappa d\xi$$

$$=1+8\pi^2L^3\int_0^1\xi^2\int_0^\infty\kappa^5Q_n(\kappa)\exp\left[-\left(\frac{\kappa^2}{k}\right)0.737\sigma_R^2Q_m^{1/6}L\xi^2\left(1-\frac{2}{3}\xi\right)\right]d\kappa d\xi$$

$$=1+8\pi^2L^3\int_0^1\xi^2\int_0^\infty\kappa^5Q_n(\kappa)\exp\left[-\left(\frac{Q_m\kappa^2}{L\kappa_m^2}\right)0.737\sigma_R^2Q_m^{1/6}L\xi^2\left(1-\frac{2}{3}\xi\right)\right]d\kappa d\xi$$

$$=1+8\pi^2L^3\int_0^1\xi^2\int_0^\infty\kappa^5Q_n(\kappa)\exp\left[-\left(\frac{\kappa^2}{\kappa_m^2}\right)0.737\sigma_R^2Q_m^{7/6}\xi^2\left(1-\frac{2}{3}\xi\right)\right]d\kappa d\xi$$

$$=1+8\pi^2L^3\int_0^1\xi^2\int_0^\infty\kappa^5\frac{0.033C_n^2\exp\left(-\kappa^2/\kappa_m^2\right)}{\left(\kappa^2+\kappa_0^2\right)^{11/6}}\exp\left[-\left(\frac{\kappa^2}{\kappa_m^2}\right)0.737\sigma_R^2Q_m^{7/6}\xi^2\left(1-\frac{2}{3}\xi\right)\right]d\kappa d\xi$$

$$=1+8\pi^2L^3 0.033C_n^2\int_0^1\xi^2\int_0^\infty\frac{\kappa^5}{\left(\kappa^2+\kappa_0^2\right)^{11/6}}\exp\left\{-\left(\frac{\kappa^2}{\kappa_m^2}\right)\left[1+0.737\sigma_R^2Q_m^{7/6}\xi^2(1-\frac{2}{3}\xi)\right]\right\}d\kappa d\xi$$

$$=1+2.606C_n^2LL^2\frac{k^2}{k^2}\int_0^1\xi^2\int_0^\infty\frac{\kappa^5}{\left(\kappa^2+\kappa_0^2\right)^{11/6}}\exp\left\{-\left(\frac{\kappa^2}{\kappa_m^2}\right)\left[1+0.737\sigma_R^2Q_m^{7/6}\xi^2\left(1-\frac{2}{3}\xi\right)\right]\right\}d\kappa d\xi$$

$$=1+2.606C_n^2k^2L\frac{L^2}{k^2}\int_0^1\xi^2\int_0^\infty\frac{\kappa^5}{\left(\kappa^2+\kappa_0^2\right)^{11/6}}\exp\left\{-\left(\frac{\kappa^2}{\kappa_m^2}\right)\left[1+0.737\sigma_R^2Q_m^{7/6}\xi^2\left(1-\frac{2}{3}\xi\right)\right]\right\}d\kappa d\xi$$

$$1+2.606(0.813)\sigma_R^2\left(\frac{k}{L}\right)^{5/6}\frac{L^2}{k^2}\int_0^1\xi^2\int_0^\infty\kappa\frac{\kappa^5}{\left(\kappa^2+\kappa_0^2\right)^{11/6}}\exp\left\{-\left(\frac{\kappa^2}{\kappa_m^2}\right)\left[1+0.737\sigma_R^2Q_m^{7/6}\xi^2\left(1-\frac{2}{3}\xi\right)\right]\right\}d\kappa d\xi$$

$$=1+2.118\sigma_R^2\left(\frac{L}{k}\right)^{7/6}\int_0^1\xi^2\int_0^\infty\frac{\kappa^5}{\left(\kappa^2+\kappa_0^2\right)^{11/6}}\exp\left\{-\left(\frac{\kappa^2}{\kappa_m^2}\right)\left[1+0.737\sigma_R^2Q_m^{7/6}\xi^2\left(1-\frac{2}{3}\xi\right)\right]\right\}d\kappa d\xi$$

$$=1+2.118\sigma_R^2\left(\frac{L}{k}\right)^{7/6}\int_0^1\xi^2\int_0^\infty\frac{\kappa^5}{\left(\kappa^2+\kappa_0^2\right)^{11/6}}\exp\left(-\alpha\frac{\kappa^2}{\kappa_m^2}\right)d\kappa d\xi$$

$$=1+2.118\sigma_R^2\left(\frac{L}{k}\right)^{7/6}\int_0^1\xi^2\int_0^\infty\frac{\kappa^5}{\left(\kappa^2+\kappa_0^2\right)^{11/6}}\exp\left(-\frac{\kappa^2}{\left(\kappa_m/\sqrt{\alpha}\right)^2}\right)d\kappa d\xi \qquad \text{(H5)}$$

where

$$Q_n = \frac{0.033C_n^2 \exp\left(-\kappa^2/\kappa_m^2\right)}{\left(\kappa^2+\kappa_0^2\right)^{11/6}}, \quad \text{(equation 3)}$$

$$C_n^2 k^2 L = 0.813\sigma_R^2 \left(\frac{k}{L}\right)^{5/6}$$

$$\alpha = 1 + 0.737\sigma_R^2 Q_m^{7/6} \xi^2 \left(1 - \frac{2}{3}\xi\right)$$

Evaluate $\int_0^\infty \kappa^5 \frac{1}{\left(\kappa^2+\kappa_0^2\right)^{11/6}} \exp\left(-\frac{\kappa^2}{\left(\kappa_m/\sqrt{\alpha}\right)^2}\right) d\kappa$

Recognizing[1]

$$\int_0^\infty \frac{\kappa^{2\mu}\exp\left(-\kappa^2/\kappa_m^2\right)}{\left(\kappa^2+\kappa_0^2\right)^{11/6}} = \frac{1}{2}\kappa_0^{2\mu-8/3}\Gamma(\mu+1/2)u(\mu+1/2;\mu-1/3;\kappa_0^2/\kappa_m^2)$$

where

$$2\mu = 5 \rightarrow \mu = 5/2$$

$$2\mu - 8/3 = 5 - 8/3 = 7/3$$
$$\mu + 1/2 = 3$$
$$\mu - 1/3 = 5/2 - 1/3 = 15/6 - 2/6 = 13/6$$

then

$$\int_0^\infty \kappa^5 \frac{1}{\left(\kappa^2+\kappa_0^2\right)^{11/6}} \exp\left\{-\frac{\kappa^2}{(\kappa_m/\sqrt{\alpha})^2}\right\} d\kappa = \frac{1}{2}\kappa_0^{7/3}\Gamma(3)u(3;13/6;\kappa_0^2/(\kappa_m/\sqrt{\alpha})^2)$$

Applying this to equation H5, then

$$\begin{aligned}
\sigma_I^2(L) &= 1 + 2.118\sigma_R^2\left(\frac{L}{k}\right)^{7/6}\int_0^1 \xi^2 \int_0^\infty \frac{\kappa^5}{\left(\kappa^2+\kappa_0^2\right)^{11/6}}\exp\left(-\frac{\kappa^2}{(\kappa_m/\sqrt{\alpha})^2}\right) d\kappa d\xi \\
&= 1 + 2.118\sigma_R^2\left(\frac{L}{k}\right)^{7/6}\int_0^1 \xi^2 \frac{1}{2}\kappa_0^{7/3}\Gamma(3)u(3;13/6;\kappa_0^2/(\kappa_m/\sqrt{\alpha})^2) d\varepsilon \\
&= 1 + 2.118\sigma_R^2\left(\frac{L}{k}\right)^{7/6}\int_0^1 \xi^2 \frac{1}{2}\kappa_0^{7/3}\Gamma(3)u(3;13/6;\kappa_0^2/(\kappa_m/\sqrt{\alpha})^2) d\varepsilon
\end{aligned}$$

$$=1+2.118\sigma_R^2\left(\frac{L}{k}\right)^{7/6}\frac{1}{2}\kappa_0^{7/3}\Gamma(3)\int_0^1 \xi^2 u(3;13/6;\kappa_0^2/(\kappa_m/\sqrt{\alpha})^2)d\varepsilon$$

$$=1+2.118\sigma_R^2\left(\frac{L}{k}\right)^{7/6}\frac{1}{2}\kappa_0^{7/3}\Gamma(3)\int_0^1 \xi^2 u(3;13/6;\kappa_0^2/\left\{\kappa_m/\sqrt{\left[1+0.737\sigma_R^2Q_m^{7/6}\xi^2\left(1-\frac{2}{3}\xi\right)\right]}\right\}^2)d\varepsilon \qquad \text{(H6)}$$

Evaluate $u(3;13/6;\kappa_0^2/\left\{\kappa_m/\sqrt{\left[1+0.737\sigma_R^2Q_m^{7/6}\xi^2\left(1-\frac{2}{3}\xi\right)\right]}\right\}^2)$

$$u(a;c;z)\cong\frac{\Gamma(1-c)}{\Gamma(1+a-c)}+\frac{\Gamma(c-1)z^{1-c}}{\Gamma(a)}$$

Therefore

$$\begin{aligned}u(3;13/6;\kappa_0^2/(\kappa_m/\sqrt{[\alpha]})^2) &\cong \frac{\Gamma(1-13/6)}{\Gamma(1+3-13/6)}+\frac{\Gamma(13/6-1)(\kappa_0^2/\{\kappa_m/\sqrt{[\alpha]}\}^2)^{1-13/6}}{\Gamma(3)}\\ &\cong \frac{\Gamma(-7/6)}{\Gamma(11/6)}+\frac{\Gamma(7/6)(\kappa_0^2/\{\kappa_m/\sqrt{[\alpha]}\}^2)^{-7/6}}{\Gamma(3)}\\ &\cong \frac{\Gamma(-7/6)}{\Gamma(11/6)}+\frac{\Gamma(7/6)(\kappa_0^2/\kappa_m^2)^{-7/6}}{\alpha^{7/6}\Gamma(3)}\\ &\cong 6.171+\frac{\Gamma(7/6)(\kappa_0^2/\kappa_m^2)^{-7/6}}{\alpha^{7/6}\Gamma(3)}\end{aligned} \qquad \text{(H7)}$$

As a result

$$\begin{aligned}\sigma_I^2(L) &\cong 1+2.118\sigma_R^2\left(\frac{L}{k}\right)^{7/6}\frac{1}{2}\kappa_0^{7/3}\Gamma(3)\int_0^1 \xi^2\{6.171+\frac{\Gamma(7/6)(\kappa_0^2/\kappa_m^2)^{-7/6}}{\alpha^{7/6}\Gamma(3)}\}d\varepsilon\\ &=1+2.118\sigma_R^2\left(\frac{L}{k}\right)^{7/6}\left(\frac{1}{2}\right)\int_0^1 \xi^2\left[\kappa_0^{7/3}\Gamma(3)6.171+\frac{\kappa_0^{7/3}\Gamma(3)\Gamma(7/6)\left(\kappa_0^2/(\kappa_m^2\right)^{-7/6}}{\alpha^{7/6}\Gamma(3)}\right]d\varepsilon\\ &=1+2.118\sigma_R^2\left(\frac{L}{k}\right)^{7/6}\left(\frac{1}{2}\right)\int_0^1 \xi^2\left[\kappa_0^{7/3}(2)6.171+\frac{\left(\kappa_m^2\right)^{7/6}\Gamma(7/6)}{\left[1+0.737\sigma_R^2Q_m^{7/6}\xi^2\left(1-\frac{2}{3}\xi\right)\right]^{7/6}}\right]d\varepsilon\\ &=1+2.118\sigma_R^2\left(\frac{L}{k}\right)^{7/6}\left(\frac{1}{2}\right)\int_0^1 \xi^2\left[12.341\kappa_0^{7/3}+\frac{\left(\kappa_m^2\right)^{7/6}\Gamma(7/6)}{\left[1+0.737\sigma_R^2Q_m^{7/6}\xi^2\left(1-\frac{2}{3}\xi\right)\right]^{7/6}}\right]d\varepsilon\end{aligned} \qquad \text{(H8)}$$

Assuming infinite outer scale $\left(\kappa_0 = 1/L_0 \cong 1/\infty \cong 0\right)$, then

$$\sigma_I^2(L) \cong 1 + 2.118\sigma_R^2\left(\frac{L}{k}\right)^{7/6}\left(\frac{1}{2}\right)\int_0^1 \xi^2 \frac{\left(\kappa_m^2\right)^{7/6}\Gamma(7/6)}{\left[1+0.737\sigma_R^2 Q_m^{7/6}\xi^2\left(1-\frac{2}{3}\xi\right)\right]^{7/6}}d\varepsilon$$

$$= 1 + \left(\frac{1}{2}\right)\Gamma(7/6)2.118\sigma_R^2\left(\frac{L\kappa_m^2}{k}\right)^{7/6}\int_0^1 \frac{\xi^2}{\left[1+0.737\sigma_R^2 Q_m^{7/6}\xi^2\left(1-\frac{2}{3}\xi\right)\right]^{7/6}}d\xi$$

$$= 1 + \left(\frac{1}{2}\right)(0.9278)2.118\sigma_R^2 Q_m^{7/6}\int_0^1 \frac{\xi^2}{\left[1+0.737\sigma_1^2 Q_m^{7/6}\xi^2\left(1-\frac{2}{3}\xi\right)\right]^{7/6}}d\xi$$

$$= 1 + 0.982\sigma_1^2 Q_m^{7/6}\int_0^1 \frac{\xi^2}{\left[1+0.737\sigma_1^2 Q_m^{7/6}\xi^2\left(1-\frac{2}{3}\xi\right)\right]^{7/6}}d\xi \qquad \text{(H9)}$$

If $\sigma_1^2 Q_m^{7/6} >> 100$, then

$$1+0.737\sigma_R^2 Q_m^{7/6}\xi^2\left(1-\frac{2}{3}\xi\right) \cong 0.737\sigma_R^2 Q_m^{7/6}\xi^2\left(1-\frac{2}{3}\xi\right)$$

$$\sigma_I^2(L) \cong 1 + 0.982\sigma_R^2 Q_m^{7/6}\int_0^1 \frac{\xi^2}{\left[0.737\sigma_R^2 Q_m^{7/6}\xi^2\left(1-\frac{2}{3}\xi\right)\right]^{7/6}}d\xi$$

$$= 1 + \frac{0.982\sigma_R^2 Q_m^{7/6}}{(0.737)^{7/6}(\sigma_R^2)^{7/6}\left(Q_m^{7/6}\right)^{7/6}}\int_0^1 \frac{\xi^{-1/3}}{\left(1-\frac{2}{3}\xi\right)^{7/6}}d\xi$$

$$= 1 + \frac{1.40}{\left(\sigma_R^2 Q_m^{7/6}\right)^{1/6}}\int_0^1 \frac{\xi^{-1/3}}{\left(1-\frac{2}{3}\xi\right)^{7/6}}d\xi \qquad \text{(H10)}$$

Applying[35]

$$\int_0^u \frac{x^{\mu-1}dx}{(1+\beta x)^{\nu}} = \frac{u^{\mu}}{\mu}[{}_2F_1(\nu,\mu;1+\mu,-\beta u)] \qquad \text{(H11)}$$

where

$$u = 1$$
$$\mu - 1 = -1/3, \rightarrow \mu = 2/3$$
$$\beta = -2/3$$
$$\nu = 7/6$$

Results in

$$\begin{aligned}
\sigma_I^2(L) &\cong 1 + \frac{1.40}{\left(\sigma_R^2 Q_m^{7/6}\right)^{1/6}} \frac{1}{2/3} [{}_2F_1(7/6, 2/3; 5/3, 2/3)] \\
&= 1 + \frac{2.10}{\left(\sigma_R^2 Q_m^{7/6}\right)^{1/6}} [{}_2F_1(7/6, 2/3; 5/3, 2/3)]] \\
&\cong 1 + \frac{2.10}{\left(\sigma_R^2 Q_m^{7/6}\right)^{1/6}} [1.618] \\
&= 1 + \frac{3.40}{(\sigma_R^2 Q_m^{7/6})^{1/6}} \qquad \text{(H12)}
\end{aligned}$$

Therefore, for a plane wave in saturated turbulence using the von Karman spectrum

$$\sigma_I^2(L) \cong 1 + \frac{3.40}{(\sigma_R^2 Q_m^{7/6})^{1/6}}, \qquad \sigma_R^2 Q_m^{7/6} >> 100$$

APPENDIX I: SATURATED TURBULENCE, MODIFIED (BUMP), PLANE WAVE

Reference equation 15, in the saturation regime, the scintillation index for an unbounded plane wave can be expressed in the form

$$\sigma_I^2(L) = 1 + 32\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa\Phi_n(\kappa)\sin^2\left[\frac{L\kappa^2}{2k}h(\xi,\xi)\right]\exp\left\{-\int_0^1 D_s\left[\frac{L\kappa}{k}h(\tau,\xi)\right]d\tau\right\}d\kappa d\xi \tag{I1}$$

For the modified spectrum (inner scale included) assume $D_s(\rho,L) \approx \frac{1}{2}D(\rho,l)$. As a result,

$$\sigma_I^2(L) \cong 1 + 32\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa Q_n(\kappa)\sin^2\left[\frac{L\kappa^2}{2k}h(\xi,\xi)\right]\exp\left\{-\int_0^1 \frac{1}{2}D_{pl}\left[\frac{L\kappa}{k}h(\tau,\xi)\right]d\tau\right\}d\kappa d\xi \tag{I2}$$

Assume the following approximation for the plane wave structure functions based on the modified spectrum[1]

$$\begin{aligned} D(\rho) &= 2.700 C_n^2 k^2 L l_0^{-1/3}\rho^2 \Bigg[\frac{1}{\left(1+0.632\rho^2/l_0^2\right)^{1/6}} + \frac{0.438}{\left(1+0.442\rho^2/l_0^2\right)^{2/3}} \\ &\quad - \frac{0.056}{\left(1+0.376\rho^2/l_0^2\right)^{3/4}} - 0.868\left(\kappa_0 l_0\right)^{1/3}\Bigg] \\ &\cong 2.700 C_n^2 k^2 L l_0^{-1/3}\rho^2\left[\frac{1}{(1+0)^{1/6}} + \frac{0.438}{(1+0)^{2/3}} - \frac{0.056}{(1+0)^{3/4}} - 0.868(0)^{1/3}\right] \\ &= 3.74 C_n^2 k^2 L l_0^{-1/3}\rho^2 \end{aligned} \tag{I3}$$

where $\rho = \left[\frac{L\kappa}{k}h(\tau,\xi)\right]$

Then

$$\begin{aligned} \sigma_I^2(L) &\cong 1 + 32\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa Q_n(\kappa)\sin^2\left[\frac{L\kappa^2}{2k}h(\xi,\xi)\right]\exp\left\{-\int_0^1 \left(\frac{1}{2}\right)\left(3.74 C_n^2 k^2 L l_0^{-1/3}\left[\frac{L\kappa}{k}h(\tau,\xi)\right]^2\right)d\tau\right\}d\kappa d\xi \\ &= 1 + 32\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa\Phi_n(\kappa)\sin^2\left[\frac{L\kappa^2}{2k}h(\xi,\xi)\right]\exp\left\{-\int_0^1 \left(\frac{1}{2}\right)3.74\left[0.813\sigma_R^2\left(\frac{k}{L}\right)^{5/6}\right]l_0^{-1/3}\left[\frac{L\kappa}{k}h(\tau,\xi)\right]^2 d\tau\right\}d\kappa d\xi \\ &= 1 + 32\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa\Phi_n(\kappa)\sin^2\left[\frac{L\kappa^2}{2k}h(\xi,\xi)\right]\exp\left\{-\int_0^1 1.021\sigma_R^2\left(\frac{L}{k}\right)\left(\frac{10.89L}{kl_0^2}\right)^{1/6}\left[\kappa h(\tau,\xi)\right]^2 d\tau\right\}d\kappa d\xi \\ &= 1 + 32\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa\Phi_n(\kappa)\sin^2\left[\frac{L\kappa^2}{2k}h(\xi,\xi)\right]\exp\left\{-\int_0^1 1.021\sigma_R^2 Q_l^{1/6}\left(\frac{L\kappa^2}{k}\right)\left[h(\tau,\xi)\right]^2 d\tau\right\}d\kappa d\xi \\ &= 1 + 32\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa\Phi_n(\kappa)\sin^2\left[\frac{L\kappa^2}{2k}h(\xi,\xi)\right]\exp\left\{-1.021\sigma_R^2 Q_l^{1/6}\left(\frac{L\kappa^2}{k}\right)\left[\int_0^\xi h(\tau,\xi)^2 d\tau + \int_\xi^1 h(\tau,\xi)^2 d\tau\right]\right\}d\kappa d\xi \end{aligned}$$

$$= 1 + 32\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa \Phi_n(\kappa) \sin^2\left[\frac{L\kappa^2}{2k} h(\xi,\xi)\right] \exp\left\{-1.021\sigma_R^2 Q_l^{1/6}\left(\frac{L\kappa^2}{k}\right)\int_0^\xi [\tau(1-\beta\xi)]^2 d\tau + \int_\xi^1 [\xi(1-\beta\tau)]^2 d\tau\right\} d\kappa d\xi$$

$$= 1 + 32\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa \Phi_n(\kappa) \sin^2\left[\frac{L\kappa^2}{2k} h(\xi,\xi)\right] \exp\left\{-1.021\sigma_R^2 Q_l^{1/6}\left(\frac{L\kappa^2}{k}\right)\int_0^\xi \tau^2 d\tau + \int_\xi^1 \xi^2 d\tau\right\} d\kappa d\xi$$

$$= 1 + 32\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa \Phi_n(\kappa) \sin^2\left[\frac{L\kappa^2}{2k} h(\xi,\xi)\right] \exp\left\{-1.021\sigma_R^2 Q_l^{1/6}\left(\frac{L\kappa^2}{k}\right)\xi^2\left(1-\frac{2}{3}\xi\right)\right\} d\kappa d\xi \qquad \text{(I4)}$$

where

$$C_n^2 k^2 L \kappa_l^{-5/3} = C_n^2 k^{7/6} k^{5/6} L^{11/6} L^{-5/6} \kappa_l^{-5/3}$$
$$= C_n^2 k^{7/6} L^{11/6}\left(\frac{k}{L\kappa_l^2}\right)^{5/6}$$
$$= 0.813\sigma_R^2 Q_l^{-5/6}$$

$$Q_l = \frac{L\kappa_l^2}{k} = \frac{L(3.3)^2}{kl_0^2} = \frac{10.89L}{kl_0^2}$$

$$h(\tau,\xi) = \tau(1-\beta\xi), \qquad \tau < \xi$$

$$h(\tau,\xi) = \xi(1-\beta\tau), \qquad \tau > \xi$$

$\beta = 0$ (for a plane wave)

Assume, for x<<1, that

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \cong x$$

$$\sin^2(x) \cong (x)^2$$

$$\sin^2\left[\frac{L\kappa^2}{2k} h(\xi,\xi)\right] \cong \left[\frac{L\kappa^2}{2k} h(\xi,\xi)\right]^2$$
$$= \left[\frac{L\kappa^2}{2k}\xi(1-\beta\xi)\right]^2$$
$$= \left[\frac{L^2\kappa^4}{4k^2}\xi^2(1-\beta\xi)^2\right]$$
$$= \left[\frac{L^2\kappa^4}{4k^2}\xi^2\right]$$

Then,

$$\sigma_I^2(L) = 1 + 32\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa \Phi_n(\kappa) \left[\frac{L^2\kappa^4}{4k^2} \xi^2 \right] \exp\left\{ -1.021\sigma_R^2 Q_l^{1/6} \left(\frac{L\kappa^2}{k} \right) \xi^2 \left(1 - \frac{2}{3}\xi \right) \right\} d\kappa d\xi$$

$$= 1 + 8\pi^2 L^3 \int_0^1 \xi^2 \int_0^\infty \kappa^5 \Phi_n(\kappa) \exp\left\{ -1.021\sigma_R^2 Q_l^{1/6} \left(\frac{L\kappa^2}{k} \right) \xi^2 \left(1 - \frac{2}{3}\xi \right) \right\} d\kappa d\xi \qquad \text{(I5)}$$

Reference equation 4, for the modified spectrum

$$\Phi_n(\kappa) = \frac{0.033Cn^2 \left[1 + 1.802\kappa/\kappa_l - 0.254(\kappa/\kappa_l)^{7/6} \right] \exp\left(-\kappa^2/\kappa_l^2 \right)}{\left(\kappa^2 + \kappa_0^2 \right)^{11/6}} \qquad \text{(I6)}$$

Assuming infinite outer scale $\kappa_0 = 1/L_0 \cong 1/\infty \cong 0$,

$$\Phi_n(\kappa) \cong 0.033 C_n^2 \kappa^{-11/3} \left[1 + 1.802\kappa/\kappa_l - 0.254(\kappa/\kappa_l)^{7/6} \right] \exp\left(-\kappa^2/\kappa_l^2 \right) \qquad \text{(I7)}$$

Equation I5 can be written as

$$\sigma_I^2(L) \cong 1 + 8\pi^2 L^3 \int_0^1 \xi^2 \int_0^\infty \kappa^5 \left\{ 0.033 C_n^2 \kappa^{-11/3} \left[1 + 1.802\kappa/\kappa_l - 0.254(\kappa/\kappa_l)^{7/6} \right] \exp\left(-\kappa^2/\kappa_l^2 \right) \right\}$$
$$\exp\left\{ -(\frac{\kappa^2}{k}) 1.021\sigma_R^2 Q_l^{1/6} L\xi^2 \left(1 - \frac{2}{3}\xi \right) \right\} d\kappa d\xi \qquad \text{(I8)}$$

This equation can be split into 3 intervals. Let $\mathrm{B}(\kappa/\kappa_l)^\alpha = 1 + 1.802\kappa/\kappa_l - 0.254(\kappa/\kappa_l)^{7/6}$.

Determine a general solution for $\mathrm{B}(\kappa/\kappa_l)^\alpha$, then solve for each interval separately.

$$\sigma_I^2(L) \cong 1 + 8\pi^2 L^3 \int_0^1 \xi^2 \int_0^\infty \kappa^5 (0.033) C_n^2 \kappa^{-11/3} \left[\mathrm{B}(\kappa/\kappa_l)^\alpha \right] \exp\left(\frac{-\kappa^2}{\kappa_l^2} \right) \exp\left\{ -\left(\frac{\kappa^2}{k} \right) 1.021\sigma_R^2 Q_l^{1/6} L\xi^2 \left(1 - \frac{2}{3}\xi \right) \right\} d\kappa d\xi$$

$$= 1 + 8\pi^2 L^3 (0.033) C_n^2 \frac{\mathrm{B}}{(\kappa_l)^\alpha} \int_0^1 \xi^2 \int_0^\infty \kappa^{4/3+\alpha} \exp\left(\frac{-\kappa^2}{\kappa_l^2} \right) \exp\left\{ -\left(\frac{\kappa^2}{k} \right) 1.021\sigma_R^2 Q_l^{1/6} L\xi^2 \left(1 - \frac{2}{3}\xi \right) \right\} d\kappa d\xi$$

$$= 1 + 8\pi^2 L^3 (0.033) C_n^2 \frac{\mathrm{B}}{(\kappa_l)^\alpha} \int_0^1 \xi^2 \int_0^\infty \kappa^{4/3+\alpha} \exp\left(\frac{-\kappa^2}{\kappa_l^2} \right) \exp\left\{ -\left(\frac{\kappa^2 Q_l}{L\kappa_l^2} \right) 1.021\sigma_R^2 Q_l^{1/6} L\xi^2 \left(1 - \frac{2}{3}\xi \right) \right\} d\kappa d\xi$$

$$= 1 + 2.606 C_n^2 k^2 L^2 \frac{L}{k^2} \frac{\mathrm{B}}{(\kappa_l)^\alpha} \int_0^1 \xi^2 \int_0^\infty \kappa^{4/3+\alpha} \exp\left\{ -\left(\frac{\kappa^2}{\kappa_l^2} \right) \left[1 + 1.021\sigma_R^2 Q_l^{7/6} \xi^2 \left(1 - \frac{2}{3}\xi \right) \right] \right\} d\kappa d\xi$$

$$=1+2.118\sigma_R^2\left(\frac{L}{k}\right)^{7/6}\frac{\mathrm{B}}{(\kappa_l)^\alpha}\int_0^1\xi^2\left[\int_0^\infty \kappa^{4/3+\alpha}\exp\left\{-\frac{1+1.021\sigma_R^2Q_l^{7/6}\xi^2\left(1-\frac{2}{3}\xi\right)}{\kappa_l^2}\kappa^2\right\}d\kappa\right]d\xi \tag{I9}$$

where

$$Cn^2k^2L=0.813\sigma_R^2\left(\frac{k}{L}\right)^{5/6}$$

Substituting $t=\kappa^2 \quad \rightarrow \kappa=t^{1/2}$, $d\kappa=\frac{1}{2}t^{-1/2}dt$

$$\sigma_I^2(L)\cong 1+2.118\sigma_R^2\left(\frac{L}{k}\right)^{7/6}\frac{\mathrm{B}}{(\kappa_l)^\alpha}\int_0^1\xi^2\left[\int_0^\infty (t^{1/2})^{4/3+\alpha}\exp\left\{-\frac{1+1.021\sigma_R^2Q_l^{7/6}\xi^2\left(1-\frac{2}{3}\xi\right)}{\kappa_l^2}t\right\}\frac{1}{2}t^{-1/2}dt\right]d\xi$$

$$=1+1.059\sigma_R^2\left(\frac{L}{k}\right)^{7/6}\frac{\mathrm{B}}{(\kappa_l)^\alpha}\int_0^1\xi^2\left[\int_0^\infty t^{1/6+\alpha/2}\exp\left\{-\frac{1+1.021\sigma_R^2Q_l^{7/6}\xi^2\left(1-\frac{2}{3}\xi\right)}{\kappa_l^2}t\right\}dt\right]d\xi \tag{I10}$$

Recognizing[1]

$$\int_0^\infty t^{x-1}\exp\{-st\}dt=\frac{\Gamma(x)}{s^x} \tag{I11}$$

Then

$$\sigma_I^2(L)\cong 1+1.059\sigma_R^2\left(\frac{L}{k}\right)^{7/6}\frac{\mathrm{B}}{(\kappa_l)^\alpha}\int_0^1\xi^2\left[\frac{\left(\kappa_l^2\right)^{\frac{7}{6}+\frac{\alpha}{2}}\Gamma\left(\frac{7}{6}+\frac{\alpha}{2}\right)}{\left[1+1.021\sigma_R^2Q_l^{7/6}\xi^2\left(1-\frac{2}{3}\xi\right)\right]^{\left(\frac{7}{6}+\frac{\alpha}{2}\right)}}\right]d\xi$$

$$=1+1.059\sigma_R^2Q_l^{7/6}B\Gamma\left(\frac{7}{6}+\frac{\alpha}{2}\right)\int_0^1\left[\frac{\xi^2}{\left[1+1.021\sigma_1^2Q_l^{7/6}\xi^2\left(1-\frac{2}{3}\xi\right)\right]^{\left(\frac{7}{6}+\frac{\alpha}{2}\right)}}\right]d\xi \tag{I12}$$

If $\sigma_1^2 Q_m^{7/6}$ >>100,

$$1+1.021\sigma_R^2 Q_l^{7/6}\xi^2\left(1-\frac{2}{3}\xi\right)\cong 1.021\sigma_R^2 Q_l^{7/6}\xi^2\left(1-\frac{2}{3}\xi\right) \tag{I13}$$

and

$$\sigma_I^2(L)\cong 1+1.059\sigma_R^2 Q_l^{7/6} B\Gamma\left(\frac{7}{6}+\frac{\alpha}{2}\right)\int_0^1\left[\frac{\xi^2}{\left[1.021\sigma_1^2 Q_l^{7/6}\xi^2\left(1-\frac{2}{3}\xi\right)\right]^{\left(\frac{7}{6}+\frac{\alpha}{2}\right)}}\right]d\xi$$

$$=1+\frac{1.059\sigma_R^2 Q_l^{7/6} B\Gamma\left(\frac{7}{6}+\frac{\alpha}{2}\right)}{\left(1.021\sigma_R^2 Q_l^{7/6}\right)^{\left(\frac{7}{6}+\frac{\alpha}{2}\right)}}\int_0^1\left[\frac{\xi^{-1/3-\alpha}}{\left(1-\frac{2}{3}\xi\right)^{\left(\frac{7}{6}+\frac{\alpha}{2}\right)}}\right]d\xi \tag{I14}$$

Reminder: $\mathrm{B}(\kappa/\kappa_l)^{\alpha} = 1+1.802\kappa/\kappa_l - 0.254(\kappa/\kappa_l)^{7/6}$

Solve for each Interval independently, then add them together:
(i.e., $\sigma_I^2(L)\cong$ 1 + term 1 + term 2 + term 3)

<u>First interval l</u>: B=1, $\alpha = 0$

interval 1 $\cong \dfrac{1.059\sigma_R^2 Q_l^{7/6}\Gamma\left(\frac{7}{6}\right)}{\left(1.021\sigma_R^2 Q_l^{7/6}\right)^{\left(\frac{7}{6}\right)}}\int_0^1\left[\frac{\xi^{-1/3}}{\left(1-\frac{2}{3}\xi\right)^{\left(\frac{7}{6}\right)}}\right]d\xi$

$$=\frac{0.9588}{\left(\sigma_R^2 Q_l^{7/6}\right)^{1/6}}\int_0^1\left[\frac{\xi^{-1/3}}{\left(1-\frac{2}{3}\xi\right)^{\left(\frac{7}{6}\right)}}\right]d\xi$$

$$\cong\frac{0.9588}{\left(\sigma_R^2 Q_l^{7/6}\right)^{1/6}}\frac{1}{2/3}[{}_2F_1(7/6,2/3;5/3,2/3)]$$

$$= \frac{1.438}{\left(\sigma_R^2 Q_l^{7/6}\right)^{1/6}}(1.618)$$

$$= \frac{2.33}{\left(\sigma_R^2 Q_l^{7/6}\right)^{1/6}} \tag{I15}$$

where[35] $\int_0^u \frac{x^{\mu-1}dx}{(1+\beta x)^\nu} = \frac{u^\mu}{\mu}[{}_2F_1(\nu,\mu;1+\mu,-\beta u)]$

<u>Second interval:</u> B= 1.802 , $\alpha = 1$

$$\text{interval 2} \cong \frac{1.059\sigma_R^2 Q_l^{7/6} B\Gamma\left(\frac{7}{6}+\frac{\alpha}{2}\right))}{\left(1.021\sigma_R^2 Q_l^{7/6}\right)]^{\left(\frac{7}{6}+\frac{\alpha}{2}\right)}} \int_0^1 \left[\frac{\xi^{-1/3-\alpha}}{\left(1-\frac{2}{3}\xi\right)^{\left(\frac{7}{6}+\frac{\alpha}{2}\right)}}\right] d\xi$$

$$= \frac{1.059\sigma_R^2 Q_l^{7/6} 1.802\Gamma\left(\frac{7}{6}+\frac{1}{2}\right)}{\left[\left(1.021\sigma_R^2 Q_l^{7/6}\right)^{\frac{7}{6}}\right]^{\frac{13}{7}}} \int_0^1 \left[\frac{\xi^{-1/3}\xi^{-1}}{\left[\left(1-\frac{2}{3}\xi\right)^{\frac{7}{6}}\right]^{\frac{13}{7}}}\right] d\xi$$

$$\cong 0 \quad \text{(Since interval 2 << interval 1)} \tag{I16}$$

<u>Third interval</u>: B=-0.254, α =7/6

$$\text{interval 3} \cong 1 + \frac{1.059\sigma_R^2 Q_l^{7/6} B\Gamma\left(\frac{7}{6}+\frac{\alpha}{2}\right)}{\left(1.021\sigma_R^2 Q_l^{7/6}\right)^{\left(\frac{7}{6}+\frac{\alpha}{2}\right)}} \int_0^1 \left[\frac{\xi^{-1/3-\alpha}}{\left(1-\frac{2}{3}\xi\right)^{\left(\frac{7}{6}+\frac{\alpha}{2}\right)}}\right] d\xi$$

$$= 1 + \frac{1.059\sigma_R^2 Q_l^{7/6}(0.254)\Gamma\left(\frac{7}{6}+\frac{7}{6}\right)}{\left[\left(1.021\sigma_R^2 Q_l^{7/6}\right)^{\frac{7}{6}}\right]^{\frac{14}{7}}} \int_0^1 \left[\frac{\xi^{-1/3}\xi^{-7/6}}{\left[\left(1-\frac{2}{3}\xi\right)^{\frac{7}{6}}\right]^{\frac{14}{7}}}\right] d\xi$$

$$\cong 0 \quad \text{(Since interval 3 << interval 1)} \tag{I17}$$

Recall that $\sigma_I^2(L) \cong 1$ + interval 1 + interval 2 + interval 3 $\cong 1 + \frac{2.33}{\left(\sigma_R^2 Q_l^{7/6}\right)^{1/6}}$. Therefore, for a plane wave in saturated turbulence using the modified (bump) spectrum:

$$\sigma_I^2(L) \cong 1 + \frac{2.33}{\left(\sigma_R^2 Q_l^{7/6}\right)^{1/6}} \qquad \text{(I18)}$$

APPENDIX J: SATURATED TURBULENCE, KOLMOGOROV, SPHERICAL WAVE

Reference equation 15, in the saturation regime, the scintillation index for an unbounded plane wave can be expressed in the form

$$\sigma_I^2(L) = 1 + 32\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa \Phi_n(\kappa) \sin^2\left[\frac{L\kappa^2}{2k} h(\xi,\xi)\right] \exp\left\{-\int_0^1 D_s\left[\frac{L\kappa}{k} h(\tau,\xi)\right] d\tau\right\} d\kappa d\xi \tag{J1}$$

For the Kolmogorov spectrum (inner scale not included) $D_s = D_{pl}$. As a result,

$$\sigma_I^2(L) = 1 + 32\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa Q_n(\kappa) \sin^2\left[\frac{L\kappa^2}{2k} h(\xi,\xi)\right] \exp\left\{-\int_0^1 D_{pl}\left[\frac{L\kappa}{k} h(\tau,\xi)\right] d\tau\right\} d\kappa d\xi \tag{J2}$$

For spherical wave, convert D_{pl} to D_{sp} (the wave structure function for a spherical wave)

Assume the following approximations for the plane wave and spherical wave structure functions based on the Kolmogorov spectrum [page 403 laser book]

$$D_{pl}(\rho) = 2.914 C_n^2 k^2 L \rho^{5/3} \tag{J3}$$

$$D_{sp}(\rho) = 1.093 C_n^2 k^2 L \rho^{5/3} \tag{J4}$$

$$D_{pl}(\rho) = \left(\frac{1.093}{1.093}\right) 2.914 C_n^2 k^2 L \rho^{5/3} = \left(\frac{2.914}{1.093}\right) 1.093 C_n^2 k^2 L \rho^{5/3} = 2.666 D_{sp}(\rho) \tag{J5}$$

where $\rho = \left[\frac{L\kappa}{k} h(\tau,\xi)\right]$

As a result,

$$\sigma_I^2(L) = 1 + 32\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa Q_n(\kappa) \sin^2\left[\frac{L\kappa^2}{2k} h(\xi,\xi)\right] \exp\left\{-\int_0^1 2.666 D_{sp}\left[\frac{L\kappa}{k} h(\tau,\xi)\right]^{5/3} d\tau\right\} d\kappa d\xi$$

$$= 1 + 32\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa Q_n(\kappa) \sin^2\left[\frac{L\kappa^2}{2k} h(\xi,\xi)\right] \exp\left\{-\int_0^1 (2.666)(1.093) C_n^2 k^2 L \left[\frac{L\kappa}{k} h(\tau,\xi)\right]^{5/3} d\tau\right\} d\kappa d\xi$$

$$= 1 + 32\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa Q_n(\kappa) \sin^2\left[\frac{L\kappa^2}{2k} h(\xi,\xi)\right] \exp\left\{-\int_0^1 (2.666)(1.093)(0.813) \sigma_R^2 (\frac{k}{L})^{5/6} \left[\frac{L\kappa}{k} h(\tau,\xi)\right]^{5/3} d\tau\right\} d\kappa d\xi$$

$$= 1 + 32\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa Q_n(\kappa) \sin^2\left[\frac{L\kappa^2}{2k} h(\xi,\xi)\right] \exp\left\{-\int_0^1 2.369 \sigma_R^2 (\frac{k}{L})^{5/6} \left[\frac{L\kappa}{k} h(\tau,\xi)\right]^{5/3} d\tau\right\} d\kappa d\xi$$

$$= 1 + 32\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa Q_n(\kappa) \sin^2\left[\frac{L\kappa^2}{2k} h(\xi,\xi)\right] \exp\left\{-\int_0^1 2.369 \sigma_R^2 \left(\frac{L}{k}\right)^{5/6} [\kappa h(\tau,\xi)]^{5/3} d\tau\right\} d\kappa d\xi$$

$$
\begin{aligned}
&= 1 + 32\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa Q_n(\kappa) \sin^2\left[\frac{L\kappa^2}{2k} h(\xi,\xi)\right] \exp\left\{ -\int_0^1 2.369\sigma_R^2 \left(\frac{L}{k}\right)^{5/6} \kappa^{5/3} \left[h(\tau,\xi)\right]^{5/3} d\tau \right\} d\kappa d\xi \\
&= 1 + 32\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa Q_n(\kappa) \sin^2\left[\frac{L\kappa^2}{2k} h(\xi,\xi)\right] \exp\left\{ -2.369\sigma_R^2 \left(\frac{L}{k}\right)^{5/6} \kappa^{5/3} \int_0^1 h(\tau,\xi)^{5/3} d\tau \right\} d\kappa d\xi \\
&= 1 + 32\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa Q_n(\kappa) \sin^2\left[\frac{L\kappa^2}{2k} h(\xi,\xi)\right] \exp\left\{ -2.369\sigma_R^2 \left(\frac{L}{k}\right)^{5/6} \kappa^{5/3} \left(\int_0^\xi h(\tau,\xi)^{5/3} d\tau + \int_\xi^1 h(\tau,\xi)^{5/3} d\tau \right) \right\} d\kappa d\xi \\
&= 1 + 32\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa Q_n(\kappa) \sin^2\left[\frac{L\kappa^2}{2k} h(\xi,\xi)\right] \exp\left\{ -2.369\sigma_R^2 \left(\frac{L}{k}\right)^{5/6} \kappa^{5/3} \left(\int_0^\xi [\tau(1-\beta\xi)]^{5/3} d\tau + \int_\xi^1 [\xi(1-\beta\tau)]^{5/3} d\tau \right) \right\} d\kappa d\xi \\
&= 1 + 32\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa Q_n(\kappa) \sin^2\left[\frac{L\kappa^2}{2k} h(\xi,\xi)\right] \exp\left\{ -2.369\sigma_R^2 \left(\frac{L}{k}\right)^{5/6} \kappa^{5/3} \left(\int_0^\xi [\tau(1-\xi)]^{5/3} d\tau + \int_\xi^1 [\xi(1-\tau)]^{5/3} d\tau \right) \right\} d\kappa d\xi \\
&= 1 + 32\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa Q_n(\kappa) \sin^2\left[\frac{L\kappa^2}{2k} h(\xi,\xi)\right] \exp\left\{ -2.369\sigma_R^2 \left(\frac{L}{k}\right)^{5/6} \kappa^{5/3} \left((1-\xi)^{5/3} \int_0^\xi \tau^{5/3} d\tau + \xi^{5/3} \int_\xi^1 (1-\tau)^{5/3} d\tau \right) \right\} d\kappa d\xi \\
&= 1 + 32\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa Q_n(\kappa) \sin^2\left[\frac{L\kappa^2}{2k} h(\xi,\xi)\right] \exp\left\{ -2.369\sigma_R^2 \left(\frac{L}{k}\right)^{5/6} \kappa^{5/3} \left((1-\xi)^{5/3} \frac{3}{8} \tau^{8/3} \Big|_0^\xi + \xi^{5/3} \frac{(-)3}{8} (1-\tau)^{8/3} \right) \right\} d\kappa d\xi \\
&= 1 + 32\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa Q_n(\kappa) \sin^2\left[\frac{L\kappa^2}{2k} h(\xi,\xi)\right] \exp\left\{ -2.369\sigma_R^2 \left(\frac{L}{k}\right)^{5/6} \kappa^{5/3} \left((1-\xi)^{5/3} \frac{3}{8} \{\xi^{8/3} - 0^{8/3}\} - \xi^{5/3} \frac{3}{8} \{(1-1)^{8/3} - (1-\xi)^{8/3}\} \right) \right\} d\kappa d\xi \\
&= 1 + 32\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa Q_n(\kappa) \sin^2\left[\frac{L\kappa^2}{2k} h(\xi,\xi)\right] \exp\left\{ -2.369\sigma_R^2 \left(\frac{L}{k}\right)^{5/6} \kappa^{5/3} \left((1-\xi)^{5/3} \frac{3}{8} \xi^{8/3} + \xi^{5/3} \frac{3}{8} \{(1-\xi)^{8/3}\} \right) \right\} d\kappa d\xi \\
&= 1 + 32\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa Q_n(\kappa) \sin^2\left[\frac{L\kappa^2}{2k} h(\xi,\xi)\right] \exp\left\{ -2.369\sigma_R^2 \left(\frac{L}{k}\right)^{5/6} \kappa^{5/3} \frac{3}{8} \xi^{5/3} (1-\xi)^{5/3} \left[\xi + (1-\xi)\right] \right\} d\kappa d\xi \\
&= 1 + 32\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa Q_n(\kappa) \sin^2\left[\frac{L\kappa^2}{2k} h(\xi,\xi)\right] \exp\left\{ -0.8884\sigma_R^2 \left(\frac{L}{k}\right)^{5/6} \kappa^{5/3} \xi^{5/3} (1-\xi)^{5/3} \right\} d\kappa d\xi \qquad \text{(J6)}
\end{aligned}
$$

where

$$
\begin{aligned}
C_n^2 k^2 L &= C_n^2 k^{7/6} k^{5/6} L^{11/6} L^{-5/6} \\
&= (C_n^2 k^{7/6} L^{11/6}) \frac{k^{5/6}}{L^{5/6}} \\
&= 0.813\sigma_R^2 \left(\frac{k}{L}\right)^{5/6}
\end{aligned}
$$

$h(\tau,\xi) = \tau(1-\beta\xi), \quad \tau < \xi$

$h(\tau,\xi) = \xi(1-\beta\tau), \quad \tau > \xi$

$\beta = 1$ spherical wave

Now, evaluate: $\sin^2\left[\frac{L\kappa^2}{2k} h(\xi,\xi)\right]$

Assume, for x<<1, that

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ... \cong x$$

$$\sin^2(x) \cong (x)^2$$

$$\sin^2\left[\frac{L\kappa^2}{2k} h(\xi,\xi)\right] \cong \left[\frac{L\kappa^2}{2k} h(\xi,\xi)\right]^2$$

$$= \left[\frac{L\kappa^2}{2k} \xi(1-\beta\xi)\right]^2$$

$$= \left[\frac{L^2\kappa^4}{4k^2} \xi^2 (1-\beta\xi)^2\right]$$

$$= \left[\frac{L^2\kappa^4}{4k^2} \xi^2 (1-\xi)^2\right]$$

Equation J6 can be re-written as

$$\sigma_I^2(L) = 1 + 32\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa Q_n(\kappa) \left[\frac{L\kappa^2}{2k}\xi(1-\xi)\right]^2 \exp\left\{-0.8884\sigma_R^2\left(\frac{L}{k}\right)^{5/6} \kappa^{5/3}\xi^{5/3}(1-\xi)^{5/3}\right\} d\kappa d\xi$$

$$= 1 + 32\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa Q_n(\kappa) \frac{L^2\kappa^4}{4k^2}\xi^2(1-\xi)^2 \exp\left\{-0.8884\sigma_R^2\left(\frac{L}{k}\right)^{5/6} \kappa^{5/3}\xi^{5/3}(1-\xi)^{5/3}\right\} d\kappa d\xi$$

$$= 1 + 8\pi^2 L^3 \int_0^1 \int_0^\infty \kappa^5 Q_n(\kappa)\xi^2(1-\xi)^2 \exp\left\{-0.8884\sigma_R^2\left(\frac{L}{k}\right)^{5/6} \kappa^{5/3}\xi^{5/3}(1-\xi)^{5/3}\right\} d\kappa d\xi$$

$$= 1 + 8\pi^2 L^3 \int_0^1 \int_0^\infty \kappa^5 \left(0.033 C_n^2 \kappa^{-11/3}\right)\xi^2(1-\xi)^2 \exp\left\{-0.8884\sigma_R^2\left(\frac{L}{k}\right)^{5/6} \kappa^{5/3}\xi^{5/3}(1-\xi)^{5/3}\right\} d\kappa d\xi$$

$$= 1 + (0.033)8\pi^2 C_n^2 L^3 \int_0^1 \int_0^\infty \kappa^{4/3}\xi^2(1-\xi)^2 \exp\left\{-0.8884\sigma_R^2\left(\frac{L}{k}\right)^{5/6} \kappa^{5/3}\xi^{5/3}(1-\xi)^{5/3}\right\} d\kappa d\xi$$

$$= 1 + (0.033)8\pi^2 C_n^2 L^3 \left(\frac{k^2}{k^2}\right) \int_0^1 \int_0^\infty \kappa^{4/3}\xi^2(1-\xi)^2 \exp\left\{-0.8884\sigma_R^2\left(\frac{L}{k}\right)^{5/6} \kappa^{5/3}\xi^{5/3}(1-\xi)^{5/3}\right\} d\kappa d\xi$$

$$= 1 + (0.033)8\pi^2 C_n^2 k^2 L \left(\frac{L^2}{k^2}\right) \int_0^1 \int_0^\infty \kappa^{4/3}\xi^2(1-\xi)^2 \exp\left\{-0.8884\sigma_R^2\left(\frac{L}{k}\right)^{5/6} \kappa^{5/3}\xi^{5/3}(1-\xi)^{5/3}\right\} d\kappa d\xi$$

$$= 1 + (0.033)8\pi^2 \left[0.813\sigma_R^2\left(\frac{k}{L}\right)^{5/6}\right] \frac{L^2}{k^2} \int_0^1 \int_0^\infty \kappa^{4/3}\left[\xi^2(1-\xi)^2\right] \exp\left\{-0.8884\sigma_R^2\left(\frac{L}{k}\right)^{5/6} \kappa^{5/3}\xi^{5/3}(1-\xi)^{5/3}\right\} d\kappa d\xi$$

$$= 1 + 2.118\sigma_R^2\left(\frac{k}{L}\right)^{5/6} \frac{L^2}{k^2} \int_0^1 \int_0^\infty \kappa^{4/3}\left[\xi^2(1-\xi)^2\right] \exp\left\{-0.8884\sigma_R^2\left(\frac{L}{k}\right)^{5/6} \kappa^{5/3}\xi^{5/3}(1-\xi)^{5/3}\right\} d\kappa d\xi$$

$$= 1 + 2.118\sigma_R^2\left(\frac{L}{k}\right)^{7/6} \int_0^1 \int_0^\infty \kappa^{4/3}\left[\xi^2(1-\xi)^2\right] \exp\left\{-0.8884\sigma_R^2\left(\frac{L}{k}\right)^{5/6} \kappa^{5/3}\xi^{5/3}(1-\xi)^{5/3}\right\} d\kappa d\xi \qquad \text{(J7)}$$

where

$\phi_n(\kappa) = 0.033 C_n^2 \kappa^{-11/3}$ (equation 1)

$$Cn^2 k^2 L = 0.813\sigma_R^2 \left(\frac{k}{L}\right)^{5/6}$$

Evaluate $\int_0^\infty \kappa^{4/3} \exp\{-s\kappa^{5/3}\} d\kappa$, where s = $0.8884\sigma_R^2 \left(\frac{L}{k}\right)^{5/6} \xi^{5/3}(1-\xi)^{5/3}$

Substituting $t = \kappa^{5/3} \quad \rightarrow \kappa = t^{3/5}, \quad d\kappa = \frac{3}{5} t^{-2/5} dt$

$$\int_0^\infty \kappa^{4/3} \exp\{-s\kappa^{5/3}\} d\kappa = \int_0^\infty (t^{3/5})^{4/3} \exp\{-st\} \frac{3}{5} t^{-2/5} dt = \frac{3}{5} \int_0^\infty t^{2/5} \exp\{-st\} dt \tag{J8}$$

Note that[1]

$$\int_0^\infty t^{x-1} \exp\{-st\} dt = \frac{\Gamma(x)}{s^x} \tag{J9}$$

where $x - 1 = \frac{2}{5} \quad \rightarrow x = \frac{7}{5}$

Then

$$\int_0^\infty \kappa^{4/3} \exp\{-s\kappa^{5/3}\} d\kappa = \frac{3}{5} \frac{\Gamma(7/5)}{\left[0.8884\sigma_R^2 \left(\frac{L}{k}\right)^{5/6} \xi^{5/3}(1-\xi)^{5/3}\right]^{7/5}}$$

$$= \frac{3}{5} \frac{0.8873}{(0.8884)^{7/5} \left(\sigma_R^2\right)^{7/5} \left[\left(\frac{L}{k}\right)^{5/6}\right]^{7/5} \left[\xi^{5/3}(1-\xi)^{5/3}\right]^{7/5}}$$

$$= \frac{0.6283}{\left(\sigma_R^2\right)^{7/5} \left[\left(\frac{L}{k}\right)^{5/6}\right]^{7/5} \left[\xi^{5/3}(1-\xi)^{5/3}\right]^{7/5}} \tag{J10}$$

As a result,

$$\sigma_I^2(L) = 1 + 2.118\sigma_R^2 \left(\frac{L}{k}\right)^{7/6} \int_0^1 \left[\xi^2(1-\xi)^2\right] \frac{0.6283}{\left(\sigma_R^2\right)^{7/5} \left[\left(\frac{L}{k}\right)^{5/6}\right]^{7/5} \left[\xi^{5/3}(1-\xi)^{5/3}\right]^{7/5}} d\xi$$

$$= 1 + \frac{1.331}{\left(\sigma_R^2\right)^{2/5}} \int_0^1 \frac{\xi^2 (1-\xi)^2}{\left[\xi^{5/3} (1-\xi)^{5/3}\right]^{7/5}} d\xi$$

$$= 1 + \frac{1.331}{\left(\sigma_R^2\right)^{2/5}} \int_0^1 \frac{\xi^2 (1-\xi)^2}{\xi^{7/3} (1-\xi)^{7/3}} d\xi$$

$$= 1 + \frac{1.331}{\left(\sigma_R^2\right)^{2/5}} \int_0^1 \frac{\xi^{-1/3}}{(1-\xi)^{1/3}} d\xi \tag{J11}$$

Recognizing that[35]

$$\int_0^u \frac{x^{\mu-1} dx}{(1+\beta x)^{\nu}} = \frac{u^{\mu}}{\mu} [{}_2F_1(\nu, \mu; 1+\mu, -\beta u)] \tag{J12}$$

where

$$u = 1$$
$$\mu - 1 = -1/3, \rightarrow \mu = 2/3, \rightarrow \mu + 1 = 5/3$$
$$\beta = -1$$
$$\nu = 1/3$$

Then

$$\sigma_I^2(L) = 1 + \frac{1.331}{\left(\sigma_R^2\right)^{2/5}} \frac{3}{2} \left({}_2F_1[1/3, 2/3; 5/3, 1]\right)$$

$$= 1 + \frac{1.331}{(\sigma_R^2)^{2/5}} \frac{3}{2} (1.369)$$

$$= 1 + \frac{2.73}{(\sigma_R^2)^{2/5}} \tag{J13}$$

Therefore, for a spherical wave in saturated turbulence using the Kolmogorov spectrum,

$$\sigma_I^2(L) = 1 + \frac{2.73}{\left(\sigma_R^2\right)^{2/5}}$$

APPENDIX K: SATURATED TURBULENCE, VON KARMAN, SPHERICAL WAVE

Reference equation 15, in the saturation regime, the scintillation index for an unbounded plane wave can be expressed in the form

$$\sigma_I^2(L) = 1 + 32\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa \Phi_n(\kappa) \sin^2\left[\frac{L\kappa^2}{2k} h(\xi,\xi)\right] \exp\left\{-\int_0^1 D_s\left[\frac{L\kappa}{k} h(\tau,\xi)\right] d\tau\right\} d\kappa d\xi \qquad \text{(K1)}$$

For the von Karman spectrum (inner scale included) assume $D_s(\rho, L) \approx \frac{1}{2} D(\rho, l)$. As a result,

$$\sigma_I^2(L) \cong 1 + 32\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa Q_n(\kappa) \sin^2\left[\frac{L\kappa^2}{2k} h(\xi,\xi)\right] \exp\left\{-\int_0^1 \frac{1}{2} D_{pl}\left[\frac{L\kappa}{k} h(\tau,\xi)\right] d\tau\right\} d\kappa d\xi \qquad \text{(K2)}$$

For spherical wave, convert D_{pl} to D_{sp} (the wave structure function for a spherical wave)

Assume the following approximations for the plane wave and spherical wave structure functions based on the von Karman spectrum[1]

$$D_{pl}(\rho) = 3.280 C_n^2 k^2 L l_0^{-1/3} \rho^2 \left[\frac{1}{\left(1 + 2.033\rho^2 / l_0^2\right)^{1/6}} - 0.715\left(\kappa_0 l_0\right)^{1/3}\right] \cong 3.280 C_n^2 k^2 L l_0^{-1/3} \rho^2 \qquad \text{(K3)}$$

$$D_{sp}(\rho) = 1.093 C_n^2 k^2 L l_0^{-1/3} \rho^2 \left[\frac{1}{\left(1 + \rho^2 / l_0^2\right)^{1/6}} - 0.715\left(\kappa_0 l_0\right)^{1/3}\right] \cong 1.093 C_n^2 k^2 L l_0^{-1/3} \rho^2 \qquad \text{(K4)}$$

$$D_{pl}(\rho) \cong \frac{3.280}{1.093} D_{sp}(\rho) = 3.00 D_{sp}(\rho) \qquad \text{(K5)}$$

where $\rho = \left[\frac{L\kappa}{k} h(\tau,\xi)\right]$

As a result,

$$\begin{aligned}
\sigma_I^2(L) &\cong 1 + 32\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa Q_n(\kappa) \sin^2\left[\frac{L\kappa^2}{2k} h(\xi,\xi)\right] \exp\left\{-\int_0^1 \left(\frac{1}{2}\right) 3.00 D_{sp}\left[\frac{L\kappa}{k} h(\tau,\xi)\right] d\tau\right\} d\kappa d\xi \\
&= 1 + 32\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa \Phi_n(\kappa) \sin^2\left[\frac{L\kappa^2}{2k} h(\xi,\xi)\right] \exp\left\{-\int_0^1 \left(\frac{1}{2}\right)(3.00)\left(1.093 C_n^2 k^2 L l_0^{-1/3}\left[\frac{L\kappa}{k} h(\tau,\xi)\right]^2\right) d\tau\right\} d\kappa d\xi \\
&= 1 + 32\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa \Phi_n(\kappa) \sin^2\left[\frac{L\kappa^2}{2k} h(\xi,\xi)\right] \exp\left\{-\int_0^1 \left(\frac{1}{2}\right)(3.00)(1.093)\left[0.813\sigma_R^2\left(\frac{k}{L}\right)^{5/6}\right] l_0^{-1/3}\left[\frac{L\kappa}{k} h(\tau,\xi)\right]^2 d\tau\right\} d\kappa d\xi \\
&= 1 + 32\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa \Phi_n(\kappa) \sin^2\left[\frac{L\kappa^2}{2k} h(\xi,\xi)\right] \exp\left\{-\int_0^1 1.33\sigma_R^2 \frac{L^{7/6}\kappa^2}{l_0^{1/3} k^{7/6}} \left[h(\tau,\xi)\right]^2 d\tau\right\} d\kappa d\xi
\end{aligned}$$

$$=1+32\pi^2k^2L\int_0^1\int_0^\infty \kappa\Phi_n(\kappa)\sin^2\left[\frac{L\kappa^2}{2k}h(\xi,\xi)\right]\exp\left\{-\int_0^1\frac{1.33}{(35.05)^{1/6}}\sigma_R^2\left(\frac{35.05L}{kl_0^2}\right)^{1/6}\left(\frac{L\kappa^2}{k}\right)[h(\tau,\xi)]^2d\tau\right\}d\kappa d\xi$$

$$=1+32\pi^2k^2L\int_0^1\int_0^\infty \kappa\Phi_n(\kappa)\sin^2\left[\frac{L\kappa^2}{2k}h(\xi,\xi)\right]\exp\left\{-0.737\sigma_R^2Q_m^{1/6}\left(\frac{L\kappa^2}{k}\right)\int_0^1[h(\tau,\xi)]^2d\tau\right\}d\kappa d\xi \qquad \text{(K6)}$$

where

$$\sigma_R^2=1.23C_n^2k^{7/6}L^{11/6}$$

$$Q_m=\frac{L\kappa_m^2}{k}$$

$$\begin{aligned}C_n^2k^2L\kappa_m^{-5/3}&=C_n^2k^{7/6}k^{5/6}L^{11/6}L^{-5/6}\kappa_m^{-5/3}\\&=C_n^2k^{7/6}L^{11/6}\left(\frac{k}{L\kappa_m^2}\right)^{5/6}\\&=0.813\sigma_R^2Q_m^{-5/6}\end{aligned}$$

$$h(\tau,\xi)=\tau(1-\beta\xi),\qquad \tau<\xi$$

$$h(\tau,\xi)=\xi(1-\beta\tau),\qquad \tau>\xi$$

For a spherical wave, $\beta=1$. Equation K6 reduces to

$$\sigma_I^2(L)=1+32\pi^2k^2L\int_0^1\int_0^\infty \kappa\Phi_n(\kappa)\sin^2\left[\frac{L\kappa^2}{2k}h(\xi,\xi)\right]\exp\left\{-0.737\sigma_R^2Q_m^{1/6}\left(\frac{L\kappa^2}{k}\right)\left[\int_0^\xi h(\tau,\xi)^2d\tau+\int_\xi^1 h(\tau,\xi)^2d\tau\right]\right\}d\kappa d\xi$$

$$=1+32\pi^2k^2L\int_0^1\int_0^\infty \kappa\Phi_n(\kappa)\sin^2\left[\frac{L\kappa^2}{2k}h(\xi,\xi)\right]\exp\left\{-0.737\sigma_R^2Q_m^{1/6}\left(\frac{L\kappa^2}{k}\right)\left[\int_0^\xi[\tau(1-\beta\xi)]^2d\tau+\int_\xi^1[[\xi(1-\beta\tau)]^2d\tau\right]\right\}d\kappa d\xi$$

$$=1+32\pi^2k^2L\int_0^1\int_0^\infty \kappa\Phi_n(\kappa)\sin^2\left[\frac{L\kappa^2}{2k}h(\xi,\xi)\right]\exp\left\{-0.737\sigma_R^2Q_m^{1/6}\left(\frac{L\kappa^2}{k}\right)\left[\int_0^\xi[\tau(1-\xi)]^2d\tau+\int_\xi^1[[\xi(1-\tau)]^2d\tau\right]\right\}d\kappa d\xi$$

$$=1+32\pi^2k^2L\int_0^1\int_0^\infty \kappa\Phi_n(\kappa)\sin^2\left[\frac{L\kappa^2}{2k}h(\xi,\xi)\right]\exp\left\{-0.737\sigma_R^2Q_m^{1/6}\left(\frac{L\kappa^2}{k}\right)\left[(1-\xi)^2\int_0^\xi\tau^2d\tau+\xi^2\int_\xi^1(1-\tau)^2d\tau\right]\right\}d\kappa d\xi$$

$$=1+32\pi^2k^2L\int_0^1\int_0^\infty \kappa\Phi_n(\kappa)\sin^2\left[\frac{L\kappa^2}{2k}h(\xi,\xi)\right]\exp\left\{-0.737\sigma_R^2Q_m^{1/6}\left(\frac{L\kappa^2}{k}\right)\left[(1-\xi)^2\frac{1}{3}\tau^3\Big|_0^\xi-\frac{1}{3}\xi^2(1-\tau)^3\Big|_\xi^1\right]\right\}d\kappa d\xi$$

$$=1+32\pi^2k^2L\int_0^1\int_0^\infty \kappa\Phi_n(\kappa)\sin^2\left[\frac{L\kappa^2}{2k}h(\xi,\xi)\right]\exp\left\{-0.737\sigma_R^2Q_m^{1/6}\left(\frac{L\kappa^2}{k}\right)\left[(1-\xi)^2\frac{1}{3}\{\xi^3-0^3\}-\frac{1}{3}\xi^2\{(1-1)^3-(1-\xi)^3\}\right]\right\}d\kappa d\xi$$

$$=1+32\pi^2k^2L\int_0^1\int_0^\infty \kappa\Phi_n(\kappa)\sin^2\left[\frac{L\kappa^2}{2k}h(\xi,\xi)\right]\exp\left\{-0.737\sigma_R^2Q_m^{1/6}\left(\frac{L\kappa^2}{k}\right)\left[(1-\xi)^2\frac{1}{3}\xi^3+\frac{1}{3}\xi^2(1-\xi)^3\right]\right\}d\kappa d\xi$$

$$=1+32\pi^2k^2L\int_0^1\int_0^\infty \kappa\Phi_n(\kappa)\sin^2\left[\frac{L\kappa^2}{2k}h(\xi,\xi)\right]\exp\left\{-0.737\sigma_R^2Q_m^{1/6}\left(\frac{L\kappa^2}{k}\right)\left[\frac{1}{3}\xi^2(1-\xi)^2[\xi+(1-\xi)]\right]\right\}d\kappa d\xi$$

$$=1+32\pi^2k^2L\int_0^1\int_0^\infty \kappa\Phi_n(\kappa)\sin^2\left[\frac{L\kappa^2}{2k}h(\xi,\xi)\right]\exp\left\{-0.737\sigma_R^2Q_m^{1/6}\left(\frac{L\kappa^2}{k}\right)\left[\frac{1}{3}\xi^2(1-\xi)^2\right]\right\}d\kappa d\xi$$

$$=1+32\pi^2k^2L\int_0^1\int_0^\infty \kappa\Phi_n(\kappa)\sin^2\left[\frac{L\kappa^2}{2k}h(\xi,\xi)\right]\exp\left\{-0.246\sigma_R^2Q_m^{1/6}\left(\frac{L\kappa^2}{k}\right)\left[\xi^2(1-\xi)^2\right]\right\}d\kappa d\xi \qquad \text{(K7)}$$

Assume, for x<<1, that

$$\sin(\text{x}) = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+... \cong x$$

$$\sin^2(x)\cong(x)^2$$

$$\sin^2\left[\frac{L\kappa^2}{2k}h(\xi,\xi)\right]\cong\left[\frac{L\kappa^2}{2k}h(\xi,\xi)\right]^2$$

$$=\left[\frac{L\kappa^2}{2k}\xi(1-\beta\xi)\right]^2$$

$$=\left[\frac{L^2\kappa^4}{4k^2}\xi^2(1-\beta\xi)^2\right]$$

$$=\left[\frac{L^2\kappa^4}{4k^2}\xi^2(1-\xi)^2\right]$$

Equation K7 can be re-written as

$$\sigma_I^2(L)=1+32\pi^2k^2L\int_0^1\int_0^\infty \kappa\Phi_n(\kappa)\left[\frac{L^2\kappa^4}{4k^2}\xi^2(1-\xi)^2\right]\exp\left\{-0.246\sigma_R^2Q_m^{1/6}\left(\frac{L\kappa^2}{k}\right)\left[\xi^2(1-\xi)^2\right]\right\}d\kappa d\xi$$

$$=1+8\pi^2L^3\int_0^1\xi^2(1-\xi^2)\int_0^\infty\kappa^5Q_n(\kappa)\exp\left\{-0.246\sigma_R^2Q_m^{1/6}\left(\frac{L\kappa^2}{k}\right)\xi^2(1-\xi)^2\right\}d\kappa d\xi$$

$$=1+8\pi^2L^3\int_0^1\xi^2(1-\xi^2)\int_0^\infty\kappa^5Q_n(\kappa)\exp\left\{-0.246\sigma_R^2Q_m^{1/6}\left(\frac{L\kappa^2Q_m}{L\kappa_m^2}\right)\xi^2(1-\xi)^2\right\}d\kappa d\xi$$

$$=1+8\pi^2L^3\int_0^1\xi^2(1-\xi^2)\int_0^\infty\kappa^5Q_n(\kappa)\exp\left\{-\left(\frac{\kappa^2}{\kappa_m^2}\right)0.246\sigma_R^2Q_m^{7/6}\xi^2(1-\xi)^2\right\}d\kappa d\xi \qquad \text{(K8)}$$

In the Von Karman Spectrum, assume $\kappa_0=1/L_0\cong1/\infty\cong0$.

Then,

$$Q_n=\frac{0.033C_n^2\exp(-\kappa^2/\kappa_m^2)}{(\kappa^2+\kappa_0^2)^{11/6}}\cong\frac{0.033C_n^2\exp\left(-\kappa^2/\kappa_m^2\right)}{(\kappa^2+0)^{11/6}}\cong0.033C_n^2\kappa^{-11/3}\exp\left(-\kappa^2/\kappa_m^2\right) \qquad \text{(K9)}$$

and

$$\sigma_I^2(L) \cong 1 + 8\pi^2 L^3 \int_0^1 \xi^2(1-\xi^2)\int_0^\infty \kappa^5 (0.033C_n^2\kappa^{-11/3})\exp\left(-\frac{\kappa^2}{\kappa_m^2}\right)\exp\left\{-\left(\frac{\kappa^2}{\kappa_m^2}\right)0.246\sigma_R^2 Q_m^{7/6}\xi^2(1-\xi)^2\right\}d\kappa d\xi$$

$$= 1 + 2.606C_n^2 k^2 L\left(\frac{L^2}{k^2}\right)\int_0^1 \xi^2(1-\xi^2)\int_0^\infty \kappa^{4/3}\exp\left\{-\frac{\kappa^2}{\kappa_m^2}\left[1+0.246\sigma_R^2 Q_m^{7/6}\xi^2(1-\xi)^2\right]\right\}d\kappa d\xi$$

$$\cong 1 + 2.606\left[0.813\sigma_R^2\left(\frac{k}{L}\right)^{5/6}\right]\frac{L^2}{k^2}\int_0^1 \xi^2(1-\xi^2)\int_0^\infty \kappa^{4/3}\exp\left\{-\frac{\kappa^2}{\kappa_m^2}\left[1+0.246\sigma_R^2 Q_m^{7/6}\xi^2(1-\xi)^2\right]\right\}d\kappa d\xi$$

$$= 1 + 2.118\sigma_R^2\left(\frac{L}{k}\right)^{7/6}\int_0^1 \xi^2(1-\xi^2)\left\{\int_0^\infty \kappa^{4/3}\exp\left\{-\left[\frac{1+0.246\sigma_R^2 Q_m^{7/6}\xi^2(1-\xi)^2}{\kappa_m^2}\right]\kappa^2\right\}d\kappa\right\}d\xi \qquad \text{(K10)}$$

where $C_n^2 k^2 L = 0.813\sigma_R^2\left(\frac{k}{L}\right)^{5/6}$

Substituting $t = \kappa^2 \quad \rightarrow \kappa = t^{1/2}, \; d\kappa = \frac{1}{2}t^{-1/2}dt$

$$\sigma_I^2(L) = 1 + 2.118\sigma_R^2\left(\frac{L}{k}\right)^{7/6}\int_0^1 \xi^2(1-\xi^2)\left[\int_0^\infty \left(t^{1/2}\right)^{4/3}\exp\left\{-\left[\frac{1+0.246\sigma_R^2 Q_m^{7/6}\xi^2(1-\xi)^2}{\kappa_m^2}\right]t\right\}\frac{1}{2}t^{-1/2}dt\right]d\xi$$

$$= 1 + 2.118\sigma_R^2\left(\frac{L}{k}\right)^{7/6}\int_0^1 \xi^2(1-\xi^2)\left[\frac{1}{2}\int_0^\infty t^{1/6}\exp\left\{-\left[\frac{1+0.246\sigma_R^2 Q_m^{7/6}\xi^2(1-\xi)^2}{\kappa_m^2}\right]t\right\}dt\right]d\xi \qquad \text{(K11)}$$

Applying[1]

$$\int_0^\infty t^{x-1}\exp\{-st\}dt = \frac{\Gamma(x)}{s^x} \qquad \text{(K12)}$$

where $x - 1 = \frac{1}{6}$

results in

$$\sigma_I^2(L) \cong 1 + 2.118\sigma_R^2\left(\frac{L}{k}\right)^{7/6}\left(\frac{1}{2}\right)\int_0^1 \xi^2(1-\xi^2)\left(\frac{(\kappa_m^2)^{7/6}}{\left[1+0.246\sigma_R^2 Q_m^{7/6}\xi^2(1-\xi)^2\right]^{7/6}}\right)\Gamma(7/6)d\xi$$

$$= 1 + 0.982\sigma_R^2\left(\frac{L\kappa_m^2}{k}\right)^{7/6}\int_0^1\left(\frac{\xi^2(1-\xi^2)}{\left[1+0.246\sigma_R^2 Q_m^{7/6}\xi^2(1-\xi)^2\right]^{7/6}}\right)d\xi \qquad \text{(K13)}$$

If $\sigma_R^2 Q_m^{7/6}$>>100, then

$$1 + 0.28\sigma_R^2 Q_m^{7/6}\xi^2(1-\xi)^2 \cong 0.246\sigma_R^2 Q_m^{7/6}\xi^2(1-\xi)^2 \qquad \text{(K14)}$$

and

$$\sigma_I^2(L) \cong 1 + 0.982\sigma_R^2 Q_m^{7/6} \int_0^1 \left(\frac{\xi^2(1-\xi^2)}{\left[0.246\sigma_R^2 Q_m^{7/6} \xi^2 (1-\xi)^2\right]^{7/6}} \right) d\xi$$

$$= 1 + \frac{0.982\sigma_R^2 Q_m^{7/6}}{(0.246)^{7/6}(\sigma_R^2)^{7/6}(Q_m^{7/6})^{7/6}} \int_0^1 \frac{\xi^{-1/3}}{(1-\xi)^{1/3}} d\xi \tag{K15}$$

Recognizing that[35]

$$\int_0^u \frac{x^{\mu-1}dx}{(1+\beta x)^\nu} = \frac{u^\mu}{\mu}[{}_2F_1(\nu, \mu; 1+\mu, -\beta u)] \tag{K16}$$

Then,

$$\sigma_I^2(L) \cong 1 + \frac{0.982\sigma_R^2 Q_m^{7/6}}{(0.246)^{7/6}(\sigma_R^2)^{7/6}(Q_m^{7/6})^{7/6}} \frac{3}{2}\{{}_2F_1[1/3, 2/3; 5/3, 1]\}$$

$$= 1 + \frac{0.982\sigma_R^2 Q_m^{7/6}}{(0.246)^{7/6}(\sigma_R^2)^{7/6}(Q_m^{7/6})^{7/6}} \frac{3}{2}(1.369)$$

$$= 1 + \frac{10.36}{\left(\sigma_R^2 Q_m^{7/6}\right)^{1/6}} \tag{K17}$$

Therefore, for a spherical wave in saturated turbulence using the von Karman spectrum

$$\sigma_I^2(L) = 1 + \frac{10.36}{\left(\sigma_R^2 Q_m^{7/6}\right)^{1/6}}, \ \sigma_R^2 Q_m^{7/6} >> 100$$

APPENDIX L: SATURATED TURBULENCE, MODIFIED (BUMP), SPHERICAL WAVE

Reference equation 15, in the saturation regime, the scintillation index for an unbounded plane wave can be expressed in the form

$$\sigma_I^2(L) = 1 + 32\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa \Phi_n(\kappa) \sin^2\left[\frac{L\kappa^2}{2k} h(\xi,\xi)\right] \exp\left\{-\int_0^1 D_s\left[\frac{L\kappa}{k} h(\tau,\xi)\right] d\tau\right\} d\kappa d\xi \quad \text{(L1)}$$

For the modified spectrum (inner scale included) assume $D_s(\rho, L) \approx \frac{1}{2} D(\rho, l)$. As a result,

$$\sigma_I^2(L) \cong 1 + 32\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa Q_n(\kappa) \sin^2\left[\frac{L\kappa^2}{2k} h(\xi,\xi)\right] \exp\left\{-\int_0^1 \frac{1}{2} D_{pl}\left[\frac{L\kappa}{k} h(\tau,\xi)\right] d\tau\right\} d\kappa d\xi \quad \text{(L2)}$$

For spherical wave, convert D_{pl} to D_{sp} (the wave structure function for a spherical wave)

Assume the following approximations for the plane wave and spherical wave structure functions based on the von Karman spectrum[1]

$$\begin{aligned} D_{sp}(\rho,l) &= 0.900 C_n^2 k^2 L l_0^{-1/3} \rho^2 [\frac{1}{(1+0.311\rho^2/l_0^2)^{1/6}} + \frac{0.438}{(1+0.183\rho^2/l_0^2)^{2/3}} \\ &\quad - \frac{0.056}{(1+0.149\rho^2/l_0^2)^{3/4}} - 0.868(\kappa_0 l_0)^{1/3}] \\ &\cong 0.900 C_n^2 k^2 L l_0^{-1/3} \rho^2 [\frac{1}{(1+0)^{1/6}} + \frac{0.438}{(1+0)^{2/3}} - \frac{0.056}{(1+0)^{3/4}} - 0.868(0)^{1/3}] \\ &= 0.900 C_n^2 k^2 L l_0^{-1/3} \rho^2 [1 + 0.438 - 0.056 - 0] \\ &= 1.24 C_n^2 k^2 L l_0^{-1/3} \rho^2 \end{aligned} \quad \text{(L3)}$$

$$\begin{aligned} D(\rho,l) &= 2.700 C_n^2 k^2 L l_0^{-1/3} \rho^2 [\frac{1}{(1+0.632\rho^2/l_0^2)^{1/6}} + \frac{0.438}{(1+0.442\rho^2/l_0^2)^{2/3}} \\ &\quad - \frac{0.056}{(1+0.376\rho^2/l_0^2)^{3/4}} - 0.868(\kappa_0 l_0)^{1/3}] \\ &\cong 2.700 C_n^2 k^2 L l_0^{-1/3} \rho^2 [\frac{1}{(1+0)^{1/6}} + \frac{0.438}{(1+0)^{2/3}} - \frac{0.056}{(1+0)^{3/4}} - 0.868(0)^{1/3}] \\ &= 3.74 C_n^2 k^2 L l_0^{-1/3} \rho^2 \end{aligned} \quad \text{(L4)}$$

$$D_{pl}(\rho) \cong \frac{3.74}{1.24} D_{sp}(\rho) = 3.00 D_{sp}(\rho) \quad \text{(L5)}$$

where $\rho = \left[\frac{L\kappa}{k} h(\tau,\xi)\right]$

As result,

$$
\begin{aligned}
\sigma_I^2(L) &\cong 1+32\pi^2k^2L\int_0^1\int_0^\infty \kappa Q_n(\kappa)\sin^2\left[\frac{L\kappa^2}{2k}h(\xi,\xi)\right]\exp\left\{-\int_0^1\left(\frac{1}{2}\right)3.00D_{sp}\left[\frac{L\kappa}{k}h(\tau,\xi)\right]d\tau\right\}d\kappa d\xi \\
&\cong 1+32\pi^2k^2L\int_0^1\int_0^\infty \kappa Q_n(\kappa)\sin^2\left[\frac{L\kappa^2}{2k}h(\xi,\xi)\right]\exp\left\{-\int_0^1\left(\frac{1}{2}\right)(3.00)\left(1.24C_n^2k^2Ll_0^{-1/3}\left[\frac{L\kappa}{k}h(\tau,\xi)\right]^2\right)d\tau\right\}d\kappa d\xi \\
&\cong 1+32\pi^2k^2L\int_0^1\int_0^\infty \kappa Q_n(\kappa)\sin^2\left[\frac{L\kappa^2}{2k}h(\xi,\xi)\right]\exp\left\{-\int_0^1\left(\frac{1}{2}\right)(3.00)(1.24)\left[0.813\sigma_R^2\left(\frac{k}{L}\right)^{5/6}\right]l_0^{-1/3}\left[\frac{L\kappa}{k}h(\tau,\xi)\right]^2d\tau\right\}d\kappa d\xi \\
&\cong 1+32\pi^2k^2L\int_0^1\int_0^\infty \kappa Q_n(\kappa)\sin^2\left[\frac{L\kappa^2}{2k}h(\xi,\xi)\right]\exp\left\{-\int_0^1 1.52\sigma_R^2\left(\frac{L}{k}\right)^{7/6}\frac{1}{l_0^{1/3}}\kappa^2\left[h(\tau,\xi)\right]^2d\tau\right\}d\kappa d\xi \\
&\cong 1+32\pi^2k^2L\int_0^1\int_0^\infty \kappa Q_n(\kappa)\sin^2\left[\frac{L\kappa^2}{2k}h(\xi,\xi)\right]\exp\left\{-\int_0^1 1.02\sigma_R^2\left(\frac{L}{k}\right)\left(\frac{10.89L}{kl_0^2}\right)^{1/6}\kappa^2\left[h(\tau,\xi)\right]^2d\tau\right\}d\kappa d\xi \\
&\cong 1+32\pi^2k^2L\int_0^1\int_0^\infty \kappa Q_n(\kappa)\sin^2\left[\frac{L\kappa^2}{2k}h(\xi,\xi)\right]\exp\left\{-\int_0^1 1.02\sigma_R^2\left(\frac{L}{k}\right)Q_l^{1/6}\kappa^2\left[h(\tau,\xi)\right]^2d\tau\right\}d\kappa d\xi \\
&\cong 1+32\pi^2k^2L\int_0^1\int_0^\infty \kappa Q_n(\kappa)\sin^2\left[\frac{L\kappa^2}{2k}h(\xi,\xi)\right]\exp\left\{-1.02\sigma_R^2Q_l^{1/6}(\frac{L\kappa^2}{k})\left[\int_0^\xi[\tau(1-\beta\xi)]^2d\tau+\int_\xi^1[\xi(1-\beta\tau)]^2d\tau\right]\right\}d\kappa d\xi \\
&\cong 1+32\pi^2k^2L\int_0^1\int_0^\infty \kappa Q_n(\kappa)\sin^2\left[\frac{L\kappa^2}{2k}h(\xi,\xi)\right]\exp\left\{-1.02\sigma_R^2Q_l^{1/6}\left(\frac{L\kappa^2}{k}\right)\left[\int_0^\xi[\tau(1-\xi)]^2d\tau+\int_\xi^1[\xi(1-\tau)]^2d\tau\right]\right\}d\kappa d\xi \\
&\cong 1+32\pi^2k^2L\int_0^1\int_0^\infty \kappa Q_n(\kappa)\sin^2\left[\frac{L\kappa^2}{2k}h(\xi,\xi)\right]\exp\left\{-1.02\sigma_R^2Q_l^{1/6}\left(\frac{L\kappa^2}{k}\right)\left[\int_0^\xi[\tau(1-\xi)]^2d\tau+\int_\xi^1[\xi(1-\tau)]^2d\tau\right]\right\}d\kappa d\xi \\
&= 1+32\pi^2k^2L\int_0^1\int_0^\infty \kappa Q_n(\kappa)\sin^2\left[\frac{L\kappa^2}{2k}h(\xi,\xi)\right]\exp\left\{-1.02\sigma_R^2\left(\frac{L\kappa^2}{k}\right)Q_l^{1/6}\left[(1-\xi)^2\int_0^\xi\tau^2d\tau+\xi^2\int_\xi^1(1-\tau)^2d\tau\right]\right\}d\kappa d\xi \\
&= 1+32\pi^2k^2L\int_0^1\int_0^\infty \kappa Q_n(\kappa)\sin^2\left[\frac{L\kappa^2}{2k}h(\xi,\xi)\right]\exp\left\{-1.02\sigma_R^2\left(\frac{L\kappa^2}{k}\right)Q_l^{1/6}\left[(1-\xi)^2\frac{1}{3}\tau^3\Big|_0^\xi-\frac{1}{3}\xi^2(1-\tau)^3\Big|_\xi^1\right]\right\}d\kappa d\xi \\
&= 1+32\pi^2k^2L\int_0^1\int_0^\infty \kappa Q_n(\kappa)\sin^2\left[\frac{L\kappa^2}{2k}h(\xi,\xi)\right]\exp\left\{-1.02\sigma_R^2\left(\frac{L\kappa^2}{k}\right)Q_l^{1/6}\left[(1-\xi)^2\frac{1}{3}\{\xi^3-0^3\}-\frac{1}{3}\xi^2\{(1-1)^3-(1-\xi)^3\}\right]\right\}d\kappa d\xi \\
&= 1+32\pi^2k^2L\int_0^1\int_0^\infty \kappa Q_n(\kappa)\sin^2\left[\frac{L\kappa^2}{2k}h(\xi,\xi)\right]\exp\left\{-1.02\sigma_R^2\left(\frac{L\kappa^2}{k}\right)Q_l^{1/6}\left[(1-\xi)^2\frac{1}{3}\xi^3+\frac{1}{3}\xi^2(1-\xi)^3\right]\right\}d\kappa d\xi \\
&= 1+32\pi^2k^2L\int_0^1\int_0^\infty \kappa Q_n(\kappa)\sin^2\left[\frac{L\kappa^2}{2k}h(\xi,\xi)\right]\exp\left\{-1.02\sigma_R^2\left(\frac{L\kappa^2}{k}\right)Q_l^{1/6}\left[\frac{1}{3}\xi^2(1-\xi)^2\left[\xi+(1-\xi)\right]\right]\right\}d\kappa d\xi \\
&= 1+32\pi^2k^2L\int_0^1\int_0^\infty \kappa Q_n(\kappa)\sin^2\left[\frac{L\kappa^2}{2k}h(\xi,\xi)\right]\exp\left\{-1.02\sigma_R^2\left(\frac{L\kappa^2}{k}\right)Q_l^{1/6}\left[\frac{1}{3}\xi^2(1-\xi)^2\right]\right\}d\kappa d\xi \\
&= 1+32\pi^2k^2L\int_0^1\int_0^\infty \kappa Q_n(\kappa)\sin^2\left[\frac{L\kappa^2}{2k}h(\xi,\xi)\right]\exp\left\{-0.34\sigma_R^2\left(\frac{L\kappa^2}{k}\right)Q_l^{1/6}\left[\xi^2(1-\xi)^2\right]\right\}d\kappa d\xi \qquad \text{(L6)}
\end{aligned}
$$

where

$$
\begin{aligned}
C_n^2k^2L\kappa_l^{-5/3} &= C_n^2k^{7/6}k^{5/6}L^{11/6}L^{-5/6}\kappa_l^{-5/3} \\
&= C_n^2k^{7/6}L^{11/6}\left(\frac{k}{L\kappa_l^2}\right)^{5/6} \\
&= 0.813\sigma_R^2Q_l^{-5/6}
\end{aligned}
$$

$$\kappa_l = 3.3 / l_0$$

$$Q_l = \frac{L\kappa_l^2}{k} = \frac{L(3.3)^2}{kl_0^2} = \frac{10.89L}{kl_0^2}$$

$$h(\tau,\xi) = \tau(1-\beta\xi), \quad \tau < \xi$$

$$h(\tau,\xi) = \xi(1-\beta\tau), \quad \tau > \xi$$

$\beta = 1$ For a spherical wave

Assume, for x<<1, that

$$\sin(\mathrm{x}) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \cong x$$

$$\sin^2(x) \cong (x)^2$$

$$\sin^2\left[\frac{L\kappa^2}{2k} h(\xi,\xi)\right] \cong \left[\frac{L\kappa^2}{2k} h(\xi,\xi)\right]^2$$

$$= \left[\frac{L\kappa^2}{2k} \xi(1-\beta\xi)\right]^2$$

$$= \left[\frac{L^2\kappa^4}{4k^2} \xi^2(1-\beta\xi)^2\right]$$

$$= \left[\frac{L^2\kappa^4}{4k^2} \xi^2(1-\xi)^2\right]$$

Equation L6 can be approximated as

$$\sigma_l^2(L) = 1 + 32\pi^2 k^2 L \int_0^1 \int_0^\infty \kappa Q_n(\kappa) \left[\frac{L^2\kappa^4}{4k^2}\xi^2(1-\xi)^2\right] \exp\left\{-0.34\sigma_R^2 Q_l^{1/6}\left(\frac{L\kappa^2}{k}\right)\xi^2(1-\xi)^2\right\} d\kappa d\xi$$

$$= 1 + 8\pi^2 L^3 \int_0^1 \xi^2(1-\xi)^2 \int_0^\infty \kappa^5 Q_n(\kappa) \exp\left\{-\left(\frac{\kappa^2}{k}\right) 0.34\sigma_R^2 Q_l^{1/6} L\xi^2(1-\xi)^2\right\} d\kappa d\xi \qquad \text{(L7)}$$

Reference equation 4, for the modified spectrum

$$\Phi_n(\kappa) = \frac{0.033C_n^2\left[1 + 1.802\kappa/\kappa_l - 0.254(\kappa/\kappa_l)^{7/6}\right]\exp\left(-\kappa^2/\kappa_l^2\right)}{\left(\kappa^2 + \kappa_0^2\right)^{11/6}} \qquad \text{(L8)}$$

Assuming infinite outer scale $\kappa_0 = 1/L_0 \cong 1/\infty \cong 0$,

$$\Phi_n(\kappa) \cong 0.033C_n^2\kappa^{-11/3}\left[1+1.802\kappa/\kappa_l - 0.254(\kappa/\kappa_l)^{7/6}\right]\exp(-\kappa^2/\kappa_l^2) \tag{L9}$$

Equation L7 can be written as

$$\sigma_I^2(L) = 1+8\pi^2L^3\int_0^1 \xi^2(1-\xi)^2\int_0^\infty \kappa^5\left\{0.033C_n^2\kappa^{-11/3}\left[1+1.802\kappa/\kappa_l - 0.254(\kappa/\kappa_l)^{7/6}\right]\exp(-\kappa^2/\kappa_l^2)\right\}$$

$$\exp\left[-\left(\frac{\kappa^2}{k}\right)0.34\sigma_R^2 Q_l^{1/6} L\xi^2(1-\xi)^2\right]d\kappa d\xi \tag{L10}$$

This equation can be split into 3 intervals. Let $\mathrm{B}(\kappa/\kappa_l)^\alpha = 1+1.802\kappa/\kappa_l - 0.254(\kappa/\kappa_l)^{7/6}$.

Determine a general solution for $\mathrm{B}(\kappa/\kappa_l)^\alpha$, then solve for each interval separately.

$$\sigma_I^2(L) = 1+8\pi^2L^3\int_0^1 \xi^2(1-\xi)^2\int_0^\infty \kappa^5\left[0.033C_n^2\kappa^{-11/3}\mathrm{B}(\kappa/\kappa_l)^\alpha\exp\left(\frac{-\kappa^2}{\kappa_l^2}\right)\right]\exp\left\{-\left(\frac{\kappa^2}{k}\right)0.34\sigma_R^2Q_l^{1/6}L\xi^2(1-\xi)^2\right\}d\kappa d\xi$$

$$= 1+8\pi^2L^3(0.033)C_n^2\frac{\mathrm{B}}{(\kappa_l)^\alpha}\int_0^1 \xi^2(1-\xi)^2\int_0^\infty \kappa^{4/3+\alpha}\exp\left(\frac{-\kappa^2}{\kappa_l^2}\right)\exp\left\{-\left(\frac{\kappa^2}{k}\right)0.34\sigma_R^2Q_l^{1/6}L\xi^2(1-\xi)^2\right\}d\kappa d\xi$$

$$= 1+8\pi^2L^3(0.033)C_n^2\frac{\mathrm{B}}{(\kappa_l)^\alpha}\int_0^1 \xi^2(1-\xi)^2\int_0^\infty \kappa^{4/3+\alpha}\exp\left(\frac{-\kappa^2}{\kappa_l^2}\right)\exp\left\{-\left(\frac{\kappa^2}{\kappa_l^2}\right)0.34\sigma_R^2Q_l^{7/6}\xi^2(1-\xi)^2\right\}d\kappa d\xi$$

$$= 1+2.606C_n^2k^2L\frac{L^2}{k^2}\frac{\mathrm{B}}{(\kappa_l)^\alpha}\int_0^1 \xi^2(1-\xi)^2\int_0^\infty \kappa^{4/3+\alpha}\exp\left\{-\left(\frac{\kappa^2}{\kappa_l^2}\right)\left[1+0.34\sigma_R^2Q_l^{7/6}\xi^2(1-\xi)^2\right]\right\}d\kappa d\xi$$

$$= 1+2.118\sigma_R^2\left(\frac{L}{k}\right)^{7/6}\frac{\mathrm{B}}{(\kappa_l)^\alpha}\int_0^1 \xi^2(1-\xi)^2\int_0^\infty \kappa^{4/3+\alpha}\exp\left\{-\left(\frac{\kappa^2}{\kappa_l^2}\right)\left[1+0.34\sigma_R^2Q_l^{7/6}\xi^2(1-\xi)^2\right]\right\}d\kappa d\xi$$

$$= 1+2.118\sigma_R^2\left(\frac{L}{k}\right)^{7/6}\frac{\mathrm{B}}{(\kappa_l)^\alpha}\int_0^1 \xi^2(1-\xi)^2\left[\int_0^\infty \kappa^{4/3+\alpha}\exp\left\{-\frac{1+0.34\sigma_R^2Q_l^{7/6}\xi^2(1-\xi)^2}{\kappa_l^2}\kappa^2\right\}d\kappa\right]d\xi \tag{L11}$$

where

$$C_n^2k^2L = 0.813\sigma_R^2\left(\frac{k}{L}\right)^{5/6}$$

Substituting $t=\kappa^2 \quad \rightarrow \kappa = t^{1/2},\ d\kappa = \frac{1}{2}t^{-1/2}dt$

Then,

$$\sigma_I^2(L) = 1+2.118\sigma_R^2\left(\frac{L}{k}\right)^{7/6}\frac{\mathrm{B}}{(\kappa_l)^\alpha}\int_0^1 \xi^2(1-\xi)^2\left[\int_0^\infty (t^{1/2})^{4/3+\alpha}\exp\left\{-\frac{1+0.34\sigma_R^2Q_l^{7/6}\xi^2(1-\xi)^2}{\kappa_l^2}t\right\}\frac{1}{2}t^{-1/2}dt\right]d\xi$$

$$= 1+1.059\sigma_R^2\left(\frac{L}{k}\right)^{7/6}\frac{\mathrm{B}}{(\kappa_l)^\alpha}\int_0^1 \xi^2(1-\xi)^2\left[\int_0^\infty t^{1/6+\alpha/2}\exp\left\{-\frac{1+0.34\sigma_R^2Q_l^{7/6}\xi^2(1-\xi)^2}{\kappa_l^2}t\right\}dt\right]d\xi \tag{L12}$$

Recognizing[1] $\int_0^\infty t^{x-1}\exp\{-st\}dt = \frac{\Gamma(x)}{s^x}$, where $x-1=\frac{1}{6}+\frac{\alpha}{2} \quad \rightarrow x=\frac{7}{6}+\frac{\alpha}{2}$, Equation L12 becomes

$$\sigma_I^2(L) = 1 + 1.059\sigma_R^2\left(\frac{L}{k}\right)^{7/6}\frac{\mathrm{B}}{(\kappa_l)^{\alpha}}\int_0^1 \xi^2(1-\xi)^2\left[\frac{(\kappa_l^2)^{\left(\frac{7}{6}+\frac{\alpha}{2}\right)}\Gamma\left(\frac{7}{6}+\frac{\alpha}{2}\right)}{\left[1+0.34\sigma_R^2 Q_l^{7/6}\xi^2(1-\xi)^2\right]^{\left(\frac{7}{6}+\frac{\alpha}{2}\right)}}\right]d\xi$$

$$= 1 + 1.059\sigma_R^2\left(\frac{L}{k}\right)^{7/6}\frac{\mathrm{B}}{(\kappa_l)^{\alpha}}(\kappa_l^2)^{\left(\frac{7}{6}+\frac{\alpha}{2}\right)}\Gamma\left(\frac{7}{6}+\frac{\alpha}{2}\right)\int_0^1\left[\frac{\xi^2(1-\xi)^2}{\left[1+0.34\sigma_R^2 Q_l^{7/6}\xi^2(1-\xi)^2\right]^{\left(\frac{7}{6}+\frac{\alpha}{2}\right)}}\right]d\xi$$

$$= 1 + 1.059\sigma_R^2\left(\frac{L\kappa_l^2}{k}\right)^{7/6} B\Gamma\left(\frac{7}{6}+\frac{\alpha}{2}\right)\int_0^1\left[\frac{\xi^2(1-\xi)^2}{\left[1+0.34\sigma_R^2 Q_l^{7/6}\xi^2(1-\xi)^2\right]^{\left(\frac{7}{6}+\frac{\alpha}{2}\right)}}\right]d\xi$$

$$= 1 + 1.059\sigma_R^2 Q_l^{7/6} B\Gamma\left(\frac{7}{6}+\frac{\alpha}{2}\right)\int_0^1\left[\frac{\xi^2(1-\xi)^2}{\left[1+0.34\sigma_R^2 Q_l^{7/6}\xi^2(1-\xi)^2\right]^{\left(\frac{7}{6}+\frac{\alpha}{2}\right)}}\right]d\xi \qquad \text{(L13)}$$

If $\sigma_R^2 Q_m^{7/6} >> 100$,

$$1 + 0.34\sigma_R^2 Q_l^{7/6}\xi^2(1-\xi)^2 \cong 0.34\sigma_R^2 Q_l^{7/6}\xi^2(1-\xi)^2 \qquad \text{(L14)}$$

and

$$\sigma_I^2(L) \cong 1 + 1.059\sigma_R^2 Q_l^{7/6} B\Gamma\left(\frac{7}{6}+\frac{\alpha}{2}\right)\int_0^1\left[\frac{\xi^2(1-\xi)^2}{\left[0.34\sigma_R^2 Q_l^{7/6}\xi^2(1-\xi)^2\right]^{\left(\frac{7}{6}+\frac{\alpha}{2}\right)}}\right]d\xi$$

$$= 1 + \frac{1.059\sigma_R^2 Q_l^{7/6} B\Gamma\left(\frac{7}{6}+\frac{\alpha}{2}\right)}{\left(0.34\sigma_R^2 Q_l^{7/6}\right)^{\left(\frac{7}{6}+\frac{\alpha}{2}\right)}}\int_0^1\left[\frac{\xi^2(1-\xi)^2}{\left[\xi^2(1-\xi)^2\right]^{\left(\frac{7}{6}+\frac{\alpha}{2}\right)}}\right]d\xi$$

$$= 1 + \frac{1.059\sigma_R^2 Q_l^{7/6} B\Gamma\left(\frac{7}{6}+\frac{\alpha}{2}\right)}{\left(0.34\sigma_R^2 Q_l^{7/6}\right)^{\left(\frac{7}{6}+\frac{\alpha}{2}\right)}}\int_0^1\left[\frac{\xi^2(1-\xi)^2}{\xi^{\left(\frac{7}{3}+\alpha\right)}(1-\xi)^{\left(\frac{7}{3}+\alpha\right)}}\right]d\xi$$

$$= 1 + \frac{1.059\sigma_R^2 Q_l^{7/6} B\Gamma\left(\frac{7}{6}+\frac{\alpha}{2}\right)}{\left(0.34\sigma_R^2 Q_l^{7/6}\right)^{\left(\frac{7}{6}+\frac{\alpha}{2}\right)}}\int_0^1\left[\frac{\xi^{2-\left(\frac{7}{3}+\alpha\right)}}{(1-\xi)^{\left(\frac{7}{3}+\alpha\right)-2}}\right]d\xi$$

$$= 1 + \frac{1.059\sigma_R^2 Q_l^{7/6} B\Gamma\left(\frac{7}{6}+\frac{\alpha}{2}\right)}{\left(0.34\sigma_R^2 Q_l^{7/6}\right)^{\left(\frac{7}{6}+\frac{\alpha}{2}\right)}}\int_0^1\left[\frac{\xi^{-\frac{1}{3}-\alpha}}{(1-\xi)^{\frac{1}{3}+\alpha}}\right]d\xi \qquad \text{(L15)}$$

Reminder: $\mathrm{B}(\kappa/\kappa_l)^{\alpha} = 1 + 1.802\kappa/\kappa_l - 0.254(\kappa/\kappa_l)^{7/6}$

Solve for each interval independently, then add them together (i.e., $\sigma_I^2(L) \cong$ 1 + interval 1 + interval 2 - interval 3)

First interval: B=1, $\alpha = 0$

$$\text{interval 1} \cong \frac{1.059\sigma_R^2 Q_l^{7/6}\Gamma\left(\frac{7}{6}\right)}{\left(0.34\sigma_R^2 Q_l^{7/6}\right)^{\left(\frac{7}{6}\right)}}\int_0^1 \left[\frac{\xi^{-\frac{1}{3}}}{(1-\xi)^{\frac{1}{3}}}\right]d\xi$$

$$= \frac{1.059\Gamma\left(\frac{7}{6}\right)}{(0.34)^{7/6}\left(\sigma_R^2 Q_l^{7/6}\right)^{1/6}}\int_0^1 \left[\frac{\xi^{-\frac{1}{3}}}{(1-\xi)^{\frac{1}{3}}}\right]d\xi$$

$$= \frac{1.059(0.9277)}{(0.34)^{7/6}\left(\sigma_R^2 Q_l^{7/6}\right)^{1/6}}\int_0^1 \left[\frac{\xi^{-\frac{1}{3}}}{(1-\xi)^{\frac{1}{3}}}\right]d\xi$$

$$= \frac{1.059(0.9277)}{(0.34)^{7/6}\left(\sigma_R^2 Q_l^{7/6}\right)^{1/6}}\int_0^1 \left[\frac{\xi^{-\frac{1}{3}}}{(1-\xi)^{\frac{1}{3}}}\right]d\xi$$

$$= \frac{3.46}{\left(\sigma_R^2 Q_l^{7/6}\right)^{1/6}}\int_0^1 \left[\frac{\xi^{-\frac{1}{3}}}{(1-\xi)^{\frac{1}{3}}}\right]d\xi$$

$$\cong \frac{3.46}{\left(\sigma_R^2 Q_l^{7/6}\right)^{1/6}}\frac{3}{2}\{{}_2F_1[1/3,2/3;5/3,1]\}$$

$$= \frac{3.46}{\left(\sigma_R^2 Q_l^{7/6}\right)^{1/6}}\frac{3}{2}(1.369)$$

$$= \frac{3.46}{\left(\sigma_R^2 Q_l^{7/6}\right)^{1/6}}\frac{3}{2}(1.369)$$

$$= \frac{7.1}{\left(\sigma_R^2 Q_l^{7/6}\right)^{1/6}} \qquad \text{(L15)}$$

where[35] $\displaystyle\int_0^u \frac{x^{\mu-1}dx}{(1+\beta x)^{\nu}} = \frac{u^{\mu}}{\mu}[{}_2F_1(\nu,\mu;1+\mu,-\beta u)]$

$u = 1$

$\mu - 1 = -1/3, \rightarrow \mu = 2/3, \rightarrow \mu + 1 = 5/3$

$\beta = -1$

$\nu = 1/3$

Second interval: B= 1.802 , $\alpha = 1$

$$\text{Term 2} \cong \frac{1.059\sigma_R^2 Q_l^{7/6} B\Gamma\left(\frac{7}{6}+\frac{\alpha}{2}\right)}{\left(0.6793\sigma_R^2 Q_l^{7/6}\right)^{\left(\frac{7}{6}+\frac{\alpha}{2}\right)}} \int_0^1 \left[\frac{\xi^{\frac{1}{3}-\alpha}}{(1-\xi)^{\frac{1}{3}+\alpha}}\right] d\xi$$

$$= \frac{1.059\sigma_R^2 Q_l^{7/6} 1.802\Gamma\left(\frac{7}{6}+\frac{1}{2}\right)}{\left(0.6793\sigma_R^2 Q_l^{7/6}\right)^{\left(\frac{7}{6}+\frac{1}{2}\right)}} \int_0^1 \left[\frac{\xi^{-\frac{1}{3}-1}}{(1-\xi)^{\frac{1}{3}+1}}\right] d\xi$$

$$\cong 0 \quad \text{(Since interval 2 << interval 1)} \qquad \text{(L16)}$$

Since interval 2 << interval 1, assume interval 2 $\cong 0$

Third interval: B=-0.254, α =7/6

$$\text{interval 3} \cong \frac{1.059\sigma_R^2 Q_l^{7/6} B\Gamma\left(\frac{7}{6}+\frac{\alpha}{2}\right)}{\left(0.6793\sigma_R^2 Q_l^{7/6}\right)^{\left(\frac{7}{6}+\frac{\alpha}{2}\right)}} \int_0^1 \left[\frac{\xi^{-\frac{1}{3}-\alpha}}{(1-\xi)^{\frac{1}{3}+\alpha}}\right] d\xi$$

$$= \frac{1.059\sigma_R^2 Q_l^{7/6} (0.254)\Gamma\left(\frac{7}{6}+\frac{7}{12}\right)}{\left(0.6793\sigma_1^2 Q_l^{7/6}\right)^{\left(\frac{7}{6}+\frac{7}{2}\right)}} \int_0^1 \left[\frac{\xi^{-\frac{1}{3}-\frac{7}{6}}}{(1-\xi)^{\frac{1}{3}+\frac{7}{6}}}\right] d\xi$$

$$\cong 0 \quad \text{(Since interval 3 << interval 1)} \qquad \text{(L17)}$$

Recall that $\sigma_I^2(L) \cong 1$ + interval 1 + interval 2 + interval 3 $\cong 1 + \frac{7.1}{\left(\sigma_R^2 Q_l^{7/6}\right)^{1/6}}$. Therefore, for a spherical wave in saturated turbulence using the modified (bump) spectrum:

$$\sigma_I^2(L) = 1 + \frac{7.1}{\left(\sigma_R^2 Q_l^{7/6}\right)^{1/6}} \qquad \text{(L18)}$$

APPENDIX M: MODERATE TURBULENCE, KOLMOGOROV, PLANE WAVE

Large-Scale Log-Irradiance Variance

Using the "effective" atmospheric spectrum, reference equations 23 and 30, the large-scale log-irradiance variance for a plane wave in the presence of a finite inner scale is[8]

$$\sigma_{\ln x}^2(L) = 8\pi^2 k^2 \int_0^L \int_0^\infty \kappa 0.033 C_n^2 \kappa^{-11/3} G_x(\kappa)\left[1-\cos\left(\frac{\kappa^2 z}{k}\right)\right] d\kappa dz \qquad \text{(M1)}$$

Let $1-\cos\left(\frac{\kappa^2 z}{k}\right) \cong \frac{1}{2}\left(\frac{\kappa^2 z}{k}\right)$ (its Taylor expansion estimate). The approximation is valid for the large-scale log-irradiance variance because filter function $G_x(\kappa)$ eliminates high spatial frequency contributions in moderate-to-strong fluctuations. Consequently

$$\begin{aligned}\sigma_{\ln x}^2(L) &\cong 8\pi^2 k^2 \int_0^L \int_0^\infty \kappa 0.033 C_n^2 \kappa^{-11/3} G_x(\kappa)\left[\frac{1}{2}\left(\frac{\kappa^2 z}{k}\right)^2\right] d\kappa dz \\ &= 4\pi^2 \int_0^L \int_0^\infty \kappa^5 0.033 C_n^2 \kappa^{-11/3} G_x(\kappa) z^2 d\kappa dz \qquad \text{(M2)}\end{aligned}$$

Substituting

$$\xi = z/L \;\rightarrow\; dz = Ld\xi$$

$$\eta = \frac{L\kappa^2}{k} \;\rightarrow d\kappa = \frac{k}{2L\kappa} d\eta$$

$$\begin{aligned}\sigma_{\ln x}^2(L) &\cong 4\pi^2 \int_0^1 \int_0^\infty \kappa^5 0.033 C_n^2 \kappa^{-11/3} G_x(\kappa) L^2 \xi^2 \frac{k}{2L\kappa} d\eta L d\xi \\ &= 2\pi^2 k L^2 \int_0^1 \int_0^\infty \kappa^4 0.033 C_n^2 \kappa^{-11/3} G_x(\kappa) \xi^2 d\eta d\xi \\ &= 0.651 k L^2 C_n^2 \int_0^1 \xi^2 d\xi \int_0^\infty \kappa^{1/3} G_x(\kappa) d\eta \\ &= 0.651 k L^2 C_n^2 \int_0^1 \xi^2 d\xi \int_0^\infty \kappa^{1/3} \exp\left(\frac{-\kappa^2}{\kappa_x^2}\right) d\eta\end{aligned}$$

$$= 0.651k^{7/6}L^{11/6}C_n^2\int_0^1 \xi^2 d\xi \int_0^\infty \frac{L^{1/6}\kappa^{1/3}}{k^{1/6}} \exp\left(\frac{-Lk\kappa^2}{Lk\kappa_x^2}\right) d\eta$$

$$= 0.651k^{7/6}L^{11/6}C_n^2\int_0^1 \xi^2 d\xi \int_0^\infty \left(\frac{L\kappa^2}{k}\right)^{1/6} \exp\left[\left(\frac{-L\kappa^2}{k}\right)\left(\frac{k}{L\kappa_x^2}\right)\right] d\eta$$

$$= 0.651k^{7/6}L^{11/6}C_n^2\int_0^1 \xi^2 d\xi \int_0^\infty \eta^{1/6} \exp\left(\frac{-\eta}{\eta_x}\right) d\eta$$

$$= 0.651\left(\frac{1.23}{1.23}\right)k^{7/6}L^{11/6}C_n^2\int_0^1 \xi^2 d\xi \int_0^\infty \eta^{1/6} \exp\left(\frac{-\eta}{\eta_x}\right) d\eta$$

$$= 0.53\sigma_R^2\int_0^1 \xi^2 d\xi \int_0^\infty \eta^{1/6} \exp\left(\frac{-\eta}{\eta_x}\right) d\eta$$

$$= 0.53\sigma_R^2\left(\frac{1}{3}\right)\int_0^\infty \eta^{1/6} \exp\left(\frac{-\eta}{\eta_x}\right) d\eta$$

$$= 0.176\sigma_R^2\int_0^\infty \eta^{1/6} \exp\left(\frac{-\eta}{\eta_x}\right) d\eta \qquad \text{(M3)}$$

where

$$G_x(\kappa) = f(\kappa d_0)\exp\left(-\kappa^2/\kappa_x^2\right)$$
$$= \exp\left(-\kappa^2/\kappa_x^2\right)$$

$$\sigma_R^2 = 1.23C_n^2 k^{7/6}L^{11/6}$$

Recognizing[1]

$$\int_0^\infty e^{-s\eta}\eta^{x-1}d\eta = \frac{\Gamma(x)}{s^x} \qquad \text{(M4)}$$

where

$$s = \frac{1}{\eta_x}$$

x-1=1/6 $\rightarrow$ x=7/6

Results in

$$\sigma^2_{\ln x}(L) = 0.176\sigma_R^2 \frac{\Gamma(7/6)}{\left(\dfrac{1}{\eta_x}\right)^{7/6}}$$

$$= 0.176{\eta_x}^{7/6}\sigma_R^2(0.9277)$$

$$= 0.16\sigma_R^2{\eta_x}^{7/6} \qquad \text{(M5)}$$

<u>Small-Scale Log-Irradiance Variance</u>

Using the "effective" atmospheric spectrum, reference equations 23 and 33, the small-scale log-irradiance variance for a plane wave in the presence of a finite inner scale is[8]

$$\sigma^2_{\ln y}(L) = 8\pi^2 k^2 \int_0^L\int_0^\infty \kappa 0.033 C_n^2 \kappa^{-11/3} G_y(\kappa)\left[1-\cos\left(\frac{\kappa^2 z}{k}\right)\right]d\kappa dz \qquad \text{(M6)}$$

In moderate to strong irradiance fluctuations, $\kappa_y >> \sqrt{k/L}$, let

$$\sigma^2_{\ln y}(L) \cong 8\pi^2 k^2 \int_0^L\int_0^\infty \kappa 0.033 C_n^2 \kappa^{-11/3} G_y(\kappa) d\kappa dz \qquad \text{(M7)}$$

Substituting

$$\xi = z/L \;\rightarrow\; dz = Ld\xi$$

$$\eta = \frac{L\kappa^2}{k} \;\rightarrow\; d\kappa = \frac{k}{2L\kappa}d\eta$$

$$\sigma^2_{\ln y}(L) \cong 8\pi^2 k^2 \int_0^1\int_0^\infty \kappa 0.033 C_n^2 \kappa^{-11/3} G_y(\kappa)\frac{k}{2L\kappa}d\eta L d\xi$$

$$= 4\pi^2 k^3 \int_0^1\int_0^\infty 0.033 C_n^2 \kappa^{-11/3} G_y(\kappa) d\eta d\xi$$

$$= 1.30 k^3 C_n^2 \int_0^1\int_0^\infty \kappa^{-11/3}\left[\frac{\kappa^{11/3}}{\left(\kappa^2+\kappa_y^2\right)^{11/6}}\right]d\eta d\xi$$

$$= 1.30 k^3 C_n^2 \int_0^\infty \left[\frac{1}{\left(\kappa^2+\kappa_y^2\right)^{11/6}}\right]d\eta$$

$$= 1.30\left(\frac{1.23}{1.23}\right)C_n^2\left(k^{7/6}k^{11/6}\right)\frac{L^{11/16}}{L^{11/16}}\int_0^\infty\left[\frac{1}{\left(\kappa^2+\kappa_y^2\right)^{11/6}}\right]d\eta$$

$$= 1.06\sigma_R^2\int_0^\infty\left[\frac{k^{11/6}}{L^{11/16}\left(\kappa^2+\kappa_y^2\right)^{11/6}}\right]d\eta$$

$$= 1.06\sigma_R^2\int_0^\infty\left(\frac{k}{L\kappa^2}+\frac{k}{L\kappa_y^2}\right)^{11/6}d\eta$$

$$= 1.06\sigma_R^2\int_0^\infty(\eta+\eta_y)^{-11/6}d\eta$$

$$= 1.06\sigma_R^2\left(\frac{-6}{5}\right)(\eta+\eta_y)^{-5/6}\Big|_{\eta=0}^{\eta=\infty}$$

$$= 1.06\sigma_R^2\left(\frac{-6}{5}\right)[(\infty+\eta_y)^{-5/6}-(0+\eta_y)^{-5/6}]$$

$$= 1.06\sigma_R^2\left(\frac{-6}{5}\right)\left[-(\eta_y)^{-5/6}\right]$$

$$= 1.272\sigma_R^2\eta_y^{-5/6} \qquad \text{(M8)}$$

where

$$G_y(\kappa)=\frac{\kappa^{11/3}}{\left(\kappa^2+\kappa_y^2\right)^{11/6}}$$

$$\sigma_R^2 = 1.23C_n^2k^{7/6}L^{11/6}$$

Asymptotic Comparisons

For constants c_1, c_2, c_3, and c_4, assume the following functional form of η_x and η_y [see Appendix S].

$$\eta_x=\frac{1}{c_1+c_2L/k\rho_o^2}\cong\begin{cases}\dfrac{1}{c_1+c_2(0)}=\dfrac{1}{c_1}, & \dfrac{L}{k\rho_0^2}<<1\\ \dfrac{1}{0+c_2L/k\rho_o^2}=\dfrac{k\rho_o^2}{c_2L}, & \dfrac{L}{k\rho_0^2}>>1\end{cases} \qquad \text{(M9)}$$

$$\eta_y = c_3 + \frac{c_4 L}{k\rho_0^2} \cong \begin{cases} c_3 + c_4(0) = c_3, & \frac{L}{k\rho_0^2} << 1 \\ 0 + \frac{c_4 L}{k\rho_0^2} = \frac{c_4 L}{k\rho_0^2}, & \frac{L}{k\rho_0^2} >> 1 \end{cases} \tag{M10}$$

Weak Turbulence

The log irradiance variance, $\sigma_{\ln I}^2$, is defined under the Rytov approximation for weak turbulence by[8]

$$\sigma_{\ln I}^2 = 8\pi^2 k^2 \int_0^L \int_0^\infty \kappa \Phi_n(\kappa) \left[1 - \cos\left(\frac{\kappa^2 z}{k}\right)\right] d\kappa dz \tag{M11}$$

where $\Phi_n(\kappa) = 0.033 C_n^2 \kappa^{-11/3}$ (zero inner scale)

Substituting

$$\xi = z / L \rightarrow dz = L d\xi$$

$$\eta = L\kappa^2 / k \rightarrow \kappa^2 = \eta k / L,$$

$$d\kappa = \frac{1}{2}\left(\frac{\eta k}{L}\right)^{-1/2}\left(\frac{k}{L}\right) d\eta = \frac{1}{2}\left(\frac{L}{\eta k}\right)^{1/2}\left(\frac{k}{L}\right) d\eta$$

$$\eta\xi = \left(\frac{z}{L}\right)\left(\frac{L\kappa^2}{k}\right) = \frac{\kappa^2 z}{k}$$

Then

$$\begin{aligned} \sigma_{\ln I}^2(L) &= 8\pi^2 k^2 \int_0^1 \int_0^\infty (\eta k / L)^{1/2} \Phi_n(\kappa) [1 - \cos(\eta\xi)] \frac{1}{2}\left(\frac{L}{\eta k}\right)^{1/2}\left(\frac{k}{L}\right) d\eta L d\xi \\ &= \frac{8\pi^2 k^3}{2} \int_0^1 \int_0^\infty \Phi_n(\kappa) [1 - \cos(\eta\xi)] d\eta d\xi \\ &= \frac{8\pi^2 k^3}{2} \int_0^1 \int_0^\infty 0.033 C_n^2 \kappa^{-11/3} [1 - \cos(\eta\xi)] d\eta d\xi \end{aligned}$$

$$= \frac{8\pi^2(0.033C_n^2)k^3}{2}\int_0^1\int_0^\infty \kappa^{-11/3}\left[1-\cos(\eta\xi)\right]d\eta d\xi$$

$$= \frac{8\pi^2(0.033)}{2}\left(\frac{1.23}{1.23}\right)C_n^2\left(k^{7/6}k^{11/6}\right)\left(\frac{L^{11/6}}{L^{11/6}}\right)\int_0^1\int_0^\infty \kappa^{-11/3}\left[1-\cos(\eta\xi)\right]d\eta d\xi$$

$$= \frac{8\pi^2(0.033)}{(2)(1.23)}(1.23C_n^2k^{7/6}L^{11/6})\int_0^1\int_0^\infty \left(\frac{k^{11/6}}{L^{11/6}\kappa^{11/3}}\right)\left[1-\cos(\eta\xi)\right]d\eta d\xi$$

$$= 1.06\sigma_R^2\int_0^1\int_0^\infty \left(\frac{k}{L\kappa^2}\right)^{11/6}\left[1-\cos(\eta\xi)\right]d\eta d\xi$$

$$= 1.06\sigma_R^2\int_0^1\int_0^\infty \eta^{-11/6}\left[1-\cos(\eta\xi)\right]d\eta d\xi \qquad \text{(M12)}$$

where

$$\sigma_R^2 = 1.23C_n^2k^{7/6}L^{11/6}$$

$$\eta = \frac{k}{L\kappa^2}$$

Using the relationship $\sin^2(u) = \frac{1}{2}\left[1-\cos(2u)\right]$, the previous equation reduces to

$$\sigma_{\ln I}^2(L) = 2.12(\sigma_R^2)\int_0^1\int_0^\infty \eta^{-11/6}\sin^2\left(\frac{\eta\xi}{2}\right)d\eta d\xi \qquad \text{(M13)}$$

Recognizing[35]

$$\int_0^\infty x^{u-1}\sin^2 ax dx = \frac{(-)\Gamma(u)\cos\frac{u\pi}{2}}{2^{u+1}a^u} \qquad \text{(M14)}$$

where

u = -5/6

a= $\frac{\xi}{2}$

leads to

$$\sigma_{\ln I}^2(L) = 2.12\sigma_R^2 \int_0^1 \frac{(-)\Gamma(-5/6)\cos\left(\frac{-5\pi}{12}\right)}{2^{1/6}\left(\frac{\xi}{2}\right)^{-5/6}} d\xi$$

$$= (-)2.12\Gamma(-5/6)\cos\left(\frac{-5\pi}{12}\right)(2^{-1/6})\left(\frac{1}{2}\right)^{5/6} \sigma_R^2 \int_0^1 \xi^{5/6} d\xi$$

$$= (-)2.12\Gamma(-5/6)(2^{-1/6})\left(\frac{1}{2}\right)^{5/6} \sigma_R^2 \left(\frac{6}{11}\right)$$

$$= (-)2.12(\text{-}6.680)(0.259)(0.89)(0.561)\left(\frac{6}{11}\right)\sigma_R^2$$

$$= \sigma_R^2 \qquad \text{(M15)}$$

Under weak irradiance fluctuations[8], $\sigma_{\ln I}^2 << 1$,

$$\sigma_{\ln I}^2 = \sigma_{\ln x}^2 + \sigma_{\ln y}^2 \qquad \text{(M16)}$$

Equating equations M13 and M15, and using equations M8 and M16, results in

$$0.15\eta_x^{7/6} + 1.272\eta_y^{-5/6} \cong 1, \quad \sigma_R^2 << 1, \quad l_0 \to 0 \qquad \text{(M17)}$$

Assuming $\eta_x \cong \eta_y$, then $\eta_x = \eta_y = 3$ is an approximate solution.

Plugging this into equations M9 and M10

$$\eta_x = \frac{1}{c_1 + c_2 L / k\rho_o^2} \cong \frac{1}{c_1 + c_2(0)} = \frac{1}{c_1} = 3$$

$$\to c_1 \cong 1/3 \qquad \text{(M18)}$$

$$\eta_y = c_3 + \frac{c_4 L}{k\rho_0^2} \cong c_3 + c_4(0) = c_3 = 3 \qquad \text{(M19)}$$

Thus, c_1 and c_3 are now established using behavior of the scintillation index in the weak turbulence regime.

Saturated Turbulence

First, perform an asymptotic comparison for small-scale turbulent cell effects. Normalized irradiance[8] can be expressed as I = xy where x is associated with large-scale turbulent eddy affects, y is associated with small-scale eddy effects, x and y are statistically independent random quantities, and $\langle x\rangle=\langle y\rangle=1$. Given these conditions, $\langle I\rangle = \langle x\rangle\langle y\rangle = 1$. The second moment of irradiance takes the form[8]

$$\langle I^2\rangle = \langle x^2\rangle\langle y^2\rangle = (1+\sigma_x^2)(1+\sigma_y^2) \quad \text{(M20)}$$

where σ_x^2 and σ_y^2 are the large scale and small scale normalized variances of x and y, respectively. Based on equations 5 and M20,the implied scintillation index is

$$\begin{aligned}\sigma_I^2 &= \left(1+\sigma_x^2\right)\left(1+\sigma_y^2\right)-1 \\ &= \sigma_x^2+\sigma_y^2+\sigma_x^2\sigma_y^2\end{aligned} \quad \text{(M21)}$$

Per equation 16, the asymptotic behavior of the scintillation index in the saturation regime is described by $\sigma_I^2 = 1+\frac{0.86}{\sigma_R^{4/5}}$, which approaches an asymptotic limit of one. Therefore, in saturated turbulence

$$\sigma_I^2 = \sigma_x^2+\sigma_y^2+\sigma_x^2\sigma_y^2 \cong 1 \quad \text{(M22)}$$

In strong fluctuations, expect the large-scale scintillation terms to die out.

$$\sigma_I^2 \approx \sigma_y^2 \cong 1 \quad \text{(M23)}$$

The small scale irradiance is given by[8]

$$\sigma_y^2 = \exp\left(\sigma_{\ln y}^2\right)-1 \quad \text{(M24)}$$

As a result, in strong fluctuations

$$\sigma_y^2 \cong 1 = \exp\left(\sigma_{\ln y}^2\right)-1 \quad \text{(M25)}$$

$$\to \sigma_{\ln y}^2 \cong \ln 2 \quad \text{(M26)}$$

The plane wave spatial coherence radius, ρ_0, when C_n^2 is constant and $l_0 << \rho << L_0$ can be approximated by[1]

$$\rho = \left(1.46 C_n^2 k^2 L\right)^{-3/5} \quad \rightarrow C_n^2 = \frac{\rho_0^{-5/3}}{1.46 k^2 L} \tag{M27}$$

Plugging this into the Rytov Variance, $\sigma_R^2 = 1.23 C_n^2 k^{7/6} L^{11/6} = 1.23\left(\frac{\rho_0^{-5/3}}{1.46 k^2 L}\right) k^{7/6} L^{11/6}$,

$$\frac{L}{k\rho_0^2} = 1.22(\sigma_R^2)^{6/5} \tag{M28}$$

In saturated turbulence, equating equation M26 to equation M8 results in

$$\begin{aligned} \ln 2 &= 1.272\sigma_R^2 (\eta_y)^{-5/6} \\ &= 1.272\sigma_R^2 \left(\frac{c_4 L}{k\rho_0^2}\right)^{-5/6} \\ &= 1.272\sigma_R^2 c_4^{-5/6}\left[1.22(\sigma_R^2)^{6/5}\right]^{-5/6} \\ &= 1.272 c_4^{-5/6}(1.22)^{-5/6} \end{aligned}$$

$$\rightarrow c_4 \cong 1.7 \tag{M29}$$

where

$\eta_y = \frac{c_4 L}{k\rho_0^2}$ (equation M10)

$\frac{L}{k\rho_0^2} = 1.22(\sigma_R^2)^{6/5}$ (equation M28)

Now determine an asymptotic comparison for large-scale turbulent cell effects. For saturated optical turbulence[8]

$$\sigma_x^2 = \exp\left(\sigma_{\ln x}^2\right) - 1 \tag{M30}$$

$$\rightarrow \exp\left(\sigma_{\ln x}^2\right) = \sigma_x^2 + 1$$

Also[8]

$$\sigma_I^2 = \exp\left(\sigma_{\ln x}^2 + \sigma_{\ln y}^2\right) - 1 \qquad \text{(M31)}$$

Therefore,

$$\begin{aligned}\sigma_I^2 &\cong \exp\left(\sigma_{\ln x}^2 + \ln 2\right) - 1 \\ &= 2\exp\left(\sigma_{\ln x}^2\right) - 1 \\ &= 2\left(\sigma_x^2 + 1\right) - 1 \\ &= 2\sigma_x^2 + 1 \end{aligned} \qquad \text{(M32)}$$

Under weak fluctuation $\left(\sigma_x^2 << 1\right)$,

$$\sigma_x^2 = \exp\left(\sigma_{\ln x}^2\right) - 1 \cong \sigma_{\ln x}^2 \qquad \text{(M33)}$$

Recall from Equation M5 that $\sigma_{\ln x}^2 = 0.16\sigma_R^2 \eta_x^{\;7/6}$

Then

$$\begin{aligned}\sigma_I^2 &\cong 2\sigma_x^2 + 1 \\ &\cong 2(0.16\sigma_R^2 \eta_x^{\;7/6}) + 1 \\ &= 0.32\sigma_R^2 \eta_x^{\;7/6} + 1 \\ &= 0.32\sigma_R^2 \left(\frac{k\rho_o^2}{c_2 L}\right)^{7/6} + 1 \\ &\cong 0.32\sigma_R^2 \left(\frac{1}{c_2 1.22\left(\sigma_R^2\right)^{6/5}}\right)^{7/6} + 1 \\ &= 1 + 0.32\sigma_R^2 \left(\frac{1}{c_2}\right)^{7/6} \left(\frac{1}{1.22}\right)^{7/6} \left(\frac{1}{\sigma_R^2}\right)^{7/5} \\ &= 1 + 0.32\sigma_R^2 \left(\frac{1}{c_2}\right)^{7/6} \left(\frac{1}{1.22}\right)^{7/6} \left(\frac{1}{\sigma_R^{14/5}}\right) \\ &= 1 + 0.25\left(\frac{1}{\sigma_R^{4/5}}\right)\left(\frac{1}{c_2}\right)^{7/6} \end{aligned} \qquad \text{(M34)}$$

where

$$\eta_x \cong \frac{k\rho_o^2}{c_2 L} \quad \text{(equation M9)}$$

$$L / k\rho_0^2 = 1.22(\sigma_R^2)^{6/5} \quad \text{(equation M28)}$$

Compare this to the existing scintillation expression (equation 16) for a plane wave in saturated turbulence.

$$\sigma_I^2 = 1 + 0.25\left(\frac{1}{\sigma_R^{4/5}}\right)\left(\frac{1}{c_2}\right)^{7/6} = 1 + \frac{0.86}{\sigma_R^{4/5}}$$

$$\rightarrow 0.25\left(\frac{1}{\sigma_R^{4/5}}\right)\left(\frac{1}{c_2}\right)^{7/6} = \left(\frac{1}{\sigma_R^{4/5}}\right)0.86$$

$$\rightarrow c_2 \cong \frac{1}{3} \qquad \text{(M35)}$$

Thus, c_2 and c_4 are now established using behavior of the scintillation index in the saturated turbulence regime.

<u>"All" Turbulence Regimes</u>

Applying constants c_1, c_2, c_3, and c_4 results in the following expressions for predicting scintillation behavior in all turbulence regimes (including inner scale effects):

$$\begin{aligned}
\sigma_{\ln x}^2 &\cong 0.16\sigma_R^2 \eta_x^{\ 7/6} \\
&= 0.16\sigma_R^2 \left(\frac{1}{c_1 + c_2 L / k\rho_o^2}\right)^{7/6} \\
&= 0.16\sigma_R^2 \left(\frac{1}{1/3 + 1/3 L / k\rho_o^2}\right)^{7/6} \\
&= 0.16\sigma_R^2 \left(\frac{3}{1 + L / k\rho_o^2}\right)^{7/6} \\
&= 0.57\sigma_R^2 \left(\frac{1}{1 + 1.22(\sigma_R^2)^{6/5}}\right)^{7/6}
\end{aligned}$$

$$\cong \begin{cases} 0.57\sigma_R^2, & \sigma_R^2 << 1 \\ \dfrac{0.45}{\sigma_R^{4/5}}, & \sigma_R^2 >> 1 \end{cases} \tag{M36}$$

and

$$\begin{aligned} \sigma_{\ln y}^2(L) &\cong 1.272\sigma_R^2\eta_y^{-5/6} \\ &= 1.272\sigma_R^2\left\{3 + 1.7\left[1.22(\sigma_R^2)^{6/5}\right]\right\}^{-5/6} \\ &= \frac{1.272\sigma_R^2}{\left(3 + 2.07\sigma_R^{12/5}\right)^{5/6}} \\ &= \frac{1.272\sigma_R^2}{\left[3\left(1 + \frac{2.07}{3}\sigma_R^{12/5}\right)\right]^{5/6}} \\ &= \frac{0.509\sigma_R^2}{\left(1 + 0.69\sigma_R^{12/5}\right)^{5/6}} \\ &\cong \begin{cases} 509\sigma_{R,}^2 & \sigma_R^2 << 1 \\ \dfrac{0.509\sigma_R^2}{\left(0.69\sigma_R^{12/5}\right)^{5/6}} = \dfrac{0.69\sigma_R^2}{\sigma_R^2} = \ln 2, & \sigma_R^2 >> 1 \end{cases} \end{aligned} \tag{M37}$$

Recall from equation M32 that $\sigma_I^2 = \exp\left(\sigma_{\ln x}^2 + \sigma_{\ln y}^2\right) - 1$

Therefore, for a plane wave in moderate turbulence using an "effective" Kolmogorov spectrum

$$\sigma_I^2 = \exp\left(\frac{0.57\sigma_R^2}{\left(1 + 1.22\sigma_R^{12/5}\right)^{7/6}} + \frac{0.509\sigma_R^2}{\left(1 + 0.69\sigma_R^{12/5}\right)^{5/6}}\right) - 1, \quad 0 \le \sigma_R^2 < \infty \tag{M38}$$

APPENDIX N: MODERATE TURBULENCE, MODIFIED (BUMP), PLANE WAVE

Large-Scale Log-Irradiance Variance

Using the "effective" atmospheric spectrum, reference equations 23 and 30, the large-scale log-irradiance variance for a plane wave in the presence of a finite inner scale is[8]

$$\sigma^2_{\ln x}(L) = 8\pi^2 k^2 \int_0^L \int_0^\infty \kappa 0.033 C_n^2 \kappa^{-11/3} G_x(\kappa)\left[1-\cos\left(\frac{\kappa^2 z}{k}\right)\right] d\kappa dz \qquad \text{(N1)}$$

Let $1-\cos\left(\frac{\kappa^2 z}{k}\right) \cong \frac{1}{2}\left(\frac{\kappa^2 z}{k}\right)$ (its Taylor expansion estimate). The approximation is valid for the large-scale log-irradiance variance because filter function $G_x(\kappa)$ eliminates high spatial frequency contributions in moderate-to-strong fluctuations. Consequently

$$\sigma^2_{\ln x}(L) \cong 8\pi^2 k^2 \int_0^L \int_0^\infty \kappa 0.033 C_n^2 \kappa^{-11/3} G_x(\kappa)\left[\frac{1}{2}\left(\frac{\kappa^2 z}{k}\right)^2\right] d\kappa dz$$

$$= 4\pi^2 \int_0^L \int_0^\infty \kappa^5 0.033 C_n^2 \kappa^{-11/3} G_x(\kappa) z^2 d\kappa dz \qquad \text{(N2)}$$

Substituting

$$\xi = z/L \rightarrow dz = Ld\xi$$

$$\eta = \frac{L\kappa^2}{k} \rightarrow d\kappa = \frac{k}{2L\kappa} d\eta$$

then

$$\sigma^2_{\ln x}(L) \cong 4\pi^2 \int_0^1 \int_0^\infty \kappa^5 0.033 C_n^2 \kappa^{-11/3} G_x(\kappa) L^2 \xi^2 \frac{k}{2L\kappa} d\eta L d\xi$$

$$= 2\pi^2 k L^2 \int_0^1 \int_0^\infty \kappa^4 0.033 C_n^2 \kappa^{-11/3} G_x(\kappa) \xi^2 d\eta d\xi$$

$$= 0.651 k L^2 C_n^2 \int_0^1 \xi^2 d\xi \int_0^\infty \kappa^{1/3} G_x(\kappa) d\eta$$

$$= 0.651 k L^2 C_n^2 \int_0^1 \xi^2 d\xi \int_0^\infty \kappa^{1/3} \exp\left[-\left(\frac{\kappa^2}{\kappa_l^2} + \frac{\kappa^2}{\kappa_x^2}\right)\right]\left[1 + \frac{1.802\kappa}{\kappa_l} - 0.254\left(\frac{\kappa}{\kappa_l}\right)^{7/6}\right] d\eta$$

$$= 0.651 k^{7/6} L^{11/6} C_n^2 \int_0^1 \xi^2 d\xi \int_0^\infty L \frac{{}^{1/6}\kappa^{1/3}}{k^{1/6}} \exp\left\{-\left[\left(\frac{Lk}{Lk}\right)\frac{\kappa^2}{\kappa_l^2} + \left(\frac{Lk}{Lk}\right)\frac{\kappa^2}{\kappa_x^2}\right]\right\}\left\{1 + 1.802\left(\frac{Lk}{Lk}\right)^{1/2}\left(\frac{\kappa}{\kappa_l}\right) - 0.254\left[\left(\frac{Lk}{Lk}\right)\left(\frac{\kappa^2}{\kappa_l^2}\right)\right]^{7/12}\right\} d\eta$$

$$= 0.651 k^{7/6} L^{11/6} C_n^2 \int_0^1 \xi^2 d\xi \int_0^\infty \left(\frac{L\kappa^2}{k}\right)^{1/6} \exp\left\{-\left[\left(\frac{L\kappa^2}{k}\right)\left(\frac{k}{L\kappa_l^2}\right) + \left(\frac{L\kappa^2}{k}\right)\left(\frac{k}{L\kappa_x^2}\right)\right]\right\}\left\{1 + 1.802\left(\frac{L\kappa^2}{k}\right)^{1/2}\left(\frac{k}{L\kappa_l^2}\right)^{1/2} - 0.254\left[\left(\frac{L\kappa^2}{k}\right)\left(\frac{k}{L\kappa_l^2}\right)\right]^{7/12}\right\} d\eta$$

$$= 0.651\left(\frac{1.23}{1.23}\right) k^{7/6} L^{11/6} C_n^2 \int_0^1 \xi^2 d\xi \int_0^\infty \left(\frac{L\kappa^2}{k}\right)^{1/6} \exp\left\{-\left[(\eta)\left(\frac{1}{Q_l}\right) + (\eta)\left(\frac{1}{\eta_x}\right)\right]\right\}\left\{1 + 1.802(\eta)^{1/2}\left(\frac{1}{Q_l}\right)^{1/2} - 0.254\left[(\eta)\left(\frac{1}{Q_l}\right)\right]^{7/12}\right\} d\eta$$

$$= 0.53\sigma_R^2 \int_0^1 \xi^2 d\xi \int_0^\infty \eta^{1/6} \exp\left[-\eta\left(\frac{1}{Q_l} + \frac{1}{\eta_x}\right)\right]\left[1 + 1.802\left(\frac{\eta}{Q_l}\right)^{1/2} - 0.254\left(\frac{\eta}{Q_l}\right)^{7/12}\right] d\eta$$

$$= 0.53\sigma_R^2 \left(\frac{1}{3}\right) \int_0^\infty \eta^{1/6} \exp\left[-\eta\left(\frac{\eta_x + Q_l}{\eta_x Q_l}\right)\right]\left[1 + 1.802\left(\frac{\eta}{Q_l}\right)^{1/2} - 0.254\left(\frac{\eta}{Q_l}\right)^{7/12}\right] d\eta$$

$$= 0.176\sigma_R^2 \int_0^\infty \eta^{1/6} \exp\left[-\eta\left(\frac{\eta_x + Q_l}{\eta_x Q_l}\right)\right]\left[1 + 1.802\left(\frac{\eta}{Q_l}\right)^{1/2} - 0.254\left(\frac{\eta}{Q_l}\right)^{7/12}\right] d\eta$$

$$= 0.176\sigma_R^2 \left\{\int_0^\infty \eta^{1/6} \exp\left[-\eta\left(\frac{\eta_x + Q_l}{\eta_x Q_l}\right)\right] d\eta + \int_0^\infty \eta^{1/6} \exp\left[-\eta\left(\frac{\eta_x + Q_l}{\eta_x Q_l}\right)\right] 1.802\left(\frac{\eta}{Q_l}\right)^{1/2} d\eta\right.$$

$$\left. - \int_0^\infty \eta^{1/6} \exp\left[-\eta\left(\frac{\eta_x + Q_l}{\eta_x Q_l}\right)\right] 0.254\left(\frac{\eta}{Q_l}\right)^{7/12} d\eta \right\}$$

$$= 0.176\sigma_R^2 \left\{\int_0^\infty \eta^{1/6} \exp\left[-\eta\left(\frac{\eta_x + Q_l}{\eta_x Q_l}\right)\right] d\eta + \frac{1.802}{Q_l^{1/2}} \int_0^\infty \eta^{2/3} \exp\left[-\eta\left(\frac{\eta_x + Q_l}{\eta_x Q_l}\right)\right] d\eta\right.$$

$$\left. - \frac{0.254}{Q_l^{7/12}} \int_0^\infty \eta^{3/4} \exp\left[-\eta\left(\frac{\eta_x + Q_l}{\eta_x Q_l}\right)\right] d\eta \right\} \qquad \text{(N3)}$$

where

$$G_x(\kappa) = f(\kappa d_0)\exp\left(-\kappa^2/\kappa_x^2\right)$$
$$= \exp\left(-\kappa^2/\kappa_l^2\right)\left[1 + 1.802\kappa/\kappa_l - 0.254(\kappa/\kappa_l)^{7/6}\right]\exp\left(-\kappa^2/\kappa_x^2\right)$$
$$= \exp\left[-\left(\frac{\kappa^2}{\kappa_l^2} + \frac{\kappa^2}{\kappa_x^2}\right)\right]\left[1 + \frac{1.802\kappa}{\kappa_l} - 0.254\left(\frac{\kappa}{\kappa_l}\right)^{7/6}\right]$$

$$Q_l = L\kappa_l^2 / k$$

$$\sigma_R^2 = 1.23 C_n^2 k^{7/6} L^{11/6}$$

Recognizing [1]

$$\int_0^{\infty} e^{-s\eta}\eta^{x-1}d\eta = \frac{\Gamma(x)}{s^x} \tag{N4}$$

where

$$s = \frac{\eta_x + Q_l}{\eta_x Q_l}$$

x-1=1/6 $\rightarrow$ x=7/6,   x-1=2/3 $\rightarrow$ x=5/3,   x-1=3/4 $\rightarrow$ x=7/4,

results in

$$\sigma_{\ln x}^2(L) = 0.176\sigma_R^2\left\{\frac{\Gamma(7/6)}{\left(\dfrac{\eta_x+Q_l}{\eta_x Q_l}\right)^{7/6}} + \frac{1.802}{Q_l^{1/2}}\frac{\Gamma(5/3)}{\left(\dfrac{\eta_x+Q_l}{\eta_x Q_l}\right)^{5/3}} - \frac{0.254}{Q_l^{7/12}}\frac{\Gamma(7/4)}{\left(\dfrac{\eta_x+Q_l}{\eta_x Q_l}\right)^{7/4}}\right\}$$

$$= 0.176\sigma_R^2\left(\frac{\eta_x Q_l}{\eta_x+Q_l}\right)^{7/6}\left\{\Gamma(7/6) + \frac{1.802}{Q_l^{1/2}\left(\dfrac{\eta_x+Q_l}{\eta_x Q_l}\right)^{1/2}}\Gamma(5/3) - \frac{0.254}{Q_l^{7/12}\left(\dfrac{\eta_x+Q_l}{\eta_x Q_l}\right)^{7/12}}\Gamma(7/4)\right\}$$

$$= 0.176\sigma_R^2\left(\frac{\eta_x Q_l}{\eta_x+Q_l}\right)^{7/6}\left\{\Gamma(7/6) + 1.802\Gamma(5/3)\left(\frac{\eta_x}{\eta_x+Q_l}\right)^{1/2} - 0.254\Gamma(7/4)\left(\frac{\eta_x}{\eta_x+Q_l}\right)^{7/12}\right\}$$

$$= 0.176\sigma_R^2\left(\frac{\eta_x Q_l}{\eta_x+Q_l}\right)^{7/6}\left\{0.9277 + 1.802(0.9028)\left(\frac{\eta_x}{\eta_x+Q_l}\right)^{1/2} - 0.254(0.9191)\left(\frac{\eta_x}{\eta_x+Q_l}\right)^{7/12}\right\}$$

$$= 0.176\sigma_R^2\left(\frac{\eta_x Q_l}{\eta_x+Q_l}\right)^{7/6}\left\{0.9277 + 1.63\left(\frac{\eta_x}{\eta_x+Q_l}\right)^{1/2} - 0.234\left(\frac{\eta_x}{\eta_x+Q_l}\right)^{7/12}\right\}$$

$$= 0.176(0.9277)\sigma_R^2\left(\frac{\eta_x Q_l}{\eta_x+Q_l}\right)^{7/6}\left\{1 + 1.76\left(\frac{\eta_x}{\eta_x+Q_l}\right)^{1/2} - 0.25\left(\frac{\eta_x}{\eta_x+Q_l}\right)^{7/12}\right\}$$

$$= 0.16\sigma_R^2\left(\frac{\eta_x Q_l}{\eta_x+Q_l}\right)^{7/6}\left\{1 + 1.75\left(\frac{\eta_x}{\eta_x+Q_l}\right)^{1/2} - 0.25\left(\frac{\eta_x}{\eta_x+Q_l}\right)^{7/12}\right\}$$

$$= 0.16\sigma_R^2\left(\left(\frac{\eta_x Q_l}{\eta_x + Q_l}\right)\left\{1 + 1.75\left(\frac{\eta_x}{\eta_x + Q_l}\right)^{1/2} - 0.25\left(\frac{\eta_x}{\eta_x + Q_l}\right)^{7/12}\right\}^{6/7}\right)^{7/6}$$

$$= 0.16\sigma_R^2\mu_x^{7/6} \qquad \text{(N5)}$$

where $\mu_x = \left(\frac{\eta_x Q_l}{\eta_x + Q_l}\right)\left\{1 + 1.75\left(\frac{\eta_x}{\eta_x + Q_l}\right)^{1/2} - 0.25\left(\frac{\eta_x}{\eta_x + Q_l}\right)^{7/12}\right\}^{6/7}$

Small-Scale Log-Irradiance Variance

Using the "effective" atmospheric spectrum, reference equations 23 and 33, the small-scale log-irradiance variance for a plane wave in the presence of a finite inner scale is[8]

$$\sigma_{\ln y}^2(L) = 8\pi^2 k^2 \int_0^L\int_0^\infty \kappa 0.033 C_n^2 \kappa^{-11/3} G_y(\kappa)\left[1 - \cos\left(\frac{\kappa^2 z}{k}\right)\right] d\kappa dz \qquad \text{(N6)}$$

In moderate to strong irradiance fluctuations, $\kappa_y >> \sqrt{k/L}$, let

$$\sigma_{\ln y}^2(L) \cong 8\pi^2 k^2 \int_0^L\int_0^\infty \kappa 0.033 C_n^2 \kappa^{-11/3} G_y(\kappa) d\kappa dz \qquad \text{(N7)}$$

Substituting

$$\xi = z/L \;\rightarrow\; dz = Ld\xi$$

$$\eta = \frac{L\kappa^2}{k} \;\rightarrow\; d\kappa = \frac{k}{2L\kappa} d\eta$$

$$\sigma_{\ln y}^2(L) \cong 8\pi^2 k^2 \int_0^1\int_0^\infty \kappa 0.033 C_n^2 \kappa^{-11/3} G_y(\kappa) \frac{k}{2L\kappa} d\eta L d\xi$$

$$= 4\pi^2 k^3 \int_0^1\int_0^\infty 0.033 C_n^2 \kappa^{-11/3} G_y(\kappa) d\eta d\xi$$

$$= 1.30 k^3 C_n^2 \int_0^1\int_0^\infty \kappa^{-11/3}\left[\frac{\kappa^{11/3}}{\left(\kappa^2 + \kappa_y^2\right)^{11/6}}\right] d\eta d\xi$$

$$= 1.30 k^3 C_n^2 \int_0^\infty \left[\frac{1}{\left(\kappa^2 + \kappa_y^2\right)^{11/6}}\right] d\eta$$

$$=1.30\left(\frac{1.23}{1.23}\right)C_n^2\left(k^{7/6}k^{11/6}\right)\frac{L^{11/16}}{L^{11/16}}\int_0^\infty\left[\frac{1}{\left(\kappa^2+\kappa_y^2\right)^{11/6}}\right]d\eta$$

$$=1.06\sigma_R^2\int_0^\infty\left[\frac{k^{11/6}}{L^{11/16}\left(\kappa^2+\kappa_y^2\right)^{11/6}}\right]d\eta$$

$$=1.06\sigma_R^2\int_0^\infty\left(\frac{k}{L\kappa^2}+\frac{k}{L\kappa_y^2}\right)^{11/6}d\eta$$

$$=1.06\sigma_R^2\int_0^\infty(\eta+\eta_y)^{-11/6}d\eta$$

$$=1.06\sigma_R^2\left(\frac{-6}{5}\right)(\eta+\eta_y)^{-5/6}\Big|_{\eta=0}^{\eta=\infty}$$

$$=1.06\sigma_R^2\left(\frac{-6}{5}\right)[(\infty+\eta_y)^{-5/6}-(0+\eta_y)^{-5/6}]$$

$$=1.06\sigma_R^2\left(\frac{-6}{5}\right)\left[-(\eta_y)^{-5/6}\right]$$

$$=1.272\sigma_R^2\eta_y^{-5/6} \qquad \text{(N8)}$$

where

$$G_y(\kappa)=\frac{\kappa^{11/3}}{\left(\kappa^2+\kappa_y^2\right)^{11/6}}$$

$$\sigma_R^2=1.23C_n^2k^{7/6}L^{11/6}$$

Asymptotic Comparisons

For constants c_1, c_2, c_3, and c_4, assume the following functional form of η_x and η_y [see Appendix S].

$$\eta_x=\frac{1}{c_1+c_2L/k\rho_o^2}\cong\begin{cases}\dfrac{1}{c_1+c_2(0)}=\dfrac{1}{c_1}, & \dfrac{L}{k\rho_0^2}<<1\\[2ex] \dfrac{1}{0+c_2L/k\rho_o^2}=\dfrac{k\rho_o^2}{c_2L}, & \dfrac{L}{k\rho_0^2}>>1\end{cases} \qquad \text{(N9)}$$

$$\eta_y = c_3 + \frac{c_4 L}{k\rho_0^2} \cong \begin{cases} c_3 + c_4(0) = c_3, & \dfrac{L}{k\rho_0^2} << 1 \\ 0 + \dfrac{c_4 L}{k\rho_0^2} = \dfrac{c_4 L}{k\rho_0^2}, & \dfrac{L}{k\rho_0^2} >> 1 \end{cases} \tag{N10}$$

Weak Turbulence

The log irradiance variance, $\sigma_{\ln I}^2$, is defined under the Rytov approximation for weak turbulence by[8]

$$\sigma_{\ln I}^2 = 8\pi^2 k^2 \int_0^L \int_0^\infty \kappa \Phi_n(\kappa)\left[1 - \cos\left(\frac{\kappa^2 z}{k}\right)\right] d\kappa dz \tag{N11}$$

where $\Phi_n(\kappa) = 0.033 C_n^2 \kappa^{-11/3}$ (zero inner scale)

Substituting

$\xi = z/L \;\rightarrow\; dz = L d\xi$

$\eta = L\kappa^2 / k \;\rightarrow \kappa^2 = \eta k / L$,

$d\kappa = \frac{1}{2}\left(\frac{\eta k}{L}\right)^{-1/2}\left(\frac{k}{L}\right) d\eta = \frac{1}{2}\left(\frac{L}{\eta k}\right)^{1/2}\left(\frac{k}{L}\right) d\eta$

$\eta\xi = \left(\frac{z}{L}\right)\left(\frac{L\kappa^2}{k}\right) = \frac{\kappa^2 z}{k}$

Then

$$\begin{aligned} \sigma_{\ln I}^2(L) &= 8\pi^2 k^2 \int_0^1 \int_0^\infty (\eta k / L)^{1/2} \Phi_n(\kappa)[1 - \cos(\eta\xi)] \frac{1}{2}\left(\frac{L}{\eta k}\right)^{1/2}\left(\frac{k}{L}\right) d\eta L d\xi \\ &= \frac{8\pi^2 k^3}{2} \int_0^1 \int_0^\infty \Phi_n(\kappa)[1 - \cos(\eta\xi)] d\eta d\xi \\ &= \frac{8\pi^2 k^3}{2} \int_0^1 \int_0^\infty 0.033 C_n^2 \kappa^{-11/3} [1 - \cos(\eta\xi)] d\eta d\xi \end{aligned}$$

$$= \frac{8\pi^2(0.033C_n^2)k^3}{2}\int_0^1\int_0^\infty \kappa^{-11/3}\left[1-\cos(\eta\xi)\right]d\eta d\xi$$

$$= \frac{8\pi^2(0.033)}{2}\left(\frac{1.23}{1.23}\right)C_n^2\left(k^{7/6}k^{11/6}\right)\left(\frac{L^{11/6}}{L^{11/6}}\right)\int_0^1\int_0^\infty \kappa^{-11/3}\left[1-\cos(\eta\xi)\right]d\eta d\xi$$

$$= \frac{8\pi^2(0.033)}{(2)(1.23)}(1.23C_n^2k^{7/6}L^{11/6})\int_0^1\int_0^\infty\left(\frac{k^{11/6}}{L^{11/6}\kappa^{11/3}}\right)\left[1-\cos(\eta\xi)\right]d\eta d\xi$$

$$= 1.06\sigma_R^2\int_0^1\int_0^\infty\left(\frac{k}{L\kappa^2}\right)^{11/6}\left[1-\cos(\eta\xi)\right]d\eta d\xi$$

$$= 1.06\sigma_R^2\int_0^1\int_0^\infty \eta^{-11/6}\left[1-\cos(\eta\xi)\right]d\eta d\xi \qquad \text{(N12)}$$

where

$$\sigma_R^2 = 1.23C_n^2k^{7/6}L^{11/6}$$

$$\eta = \frac{k}{L\kappa^2}$$

Using the relationship $\sin^2(u) = \frac{1}{2}\left[1-\cos(2u)\right]$, the previous equation reduces to

$$\sigma_{\ln I}^2(L) = 2.12(\sigma_R^2)\int_0^1\int_0^\infty \eta^{-11/6}\sin^2\left(\frac{\eta\xi}{2}\right)d\eta d\xi \qquad \text{(N13)}$$

Recognizing[35]

$$\int_0^\infty x^{u-1}\sin^2 ax dx = \frac{(-)\Gamma(u)\cos\frac{u\pi}{2}}{2^{u+1}a^u} \qquad \text{(N14)}$$

where

u = -5/6

a= $\frac{\xi}{2}$

leads to

$$\sigma^2_{\ln I}(L) = 2.12\sigma^2_R \int_0^1 \frac{(-)\Gamma(-5/6)\cos\left(\frac{-5\pi}{12}\right)}{2^{1/6}\left(\frac{\xi}{2}\right)^{-5/6}} d\xi$$

$$= (-)2.12\Gamma(-5/6)\cos\left(\frac{-5\pi}{12}\right)(2^{-1/6})\left(\frac{1}{2}\right)^{5/6} \sigma^2_R \int_0^1 \xi^{5/6} d\xi$$

$$= (-)2.12\Gamma(-5/6)(2^{-1/6})\left(\frac{1}{2}\right)^{5/6} \sigma^2_R \left(\frac{6}{11}\right)$$

$$= (-)2.12(\text{-}6.680)(0.259)(0.89)(0.561)\left(\frac{6}{11}\right)\sigma^2_R$$

$$= \sigma^2_R \qquad \text{(N15)}$$

Assume in equation N5 that $l_0 \to 0$, $Q_l \to \infty$

$$\sigma^2_{\ln x} = 0.16\sigma^2_R \left(\frac{\eta_x Q_l}{\eta_x + Q_l}\right)^{7/6} \left\{1 + 1.75\left(\frac{\eta_x}{\eta_x + Q_l}\right)^{1/2} - 0.25\left(\frac{\eta_x}{\eta_x + Q_l}\right)^{7/12}\right\}$$

$$\cong 0.16\sigma^2_R \left(\frac{\eta_x Q_l}{Q_l}\right)^{7/6} (1 + 0 - 0)$$

$$\cong 0.16\sigma^2_R \eta_x^{7/6} \qquad \text{(N16)}$$

Under weak irradiance fluctuations[8], $\sigma^2_{\ln I} << 1$,

$$\sigma^2_{\ln I} = \sigma^2_{\ln x} + \sigma^2_{\ln y} \qquad \text{(N17)}$$

Equating equations N15 and N17, and using equations N8 and N16, results in

$$0.16\eta_x^{7/6} + 1.272\eta_y^{-5/6} \cong 1, \quad \sigma^2_R << 1, \quad l_0 \to 0 \qquad \text{(N18)}$$

Assuming $\eta_x \cong \eta_y$, then 2.61 is an approximate solution.

Plugging this into equations N9 and N10

$$\eta_x = \frac{1}{c_1 + c_2 L / k\rho_o^2} \cong \frac{1}{c_1 + c_2(0)} = \frac{1}{c_1} = 2.61$$

$$\to c_1 \cong 0.38 \qquad \text{(N19)}$$

$$\eta_y = c_3 + \frac{c_4 L}{k\rho_0^2} \cong c_3 + c_4(0) = c_3 = 2.61 \cong 3 \tag{N20}$$

Thus, c_1 and c_3 are now established using behavior of the scintillation index in the weak turbulence regime.

<u>Saturated Turbulence</u>

First, perform an asymptotic comparison for small-scale turbulent cell effects. Normalized irradiance[8] can be expressed as I = *xy* where *x* is associated with large-scale turbulent eddy affects, *y* is associated with small-scale eddy effects, *x* and *y* are statistically independent random quantities, and $<x>=<y>=1$. Given these conditions, $<I> = <x><y> = 1$. The second moment of irradiance takes the form[8]

$$<I^2> = <x^2><y^2> = (1+\sigma_x^2)(1+\sigma_y^2) \tag{N21}$$

where σ_x^2 and σ_y^2 are the large scale and small scale normalized variances of *x* and *y*, respectively. Based on equations 5 and N21,the implied scintillation index is

$$\begin{aligned}\sigma_I^2 &= \left(1+\sigma_x^2\right)\left(1+\sigma_y^2\right)-1 \\ &= \sigma_x^2 + \sigma_y^2 + \sigma_x^2\sigma_y^2\end{aligned} \tag{N22}$$

Per equation 18, the asymptotic behavior of the scintillation index in the saturation regime is described by $\sigma_I^2(L) = 1 + \frac{2.39}{\left(\sigma_R^2 Q_l^{7/6}\right)^{1/6}}$, $\sigma_R^2 Q_l^{7/6} >> 100$ which approaches an asymptotic limit of one. Therefore, in saturated turbulence

$$\sigma_I^2 = \sigma_x^2 + \sigma_y^2 + \sigma_x^2\sigma_y^2 \cong 1 \tag{N23}$$

In strong fluctuations, expect the large-scale scintillation terms to die out.

$$\sigma_I^2 \approx \sigma_y^2 \cong 1 \tag{N24}$$

The small scale irradiance is given by[8]

$$\sigma_y^2 = \exp\left(\sigma_{\ln y}^2\right) - 1 \tag{N25}$$

As a result, in strong fluctuations

$$\sigma_y^2 \cong 1 = \exp\left(\sigma_{\ln y}^2\right) - 1 \tag{N26}$$

$$\rightarrow \sigma_{\ln y}^2 \cong \ln 2 \tag{N27}$$

The plane wave spatial coherence radius, ρ_0, when C_n^2 is constant and $l_0 << \rho << L_0$ can be approximated by[1]

$$\rho = \left(1.46 C_n^2 k^2 L\right)^{-3/5} \quad \rightarrow C_n^2 = \frac{\rho_0^{-5/3}}{1.46 k^2 L} \tag{N28}$$

Plugging this into the Rytov Variance, $\sigma_R^2 = 1.23 C_n^2 k^{7/6} L^{11/6} = 1.23\left(\frac{\rho_0^{-5/3}}{1.46 k^2 L}\right) k^{7/6} L^{11/6}$,

$$\frac{L}{k\rho_0^2} = 1.22(\sigma_R^2)^{6/5} \tag{N29}$$

In saturated turbulence, equating equation N27 to equation N8 results in

$$\begin{aligned} \ln 2 &= 1.272\sigma_R^2 (\eta_y)^{-5/6} \\ &= 1.272\sigma_R^2 \left(\frac{c_4 L}{k\rho_0^2}\right)^{-5/6} \\ &= 1.272\sigma_R^2 c_4^{-5/6}\left[1.22(\sigma_R^2)^{6/5}\right]^{-5/6} \\ &= 1.272 c_4^{-5/6}(1.22)^{-5/6} \end{aligned}$$

$$\rightarrow c_4 \cong 1.7 \tag{N30}$$

where

$\eta_y = \dfrac{c_4 L}{k\rho_0^2}$ (equation N10)

$$\frac{L}{k\rho_0^2} = 1.22(\sigma_R^2)^{6/5} \quad \text{(equation N29)}$$

Now determine an asymptotic comparison for large-scale turbulent cell effects. For saturated optical turbulence[8]

$$\sigma_x^2 = \exp\left(\sigma_{\ln x}^2\right) - 1 \tag{N31}$$

$$\rightarrow \exp\left(\sigma_{\ln x}^2\right) = \sigma_x^2 + 1$$

Also[8]

$$\sigma_I^2 = \exp\left(\sigma_{\ln x}^2 + \sigma_{\ln y}^2\right) - 1 \tag{N32}$$

Therefore,

$$\begin{aligned} \sigma_I^2 &\cong \exp\left(\sigma_{\ln x}^2 + \ln 2\right) - 1 \\ &= 2\exp\left(\sigma_{\ln x}^2\right) - 1 \\ &= 2\left(\sigma_x^2 + 1\right) - 1 \\ &= 2\sigma_x^2 + 1 \end{aligned} \tag{N33}$$

Under weak fluctuation $\left(\sigma_x^2 << 1\right)$,

$$\sigma_x^2 = \exp\left(\sigma_{\ln x}^2\right) - 1 \cong \sigma_{\ln x}^2 \tag{N34}$$

Note in equation N16 that $\mu_x \rightarrow \eta_x$ as $l_0 \rightarrow 0$ (i.e., $l_0 \rightarrow 0$ and $L_0 \rightarrow \infty$). Therefore, for determining c_2, assume

$$\sigma_{\ln x}^2 = 0.16\sigma_R^2 \mu_x^{\ 7/6} \cong 0.16\sigma_R^2 \eta_x^{\ 7/6} \tag{N35}$$

Then

$$\begin{aligned} \sigma_I^2 &\cong 2\sigma_x^2 + 1 \\ &\cong 2(0.16\sigma_R^2 \eta_x^{\ 7/6}) + 1 \\ &= 0.32\sigma_R^2 \eta_x^{\ 7/6} + 1 \end{aligned} \tag{N36}$$

For plane wave von Karman spectrum, with inner scale effects, the plane wave spatial coherence radius is[1] $\rho_0 = (1.87 C_n^2 k^2 L l_0^{-1/3})^{-1/2}$

As a result:

$$
\begin{aligned}
L/k\rho_0^2 &= \frac{L}{k}\rho_0^{-2} \\
&= \frac{L}{k}\left[\left(1.87C_n^2k^2Ll_0^{-1/3}\right)^{-1/2}\right]^{-2} \\
&= 1.87\left(\frac{1.23k^{1/6}}{1.23k^{1/6}}\right)C_n^2kL^2l_0^{-1/3} \\
&= 1.52\sigma_R^2\left(\frac{10.89L}{10.89kl_0^2}\right)^{1/6} \\
&= \frac{1.52}{(10.89)^{1/6}}\sigma_R^2\left(\frac{10.89L}{kl_0^2}\right)^{1/6} \\
&= 1.02\sigma_R^2Q_l^{1/6} \qquad \text{(N37)}
\end{aligned}
$$

where

$$\sigma_R^2 = 1.23C_n^2k^{7/6}L^{11/6}$$

$$Q_l = \frac{L\kappa_l^2}{k} = \frac{L(3.3)^2}{kl_0^2} = \frac{10.89L}{kl_0^2}$$

Reference equation N9, assuming $\frac{L}{k\rho_0^2} >> 1$

$$\eta_x \cong \frac{1}{c_2 1.02\sigma_R^2Q_l^{1/6}} \qquad \text{(N38)}$$

and

$$
\begin{aligned}
\sigma_I^2 &\cong 0.32\sigma_R^2\left(\frac{1}{c_2 1.02\sigma_R^2Q_l^{1/6}}\right)^{7/6} + 1 \\
&= 1 + 0.32\sigma_R^2\left(\frac{1}{c_2}\right)^{7/6}\left(\frac{1}{1.02}\right)^{7/6}\left(\frac{1}{\sigma_R^2}\right)^{7/6}\left(\frac{1}{Q_l^{1/6}}\right)^{7/6} \\
&= 1 + 0.31\sigma_R^2\left(\frac{1}{c_2}\right)^{7/6}\left(\frac{1}{\sigma_R^{7/3}}\right)\left(\frac{1}{Q_l^{7/36}}\right) \\
&= 1 + 0.31\left(\frac{1}{c_2}\right)^{7/6}\left(\frac{1}{\sigma_R^{1/3}}\right)\left(\frac{1}{Q_l^{7/36}}\right) \qquad \text{(N39)}
\end{aligned}
$$

Compare this to the existing scintillation expression (equation 18) for a plane wave in saturated turbulence.

$$\sigma_I^2 = 1 + 0.31\left(\frac{1}{c_2}\right)^{7/6}\left(\frac{1}{\sigma_R^{1/3}}\right)\left(\frac{1}{Q_l^{7/36}}\right) = 1 + \frac{2.39}{\left(\sigma_R^2 Q_l^{7/6}\right)^{1/6}}$$

$$\rightarrow\ 0.31\left(\frac{1}{c_2}\right)^{7/6}\left(\frac{1}{\sigma_R^{1/3}}\right)\left(\frac{1}{Q_l^{7/36}}\right) = \frac{2.39}{(\sigma_R^2 Q_l^{7/6})^{1/6}}$$

$$\rightarrow\ c_2 = 0.17 \qquad \text{(N40)}$$

Thus, c_2 and c_4 are now established using behavior of the scintillation index in the saturated turbulence regime.

<u>"All" Turbulence Regimes</u>

Applying constants c_1, c_2, c_3, and c_4 results in the following expressions for predicting scintillation behavior in all turbulence regimes (including inner scale effects)

$$\begin{aligned}\eta_x &= \frac{1}{c_1 + c_2 L / k\rho_o^2} \\ &\cong \frac{1}{0.38 + 0.17 L / k\rho_o^2} \\ &= \frac{1}{0.38 + 0.17(1.02\sigma_R^2 Q_l^{1/6})} \\ &= \frac{2.61}{1 + 0.45\sigma_R^2 Q_l^{1/6}} \qquad \text{(N41)}\end{aligned}$$

and

$$\sigma_{\ln x}^2 \cong 0.16\sigma_R^2\left(\frac{\eta_x Q_l}{\eta_x + Q_l}\right)^{7/6}\left[1 + 1.76\left(\frac{\eta_x}{\eta_x + Q_l}\right)^{1/2} - 0.25\left(\frac{\eta_x}{\eta_x + Q_l}\right)^{7/12}\right]$$

$$= 0.16\sigma_R^2\left(\frac{\left(\frac{2.61}{1+0.45\sigma_R^2 Q_l^{1/6}}\right)Q_l}{\left(\frac{2.61}{1+0.45\sigma_R^2 Q_l^{1/6}}\right)+Q_l}\right)^{7/6}\left[1 + 1.76\left(\frac{\left(\frac{2.61}{1+0.45\sigma_R^2 Q_l^{1/6}}\right)}{\left(\frac{2.61}{1+0.45\sigma_R^2 Q_l^{1/6}}\right)+Q_l}\right)^{1/2} - 0.25\left(\frac{\left(\frac{2.61}{1+0.45\sigma_R^2 Q_l^{1/6}}\right)}{\left(\frac{2.61}{1+0.45\sigma_R^2 Q_l^{1/6}}\right)+Q_l}\right)^{7/12}\right]$$

$$= 0.16\sigma_R^2 \left(\frac{2.61 Q_l}{2.61 + Q_l \left(1 + 0.45\sigma_R^2 Q_l^{1/6}\right)} \right)^{7/6} \left[1 + 1.76 \left(\frac{2.61}{2.61 + Q_l \left(1 + 0.45\sigma_R^2 Q_l^{1/6}\right)} \right)^{1/2} - 0.25 \left(\frac{2.61}{2.61 + Q_l \left(1 + 0.45\sigma_R^2 Q_l^{1/6}\right)} \right)^{7/12} \right]$$

$$= 0.16\sigma_R^2 \left(\frac{2.61 Q_l}{2.61 + Q_l + 0.45\sigma_R^2 Q_l^{7/6}} \right)^{7/6} \left[1 + 1.76 \left(\frac{2.61}{2.61 + Q_l + 0.45\sigma_R^2 Q_l^{7/6}} \right)^{1/2} - 0.25 \left(\frac{2.61}{2.61 + Q_l + 0.45\sigma_R^2 Q_l^{7/6}} \right)^{7/12} \right] \quad \text{(N42)}$$

From equations N10 and N29

$$\begin{aligned} \eta_y &= c_3 + \frac{c_4 L}{k\rho_0^2} \\ &= 3 + \frac{1.7L}{k\rho_0^2} \\ &= 3 + 1.7\left[1.22(\sigma_R^2)^{6/5}\right] \\ &= 3 + 2.07(\sigma_R^2)^{6/5} \\ &= 3\left(1 + 0.69\sigma_R^{12/5}\right) \end{aligned} \quad \text{(N43)}$$

$$\begin{aligned} \sigma_{\ln y}^2(L) &= 1.272\sigma_R^2 \eta_y^{-5/6} \\ &= 1.272\sigma_R^2 \left[3\left(1 + 0.69\sigma_R^{12/5}\right)\right]^{-5/6} \\ &\cong \frac{0.509\sigma_R^2}{\left(1 + 0.69\sigma_R^{12/5}\right)^{5/6}} \end{aligned} \quad \text{(N44)}$$

To apply inner-scale effects, let (ref. equation 10, plane wave in weak turbulence)

$$\sigma_I^2 = \sigma_p^2 = 3.86\sigma_R^2 \left[\left(1 + \frac{1}{Q_l^2}\right)^{11/12} \sin\left(\frac{11}{6}\tan^{-1} Q_l\right) + \frac{1.507}{\left(1 + Q_l^2\right)^{1/4}} \sin\left(\frac{4}{3}\tan^{-1} Q_l\right) - \frac{0.273}{\left(1 + Q_l^2\right)^{7/24}} \sin\left(\frac{5}{4}\tan^{-1} Q_l\right) - 3.50 Q_l^{-5/6} \right] \quad \text{(N45)}$$

impose

$$\sigma_{\ln y}^2 \cong \frac{0.509\sigma_R^2}{(1 + 0.69\sigma_R^{12/5})^{5/6}} \cong 0.51\sigma_R^2 = 0.51\sigma_p^2 \quad \text{(N46)}$$

Recall that $\sigma_{\ln y}^2(L) = 1.272\sigma_R^2 \eta_y^{-5/6}$. Then,

$$\eta_y \cong \left(\frac{1.272\sigma_R^2}{0.51\sigma_p^2} \right)^{6/5} \cong 3\left(\frac{\sigma_R^2}{\sigma_p^2} \right)^{6/5} \quad \text{(N47)}$$

Reference equation N44, under strong fluctuations, inner-scale effects tend to diminish such that

$$\eta_y = 3(0.69)\sigma_R^{12/5}$$

$$= 2.07\left(\sigma_R^2\right)^{6/5} \qquad \text{(N48)}$$

As a result, for all fluctuations

$$\eta_y \cong 3\left(\frac{\sigma_R^2}{\sigma_p^2}\right)^{6/5} + 2.07\left(\sigma_R^2\right)^{6/5}$$

$$= 3\left(\frac{\sigma_R}{\sigma_p}\right)^{12/5}(1 + 0.69\sigma_p^{12/5}) \qquad \text{(N49)}$$

As a test of this expression, notice that under strong fluctuations

$\eta_y \cong 3\left(\frac{\sigma_R^2}{\sigma_p^2}\right)^{6/5}\left(0.69\sigma_p^{12/5}\right) = 2.07\left(\sigma_R^2\right)^{6/5}$ is in agreement with equation N48.

The small-scale log-irradiance variance becomes

$$\sigma_{\ln y}^2(L) \cong 1.272\sigma_R^2\left[3\left(\frac{\sigma_R}{\sigma_P}\right)^{12/5}\left(1 + 0.69\sigma_P^{12/5}\right)\right]^{-5/6}$$

$$= \frac{0.51\sigma_P^2}{\left(1 + 0.69\sigma_P^{12/5}\right)^{5/6}} \qquad \text{(N49)}$$

Recall from equation N33 that $\sigma_I^2 = \exp\left(\sigma_{\ln x}^2 + \sigma_{\ln y}^2\right) - 1$

Therefore, for a plane wave in moderate turbulence using an "effective" modified (bump) spectrum

$$\sigma_I^2 = \exp\left(\sigma_{\ln x}^2 + \frac{0.51\sigma_p^2}{\left(1 + 0.69\sigma_p^{12/5}\right)^{5/6}}\right) - 1 \qquad \text{(N50)}$$

where

$$\sigma_{\ln x}^2 = 0.16\sigma_R^2\left(\frac{2.61Q_l}{2.61 + Q_l + 0.45\sigma_R^2 Q_l^{7/6}}\right)^{7/6}\left[1 + 1.76\left(\frac{2.61}{2.61 + Q_l + 0.45\sigma_R^2 Q_l^{7/6}}\right)^{1/2} - 0.25\left(\frac{2.61}{2.61 + Q_l + 0.45\sigma_R^2 Q_l^{7/6}}\right)^{7/12}\right]$$

$$\sigma_p^2 = 3.86\sigma_R^2\left\{(1 + \frac{1}{Q_l^2})^{11/12}\left[\sin[\frac{11}{6}\tan^{-1} Q_l] + \frac{1.507}{(1 + Q_l^2)^{1/4}}\sin[\frac{4}{3}\tan^{-1} Q_l] - \frac{0.273}{(1 + Q_l^2)^{7/24}}\sin[\frac{5}{4}\tan^{-1} Q_l]\right] - 3.50Q_l^{-5/6}\right\}$$

APPENDIX O: MODERATE TURBULENCE, KOLMOGOROV, SPHERICAL WAVE

Large-Scale Log-Irradiance Variance

Using the "effective" atmospheric spectrum (equation 23), the large-scale log-irradiance variance for a spherical wave in the presence of a finite inner scale is[8]

$$\sigma^2_{\ln x} = 8\pi^2 k^2 \int_0^L \int_0^\infty \kappa 0.033 C_n^2 \kappa^{-11/3} G_x(\kappa) \left\{ 1 - \cos\left[\frac{\kappa^2}{k} z \left(1 - \frac{z}{L} \right) \right] \right\} d\kappa dz \qquad \text{(O1)}$$

Let $1 - \cos\left(\frac{\kappa^2 z}{k} \right) \cong \frac{1}{2}\left(\frac{\kappa^2 z}{k} \right)$ (its Taylor expansion estimate). The approximation is valid for the large-scale log-irradiance variance because filter function $G_x(\kappa)$ eliminates high spatial frequency contributions in moderate-to-strong fluctuations. Consequently

$$\begin{aligned}
\sigma^2_{\ln x} &= 8\pi^2 k^2 \int_0^L \int_0^\infty \kappa 0.033 C_n^2 \kappa^{-11/3} G_x(\kappa) \left(1 - \left\{ 1 - \frac{1}{2}\left[\frac{\kappa^2}{k} z \left(1 - \frac{z}{L} \right) \right]^2 \right\} - \right) d\kappa d\xi \\
&= 8\pi^2 k^2 \int_0^L \int_0^\infty \kappa 0.033 C_n^2 \kappa^{-11/3} G_x(\kappa) \frac{1}{2}\left[\frac{\kappa^2}{k} z (1 - \frac{z}{L}) \right]^2 d\kappa d\xi \\
&= 4\pi^2 \int_0^L \int_0^\infty \kappa 0.033 C_n^2 \kappa^{-11/3} G_x(\kappa) \kappa^4 z^2 \left(1 - \frac{z}{L} \right)^2 d\kappa dz \qquad \text{(O2)}
\end{aligned}$$

Substituting

$$\xi = z/L \;\rightarrow\; dz = L d\xi$$

$$\eta = \frac{L\kappa^2}{k} \;\rightarrow\; d\kappa = \frac{k}{2L\kappa} d\eta$$

$$\begin{aligned}
\sigma^2_{\ln x} &= 4\pi^2 \int_0^1 \int_0^\infty \kappa 0.033 C_n^2 \kappa^{-11/3} G_x(\kappa) \kappa^4 L^2 \xi^2 \left(1 - \frac{L\xi}{L} \right)^2 \frac{k}{2L\kappa} d\eta L d\xi \\
&= 2\pi^2 k L^2 \int_0^1 \int_0^\infty 0.033 C_n^2 \kappa^{-11/3} G_x(\kappa) \kappa^4 \xi^2 (1 - \xi)^2 d\eta d\xi \\
&= 0.651 k L^2 C_n^2 \int_0^1 \int_0^\infty \kappa^{1/3} G_x(\kappa) \xi^2 (1 - \xi)^2 d\eta d\xi \\
&= 0.651 k L^2 C_n^2 \int_0^1 \xi^2 (1 - \xi)^2 dz \int_0^\infty \kappa^{1/3} \exp\left(\frac{-\kappa^2}{\kappa_x^2} \right) d\eta \\
&= 0.651 k^{7/6} L^{11/6} C_n^2 \int_0^1 \xi^2 (1 - \xi)^2 d\xi \int_0^\infty \frac{L^{1/6}}{k^{1/6}} \kappa^{1/3} \exp\left(\frac{-Lk\kappa^2}{Lk\kappa_x^2} \right) d\eta
\end{aligned}$$

$$= 0.651k^{7/6}L^{11/6}C_n^2\int_0^1 \xi^2(1-\xi)^2\,d\xi\int_0^\infty\left(\frac{L\kappa^2}{k}\right)^{1/6}\exp\exp\left(\frac{-L\kappa^2}{k}\right)\left(\frac{k}{L\kappa_x^2}\right)d\eta$$

$$= 0.651k^{7/6}L^{11/6}C_n^2\int_0^1 \xi^2(1-\xi)^2\,d\xi\int_0^\infty \eta^{1/6}\exp\left(\frac{-\eta}{\eta_x}\right)dn$$

$$= 0.651\left(\frac{1.23}{1.23}\right)k^{7/6}L^{11/6}C_n^2\int_0^1 \xi^2(1-\xi)^2\,d\xi\int_0^\infty \eta^{1/6}\exp\left(\frac{-\eta}{\eta_x}\right)dn$$

$$= 0.539\sigma_R^2\int_0^1 \xi^2(1-\xi)^2\,d\xi\int_0^\infty \eta^{1/6}\exp\left(\frac{-\eta}{\eta_x}\right)dn$$

$$= 0.539\sigma_R^2\int_0^1 (\xi-\xi^2)^2\,d\xi\int_0^\infty \eta^{1/6}\exp\left(\frac{-\eta}{\eta_x}\right)dn \qquad \text{(O3)}$$

where

$$G_x(\kappa) = f(\kappa d_0)\exp\left(-\kappa^2/\kappa_x^2\right) = \exp\left(-\kappa^2/\kappa_m^2\right)$$

$$\sigma_R^2 = 1.23C_n^2k^{7/6}L^{11/6}$$

Using a binomial series expansion $(x+y)^n \cong x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2$,

where $x=\xi$, $y=-\xi^2$, $n=2$, and $\left(\xi-\xi^2\right)^2 \cong \xi^2 - 2\xi^3 + \xi^4$, results in

$$\sigma_{\ln x}^2(L) \cong 0.53\sigma_R^2\int_0^1\left(\xi^2 - 2\xi^3 + \xi^4\right)d\xi\int_0^\infty \eta^{1/6}\exp\left(\frac{-\eta}{\eta_x}\right)d\eta$$

$$= 0.53\sigma_R^2\left(\frac{1}{3}-\frac{2}{4}+\frac{1}{5}\right)\int_0^\infty \eta^{1/6}\exp\left(\frac{-\eta}{\eta_x}\right)d\eta$$

$$= 0.53\sigma_R^2\left(\frac{20}{60}-\frac{30}{60}+\frac{12}{60}\right)\int_0^\infty \eta^{1/6}\exp\left(\frac{-\eta}{\eta_x}\right)d\eta$$

$$= 0.53\sigma_R^2\left(\frac{1}{30}\right)\int_0^\infty \eta^{1/6}\exp\left(\frac{-\eta}{\eta_x}\right)d\eta$$

$$= 0.018\sigma_R^2\int_0^\infty \eta^{1/6}\exp\left(\frac{-\eta}{\eta_x}\right)d\eta \qquad \text{(O4)}$$

Recognizing[1]

$$\int_0^\infty e^{-s\eta}\eta^{x-1}d\eta = \frac{\Gamma(x)}{s^x} \tag{O5}$$

where

$s = \dfrac{1}{\eta_x}$, x-1=1/6 $\rightarrow$ x=7/6

x-1=1/6 $\rightarrow$ x=7/6

Equation O4 can be written as

$$\begin{aligned}\sigma^2_{\ln x}(L) &= 0.018\sigma_R^2 \frac{\Gamma(7/6)}{\left(\dfrac{1}{\eta_x}\right)^{7/6}} \\ &= 0.018\sigma_R^2\eta_x^{7/6}\Gamma(7/6) \\ &= 0.018\sigma_R^2\eta_x^{7/6}(0.9277) \\ &= 0.016\sigma_R^2\eta_x^{7/6}\end{aligned} \tag{O6}$$

Small-scale log-irradiance variance

Using the "effective" atmospheric spectrum (equation 23), the small-scale log-irradiance variance for a spherical wave in the presence of a finite inner scale is[8]

$$\sigma^2_{\ln y} = 8\pi^2k^2\int_0^L\int_0^\infty \kappa 0.033C_n^2\kappa^{-11/3}G_y(\kappa)\left\{1-\cos\left[\frac{\kappa^2}{k}z\left(1-\frac{z}{L}\right)\right]\right\}d\kappa dz \tag{O7}$$

In moderate-to-strong irradiance fluctuations, $\kappa_y >> \sqrt{k/L}$, let

$$\sigma^2_{\ln y} \cong 8\pi^2k^2\int_0^L\int_0^\infty \kappa 0.033C_n^2\kappa^{-11/3}G_y(\kappa)d\kappa dz \tag{O8}$$

Substituting

$\xi = z/L \rightarrow dz = Ld\xi$

$\eta = \dfrac{L\kappa^2}{k} \rightarrow d\kappa = \dfrac{k}{2L\kappa}d\eta$

$$
\begin{aligned}
\sigma_{\ln y}^2(L) &\cong 8\pi^2 k^2 \int_0^1\int_0^\infty \kappa 0.033 C_n^2 \kappa^{-11/3} G_y(\kappa)\frac{k}{2L\kappa}\, d\eta L d\xi \\
&= 4\pi^2 k^3 \int_0^1\int_0^\infty 0.033 C_n^2 \kappa^{-11/3} G_y(\kappa) d\eta d\xi \\
&= 1.30 k^3 C_n^2 \int_0^1\int_0^\infty \kappa^{-11/3}\left[\frac{\kappa^{11/3}}{\left(\kappa^2+\kappa_y^2\right)^{11/6}}\right] d\eta d\xi \\
&= 1.30 k^3 C_n^2 \int_0^\infty \left[\frac{1}{\left(\kappa^2+\kappa_y^2\right)^{11/6}}\right] d\eta \\
&= 1.30\left(\frac{1.23}{1.23}\right) C_n^2 \left(k^{7/6} k^{11/6}\right)\frac{L^{11/16}}{L^{11/16}} \int_0^\infty \left[\frac{1}{\left(\kappa^2+\kappa_y^2\right)^{11/6}}\right] d\eta \\
&= 1.06\sigma_R^2 \int_0^\infty \left[\frac{k^{11/6}}{L^{11/16}\left(\kappa^2+\kappa_y^2\right)^{11/6}}\right] d\eta \\
&= 1.06\sigma_R^2 \int_0^\infty \left(\frac{k}{L\kappa^2}+\frac{k}{L\kappa_y^2}\right)^{11/6} d\eta \\
&= 1.06\sigma_R^2 \int_0^\infty (\eta+\eta_y)^{-11/6} d\eta \\
&= 1.06\sigma_R^2 \left(\frac{-6}{5}\right)(\eta+\eta_y)^{-5/6}\Big|_{\eta=0}^{\eta=\infty} \\
&= 1.06\sigma_R^2 \left(\frac{-6}{5}\right)[(\infty+\eta_y)^{-5/6}-(0+\eta_y)^{-5/6}] \\
&= 1.06\sigma_R^2 \left(\frac{-6}{5}\right)\left[-(\eta_y)^{-5/6}\right] \\
&= 1.272\sigma_R^2 \eta_y^{-5/6} \qquad \text{(O9)}
\end{aligned}
$$

where

$$
G_y(\kappa)=\frac{\kappa^{11/3}}{\left(\kappa^2+\kappa_y^2\right)^{11/6}}
$$

$$
\sigma_R^2 = 1.23 C_n^2 k^{7/6} L^{11/6}
$$

<u>Asymptotic Comparisons</u>

For constants c_1, c_2, c_3, and c_4, assume the following functional form of η_x and η_y [see Appendix S].

$$\eta_x = \frac{1}{c_1 + c_2 L / k\rho_o^2} \cong \begin{cases} \dfrac{1}{c_1 + c_2(0)} = \dfrac{1}{c_1}, & \dfrac{L}{k\rho_0^2} << 1 \\ \dfrac{1}{0 + c_2 L / k\rho_o^2} = \dfrac{k\rho_o^2}{c_2 L}, & \dfrac{L}{k\rho_0^2} >> 1 \end{cases} \tag{O10}$$

$$\eta_y = c_3 + \frac{c_4 L}{k\rho_0^2} \cong \begin{cases} c_3 + c_4(0) = c_3, & \dfrac{L}{k\rho_0^2} << 1 \\ 0 + \dfrac{c_4 L}{k\rho_0^2} = \dfrac{c_4 L}{k\rho_0^2}, & \dfrac{L}{k\rho_0^2} >> 1 \end{cases} \tag{O11}$$

Weak Turbulence

Reference equation 12 and appendix D, for a spherical wave in weak turbulence using the Kolmogorov spectrum

$$\sigma_I^2(L) = 0.4\sigma_R^2 \tag{O12}$$

Weak turbulence, assuming $\sigma_I^2 << 1$, results in

$$\sigma_I^2 = \exp\left(\sigma_{\ln I}^2\right) - 1 \cong \sigma_{\ln I}^2 \tag{O13}$$

Under weak irradiance fluctuations[8], $\sigma_{\ln I}^2 << 1$

$$\sigma_{\ln I}^2 = \sigma_{\ln x}^2 + \sigma_{\ln y}^2 \tag{O14}$$

Equating equations O12 through O13 and using equations O6 and O9

$$0.016\eta_x^{7/6} + 1.272\eta_y^{-5/6} \cong 0.4 \tag{O15}$$

Assuming $\eta_x \cong \eta_y$, then $\eta_x = \eta_y = 8$ is an approximate solution

Plugging this into equations O10 and O11

$$\eta_x = \frac{1}{c_1 + c_2 L / k\rho_o^2} \cong \frac{1}{c_1 + c_2(0)} = \frac{1}{c_1} = 8$$

$$\rightarrow c_1 \cong 1/8 \tag{O16}$$

$$\eta_y = c_3 + \frac{c_4 L}{k\rho_0^2} \cong c_3 + c_4(0) = c_3 = 8 \tag{O17}$$

Thus, c_1 and c_3 are now established using behavior of the scintillation index in the weak turbulence regime.

<u>Saturated Turbulence</u>

First, perform an asymptotic comparison for small-scale turbulent cell effects. Normalized irradiance[8] can be expressed as I = *xy* where *x* is associated with large-scale turbulent eddy affects, *y* is associated with small-scale eddy effects, *x* and *y* are statistically independent random quantities, and $\langle x\rangle=\langle y\rangle=1$. Given these conditions, $\langle I\rangle = \langle x\rangle\langle y\rangle = 1$. The second moment of irradiance takes the form[8]

$$\langle I^2\rangle = \langle x^2\rangle\langle y^2\rangle = (1+\sigma_x^2)(1+\sigma_y^2) \tag{O18}$$

where σ_x^2 and σ_y^2 are the large scale and small scale normalized variances of *x* and *y*, respectively. Based on equations 5 and O19,the implied scintillation index is

$$\begin{aligned}\sigma_I^2 &= \left(1+\sigma_x^2\right)\left(1+\sigma_y^2\right) - 1 \\ &= \sigma_x^2 + \sigma_y^2 + \sigma_x^2\sigma_y^2\end{aligned} \tag{O19}$$

Per equation 19, the asymptotic behavior of the scintillation index in the saturation regime is described by $\sigma_I^2(L) = 1 + \frac{2.73}{\left(\sigma_R^2\right)^{2/5}}$ which approaches an asymptotic limit of one. Therefore, in saturated turbulence

$$\sigma_I^2 = \sigma_x^2 + \sigma_y^2 + \sigma_x^2\sigma_y^2 \cong 1 \tag{O20}$$

In strong fluctuations, expect the large-scale scintillation terms to die out.

$$\sigma_I^2 \approx \sigma_y^2 \cong 1 \tag{O21}$$

The small scale irradiance is given by[8]

$$\sigma_y^2 = \exp\left(\sigma_{\ln y}^2\right) - 1 \tag{O22}$$

As a result, in strong fluctuations

$$\sigma_y^2 \cong 1 = \exp\left(\sigma_{\ln y}^2\right) - 1 \tag{O23}$$

$$\rightarrow \sigma_{\ln y}^2 \cong \ln 2 \tag{O24}$$

The spherical wave spatial coherence radius, ρ_0, when C_n^2 is constant and $l_0 << \rho << L_0$ can be approximated by[1]

$$\rho_0 = \left(0.55 C_n^2 k^2 L\right)^{-3/5} \rightarrow C_n^2 = \frac{\rho_0^{-5/3}}{0.55 k^2 L} \tag{O25}$$

Plugging this into the Rytov Variance, $\sigma_R^2 = 1.23 C_n^2 k^{7/6} L^{11/6}$ results in

$$\sigma_R^2 = 1.23\left(\frac{\rho_0^{-5/3}}{1.46 k^2 L}\right) k^{7/6} L^{11/6}$$

$$\rightarrow \frac{L}{k\rho_0^2} = 0.38(\sigma_R^2)^{6/5} \tag{O26}$$

In saturated turbulence, equating equation O24 to equation O9 results in

$$\begin{aligned}
\ln 2 &= 1.272\sigma_R^2(\eta_y)^{-5/6} \\
&= 1.272\sigma_R^2\left(\frac{c_4 L}{k\rho_0^2}\right)^{-5/6} \\
&= 1.272\sigma_R^2 c_4^{-5/6}\left[0.38(\sigma_R^2)^{6/5}\right]^{-5/6} \\
&= 1.272 c_4^{-5/6}(0.38)^{-5/6}
\end{aligned}$$

$$\rightarrow c_4 \cong 5.45 \tag{O27}$$

where

$$\eta_y = \frac{c_4 L}{k\rho_0^2} \quad \text{(equation O11)}$$

$$\frac{L}{k\rho_0^2} = 0.38(\sigma_R^2)^{6/5} \quad \text{(equation O26)}$$

Now determine an asymptotic comparison for large-scale turbulent cell effects. For saturated optical turbulence[8]

$$\sigma_x^2 = \exp\left(\sigma_{\ln x}^2\right) - 1 \tag{O28}$$

$$\rightarrow \exp\left(\sigma_{\ln x}^2\right) = \sigma_x^2 + 1$$

Also[8]

$$\sigma_I^2 = \exp\left(\sigma_{\ln x}^2 + \sigma_{\ln y}^2\right) - 1 \tag{O29}$$

Therefore,

$$\begin{aligned} \sigma_I^2 &\cong \exp\left(\sigma_{\ln x}^2 + \ln 2\right) - 1 \\ &= 2\exp\left(\sigma_{\ln x}^2\right) - 1 \\ &= 2\left(\sigma_x^2 + 1\right) - 1 \\ &= 2\sigma_x^2 + 1 \end{aligned} \tag{O30}$$

Under weak fluctuation $\left(\sigma_x^2 << 1\right)$,

$$\sigma_x^2 = \exp\left(\sigma_{\ln x}^2\right) - 1 \cong \sigma_{\ln x}^2 \tag{O31}$$

Equating equation O30 to O31 and using equations O6 and O29,

$$\begin{aligned} \sigma_I^2 &\cong 2\sigma_x^2 + 1 \\ &= 2\left(0.016\sigma_R^2 \eta_x{}^{7/6}\right) + 1 \\ &= 0.032\sigma_R^2 \left(\frac{k\rho_o^2}{c_2 L}\right)^{7/6} + 1 \end{aligned}$$

$$= 0.032\sigma_R^2\left(\frac{1}{c_2 0.38(\sigma_R^2)^{6/5}}\right)^{7/6} + 1$$

$$= 1 + 0.093\left(\frac{1}{\sigma_R^{4/5}}\right)\left(\frac{1}{c_2}\right)^{7/6} \qquad \text{(O32)}$$

where

$\eta_x \cong \dfrac{k\rho_o^2}{c_2 L}$ (equation O10)

$L / k\rho_0^2 = 0.38\left(\sigma_R^2\right)^{6/5}$ (equation O26)

Compare this to the existing scintillation expression (equation 19) for a spherical wave in saturated turbulence.

$$1 + 0.093\left(\frac{1}{\sigma_R^{4/5}}\right)\left(\frac{1}{c_2}\right)^{7/6} = 1 + \frac{2.73}{\sigma_R^{4/5}}$$

$$\rightarrow 0.093\left(\frac{1}{\sigma_R^{4/5}}\right)\left(\frac{1}{c_2}\right)^{7/6} = \left(\frac{1}{\sigma_R^{4/5}}\right)2.73$$

$$\rightarrow c_2 = 0.055 \qquad \text{(O33)}$$

Thus, c_2 and c_4 are now established using behavior of the scintillation index in the saturated turbulence regime.

“All” Turbulence Regimes

Applying constants c_1, c_2, c_3, and c_4 results in the following expressions for predicting scintillation behavior in all turbulence regimes (including inner scale effects)

$$\sigma_{\ln x}^2 \cong 0.016\sigma_R^2 \eta_x^{7/6}$$

$$= 0.016\sigma_R^2\left(\frac{1}{1/8 + 0.055L / k\rho_o^2}\right)^{7/6}$$

$$= 0.016\sigma_R^2\left(\frac{8}{1 + 0.441L / k\rho_o^2}\right)^{7/6}$$

$$= \frac{0.17\sigma_R^2}{\left(1 + 0.441L / k\rho_o^2\right)^{7/6}}$$

$$= \frac{0.17\sigma_R^2}{\left\{1 + 0.441\left[0.38\left(\sigma_R^2\right)^{6/5}\right]\right\}^{7/6}}$$

$$= \frac{0.17\sigma_R^2}{(1 + 0.167\sigma_R^{12/5})^{7/6}}$$

$$\cong \begin{cases} 0.17\sigma_R^2, & \sigma_R^2 << 1 \\ \dfrac{0.17\sigma_R^2}{(0.167\sigma_R^{12/5})^{7/6}} = \dfrac{1.37}{\sigma_R^{4/5}}, & \sigma_R^2 >> 1 \end{cases} \qquad \text{(O34)}$$

and

$$\sigma_{\ln y}^2(L) \cong 1.272\sigma_R^2 \eta_y^{-5/6}$$

$$= 1.272\sigma_R^2 \left(c_3 + \frac{c_4 L}{k\rho_0^2} \right)^{-5/6}$$

$$= 1.272\sigma_R^2 \left[8 + 5.45(0.38)\left(\sigma_R^2\right)^{6/5}\right]^{-5/6}$$

$$= \frac{1.272\sigma_R^2}{\left(8 + 2.07\sigma_R^{12/5}\right)^{5/6}}$$

$$= \frac{1.272\sigma_R^2}{8^{5/6}\left(1 + 0.259\sigma_R^{12/5}\right)^{5/6}}$$

$$= \frac{0.225\sigma_R^2}{\left(1 + 0.259\sigma_R^{12/5}\right)^{5/6}}$$

$$\cong \begin{cases} 0.225\sigma_{R,}^2 & \sigma_R^2 << 1 \\ \dfrac{0.225\sigma_R^2}{\left(0.259\sigma_R^{12/5}\right)^{5/6}} = \dfrac{0.69\sigma_R^2}{\sigma_R^2} = \ln 2, & \sigma_R^2 >> 1 \end{cases} \qquad \text{(O35)}$$

Recall from equation O30 that $\sigma_I^2 = \exp\left(\sigma_{\ln x}^2 + \sigma_{\ln y}^2\right) - 1$

Therefore, for a plane wave in moderate turbulence using an "effective" Kolmogorov spectrum

Therefore, for a spherical wave in moderate turbulence using an "effective" Kolmogorov Spectrum

$$\sigma_I^2 = \exp\left(\frac{0.17\sigma_R^2}{\left(1 + 0.167\sigma_R^{12/5}\right)^{7/6}} + \frac{0.225\sigma_R^2}{\left(1 + 0.259\sigma_R^{12/5}\right)^{5/6}} \right) - 1, \qquad 0 \le \sigma_R^2 < \infty << 1 \qquad \text{(O36)}$$

APPENDIX P: MODERATE TURBULENCE, MODIFIED (BUMP), SPHERICAL WAVE

Large-Scale Log-Irradiance Variance

Using the "effective" atmospheric spectrum (equation 23), the large-scale log-irradiance variance for a spherical wave in the presence of a finite inner scale is[8]

$$\sigma_{\ln x}^2 = 8\pi^2 k^2 \int_0^L \int_0^\infty \kappa 0.033 C_n^2 \kappa^{-11/3} G_x(\kappa) \left\{1-\cos\left[\frac{\kappa^2}{k} z\left(1-\frac{z}{L}\right)\right]\right\} d\kappa dz \tag{P1}$$

Let $1-\cos\left(\frac{\kappa^2 z}{k}\right) \cong \frac{1}{2}\left(\frac{\kappa^2 z}{k}\right)$ (its Taylor expansion estimate). The approximation is valid for the large-scale log-irradiance variance because filter function $G_x(\kappa)$ eliminates high spatial frequency contributions in moderate-to-strong fluctuations. Consequently

$$\begin{aligned}\sigma_{\ln x}^2 &= 8\pi^2 k^2 \int_0^L \int_0^\infty \kappa 0.033 C_n^2 \kappa^{-11/3} G_x(\kappa) \left(1-\left\{1-\frac{1}{2}\left[\frac{\kappa^2}{k} z\left(1-\frac{z}{L}\right)\right]^2\right\}-\right) d\kappa d\xi \\ &= 8\pi^2 k^2 \int_0^L \int_0^\infty \kappa 0.033 C_n^2 \kappa^{-11/3} G_x(\kappa) \frac{1}{2}\left[\frac{\kappa^2}{k} z(1-\frac{z}{L})\right]^2 d\kappa d\xi \\ &= 4\pi^2 \int_0^L \int_0^\infty \kappa 0.033 C_n^2 \kappa^{-11/3} G_x(\kappa) \kappa^4 z^2 \left(1-\frac{z}{L}\right)^2 d\kappa dz \end{aligned} \tag{P2}$$

Substituting

$$\xi = z/L \rightarrow dz = Ld\xi$$

$$\eta = \frac{L\kappa^2}{k} \rightarrow d\kappa = \frac{k}{2L\kappa} d\eta$$

then

$$\begin{aligned}\sigma_{\ln x}^2 &= 4\pi^2 \int_0^1 \int_0^\infty \kappa 0.033 C_n^2 \kappa^{-11/3} G_x(\kappa) \kappa^4 L^2 \xi^2 \left(1-\frac{L\xi}{L}\right)^2 \frac{k}{2L\kappa} d\eta L d\xi \\ &= 2\pi^2 k L^2 \int_0^1 \int_0^\infty 0.033 C_n^2 \kappa^{-11/3} G_x(\kappa) \kappa^4 \xi^2 (1-\xi)^2 d\eta d\xi \\ &= 0.651 k L^2 C_n^2 \int_0^1 \xi^2 (1-\xi)^2 d\xi \int_0^\infty \kappa^{1/3} G_x(\kappa) d\eta \\ &= 0.651 k L^2 C_n^2 \int_0^1 \xi^2 (1-\xi)^2 d\xi \int_0^\infty \kappa^{1/3} \exp\left[-\left(\frac{\kappa^2}{\kappa_l^2}+\frac{\kappa^2}{\kappa_x^2}\right)\right]\left[1+\frac{1.802\kappa}{\kappa_l} - 0.254\left(\frac{\kappa}{\kappa_l}\right)^{7/6}\right] d\eta \end{aligned}$$

$$= 0.651 k^{7/6} L^{11/6} C_n^2 \int_0^1 \xi^2 (1-\xi)^2 \, d\xi \int_0^\infty L \frac{^{1/6}\kappa^{1/3}}{k^{1/6}} \exp\left\{-\left[\left(\frac{Lk}{Lk}\right)\frac{\kappa^2}{\kappa_l^2} + \left(\frac{Lk}{Lk}\right)\frac{\kappa^2}{\kappa_x^2}\right]\right\}\left\{1 + 1.802\left(\frac{Lk}{Lk}\right)^{1/2}\left(\frac{\kappa}{\kappa_l}\right) - 0.254\left[\left(\frac{Lk}{Lk}\right)\left(\frac{\kappa^2}{\kappa_l^2}\right)\right]^{7/12}\right\} d\eta$$

$$= 0.651 k^{7/6} L^{11/6} C_n^2 \int_0^1 \xi^2 (1-\xi)^2 \, d\xi \int_0^\infty \left(\frac{L\kappa^2}{k}\right)^{1/6} \exp\left\{-\left[\left(\frac{L\kappa^2}{k}\right)\left(\frac{k}{L\kappa_l^2}\right) + \left(\frac{L\kappa^2}{k}\right)\left(\frac{k}{L\kappa_x^2}\right)\right]\right\}\left\{1 + 1.802\left(\frac{L\kappa^2}{k}\right)^{1/2}\left(\frac{k}{L\kappa_l^2}\right)^{1/2} - 0.254\left[\left(\frac{L\kappa^2}{k}\right)\left(\frac{k}{L\kappa_l^2}\right)\right]^{7/12}\right\} d\eta$$

$$= 0.651 \left(\frac{1.23}{1.23}\right) k^{7/6} L^{11/6} C_n^2 \int_0^1 \xi^2 (1-\xi)^2 \, d\xi \int_0^\infty \left(\frac{L\kappa^2}{k}\right)^{1/6} \exp\left\{-\left[(\eta)\left(\frac{1}{Q_l}\right) + (\eta)\left(\frac{1}{\eta_x}\right)\right]\right\}\left\{1 + 1.802(\eta)^{1/2}\left(\frac{1}{Q_l}\right)^{1/2} - 0.254\left[(\eta)\left(\frac{1}{Q_l}\right)\right]^{7/12}\right\} d\eta$$

$$= 0.53\sigma_R^2 \int_0^1 \xi^2 (1-\xi)^2 \, d\xi \int_0^\infty \eta^{1/6} \exp\left[-\eta\left(\frac{1}{Q_l} + \frac{1}{\eta_x}\right)\right]\left[1 + 1.802\left(\frac{\eta}{Q_l}\right)^{1/2} - 0.254\left(\frac{\eta}{Q_l}\right)^{7/12}\right] d\eta$$

$$= 0.53\sigma_R^2 \int_0^1 \xi^2 (1-\xi)^2 \, d\xi \int_0^\infty \eta^{1/6} \exp\left[-\eta\left(\frac{\eta_x + Q_l}{\eta_x Q_l}\right)\right]\left[1 + 1.802\left(\frac{\eta}{Q_l}\right)^{1/2} - 0.254\left(\frac{\eta}{Q_l}\right)^{7/12}\right] d\eta$$

$$= 0.53\sigma_R^2 \int_0^1 \xi^2 (1-\xi)^2 \, d\xi \int_0^\infty \eta^{1/6} \exp\left[-\eta\left(\frac{\eta_x + Q_l}{\eta_x Q_l}\right)\right]\left[1 + 1.802\left(\frac{\eta}{Q_l}\right)^{1/2} - 0.254\left(\frac{\eta}{Q_l}\right)^{7/12}\right] d\eta$$

$$= 0.53\sigma_R^2 \int_0^1 \xi^2 (1-\xi)^2 \, d\xi \left\{ \int_0^\infty \eta^{1/6} \exp\left[-\eta\left(\frac{\eta_x + Q_l}{\eta_x Q_l}\right)\right] d\eta \right.$$

$$+ \int_0^\infty \eta^{1/6} \exp\left[-\eta\left(\frac{\eta_x + Q_l}{\eta_x Q_l}\right)\right] 1.802 \left(\frac{\eta}{Q_l}\right)^{1/2} d\eta$$

$$\left. - \int_0^\infty \eta^{1/6} \exp\left[-\eta\left(\frac{\eta_x + Q_l}{\eta_x Q_l}\right)\right] 0.254 \left(\frac{\eta}{Q_l}\right)^{7/12} d\eta \right\}$$

$$= 0.53\sigma_R^2 \int_0^1 (\xi - \xi^2)^2 \, d\xi \left\{ \int_0^\infty \eta^{1/6} \exp\left[-\eta\left(\frac{\eta_x + Q_l}{\eta_x Q_l}\right)\right] d\eta + \frac{1.802}{Q_l^{1/2}} \int_0^\infty \eta^{2/3} \exp\left[-\eta\left(\frac{\eta_x + Q_l}{\eta_x Q_l}\right)\right] d\eta \right.$$

$$\left. - \frac{0.254}{Q_l^{7/12}} \int_0^\infty \eta^{3/4} \exp\left[-\eta\left(\frac{\eta_x + Q_l}{\eta_x Q_l}\right)\right] d\eta \right\} \tag{P3}$$

where

$$G_x(\kappa) = f(\kappa l_0) \exp(-\kappa^2/\kappa_x^2)$$
$$= \exp(-\kappa^2/\kappa_l^2)\left[1 + 1.802\kappa/\kappa_l - 0.254(\kappa/\kappa_l)^{7/6}\right]\exp(-\kappa^2/\kappa_x^2)$$
$$= \exp\left[-\left(\frac{\kappa^2}{\kappa_l^2} + \frac{\kappa^2}{\kappa_x^2}\right)\right]\left[1 + \frac{1.802\kappa}{\kappa_l} - 0.254\left(\frac{\kappa}{\kappa_l}\right)^{7/6}\right]$$

$$\sigma_R^2 = 1.23 C_n^2 k^{7/6} L^{11/6}$$

$$Q_l = L\kappa_l^2 / k$$

Evaluate $\int_0^1 \left(\xi - \xi^2\right)^2 d\xi$

Using a binomial series expansion $(x+y)^n \cong x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2$,

where $x = \xi$, $y = -\xi^2$, $n = 2$, and $(\xi - \xi^2)^2 \cong \xi^2 - 2\xi^3 + \xi^4$, results in

$$\begin{aligned}\int_0^1 \left(\xi - \xi^2\right)^2 d\xi &\cong \int_0^1 (\xi^2 - 2\xi^3 + \xi^4) d\xi \\ &= \frac{1}{3} - \frac{2}{4} + \frac{1}{5} \\ &= \frac{1}{30} \end{aligned} \tag{P4}$$

Then,

$$\begin{aligned}\sigma_{\ln x}^2 &\cong 0.53\sigma_R^2 \left(\frac{1}{30}\right)\left\{\int_0^\infty \eta^{1/6} \exp\left[-\eta\left(\frac{\eta_x + Q_l}{\eta_x Q_l}\right)\right] d\eta + \frac{1.802}{Q_l^{1/2}}\int_0^\infty \eta^{2/3} \exp\left[-\eta\left(\frac{\eta_x + Q_l}{\eta_x Q_l}\right)\right] d\eta \right.\\ &\quad \left. - \frac{0.254}{Q_l^{7/12}}\int_0^\infty \eta^{3/4} \exp\left[-\eta\left(\frac{\eta_x + Q_l}{\eta_x Q_l}\right)\right] d\eta \right\} \\ &= 0.018\sigma_R^2 \left\{\int_0^\infty \eta^{1/6} \exp\left[-\eta\left(\frac{\eta_x + Q_l}{\eta_x Q_l}\right)\right] d\eta + \frac{1.802}{Q_l^{1/2}}\int_0^\infty \eta^{2/3} \exp\left[-\eta\left(\frac{\eta_x + Q_l}{\eta_x Q_l}\right)\right] d\eta \right.\\ &\quad \left. - \frac{0.254}{Q_l^{7/12}}\int_0^\infty \eta^{3/4} \exp\left[-\eta\left(\frac{\eta_x + Q_l}{\eta_x Q_l}\right)\right] d\eta \right\}\end{aligned} \tag{P5}$$

Recognizing[1]

$$\int_0^\infty e^{-s\eta}\eta^{x-1} d\eta = \frac{\Gamma(x)}{s^x} \tag{P6}$$

where

$$s = \frac{\eta_x + Q_m}{\eta_x Q_m}$$

x-1=1/6 $\rightarrow$ x=7/6,   x-1=2/3 $\rightarrow$ x=5/3,   x-1=3/4 $\rightarrow$ x=7/4,

Equation P5 can be written as

$$\sigma_{\ln x}^{2}(L) = 0.018\sigma_R^2\left(\frac{\Gamma(7/6)}{\left(\frac{\eta_x+Q_l}{\eta_x Q_l}\right)^{7/6}} + \frac{1.802}{Q_l^{1/2}}\frac{\Gamma(5/3)}{\left(\frac{\eta_x+Q_l}{\eta_x Q_l}\right)^{5/3}} - \frac{0.254}{Q_l^{7/12}}\frac{\Gamma(7/4)}{\left(\frac{\eta_x+Q_l}{\eta_x Q_l}\right)^{7/4}}\right)$$

$$= 0.018\sigma_R^2\left(\frac{\Gamma(7/6)}{\left(\frac{\eta_x+Q_l}{\eta_x Q_l}\right)^{7/6}} + \frac{1.802}{Q_l^{1/2}}\frac{\Gamma(5/3)}{\left(\frac{\eta_x+Q_l}{\eta_x Q_l}\right)^{7/6}\left(\frac{\eta_x+Q_l}{\eta_x Q_l}\right)^{3/6}} - \frac{0.254}{Q_l^{7/12}}\frac{\Gamma(7/4)}{\left(\frac{\eta_x+Q_l}{\eta_x Q_l}\right)^{7/6}\left(\frac{\eta_x+Q_l}{\eta_x Q_l}\right)^{7/12}}\right)$$

$$= 0.018\sigma_R^2\left(\frac{\eta_x Q_l}{\eta_x+Q_l}\right)^{7/6}\left(\Gamma(7/6) + \frac{1.802}{Q_l^{1/2}\left(\frac{\eta_x+Q_l}{\eta_x Q_l}\right)^{1/2}}\Gamma(5/3) - \frac{0.254}{Q_l^{7/12}\left(\frac{\eta_x+Q_l}{\eta_x Q_l}\right)^{7/12}}\Gamma(7/4)\right)$$

$$= 0.018\sigma_R^2\left(\frac{\eta_x Q_l}{\eta_x+Q_l}\right)^{7/6}\left(\Gamma(7/6) + \frac{1.802}{\left(\frac{\eta_x+Q_l}{\eta_x}\right)^{1/2}}\Gamma(5/3) - \frac{0.254}{\left(\frac{\eta_x+Q_l}{\eta_x}\right)^{7/12}}\Gamma(7/4)\right)$$

$$= 0.018\sigma_R^2\left(\frac{\eta_x Q_l}{\eta_x+Q_l}\right)^{7/6}\left(\Gamma(7/6) + 1.802\Gamma(5/3)\left(\frac{\eta_x}{\eta_x+Q_l}\right)^{1/2} - 0.254\Gamma(7/4)\left(\frac{\eta_x}{\eta_x+Q_l}\right)^{7/12}\right)$$

$$= 0.018\sigma_R^2\left(\frac{\eta_x Q_l}{\eta_x+Q_l}\right)^{7/6}\left(0.9277 + 1.802(0.9028)\left(\frac{\eta_x}{\eta_x+Q_l}\right)^{1/2} - 0.254(0.9191)\left(\frac{\eta_x}{\eta_x+Q_l}\right)^{7/12}\right)$$

$$= 0.018\sigma_R^2\left(\frac{\eta_x Q_l}{\eta_x+Q_l}\right)^{7/6}\left(0.9277 + 1.63\left(\frac{\eta_x}{\eta_x+Q_l}\right)^{1/2} - 0.234\left(\frac{\eta_x}{\eta_x+Q_l}\right)^{7/12}\right)$$

$$= 0.018(0.9277)\sigma_R^2\left(\frac{\eta_x Q_l}{\eta_x+Q_l}\right)^{7/6}\left(1 + 1.76\left(\frac{\eta_x}{\eta_x+Q_l}\right)^{1/2} - 0.25\left(\frac{\eta_x}{\eta_x+Q_l}\right)^{7/12}\right)$$

$$= 0.016\sigma_R^2\left(\frac{\eta_x Q_l}{\eta_x+Q_l}\right)^{7/6}\left(1 + 1.76\left(\frac{\eta_x}{\eta_x+Q_l}\right)^{1/2} - 0.25\left(\frac{\eta_x}{\eta_x+Q_l}\right)^{7/12}\right)$$

$$= 0.016\sigma_R^2\left\{\left(\frac{\eta_x Q_l}{\eta_x+Q_l}\right)\left(1 + 1.76\left(\frac{\eta_x}{\eta_x+Q_l}\right)^{1/2} - 0.25\left(\frac{\eta_x}{\eta_x+Q_l}\right)^{7/12}\right)^{6/7}\right\}^{7/6}$$

$$= 0.016\,\sigma^2\mu_x^{7/6} \qquad \text{(P7)}$$

where $\mu_x = \left(\frac{\eta_x Q_l}{\eta_x + Q_l}\right)\left(1 + 1.76\left(\frac{\eta_x}{\eta_x + Q_l}\right)^{1/2} - 0.25\left(\frac{\eta_x}{\eta_x + Q_l}\right)^{7/12}\right)^{6/7}$

Small-scale log-irradiance variance

Using the "effective" atmospheric spectrum (equation 23), the small-scale log-irradiance variance for a spherical wave in the presence of a finite inner scale is[8]

$$\sigma^2_{\ln y} = 8\pi^2 k^2 \int_0^L \int_0^\infty \kappa 0.033 C_n^2 \kappa^{-11/3} G_y(\kappa)\left\{1 - \cos\left[\frac{\kappa^2}{k} z\left(1 - \frac{z}{L}\right)\right]\right\} d\kappa dz \qquad \text{(P8)}$$

In moderate-to-strong irradiance fluctuations, $\kappa_y >> \sqrt{k/L}$, let

$$\sigma^2_{\ln y} \cong 8\pi^2 k^2 \int_0^L \int_0^\infty \kappa 0.033 C_n^2 \kappa^{-11/3} G_y(\kappa) d\kappa dz \qquad \text{(P9)}$$

Substituting

$$\xi = z/L \rightarrow dz = Ld\xi$$

$$\eta = \frac{L\kappa^2}{k} \rightarrow d\kappa = \frac{k}{2L\kappa} d\eta$$

$$\begin{aligned}
\sigma^2_{\ln y}(L) &\cong 8\pi^2 k^2 \int_0^1 \int_0^\infty \kappa 0.033 C_n^2 \kappa^{-11/3} G_y(\kappa) \frac{k}{2L\kappa} d\eta L d\xi \\
&= 4\pi^2 k^3 \int_0^1 \int_0^\infty 0.033 C_n^2 \kappa^{-11/3} G_y(\kappa) d\eta d\xi \\
&= 1.30 k^3 C_n^2 \int_0^1 \int_0^\infty \kappa^{-11/3}\left[\frac{\kappa^{11/3}}{\left(\kappa^2 + \kappa_y^2\right)^{11/6}}\right] d\eta d\xi \\
&= 1.30 k^3 C_n^2 \int_0^\infty \left[\frac{1}{\left(\kappa^2 + \kappa_y^2\right)^{11/6}}\right] d\eta \\
&= 1.30\left(\frac{1.23}{1.23}\right) C_n^2 \left(k^{7/6} k^{11/6}\right) \frac{L^{11/16}}{L^{11/16}} \int_0^\infty \left[\frac{1}{\left(\kappa^2 + \kappa_y^2\right)^{11/6}}\right] d\eta \\
&= 1.06 \sigma_R^2 \int_0^\infty \left[\frac{k^{11/6}}{L^{11/16}\left(\kappa^2 + \kappa_y^2\right)^{11/6}}\right] d\eta
\end{aligned}$$

$$
\begin{aligned}
&= 1.06\sigma_R^2 \int_0^\infty \left(\frac{k}{L\kappa^2} + \frac{k}{L\kappa_y^2} \right)^{11/6} d\eta \\
&= 1.06\sigma_R^2 \int_0^\infty (\eta + \eta_y)^{-11/6} d\eta \\
&= 1.06\sigma_R^2 \left(\frac{-6}{5} \right) (\eta + \eta_y)^{-5/6} \Big|_{\eta=0}^{\eta=\infty} \\
&= 1.06\sigma_R^2 \left(\frac{-6}{5} \right) [(\infty + \eta_y)^{-5/6} - (0 + \eta_y)^{-5/6}] \\
&= 1.06\sigma_R^2 \left(\frac{-6}{5} \right) \left[-(\eta_y)^{-5/6} \right] \\
&= 1.272\sigma_R^2 \eta_y^{-5/6}
\end{aligned} \qquad \text{(P10)}
$$

where

$$
G_y(\kappa) = \frac{\kappa^{11/3}}{\left(\kappa^2 + \kappa_y^2\right)^{11/6}}
$$

$$
\sigma_R^2 = 1.23 C_n^2 k^{7/6} L^{11/6}
$$

Asymptotic Comparisons

For constants c_1, c_2, c_3, and c_4, assume the following functional form of η_x and η_y [see Appendix S].

$$
\eta_x = \frac{1}{c_1 + c_2 L / k\rho_o^2} \cong \begin{cases} \dfrac{1}{c_1 + c_2(0)} = \dfrac{1}{c_1}, & \dfrac{L}{k\rho_0^2} << 1 \\ \dfrac{1}{0 + c_2 L / k\rho_o^2} = \dfrac{k\rho_o^2}{c_2 L}, & \dfrac{L}{k\rho_0^2} >> 1 \end{cases} \qquad \text{(P11)}
$$

$$
\eta_y = c_3 + \frac{c_4 L}{k\rho_0^2} \cong \begin{cases} c_3 + c_4(0) = c_3, & \dfrac{L}{k\rho_0^2} << 1 \\ 0 + \dfrac{c_4 L}{k\rho_0^2} = \dfrac{c_4 L}{k\rho_0^2}, & \dfrac{L}{k\rho_0^2} >> 1 \end{cases} \qquad \text{(P12)}
$$

Weak Turbulence

Assume in equations 14 and P7 that $l_0 \to 0$, $Q_l \to \infty$

Then

$$\begin{aligned}
\sigma_I^2 &= 3.86\sigma_R^2\left\{0.40\left(1+\frac{9}{Q_l^2}\right)^{11/12}\left[\sin\left(\frac{11}{6}\tan^{-1}\frac{Q_l}{3}\right)+\frac{2.61}{\left(9+Q_l^2\right)^{1/4}}\sin\left(\frac{4}{3}\tan^{-1}\frac{Q_l}{3}\right)-\frac{0.52}{\left(9+Q_l^2\right)^{7/24}}\sin\left(\frac{5}{4}\tan^{-1}\frac{Q_l}{3}\right)\right]-3.50Q_l^{-5/6}\right\} \\
&= 3.86\sigma_R^2\left\{0.40\left(1+\frac{9}{\infty}\right)^{11/12}\left[\sin\left(\frac{11}{6}\tan^{-1}\frac{\infty}{3}\right)+\frac{2.61}{\left(9+\infty\right)^{1/4}}\sin\left(\frac{4}{3}\tan^{-1}\frac{\infty}{3}\right)-\frac{0.52}{\left(9+\infty\right)^{7/24}}\sin\left(\frac{5}{4}\tan^{-1}\frac{\infty}{3}\right)\right]-\frac{3.50}{\infty}\right\} \\
&= 3.86\sigma_R^2\left\{0.40(1)^{11/12}\left[\sin\left[\frac{11}{6}\left(\frac{\pi}{2}\right)\right]+0-0\right]-0\right\} \\
&= 3.86\sigma_R^2(0.40)(0.258) \\
&= 0.4\sigma_R^2
\end{aligned} \tag{P13}$$

and

$$\begin{aligned}
\sigma_{\ln x}^2(L) &= 0.016\sigma_R^2\left\{\left(\frac{\eta_x Q_l}{\eta_x+Q_l}\right)\left(1+1.76\left(\frac{\eta_x}{\eta_x+Q_l}\right)^{1/2}-0.25\left(\frac{\eta_x}{\eta_x+Q_l}\right)^{7/12}\right)^{6/7}\right\}^{7/6} \\
&\cong 0.016\sigma_R^2\left[\left(\frac{\eta_x Q_l}{Q_l}\right)(1+0-0)^{6/7}\right]^{7/6} \\
&= 0.016\sigma_R^2\eta_x^{7/6}
\end{aligned} \tag{P14}$$

Weak turbulence, assuming $\sigma_I^2 << 1$, results in

$$\sigma_I^2 = \exp\left(\sigma_{\ln I}^2\right)-1 \cong \sigma_{\ln I}^2 \tag{P15}$$

Under weak irradiance fluctuations[8], $\sigma_{\ln I}^2 << 1$,

$$\sigma_{\ln I}^2 = \sigma_{\ln x}^2 + \sigma_{\ln y}^2 \tag{P16}$$

Equating equations P13 and P15 and using equations P10 and P14

$$0.016\eta_x^{7/6} + 1.272\eta_y^{-5/6} \cong 0.4 \tag{P17}$$

Assuming $\eta_x \cong \eta_y$, then $\eta_x = \eta_y = 8$ is an approximate solution.

Plugging this into equations P10 and P11.

$$\eta_x = \frac{1}{c_1 + c_2 L / k\rho_o^2} \cong \frac{1}{c_1 + c_2(0)} = \frac{1}{c_1} = 8$$

$$\rightarrow c_1 \cong 1/8 \tag{P18}$$

$$\eta_y = c_3 + \frac{c_4 L}{k\rho_0^2} \cong c_3 + c_4(0) = c_3 = 8 \tag{P19}$$

Thus, c_1 and c_3 are now established using behavior of the scintillation index in the weak turbulence regime.

Saturated Turbulence

First, perform an asymptotic comparison for small-scale turbulent cell effects. Normalized irradiance[8] can be expressed as I = *xy* where *x* is associated with large-scale turbulent eddy affects, *y* is associated with small-scale eddy effects, *x* and *y* are statistically independent random quantities, and $\langle x \rangle = \langle y \rangle = 1$. Given these conditions, $\langle I \rangle = \langle x \rangle \langle y \rangle = 1$. The second moment of irradiance takes the form[8]

$$\langle I^2 \rangle = \langle x^2 \rangle \langle y^2 \rangle = (1 + \sigma_x^2)(1 + \sigma_y^2) \tag{P20}$$

where σ_x^2 and σ_y^2 are the large scale and small scale normalized variances of *x* and *y*, respectively. Based on equations 5 and P20, the implied scintillation index is

$$\begin{aligned} \sigma_I^2 &= (1 + \sigma_x^2)(1 + \sigma_y^2) - 1 \\ &= \sigma_x^2 + \sigma_y^2 + \sigma_x^2 \sigma_y^2 \end{aligned} \tag{P21}$$

Per equation 21, the asymptotic behavior of the scintillation index in the saturation regime is described by $\sigma_I^2 = 1 + 7.65 / \left(\sigma_R^2 Q_l^{7/6}\right)^{1/6}$ which approaches an asymptotic limit of one. Therefore, in saturated turbulence

$$\sigma_I^2 = \sigma_x^2 + \sigma_y^2 + \sigma_x^2 \sigma_y^2 \cong 1 \tag{P22}$$

In strong fluctuations, expect the large-scale scintillation terms to die out.

$$\sigma_I^2 \approx \sigma_y^2 \cong 1 \tag{P23}$$

The small scale irradiance is given by[8]

$$\sigma_y^2 = \exp\left(\sigma_{\ln y}^2\right) - 1 \tag{P24}$$

As a result, in strong fluctuations

$$\sigma_y^2 \cong 1 = \exp\left(\sigma_{\ln y}^2\right) - 1$$

$$\rightarrow \sigma_{\ln y}^2 \cong \ln 2 \tag{P25}$$

The spherical wave spatial coherence radius, ρ_0, when C_n^2 is constant and $l_0 << \rho << L_0$ can be approximated by[1]

$$\rho_0 = \left(0.55 C_n^2 k^2 L\right)^{-3/5} \rightarrow C_n^2 = \frac{\rho_0^{-5/3}}{0.55 k^2 L} \tag{P26}$$

Plugging this into the Rytov Variance, $\sigma_R^2 = 1.23 C_n^2 k^{7/6} L^{11/6}$ results in

$$\sigma_R^2 = 1.23\left(\frac{\rho_0^{-5/3}}{1.46 k^2 L}\right) k^{7/6} L^{11/6}$$

$$\rightarrow \frac{L}{k\rho_0^2} = 0.38(\sigma_R^2)^{6/5} \tag{P27}$$

In saturated turbulence, equating equation P25 to equation P10 results in

$$\begin{aligned} \ln 2 &= 1.272\sigma_R^2 \eta_y^{-5/6} \\ &= 1.272\sigma_R^2 \left(\frac{c_4 L}{k\rho_0^2}\right)^{-5/6} \\ &= 1.272\sigma_R^2 c_4^{-5/6}\left[0.38(\sigma_R^2)^{6/5}\right]^{-5/6} \\ &= 1.272 c_4^{-5/6}(0.38)^{-5/6} \end{aligned}$$

$$\rightarrow c_4 \cong 5.45 \tag{P28}$$

where

$$\eta_y = \frac{c_4 L}{k\rho_0^2} \quad \text{(equation P12)}$$

$$\frac{L}{k\rho_0^2} = 0.38(\sigma_R^2)^{6/5} \quad \text{(equation P27)}$$

Recall from equation P13 that $\sigma_{\ln x}^2 \cong 0.016\sigma_R^2 \eta_x{}^{7/6}$ as $l_0 \to 0$.

As a result

$$\begin{aligned} \sigma_I^2 &\cong 2\sigma_x^2 + 1 \\ &\cong 2\left(0.016\sigma_R^2 \eta_x{}^{7/6}\right) + 1 \\ &= 0.032\sigma_R^2 \eta_x{}^{7/6} + 1 \end{aligned} \quad \text{(P29)}$$

For spherical wave von Karman spectrum, with inner scale effects, the plane wave spatial coherence radius is[1] $\rho = (0.62 C_n^2 k^2 L l_0^{-1/3})^{-1/2}$ for $\rho << l_0$

$$\begin{aligned} L/k\rho_0^2 &= \frac{L}{k}\rho_0^{-2} \\ &= \frac{L}{k}\left[\left(0.62 C_n^2 k^2 L l_0^{-1/3}\right)^{-1/2}\right]^{-2} \\ &= 0.62 C_n^2 k L^2 l_0^{-1/3} \\ &= 0.62\left(\frac{1.23k^{1/6}}{1.23k^{1/6}}\right) C_n^2 k L^2 l_0^{-1/3} \\ &= 0.504\sigma_R^2 \left(\frac{10.89L}{10.89kl_0^2}\right)^{1/6} \\ &= 0.339\sigma_R^2 Q_l^{1/6} \end{aligned} \quad \text{(P30)}$$

where $Q_l = \frac{L\kappa_l^2}{k} = \frac{L(3.3)^2}{kl_0^2} = \frac{10.89L}{kl_0^2}$

Reference equation P11, assuming $\frac{L}{k\rho_0^2} >> 1$, then

$$\eta_x \cong \frac{k\rho_o^2}{c_2 L} = \frac{1}{c_2 0.339\sigma_R^2 Q_m^{1/6}}$$

and

$$\begin{aligned}
\sigma_I^2 &\cong 0.032\sigma_R^2 \eta_x{}^{7/6} + 1 \\
&= 0.032\sigma_R^2 \left(\frac{1}{c_2 0.339\sigma_R^2 Q_l^{1/6}} \right)^{7/6} + 1 \\
&= 1 + 0.113 \left(\frac{1}{c_2} \right)^{7/6} \left(\frac{1}{\sigma_R^{1/3}} \right) \left(\frac{1}{Q_l^{7/36}} \right)
\end{aligned} \qquad \text{(P31)}$$

Compare this to the existing scintillation expression (equation 21) for a spherical wave in saturated turbulence.

$$\begin{aligned}
\sigma_I^2 &= 1 + 0.113 \left(\frac{1}{c_2} \right)^{7/6} \left(\frac{1}{\sigma_R^{1/3}} \right) \left(\frac{1}{Q_l^{7/36}} \right) = 1 + \frac{7.65}{\left(\sigma_R^2 Q_l^{7/6} \right)^{1/6}} \\
0.113 &\left(\frac{1}{c_2} \right)^{7/6} \left(\frac{1}{\sigma_R^{1/3}} \right) \left(\frac{1}{Q_l^{7/36}} \right) = \frac{7.65}{\left(\sigma_R^2 Q_l^{7/6} \right)^{1/6}} \\
c_2 &= 0.027
\end{aligned} \qquad \text{(P32)}$$

Thus, c_2 and c_4 are now established using behavior of the scintillation index in the saturated turbulence regime.

<u>"All" Turbulence Regimes</u>

Applying constants c_1, c_2, c_3, and c_4 results in the following expressions for predicting scintillation behavior in all turbulence regimes (including inner scale effects)

$$\begin{aligned}
\eta_x &= \frac{1}{c_1 + c_2 L / k\rho_o^2} \\
&\cong \frac{1}{1/8 + 0.027 L / k\rho_o^2} \\
&= \frac{8}{1 + 0.216\left(0.339\sigma_R^2 Q_l^{1/6}\right)}
\end{aligned}$$

$$= \frac{8}{1 + 0.073\sigma_R^2 Q_l^{1/6}} \tag{P33}$$

$$\frac{\eta_x Q_l}{\eta_x + Q_l} = \frac{\left(\frac{8}{1 + 0.073\sigma_R^2 Q_l^{1/6}}\right) Q_l}{\left(\frac{8}{1 + 0.073\sigma_R^2 Q_l^{1/6}}\right) + Q_l}$$

$$= \frac{8Q_l}{8 + Q_l\left(1 + 0.073\sigma_R^2 Q_l^{1/6}\right)}$$

$$= \frac{8Q_l}{8 + Q_l + 0.073\sigma_R^2 Q_l^{7/6}} \tag{P34}$$

and

$$\sigma_{\ln x}^2 = 0.016\sigma_R^2 \left(\frac{\eta_x Q_l}{\eta_x + Q_l}\right)^{7/6} \left(1 + 1.76\left(\frac{\eta_x}{\eta_x + Q_l}\right)^{1/2} - 0.25\left(\frac{\eta_x}{\eta_x + Q_l}\right)^{7/12}\right)$$

$$= 0.016\sigma_R^2 \left(\frac{8Q_l}{8 + Q_l + 0.073\sigma_1^2 Q_l^{7/6}}\right)^{7/6} \left[1 + 1.76\left(\frac{8Q_l}{8 + Q_l + 0.073\sigma_1^2 Q_l^{7/6}}\right)^{1/2} - 0.25\left(\frac{8Q_l}{8 + Q_l + 0.073\sigma_1^2 Q_l^{7/6}}\right)^{7/12}\right] \tag{P35}$$

From equations P12 and P27

$$\eta_y = c_3 + \frac{c_4 L}{k\rho_0^2}$$

$$= 8 + \frac{5.45L}{k\rho_0^2}$$

$$= 8 + 5.45\left[0.38(\sigma_R^2)^{6/5}\right]$$

$$= 8 + 2.07(\sigma_R^2)^{6/5} \tag{P37}$$

For a spherical wave in weak turbulence (equation 14), let

$$\sigma_I^2 = \sigma_s^2 = 3.86\sigma_R^2 \left\{0.40\left(1 + \frac{9}{Q_l^2}\right)^{11/12}\left[\sin\left(\frac{11}{6}\tan^{-1}\frac{Q_l}{3}\right) + \frac{2.61}{\left(9 + Q_l^2\right)^{1/4}}\sin\left(\frac{4}{3}\tan^{-1}\frac{Q_l}{3}\right) - \frac{0.52}{\left(9 + Q_l^2\right)^{7/24}}\sin\left(\frac{5}{4}\tan^{-1}\frac{Q_l}{3}\right)\right] - 3.50Q_l^{-5/6}\right\} \tag{P36}$$

Under strong fluctuations, inner-scale effects tend to diminish such that

$$\eta_y = 2.07\left(\sigma_R^2\right)^{6/5}$$

$$= 3(0.69)\sigma_R^{12/5} \tag{P38}$$

For all fluctuations, impose the following expression

$$\eta_y = 3\left(\frac{\sigma_R^2}{\sigma_S^2}\right)^{6/5}(1 + 0.69\sigma_S^{12/5}) \tag{P39}$$

As a test of this expression, notice that under strong fluctuations

$$\eta_y \cong 3\left(\frac{\sigma_R^2}{\sigma_S^2}\right)^{6/5}\left(0.69\sigma_S^{12/5}\right) = 3(0.69)\sigma_R^{12/5} = 2.07\left(\sigma_R^2\right)^{6/5}$$ in agreement with equation P41.

The small-scale log-irradiance variance becomes

$$\sigma_{\ln y}^2(L) \cong 1.272\sigma_R^2\left[3\left(\frac{\sigma_R}{\sigma_s}\right)^{12/5}\left(1+0.69\sigma_s^{12/5}\right)\right]^{-5/6}$$

$$= \frac{0.51\sigma_s^2}{\left(1+0.69\sigma_s^{12/5}\right)^{5/6}} \qquad \text{(P40)}$$

Scintillation can be expressed as[1]

$$\sigma_I^2 = \exp\left(\sigma_{\ln x}^2 + \sigma_{\ln y}^2\right) - 1 \qquad \text{(P41)}$$

Therefore, for a spherical wave in moderate turbulence using an "effective" von Karman spectrum

$$\sigma_I^2 = \exp\left(\sigma_{\ln x}^2 + \frac{0.51\sigma_s^2}{(1+0.69\sigma_s^{12/5})^{5/6}}\right) - 1 \qquad \text{(P42)}$$

where

$$\sigma_{\ln x}^2 = 0.016\sigma_R^2\left(\frac{8Q_l}{8+Q_l+0.073\sigma_1^2 Q_l^{7/6}}\right)^{7/6}\left[1+1.76\left(\frac{8Q_l}{8+Q_l+0.073\sigma_1^2 Q_l^{7/6}}\right)^{1/2} - 0.25\left(\frac{8Q_l}{8+Q_l+0.073\sigma_1^2 Q_l^{7/6}}\right)^{7/12}\right]$$

$$\sigma_s^2 = 3.86\sigma_R^2\left\{0.40\left(1+\frac{9}{Q_l^2}\right)^{11/12}\left[\sin\left(\frac{11}{6}\tan^{-1}\frac{Q_l}{3}\right) + \frac{2.61}{\left(9+Q_l^2\right)^{1/4}}\sin\left(\frac{4}{3}\tan^{-1}\frac{Q_l}{3}\right) - \frac{0.52}{\left(9+Q_l^2\right)^{7/24}}\sin\left(\frac{5}{4}\tan^{-1}\frac{Q_l}{3}\right)\right] - 3.50Q_l^{-5/6}\right\}$$

APPENDIX Q: MODERATE TURBULENCE, VON KARMAN, PLANE WAVE

Large-Scale Log-Irradiance Variance

Using the "effective" atmospheric spectrum, reference equations 23 and 30, the large-scale log-irradiance variance for a plane wave in the presence of a finite inner scale is[8]

$$\sigma_{\ln x}^2(L) = 8\pi^2 k^2 \int_0^L \int_0^\infty \kappa 0.033 C_n^2 \kappa^{-11/3} G_x(\kappa) \left[1 - \cos\left(\frac{\kappa^2 z}{k}\right)\right] d\kappa dz \qquad \text{(Q1)}$$

Let $1 - \cos\left(\frac{\kappa^2 z}{k}\right) \cong \frac{1}{2}\left(\frac{\kappa^2 z}{k}\right)$ (its Taylor expansion estimate). The approximation is valid for the large-scale log-irradiance variance because filter function $G_x(\kappa)$ eliminates high spatial frequency contributions in moderate-to-strong fluctuations. Consequently

$$\sigma_{\ln x}^2(L) \cong 8\pi^2 k^2 \int_0^L \int_0^\infty \kappa 0.033 C_n^2 \kappa^{-11/3} G_x(\kappa) \left[\frac{1}{2}\left(\frac{\kappa^2 z}{k}\right)^2\right] d\kappa dz$$

$$= 4\pi^2 \int_0^L \int_0^\infty \kappa^5 0.033 C_n^2 \kappa^{-11/3} G_x(\kappa) z^2 d\kappa dz \qquad \text{(Q2)}$$

Substituting

$$\xi = z / L \;\rightarrow\; dz = L d\xi$$

$$\eta = \frac{L\kappa^2}{k} \;\rightarrow d\kappa = \frac{k}{2L\kappa} d\eta$$

$$\sigma_{\ln x}^2(L) \cong 4\pi^2 \int_0^1 \int_0^\infty \kappa^5 0.033 C_n^2 \kappa^{-11/3} G_x(\kappa) L^2 \xi^2 \frac{k}{2L\kappa} d\eta L d\xi$$

$$= 2\pi^2 k L^2 \int_0^1 \int_0^\infty \kappa^4 0.033 C_n^2 \kappa^{-11/3} G_x(\kappa) \xi^2 d\eta d\xi$$

$$= 0.651 k L^2 C_n^2 \int_0^1 \xi^2 d\xi \int_0^\infty \kappa^{1/3} G_x(\kappa) d\eta$$

$$= 0.651 k L^2 C_n^2 \int_0^1 \xi^2 d\xi \int_0^\infty \kappa^{1/3} \exp\left(\frac{-\kappa^2}{\kappa_m^2}\right) \exp\left(\frac{-\kappa^2}{\kappa_x^2}\right) d\eta$$

$$= 0.651 k^{7/6} L^{11/6} C_n^2 \int_0^1 \xi^2 d\xi \int_0^\infty \frac{L^{1/6}\kappa^{1/3}}{k^{1/6}} \exp\left[-\left(\frac{Lk}{Lk}\right)\frac{\kappa^2}{\kappa_m^2}\right] \exp\left[-\left(\frac{Lk}{Lk}\right)\frac{\kappa^2}{\kappa_x^2}\right] d\eta$$

$$= 0.651 k^{7/6} L^{11/6} C_n^2 \int_0^1 \xi^2 d\xi \int_0^\infty \left(\frac{L\kappa^2}{k}\right)^{1/6} \exp\left[-\left(\frac{L\kappa^2}{k}\right)\frac{k}{L\kappa_m^2}\right] \exp\left[\left(\frac{-L\kappa^2}{k}\right)\left(\frac{k}{L\kappa_x^2}\right)\right] dn$$

$$= 0.651\left(\frac{1.23}{1.23}\right) k^{7/6} L^{11/6} C_n^2 \int_0^1 \xi^2 d\xi \int_0^\infty \eta^{1/6} \exp\left(\frac{-\eta}{Q_m}\right) \exp\left(\frac{-\eta}{\eta_x}\right) d\eta$$

$$= 0.53\sigma_R^2 \int_0^1 \xi^2 d\xi \int_0^\infty \eta^{1/6} \exp\left(\frac{-\eta}{Q_m} - \frac{\eta}{\eta_x}\right)$$

$$= 0.53\sigma_R^2 \left(\frac{1}{3}\right) \int_0^\infty \eta^{1/6} \exp\left(\frac{-\eta}{Q_m} - \frac{\eta}{\eta_x}\right) d\eta$$

$$= 0.176\sigma_R^2 \int_0^\infty \eta^{1/6} \exp\left(\frac{-\eta}{Q_m} - \frac{\eta}{\eta_x}\right) d\eta$$

$$= 0.176\sigma_R^2 \int_0^\infty \eta^{1/6} \exp\left[-\eta\left(\frac{\eta_x + Q_m}{\eta_x Q_m}\right)\right] d\eta \qquad \text{(Q3)}$$

where

$$G_x(\kappa) = f(\kappa l_0)\exp\left(-\kappa^2/\kappa_x^2\right)$$
$$= \exp\left(-\kappa^2/\kappa_m^2\right)\exp\left(-\kappa^2/\kappa_x^2\right)$$

$$Q_m = L\kappa_m^2/k$$

$$\frac{\kappa^2}{\kappa_m^2} = \left(\frac{L\kappa^2}{k}\right)\left(\frac{k}{L\kappa_m^2}\right) = \frac{\eta}{Q_m}$$

$$\sigma_R^2 = 1.23 C_n^2 k^{7/6} L^{11/6}$$

Recognizing[1]

$$\int_0^\infty e^{-s\eta}\eta^{x-1} d\eta = \frac{\Gamma(x)}{s^x} \qquad \text{(Q4)}$$

where

$$s = \frac{\eta_x + Q_m}{\eta_x Q_m}$$

x-1=1/6 $\rightarrow$ x=7/6

results in

$$\sigma_{\ln x}^2(L) = 0.176\sigma_R^2 \frac{\Gamma(7/6)}{\left(\frac{\eta_x + Q_m}{\eta_x Q_m}\right)^{7/6}}$$

$$= 0.176\sigma_R^2 \left(\frac{\eta_x Q_m}{\eta_x + Q_m}\right)^{7/6} \Gamma(7/6)$$

$$= 0.176\sigma_R^2 \left(\frac{\eta_x Q_m}{\eta_x + Q_m}\right)^{7/6} (0.9277)$$

$$= 0.16\sigma_R^2 \left(\frac{\eta_x Q_m}{\eta_x + Q_m}\right)^{7/6}$$

$$= 0.16\sigma_R^2 \mu_x \qquad \text{(Q5)}$$

where $\mu_x = \left(\frac{\eta_x Q_m}{\eta_x + Q_m}\right)^{7/6}$

Small-Scale Log-Irradiance Variance

Using the "effective" atmospheric spectrum, reference equations 23 and 33, the small-scale log-irradiance variance for a plane wave in the presence of a finite inner scale is[8]

$$\sigma_{\ln y}^2(L) = 8\pi^2 k^2 \int_0^L \int_0^\infty \kappa 0.033 C_n^2 \kappa^{-11/3} G_y(\kappa) \left[1 - \cos\left(\frac{\kappa^2 z}{k}\right)\right] d\kappa dz \qquad \text{(Q6)}$$

In moderate to strong irradiance fluctuations, $\kappa_y >> \sqrt{k/L}$, let

$$\sigma_{\ln y}^2(L) \cong 8\pi^2 k^2 \int_0^L \int_0^\infty \kappa 0.033 C_n^2 \kappa^{-11/3} G_y(\kappa) d\kappa dz \qquad \text{(Q7)}$$

Substituting

$$\xi = z/L \rightarrow dz = Ld\xi$$

$$\eta = \frac{L\kappa^2}{k} \rightarrow d\kappa = \frac{k}{2L\kappa} d\eta$$

$$\sigma_{\ln y}^2(L) \cong 8\pi^2 k^2 \int_0^1 \int_0^\infty \kappa 0.033 C_n^2 \kappa^{-11/3} G_y(\kappa) \frac{k}{2L\kappa} d\eta L d\xi$$

$$= 4\pi^2 k^3 \int_0^1 \int_0^\infty 0.033 C_n^2 \kappa^{-11/3} G_y(\kappa) d\eta d\xi$$

$$= 1.30 k^3 C_n^2 \int_0^1 \int_0^\infty \kappa^{-11/3} \left[\frac{\kappa^{11/3}}{\left(\kappa^2 + \kappa_y^2\right)^{11/6}} \right] d\eta d\xi$$

$$= 1.30 k^3 C_n^2 \int_0^\infty \left[\frac{1}{\left(\kappa^2 + \kappa_y^2\right)^{11/6}} \right] d\eta$$

$$= 1.30 \left(\frac{1.23}{1.23} \right) C_n^2 \left(k^{7/6} k^{11/6} \right) \frac{L^{11/16}}{L^{11/16}} \int_0^\infty \left[\frac{1}{\left(\kappa^2 + \kappa_y^2\right)^{11/6}} \right] d\eta$$

$$= 1.06 \sigma_R^2 \int_0^\infty \left[\frac{k^{11/6}}{L^{11/16} \left(\kappa^2 + \kappa_y^2\right)^{11/6}} \right] d\eta$$

$$= 1.06 \sigma_R^2 \int_0^\infty \left(\frac{k}{L\kappa^2} + \frac{k}{L\kappa_y^2} \right)^{11/6} d\eta$$

$$= 1.06 \sigma_R^2 \int_0^\infty (\eta + \eta_y)^{-11/6} d\eta$$

$$= 1.06 \sigma_R^2 \left(\frac{-6}{5} \right) (\eta + \eta_y)^{-5/6} \Big|_{\eta=0}^{\eta=\infty}$$

$$= 1.06 \sigma_R^2 \left(\frac{-6}{5} \right) [(\infty + \eta_y)^{-5/6} - (0 + \eta_y)^{-5/6}]$$

$$= 1.06 \sigma_R^2 \left(\frac{-6}{5} \right) \left[-(\eta_y)^{-5/6} \right]$$

$$= 1.272 \sigma_R^2 \eta_y^{-5/6} \qquad \text{(Q8)}$$

where

$$G_y(\kappa) = \frac{\kappa^{11/3}}{\left(\kappa^2 + \kappa_y^2\right)^{11/6}}$$

$$\sigma_R^2 = 1.23 C_n^2 k^{7/6} L^{11/6}$$

<u>Asymptotic Comparisons</u>

For constants c_1, c_2, c_3, and c_4, assume the following functional form of η_x and η_y [see Appendix S].

$$\eta_x = \frac{1}{c_1 + c_2 L / k\rho_o^2} \cong \begin{cases} \dfrac{1}{c_1 + c_2(0)} = \dfrac{1}{c_1}, & \dfrac{L}{k\rho_0^2} << 1 \\ \dfrac{1}{0 + c_2 L / k\rho_o^2} = \dfrac{k\rho_o^2}{c_2 L}, & \dfrac{L}{k\rho_0^2} >> 1 \end{cases} \qquad \text{(Q9)}$$

$$\eta_y = c_3 + \frac{c_4 L}{k\rho_0^2} \cong \begin{cases} c_3 + c_4(0) = c_3, & \dfrac{L}{k\rho_0^2} << 1 \\ 0 + \dfrac{c_4 L}{k\rho_0^2} = \dfrac{c_4 L}{k\rho_0^2}, & \dfrac{L}{k\rho_0^2} >> 1 \end{cases} \qquad \text{(Q10)}$$

Weak Turbulence

The log irradiance variance, $\sigma_{\ln I}^2$, is defined under the Rytov approximation for weak turbulence by[8]

$$\sigma_{\ln I}^2 = 8\pi^2 k^2 \int_0^L \int_0^\infty \kappa \Phi_n(\kappa) \left[1 - \cos\left(\frac{\kappa^2 z}{k} \right) \right] d\kappa dz \qquad \text{(Q11)}$$

where

$$\Phi_n(\kappa) = 0.033 C_n^2 \kappa^{-11/3} \quad \text{(zero inner scale)}$$

Substituting

$$\xi = z / L \;\rightarrow\; dz = L d\xi$$

$$\eta = L\kappa^2 / k \;\rightarrow \kappa^2 = \eta k / L,$$

$$d\kappa = \frac{1}{2}\left(\frac{\eta k}{L} \right)^{-1/2} \left(\frac{k}{L} \right) d\eta = \frac{1}{2}\left(\frac{L}{\eta k} \right)^{1/2} \left(\frac{k}{L} \right) d\eta$$

$$\eta\xi = \left(\frac{z}{L} \right)\left(\frac{L\kappa^2}{k} \right) = \frac{\kappa^2 z}{k}$$

Then

$$\sigma^2_{\ln I}(L) = 8\pi^2 k^2 \int_0^1\int_0^\infty (\eta k/L)^{1/2}\Phi_n(\kappa)[1-\cos(\eta\xi)]\frac{1}{2}\left(\frac{L}{\eta k}\right)^{1/2}\left(\frac{k}{L}\right)d\eta L d\xi$$

$$= \frac{8\pi^2 k^3}{2}\int_0^1\int_0^\infty \Phi_n(\kappa)[1-\cos(\eta\xi)]d\eta d\xi$$

$$= \frac{8\pi^2 k^3}{2}\int_0^1\int_0^\infty 0.033 C_n^2 \kappa^{-11/3}[1-\cos(\eta\xi)]d\eta d\xi$$

$$= \frac{8\pi^2(0.033C_n^2)k^3}{2}\int_0^1\int_0^\infty \kappa^{-11/3}[1-\cos(\eta\xi)]d\eta d\xi$$

$$= \frac{8\pi^2(0.033)}{2}\left(\frac{1.23}{1.23}\right)C_n^2\left(k^{7/6}k^{11/6}\right)\left(\frac{L^{11/6}}{L^{11/6}}\right)\int_0^1\int_0^\infty \kappa^{-11/3}[1-\cos(\eta\xi)]d\eta d\xi$$

$$= \frac{8\pi^2(0.033)}{(2)(1.23)}(1.23C_n^2 k^{7/6}L^{11/6})\int_0^1\int_0^\infty \left(\frac{k^{11/6}}{L^{11/6}\kappa^{11/3}}\right)[1-\cos(\eta\xi)]d\eta d\xi$$

$$= 1.06\sigma_R^2\int_0^1\int_0^\infty \left(\frac{k}{L\kappa^2}\right)^{11/6}[1-\cos(\eta\xi)]d\eta d\xi$$

$$= 1.06\sigma_R^2\int_0^1\int_0^\infty \eta^{-11/6}[1-\cos(\eta\xi)]d\eta d\xi \qquad \text{(Q12)}$$

where

$$\sigma_R^2 = 1.23C_n^2 k^{7/6}L^{11/6}$$

$$\eta = \frac{k}{L\kappa^2}$$

Using the relationship $\sin^2(u) = \frac{1}{2}[1-\cos(2u)]$, the previous equation reduces to

$$\sigma^2_{\ln I}(L) = 2.12(\sigma_R^2)\int_0^1\int_0^\infty \eta^{-11/6}\sin^2\left(\frac{\eta\xi}{2}\right)d\eta d\xi \qquad \text{(Q13)}$$

Recognizing[35]

$$\int_0^\infty x^{u-1}\sin^2 ax dx = \frac{(-)\Gamma(u)\cos\frac{u\pi}{2}}{2^{u+1}a^u} \qquad \text{(Q14)}$$

where

u = -5/6

$$a=\frac{\xi}{2}$$

leads to

$$\sigma^2_{\ln I}(L) = 2.12\sigma_R^2 \int_0^1 \frac{(-)\Gamma(-5/6)\cos\left(\frac{-5\pi}{12}\right)}{2^{1/6}\left(\frac{\xi}{2}\right)^{-5/6}} d\xi$$

$$= (-)2.12\Gamma(-5/6)\cos\left(\frac{-5\pi}{12}\right)(2^{-1/6})\left(\frac{1}{2}\right)^{5/6} \sigma_R^2 \int_0^1 \xi^{5/6} d\xi$$

$$= (-)2.12\Gamma(-5/6)(2^{-1/6})\left(\frac{1}{2}\right)^{5/6} \sigma_R^2 \left(\frac{6}{11}\right)$$

$$= (-)2.12(\text{-}6.680)(0.259)(0.89)(0.561)\left(\frac{6}{11}\right)\sigma_R^2$$

$$= \sigma_R^2 \qquad \text{(Q15)}$$

Assume in equation Q5 that $l_0 \to 0$, $Q_m \to \infty$

$$\sigma^2_{\ln x}(L) = 0.16\sigma_R^2 \left(\frac{\eta_x Q_m}{\eta_x + Q_m}\right)^{7/6}$$

$$\cong 0.16\sigma_R^2 \left(\frac{\eta_x Q_m}{Q_m}\right)^{7/6}$$

$$\cong 0.16\sigma_R^2 \eta_x^{7/6} \qquad \text{(Q16)}$$

Under weak irradiance fluctuations[8], $\sigma^2_{\ln I} << 1$,

$$\sigma^2_{\ln I} = \sigma^2_{\ln x} + \sigma^2_{\ln y} \qquad \text{(Q17)}$$

Equating equations Q15 through Q17, and using equations Q8 and Q16, results in

$$0.16\eta_x^{7/6} + 1.272\eta_y^{-5/6} \cong 1, \quad \sigma_R^2 << 1, \quad l_0 \to 0 \qquad \text{(Q18)}$$

Assuming $\eta_x \cong \eta_y$, then $\eta_x = \eta_y = 3$ is an approximate solution.

Plugging this into equations Q9 and Q10

$$\eta_x = \frac{1}{c_1 + c_2 L / k\rho_o^2} \cong \frac{1}{c_1 + c_2(0)} = \frac{1}{c_1} = 3$$

$$\rightarrow c_1 \cong 1/3 \qquad \text{(Q19)}$$

$$\eta_y = c_3 + \frac{c_4 L}{k\rho_0^2} \cong c_3 + c_4(0) = c_3 = 3 \qquad \text{(Q20)}$$

Thus, c_1 and c_3 are now established using behavior of the scintillation index in the weak turbulence regime.

Saturated Turbulence

First, perform an asymptotic comparison for small-scale turbulent cell effects. Normalized irradiance[8] can be expressed as I = *xy* where *x* is associated with large-scale turbulent eddy affects, *y* is associated with small-scale eddy effects, *x* and *y* are statistically independent random quantities, and $<x>=<y>=1$. Given these conditions, $<I> = <x><y> = 1$. The second moment of irradiance takes the form[8]

$$<I^2> = <x^2><y^2> = (1+\sigma_x^2)(1+\sigma_y^2) \qquad \text{(Q21)}$$

where σ_x^2 and σ_y^2 are the large scale and small scale normalized variances of *x* and *y*, respectively. Based on equations 5 and Q21, the implied scintillation index is

$$\sigma_I^2 = \left(1+\sigma_x^2\right)\left(1+\sigma_y^2\right) - 1$$

$$= \sigma_x^2 + \sigma_y^2 + \sigma_x^2\sigma_y^2 \qquad \text{(Q22)}$$

Per equation 17, the asymptotic behavior of the scintillation index in the saturation regime is described by $\sigma_I^2 = 1 + 3.40/\left(\sigma_R^2 Q_m^{7/6}\right)^{1/6}$, $\sigma_R^2 Q_m^{7/6} >> 100$ which approaches an asymptotic limit of one. Therefore, in saturated turbulence

$$\sigma_I^2 = \sigma_x^2 + \sigma_y^2 + \sigma_x^2\sigma_y^2 \cong 1 \qquad \text{(Q23)}$$

In strong fluctuations, expect the large-scale scintillation terms to die out.

$$\sigma_I^2 \approx \sigma_y^2 \cong 1 \tag{Q24}$$

The small scale irradiance is given by[8]

$$\sigma_y^2 = \exp\left(\sigma_{\ln y}^2\right) - 1 \tag{Q25}$$

As a result, in strong fluctuations

$$\sigma_y^2 \cong 1 = \exp\left(\sigma_{\ln y}^2\right) - 1 \tag{Q26}$$

$$\rightarrow \sigma_{\ln y}^2 \cong \ln 2 \tag{Q27}$$

The plane wave spatial coherence radius, ρ_0, when C_n^2 is constant and $l_0 << \rho << L_0$ can be approximated by[1]

$$\rho = \left(1.46 C_n^2 k^2 L\right)^{-3/5} \quad \rightarrow C_n^2 = \frac{\rho_0^{-5/3}}{1.46 k^2 L} \tag{Q28}$$

Plugging this into the Rytov Variance, $\sigma_R^2 = 1.23 C_n^2 k^{7/6} L^{11/6} = 1.23\left(\frac{\rho_0^{-5/3}}{1.46 k^2 L}\right) k^{7/6} L^{11/6}$,

$$\frac{L}{k\rho_0^2} = 1.22(\sigma_R^2)^{6/5} \tag{Q29}$$

In saturated turbulence, equating equation Q27 to equation Q8 results in

$$\begin{aligned} \ln 2 &= 1.272\sigma_R^2 (\eta_y)^{-5/6} \\ &= 1.272\sigma_R^2 \left(\frac{c_4 L}{k\rho_0^2}\right)^{-5/6} \\ &= 1.272\sigma_R^2 c_4^{-5/6}\left[1.22(\sigma_R^2)^{6/5}\right]^{-5/6} \\ &= 1.272 c_4^{-5/6}(1.22)^{-5/6} \end{aligned}$$

$$\rightarrow c_4 \cong 1.7 \tag{Q30}$$

where

$$\eta_y = \frac{c_4 L}{k\rho_0^2} \quad \text{(equation Q10)}$$

$$\frac{L}{k\rho_0^2} = 1.22(\sigma_R^2)^{6/5} \quad \text{(equation Q29)}$$

Now determine an asymptotic comparison for large-scale turbulent cell effects. For saturated optical turbulence[8]

$$\sigma_x^2 = \exp\left(\sigma_{\ln x}^2\right) - 1 \tag{Q31}$$

$$\rightarrow \exp\left(\sigma_{\ln x}^2\right) = \sigma_x^2 + 1$$

Also[8] $$\sigma_I^2 = \exp\left(\sigma_{\ln x}^2 + \sigma_{\ln y}^2\right) - 1 \tag{Q32}$$

Therefore,

$$\begin{aligned} \sigma_I^2 &\cong \exp\left(\sigma_{\ln x}^2 + \ln 2\right) - 1 \\ &= 2\exp\left(\sigma_{\ln x}^2\right) - 1 \\ &= 2\left(\sigma_x^2 + 1\right) - 1 \\ &= 2\sigma_x^2 + 1 \end{aligned} \tag{Q33}$$

Under weak fluctuation $\left(\sigma_x^2 << 1\right)$,

$$\sigma_x^2 = \exp\left(\sigma_{\ln x}^2\right) - 1 \cong \sigma_{\ln x}^2 \tag{Q34}$$

Note in equation Q16 that $\mu_x \rightarrow \eta_x$ as $l_0 \rightarrow 0$ (i.e., $l_0 \rightarrow 0$ and $L_0 \rightarrow \infty$). Therefore, for determining c_2, assume

$$\sigma_{\ln x}^2 = 0.16\sigma_R^2 \mu_x{}^{7/6} \cong 0.16\sigma_R^2 \eta_x{}^{7/6} \tag{Q35}$$

Then

$$\begin{aligned} \sigma_I^2 &\cong 2\sigma_x^2 + 1 \\ &\cong 2(0.16\sigma_R^2 \eta_x{}^{7/6}) + 1 \\ &= 0.32\sigma_R^2 \eta_x{}^{7/6} + 1 \end{aligned} \tag{Q36}$$

For plane wave von Karman spectrum, with inner scale effects, the plane wave spatial coherence radius is[1] $\rho_0 = \left(1.64 C_n^2 k^2 L l_0^{-1/3}\right)^{-1/2}$.

$$
\begin{aligned}
L / k\rho_0^2 &= \frac{L}{k}\rho_0^{-2} \\
&= \frac{L}{k}\left[(1.64 C_n^2 k^2 L l_0^{-1/3})^{-1/2}\right]^{-2} \\
&= 1.64 C_n^2 k L^2 l_0^{-1/3} \\
&= \frac{1.64}{1.23}(1.23 C_n^2 k^{7/6} L^{11/6})\left(\frac{L^{1/6}}{k^{1/6} l_0^{1/3}}\right) \\
&= \frac{1.33}{(35.05)^{1/6}}\sigma_R^2\left(\frac{35.05 L}{k l_0^2}\right)^{1/6} \\
&= 0.737\sigma_R^2 Q_m^{1/6} \qquad \text{(Q37)}
\end{aligned}
$$

Reference equation Q9, assuming $\frac{L}{k\rho_0^2} >> 1$

$$
\eta_x = \frac{1}{c_1 + c_2 L / k\rho_o^2} \cong \frac{1}{c_2 L / k\rho_o^2} = \frac{1}{c_2 0.737\sigma_R^2 Q_m^{1/6}} \qquad \text{(Q38)}
$$

and

$$
\begin{aligned}
\sigma_I^2 &\cong 0.32\sigma_R^2\left(\frac{1}{c_2 0.737\sigma_R^2 Q_m^{1/6}}\right)^{7/6} + 1 \\
&= 1 + 0.32\sigma_R^2\left(\frac{1}{c_2}\right)^{7/6}\left(\frac{1}{0.737}\right)^{7/6}\left(\frac{1}{\sigma_R^2}\right)^{7/6}\left(\frac{1}{Q_m^{1/6}}\right)^{7/6} \\
&= 1 + 0.46\sigma_R^2\left(\frac{1}{c_2}\right)^{7/6}\left(\frac{1}{\sigma_R^{7/3}}\right)\left(\frac{1}{Q_m^{7/36}}\right) \\
&= 1 + 0.46\left(\frac{1}{c_2}\right)^{7/6}\left(\frac{1}{\sigma_R^{1/3}}\right)\left(\frac{1}{Q_m^{7/36}}\right) \qquad \text{(Q39)}
\end{aligned}
$$

Compare this to the existing scintillation expression (equation 17) for a plane wave in saturated turbulence.

$$1+0.46\left(\frac{1}{c_2}\right)^{7/6}\left(\frac{1}{\sigma_R^{1/3}}\right)\left(\frac{1}{Q_m^{7/36}}\right)=1+\frac{3.40}{\left(\sigma_R^2 Q_m^{7/6}\right)^{1/6}}$$

$$\rightarrow\ 0.46\left(\frac{1}{c_2}\right)^{7/6}\left(\frac{1}{\sigma_R^{1/3}}\right)\left(\frac{1}{Q_m^{7/36}}\right)=\frac{3.40}{\left(\sigma_R^2 Q_m^{7/6}\right)^{1/6}}$$

$$\rightarrow\ c_2^{7/6}=\frac{0.46}{3.40}\left(\frac{(\sigma_R^2)^{1/6}}{\sigma_R^{1/3}}\right)\left(\frac{(Q_m^{7/6})^{1/6}}{Q_m^{7/36}}\right)$$

$$\rightarrow\ c_2=0.18 \qquad \text{(Q40)}$$

Thus, c_2 and c_4 are now established using behavior of the scintillation index in the saturated turbulence regime.

<u>"All" Turbulence Regimes</u>

Applying constants c_1, c_2, c_3, and c_4 results in the following expressions for predicting scintillation behavior in all turbulence regimes (including inner scale effects)

$$\begin{aligned}\eta_x &= \frac{1}{c_1+c_2 L/k\rho_o^2}\\ &\cong \frac{1}{1/3+0.18L/k\rho_o^2}\\ &= \frac{1}{1/3+0.18(0.737\sigma_R^2 Q_m^{1/6})}\\ &= \frac{3}{1+0.40\sigma_R^2 Q_m^{1/6}}\end{aligned} \qquad \text{(Q41)}$$

and

$$\begin{aligned}\sigma_{\ln x}^2 &\cong 0.16\sigma_R^2\left(\frac{\eta_x Q_m}{\eta_x+Q_m}\right)^{7/6}\\ &= 0.16\sigma_R^2\left[\frac{\left(\frac{3}{1+0.40\sigma_R^2 Q_m^{1/6}}\right)Q_m}{\left(\frac{3}{1+0.40\sigma_R^2 Q_m^{1/6}}\right)+Q_m}\right]^{7/6}\end{aligned}$$

$$= 0.16\sigma_R^2 \left[\frac{3Q_m}{3 + Q_m\left(1 + 0.40\sigma_R^2 Q_m^{1/6}\right)} \right]^{7/6}$$

$$= 0.16\sigma_R^2 \left(\frac{3Q_m}{3 + Q_m + 0.40\sigma_R^2 Q_m^{7/6}} \right)^{7/6} \qquad \text{(Q42)}$$

From equations Q10 and Q29

$$\begin{aligned} \eta_y &= c_3 + \frac{c_4 L}{k\rho_0^2} \\ &= 3 + \frac{1.7L}{k\rho_0^2} \\ &= 3 + 1.7\left[1.22(\sigma_R^2)^{6/5}\right] \\ &= 3 + 2.07(\sigma_R^2)^{6/5} \\ &= 3\left(1 + 0.69\sigma_R^{12/5}\right) \end{aligned} \qquad \text{(Q43)}$$

$$\begin{aligned} \sigma_{\ln y}^2(L) &= 1.272\sigma_R^2 \eta_y^{-5/6} \\ &= 1.272\sigma_R^2 \left[3\left(1 + 0.69\sigma_R^{12/5}\right)\right]^{-5/6} \\ &\cong \frac{0.509\sigma_R^2}{\left(1 + 0.69\sigma_R^{12/5}\right)^{5/6}} \end{aligned} \qquad \text{(Q44)}$$

For a plane wave in weak turbulence (equation 10), let

$$\sigma_I^2 = \sigma_p^2 = 3.86\sigma_R^2 \left\{ \left(1 + \frac{1}{Q_m^2}\right)^{11/12} \sin\left(\frac{11}{6}\tan^{-1} Q_m\right) - \frac{11}{6} Q_m^{-5/6} \right\} \qquad \text{(Q45)}$$

To apply inner-scale effects to the von Karman derived scintillation index, impose

$$\sigma_{\ln y}^2 \cong \frac{0.509\sigma_R^2}{(1 + 0.69\sigma_R^{12/5})^{5/6}} \cong 0.51\sigma_R^2 = 0.51\sigma_p^2 \qquad \text{(Q46)}$$

Recall that $\sigma_{\ln y}^2(L) = 1.272\sigma_R^2 \eta_y^{-5/6}$. Then,

$$\eta_y \cong \left(\frac{1.272\sigma_R^2}{0.51\sigma_p^2} \right)^{6/5} \cong 3\left(\frac{\sigma_R^2}{\sigma_p^2} \right)^{6/5} \qquad \text{(Q47)}$$

Reference equation Q44, under strong fluctuations, inner-scale effects tend to diminish such that

$$\eta_y = 3(0.69)\sigma_R^{12/5}$$
$$= 2.07\left(\sigma_R^2\right)^{6/5} \quad \text{(Q48)}$$

As a result, for all fluctuations

$$\eta_y \cong 3\left(\frac{\sigma_R^2}{\sigma_p^2}\right)^{6/5} + 2.07\left(\sigma_R^2\right)^{6/5}$$
$$= 3\left(\frac{\sigma_R}{\sigma_p}\right)^{12/5}(1+0.69\sigma_p^{12/5}) \quad \text{(Q49)}$$

As a test of this expression, notice that under strong fluctuations

$\eta_y \cong 3\left(\frac{\sigma_R^2}{\sigma_p^2}\right)^{6/5}\left(0.69\sigma_p^{12/5}\right) = 2.07\left(\sigma_R^2\right)^{6/5}$ is in agreement with equation Q48.

The small-scale log-irradiance variance becomes

$$\sigma_{\ln y}^2(L) \cong 1.272\sigma_R^2\left[3\left(\frac{\sigma_R}{\sigma_P}\right)^{12/5}\left(1+0.69\sigma_P^{12/5}\right)\right]^{-5/6}$$
$$= \frac{0.51\sigma_P^2}{\left(1+0.69\sigma_P^{12/5}\right)^{5/6}} \quad \text{(Q50)}$$

Recall from equation Q33 that $\sigma_I^2 = \exp\left(\sigma_{\ln x}^2 + \sigma_{\ln y}^2\right) - 1$

Therefore, for a plane wave in moderate turbulence using an "effective" von Karman spectrum

$$\sigma_I^2 \cong \exp\left[0.16\sigma_R^2\left(\frac{3Q_m}{3+Q_m+0.40\sigma_R^2 Q_m^{7/6}}\right)^{7/6} + \frac{0.51\sigma_p^2}{\left(1+0.69\sigma_p^{12/5}\right)^{5/6}}\right] - 1 \quad \text{(Q51)}$$

where $\sigma_p^2 = 3.86\sigma_R^2\left[\left(1+\frac{1}{Q_m^2}\right)^{11/12}\sin\left(\frac{11}{6}\tan^{-1}Q_m\right) - \frac{11}{6}Q_m^{-5/6}\right]$

APPENDIX R: MODERATE TURBULENCE, VON KARMAN, SPHERICAL WAVE

Large-Scale Log-Irradiance Variance

Using the "effective" atmospheric spectrum (equation 23), the large-scale log-irradiance variance for a spherical wave in the presence of a finite inner scale is[8]

$$\sigma^2_{\ln x} = 8\pi^2 k^2 \int_0^L \int_0^\infty \kappa 0.033 C_n^2 \kappa^{-11/3} G_x(\kappa) \left\{ 1 - \cos\left[\frac{\kappa^2}{k} z \left(1 - \frac{z}{L} \right) \right] \right\} d\kappa dz \qquad \text{(R1)}$$

Let $1-\cos\left(\frac{\kappa^2 z}{k}\right) \cong \frac{1}{2}\left(\frac{\kappa^2 z}{k}\right)$ (its Taylor expansion estimate). The approximation is valid for the large-scale log-irradiance variance because filter function $G_x(\kappa)$ eliminates high spatial frequency contributions in moderate-to-strong fluctuations. Consequently

$$\begin{aligned}
\sigma^2_{\ln x} &= 8\pi^2 k^2 \int_0^L \int_0^\infty \kappa 0.033 C_n^2 \kappa^{-11/3} G_x(\kappa) \left(1 - \left\{ 1 - \frac{1}{2}\left[\frac{\kappa^2}{k} z \left(1 - \frac{z}{L} \right) \right]^2 \right\} - \right) d\kappa dz \\
&= 8\pi^2 k^2 \int_0^L \int_0^\infty \kappa 0.033 C_n^2 \kappa^{-11/3} G_x(\kappa) \frac{1}{2} \left[\frac{\kappa^2}{k} z (1 - \frac{z}{L}) \right]^2 d\kappa dz \\
&= 4\pi^2 \int_0^L \int_0^\infty \kappa 0.033 C_n^2 \kappa^{-11/3} G_x(\kappa) \kappa^4 z^2 \left(1 - \frac{z}{L} \right)^2 d\kappa dz \qquad \text{(R2)}
\end{aligned}$$

Substituting

$$\xi = z / L \;\rightarrow\; dz = L d\xi$$

$$\eta = \frac{L\kappa^2}{k} \;\rightarrow\; d\kappa = \frac{k}{2L\kappa} d\eta$$

then

$$\begin{aligned}
\sigma^2_{\ln x} &= 4\pi^2 \int_0^1 \int_0^\infty \kappa 0.033 C_n^2 \kappa^{-11/3} G_x(\kappa) \kappa^4 L^2 \xi^2 \left(1 - \frac{L\xi}{L} \right)^2 \frac{k}{2L\kappa} d\eta L d\xi \\
&= 2\pi^2 k L^2 \int_0^1 \int_0^\infty 0.033 C_n^2 \kappa^{-11/3} G_x(\kappa) \kappa^4 \xi^2 (1-\xi)^2 d\eta d\xi \\
&= 0.651 k L^2 C_n^2 \int_0^1 \int_0^\infty \kappa^{1/3} G_x(\kappa) \xi^2 (1-\xi)^2 d\eta d\xi \\
&= 0.651 k L^2 C_n^2 \int_0^1 \xi^2 (1-\xi)^2 dz \int_0^\infty \kappa^{1/3} \exp\left(-\frac{\kappa^2}{\kappa_m^2} \right) \exp\left(\frac{-\kappa^2}{\kappa_x^2} \right) d\eta
\end{aligned}$$

$$= 0.651 k^{7/6} L^{11/6} C_n^2 \int_0^1 \xi^2 (1-\xi)^2 d\xi \int_0^\infty \frac{L^{1/6}}{k^{1/6}} \kappa^{1/3} \exp\left(-\frac{\kappa^2}{\kappa_m^2}\right) \exp\left(\frac{-Lk\kappa^2}{Lk\kappa_x^2}\right) d\eta$$

$$= 0.651 k^{7/6} L^{11/6} C_n^2 \int_0^1 \xi^2 (1-\xi)^2 d\xi \int_0^\infty \left(\frac{L\kappa^2}{k}\right)^{1/6} \exp\left(-\frac{\kappa^2}{\kappa_m^2}\right) \exp\left(\frac{-L\kappa^2}{k}\right)\left(\frac{k}{L\kappa_x^2}\right) d\eta$$

$$= 0.651 k^{7/6} L^{11/6} C_n^2 \int_0^1 \xi^2 (1-\xi)^2 d\xi \int_0^\infty \eta^{1/6} \exp\left(\frac{-\eta}{Q_m}\right) \exp\left(\frac{-\eta}{\eta_x}\right) dn$$

$$= 0.651 \left(\frac{1.23}{1.23}\right) k^{7/6} L^{11/6} C_n^2 \int_0^1 \xi^2 (1-\xi)^2 d\xi \int_0^\infty \eta^{1/6} \exp\left(\frac{-\eta}{Q_m}\right) \exp\left(\frac{-\eta}{\eta_x}\right) dn$$

$$= 0.539 \sigma^2 \int_0^1 \xi^2 (1-\xi)^2 d\xi \int_0^\infty \eta^{1/6} \exp\left(\frac{-\eta}{Q_m}\right) \exp\left(\frac{-\eta}{\eta_x}\right) dn$$

$$= 0.539 \sigma_R^2 \int_0^1 (\xi-\xi^2)^2 d\xi \int_0^\infty \eta^{1/6} \exp\left(\frac{-\eta}{Q_m}\right) \exp\left(\frac{-\eta}{\eta_x}\right) dn \qquad \text{(R3)}$$

where

$$G_x(\kappa) = f(\kappa l_0) \exp\left(-\kappa^2/\kappa_x^2\right)$$
$$= \exp\left(-\kappa^2/\kappa_m^2\right) \exp\left(-\kappa^2/\kappa_x^2\right)$$

$$Q_m = L\kappa_m^2 / k$$

$$\frac{\kappa^2}{\kappa_m^2} = \left(\frac{L\kappa^2}{k}\right)\left(\frac{k}{L\kappa_m^2}\right) = \frac{\eta}{Q_m}$$

$$\sigma_R^2 = 1.23 C_n^2 k^{7/6} L^{11/6}$$

Using a binomial series expansion $(x+y)^n \cong x^n + nx^{n-1}y + \frac{n(n-1)}{2!} x^{n-2} y^2$,

where $x = \xi$, $y = -\xi^2$, $n = 2$, and $(\xi - \xi^2)^2 \cong \xi^2 - 2\xi^3 + \xi^4$, results in

$$\sigma_{\ln x}^2(L) \cong 0.53 \sigma_R^2 \int_0^1 (\xi^2 - 2\xi^3 + \xi^4) d\xi \int_0^\infty \eta^{1/6} \exp\left(\frac{-\eta}{Q_m}\right) \exp\left(\frac{-\eta}{\eta_x}\right) d\eta$$

$$= 0.53 \sigma_R^2 \left(\frac{1}{3} - \frac{2}{4} + \frac{1}{5}\right) \int_0^\infty \eta^{1/6} \exp\left(\frac{-\eta}{Q_m}\right) \exp\left(\frac{-\eta}{\eta_x}\right) d\eta$$

$$= 0.53 \sigma_R^2 \left(\frac{20}{60} - \frac{30}{60} + \frac{12}{60}\right) \int_0^\infty \eta^{1/6} \exp\left(\frac{-\eta}{Q_m}\right) \exp\left(\frac{-\eta}{\eta_x}\right) d\eta$$

$$= 0.53\sigma_R^2\left(\frac{1}{30}\right)\int_0^\infty \eta^{1/6}\exp\left(\frac{-\eta}{Q_m}\right)\exp\left(\frac{-\eta}{\eta_x}\right)d\eta$$

$$= 0.018\sigma_R^2\int_0^\infty \eta^{1/6}\exp\left(\frac{-\eta}{Q_m}\right)\exp\left(\frac{-\eta}{\eta_x}\right)d\eta$$

$$= 0.018\sigma_R^2\int_0^\infty \eta^{1/6}\exp\left(\frac{-\eta}{Q_m}+\frac{-\eta}{\eta_x}\right)d\eta$$

$$= 0.018\sigma_R^2\int_0^\infty \eta^{1/6}\exp\left[-\eta\left(\frac{1}{Q_m}+\frac{1}{\eta_x}\right)\right]d\eta$$

$$= 0.018\sigma_R^2\int_0^\infty \eta^{1/6}\exp\left[-\eta\left(\frac{\eta_x+Q_m}{\eta_x Q_m}\right)\right]d\eta \tag{R4}$$

Recognizing[1]

$$\int_0^\infty e^{-s\eta}\eta^{x-1}d\eta = \frac{\Gamma(x)}{s^x} \tag{R5}$$

where

$$s = \frac{\eta_x+Q_m}{\eta_x Q_m}$$

x-1=1/6 $\rightarrow$ x=7/6

Equation R4 can be written as

$$\sigma_{\ln x}^2(L) = 0.018\sigma_R^2\frac{\Gamma(7/6)}{\left(\frac{\eta_x+Q_m}{\eta_x Q_m}\right)^{7/6}}$$

$$= 0.018\sigma_R^2\left(\frac{\eta_x Q_m}{\eta_x+Q_m}\right)^{7/6}\Gamma(7/6)$$

$$= 0.018\sigma_R^2\left(\frac{\eta_x Q_m}{\eta_x+Q_m}\right)^{7/6}(0.9277)$$

$$= 0.016\sigma_R^2\left(\frac{\eta_x Q_m}{\eta_x+Q_m}\right)^{7/6}$$

$$= 0.016\sigma_R^2\mu_x^{7/6} \tag{R6}$$

where $\mu_x = \left(\frac{\eta_x Q_m}{\eta_x + Q_m} \right)^{7/6}$

Small-scale log-irradiance variance

Using the "effective" atmospheric spectrum (equation 23), the small-scale log-irradiance variance for a spherical wave in the presence of a finite inner scale is[8]

$$\sigma^2_{\ln y} = 8\pi^2 k^2 \int_0^L \int_0^\infty \kappa 0.033 C_n^2 \kappa^{-11/3} G_y(\kappa) \left\{ 1 - \cos\left[\frac{\kappa^2}{k} z \left(1 - \frac{z}{L} \right) \right] \right\} d\kappa dz \qquad \text{(R7)}$$

In moderate-to-strong irradiance fluctuations, $\kappa_y >> \sqrt{k/L}$, let

$$\sigma^2_{\ln y} \cong 8\pi^2 k^2 \int_0^L \int_0^\infty \kappa 0.033 C_n^2 \kappa^{-11/3} G_y(\kappa) d\kappa dz \qquad \text{(R8)}$$

Substituting

$$\xi = z / L \rightarrow dz = Ld\xi$$

$$\eta = \frac{L\kappa^2}{k} \rightarrow d\kappa = \frac{k}{2L\kappa} d\eta$$

$$\begin{aligned}
\sigma^2_{\ln y}(L) &\cong 8\pi^2 k^2 \int_0^1 \int_0^\infty \kappa 0.033 C_n^2 \kappa^{-11/3} G_y(\kappa) \frac{k}{2L\kappa} d\eta L d\xi \\
&= 4\pi^2 k^3 \int_0^1 \int_0^\infty 0.033 C_n^2 \kappa^{-11/3} G_y(\kappa) d\eta d\xi \\
&= 1.30 k^3 C_n^2 \int_0^1 \int_0^\infty \kappa^{-11/3} \left[\frac{\kappa^{11/3}}{\left(\kappa^2 + \kappa_y^2 \right)^{11/6}} \right] d\eta d\xi \\
&= 1.30 k^3 Cn^2 \int_0^\infty \left[\frac{1}{\left(\kappa^2 + \kappa_y^2 \right)^{11/6}} \right] d\eta \\
&= 1.30 \left(\frac{1.23}{1.23} \right) C_n^2 \left(k^{7/6} k^{11/6} \right) \frac{L^{11/16}}{L^{11/16}} \int_0^\infty \left[\frac{1}{\left(\kappa^2 + \kappa_y^2 \right)^{11/6}} \right] d\eta \\
&= 1.06 \sigma_R^2 \int_0^\infty \left[\frac{k^{11/6}}{L^{11/16} \left(\kappa^2 + \kappa_y^2 \right)^{11/6}} \right] d\eta
\end{aligned}$$

$$
\begin{aligned}
&= 1.06\sigma_R^2 \int_0^\infty \left(\frac{k}{L\kappa^2} + \frac{k}{L\kappa_y^2} \right)^{11/6} d\eta \\
&= 1.06\sigma_R^2 \int_0^\infty (\eta + \eta_y)^{-11/6} d\eta \\
&= 1.06\sigma_R^2 \left(\frac{-6}{5} \right) (\eta + \eta_y)^{-5/6} \Big|_{\eta=0}^{\eta=\infty} \\
&= 1.06\sigma_R^2 \left(\frac{-6}{5} \right) [(\infty + \eta_y)^{-5/6} - (0 + \eta_y)^{-5/6}] \\
&= 1.06\sigma_R^2 \left(\frac{-6}{5} \right) \left[-(\eta_y)^{-5/6} \right] \\
&= 1.272\sigma_R^2 \eta_y^{-5/6} \qquad \text{(R9)}
\end{aligned}
$$

where

$$G_y(\kappa) = \frac{\kappa^{11/3}}{\left(\kappa^2 + \kappa_y^2\right)^{11/6}}$$

$$\sigma_R^2 = 1.23 C_n^2 k^{7/6} L^{11/6}$$

Asymptotic Comparisons

For constants c_1, c_2, c_3, and c_4, assume the following functional form of η_x and η_y [see Appendix S].

$$
\eta_x = \frac{1}{c_1 + c_2 L / k\rho_o^2} \cong \begin{cases} \dfrac{1}{c_1 + c_2(0)} = \dfrac{1}{c_1}, & \dfrac{L}{k\rho_0^2} << 1 \\ \dfrac{1}{0 + c_2 L / k\rho_o^2} = \dfrac{k\rho_o^2}{c_2 L}, & \dfrac{L}{k\rho_0^2} >> 1 \end{cases} \qquad \text{(R10)}
$$

$$
\eta_y = c_3 + \frac{c_4 L}{k\rho_0^2} \cong \begin{cases} c_3 + c_4(0) = c_3, & \dfrac{L}{k\rho_0^2} << 1 \\ 0 + \dfrac{c_4 L}{k\rho_0^2} = \dfrac{c_4 L}{k\rho_0^2}, & \dfrac{L}{k\rho_0^2} >> 1 \end{cases} \qquad \text{(R11)}
$$

Weak Turbulence

Assume in equations 13 and R6 that $l_0 \to 0$, $Q_m \to \infty$

$$\begin{aligned}\sigma_{I,l}^2(L) &= 3.86\sigma_R^2\left[\left\{0.4(1+0)^{11/12}\sin\left[(11/6)\tan^{-1}(\infty)\right]\right\}-\frac{11}{6}(0)\right]\\ &= 3.86\sigma_R^2\left\{0.4\sin\left[(11/6)(\pi/2)\right]\right\}\\ &= 0.4\sigma_R^2 \end{aligned} \quad \text{(R12)}$$

and

$$\begin{aligned}\sigma_{\ln x}^2(L) &= 0.016\sigma_R^2\left(\frac{\eta_x Q_m}{\eta_x + Q_m}\right)^{7/6}\\ &\cong 0.016\sigma_R^2\left(\frac{\eta_x Q_m}{Q_m}\right)^{7/6}\\ &= 0.016\sigma_R^2\eta_x^{7/6}\end{aligned} \quad \text{(R13)}$$

Weak turbulence, assuming $\sigma_I^2 << 1$, results in

$$\sigma_I^2 = \exp\left(\sigma_{\ln I}^2\right) - 1 \cong \sigma_{\ln I}^2 \quad \text{(R14)}$$

Under weak irradiance fluctuations[8], $\sigma_{\ln I}^2 << 1$,

$$\sigma_{\ln I}^2 = \sigma_{\ln x}^2 + \sigma_{\ln y}^2 \quad \text{(N15)}$$

Equating equations R12 and R14 and using equations R9 and R13

$$0.016\eta_x^{7/6} + 1.272\eta_y^{-5/6} \cong 0.4 \quad \text{(R16)}$$

Assuming $\eta_x \cong \eta_y$, then $\eta_x = \eta_y = 8$ is an approximate solution

Plugging this into equations R10 and R11

$$\eta_x = \frac{1}{c_1 + c_2 L / k\rho_o^2} \cong \frac{1}{c_1 + c_2(0)} = \frac{1}{c_1} = 8$$

$$\to c_1 \cong 1/8 \quad \text{(R17)}$$

$$\eta_y = c_3 + \frac{c_4 L}{k\rho_0^2} \cong c_3 + c_4(0) = c_3 = 8 \quad \text{(R18)}$$

Thus, c_1 and c_3 are now established using behavior of the scintillation index in the weak turbulence regime.

Saturated Turbulence

First, perform an asymptotic comparison for small-scale turbulent cell effects. Normalized irradiance[8] can be expressed as I = *xy* where *x* is associated with large-scale turbulent eddy affects, *y* is associated with small-scale eddy effects, *x* and *y* are statistically independent random quantities, and $<x>=<y>=1$. Given these conditions, $<I> = <x><y> = 1$. The second moment of irradiance takes the form[8]

$$<I^2> = <x^2><y^2> = (1+\sigma_x^2)(1+\sigma_y^2) \qquad \text{(R19)}$$

where σ_x^2 and σ_y^2 are the large scale and small scale normalized variances of *x* and *y*, respectively. Based on equations 5 and R19, the implied scintillation index is

$$\sigma_I^2 = \left(1+\sigma_x^2\right)\left(1+\sigma_y^2\right)-1$$

$$= \sigma_x^2 + \sigma_y^2 + \sigma_x^2\sigma_y^2 \qquad \text{(R20)}$$

Per equation 20, the asymptotic behavior of the scintillation index in the saturation regime is described by $\sigma_I^2 = 1 + 10.36/\left(\sigma_R^2 Q_m^{7/6}\right)^{1/6}$, $\sigma_R^2 Q_m^{7/6}$ >>100 which approaches an asymptotic limit of one. Therefore, in saturated turbulence

$$\sigma_I^2 = \sigma_x^2 + \sigma_y^2 + \sigma_x^2\sigma_y^2 \cong 1 \qquad \text{(R21)}$$

In strong fluctuations, expect the large-scale scintillation terms to die out.

$$\sigma_I^2 \approx \sigma_y^2 \cong 1 \qquad \text{(R22)}$$

The small scale irradiance is given by[8]

$$\sigma_y^2 = \exp\left(\sigma_{\ln y}^2\right) - 1 \qquad \text{(R23)}$$

As a result, in strong fluctuations

$$\sigma_y^2 \cong 1 = \exp\left(\sigma_{\ln y}^2\right) - 1 \qquad \text{(R24)}$$

$$\rightarrow \sigma_{\ln y}^2 \cong \ln 2 \qquad \text{(R25)}$$

The spherical wave spatial coherence radius, ρ_0, when C_n^2 is constant and $l_0 << \rho << L_0$ can be approximated by[1]

$$\rho_0 = \left(0.55 C_n^2 k^2 L\right)^{-3/5} \rightarrow C_n^2 = \frac{\rho_0^{-5/3}}{0.55 k^2 L} \qquad \text{(R26)}$$

Plugging this into the Rytov Variance, $\sigma_R^2 = 1.23 Cn^2 k^{7/6} L^{11/6}$ results in

$$\sigma_R^2 = 1.23\left(\frac{\rho_0^{-5/3}}{1.46 k^2 L}\right) k^{7/6} L^{11/6}$$

$$\rightarrow \frac{L}{k\rho_0^2} = 0.38(\sigma_R^2)^{6/5} \qquad \text{(R27)}$$

In saturated turbulence, equating equation R25 to equation R9 results in

$$\begin{aligned} \ln 2 &= 1.272\sigma_R^2(\eta_y)^{-5/6} \\ &= 1.272\sigma_R^2\left(\frac{c_4 L}{k\rho_0^2}\right)^{-5/6} \\ &= 1.272\sigma_R^2 c_4^{-5/6}\left[0.38(\sigma_R^2)^{6/5}\right]^{-5/6} \\ &= 1.272 c_4^{-5/6}(0.38)^{-5/6} \end{aligned}$$

$$\rightarrow c_4 \cong 5.45 \qquad \text{(R28)}$$

where

$\eta_y = \dfrac{c_4 L}{k\rho_0^2}$ (equation R11)

$$\frac{L}{k\rho_0^2} = 0.38(\sigma_R^2)^{6/5} \quad \text{(equation R27)}$$

Now determine an asymptotic comparison for large-scale turbulent cell effects. For saturated optical turbulence[8]

$$\sigma_x^2 = \exp\left(\sigma_{\ln x}^2\right) - 1 \tag{R29}$$

$$\rightarrow \exp\left(\sigma_{\ln x}^2\right) = \sigma_x^2 + 1$$

Also[8]

$$\sigma_I^2 = \exp\left(\sigma_{\ln x}^2 + \sigma_{\ln y}^2\right) - 1 \tag{R30}$$

Therefore,

$$\begin{aligned} \sigma_I^2 &\cong \exp\left(\sigma_{\ln x}^2 + \ln 2\right) - 1 \\ &= 2\exp\left(\sigma_{\ln x}^2\right) - 1 \\ &= 2\left(\sigma_x^2 + 1\right) - 1 \\ &= 2\sigma_x^2 + 1 \end{aligned} \tag{R31}$$

Recall from equation R13 that $\sigma_{\ln x}^2 \cong 0.016\sigma_R^2 {\eta_x}^{7/6}$ as $l_0 \rightarrow 0$.

As a result

$$\begin{aligned} \sigma_I^2 &\cong 2\sigma_x^2 + 1 \\ &\cong 2\left(0.016\sigma_R^2 {\eta_x}^{7/6}\right) + 1 \\ &= 0.032\sigma_R^2 {\eta_x}^{7/6} + 1 \end{aligned} \tag{R32}$$

Under weak fluctuation $\left(\sigma_x^2 << 1\right)$,

$$\sigma_x^2 = \exp\left(\sigma_{\ln x}^2\right) - 1 \cong \sigma_{\ln x}^2 \tag{R33}$$

For spherical wave von Karman spectrum, with inner scale effects, the plane wave spatial coherence radius is[1] $\rho = (0.55 C_n^2 k^2 L l_0^{-1/3})^{-1/2}$ for $\rho << l_0$

$$L / k\rho_0^2 = \frac{L}{k}\rho_0^{-2}$$

$$= \frac{L}{k}\left[\left(0.55C_n^2 k^2 L l_0^{-1/3}\right)^{-1/2}\right]^{-2}$$

$$= \frac{L}{k} 0.55C_n^2 k^2 L l_0^{-1/3}$$

$$= 0.55\left(\frac{1.23k^{1/6}}{1.23k^{1/6}}\right) C_n^2 k L^2 l_0^{-1/3}$$

$$= 0.447\sigma_R^2 \left(\frac{35.046L}{35.046kl_0^2}\right)^{1/6}$$

$$= \frac{0.447}{(35.046)^{1/6}} \sigma_R^2 \left(\frac{10.89L}{kl_0^2}\right)^{1/6}$$

$$= 0.247\sigma_R^2 Q_m^{1/6} \qquad \text{(R34)}$$

Reference equation R10, assuming $\frac{L}{k\rho_0^2}$ >>1, then

$$\eta_x \cong \frac{k\rho_o^2}{c_2 L} = \frac{1}{c_2 0.247\sigma_R^2 Q_m^{1/6}}$$

and

$$\sigma_I^2 \cong 0.032\sigma_R^2 \eta_x{}^{7/6} + 1$$

$$= 0.032\sigma_R^2 \left(\frac{1}{c_2 (0.247)\sigma_R^2 Q_m^{1/6}}\right)^{7/6} + 1$$

$$= 1 + 0.032\sigma_R^2 \left(\frac{1}{c_2}\right)^{7/6} \left(\frac{1}{0.247}\right)^{7/6} \left(\frac{1}{\sigma_R^2}\right)^{7/6} \left(\frac{1}{Q_m^{1/6}}\right)^{7/6}$$

$$= 1 + 0.163\sigma_R^2 \left(\frac{1}{c_2}\right)^{7/6} \left(\frac{1}{\sigma_R^{7/3}}\right)\left(\frac{1}{Q_m^{7/36}}\right)$$

$$= 1 + 0.163\left(\frac{1}{c_2}\right)^{7/6} \left(\frac{1}{\sigma_R^{1/3}}\right)\left(\frac{1}{Q_m^{7/36}}\right) \qquad \text{(R35)}$$

Compare this to the existing scintillation expression (equation 20) for a spherical wave in saturated turbulence.

$$\sigma_I^2 = 1 + 0.163\left(\frac{1}{c_2}\right)^{7/6} \left(\frac{1}{\sigma_R^{1/3}}\right)\left(\frac{1}{Q_m^{7/36}}\right) = 1 + \frac{10.36}{\left(\sigma_R^2 Q_m^{7/6}\right)^{1/6}}$$

$$\rightarrow 0.163\left(\frac{1}{c_2}\right)^{7/6}\left(\frac{1}{\sigma_R^{1/3}}\right)\left(\frac{1}{Q_m^{7/36}}\right)=\frac{10.36}{\left(\sigma_R^2 Q_m^{7/6}\right)^{1/6}}$$

$$\rightarrow \frac{0.163}{10.36}\left(\sigma_R^2 Q_m^{7/6}\right)^{1/6}\left(\frac{\left(\sigma_R^2\right)^{1/6}}{\sigma_R^{1/3}}\right)\left(\frac{\left(Q_m^{7/6}\right)^{1/6}}{Q_m^{7/36}}\right)=c_2^{7/6}$$

$$\rightarrow c_2 = 0.0285 \qquad \text{(R36)}$$

Thus, c_2 and c_4 are now established using behavior of the scintillation index in the saturated turbulence regime.

"All" Turbulence Regimes

Applying constants c_1, c_2, c_3, and c_4 results in the following expressions for predicting scintillation behavior in all turbulence regimes (including inner scale effects)

$$\begin{aligned}\eta_x &= \frac{1}{c_1 + c_2 L / k\rho_o^2} \\ &\cong \frac{1}{1/8 + 0.0285 L / k\rho_o^2} \\ &= \frac{8}{1 + 0.228 L / k\rho_o^2} \\ &= \frac{8}{1 + 0.228\left(0.247\sigma_R^2 Q_m^{1/6}\right)} \\ &= \frac{8}{1 + 0.056\sigma_R^2 Q_m^{1/6}} \end{aligned} \qquad \text{(R37)}$$

and

$$\begin{aligned}\sigma_{\ln x}^2 &= 0.016\sigma_{R1}^2 \mu_x^{7/6} \\ &= 0.016\sigma_R^2\left(\frac{8Q_m}{8 + Q_m + 0.056\sigma_R^2 Q_m^{7/6})}\right)^{7/6}\end{aligned} \qquad \text{(R38)}$$

where

$$\mu_x = \frac{\eta_x Q_m}{\eta_x + Q_m}$$

$$= \frac{\left(\dfrac{8}{1+0.056\sigma_R^2 Q_m^{1/6}} Q_m\right)}{\left(\dfrac{8}{1+0.056\sigma_R^2 Q_m^{1/6}} + Q_m\right)}$$

$$= \frac{8Q_m}{8+Q_m+0.056\sigma_R^2 Q_m^{7/6}}$$

From equation R11 and R27

$$\begin{aligned}\eta_y &= c_3 + \frac{c_4 L}{k\rho_0^2} \\ &= 8 + \frac{5.45L}{k\rho_0^2} \\ &= 8 + 5.45\left[0.38(\sigma_R^2)^{6/5}\right] \\ &= 8 + 2.07(\sigma_R^2)^{6/5}\end{aligned} \tag{R39}$$

For a spherical wave in weak turbulence (equation 9), let

$$\sigma_I^2 = \sigma_s^2 = 3.86\sigma_R^2\left[\left\{0.4\left(1+\frac{9}{Q_m^2}\right)^{11/12}\sin\left[(11/6)\tan^{-1}\left(\frac{Q_m}{3}\right)\right]\right\} - \frac{11}{6}Q_m^{-5/6}\right] \tag{R40}$$

Under strong fluctuations, inner-scale effects tend to diminish such that

$$\begin{aligned}\eta_y &= 2.07\left(\sigma_R^2\right)^{6/5} \\ &= 3(0.69)\sigma_R^{12/5}\end{aligned} \tag{R41}$$

For all fluctuations, impose the following expression

$$\eta_y = 3\left(\frac{\sigma_R^2}{\sigma_S^2}\right)^{6/5}(1+0.69\sigma_S^{12/5}) \tag{R42}$$

As a test of this expression, notice that under strong fluctuations

$\eta_y \cong 3\left(\dfrac{\sigma_R^2}{\sigma_S^2}\right)^{6/5}\left(0.69\sigma_S^{12/5}\right) = 3(0.69)\sigma_R^{12/5} = 2.07\left(\sigma_R^2\right)^{6/5}$ in agreement with equation R41.

The small-scale log-irradiance variance becomes

$$\sigma_{\ln y}^2(L) \cong 1.272\sigma_R^2\left[3\left(\frac{\sigma_R}{\sigma_s}\right)^{12/5}\left(1+0.69\sigma_s^{12/5}\right)\right]^{-5/6}$$

$$= \frac{0.51\sigma_s^2}{\left(1+0.69\sigma_s^{12/5}\right)^{5/6}} \qquad \text{(R43)}$$

Recall from equation R31 that $\sigma_I^2 = \exp\left(\sigma_{\ln x}^2 + \sigma_{\ln y}^2\right) - 1$

Therefore, for a spherical wave in moderate turbulence using an "effective" von Karman spectrum

$$\sigma_I^2 = \exp\left(\sigma_{\ln x}^2 + \frac{0.51\sigma_s^2}{\left(1+0.69\sigma_s^{12/5}\right)^{5/6}}\right) - 1 \qquad \text{(R44)}$$

where

$$\sigma_{\ln x}^2 = 0.016\sigma_R^2\left(\frac{8Q_m}{8+Q_m+0.056\sigma_R^2 Q_m^{7/6})}\right)^{7/6}$$

$$\sigma_s^2 = 3.86\sigma_R^2\left[\left\{0.4\left(1+\frac{9}{Q_m^2}\right)^{11/12}\sin\left[(11/6)\tan^{-1}\left(\frac{Q_m}{3}\right)\right]\right\} - \frac{11}{6}Q_m^{-5/6}\right]$$

APPENDIX S: ASYMPTOTIC COMPARISONS

This appendix rederives equations that were developed by Andrews, Phillips, and Hopen[8,14] for performing asymptotic comparisons of known scintillation behavior in weak and turbulent regimes to create scintillation index models for all turbulence regimes.

In the absence of inner and outer scale effects, irradiance is mainly affected by cell sizes l_1, l_2, l_3 and corresponding wave numbers κ_1, κ_2, κ_3, where $\kappa \cong 1/l$.

Spatial coherence radius $$l_1 = \frac{1}{\kappa_1} \cong \rho_0 \qquad \text{(S1)}$$

Fresnel zone size $$l_2 = \frac{1}{k_2} \cong \sqrt{L/k} \qquad \text{(S2)}$$

Scattering disk $$l_3 = \frac{1}{k_3} \cong L/k\rho_0 \qquad \text{(S3)}$$

In weak fluctuations, irradiance is most affected by cell sizes proportional to the Fresnel zone, $\sqrt{L/k}$. In moderate-to-strong fluctuation regimes, irradiance is most affected by cell sizes proportional to the scattering disk, $L/k\rho_0$.

In terms of ratio of areas

$$\kappa_1 \cong \frac{1}{\rho_0} \to \frac{L\kappa_1^2}{k} \cong \frac{L}{k\rho_0^2} \qquad \text{(S4)}$$

$$\kappa_2 \cong \sqrt{\frac{k}{L}} \to \frac{L\kappa_2^2}{k} \cong 1 \qquad \text{(S5)}$$

$$\kappa_3 \cong \frac{k\rho_0}{L} \to \frac{L\kappa_3^2}{k} \cong \frac{k\rho_0^2}{L} \qquad \text{(S6)}$$

In weak fluctuations $\left(\frac{L}{k\rho_0^2} < 1\right)$, $\kappa_x \cong \kappa_2$ and $\kappa_y \cong \kappa_2$. In strong fluctuations $\left(\frac{L}{k\rho_0^2} > 1\right)$, $\kappa_x \cong \kappa_3$ and $\kappa_y \cong \kappa_3$.

Assume that at any distance L into the random medium there exists an effective scattering disk $1/\kappa_x = L/kl_x$ and an effective correlation width l_y that identify the spatial frequencies κ_x and κ_y. These scale sizes are defined, respectively, by

$$\frac{1}{\kappa_x} = \frac{L}{kl_x} \cong \begin{cases} \sqrt{L/k}, \dfrac{L}{k\rho_0^2} << 1 \\ L/k\rho_0, \dfrac{L}{k\rho_0^2} >> 1 \end{cases} \tag{S7}$$

$$\frac{1}{\kappa_y} = \frac{L}{kl_x} \cong \begin{cases} \sqrt{L/k}, \dfrac{L}{k\rho_0^2} << 1 \\ \rho_0, \dfrac{L}{k\rho_0^2} >> 1 \end{cases} \tag{S8}$$

As a result, for some constants c_1, c_2, c_3, and c_4 to be determined on the basis of known asymptotic behavior of the scintillation index, expect

$$\frac{1}{\kappa_x^2} = \left(\frac{L}{kl_x}\right)^2 \cong \frac{c_1 L}{k} + c_2\left(\frac{L}{k\rho_0}\right)^2$$

$$\rightarrow \kappa_x^2 = \frac{1}{\dfrac{c_1 L}{k} + c_2\left(\dfrac{L}{k\rho_0}\right)^2} \quad \text{for all } \frac{L}{k\rho_0^2} \tag{S9}$$

$$\kappa_y^2 = \left(\frac{kl_x}{L}\right)^2 \cong \frac{c_3}{L/k} + \frac{c_4}{\rho_0^2}, \quad \text{for all} \frac{L}{k\rho_0^2} \tag{S10}$$

let $\eta_x = \dfrac{L\kappa_x^2}{k}$, $\eta_y = \dfrac{L\kappa_y^2}{k}$

$$\eta_x = \frac{L\kappa_x^2}{k} = \frac{L}{k}(\kappa_x^2) = \frac{L}{k}\left[\frac{1}{\dfrac{c_1 L}{k} + c_2\left(\dfrac{L}{k\rho_0}\right)^2}\right]$$

$$= \frac{1}{c_1 + c_2 L / k\rho_o^2} \tag{S11}$$

$$\rightarrow \ \eta_x \cong \begin{cases} \dfrac{1}{c_1 + c_2(0)} = \dfrac{1}{c_1}, & \dfrac{L}{k\rho_0^2} << 1 \\ \dfrac{1}{0 + c_2 L / k\rho_o^2} = \dfrac{k\rho_o^2}{c_2 L}, & \dfrac{L}{k\rho_0^2} >> 1 \end{cases}$$

$$\eta_y = \frac{L\kappa_y^2}{k} = \frac{L}{k}\left(\kappa_y^2\right) = \frac{L}{k}\left(\frac{c_3}{L/k} + \frac{c_4}{\rho_0^2}\right) = c_3 + \frac{c_4 L}{k\rho_0^2} \tag{S12}$$

$$\rightarrow \ \eta_y \cong \begin{cases} c_3 + c_4(0) = c_3, & \dfrac{L}{k\rho_0^2} << 1 \\ 0 + \dfrac{c_4 L}{k\rho_0^2} = \dfrac{c_4 L}{k\rho_0^2}, & \dfrac{L}{k\rho_0^2} >> 1 \end{cases}$$

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