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Fractal Interpolation

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FRACTAL INTERPOLATION

by

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ABSTRACT

This thesis is devoted to a study about Fractals and Fractal Polynomial Interpolation. Fractal Interpolation is a great topic with many interesting applications, some of which are used in everyday lives such as television, camera, and radio. The thesis is comprised of eight chapters. Chapter one contains a brief introduction and a historical account of fractals. Chapter two is about polynomial interpolation processes such as Newton's, Hermite, and Lagrange. Chapter three focuses on iterated function systems. In this chapter I report results contained in Barnsley's paper, Fractal Functions and Interpolation. I also mention results on iterated function system for fractal polynomial interpolation. Chapters four and five cover fractal polynomial interpolation and fractal interpolation of functions studied by Navascués. Chapter five and six are the generalization of Hermite and Lagrange functions using fractal interpolation. As a concluding chapter we look at the current applications of fractals in various walks of life such as physics and finance and its prospects for the future.

This thesis is dedicated to everyone who helped me get to where I am today.

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CHAPTER 1: INTRODUCTION

1.1 Introduction to Fractals

People are always fascinated by interesting shapes and often wonder how it is possible to create beautiful images and scenes like those seen in the movies. There is a lot of interest in the phenomenon called Star Wars. George Lucas is a pioneer of the movie industry due to the original Star Wars movies, which contained special effects like never seen before. In the film Return of the Jedi we see a major weapon the size of a small moon called the “Death Star.” Fractals were used to create the outline of this magnificent weapon. Fractals were also used in many other notable Hollywood movies such as Apollo 13, The Perfect Storm and Titanic. The following are pictures from the movies, The Return of the Jedi and Apollo 13.

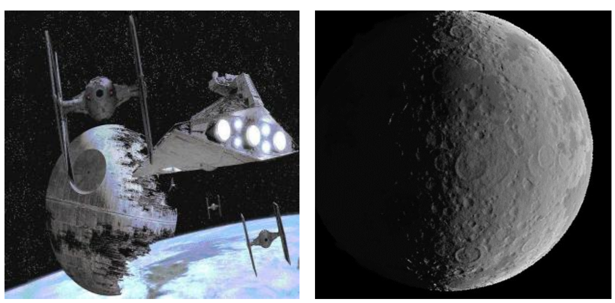

Figure 1: Star wars and Apollo 13

The field of fractals is fascinating. Fractals are not only man made but also be seen in nature. When you see a branch in a tree, which is similar to the tree, then the tree is an example of a fractal. Other examples include mountains, flowers, lightning strikes, rivers, coastlines and seashells. The following are pictures of a cloud and a fern leaf which are examples of a fractal as the exhibit self-similarity.

 

Figure 2: Pictures of a cloud and a fern leaf

The mathematics of fractals began to take shape in the 17th century when Leibniz considered the idea of straight lines being self-similar. Leibniz made the error of thinking that only straight lines were self-similar. It was in 1872 that Karl Weierstrass gave an example of a function which was continuous everywhere, but nowhere differentiable. In 1902, Helge von Koch gave a geometric definition of Weierstrass's analytical definition of the non-differentiable function and this is now called the Koch snowflake. Such self-similar recursive properties of functions in the complex plane were further investigated by Henri Poinćare, Felix Klein, Pierre Fatou and Gaston Julia. Finally, in 1975 Benoit Mandelbrot coined the term "fractals", whose Latin meaning stands for "broken" or "fractured".

Two of the most important properties of fractals are self-similarity and non-integer dimension. Self-similarity is the property where a small partition of an object is similar to the whole object. For example, a bark of a tree is similar to the tree itself and a floret of cauliflower is similar to the whole cauliflower. Self-similarity can be explained using the power law which is

$$t = c \cdot h^d$$

It is called the power law because t changes as if it was a power of h . Taking logarithm of both sides of the equality sign we obtain

$$\log t = d \cdot \log h + \log c$$

Which implies that

$$d = \frac{\log(t / c)}{\log h}$$

Fractal dimension can be obtained using a relationship between the number of copies and the scale factor. For example, if a line segment is cut into four equal pieces then the fractal dimension would be $d = -\frac{\log c}{\log r} = -\frac{\log 4}{\log(1/4)}$, where c is the number of copies and r is the scale factor.

1.2 Interpolation Process

People are often in search of a good digital camera. A camera with a higher mega pixel level will have more resolution and clarity. It is interpolation that lets the camera obtain the maximum level of mega pixel, which makes the images sharper. Other uses of interpolation include estimating a predicted value for the temperature at a grid point from data from weather stations located in its neighborhood and predicting the price of stocks from its past behavior from a time series data.

In mathematics, interpolation process is the computation of values between the ones that are known or tabulated using the surrounding points or values. According to Springer online reference works, "Interpolation is a process of obtaining a sequence of interpolation functions $\{f_n(z)\}$ for some indefinitely growing number of interpolation conditions." The aim of the interpolation process is to approximate by means of interpolation functions $f_n(z)$ of an initial function $f(z)$ about which there is incomplete information or which is complicated to deal with directly. Some of the famous interpolation processes are Newton's divided difference interpolation, Aitken's interpolation, Lagrange interpolating polynomial, Bessel's interpolating formula and Gauss's interpolating formula.

CHAPTER 2: POLYNOMIAL INTERPOLATION METHODS

2.1 Polynomial Interpolation

Table 1: $n+1$ Data Points (x_i, y_i)

x	x_0	x_1	. . . . . .	x_n
y	y_0	y_1	. . . . . .	y_n

If $n+1$ data points such as in Table 1 is given, then our goal is to find a polynomial p of lowest possible degree for which

$$p(x_i) = y_i \qquad (0 \le i \le n)$$

Such a polynomial is called an interpolating polynomial and a fundamental result is:

Theorem 1: If $x_0, x_1, ..., x_n$ are $(n+1)$ distinct real numbers, and $y_0, y_1, \ldots, y_n$ are $(n+1)$ arbitrary values, then there is a unique polynomial p_n of degree at most n such that

$$p_n(x_i) = y_i \qquad (0 \le i \le n)$$

The proof of this theorem can be found in [7]

2.2 Newton's Interpolation Polynomial

Let us assume that the function $f(x)$ is known at several values of x such and in Table 1. It is not assumed that the x's are evenly spaced or that the values are arranged in a particular order. Consider the n^{th} degree polynomial

$$P_n(x) = a_0 + (x - x_0)a_1 + (x - x_0)(x - x_1)a_2 + \cdots + (x - x_0)(x - x_1)\cdots(x - x_{n-1})a_n$$

If the a_i's are chosen such that $P_n(x) = f(x)$ at the $n+1$ known points, $(x_i, f_i), i = 0, \cdots, n$, then $P_n(x)$ is an interpolating polynomial where the a_i's are determined using the Newton's divided difference tables [5].

2.3 Newton's Divided Difference Interpolation.

Let $\prod_n(x) = \prod_{k=0}^{n}(x - x_k)$ then $f(x) = f_0 + \sum_{k=1}^{n}\prod_{k-1}(x) f[x_0, x_1, \cdots, x_k] + R_n$

Here $f[x_0, x_1, \cdots, x_k]$ is the divided difference of f at $[x_0, x_1, \cdots, x_k]$ and the remainder is

$$R_n(x) = \prod_n(x)[x_0, x_1, \cdots, x_n] = \prod_n(x)\frac{f^{(n+1)}(\xi)}{n+1\ !} \text{ for } x_0 < \xi < x_n$$

According Gerald and Wheately [5], using the standard notation, a divided difference table is shown in standard form as follows

Table 2: Newton's Divided Difference Table

x_i	f_i	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$	$f[x_i, x_{i+1}, x_{i+2}, x_{i+3}]$
x_0	f_0	$f[x_0, x_1]$	$f[x_0, x_1, x_2]$	$f[x_0, x_1, x_2, x_3]$
x_1	f_1	$f[x_1, x_2]$	$f[x_1, x_2, x_3]$	$f[x_1, x_2, x_3, x_4]$
x_2	f_2	$f[x_2, x_3]$	$f[x_2, x_3, x_4]$	

x_3	f_3	$f[x_3, x_4]$		
x_4	f_4			

Here,

$$f[x_0, x_1, ..., x_n] = \frac{f[x_1, x_2, ..., x_n] - f[x_0, x_1, ..., x_{n-1}]}{x_n - x_0}$$

Hence,

$$P_n(x) = f[x_0] + (x - x_0) f[x_0, x_1] + (x - x_0)(x - x_1) f[x_0, ..., x_2] + ... + (x - x_0)(x - x_1)...(x - x_{n-1}) f[x_0, x_1, ..., x_n]$$

2.4 Lagrange Interpolation Polynomial.

An alternative method of expressing the interpolating polynomial P is of the form

$$p(x) = y_0 \ell_0(x) + y_1 \ell_1(x) + \cdots + y_n \ell_n(x) = \sum_{k=0}^{n} y_k \ell_k(x)$$

Where $\ell_0, \ell_1, \cdots, \ell_n$ represent polynomials that depend on the nodes $x_0, x_1, \cdots, x_n$. Let,

$$\delta_{ij} = p_n(x_j) = \sum_{k=0}^{n} y_k \ell_k(x_j) = \ell_i(x_j)$$

Here, δ_{ki} is the Delta function where $\delta_{ki} = 1$ if $k = i$ and $\delta_{ki} = 0$ if $k \neq i$. Consider ℓ_0 to be a polynomial of degree *n*, which takes on the value of 0 at $x_1, x_2, \cdots, x_n$ and 1 at x_0.

$$\ell_0(x) = c(x - x_1)(x - x_2)\cdots(x - x_n) = c\prod_{j=1}^{n}(x - x_j)$$

By putting $x = x_0$ we obtain

$$1 = c\prod_{j=1}^{n}(x_0 - x_j)$$

Therefore,

$$c = \prod_{j=1}^{n} (x_0 - x_j)^{-1}$$

and

$$\ell_0(x) = c \prod_{j=1}^{n} \frac{x - x_j}{x_0 - x_j}$$

Each of the ℓ_i's, $i = 1, 2, ..., n$, are obtained by a similar reasoning. In general

$$\ell_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j} \qquad (0 \leq i \leq n)$$

For the set of nodes $x_0, x_1, \cdots, x_n$, these polynomials are called the cardinal function. The cardinal functions together with

$$p(x) = y_0 \ell_0(x) + y_1 \ell_1(x) + \cdots + y_n \ell_n = \sum_{k=0}^{n} y_k \ell_k(x) \tag{2.1}$$

yields the Lagrange form of the interpolation polynomials.

2.5 Hermite Interpolation

A loose definition of Hermite Interpolation is the interpolation of a function and its derivatives at a set of nodes. The simpler interpolation where no derivatives are interpolated is often referred to as Lagrange interpolation. Kincaid and Cheney [7] provide a useful example in their book Numerical Analysis where they require a polynomial of least degree that interpolates a function f and its derivative f' at two distinct points x_0 and x_1. The polynomial which we seek will satisfy

$$p(x_i) = f(x_i) \quad p'(x_i) = f'(x_i) \qquad (i = 0, 1)$$

There are four conditions and therefore it is reasonable to seek a solution in $\prod_3$, a linear space of all polynomials of degree at most three. An element of $\prod_3$ has four coefficients. Instead of writing $p(x)$ in terms of $1, x, x^2, x^3$, we write

$$p(x) = a + b(x - x_0) + c(x - x_0)^2 + d(x - x_0)^2(x - x_1)$$

This leads to

$$p'(x) = b + 2c(x - x_0) + 2d(x - x_0)(x - x_1) + d(x - x_0)^2$$

The four conditions on p can be written as

$f(x_0) = a$

$f'(x_0) = b$

$f(x_1) = a + bh + ch^2$ $\qquad\qquad (h = x_1 - x_0)$

$f'(x_1) = b + 2ch + dh^2$

In a Hermite problem it is assumed that whenever a derivative $p^j(x_i)$ is to be prescribed (at a node x_i), then $p^{(j-1)}(x_i), p^{(j-2)}(x_i), \cdots, p'(x_i)$, and $p(x_i)$ will also be prescribed. Let $x_0, x_1, x_2, \cdots, x_n$ be the nodes and let the following interpolation conditions be given at the node x_i

$$p^j(x_i) = c_{ij} \qquad\qquad (0 \le j \le k_i - 1, 0 \le i \le n)$$

The total conditions on p is denoted by $m+1$ and therefore

$$m + 1 = k_0 + k_1 + \cdots + k_n$$

Theorem 2 [7]: There exists a unique polynomial p in $\prod_m$ fulfilling the Hermite interpolation condition

$$p^j(x_i) = c_{ij} \qquad\qquad (0 \le j \le k_i - 1, 0 \le i \le n)$$

When there is only one node, we require a polynomial p of degree k for which

$$p^{j}(x_0) = c_{0j} \qquad (0 \le j \le k)$$

then the solution is the Taylor polynomial.

$$p(x) = c_{00} + c_{01}(x - x_0) + \cdots + \frac{c_{0k}}{k!}(x - x_0)^k$$

Hermite interpolations problems can also be solved using Newton's divided difference method and Lagrange interpolation formula [7].

2.6 Error in Polynomial Interpolation

Theorem 3 [7]: Let f be a function in $C^{n+1}[a,b]$, and let p be a polynomial of degree $\le n$ that interpolates the function at $n+1$ distinct points $x_0, x_1, x_2, \cdots, x_n$ in the interval $[a,b]$. To each x in $[a,b]$ there corresponds a point in ξ_x $[a,b]$ such that

$$f(x) - p(x) = \frac{1}{(n+1)!} f^{n+1}(\xi_x) \prod_{i=0}^{n} (x - x_i)$$

CHAPTER 3: ITERATED FUNCTION SYSTEMS

Definition 1: [6] A metric space is a pair (M, d) where *M* is a non-empty set and $d : M \times M \to R$ is a real valued function called a metric on *M*, with the following properties

i) Positive definite, i.e. $\forall x, y \in M, d(x, y) \geq 0$

ii) Symmetric, i.e. $\forall x, y \in M, d(x, y) = d(y, x)$

iii) Triangle inequality, i.e. $\forall x, y, z \in M, d(x, y) \leq d(x, z) + d(z, y)$

Definition 2 [6]: Let (M, d) be a metric space and let *a* be the family of all closed subsets of *M*. For $r > 0$ and *A* in *a*, let $V_r(A) = \{m : d(m, a) < r\}$, and definite for members *A* and *B* of *a*, $d'(A, B) = \inf\{r : A \subset V_r(B)$ and $B \subset V_r(A)\}$. Here, d' is the well-known Hausdörff metric.

Let *M* be a compact metric space and *H* be the set of all nonempty closed subsets of *M*. Then *H* is a compact metric space with the Hausdörff metric. Note that *A* and *B* are subsets of *M*.

Let $w_n = M \to M$ for $n \in \{1, 2, ..., N\}$ be continuous.

$$\{M, w_n : n = 1, 2, ..., N\} \tag{3.1}$$

is called an iterated function system (IFS).

Consider $w_n(A) = \{w_n(x) : x \in A\}$. Define $W : H \to H$ by

$$W(A) = w_1(A) \cup w_2(A) \cup ... \cup w_N(A) = \bigcup_n w_n(A) \text{ for } A \in H \tag{3.2}$$

Any set $G \subset H$ such that

$$W(G) = G \tag{3.3}$$

is called an *attractor* for the IFS.

According to Barnsley [1] the IFS always admits at least one attractor. An IFS is called hyperbolic if, for some s, $0 \le s < 1$ and $n \in \{1,2,...,N\}$,

$$d(w_n(x), w_n(y)) \le s \cdot d(x, y), \ \forall \ \ x, y \in M \tag{3.4}$$

In this case W is a contraction mapping which obeys

$$h(W(A), W(B)) \le s \cdot h(A, B), \ \forall \ \ A, B \in H$$

Also, W admits a unique attractor. Barnsley in [1] explains how to find this unique attractor.

Given a set of data points $\{(x_i, y_i) \in I \times R : i = 0,1,...,N\}$, where $I = [x_0, x_N] \subset R$ is a closed interval, see Fig. 3.

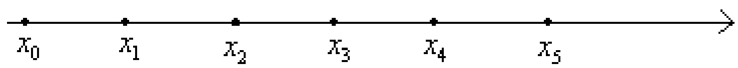

Figure 3: $I = [x_0, x_N] \subset R$

The functions that we are concerned with are functions $f : I \to R$ which interpolate the data $\{y_i : 1 = 0,1,2,...,N\}$ such that

$$f(x_i) = y_i, \ i = 0,1,2,..,N \tag{3.5}$$

as seen in Fig. 4

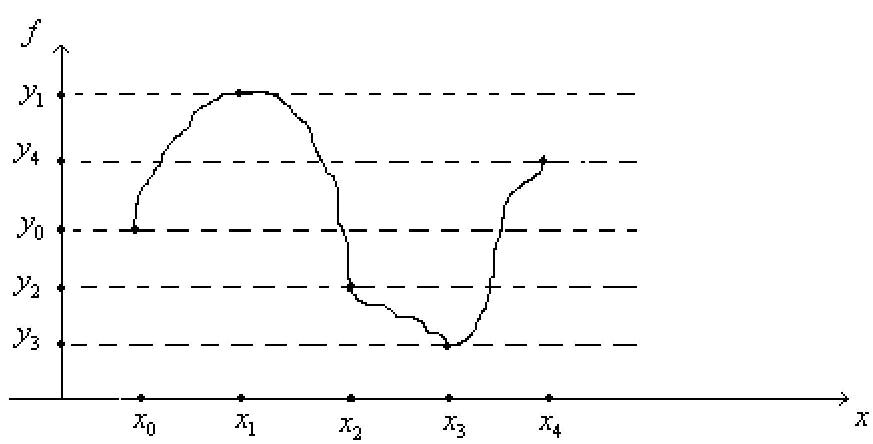

Figure 4: $f : I \to R$

The graphs of these functions $G = \{x, f(x) : x \in I\}$ are attractors of the IFS. In other words there exists a compact subset M of $I \times R$, and a collection of continuous mappings $w_n : M \to M$ such that the unique attractor of the IFS is G. Barnsley in [1] refers to such functions as the Fractal Interpolation Functions (FIF).

Here we are working with the compact metric space $M = I \times [a,b]$ for some $-\infty < a < b < \infty$, with the Euclidean metric or an equivalent metric viz.

$$d((c_1, d_1), (c_2, d_2)) = Max\{|c_1 - c_2|, |d_1 - d_2|\}$$

Assign $I_n = [x_{n+1}, x_n]$ and let $L_n : I \to I_n$ for $n \in \{1, 2, ..., N\}$ be contractive homeomorphisms such that

$$L_n(x_0) = x_{n-1}, \; L_n(x_N) = x_n, \tag{3.6}$$

$$|L_n(c_1) - L_n(c_2)| \le l\,|c_1 - c_2| \qquad \forall c_1, c_2 \in I,$$

for some $l, 0 \le l < 1$. Also, let the mappings $F_n : M \to [a,b]$ be continuous for some $0 \le q < 1$ satisfying

$$F_n(x_0, y_0) = y_{n-1}, \; F_n(x_N, y_N) = y_n,$$

$$|F_n(c, d_1) - F_n(c, d_2)| \le q \cdot |d_1 - d_2|,$$

for all $c \in I$, $d_1, d_2 \in [a,b]$, and $n \in \{1, 2, ..., N\}$.

We now define functions $w_n : M \to M$ for $n \in \{1, 2, ..., N\}$ by

$$w_n(x, y) = (L_n(x), F_n(x, y)), \; n = 1, 2, ..., N \tag{3.7}$$

Here, $\{M, w_n : n \in \{1, 2, ..., N\}$ is an IFS, but this may not be hyperbolic.

Theorem 4 [1]: The iterated function system (IFS) $\{M, w_n : n = 1, 2, .., N\}$ admits a unique attractor *G*, which is the graph of a continuous function $f : I \rightarrow [a, b]$, and that function satisfies (3.5)

Proof: Let *G* be any attractor of the IFS. Let

$$\hat{I} = \{x \in I : (x, y) \in G \text{ for some } y \in [a, b]\}.$$

Note that $I = [x_0, x_N] \subset R$ is a closed interval, see Fig 1.

From (3.2) and (3.3) $G = \bigcup_n w_n \ \ G$, and it follows that $\hat{I} = \bigcup_n L_n(\hat{I})$.

$\{I, L_n : n = 1, 2, ..., N\}$ is a hyperbolic IFS (3.4) whose unique attractor is *I*.

Hence,

$$\hat{I} = I = [x_0, x_n]$$

To show that *G* is the graph of the function $f : I \rightarrow [a, b]$, we start off by proving that there is only one *y*-value corresponding to each *x*-value (definition of a function).

Consider the *x*-values $\{x_0, x_1, ..., x_N\}$. Let,

$$S_i = \{(x, y) \in G : x = x_i\} \text{ for } i \in \{0, 1, ..., N\}.$$

Note that from (3.7)

$$w_n(s_0) = (L_n(x), F_n(S_0))$$

$$w_n(s_0) = (x_{n-1}, y_{n-1})$$

Therefore, $w_n(S_0) \cap S_0 = 0$ for $n \neq 1$, we should have $w_1(S_0) = S_0$, but w_1 is a strict contraction on the compact metric space S_0, so $S_0 = (x_0, y_0)$ and similarly $S_N = (x_N, y_N)$.

For $I \in \{1, 2, ..., N-1\}$ the only points which can map to S_i are S_0 (under w_{i-1}) and S_N (under w_i). Therefore,

$$S_i = w_{i-1}(S_0) \cup w_i(S_N) = (x_i, y_i) \tag{3.8}$$

Let,

$$\delta = Max\{|s-t| : (x,s),(x,t) \in G, x \in I\}$$

Due to the compactness of *G*, the maximum is achieved at some pair of points $(\hat{x}, s)$ and $(\hat{x}, t)$ in *G*, with $\delta = |s-t|$. From (3.7) it can be assumed that $\hat{x} \in (x_{n-1}, x_n)$ for some *n*.

But, there exists two points in *G*,

$$(L_n^{-1}(\hat{x}), u) \text{ and } L_n^{-1}(\hat{x}, v)$$

with,

$$s = F_n(L_n^{-1}(\hat{x}), u) \text{ and } t = F_n(L_n^{-1}(\hat{x}), v)$$

Hence,

$$\delta = |s-t| = \left|F_n(L_n^{-1}(\hat{x}), u) - F_n(L_n^{-1}(\hat{x}), v)\right|$$

$$\le q \cdot |u-v| \le q \cdot \delta$$

with $0 \le q < 1$, hence $\delta = 0$. Therefore, *G* is the graph of the function $f : I \to [a,b]$ which satisfies $f(x_i) = y_i$. *G* is unique because the union of two attractors is still an attractor. To prove that $f(x)$ is continuous, let $C(I)$ denote the Banach space of all continuous real-valued function $g : I \to R$. Define a norm $|g|_\infty = Max\{|g(x)| : x \in I\}$. Let us define a contraction mapping $T : C_0(I) \to C_0(I)$, where $C(I)$ consists of those $g \in C(I)$ such that $g : I \to [a,b)$, and which obeys $g(x_0) = y_0$ and $g(x_N) = y_N$. Let $C_0(I) \subset C(I)$.

$$(Tg)(x) = \left|F_n(L_n^{-1}(x)), g(L_n^{-1}(x))\right| \text{ when } x \in I_n, n = 1, 2, ..., N.$$

Using the definition of the norm

$$
\begin{aligned}
\left|Th - Tg\right|_{\infty} &= Max\{\left|F_n(L_n^{-1}(x)), h(L_n^{-1}(x)) - F_n(L_n^{-1}(x)), g(L_n^{-1}(x))\right|\} \\
&\leq Max\{q \cdot \left|h(L_n^{-1}(x)) - g(L_n^{-1}(x))\right|\} \\
&\leq q \cdot \left|h - g\right|_{\infty}
\end{aligned}
$$

Hence we know that T has a unique fixed point $\hat{h} \in C_0(I)$ such that $\hat{h}$ is the attractor of the IFS. Hence the function $f = \hat{h}$ is continuous. This completes the proof of the theorem.

CHAPTER 4: FRACTAL INTERPOLATION FUNCTIONS

In this chapter we shall consider the problem of Fractal Interpolation. Initial results in this direction were obtained by Barnsley [2]. Before we go to more recent results on this topic, we consider necessary definitions and some of the known results which will be used in subsequent chapters.

Definition 3 [2]: A data set is a set of points of the form

$$\{(x_i, f_i) \in \mathbb{R}^2 : i = 0,1,2,..., N\}$$

where,

$$x_0 < x_1 < x_2 < ... < x_N$$

This set of data points has an interpolation function corresponding to it which is a continuous function $f : [x_0, x_N] \to \mathbb{R}$ such that

$$f(x_i) = F_i \text{ for } i = 0,1,2,.., N$$

Here, the points $(x_i, f_i) \in \mathbb{R}^2$ are called the interpolation points and the function *f* interpolates the data points. The graph of the function *f* passes through the interpolation points.

For example, let $\{(x_i, f_i) \in \mathbb{R}^2 : i = 0,1,2,..., N\}$ denote a set of data. Let $f : [x_0, x_N] \to \mathbb{R}$ denote the unique continuous function which passes through the interpolation points and which is linear on each of the subintervals $[x_{i-1}, x_i]$.

That is

$$f(x) = f_{i-1} + \frac{(x - x_{i-1})}{(x_i - x_{i-1})}(F_i - F_{i-1}) \text{ for } x \in [x_{i-1}, x_i], i = 1,2,..., N$$

The function $f(x)$ is called the piecewise linear interpolation function. The graph of $f(x)$ is illustrated in Fig. 5

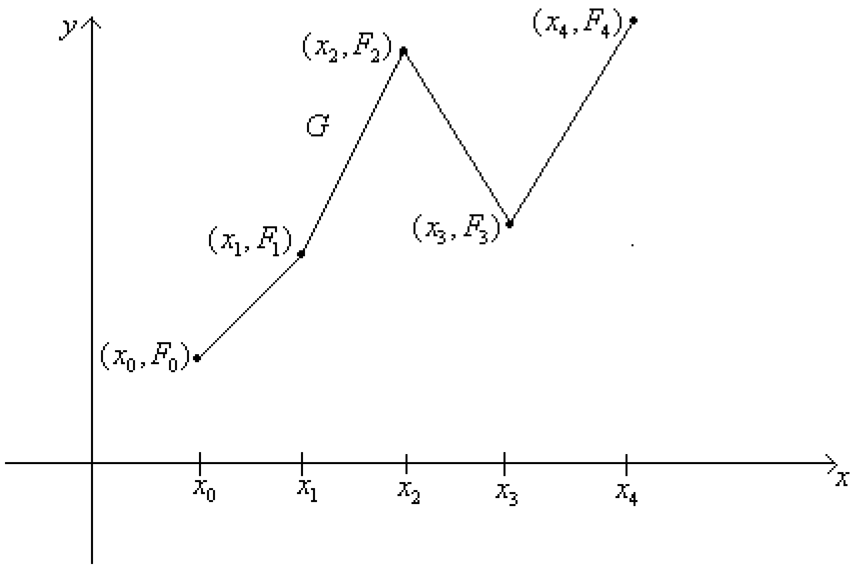

Figure 5: Graph of the Piecewise Linear Interpolation Function

The figure above is the graph of the piecewise linear interpolation function $f(x)$ through the linear interpolation points $\{(x_i, F_i) : i = 0,1,2,3,4\}$. This graph is also the attractor of an Iterated Function System of the form $\{w_n, n = 1,2,3,4\}$ where the maps are affine. The term affine stands for everything that is related to the geometry of affine spaces. A co-ordinate system for the *n*-dimensional affine spaces $\mathbb{R}^n$ is determined by any basis of *n* vectors, which are not essentially orthonormal. Therefore, the resulting axes are not necessarily mutually perpendicular nor have the same unit measure.

Here,

$$w_n \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_n & 0 \\ c_n & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e_n \\ f_n \end{pmatrix}$$

Where,

$$a_n = \frac{(x_n - x_{n-1})}{(x_N - x_0)}, \qquad e_n = \frac{(x_N x_{n-1} - x_0 x_n)}{(x_N - x_0)}$$

$$c_n = \frac{(F_n - F_{n-1})}{(x_N - x_0)}, \qquad f_n = \frac{(x_N F_{n-1} - x_0 F_n)}{(x_N - x_0)}$$

for $n = 0,1,2,..,N$.

Let $t_0 < t_1 < ... < t_N$ be real numbers, and consider $I = [t_0, t_N] \subset \mathbb{R}$, the closed interval that contains $t_0 < t_1 < ... < t_N$. Given the set of data points $\{(t_n, x_n) \in I \times \mathbb{R} : n = 0,1,2,...,N\}$. Consider $I_n = [t_{n-1}, t_n]$ and let $L_n : I \to I_n, n \in \{1,2,...,N\}$ be contractive homeomorphisms such that $L_n(t_0) = t_{n-1}$, $L_n(t_N) = t_n$, $|L_n(c_1) - L_n(c_2)| \le l|c_1 - c_2| \forall c_1, c_2 \in I$ and for some $0 \le l \le 1$. Also, let the mappings $F_n : F \to \mathbb{R}$ be continuous for some $-1 \le \alpha_n < 1$ satisfying $F_n(t_0, x_0) = x_{n-1}$, $F_n(t_N, x_N) = x_n$, $|F_n(t,x) - F_n(t,y)| \le \alpha_n |x - y|$, for all $-\infty < c < d < \infty$, $-1 < \alpha_n < 1$, and $n \in \{1,2,...,N\}$. In the precious chapter we defined IFS $w_n(t,x) = (L_n(t), F_n(t,x))$, $n = 1,2,...,N$. From Theorem 4 we know that the IFS admits a unique attractor G which is the graph of a continuous function $f : I \to \mathbb{R}$ which satisfies $f(t_n) = x_n \forall n = 0,1,2,...,N$. This function is called the fractal interpolation function (FIF) corresponding to $\{(L_n(t), F_n(t,x))\}_{n=1}^{N}$. Let $\mathcal{G}$ be the set of continuous functions $f : [t_0, t_N] \to [c,d]$ such that $f(t_0) = x_0$ and $f(t_N) = x_N$. We know that $\mathcal{G}$ is a complete metric space with respect to the uniform norm $(\mathcal{G}, \|\cdot\|_\infty)$. Define $T : \mathcal{G} \to \mathcal{G}$ by

$$(Tf)(t) = F_n\left(L_n^{-1}(t), f \circ L_n^{-1}(t)\right) \ \forall t \in [t_{n-1}, t_n], n = 1,2,...,N.$$

T is a contraction mapping on $\mathcal{G}$ follows from

$$\|Tf - Tg\|_\infty \le |\alpha|_\infty \, \|t - g\|_\infty \text{, where } |a|_\infty = \max \{|a_n|, n = 1, 2, ..., N\}$$

and $|\alpha|_\infty < 1$ since $-1 < \alpha_n < 1$. As a consequence T has a unique fixed point on $\mathcal{G}$ from

the Banach fixed point theorem. Thus $\exists\ f \in \mathcal{G}$ such that $(Tf)(t) = f(t)\ \forall t \in [t_0, t_N]$. This

function $f(t)$ is a Fractal Interpolation Function corresponding to the IFS

$w_n(t, x) = (L_n(t), F_n(t, x))$.

Also, $f : I \to \mathbb{R}$ is a unique function that satisfies the following equations

$$f(L_n(t)) = F_n(t, f(t)) \quad \forall n = 1, 2, ..., N, t \in I$$

or

$$f(t) = F_n(L_n^{-1}(t), f \circ L_n^{-1}(t)) \quad \forall n = 1, 2, ..., N, t \in I_n = [t_{n-1}, t_n] \qquad (4.1)$$

CHAPTER 5: FRACTAL POLYNOMIAL INTERPOLATION

Until recently, interpolation and approximations have been carved out with the aid of smooth functions. These functions are sometimes infinitely differentiable. Unfortunately, in the real world we deal with signals which do not posses such smooth qualities. Signals recorded with respect to time suggest "original functions with, abrupt changes, whose derivatives posses sharp steps or even do not exist at all," (Navascués [13].) The fractal interpolation functions (FIF) are considered to be an important advance in this field because the interpolants of the FIF are not necessarily differentiable over a set and in certain cases are not even point-wise differentiable. "They appear ideally suited for the approximation of naturally occurring functions which display some kind of geometric self-similarity under magnification," (Barnsley, [10].)

Navascués [13], in her paper, proposes to create a base for fractal interpolants which are perturbations of polynomials whose aim is to define a non-smooth fractal version of conventional interpolations. A complete description of the frequency domain of the fractal functions is obtained by means of their Fourier Transform. "This fact is particularly important because such functions are defined implicitly in the time domain by a functional equation," (Navascués [13].)

Let us review the IFS (3.1) which admits a unique attractor G, which is the graph of a continuous function $f: I \rightarrow R$ that obeys $f(t_n) = x_n$, $i = 0,1,2,..,N$. This function is called a fractal interpolation function corresponding to $\{(L_n(t), F_n(t,x))\}_{n=1}^{N}$. Till this day, the most studied fractal interpolation function (FIF) has been defined by the iterated function system (IFS) [13]

$$\begin{cases} L_n(t) = a_n(t) + b_n \\ F_n(t,x) = \alpha_n x + q_n(t) \end{cases} \tag{5.1}$$

where a_n is a vertical scaling factor for the transformation w_n and $\alpha = (\alpha_1, \alpha_2, ..., \alpha_N)$ is the scale vector of the IFS.

Michael Barnsley, who is a pioneer in the use of data points to create fractal functions [1], proposes a generalization of a continuous function *h* by means of a fractal interpolation using the IFS (5.1) with a polynomial $q_n(t) = h \circ L_n(t) - \alpha_n b(t)$. Here *b* is a continuous function such that $b(t_0) = x_0$, $b(t_N) = x_N$. Here, the case that we will be looking at is $b = h \circ c$, with the function *c* being a continuous and increasing function $c(t_0) = t_0, c(t_N) = t_N$. As an example, Navascués in her paper [13] considers the family $c(t) = (e^{\lambda t} - 1)/(e^{\lambda} - 1), \lambda > 0$ on an interval $[0,1]$.

Proposition 1 [13]: Let $h: I = [a,b] \to R$, be continuous, $\Delta : a = t_0 < t_1 < ... < t_N = b$, be a partition of $[a,b]$, $N > 1$, $\alpha \in R^N$ and such that $|\alpha|_\infty < 1$. The IFS (3.1) where $a_n = (t_n - t_{n-1})/(t_N - t_0)$, $b_n = (t_N t_{n-1} - t_0 t_N)/(t_N - t_0)$, $q_n(t) = h \circ L_n(t) - \alpha_n h \circ c(t)$ and *c* an increasing continuous function such that $c(t_0) = t_0$; $c(t_N) = t_N$, defines an FIF $h^\alpha(t_n) = h(t_n)$ for all $n = 0,1,...,N$.

Proof: First step is to check and see whether the conditions for L_n and F_n are satisfied. We look at (3.6) with $L_n(t_0) = t_{n-1}$, $L_n(t_N) = t_n$ $n \in \{1,2,...,N\}$, where $L_n : I \to I_n$ are contractive homeomorphisms and $x_n = h(t_n)$. We have

$$\begin{aligned} F_n(t_0, x_0) &= \alpha_n x_0 + q_n(t_0) \\ &= \alpha_n x_0 + h \circ L_n(t_0) - \alpha_n h \circ c(t_0) \\ &= \alpha_n x_0 + h(t_{n-1}) - \alpha_n h(t_0) \\ &= \alpha_n x_0 + x_{n-1} - \alpha_n x_0 \\ &= x_{n-1} \end{aligned}$$

and

$$\begin{aligned} F_n(t_N, x_N) &= \alpha_n x_N + q_n(t_N) \\ &= \alpha_n x_N + h \circ L_n(t_N) - \alpha_n h \circ c(t_N) \\ &= \alpha_n x_N + h \circ t_N - \alpha_n h \circ t_N \\ &= \alpha_n x_N + h(t_N) - \alpha_n x_N \\ &= x_N \end{aligned} .$$

F_n is uniformly Lipschitz, $|F_n(c, d_1) - F_n(c, d_2)| \le |\alpha|_\infty |d_1 - d_2|$, in the second variable

with constant $|\alpha|_\infty < 1$.

Define $T_\alpha : G \to G$, where $G = \{g \in C[a,b] : g([a,b]) \subset [c,d], g(a) = x_0, g(b) = x_N\}$,

according to $T_\alpha f(t) = F_n(L_n^{-1}(t), f \circ L_n^{-1}(t))$. According to Theorem 4 T_α admits a unique

attractor, h^α such that

$$h^\alpha(t) = F_n(L_n^{-1}(t), h^\alpha \circ L_n^{-1}(t)),\ t \in I_n .$$

Using (5.1)

$$h^\alpha(t) = \alpha_n h^\alpha \circ L_n^{-1}(t) + q_n \circ L_n^{-1}(t)$$

Substituting into the above $q_n(t) = h \circ L_n(t) - \alpha_n h \circ c(t)$, we obtain

$$\begin{aligned} h^\alpha(t) &= \alpha_n h^\alpha \circ L_n^{-1}(t) + (h \circ L_n(t) - \alpha_n h \circ c(t)) \circ L_n^{-1}(t) \\ &= \alpha_n h^\alpha \circ L_n^{-1}(t) + h - \alpha_n h \circ c(t) \circ L_n^{-1}(t) \\ &= h(t) + \alpha_n (h^\alpha - h \circ c(t)) \circ L_n^{-1}(t) \end{aligned} \tag{5.2}$$

h^α passes through the points (t_n, x_n) since,

$$
\begin{aligned}
h^{\alpha}(t_n) &= F_n(L_n^{-1}(t_n), h^{\alpha}\circ L_n^{-1}(t_n)) \\
&= F_n(t_N, x_N) \\
&= x_n h^{\alpha}(t_n) = F_n(L_n^{-1}(t_n), h^{\alpha}\circ L_n^{-1}(t_n)) \\
&= F_n(t_N, x_N) \\
&= x_n
\end{aligned}
\tag{5.3}
$$

Note that since $L_n(t_N) = t_n$, we have $L_n^{-1}(t_n) = t_N$ and $h^{\alpha}\circ L_n^{-1}(t_n) = h^{\alpha}(t_N) = x_N$.

As an example let us consider the graph of the function $h(t) = t\cos\left(\frac{\pi}{2t}\right)$. The first figure is the original graph of the function $h(t)$. The second figure is the graph of the corresponding $\alpha -$fractal function with the partition $\Delta : 0 < \frac{1}{8} < \frac{1}{7} < \frac{1}{6} < \frac{1}{5} < \frac{1}{4} < \frac{1}{3} < \frac{1}{2} < 1$

Let us consider the case where $c(t) = x^2$, a quadratic in the interval $[0,1]$. Note that $c(0) = 0, c(1) = 1$. Let $\alpha_n = 0.2\ \forall\ n = 1,...,8$. The third figure is the graph obtained from the Lagrange Polynomial Interpolation of the function $h(t)$. All of the graphs below are generated using Maple 10.

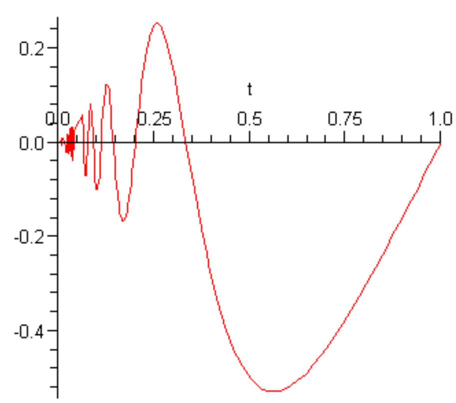

Figure 6: Graph of the Function *h*

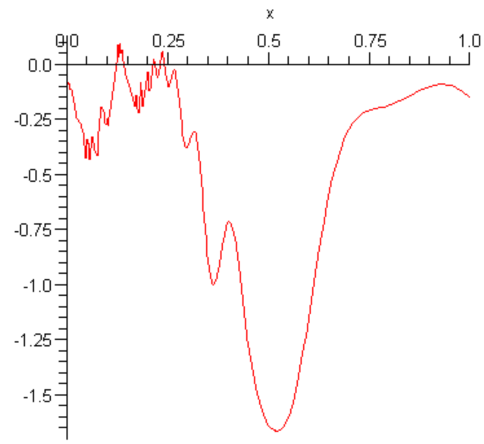

Figure 7: Graph of the α - *fractal function* of *h*

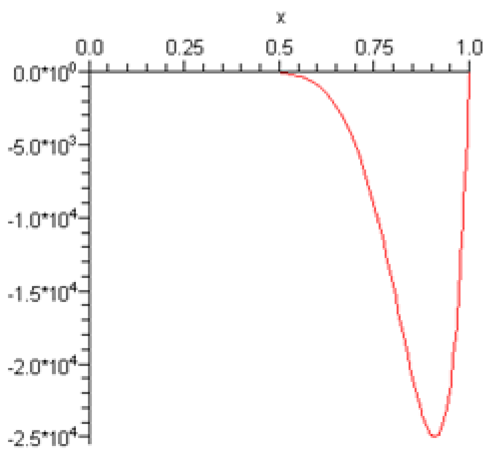

Figure 8: Graph of the Lagrange Interpolation polynomial of *h*

The graphs above were generated using the following Maple 10 code.

$with(plots):$

$h(t) := t \cdot cos\left(\frac{\pi}{2 \cdot t}\right)$

$n := [1..8]:$

$t_0 := 0:$

$t_1 := \frac{1}{8}:$

$t_2 := \frac{1}{7}:$

$t_3 := \frac{1}{6}:$

$t_4 := \frac{1}{5}:$

$t_5 := \frac{1}{6}:$

$t_6 := \frac{1}{7}:$

$t_7 := \frac{1}{8}:$

$t_N := 1:$

$a_n := (t_n - t_{n-1}):$

$b_n := (t_N \cdot t_{n-1} - t_0 \cdot t_n):$

Find a_n and b_n for each *n*.

$$plot\left(\frac{1}{8}\cdot x\cdot cos\left(\frac{\pi}{2\cdot x}\right) + \left(\frac{1}{8}\cdot x + 0\right)\cdot cos\left(\frac{\pi}{2\cdot\left(\frac{1}{8}\cdot x + 0\right)}\right) - 0.2\cdot x^2\cdot cos\left(\frac{\pi}{2\cdot x^2}\right)\right.$$

$$+ \frac{1}{56}\cdot x\cdot cos\left(\frac{\pi}{2\cdot x}\right) + \left(\frac{1}{56}\cdot x + \frac{1}{8}\right)\cdot cos\left(\frac{\pi}{2\cdot\left(\frac{1}{56}\cdot x + \frac{1}{8}\right)}\right) - 0.2\cdot x^2\cdot cos\left(\frac{\pi}{2\cdot x^2}\right)$$

$$+ \frac{1}{42}\cdot x\cdot cos\left(\frac{\pi}{2\cdot x}\right) + \left(\frac{1}{42}\cdot x + \frac{1}{7}\right)\cdot cos\left(\frac{\pi}{2\cdot\left(\frac{1}{42}\cdot x + \frac{1}{7}\right)}\right) - 0.2\cdot x^2\cdot cos\left(\frac{\pi}{2\cdot x^2}\right)$$

$$+ \frac{1}{30}\cdot x\cdot cos\left(\frac{\pi}{2\cdot x}\right) + \left(\frac{1}{30}\cdot x + \frac{1}{6}\right)\cdot cos\left(\frac{\pi}{2\cdot\left(\frac{1}{30}\cdot x + \frac{1}{6}\right)}\right) - 0.2\cdot x^2\cdot cos\left(\frac{\pi}{2\cdot x^2}\right)$$

$$+ \frac{-1}{30}\, x\, cos\left(\frac{\pi}{2\cdot x}\right) + \left(\frac{-1}{30}\cdot x + \frac{1}{5}\right) cos\left(\frac{\pi}{2\cdot\left(\frac{-1}{30}\cdot x + \frac{1}{5}\right)}\right) - 0.2\cdot x^2\cdot cos\left(\frac{\pi}{2\cdot x^2}\right)$$

$$+ \frac{-1}{42}\, x\, cos\left(\frac{\pi}{2\cdot x}\right) + \left(\frac{-1}{42}\cdot x + \frac{1}{6}\right) cos\left(\frac{\pi}{2\cdot\left(\frac{-1}{42}\cdot x + \frac{1}{6}\right)}\right) - 0.2\cdot x^2\cdot cos\left(\frac{\pi}{2\cdot x^2}\right)$$

$$+ \frac{-1}{56}\, x\, cos\left(\frac{\pi}{2\cdot x}\right) + \left(\frac{-1}{56}\cdot x + \frac{1}{7}\right) cos\left(\frac{\pi}{2\cdot\left(\frac{-1}{56}\cdot x + \frac{1}{7}\right)}\right) - 0.2\, x^2\cdot cos\left(\frac{\pi}{2\cdot x^2}\right)$$

$$\left. + \frac{5}{6}\, x\, cos\left(\frac{\pi}{2\cdot x}\right) + \left(\frac{5}{6}\cdot x + \frac{1}{8}\right) cos\left(\frac{\pi}{2\cdot\left(\frac{5}{6}\cdot x + \frac{1}{8}\right)}\right) - 0.2\, x^2\cdot cos\left(\frac{\pi}{2\cdot x^2}\right), x = 0..1\right)$$

$with(CurveFitting):$

$$P := PolynomialInterpolation\left(\left[[0,0], \left[\frac{1}{8}, \frac{1}{8}\cdot cos\left(\frac{\pi}{\frac{2\cdot 1}{8}}\right)\right], \left[\frac{1}{7}, \frac{1}{7}\cdot cos\left(\frac{\pi}{\frac{2\cdot 1}{7}}\right)\right], \left[\frac{1}{6}, \frac{1}{6}\cdot cos\left(\frac{\pi}{\frac{2\cdot 1}{6}}\right)\right], \left[\frac{1}{5}, \frac{1}{5}\cdot cos\left(\frac{\pi}{\frac{2\cdot 1}{5}}\right)\right], \left[\frac{1}{4}, \frac{1}{4}\cdot cos\left(\frac{\pi}{\frac{2\cdot 1}{4}}\right)\right], \left[\frac{1}{3}, \frac{1}{3}\cdot cos\left(\frac{\pi}{\frac{2\cdot 1}{3}}\right)\right], \left[\frac{1}{2}, \frac{1}{2}\cdot cos\left(\frac{\pi}{\frac{2\cdot 1}{2}}\right)\right], \left[1, \frac{1}{1}\cdot cos\left(\frac{\pi}{\frac{2\cdot 1}{1}}\right)\right]\right], x, form = Lagrange\right)$$

Definition 4 [10]: Let $h \in C(I)$, Δ, *c*, and α be as in Proposition 1. The FIF $h^{\alpha}_{\Delta c}$ defined in this proposition is termed α - *fractal function* of *h* with respect to Δ and *c*. Define the α -fractal operator (attractor) respect to Δ and *c* by

$$O^{\alpha}_{\Delta,c} : C(I) \to C(I)$$

$$h \to h^{\alpha}$$

Definition 5 [10]: An α - *fractal polynomial* is an element $p^{\alpha}(t) \in C(I)$ such that there is a polynomial $p \in P[a,b]$ with $O^{\alpha}(p) = p^{\alpha}$. If *p* has degree *m*, then p^{α} is an α - fractal polynomial of degree *m*.

In the definition above, $P_m[a,b]$ is a set of polynomials of degree less than or equal to *m* on the interval $I = [a,b]$ and $P[a,b] = \bigcup_{m-1}^{\infty} P_m[a,b]$. $\{1, t, t^2, ...\}$ constitutes a basis of $P[a,b]$. For notation purposes Navascués and Sebastián in [13] assume that $P^{\alpha}_m[a,b] = O^{\alpha}(P_m[a,b], P^{\alpha}[a,b]) = O^{\alpha}(P[a,b])$.

Let us consider another example to emphasize the importance of fractal polynomial interpolation. I used Maple 10 to plot the graph of a polynomial and a graph of the fractal interpolation polynomial using a program created by Ken Monks [10]. The graph in Figure 6 is the graph of a function $f(x) = \sum_{k=1}^{10} \frac{1}{2^k} \sin(6^k x)$ on in interval $[0,1]$. The graph in Figure 7 is the graph of the same function interpolated using the Lagrange interpolation formula in the interval $[0,1]$ created using four partitions and the one in Figure 8 is the graph of the fractal function on the same interval created using four partitions.

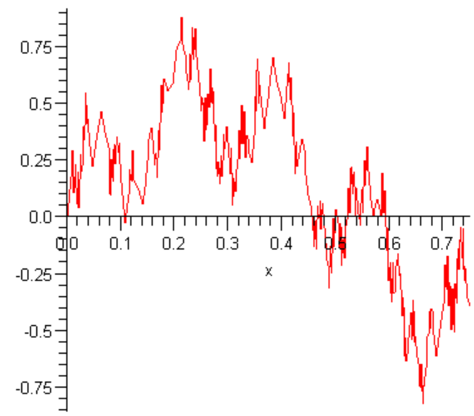

Figure 9: Graph of the function *f*

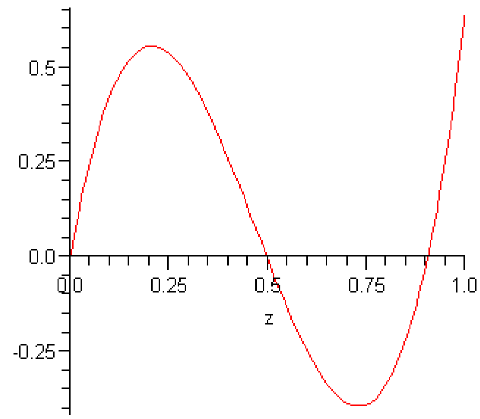

Figure 10: Graph of the Lagrange interpolation polynomial of *f*

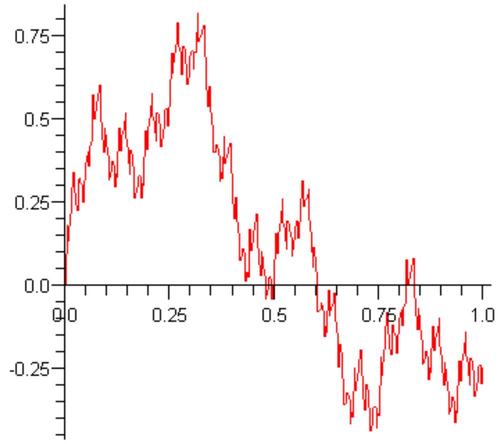

Figure 11: Graph of the fractal function

An example of Lagrange polynomial interpolation is given in chapter 7. The graphs above were drawn using the following commands in Maple 10 respectively.

$$with(CurveFitting):$$

$$plot\left(\sum_{k=1}^{10} \frac{1}{2^k} \cdot \sin\left(6^k \cdot x\right),\ x = 0..\frac{3}{4}\right)$$

$$P := PolynomialInterpolation\left(\left[0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right], \left[\sum_{k=1}^{10} \frac{1}{2^k} \cdot \sin\left(6^k \cdot 0\right), \sum_{k=1}^{10} \frac{1}{2^k} \cdot \sin\left(6^k \cdot \frac{1}{4}\right), \sum_{k=1}^{10} \frac{1}{2^k} \cdot \sin\left(6^k \cdot \frac{1}{2}\right), \sum_{k=1}^{10} \frac{1}{2^k} \cdot \sin\left(6^k \cdot \frac{3}{4}\right), \sum_{k=1}^{10} \frac{1}{2^k} \cdot \sin\left(6^k \cdot 1\right)\right],\ z,\ form = Lagrange\right):$$

$$plot(P, z = 0..1)$$

$$mb := BarnsInt\left(\left[\left[0, \sum_{k=1}^{10} \frac{\sin\left(\left(0 \cdot 6^k\right)\right)}{2^k}\right], \left[\frac{1}{4}, \sum_{k=1}^{10} \frac{\sin\left(\frac{1}{4}\, 6^k\right)}{2^k}\right], \left[\frac{1}{2}, \sum_{k=1}^{10} \frac{\sin\left(\frac{1}{2}\, 6^k\right)}{2^k}\right], \right]\right)$$

$$mb := BarnsInt\left(\left[\left[\left[0, \sum_{k=1}^{10} \frac{\sin((0 \cdot 6^k))}{2^k}\right], \left[\frac{1}{4}, \sum_{k=1}^{10} \frac{\sin\left(\frac{1}{4} 6^k\right)}{2^k}\right],\right.\right.\right.$$
$$\left.\left.\left[\frac{1}{2}, \sum_{k=1}^{10} \frac{\sin\left(\frac{1}{2} 6^k\right)}{2^k}\right],\right]\right]\left(\left[\left[\left[\frac{3}{4} \sum_{k=1}^{10} \frac{\sin\left(\frac{3}{4} 6^k\right)}{2^k}\right], \left[1, \sum_{k=1}^{10} \frac{\sin(6^k)}{2^k}\right]\right]\right],\right.$$

$$\left.[0.5, 0.5, 0.5, 0.5]\right] : IFSCurve(0, mb)\ IFSCurve\left(\frac{1}{4}, mb\right);$$
$$IFSCurve\left(\frac{1}{2}, mb\right);\ IFSCurve\left(\frac{3}{4}, mb\right);\ IFSCurve(1, mb);$$
$$DrawIFSCurve(mb, 400, color = red)$$

Notice that the Lagrange interpolation gave a smooth curve which did not highlight the finer details in the graph. The stock market and weather data fluctuates by the minute and the fractal functions are ideal to capture the minute details and predict the future up to a certain extent.

The procedure for fitting a real world data utilizing Fractal Interpolation Functions is described in the following paragraph.

First of all let us consider the set of data t_i^*, y_i^* $, i = 0,1,...,M$. The FIF is built as a perturbation of an interpolant *g* of a subset of the data. Let the subset of t_i^*, y_i^* $, i = 0,1,...,M$ be $P =$ t_n, x_n $, n = 0,1,...,N$ where $t_0, x_0 = t_0^*, x_0^*$ and $t_N, x_N = t_M^*, x_M^*$. Let the interpolant *g* be a function that passes through *P*. Consider the IFS (5.1) with $a_n = (t_n - t_{n-1}) / (t_N - t_0)$, $b_n = (t_N t_{n-1} - t_0 t_N) / (t_N - t_0)$ and $q_n(t) = h \circ L_n(t) - \alpha_n b(t)$, here *b* is continuous and $b(t_0) = x_0$ and $b(t_N) = x_N$. Also, let h^α be the corresponding FIF.

Let $\left\{\left(\bar{t}_j^n, \bar{x}_j^n\right), j = 1, 2, ..., m^{(n)}\right\}$ be the intermediate points in $I_n = \; t_{n-1}, t_n \;$ not in P:

$t_{n-1} \le \bar{t}_n^j \le t_n \;\; \forall \;\; j = 1, ..., m^{(n)}$. According to (5.2), $h^\alpha(t) = h(t) + \alpha_n (h^\alpha - h \circ c(t)) \circ L_n^{-1}(t)$.

Adding the condition $h^\alpha\left(\bar{t}_n^j\right) = \bar{x}_n^j$ to (5.2) yields

$$\bar{x}_j^n = h\left(\bar{t}_j^n\right) + \alpha_n \;\; h^\alpha - b \;\; \circ L_n^{-1}\left(\bar{t}_j^n\right)$$

And

$$\bar{x}_j^n \simeq h\left(\bar{t}_j^n\right) + \alpha_n \;\; h - b \;\; \circ L_n^{-1}\left(\bar{t}_j^n\right).$$

Choosing α_n such that the sum of the square residuals is minimum yields

$$\min E \;\; \alpha_n \;\; = \sum_{j=1}^{m^{(n)}} \left(h\left(\bar{t}_j^n\right) - \bar{x}_j^n + \alpha_n \;\; h - b \;\; \circ L_n^{-1}\left(\bar{t}_j^n\right)\right)^2$$

Solving the above equation for $E^{'} \;\; \alpha_n \;\; = 0$ yields

$$\alpha_n \;\; = \frac{\sum_{j=1}^{m^{(n)}} \left(h\left(\bar{t}_j^n\right) - \bar{x}_j^n \right)\left(h\left(L_n^{-1}\left(\bar{t}_j^n\right)\right) - b\left(L_n^{-1}\left(\bar{t}_j^n\right)\right)\right)}{\sum_{j=1}^{m^{(n)}} \left(h\left(L_n^{-1}\left(\bar{t}_j^n\right)\right) - b\left(L_n^{-1}\left(\bar{t}_j^n\right)\right)\right)^2}$$

Let $\varepsilon = \left(h\left(\bar{t}_j^n\right) - \bar{x}_j^n \right), ..., \left(h\left(\bar{t}_{m^{(n)}}^n\right) - \bar{x}_{m^{(n)}}^n \right)$ and

$$\varepsilon_b = h\left(L_n^{-1}\left(\bar{t}_j^n\right)\right) - b\left(L_n^{-1}\left(\bar{t}_j^n\right)\right), ..., h\left(L_n^{-1}\left(\bar{t}_{m^{(n)}}^n\right)\right) - b\left(L_n^{-1}\left(\bar{t}_{m^{(n)}}^n\right)\right)$$

Substituting these back in α_n yields

$$\left|\alpha_n\right| = \frac{\left|\varepsilon \cdot \varepsilon_b\right|}{\left|\varepsilon_b\right|^2}$$

According to Navascués [14], “If the interpolant *h* converges to the original function when the diameter of the partition tends to 0, we get $\left(h\left(\bar{t}_j\right) - x_j\right) \to 0$ and $\varepsilon \to 0$.”

Therefore, we can choose α_n such that $\left|\alpha_n\right| < 1, \forall\, n = 1, 2, .., N$. The function *h* together with the scale vector α determine the fitting curve h^α.

CHAPTER 6: HETMITE FRACTAL POLYNOMIAL INTERPOLATION

Section 2.5 gives a brief introduction to Hermite interpolation which can also be obtained using Newton's divided difference table and Lagrange interpolation. In [11] M.A. Navascués and M.V. Sebastián give an introduction to obtaining a Hermite interpolation function by means of fractal interpolation. In order to obtain the Hermite interpolation function let us look at a couple of theorems by Barnsley.

Theorem 5 [4]: Let $t_0 < t_1 < ... < t_N$ and $L_n(t)$, $n = 1, 2, ..., N$, the affine function $L_n(t) = a_n t + b_n$ satisfying the expressions in (3.6). Let $a_n = L_n^{-1}(t) = \dfrac{t_n - t_{n-1}}{t_N - t_0}$

and $F_n(t, x) = \alpha_n x + q_n(t)$, $n = 1, 2, ..., N$ satisfying $F_n(t_0, x_0) = x_{n-1}$, $F_n(t_N, x_N) = x_n$,

$\left|F_n(c, d_1) - F_n(c, d_2)\right| \le \alpha \cdot \left|d_1 - d_2\right|$ (check Proposition 1). Suppose for some integer $p \ge 0$,

$\left|\alpha_n\right| < a_n^p$ and $q_n \in C^p[t_0, t_N]$, $n = 1, 2, ..., N$. Let

$$F_{nk}(t, x) = \frac{\alpha_n x + q_n^{(k)}(t)}{a_n^k}, k = 1, 2, ..., p$$

$$x_{0,k} = \frac{q_1^{(k)}(t_0)}{a_1^k - \alpha_1}, x_{N,k} = \frac{q_N^{(k)}(t_N)}{a_N^k - \alpha_N}, k = 1, 2, ..., p$$

If $F_{n-1,k}(t_N, x_{N,k}) = F_{nk}(t_0, x_{0,k})$ with $n = 2, 3, ..., N$, then $\{(L_n(t), F_n(t, x))\}_{n=1}^N$

determines a FIF $f \in C^p[t_0, t_N]$ and $f^{(k)}$ is the FIF determined by $\{(L_n(t), F_{nk}(t, x))\}_{n=1}^N$

for $k = 1, 2, ..., p$.

The above theorem leads us to expect the Hermite fractal interpolation problems can be solved uniquely and assures the existence of a differentiable FIF. This FIF has $p+1$ derivatives prescribed at $((t_n, x_{nk}); n = 0, 1, ..., N; k = 0, 1, ..., p)$.

Theorem 6 [11]: Let $N \geq 1, p \in \mathbb{N}, t_0 < t_1 < ... < t_N$ and $x_{nk}; n = 0,1..., N; k = 0,1,..., p$ by given. Let $\alpha_1, \alpha_2, ..., \alpha_N$ be real numbers such that $|\alpha_n| < a_n^p \forall n = 1,2,..., N$, with $a_n = \frac{t_n - t_{n-1}}{t_N - t_0}$. Then there exists precisely one function of fractal interpolation $f \in C^p$ defined by an IFS given by:

$$L_n(t) = a_n(t) + b_n$$

$$F_n(t, x) = \alpha_n x + q_n(t)$$

Where $q_n(t) \forall n = 1,2,.., N$ are polynomials of degree at most 2p+1, such that $f^{(k)}(t_n) = x_{nk}$ for $n = 0,1,..., N; k = 0,1,...p$.

Proof: Consider $a_n = \frac{t_n - t_{n-1}}{t_N - t_0}$, $b_n = \frac{t_N t_{n-1} - t_0 t_n}{t_n - t_0}$ and define

$$F_{nk}\ \ t, x\ \ = \frac{\alpha_n x + q_n^k\ \ t}{a_n^k} \tag{6.1}$$

for $0 \leq k \leq p$ with the degree of q_n, deg $q_n\ \ = 2p + 1$.

The polynomial $q_n\ \ t$ is computed as solution of the system of equations $0 \leq k \leq p$

$$\begin{cases} F_{nk}\ \ t_0, x_{0k}\ \ = \dfrac{\alpha_n x_{0k} + q_n^{\ k}\ \ t_0}{a_n^k} = x_{n-1,k}, \\ \\ F_{nk}\ \ t_N, x_{Nk}\ \ = \dfrac{\alpha_n x_{Nk} + q_n^{\ k}\ \ t_N}{a_n^k} = x_{nk} \end{cases} \tag{6.2}$$

The $2p + 2$ unknowns of the above equation are the coefficients of $q_n\ \ t$. Solving the above equation for $q_n\ \ t$, we obtain

$$\begin{cases} \left(q_n \circ L_n^{-1}\right)^{(k)}\left(t_{n-1}\right) = \dfrac{1}{a_n^k} q_n^{(k)}\left(t_0\right) = x_{n-1,k} - \dfrac{\alpha_n x_{0k}}{a_n^k}, \\ \left(q_n \circ L_n^{-1}\right)^{(k)}\left(t_n\right) = \dfrac{1}{a_n^k} q_n^{(k)}\left(t_N\right) = x_{nk} - \dfrac{\alpha_n x_{Nk}}{a_n^k} \end{cases}$$

for $0 \le k \le p$.

The function $q_n \circ L_n^{-1}(t)$ is a polynomial with a degree of at most $2p+1$ and whose derivatives up to order p at t_{n-1} and t_n are $x_{n-1,k} - \dfrac{\alpha_n x_{0k}}{a_n^k}$ and $x_{nk} - \dfrac{\alpha_n x_{Nk}}{a_n^k}$. Thus it can be concluded that $q_n \circ L_n^{-1}(t)$ is a Hermite interpolating polynomial in the interval $[t_{n-1}, t_n]$. This Hermite polynomial exists and is unique [8] and thus it can be deduced that $q_n(t)$ exists and is unique.

Let us verify that the functions defined by (6.1) equation reference goes here satisfies the hypothesis of Barnsley and Harrington's Theorem 5.

From (6.2) we know that $F_{nk}(t_0, x_{0k}) = x_{n-1,k} = F_{n-1,k}(t_N, x_{Nk})$ $\forall n = 2,3,...,N$. Therefore, we know that the hypothesis is satisfied and thus from Theorem 5, we can guarantee the existence of $f \in C^p$ such that $f^{(k)}$ is the FIF defined by the IFS $\{(L_n(t), F_n(t,x))\}_{n=1}^N$. As a consequence, we know that $f^{(k)}$ is the fixed point of a contraction mapping $T_k : M_k \to M_k$.

Similar to Chapter 3, page 15. Let us define a norm $\left|f-g\right|_\infty = Max\{\left|f(t) - g(t)\right| : x \in I\}$. Let us define a contraction mapping $T : M \to M$, where M consists of those $f \in M$ such that $f : [t_0, t_N] \to [c, d]$, and which obeys $f(t_0) = x_0$ and $t(t_N) = x_N$. Let us define the mapping T by

$$(Tf)(t) = F_n\left(L_n^{-1}(t), f \circ L_n^{-1}(t)\right) \quad \text{when } t \in [t_{n-1}, t_n].$$

Using the definition of the norm

$$|Tf - Tg|_\infty = Max\{|F_n(L_n^{-1}(x)), f(L_n^{-1}(x)) - F_n(L_n^{-1}(x)), g(L_n^{-1}(x))|\}$$

$$\leq Max\left(|\alpha| \cdot |f(L_n^{-1}(x)) - g(L_n^{-1}(x))|\right)$$

$$\leq |\alpha|_\infty \cdot |f - g|_\infty$$

Hence we know that T has a unique fixed point f^k of $T_k : M_k \to M_k$ and the function f^k is continuous.

$$T_k g(t) = F_{nk}\left(L_n^{-1}(t), g \circ L_n^{-1}(t)\right) \quad \forall t \in [t_{n-1}, t_n]$$

$$f^k(t_0) = x_{0k}, f^k(t_N) = x_{Nk} \qquad (6.3)$$

From (6.2) and (6.3) we can conclude that

$$f^k(t_n) = F_{nk}\left(L_n^{-1}(t_n), f^k\left(L_n^{-1}(t_n)\right)\right) = F_{nk}\left(t_N, f^k(t_N)\right) = F_{nk}(t_N, x_{Nk})$$

$$\forall n = 0,1,2,...,N \text{ and } k = 0,1,2,...,p$$

The function f generalizes the Hermite functions as $\alpha_n = 0 \ \forall \ n = 1,2,...,N$. $f \in C^p$ and $f(t) = F_{n0}\left(L_n^{-1}(t), f \circ L_n^{-1}(t)\right) = q_n \circ L_n^{-1}(t)$ if $t \in [t_{n-1}, t_n]$. f is a polynomial of degree less than or equal to $2p+1$ in the interval $I_n = [t_{n-1}, t_n]$ and as a consequence f is a Hermite function since it satisfies all the conditions prescribed in section 2.5.

This result implies that the IFS in Theorem 6,

$$L_n(t) = a_n(t) + b_n$$
$$F_n(t, x) = \alpha_n x + q_n(t),$$

can be called the Hermite Fractal Interpolation Function.

CHAPTER 7: LAGRANGE FRACTAL POLYNOMIAL INTERPOLATION

As a review let us look at the Lagrange polynomials given by equation (2.1), which is $p(t) = x_0\ell_0(t) + x_1\ell_1(t) + \cdots + x_N\ell_N = \sum_{i=0}^{N} x_i\ell_{i,N}(t)$, where

$\ell_{i,N}(t) = \frac{(t-t_0)(t-t_1)\cdots(t-t_{i-1})(t-t_{i+1})\cdots(t-t_N)}{(t_i-t_0)(t_i-t_1)\cdots(t_i-t_{i-1})(t_i-t_{i+1})\cdots(t_i-t_N)} = \prod_{\substack{j=0\\ j\neq i}}^{N} \frac{t-t_j}{t_i-t_j}$. As an example let us interpolate $f(x) = x^4$ from $[1,4]$. The Lagrange polynomial $L(x)$ is given below.

Table 3: Example of Lagrange Interpolation

x_i	$f(x_i)$
$x_0 = 1$	$f(x_0) = 1$
$x_1 = 2$	$f(x_1) = 16$
$x_2 = 3$	$f(x_2) = 81$
$x_3 = 4$	$f(x_3) = 256$

$$L(x) = f(x_0)\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} + f(x_1)\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}$$

$$+ f(x_2)\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} + f(x_3)\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}$$

$$L(x) = 1\frac{(x-2)(x-3)(x-4)}{(1-2)(1-3)(1-4)} + 16\frac{(x-1)(x-3)(x-4)}{(2-1)(2-3)(2-4)}$$

$$+81\frac{(x-1)(x-2)(x-4)}{(3-1)(3-2)(3-4)} + 256\frac{(x-1)(x-2)(x-3)}{(4-1)(4-2)(4-3)}$$

$$= 10x^3 - 35x^2 + 50x - 24$$

Definition 6[13]: Navascués and Sebastián define the α - fractal interpolant of the Lagrange polynomial as

$$p_N^\alpha(t) = O^\alpha(p_N) = \sum_{i=0}^{N} x_i \ell_{i,N}^\alpha(t)$$

where,

Δ : The partition.

$\ell_{i,N}^\alpha$: α - fractal polynomial of $\ell_{i,N}$ with respect to the partition Δ.

$p_N^\alpha(t)$: A function which passes through the points (t_n, x_n) as in Proposition 1 on page 22.

Let L_N represent a Lagrange operator that assigns an interpolant polynomial to a function *f* with respect to $\{(t_n, f(t_n))\}_{n=0}^N$, then $p_N^\alpha(t) = O^\alpha \circ L_n(f)$. The basis polynomials of the Lagrange operator $\ell_{i,N}(t)$ are orthogonal with respect to the norm

$\left\langle \ell_{i,N}, \ell_{j,N} \right\rangle = \sum_{n=0}^{N} \ell_{i,N}(t_n) \ell_{j,N}(t_n)$. The same property is true for $\ell_{i,N}^\alpha$. Therefore,

$\left\langle \ell_{i,N}^\alpha, \ell_{j,N}^\alpha \right\rangle = \sum_{n=0}^{N} \delta_i^n \delta_j^n$, where δ_i^n is the Delta function (the definition of the Delta function can be found in page 7).

If $p^\alpha \in P_N^\alpha[a,b]$, by the linearity of the operator O^α (see page 406 of [13] Corollary 1) $p^\alpha = \sum_{i=0}^{N} \lambda_i \ell_{i,N}^\alpha$. Also, the orthogonality of $\ell_{i,N}^\alpha$ implies linear independence and therefore, $\{\ell_{i,N}^\alpha\}$ constitutes a basis for the space of α -fractal polynomials for the space $P_N^{\alpha}[a,b]$ on the partition Δ. $P_N^{\alpha}[a,b]$ has a finite dimension which allows us to obtain a p^{α^*} for each $h \in C[a,b]$ such that $\left\| h - p^{a^*} \right\|_{\infty} = \inf \left\| h - p^{a^*} \right\|_{\infty}$; $p^{\alpha} \in P_N^{\alpha}[a,b]$

Utilizing the fractal polynomials in the Lagrange interpolation gives us the advantage of obtaining a non-smooth version of the Lagrange polynomials which will highlight the finer details of the graph.

CHAPTER 8: USES OF FRACTALS

One of the many uses of fractal geometry is the generation of programs that create images of clouds, trees, landscapes and the coastal line on the computer screen, fractals have many other applications. We have already went over the use of fractals in science fiction movies. Fractals together with knowledge of ecosystem are also used to determine the spread of smoke, acid rain, and other air borne or water borne toxicants. Fractal interpolation also provides a good representation of economic time series such as the stock market fluctuation and weather data. With the current economic crisis we need some new models that take into account many more variables and that provide more accurate interpretation of the future behavior and I think fractal polynomial interpolation can play a significant part in that. The financial markets are churning with the sub-prime home loan crisis in the US and the global banking system. The huge derivatives overhang is a problem that requires global cooperation to manage. Hike in food prices and increase of poverty. The Governments together with the Central Banks and Commercial Banks need to come up with a solution for these global problems. Modeling the problem involves several interdependent factors mostly in non-linear relationships- the model, in addition to traditional quantitative data, has to incorporate intangible qualitative factors such as political policies, changing food habits, subsidies and retail supply chain all of which are dynamically changing. We need to develop holistic models which are driven by the problem against the traditional approach wherein the problem is mutilated to suit known models such as OR resulting in short-term non-sustainable solutions. Benoit Mandelbrot, the father of fractals, has taken on the stock market and he is analyzing these problems."Markets, like oceans, have turbulence," he said. "Some days the change in markets is very small, and some days it moves in a huge leap. Only fractals can explain this kind of random change." With the current economic crisis we need some new models that take into account many more variables and that provide more accurate interpretation of the future behavior and fractal polynomial interpolation can play a significant part in that.

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