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Price Discovery In The U.S. Bond Market Trading Strategies And The Cost Of Liquidity

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PRICE DISCOVERY IN THE U.S BOND MARKET
TRADING STRATEGIES AND THE COST OF LIQUIDITY

by

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Abstract

The world bond market is nearly twice as large as the equity market. The goal of this dissertation is to study the dynamics of bond price. Among the liquidity risk, interest rate risk and default risk, this dissertation will focus on the liquidity risk and trading strategy. Under the mathematical frame of stochastic control, we model price setting in U.S. bond markets where dealers have multiple instruments to smooth inventory imbalances. The difficulty in obtaining the optimal trading strategy is that the optimal strategy and value function depend on each other, and the corresponding HJB equation is nonlinear. To solve this problem, we derived an approximate optimal explicit trading strategy. The result shows that this trading strategy is better than the benchmark central symmetric trading strategy.

To my husband Grady, and my dear daughter Mollie

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CHAPTER 1

INTRODUCTION

1.1   Motivation

Bonds are an important source of financing for governments and corporations. As of 2009, the size of the outstanding U.S. bond market debt was $31.2 trillion, according to Bank for International Settlements. Nearly all of the $822 billion average daily trading volume in the U.S. bond market takes place between broker-dealers and large institutions in a decentralized, over-the-counter (OTC) market.

In the OTC market, typically, dealers act as counterparties: they buy from public sellers and sell to public buyers. Dealers quote a pair of bid and ask prices to customers and have the obligation to buy or sell, respectively, at the quoted prices if their clients wish to exchange at the quoted price. Dealers provide market liquidity and make their profit from the difference between their bid and ask (buying and selling) prices and a service charge. Their objective is to profit from the spread between bid and ask prices, not from price movements. In that regard, they are different from ordinary investors, who seek to profit by betting on how the price moves.

By virtue of their central position and role as price setters, dealers are a logical starting point for an exploriation of how prices are actually determined inside the "black box." Dealers' trading strategies contribute to price formation. Dealers face an inventory risk when receiving consecutive trades in the same direction. For example, when a dealer's inventory position is large and positive, it is potentially very risky because if the bond depreciates due to the interest rate change or default, the dealer will lose a considerable amount. For a risk-averse dealer, this is certainly undesirable. Thus, they can adjust bid and ask prices to induce more buy orders rather than sell orders and bring their inventory position back to zero. However, if dealers lower the ask price below the previous bid price at which they bought the bond, they may lose money. To avoid such situations, dealers may consider selling a certain amount bond to other dealers or dealer brokes.

The goal of this dissertation is to model bond price determination in the over-the-counter market. Essentially, the dealers are profit-maximizers, who control their buying and selling. We modeled the activities of dealers under the framework of stochastical control. Based on this model, we will answer the following questions:

1. What is the optimal price quoting strategy for a dealer that maximizes profit? Does it exist? If it exists, is it explicit?

2. What is the comparison with the other strategies, such as a frozen strategy or a benchmark symmetric strategy?

1.2   Literature Review

One set of papers, including [Garman 1976], [Amihud and Mendelson 1980], [Madhaven and Smidt 1993], [Avellaneda 2008], models the pricing and inventory behavior of risk-averse dealers, usually assuming either monopolistic dealers or ignoring the interactions among dealers' quotes.

Garman (1976) first presented a rigorous stochastic model of the dealers' market, exploring the nature of possible failure that the dealer's inventory of stock or cash becomes zero. Based on Garman's framework, Amihud and Mendelson (1980) proved that the bid and ask prices are monotone decreasing functions of the inventory level and that the spread (the difference between the ask and bid price) is increasing in distance from the preferred position. Due to the complexity of the model, neither Garman nor Amihud were able to give the closed form solutions to their models. Later, Madhaven and Smidt(1993) derived the quoted price as a function of inventory deviation from preferred level in a similar framework. Recent work by Avellaneda and Stoikov (2008) has also focused on the optimal trading strategy.

However, the above-mentioned models of price formation assumed that bid and ask prices are the only instruments by which a dealer can adjust inventory levels. Indeed, in dealers' market, dealers may pass around their imbalance of inventory. Without concerning the inter-dealer trading, the model may fail to be realistic. Since the liquidity in bond markets is very limited, the optimal strategy would be not to buy and to sell at a unreasonablely low price.

Ho and Stoll (1983) used a framework that permits inter-dealer trading, although it does not arise in the model solution. Lyons (1997) developed a simultaneous trade model of the spot foreign exchange market, called the "hot potato" model, in which dealers trade with dealers, passing around inventory imbalances, which is the basis of this dissertation.

The second set of papers, including Hasbrouck and Sofianos (1993), and Hansch and Neuberger (1996), studied the empirical trading behavior of dealers. Hasbrouck did empirical analysis of New York Stock Exchange (NYSE) specialists. His study examined a comprehensive sample of quote, trade, and inventory data. He studied 138 stocks from November 1988 through August 1990. The avarage daily closing inventory was 118.97 (hundred shares) with a standard deviation of 605.44 (hundred shares). The overall average holding period was only 0.84 days. His evidence suggested that dealers do have impact on the market. In his study, he also found out that dealers have different levels of risk aversion, which leads to different trading behaviors. Even though Hasbrouck studies NYSE specialists, not dealers in bond market, one thing is in common: both study traders' behavior under the same objective. Hansch and Neuberger (1998) used a rich database from the London Stock Exchange, which allows them to observe market maker inventories directly, something previous studies have not been able to do. Their results showed that dealers will not protect themselves from informed customers with wide spreads, but will instead seek to attract them with narrow spreads.

This dissertation attempts to extend existing models of dealers' price setting behavior to capture the essential features of markets with inter-dealer trading in an empirically tractable way.

CHAPTER 2

MATHEMATICAL PRELIMINARIES

This chapter collects all the concepts, theorems and examples that are needed in later chapters. For the reader's convenience, proofs are given for some theorems.

2.1   Stochastic Process

Definition 1 (Oksendal 1998) *Given $(\Omega, \mathcal{F}, \mathbb{P})$, (1) a stochastic process X_t is a collection $\{X_t : t \in I\}$ of random variables where the index t belongs to some index set I. We call that $\{X_t\}$ is a continuous-time stochastic process if I is an interval in R, or discrete-time stochastic process if I is a subset of $\{0, 1, 2, \cdots, n, \cdots\}$. We also call $t \to X_t(\omega)$ the sample path of the stochastic process X_t.*

(2) a stochastic process $\{X_t, t \geq 0\}$ is called an independent increment process if for any n and any $0 \leq t_0 < t_1 < \cdots < t_n$, the increments $X_0, X_{t_1} - X_{t_0}, \cdots, X_{t_n} - X_{t_{n-1}}$ are independent.

(3) Let an $\mathbb{R}^n$-valued stochastic process $X(\cdot)$ be defined on a filtered probability space $(\Omega, \mathcal{F}, F = (\mathcal{F}_t, t \geq 0), \mathbb{P})$. If for any $t \geq 0$ and $s > t$

$$\mathbb{P}(X(s) \in B | \mathcal{F}_t) = \mathbb{P}(X(s) \in B | X(t)), \quad \forall B \in \mathcal{B}(\mathbb{R}^n)$$

process $X(\cdot)$ is called a Markov process, where $\mathcal{B}(\mathbb{R}^n)$ is the Borel σ-fields of $\mathbb{R}^n$.

Theorem 2 *An independent increment process is a Markov process.*

Definition 3 *A stochastic process $\{B_t, t \geq 0\}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a Brownian motion if :*

1. *$B_0 = 0$*

2. *B_t is an independent increment process*

3. *for $0 \leq s < t$, the increment $B_t - B_s$ has a normal distribution $N(0, t - s)$*

Brownian motion is an independent increment process, therefore it is a Markov process.

Definition 4 *A stochastic process $\{N_t, t \geq 0\}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a Poisson process of rate λ if it verifies the following properties:*

1. *$N_0 = 0$*

2. *N_t is an independent increment process*

3. for $0 \leq s < t$, the increment $N_t - N_s$ has a Poisson distribution with rate $\lambda(t-s)$, that

   is,

$$P(N_t - N_s = k) = e^{-\lambda(t-s)} \frac{(\lambda(t-s))^k}{k!}, k = 0, 1, 2, \cdots$$

where $k = 0, 1, 2, \cdots ,$.

Poisson process is an independent increment prosess, therefore it is a Markov process.

Theorem 5 *Let L_t and M_t be two independent Poisson processes with respective rates λ and μ. Then the process $N_t = L_t + M_t$, called the superposition of the processes L_t and M_t, is a Poisson process of rate $\lambda + \mu$.*

Theorem 6 *Let N_t be a Poisson process with rate λ. Let Y_n be a sequence of independent Bernoulli random variables with parameter $p \in (0, 1)$, independent of N_t. Set*

$$M_t = Y_1 + Y_2 + \cdots + Y_{N_t}$$

and

$$L_t = N_t - M_t.$$

Then the processes L_t and M_t are independent Poisson processes with respective rates λp and $\lambda(1-p)$.

Definition 7 *A stochastic process $\{C_t, t \geq 0\}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a compound Poisson process if it is of the form*

$$C_t = \sum_{n=0}^{N_t} X_n,$$

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where $X_n, n = 0, 1, \cdots$ are indepent identically distributed and N_t is a Poisson process.

Theorem 8 Wald Equation *Let $X_n; n \in \mathbb{N}$ be an infinite sequence of real-valued, finite-mean random variables and let N be a nonnegative integer-valued random variable. Assume that*

(i) N has finite expectation,

(ii) $X_n; n \in \mathbb{N}$ all have the same expectation,

(iii) $E[X_n 1_{N \geq n}] = E[X_n] P(N \geq n)$ for every natural number n, and

(iv) the series $\sum_{n=1}^{\infty} \mathrm{E}\big[|X_n| 1_{\{N \geq n\}}\big]$ is convergent.

Then the random sum $S := \sum_{n=1}^{N} X_n$ is integrable and

$$\mathrm{E}[S] = \mathrm{E}[N]\, \mathrm{E}[X_1],$$

which is called the Wald equation

Condition (iii) in the above theorem means that for all n, N is not necessary to be independent of X_n for the Wald equation holds, but $\{N \geq n\}$ is independent of X_n.

For a compound Poisson process, due to the independence of N_t and X_n, $\{N_t \geq n\}$ is independent of X_n for all $n \geq 0$. Thus, the Wald equation holds.

2.2 Generator of a Process

Let an $\mathbb{R}^n$-valued Markov process $X(\cdot)$ be defined on a filtered probability space $(\Omega, \mathcal{F}, F = (\mathcal{F}_t, t \geq 0), \mathbb{P})$. The transition distribution of $X(\cdot)$ is defined as follows:

$$\hat{P}(t, x, s, B) = \mathbb{P}(X(s) \in B | X(t) = x), \;\; 0 \leq t \leq s, \; B \in \mathcal{B}(\mathbb{R}^n), x \in \mathbb{R}^n$$

Further, for any $\phi(\cdot) \in C(\mathbb{R}^n; \mathbb{R}) \equiv \{\phi : \mathbb{R}^n \to \mathbb{R} | \; \phi(\cdot) \text{ is continuous}\}$, we define

$$E(\phi(X(s; t, x))) = \int_{\mathbb{R}^n} \phi(y) \hat{P}(t, x, s, dy), \;\; 0 \leq t \leq s,$$

as long as the right-hand side exists.

Definition 9 *Suppose $X(\cdot)$ is an $\mathbb{R}^n$-valued process. Let*

$$\mathcal{D}(\mathbb{A}) = \left\{ \phi \in C([0, T] \times R^n) | \lim_{h \to 0^+} \frac{E\phi(t + h, X(t + h; t, x)) - \phi(t, x)}{h} \; exists \right\},$$

and define

$$(\mathbb{A}\phi)(t, x) = \lim_{h \to 0^+} \frac{E\phi(t + h, X(t + h; t, x)) - \phi(t, x)}{h}, \;\; \forall \phi \in \mathcal{D}(\mathbb{A}),$$

We call $\mathbb{A}$ the Backward Evolution Operator of $X(\cdot)$.

Theorem 10 *(Dynkin's formula [Fleming 2006]) For $t < s$,*

$$E\phi(s, X(s; t, x)) - \phi(t, x) = E \int_t^s A\phi(r, X(r; t, x)) dr, \;\; \forall \phi \in \mathcal{D}(A), 0 \leq t \leq s$$

Definition 11 *Suppose $X(\cdot)$ is an $\mathbb{R}^n$-valued stochastic process. Let*

$$\mathcal{D}(\mathbb{L}) = \left\{ f \in C(\mathbb{R}^n)| \lim_{h \to 0^+} \frac{E[f(X(t+h;t,x)) - f(x)]}{h} \ exists \right\}.$$

Define

$$\mathbb{L}f(x) = \lim_{h \downarrow 0} \frac{E[f(X(t+h;t,x)) - f(x)]}{h}, \quad \forall f \in \mathcal{D}(\mathbb{L}).$$

We call $\mathbb{L}$ the generator of $X(\cdot)$.

Example 12 (Fleming 2006) *The generator of a Poisson process, with rate $\lambda > 0$ is*

$$(\mathbb{L})f(x) = \lambda[f(x+1) - f(x)]$$

Example 13 (Fleming 2006) *The generator of a compound Poisson process*

$$X(s) = x + \sum_{i=1}^{N_s} J_i,$$

where N_s is a Poisson process with rate $\lambda(s,x)$ at time s, is

$$\mathbb{L}f(x) = \lambda(t,x) \int_{R^n} (f(x+y) - f(x))\Pi(t,x,dy), \ \forall f \in \mathcal{D}(\mathbb{L})$$

where $\Pi(t,x,dy)$ is the density function of jump size conditional on $X(t) = x$

It is known that [Fleming 2006] the relation between backward evolution operator $\mathbb{A}$ and generator $\mathbb{L}$ is given by

$$\mathbb{A}\phi = \phi_t + \mathbb{L}\phi(t,\cdot), \quad \forall f \in \mathcal{D}(\mathbb{A})$$

2.3   Ito's Lemma and Feynman-Kac Formula

In its simplest form, Ito's lemma states the following: let X_t be a solution of the following stochastic differential equation

$$dX_t = \mu_t \, dt + \sigma_t \, dB_t. \tag{2.1}$$

such an X_t is called an Ito drift-diffusion process. Let $f(t, x)$ be a twice differentiable function of two real variables t and x, the following holds:

$$df(t, X_t) = \left(\frac{\partial f}{\partial t} + \mu_t \frac{\partial f}{\partial x} + \frac{\sigma_t^2}{2} \frac{\partial^2 f}{\partial x^2} \right) dt + \sigma_t \frac{\partial f}{\partial x} \, dB_t, \tag{2.2}$$

where the last term is the differential form of Ito integral. This immediately implies that $f(t, X)$ itself is an Ito drift-diffusion process.

The Feynman-Kac formula establishes a link between parabolic partial differential equations (PDEs) and the expected values. It offers a method of solving certain PDEs by simulating random paths of a stochastic process. Conversely, an important class of expectations of random processes can be computed by deterministic methods.

Suppose X_t solves the scalar stochastic differential equation

$$dX_t = b(X_t, t)dt + \sigma(X_t, t)dB_t$$

and let

$$u(x, t) = E[f(X_T)|X_t = x]$$

be the expected value of $f(X_T)$, given that $X_t = x$. Then u solves the PDE,

$$\begin{cases} \frac{\partial u}{\partial t} + b(x,t)\frac{\partial u}{\partial x} + \frac{1}{2}\sigma^2(x,t)\frac{\partial^2 u}{\partial x^2} = 0 \\ u(x,T) = f(x) \end{cases} \tag{2.3}$$

The following example will be used in later chapter.

Example 14 *Assume that $dS_t = \sigma S_t dB_t$ where σ is a constant, then*

$$v(s,t) = E(x + qS_T - \gamma(qS_T)^2|S_t = s)$$

is a solution to the following PDE:

$$\begin{cases} u_t + \frac{1}{2}\sigma^2 s^2 u_{ss} = 0, \quad t \in [0,T], s \in \mathbb{R}^+ \\ u(q,s,T) = x + qs - \gamma q^2 s^2 \end{cases} \tag{2.4}$$

2.4   Dynamic Programming

We now consider problems in which the Markov process $X(\cdot)$ is actively infulenced by a control $u(\cdot) \in \mathcal{U}[t,T]$ where

$$\mathcal{U}[t,T] = \{u : [t,T] \times \mathbb{R}^n \to \mathbb{U}\}$$

More precisely, for any $(t,x) \in [0,T] \times \mathbb{R}^n$, $u(\cdot) \in \mathcal{U}[t,T]$, $X(\cdot) = X(\cdot;t,x,u(\cdot))$ is defined on $[t,T]$. For convenience, let us call $X(\cdot;t,x,u(\cdot))$ a controlled Markov process.

Define the performance function:

$$J(t,x,u(\cdot)) = E\{W(X(T;t,x,u(\cdot)))|X_t = x\} \tag{2.5}$$

Then define the value function:

$$\begin{cases} V(t,x) = \sup_{u(\cdot)\in\mathcal{U}[t,T]} J(t,x,u(\cdot)) \\ V(T,x) = W(x) \end{cases} \tag{2.6}$$

Theorem 15 *Bellman's Priciple of Optimality. Let $X(s;t,x,u(\cdot))$ be a controlled Markov process. If $W : \mathbb{R}^n \to \mathbb{R}$ is continuous, then*

$$V(t,x) = \sup_{u(\cdot)\in\mathcal{U}[t,s]} E\{V(s,X(s;t,x,u(\cdot)))\}, \quad 0 \le t < s \le T \tag{2.7}$$

Proof: For all $u(\cdot) \in \mathcal{U}[t,T]$ and $0 \le t < s \le T$, by the property of conditional expectation,

$$\begin{aligned} J(t,x,u(\cdot)) &= E_t\{W(X(T;t,x,u(\cdot)))\} \\ &= E_t\{E(W(X(T;t,x,u(\cdot)))|\mathcal{F}_s)\} \\ &= E_t\{E(W(X(T;t,x,u(\cdot)))|X(s;t,x,u(\cdot)))\} \\ &= E_t\{E(W(X(T;s,X(s;t,x,u(\cdot)),u(\cdot))))\} \\ &= E_t\{J(s,X(s;t,x,u(\cdot)),u(\cdot))\} \end{aligned} \tag{2.8}$$

The third equation holds because $X(\cdot)$ has the Markov property. Since

$$J(s,X(s;t,x,u(\cdot)),u(\cdot)) \le V(s,X(s;t,x,u(\cdot))),$$

we have

$$J(t,x,u(\cdot)) \le E_t\{V(s,X(s;t,x,u(\cdot)))\} \le \sup_{u(\cdot)\in\mathcal{U}[t,s]} E_t\{V(s,X(s;t,x,u(\cdot)))\}$$

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and this implies

$$V(t,x) \leq \sup_{u(\cdot) \in \mathcal{U}[t,s]} E_t\{V(s, X(s; t, x, u(\cdot))\} \tag{2.9}$$

Next, for any $\epsilon > 0$ and any $u(\cdot) \in \mathcal{U}[t, s]$, there exists a $u_\epsilon(\cdot) \in \mathcal{U}[s, T]$, depending on ϵ

and $u(\cdot)$, such that

$$V(s, X(s; t, x, u(\cdot))) - \epsilon \leq J(s, X(s; t, x, u(\cdot)), u_\epsilon(\cdot))$$

For any $u(\cdot) \in \mathcal{U}[t, T]$, we now construct a control

$$\tilde{u}(s) = \begin{cases} u(s) & \text{if } r \in [t, s), \\ u_\epsilon(r) & \text{if } r \in [s, T] \end{cases}$$

Then $\tilde{u}(\cdot) \in \mathcal{U}[t, T]$, which is still depending on ϵ and $u(\cdot) \in \mathcal{U}[t, s)$, and

$$\begin{aligned} V(t, x) &\geq J(t, x, \tilde{u}(\cdot)) \\ &= E_t\{J(s, X(s; t, x, u(\cdot)), u_\epsilon(\cdot))\} \\ &\geq E_t\{V(s, X(s; t, x, u(\cdot))) - \epsilon)\}, \quad \forall u(\cdot) \in \mathcal{U}[t, s] \end{aligned} \tag{2.10}$$

Consequently,

$$V(t, x) \geq \sup_{u(\cdot) \in \mathcal{U}[t,s]} E_t\{V(s, X(s; t, x, u(\cdot)))\} - \epsilon \tag{2.11}$$

Since $\epsilon > 0$ is arbitrary, we must have

$$V(t, x) \geq \sup_{u(r), t \leq r \leq s} E_t\{V(s, X(s; t, x, u))\} \tag{2.12}$$

From (2.9) and (2.12), we get $V(t, x) = \sup_{u(\cdot) \in \mathcal{U}[t,s]} E\{V(s, X(s; t, x, u(\cdot)))\}$

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2.5 Heat Equation

We consider the following standard inhomogeneous heat equation:

$$\begin{cases} u_t = \frac{1}{2}\sigma^2 u_{xx} + f(u), & t \in [0, T] \\ u(0, x) = g(x) \end{cases} \tag{2.13}$$

Theorem 16 *If $f : X \to X$ satisfies the Lipschitz condition,*

$$|f(u) - f(\hat{u})| \le L|u - \hat{u}|, \quad \forall u, \hat{u} \in X, \quad \text{for some constant } L > 0, \tag{2.14}$$

then equation (2.13) admits a unique solution which coincides with that of the following integral equation:

$$u(t, x) = e^{t\Delta} g(x) + \int_0^t e^{(t-s)\Delta} f(u(s, x)) ds \tag{2.15}$$

where

$$e^{t\Delta} g(x) = \int_{R^n} G(t, x - y) g(y) dy,$$

$$G(t, x) = \frac{1}{\sqrt{2\pi t}\sigma} e^{-\frac{x^2}{2\sigma^2 t}} \tag{2.16}$$

Next, we consider the following problem:

$$\begin{cases} u_t + \frac{1}{2}\sigma^2 x^2 u_{xx} + f(u) = 0, & t \in [0, T], x > 0 \\ u(T, x) = g(x), & x > 0 \\ u(t, 0) = 0, & t \in [0, T] \end{cases} \tag{2.17}$$

Theorem 17 *The initial value problem (2.17) is equivalent to the following integral equation:*

$$v(\tau, z) = e^{\tau\Delta} g(e^z) + \int_0^\tau e^{(\tau-s)\Delta} f(v(s, z)) ds, (\tau, z) \in [0, T] \times \mathbb{R}. \tag{2.18}$$

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where

$$\tau = T - t$$

$$z = \ln x - \frac{1}{2}\sigma^2\tau$$

$$e^{\tau\Delta}g(x) = \int_{R^n} G(\tau, x - y)g(y)dy,$$

$$G(\tau, x) = \frac{1}{\sqrt{2\pi\tau}\sigma}e^{-\frac{x^2}{2\sigma^2\tau}}.$$

(2.19)

Proof: Let

$$\tau = T - t, \quad z = \ln x - \frac{1}{2}\sigma^2\tau \tag{2.20}$$

We denote

$$v(\tau, z) = u(T - \tau, e^{z + \frac{1}{2}\sigma^2\tau}),$$

which gives

$$v_\tau(\tau, z) = -u_t(t, x) + \frac{1}{2}\sigma^2 x u_x$$

$$v_{zz}(\tau, x) = x^2 u_{xx} + x u_x$$

(2.21)

Subsituting (2.20) and (2.21) back into (2.17) yields

$$\begin{cases} v_\tau = \frac{1}{2}\sigma^2 v_{zz} + f(v), & z \in \mathbb{R} \\ v(0, z) = g(e^z) \end{cases} \tag{2.22}$$

By applying the result of theorem (16), we obtained our conclusion.

In a later chapter, we are interested in the following homogeneous heat equation.

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Example 18

$$
\begin{cases}
u_t + \frac{1}{2}\sigma^2 s^2 u_{ss} = 0, \quad t \in [0,T], s > 0, q > 0 \\
u(T,q,s) = qs - \gamma q^2 s^2
\end{cases}
\tag{2.23}
$$

Let

$$
\tau = T - t, \quad z = lns - \frac{1}{2}\sigma^2\tau
\tag{2.24}
$$

we denote

$$
v(\tau, q, z) = u(q, e^{z+\frac{1}{2}\sigma^2\tau}, T - \tau).
$$

After differentiating both sides in terms of τ and z, respectively, we have

$$
\begin{aligned}
v_\tau &= u_s \frac{ds}{d\tau} + u_t \frac{dt}{d\tau} = \frac{1}{2}\sigma^2 s u_s - u_t \\
v_z &= u_s \frac{ds}{dz} = s u_s \\
v_{zz} &= s^2 u_{ss} + s u_s
\end{aligned}
\tag{2.25}
$$

Therefore, equation (2.23) becomes

$$
\begin{cases}
v_\tau = \frac{1}{2}\sigma^2 v_{zz}, \quad \tau \in [0,T], z \in \mathbb{R}, q \geq 0 \\
v(\tau, q, z) = qe^z - \gamma q^2 e^{2z}
\end{cases}
\tag{2.26}
$$

The solution of equation (2.26) is

$$
\begin{aligned}
&\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\tau}\sigma} e^{-\frac{(z-y)^2}{2\sigma^2\tau}}(qe^y - \gamma q^2 e^{2y})dy \\
&= qe^{-\frac{z^2-(\sigma^2\tau+z)^2}{2\sigma^2\tau}} - \gamma q^2 e^{-\frac{z^2-(2\sigma^2\tau+z)^2}{2\sigma^2\tau}} \\
&= qe^{\frac{\sigma^2\tau+2z}{2}} - \gamma q^2 e^{2(z+\frac{1}{2}\sigma^2\tau)} \\
&= qe^{lns} - \gamma q^2 e^{lns^2} e^{\sigma^2(T-t)} \\
&= qs - \gamma q^2 s^2 e^{\sigma^2(T-t)}
\end{aligned}
\tag{2.27}
$$

Also, the solution of the following inhomogeneous heat equation gives an insight into problems found in chapter 4.

Example 19

$$\begin{cases} u_t + \frac{1}{2}\sigma^2 s^2 u_{ss} + M = 0, & t \in [0, T], s > 0, q > 0 \\ u(T, q, s) = 0 \end{cases} \tag{2.28}$$

Let

$$\tau = T - t, \quad z = \ln s - \frac{1}{2}\sigma^2 \tau \tag{2.29}$$

and we denote

$$v(\tau, z) = u(q, e^{z + \frac{1}{2}\sigma^2 \tau}, T - \tau).$$

Then equation (2.28) becomes

$$\begin{cases} v_\tau = \frac{1}{2}\sigma^2 v_{zz} + M, t \in [0, T], z \in \mathbb{R} \\ v(0, q, z) = 0 \end{cases} \tag{2.30}$$

The solution of equation (2.30) is

$$\begin{aligned} v(\tau, q, z) &= \int_0^\tau \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi(\tau - l)}\sigma} e^{-\frac{(z-y)^2}{2\sigma^2(\tau-l)}} M \, dy \, dl \\ &= \int_0^\tau M \, dl \\ &= M\tau \end{aligned} \tag{2.31}$$

That is, $u(q, s, t) = M(T - t)$

CHAPTER 3

EMPIRICAL ANALYSIS

In order to build a realistic model, we will take a look at some basic facts in the bond markets. First we briefly review the bond market in U.S. Then, we explore the real trading data provided by Trade Reporting and Compliance Engine (TRACE) [1]. The data will serve to help calibrate our model in simulation. Lastly, we provide the evidence that dealers do adjust their inventory through controlling the bid and ask prices, which is the reason why we build the model under the frame of stochastical control.

3.1 Bond Market Introduction

3.1.1 U.S. Bond Market

The most striking observation is that the world's stock and bond markets, in aggregate, have a market value in U.S dollar terms of more than $125 trillion in 2009, or approximately two times the value of the world's economic output, estimated at approximately $61 trillion in 2008. It is also interesting to note that the world bond market exceeds the world stock

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[1]http://cxa.marketwatch.com/finra/BondCenter/Default.aspx

market in size by a factor of nearly 2 to 1. U.S. bonds account for $31,172$ billion [2], a portion of the global bond market at nearly 38%, despite the fact that U.S. economic output accounts for less than one-fourth of global output. Due to its size, importance and easily accessible and free trading data, we focus on modeling the U.S. bond market.

Table (3.1) shows that the largest segment of the U.S. bond market is the mortgage-backed bond market, and the second largest segment is the U.S. Treasury [3].

Table 3.1: U.S. Bond Market Debt Outstanding As of 30 June 2009, in billions of dollars

U.S. Treasury	6,927.8
Agencies of the U.S.	2,972.4
State and Municipal	2,726.8
Corpotate	6,778.4
Money Market	3,430.3
Mortgage-backed	8,948.7
Asset-backed	2,533.6

Most bonds are issued by one of four groups: the U.S. government, a U.S. government-sponsored agency, state and local governments, or corporations.

The bonds issued by the U.S. government are called treasurys and are grouped into three categories based on their time to maturity: treasury bills, treasury notes and treasury bonds.

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[2]Data is provided by The Bank for International Settlements (BIS)

[3]This Data is provided by the Securities Industry and Financial Markets Association (SIFMA). The total U.S. debt outstanding as of 31 March 2009 was $31.2 trillion according to BIS, compared to $34.2 trillion according to SIFMA

Treasury bills mature from 28 days to one year. Treasury notes mature from 1 to 10 years. Treasury bonds mature from 10 to 30 years. Treasurys are widely regarded as the safest bond investments because they are backed by the full faith and credit of the U.S. government. The income earned from treasurys is exempt from state and local taxes.

Agency bonds are issued by U.S. government-sponsored agencies. The offerings of these agencies are backed by the U.S. government, but not guaranteed by the full faith and credit government since the agencies are private entities. Such agencies have been set up in order to allow certain groups of people to access low cost financing, especially students and first-time home buyers. Some prominent issuers of agency bonds are the Student Loan Marketing Association (Sallie Mae), the Federal National Mortgage Association (Fannie Mae) and the Federal Home Loan Mortgage Corporation (Freddie Mac). Agency bonds are usually exempt from state and local taxes, but not federal tax.

The bonds issued by state and local government are called municipal bonds. Municipal bonds are a step-up on the risk scale from Treasurys, but they make up for it in tax trickery. Many munis are exampt from city, state and federal taxes (triple tax-free). Because tax-free income is so enticing to high-income investors, triple tax-free munis generally offer a lower coupon rate than equivalent taxable bonds.

Corporate bonds, issued by corporations, are generally the riskiest fixed-income securities of all because companies, even large, stable ones, are much more susceptible than governments to economic problems, mismanagement and competition. Cities do go bankrupt, but it's infrequent. Not so rare is the once-proud company brought low by foreign rivals or

management missteps. Pan Am, LTV Steel and the Chrysler bankruptcies of 1979 are the facts.

Corporate bonds come in several maturities: short term, one to five years; intermediate term, five to 15 years; long term, longer than 15 years.

The credit quality of companies and governments is closely monitored by two major debt-rating agencies: Standard & Poor's and Moody's. They assign credit ratings based on the entity's perceived ability to pay its debts over time. Those ratings, expressed as letters (Aaa, Aa, A, etc.), help determine the interest rate that a company or government has to pay.

Corporations, of course, do many things to keep their credit ratings high. The difference between an A rating and a BBB rating can mean millions of dollars in extra interest paid. But even companies with less-than-investment-grade (B and below) ratings issue bonds. These securities, known as high-yield, or junk bonds, are generally too speculative for the average investor, but they can provide higher return with greater risk.

Zero-coupon bonds are fixed-income securities that do not make periodic interest payments like regular bonds. Instead, the bond is sold at a deep discount to its face value and at maturity, the bondholder collects all of the compounded interest, plus the principal. Zeros are usually priced aggressively and are useful for investors who are looking for a set payout on a given date, instead of a stream of payments that they have to figure out where to invest elsewhere.

Table 3.2: U.S. Bond Types, Tax Exemption and Risk Level

Types	Issuer	Tax Exemption	Risk
Treasureies	U.S. Government	State and Local	Safest
Agency	U.S.Government-sponsored Agency	State and Local	Safer
Municipal	State and Local Government	Federal, State and Local	Safe
Corporate	Corporate	None	Risky

3.1.2 Features of Bonds

The most important features of a bond are:

1. nominal, par or face value: the amount on which the issuer pays interest, and which, most commonly, has to be repaid at the end of the term. Usually, the face value is 1,000 per bond.

2. issue price: the price at which investors buy the bonds when they are first issued, which will typically be approximately equal to the nominal amount.

3. maturity date: the date on which the issuer has to repay the nominal amount. As long as all payments have been made, the issuer has no more obligation to the bond holders after the maturity date.

4. coupon rate: the interest rate that the issuer pays to the bond holders. Usually this rate is fixed throughout the life of the bond. It can also vary with a money market index.

5. callability: the right given to the issuer to repay the bond before the maturity date on the call dates. These bonds are referred to as callable bonds. Most callable bonds allow the issuer to repay the bond at par. With some bonds, the issuer has to pay a premium, the so called call premium. This is mainly the case for high-yield bonds. These have very strict covenants, restricting the issuer in its operations. To be free from these covenants, the issuer can repay the bonds early, but only at a high cost.

6. putability: the right given to the holder to force the issuer to repay the bond before the maturity date on the put dates.

3.1.3 Risks and Pricing

A bond is not risk-free. Its inherent risks are liquidity risk, default risk and interest rate risk.

Liquidity risk describes the danger that when investors need to sell a bond, investors will not be able to. The simple truth is that when a bond is sold on the secondary market, there is not always a buyer. The market for bonds is considerably more illiquid than for stocks. The most activly traded bond has only about 10 trades a day on average. For an illiquid bond, it only has 1 trade a month.

Default risk is that bond issuer fails to pay the payments to the bond holder. A bond is nothing more than a promise to repay the debt holder. And promises are made to be broken.

Treasury bond are guaranteed by the full faith and credit of the federal government. But Municipal Bond and corporate bond are possible to default.

Interest rate risk is that the bond price may go down due to the change of interest rate. If investors want to sell it, they may loose money. Bond prices have an inverse relationship to interest rates. When one rises, the other falls. However, if investors hold a security until maturity, interest rate risk is not a factor.

The higher the risk, the higher the return, the principle of pricing. In general, the risk for Treasury bond is lowest, then the municipal bond, and the highest risk bond is a corporate bond. In figure (3.1), the x-axis stands for the maturity of bond and the y-axis stands for the yields. The upper curve is the AAA corporate bond, treasury bond is the lower curve, and the circle dots are for the municiple bond. The data is collected on Dec.2010 [1].

Bond pricing is another branch of the literature different from the focus of this dissertation. Bond pricing computes the premium of risk, or the true value. This dissertation focuses on the trading strategy of dealer, however, dealer's trading strategy is based on the true price. The methods of bond pricing are introduced in Appendix B.

3.2   The Data

In this section, we explore the statistic of data, which will be useful for chosing the right parameters in the simulation of later chapters. The time series data used consists of trade-

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[1]http://finance.yahoo.com/bonds

Figure 3.1: Yield curve of Treasury, Municipal and Corporate Bonds

Yield Curve
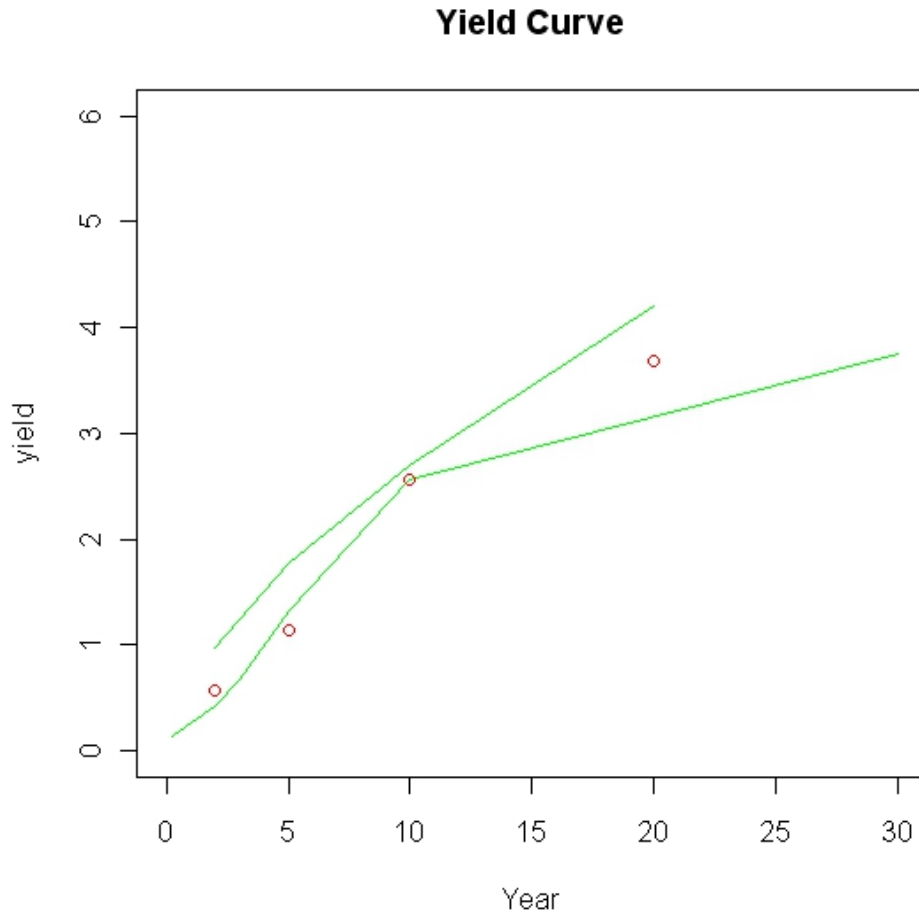
by-trade information from Trade Reporting and Compliance Engine (TRACE). TRACE is the FINRA [0] developed vehicle that facilitates the mandatory reporting of over-the-counter secondary market transactions in eligible fixed income securities. All broker/dealers who are FINRA member firms have an obligation to report transactions in corporate bonds to TRACE under an SEC approved set of rules. Current TRACE reporting time is 15 minutes.

[0]Financial Industry Regulatory Authority

TRACE started operation in July 2002.

We choose one bond as an example. In Figure (3.2), it is the trading data of bond issued by Bank of America [1] from Jan. 1 of 2009 to Jan.1 of 2010. Letter "S" stands for the price of selling to customer, "B" for buying from customers and "D" for inter-dealer trading. We can see that "B<D<S", that is, the inter-dealer price is always between the buying and selling prices.

Figure 3.2: BAC bond trading data: time period 1/1/2009 - 1/1/2010

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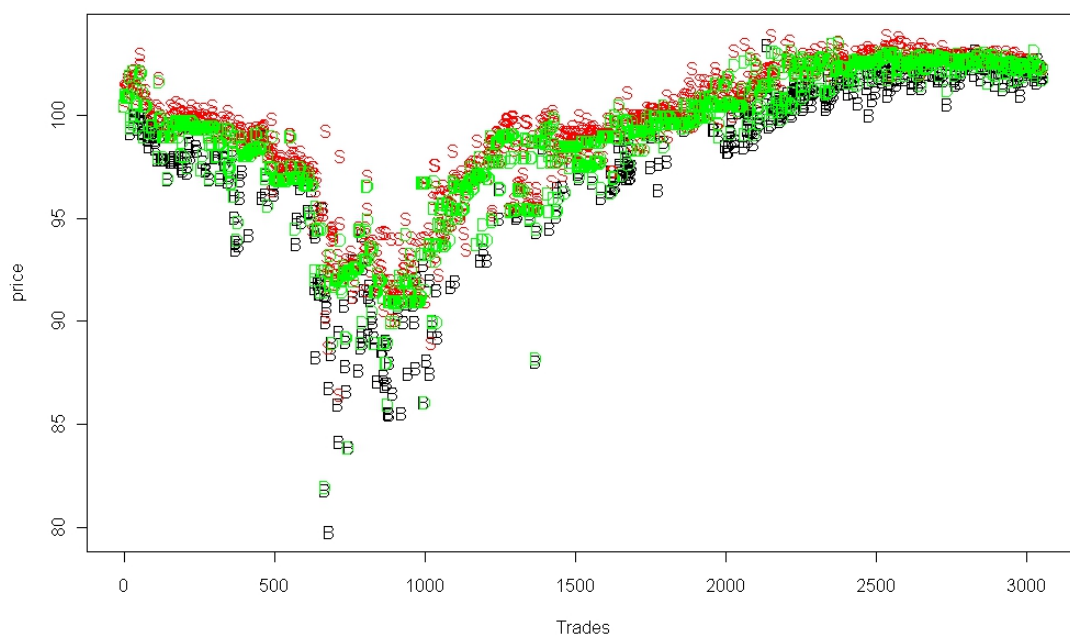
[1]CUSIP: 060505BP8. Security Category: Corporate. Offer date:08/23/2004. Maturity Date: 10/01/2010. Coupon Rate: 4.250%. Coupon Type: Fixed. Pay Frequency: Semi-annually. Offer Price: $99.478 . Offer Size: $750,000,000.00 . S&P rating:A.

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In Table (3.3), some statistics of the data above are given. There are about 3000 trades during 1/1/2009 to 1/1/2010. Regardless the order types, the maximal order arrival rate is 77 on 4/16/2009, the median order size is 25,000 dollars, and the median price is 99.50 dollars.

Table 3.3: BAC Bond Trading Data Statistic, time period 1/1/2009 - 1/1/2010

Date	quantity	Price
4/16/2009: 77	Min. : 1000	Min. : 79.80
5/12/2009: 68	1st Qu.: 10000	1st Qu.: 97.15
3/17/2009: 52	Median : 25000	Median : 99.50
4/15/2009: 48	Mean : 114931	Mean : 98.85
4/2/2009 : 48	3rd Qu.: 50000	3rd Qu.: 101.89
2/17/2009: 43	Max. : 5000000	Max. : 103.92

There are 518 trades that buy from the customers, summarized in Table (3.4). The maximal order arrival rates is 16 on 1/20/2009, the median trade quantity of buying from the customer is 30,000 dollars, and the median price is 99.64 dollars.

There are 1172 trades that sell to the customers and 1366 trades that take place inter-dealers, shown in Table (3.5) and Table (3.6), respectively.

Table 3.4: The Statistic of Buying, time period 1/1/2009 - 1/1/201

Date	quantity	Price
1/20/2009: 16	Min. : 1000	Min. : 79.80
2/20/2009: 9	1st Qu.: 15000	1st Qu.: 96.90
2/19/2009: 7	Median : 30000	Median : 99.64
2/25/2009: 7	Mean : 224921	Mean : 98.31
2/26/2009: 7	3rd Qu.: 100000	3rd Qu.:101.80
2/4/2009 : 7	Max. :5000000	Max. :103.43

3.3   Empirical Results

3.3.1   Inventory adjustment

To test if dealers adjust their inventories, we check the total quantity buying from customers, total quantity selling to customers and total quantity trading between dealers, where quantity is counted by the traded par value.

To avoid a tedious data display, we only elaborate one statistic in Table 3.7, bond issued by Bank of America with symbol BAC.

From the Table 3.7, buying from customers is slightly more than selling to customers in the first 500 trades, however, there is significant jump after that: Selling to customers is

Table 3.5: The statistic of Selling

Date	quantity	Price
5/12/2009: 38	Min. : 1000	Min. : 86.45
4/16/2009: 33	1st Qu.: 10000	1st Qu.: 97.59
2/13/2009: 26	Median : 25000	Median : 99.62
3/17/2009: 25	Mean : 121823	Mean : 99.14
4/15/2009: 25	3rd Qu.: 50000	3rd Qu.:101.75
5/5/2009 : 25	Max. :5000000	Max. :103.92

much higher than buying from customers between 500 and 2000 trades, and in the last 1000 trades, buying and selling are about even.

In order to scale the trading volume, we divide the volume by the offersize, obtaining the percentage in terms of the offersize, which is shown in Table (3.8). The volume selling to the customers is about twice as much as that buying from the customers between 500 and 2000 trades. In addition, we test the correlation of buying from customers, selling the customers and inter-dealer trades. Buying from customers is highly correlated with inter-dealer trades.

Thus far, we know that a dealer does control his inventory. How does the dealer control his inventory? We suspect that the dealer controls his inventory through adjusting the price. In next section, we will test our hypothesis.

Table 3.6: The Statistic of Inter-dealer trades

Date	quantity	Price
4/16/2009: 41	Min. : 1000	Min. : 82.00
5/12/2009: 29	1st Qu.: 10000	1st Qu.: 97.00
3/17/2009: 25	Median : 25000	Median : 99.39
4/2/2009 : 25	Mean : 67308	Mean : 98.80
3/12/2009: 21	3rd Qu.: 50000	3rd Qu.:102.03
3/16/2009: 21	Max. :3030000	Max. :103.53

Table 3.7: Trading Volume Dynamics, in millions of dollars. BAC: period 1/1/2009-1/1/2010

Trades type	1-500	501-1000	1001-1500	1501-2000	2001-2500	2501-3000
Buy from customers	19.182	19.237	9.563	10.854	18.007	35.638
Sell to customers	15.535	30.435	15.157	25.599	17.694	35.295
Inter-dealer trades	15.164	14.533	8.889	13.555	11.593	24.928

3.3.2 Price Determination

First, we check the correlation of prices. In Table (3.10), both the prices buying from the customers and the price selling to the customers are highly correlated with the price of inter-dealer trading. They three co-move up and down. However, it shows a weak correlation between price and trading volume in Table (3.11). This seems to be contrary to what we

Table 3.8: Trading Volume Dynamics, percentage of offersize.BAC: period 1/1/2009-1/1/2010

Trades type	1-500	501-1000	1001-1500	1501-2000	2001-2500	2501-3000
Buy from customers	2.5576	2.5649	1.2751	1.4472	2.4009	4.7517
Sell to customers	2.0713	4.058	2.0209	3.4132	2.3592	4.706
Inter-dealer trades	2.0219	1.9377	1.1852	1.8073	1.5457	3.3237

Table 3.9: Correlation of Monthly Trading Volume

Correlation	Buy from customers	Sell to customers	Inter-dealer
Buy from customers	1	0.6213955	0.9198096
Sell to customers	0.6213955	1	0.6686847
Inter-dealer trades	0.9198096	0.6686847	1

expect. To understand this puzzle, a further exploration is done by checking the correlation between the price and the accumulated trading volume, shown in Figure (3.3).

From Figure (3.3), we can tell that

1. The accumulated trading volume fluctuates around zero.

2. During the big downturn period, the variance of the accumulated trading volume is significantly larger than the other periods.

3. During the big downturn period, the buying trades take place at a very low price, which indicate the desire not to buy.

Table 3.10: Correlation of Weekly Average Price

Correlation	Buy from customers	Sell to customers	Inter-dealer
Buy from customers	1	0.9534928	0.9602006
Sell to customers	0.9534928	1	0.9944935
Inter-dealer Sell	0.9602006	0.9944935	1

Table 3.11: Correlation of Average Price and Trading Volume

Correlation	Buy Volume	Sell Volume	Inter-dealer volume	Imbalance
Price Buy	0.057	-0.283	0.265	0.486
Price Sell	0.105	-0.245	0.285	0.492
Price Inter-dealer	0.077	-0.244	0.274	0.454

The information from the accumulated trading volume tells us that the price weakly responds to the volume of each single trade, however, the price strongly responds to the accumulated trading volume associated with the dealer's inventory.

3.3.3 Summary

The following are our findings.

1. The retail prices (bid and ask) are positively strongly correlated with inter-dealer price.

2. The price weakly corresponds to each single trading volume.

Figure 3.3: Price and Accumulated Trading Volume

Price and Accumulated Trading Volume
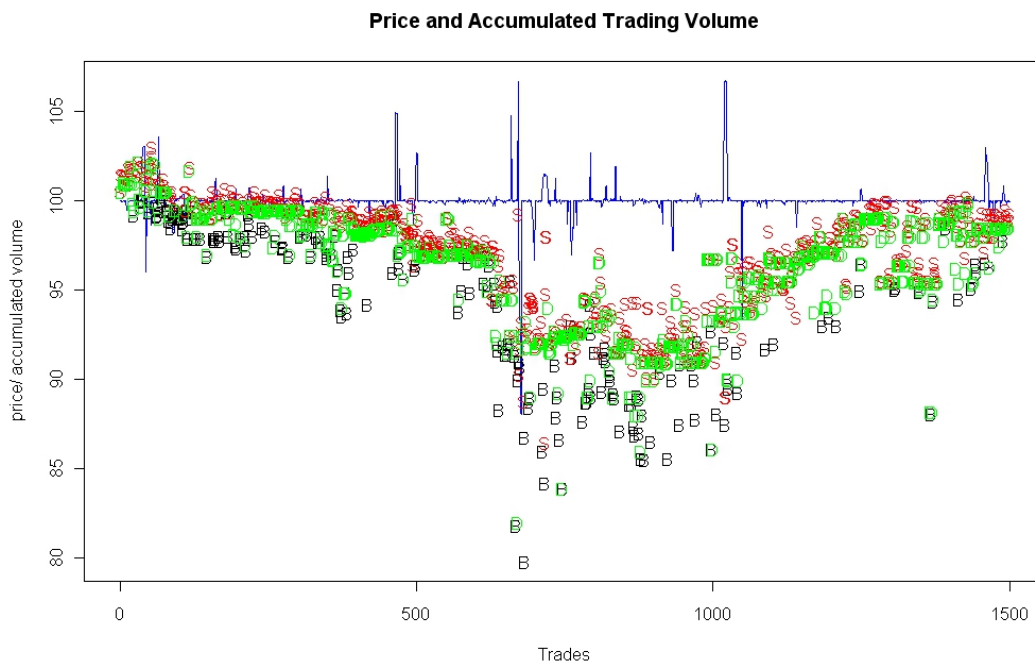
3. The bid and ask prices are not central symmetric to the inter-dealer trading price.

4. The accumulated trading volume fluctuates around zero. The price corresponds to the significant fluctuation of the accumulated trading volume. This is the evidence that the dealers control their inventory thought adjusting their quoting prices.

All of these findings will be explained by the results of our model later.

CHAPTER 4

CONTINUOUS TIME MODEL WITH INTER-DEALER

TRADING

A bond dealer's objective is to maximize his profit from providing liquidity and avoiding the inventory risk at the same time. Lyons (1997) introduced a simultaneous trade model of the spot foreign exchange market in which dealers trade with other dealers. We adopt a similar idea for bond market trading since inter-dealer trading exists in bond market, as our empirical analysis shows in Chapter 3. In our model, bid price, ask price and the amount traded with other dealers are three instruments for the dealer to achieve his objective.

4.1   The Model

In this section, we will describe the model and settings.

The time horizon. We consider a time horizon of one day or a short time period $[0, T]$, which is far away from the issuance date and the maturity date of the bond.

The market dynamics. We assume that the random source governing the true price $S(\cdot)$ of the bond is exogenous, and $S(\cdot)$ follows the following stochastic differential equation:

$$\begin{cases} dS(\tau) = \sigma S(\tau)dB(\tau), \tau \geq 0 \\ S(t) = s \end{cases}$$

(4.1)

where $B(\tau)$ is a one dimensional standard Brownian motion and $\sigma > 0$ is a constant. The dealer measures the risk of his inventory based on the true price $S(\cdot)$.

The dealer's state variables and controls. The dynamics of the dealer's cash and inventory are described by the following equations:

$$\begin{cases} C(\tau) = x + \sum_{i=1}^{N_\tau^s}(S(\tau_i) + \delta^s(\tau_i))V_i^s - \sum_{i=1}^{N_\tau^b}((S(\tau_i) - \delta^b(\tau_i)))V_i^b + \sum_{i=1}^{N_\tau^d}(S(\tau_i) + \delta^d(\tau_i))v^d(\tau_i) \\ Q(\tau) = q + \sum_{i=1}^{N_\tau^b}V_i^b - \sum_{i=1}^{N_\tau^s}V_i^s - \sum_{i=1}^{N_\tau^d}v^d(\tau_i), \quad \tau \in [t, T] \\ C(t) = x, Q(t) = q \end{cases}$$

(4.2)

In the above,

- $C(\tau)$ is the cash position at time τ.

- $Q(\tau)$ is the inventory position at time τ.

- $\delta^b(\tau)$ is the spread of the buying price to customers at time τ. In another words, $S(\tau) - \delta^b(\tau)$ is the price at which bonds are purchased from customers.

- $\delta^s(\tau)$ is the spread of the selling price from customers at time τ. In another word, $S(\tau) + \delta^s(\tau)$ is the price at which bonds are sold to customers.

- $\delta^d(\tau)$ is the spread of the inter-dealer price at time τ, which is oberserved from the market. In another word, $S(\tau) + \delta^d(\tau)$ is the price traded between dealers.

- V_i^s is the volume of bonds sold to customers at time τ_i. $V_i^s{}_{i \geq 1}$ is a sequence of i.i.d random variables.

- V_i^b is the volume of bonds purchased from customers at time τ_i. $V_i^b{}_{i \geq 1}$ is a sequence of i.i.d random variables.

- $v^d(\tau)$ is the volume of bonds bought or sold in inter-dealer market at time τ.

- N_τ^b and N_τ^s are Poisson processes with intensities λ^b and λ^s, which depend on δ^b and δ^s. We assume that N_τ^b and N_τ^s are independent.

- N_τ^d is a Poisson process with intensities λ^d, which is a constant. Process N_τ^d is independent of N_τ^s and N_τ^b.

The wealth process of the dealer is given by

$$W(\tau) = C(\tau) + S(\tau)Q(\tau), \quad \tau \in [t, T] \tag{4.3}$$

The objective. The objective of the dealer is to maximize his expected profit from transactions given the uncertainty in the security's value. Therefore, we will use the following objective function:

$$J(t, x, q, s; u(\cdot)) = E_t(W(T) - \gamma(Q(T)S(T))^2), \tag{4.4}$$

where $\gamma > 0$ is a fixed constant, risk aversion coefficient, $u(\cdot) = (\delta^b(\cdot), \delta^s(\cdot), v^d(\cdot))$ is the control processes. The value function is defined by

$$v(t, x, q, s) = \max_{u(\cdot) \in \mathcal{U}[t,T]} J(t, x, q, s; u(\cdot)), \tag{4.5}$$

with $\mathcal{U}[t, T]$ being a set of all admissible controls $u(\cdot) = (\delta^b(\cdot), \delta^s(\cdot), v^d(\cdot))$.

4.2   Frozen Strategy and Reservation Price

A frozen strategy is defined as one in which the dealer holds his inventory until the terminal time T. In this case, the dealer takes $\delta^b(\tau) = \delta_0^b$, $\delta^s(\tau) = \delta_0^s$, and $v^d(\tau) = 0$ such that

$$\lambda^b(\delta^b(\tau), \delta^s(\tau)) = \lambda^b(\delta_0^b, \delta_0^s) = 0,$$
$$\tag{4.6}$$
$$\lambda^s(\delta^b(\tau), \delta^s(\tau)) = \lambda^s(\delta_0^b, \delta_0^s) = 0,$$

and we denote $u_0(\cdot) = (\delta_0^b, \delta_0^s, 0)$. Under the control $u_0(\cdot)$, the dealer's cash and inventory position remain as constants on $[t, T]$, that is, $Q(\tau) = q$, $C(\tau) = x$, $\tau \in [t, T]$.

For convenience, we refer to $v^0(t, x, q, s)$ as the frozen value function,

$$v^0(t, x, q, s) \equiv J(t, x, q, s; u_0(\cdot)) = E_t(x + qS_T - \gamma(qS_T)^2), \quad \forall (t, x, q, s) \in [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+.$$
$$\tag{4.7}$$

The following proposition gives a representation of the frozen value function $v^0(t, x, q, s)$.

Proposition 20

$$v^0(t, x, q, s) = x + qs - \gamma q^2 s^2 e^{\sigma^2(T-t)} \tag{4.8}$$

Proof: Applying Ito's Lemma to $\ln S(\tau)$ with $S(\tau)$ being the solution of (4.1), we have

$$d\ln S(\tau) = \frac{1}{S(\tau)}dS(\tau) + \frac{1}{2}(-\frac{1}{S^2(\tau)})S^2(\tau)\sigma^2 d\tau = -\frac{1}{2}\sigma^2 d\tau + \sigma dB(\tau) \qquad (4.9)$$

Therefore,

$$\ln \frac{S(T)}{s} = -\frac{1}{2}\sigma^2(T-t) + \int_t^T \sigma dB(\tau) = -\frac{1}{2}\sigma^2(T-t) + \sigma(B(T) - B(t)) \qquad (4.10)$$

Noticing that $\frac{S(T)}{s}$ is log normal distributed with mean $-\frac{1}{2}\sigma^2(T-t)$ and variance $\sigma^2(T-t)$,

one has

$$
\begin{aligned}
E(\frac{S(T)}{s}) &= e^{-\frac{1}{2}\sigma^2(T-t)+\frac{1}{2}\sigma^2(T-t)} = 1 \\
E(\frac{S(T)}{s})^2 &= (e^{\sigma^2(T-t)} - 1)e^{-2\frac{1}{2}\sigma^2(T-t)+\sigma^2(T-t)} + 1 = e^{\sigma^2(T-t)}
\end{aligned}
\qquad (4.11)
$$

Therefore,

$$
\begin{aligned}
v^0(t,x,q,s) &= E_t(W(T) - \gamma(Q(T)S(T))^2) \\
&= E_t(x + qS(T) - \gamma(qS(T))^2) = x + qs - \gamma q^2 s^2 e^{\sigma^2(T-t)}
\end{aligned}
\qquad (4.12)
$$

proving (4.8)

Reservation price [Avellaneda 2008] is the price that would make the dealer indifferent between his current inventory and his current inventory plus or minus 1. The following is an extension of this notion.

Definition 21 *Let v be the objective function of the dealer, the reservation bid price at (t, q, s, y), denoted by $r^b(t, q, s, y)$, is given by the relation*

$$v(t, x - r^b(t, q, s, y)y, q + y, s) = v(t, x, q, s)$$

The reservation ask price at (t, q, s, y), denoted by $r^s(t, q, s, y)$, is given by

$$v(t, x + r^s(t, q, s, y)y, q - y, s) = v(t, x, q, s)$$

Note that the case $y = 1$ is the notion introduced in [Avellaneda 2008]. The following results give the explicit formulas for $r^b(t, q, s, y)$ and $r^s(t, q, s, y)$

Proposition 22 *For the value funtion $v^0(x, q, s, t) = x + qs - \gamma q^2 s^2 e^{\sigma^2(T-t)}$, the reservation bid and ask prices at (t, q, s, y) are given by*

$$
\begin{aligned}
r^b(t, q, s, y) &= s - (y + 2q)\gamma s^2 e^{\sigma^2(T-t)} \\
r^s(t, q, s, y) &= s + (y - 2q)\gamma s^2 e^{\sigma^2(T-t)}
\end{aligned}
\tag{4.13}
$$

Proof: By definition (21) and (4.8),

$$
\begin{aligned}
0 =& v^0(t, x - r^b(t, q, s, y)y, s, q + y) - v^0(t, x, q, s) \\
=& x - r^b(t, q, s, y)y + (q + y)s - \gamma(q + y)^2 s^2 e^{\sigma^2(T-t)} - (x + qs - \gamma q^2 s^2 e^{\sigma^2(T-t)}) \quad (4.14) \\
=& - r^b(t, q, s, y)y + ys - (2qy + y^2)\gamma s^2 e^{\sigma^2(T-t)}
\end{aligned}
$$

This leads to the first formula in (4.13) Similarly, we can show the second formula in (4.13).

If the dealer quotes the bid price $S(t) - \delta^b(t) < r^b(t, q, s, y)$ and ask price $S(t) + \delta^s(t) > r^s$, his value function $v > v^0$. This implies the fozen strategy $u_0(\cdot) = (\delta_0^b, \delta_0^s, 0) = (s - r^b, r^s - s, 0)$.

It is easy to see that v^0 satisfies the following PDE:

$$
\begin{cases}
w_t + \frac{1}{2}\sigma^2 s^2 w_{ss} = 0, \quad t \in [0, T], s \in \mathbb{R}^+ \\
w(q, s, T) = x + qs - \gamma q^2 s^2
\end{cases}
\tag{4.15}
$$

This inspires us to break down the problems in the model in Section 1. We will use this result to construct the value function of the model in section 6.

4.3   A Symmetric Strategy

In this section, we consider a benchmark strategy that is symmetric around the true price, regardless of the inventory, and inter-dealer trading is not considered. More precisely, a trading strategy $u(\cdot) = (\delta^b(\cdot), \delta^s(\cdot), v^d(\cdot))$ is called a symmetric strategy if $\delta^b(\tau) = \delta^s(\tau) = \delta$, $v^d(\tau) = 0$, $\tau \in [t, T]$. Further, we assume that the arrival rates satisfy the following

$$\lambda(\delta) = \lambda^s(\delta) = \lambda^b(\delta) = Ae^{-k\delta},$$

where A, k are positive constants.

Lemma 23 Let $u_\delta(\cdot) = (\delta, \delta, 0)$. If $E(V_i^b) = E(V_i^s) = \bar{v}$ and $\frac{E(V_i^b)^2}{\bar{v}} = \frac{E(V_i^s)^2}{\bar{v}} = \epsilon$, then

$$J(t, x, q, s; u_\delta(\cdot)) = x + qs - \gamma q^2 s^2 e^{\sigma^2(T-t)} + 2Ae^{-k\delta}(T-t)\bar{v}(\delta - \gamma \epsilon s^2 e^{\sigma^2(T-t)}) \qquad (4.16)$$

When $\delta = \bar{\delta} \equiv 1/k + \gamma \epsilon s^2 e^{\sigma^2(T-t)}$, $J(t, x, q, s; u_\delta(\cdot))$ reaches its maximum.

Proof: Under $u_\delta(\tau) = (\delta, \delta, 0)$, $\tau \in [t, T]$, $Q(\tau)$ and $C(\tau)$ are compound Poisson processes. By the Wald equation,

$$E_t(Q_T) = q + E_t(N_T^b)E_t(V_1^b) - E_t(N_T^s)E_t(V_1^s)$$

$$= q + \lambda^b(\delta)(T-t)\bar{v} - \lambda^s(\delta)(T-t)\bar{v} \qquad (4.17)$$

$$= q$$

since $E(\sum_{i=1}^{N_{T-t}^b} V_i^b - \sum_{i=1}^{N_{T-t}^s} V_i^s) = 0,$

$$\begin{aligned}
E_t(Q_T)^2 &= E(q + \sum_{i=1}^{N_{T-t}^b} V_i^b - \sum_{i=1}^{N_{T-t}^s} V_i^s)^2 \\
&= q^2 + E(\sum_{i=1}^{N_{T-t}^b} V_i^b)^2 + E(\sum_{i=1}^{N_{T-t}^s} V_i^s)^2 - 2E(\sum_{i=1}^{N_{T-t}^b} V_i^b)(\sum_{i=1}^{N_{T-t}^b} V_i^b) \\
&= q^2 + E(N_{T-t}^b)E(V_1^b)^2 + E((N_{T-t}^b)^2 - N_{T-t}^b)(E(V_1^b))^2 \\
&\quad + E(N_{T-t}^s)E(V_1^s)^2 + E((N_{T-t}^s)^2 - N_{T-t}^s)(E(V_1^s))^2 \\
&\quad - 2\lambda^b(\delta)(T-t)\bar{v}\lambda^s(\delta)(T-t)\bar{v} \\
&= q^2 + 2\lambda(T-t)\epsilon\bar{v}
\end{aligned}$$

(4.18)

Similarly, we have

$$\begin{aligned}
E_t(C(T)) &= E(x + \sum_{i=1}^{N_{T-t}^s} (S_{t_i} + \delta)V_i^s - \sum_{i=1}^{N_{T-t}^b} (S_{t_i} - \delta)V_i^b) \\
&= x + E(N_{T-t}^s)(s+\delta)\bar{v} - E(N_{T-t}^b)(s-\delta)\bar{v} \\
&= x + 2\lambda(T-t)\bar{v}\delta
\end{aligned}$$

(4.19)

Therefore,

$$\begin{aligned}
J(t,x,q,s;u_\delta(\cdot)) &= E_t(C(T) + S_T Q_T - \gamma(S_T Q_T)^2) \\
&= x + 2\lambda(T-t)\bar{v}\delta + qs - \gamma(q^2 + 2\lambda(T-t)\epsilon\bar{v})s^2 e^{\sigma^2(T-t)} \\
&= x + qs - \gamma q^2 s^2 e^{\sigma^2(T-t)} + 2\lambda(T-t)\bar{v}(\delta - \gamma\epsilon s^2 e^{\sigma^2(T-t)}) \\
&= v^0(t,x,q,s) + 2Ae^{-k\delta}(T-t)\bar{v}(\delta - \gamma\epsilon s^2 e^{\sigma^2(T-t)})
\end{aligned}$$

(4.20)

This gives (4.16). Now, we maximize $\delta \rightarrow 2Ae^{-k\delta}(T-t)\bar{v}(\delta - \gamma\epsilon s^2 e^{\sigma^2(T-t)}) \equiv g(\delta)$. To this end, we let

$$0 = g'(\delta) = -2kAe^{-k\delta}(T-t)\bar{v}(\delta - \gamma\epsilon s^2 e^{\sigma^2(T-t)}) + 2Ae^{-k\delta}(T-t)\bar{v} \qquad (4.21)$$

Solving (4.21) for δ, we obtain $\bar{\delta} = 1/k + \gamma\epsilon s^2 e^{\sigma^2(T-t)}$. It is easy to show that $g''(\bar{\delta}) < 0$.

$$\sup_{\delta > 0} J(t, x, q, s, u_\delta) = x + qs - \gamma q^2 s^2 e^{\sigma^2(T-t)} + \frac{2A\bar{v}(T-t)}{ke}e^{-k\gamma\epsilon s^2 e^{\sigma^2(T-t)}}$$

$$= v^0 + m(t), \qquad (4.22)$$

where $m(t) = \frac{2A\bar{v}(T-t)}{ke}e^{-k\gamma\epsilon s^2 e^{\sigma^2(T-t)}}$.

It is interesting to see that $m(t) > 0$, $t < T$, decreasing function in t.

Another meaningful result that one can obtain from Lemma (23) is that δ is an increasing function in σ. This suggests that the larger the volatility of the true price is, the larger the dealers' quoting spread.

4.4   HJB Equation

We have calculated a special case in the previous section in which the optimal quoting prices can be obtained through calculating the objective function directly. However, it is difficult to calculate the objective function when δ^s and δ^b are not constant over time, so that we can not obtain the optimal quoting price. Therefore, we need to explore some other ways. In

this section, we apply Bellman's principle to obtain the HJB equation satisfied by the value function. We first derive the HJB equation for the model introduced in section (4.1).

Proposition 24 *Let the processes $C(\tau), Q(\tau), S(\tau)$ be determined by (4.1) and (4.2) Suppose $f \in C^2(R^4, R)$, then, for any $u = (\delta^b, \delta^s, v^d) \in \mathbb{U}$*

$$
\begin{aligned}
\mathbb{A}^u f(t, x, q, s) =& f_t + \frac{1}{2} s^2 \sigma^2 f_{ss} \\
& + \lambda^s(\delta^b, \delta^s) \int_{\mathbb{R}} \{f(t, x + (s + \delta^s)y, q - y, s) - f(t, x, q, s)\} F(dy) \\
& + \lambda^b(\delta^b, \delta^s) \int_{\mathbb{R}} \{f(x - (s - \delta^b)y, q + y, s, t) - f(x, q, s, t)\} F(dy) \\
& + \lambda^d \{f(t, x + (s + \delta^d)v^d, q - v^d, s) - f(t, x, q, s)\}
\end{aligned}
\tag{4.23}
$$

where $\mathbb{A}^u$ is the backward evolution operator of $C(\tau), Q(\tau), S(\tau)$ (Definition (9)), F^b, F^s are the distribution functions of V_i^b, V_i^s respectively.

Proof: In $(t, t + \Delta t)$, by applying Taylor expansion, we have

$$f(t + \Delta t, C(t + \Delta t; t, x), Q(t + \Delta t; t, q), S(t + \Delta t; t, s)) - f(t, x, q, s)$$

$$= f(t + \Delta t, C(t + \Delta t; t, x), Q(t + \Delta t; t, q), S(t + \Delta t; t, s)) - f(t + \Delta t, x, q, S(t + \Delta t; t, s))$$

$$+ f(t + \Delta t, x, q, S(t + \Delta t; t, s)) - f(t + \Delta t, x, q, s)$$

$$+ f(t + \Delta t, x, q, s) - f(t, x, q, s)$$

$$= f_t(t, x, q, s) \Delta t + f_s(t + \Delta t, x, q, s)(S(t + \Delta t) - s) + \frac{1}{2} f_{ss}(t + \Delta t, x, q, s)(S(t + \Delta t) - s)^2$$

$$+ f(t + \Delta t, C(t + \Delta t; t, x), Q(t + \Delta t; t, q), S(t + \Delta t; t, s)) - f(t + \Delta t, q, S(t + \Delta t; t, s))$$

$$+ o(\Delta t) + o((S(t + \Delta t) - s)^2)$$

$$\tag{4.24}$$

For the term $f_s(t + \Delta t, x, q, s)(S(t + \Delta t) - s)$, we notice that

$$E_t(S(t + \Delta t) - s) = 0,$$

hence we have

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} E_t\{f_s(t + \Delta t, x, q, s)(S(t + \Delta t) - s)\} = 0 \qquad (4.25)$$

For the third term,

$$E_t(S(t + \Delta t) - s)^2 = s^2(e^{\sigma^2 \Delta t} - 1),$$

therefore,

$$
\begin{aligned}
&\lim_{\Delta t \to 0} \frac{1}{\Delta t} E_t\{\frac{1}{2} f_{ss}(t + \Delta t, x, q, s)(S(t + \Delta t) - s)^2\} \\
&= \lim_{\Delta t \to 0} 1/2 f_{ss}(t + \Delta t, x, q, s) s^2 \frac{e^{\sigma^2 \Delta t} - 1}{\Delta t} \\
&= \frac{1}{2} \sigma^2 s^2 f_{ss}(x, q, s, t)
\end{aligned}
\qquad (4.26)
$$

For the last term,

$$f(t + \Delta t, C(t + \Delta t; t, x), Q(t + \Delta t; t, q), S(t + \Delta t; t, s)) - f(t + \Delta t, q, S(t + \Delta t; t, s)),$$

we can not apply Taylor expansion since $C(t)$ and $Q(t)$ are not continuous processes.

Note that Poisson processes N_t^s, N_t^b, N_t^d are independent. In a short enough time period $(t, t + \Delta t)$, there is only one order, selling, buying, or inter-dealer, coming in with probability $(\lambda^s + \lambda^b + \lambda^d)\Delta t$, and there is two or more orders coming in with probability $O((\Delta t)^2)$.

Denote $N_t = N_t^s + N_t^b + N_t^d$.

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} E_t(f(t + \Delta t, C(t + \Delta t; t, x), Q(t + \Delta t; t, q), S(t + \Delta t)) - f(t + \Delta t, x, q, S(t + \Delta t)))$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \sum_{n=0}^{\infty} E_t(f(t + \Delta t, C(t + \Delta t; t, x), Q(t + \Delta t; t, q), S(t + \Delta t))$$

$$- f(t + \Delta t, x, q, S(t + \Delta t))|N_{t+\Delta t} = n)P(N_{t+\Delta t} = n)$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \{\lambda^s(\delta^b, \delta^s)\Delta t \int_{\mathbb{R}} \{f(t, x + (s + \delta^s)y, q - y, s) - f(t, x, q, s)\}F^s(dy)$$

$$+ \lambda^b(\delta^b, \delta^s)\Delta t \int_{\mathbb{R}} \{f(t, x - (s - \delta^b)y, q + y, s) - f(t, x, q, s)\}F^b(dy)$$

$$+ \lambda^d \Delta t \{f(t, x + (s + \delta^d)v^d, q - v^d, s) - f(t, x, q, s)\}$$

$$+ \sum_{n=2}^{\infty} E_t(f(t + \Delta t, C(t + \Delta t; t, x), Q(t + \Delta t; t, q), S(t + \Delta t))$$

$$- f(t + \Delta t, x, q, S(t + \Delta t))|N_{t+\Delta t} = n)\frac{e^{-(\lambda^b + \lambda^s + \lambda^d)\Delta t}((\lambda^b + \lambda^s + \lambda^d)\Delta t)^n}{n!}\}$$

$$= \lambda^s(\delta^b, \delta^s) \int_{\mathbb{R}} \{f(t, x + (s + \delta^s)y, q - y, s) - f(t, x, q, s)\}F^s(dy)$$

$$+ \lambda^b(\delta^b, \delta^s) \int_{\mathbb{R}} \{f(t, x - (s - \delta^b)y, q + y, s) - f(t, x, q, s)\}F^b(dy)$$

$$+ \lambda^d \{f(t, x + (s + \delta^d)v^d, q - v^d, s) - f(t, x, q, s)\}$$

$$(4.27)$$

Combining (4.24),(4.25),(4.26)and (4.27), we obtain our conclusion.

Corollary 25 *If $f \in C^2(R^4, R)$ and $\lambda^s + \lambda^b + \lambda^d \leq M$, then,*

$$\lim_{\Delta t \to 0} \sup_{u \in \mathbb{U}} \frac{1}{\Delta t} E_t\{f(t + \Delta t, C(t + \Delta t; t, x), Q(t + \Delta t; t, q), S(t + \Delta t; t, s)) - f(x, q, s, t)\}$$

$$= \sup_{u \in \mathbb{U}} A^u f(x, q, s, t), \quad uniformly \ in \ u \equiv (\delta^b, \delta^s, v^d) \in \mathbb{U}$$

$$(4.28)$$

Proof: By (4.24) and (4.27), in $(t, t + \Delta t)$, we have

$$\frac{1}{\Delta t} E_t\{f(C(t + \Delta t; t, x), Q(t + \Delta t; t, q), S(t + \Delta t; t, s), t + \Delta t) - f(x, q, s, t)\}$$

$$= f_t(x, q, s, t) + \frac{1}{2} f_{ss}(x, q, s, t + \Delta t) s^2 \frac{e^{\sigma^2 \Delta t - 1}}{\Delta t}$$

$$+ \frac{1}{\Delta t} E_t\{f(C(t + \Delta t; t, x), Q(t + \Delta t; t, q), S(t + \Delta t; t, s), t + \Delta t) - f(x, q, S(t + \Delta t; t, s), t + \Delta t)$$

$$+ o(1)\}$$

$$= f_t(x, q, s, t) + \frac{1}{2} f_{ss}(x, q, s, t + \Delta t) s^2 \frac{e^{\sigma^2 \Delta t - 1}}{\Delta t}$$

$$+ \lambda^s(\delta^b, \delta^s) \int_{\mathbb{R}} E_t\{f(x + (s + \delta^s)y, q - y, S(t + \Delta t; t, s), t + \Delta t)$$

$$- f(x, q, S(t + \Delta t; t, s), t + \Delta t)\} F^s(dy)$$

$$+ \lambda^b(\delta^b, \delta^s) \int_{\mathbb{R}} E_t\{f(x - (s - \delta^b)y, q + y, S(t + \Delta t; t, s), t + \Delta t)$$

$$- f(x, q, S(t + \Delta t; t, s), t + \Delta t)\} F^b(dy)$$

$$+ \lambda^d E_t\{f(x + (s + \delta^d)v^d, q - v^d, S(t + \Delta t; t, s), t + \Delta t) - f(x, q, S(t + \Delta t; t, s), t + \Delta t)\}\}$$

$$+ o(1) + O(\Delta t)$$

$$(4.29)$$

Taking the supremum over $u \equiv (\delta^b, \delta^s, v^d) \in \mathbb{U}$ and sending $\Delta t \to 0$, we obtain our conclusion.

Theorem 26 *If the value function is in $C^2(R^4, R)$, then it satisfies the HJB eqution*

$$\begin{cases} v_t + \frac{1}{2} s^2 \sigma^2 v_{ss} + H(t, v(t, x, q, s)) = 0, t \in [0, T] \\ v(T, x, q, s) = x + qs - \gamma q^2 s^2, \quad (x, q, s) \in \mathbb{R}^2 \times \mathbb{R}_+ \\ v(t, x, q, 0) = x, \quad (t, x, q) \in [0, T] \times \mathbb{R}^2 \end{cases}$$

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where

$$
\begin{aligned}
&H(t, v(t, x, q, s)) \\
&= \sup_{u \in \mathbb{U}} \{ \lambda^s(\delta^s, \delta^b) \int_{\mathbb{R}} \{ v(t, x + (s + \delta^s)y, q - y, s) - v(t, x, q, s) \} F(dy) \\
&+ \lambda^b(p^a, \delta^b) \int_{\mathbb{R}} \{ v(t, x - (s - \delta^b)y, q + y, s) - v(t, x, q, s) \} F(dy) \\
&+ \lambda^d \{ v(t, x + (s + \delta^d)v^d, q - v^d, s) - v(t, x, q, s) \} \}
\end{aligned}
\tag{4.30}
$$

Proof: By the property of conditional expectation, we have

$$
E_t(W_T - \gamma(Q_T S_T)^2) = E_t(E_{t+\Delta t}(W_T - \gamma(Q_T S_T)^2)) \tag{4.31}
$$

By the Bellman's priciple,

$$
\begin{aligned}
v(t, x, q, s) &= \sup_{u(\cdot) \in \mathcal{U}[t,T]} (E_t(W_T - \gamma(Q_T S_T)^2)) \\
&= \sup_{u(\cdot) \in \mathcal{U}[t,T]} E_t(E_{t+\Delta t}(W_T - \gamma(Q_T S_T)^2)) \\
&= \sup_{u(\cdot) \in \mathcal{U}[t,t+\Delta t]} E_t(\sup_{u(\cdot) \in \mathcal{U}[t+\Delta t,T]} E_{t+\Delta t}(W_T - \gamma(Q_T S_T)^2)) \\
&= \sup_{u(\cdot) \in \mathcal{U}[t,t+\Delta t]} E_t(v(t + \Delta t, C(t + \Delta t; t, x), Q(t + \Delta t; t, q), S(t + \Delta t; t, s)))
\end{aligned}
\tag{4.32}
$$

Fix $(t, x, q, s) \in [t, T] \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+$, for any $u \in \mathbb{U}$, we have

$$
v(t, x, q, s) \geq E_t(v(t + \Delta t, C(t + \Delta t; t, x, u), Q(t + \Delta t; t, q, u), S(t + \Delta t; t, s))) \tag{4.33}
$$

Consequently, by substracting $v(t, x, q, s)$ both sides and letting $\Delta t \to 0$, we have

$$
0 \geq \frac{1}{\Delta t} E_t(v(t + \Delta t, C(t + \Delta t; t, x, u), Q(t + \Delta t; t, q, u), S(t + \Delta t; t, s)) - v(t, x, q, s))
$$

$$
0 \geq \mathbb{A}^u v(t, x, q, s), \quad \forall u \in \mathbb{U}
$$

$$
\tag{4.34}
$$

Consequently,

$$\sup_{u\in\mathbb{U}} \mathbb{A}^u v(x,q,s,t) \leq 0. \tag{4.35}$$

On the other hand, for any $\zeta > 0$, and Δt small enough, there exists a $u_{\zeta,\Delta t} \in \mathcal{U}[t,T]$ such that

$$v(t,x,q,s) - \zeta\Delta t \leq E_t\{v(t+\Delta t, C(t+\Delta t; t,x,u_{\zeta,\Delta t}), Q(t+\Delta t; t,q,u_{\zeta,\Delta t}), S(t+\Delta t; t,s))\}$$

$$-\zeta \leq \frac{1}{\Delta t} \sup_{u\in\mathbb{U}} E_t\{vt+\Delta t, (C(t+\Delta t; t,x,u), Q(t+\Delta t; t,q,u), S(t+\Delta t; t,s))$$

$$- v(t,x,q,s)\}$$

$$\rightarrow \sup_{u\in\mathbb{U}} \mathbb{A}^u v(x,q,s,t)$$

$$\tag{4.36}$$

In proving the last limit above, we used (4.28). Combining (4.35) and (4.36), since $\zeta > 0$ is arbitrary, we obtain our conclusion.

4.5 Existence and Uniqueness of the Solution to the HJB Equation

For any $v(\cdot) \in C([0,T] \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+)$, define operator $\mathbb{T} : C([0,T] \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+) \to C([0,T] \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+)$ by

$$
\begin{aligned}
&(\mathbb{T}v)(t,x,q,s)) \\
&= \sup_{u \in \mathbb{U}} \{ \lambda^s(\delta^b, \delta^s) \int_{\mathbb{R}} \{ v(t, x + (s + \delta^s)y, q - y, s) - v(t,x,q,s) \} F^s(dy) \\
&\quad + \lambda^b(\delta^b, \delta^s) \int_{\mathbb{R}} \{ v(t, x - (s - \delta^b)y, q + y, s) - v(t,x,q,s) \} F^b(dy) \\
&\quad + \lambda^d \{ v(t, x + (s + \delta^d)v^d, q - v^d, s) - v(t,x,q,s) \} \}
\end{aligned}
\tag{4.37}
$$

Theorem 27 *Suppose*

$$
\lambda^s(\delta^b, \delta^s) + \lambda^b(\delta^b, \delta^s) + \lambda^d \le M, \quad \forall \delta^b, \delta^s \in \mathbb{R}^+,
$$

then operator $\mathbb{T}$ is bounded.

Proof: For any $v(\cdot) \in C([0,T] \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+)$,

$$
\begin{aligned}
\| (\mathbb{T}v)(\cdot) \|_\infty &\le 2(\lambda^s + \lambda^b + \lambda^d) \, \| v(\cdot) \|_\infty \\
&\le 2M \, \| v(\cdot) \|_\infty
\end{aligned}
\tag{4.38}
$$

Therefore, operator $\mathbb{T}$ is bounded.

Rewrite the HJB equation as follows:

$$\begin{cases} v_t + \frac{1}{2}s^2\sigma^2 v_{ss} + (\mathbb{T}v)(t,x,q,s) = 0 \\[2mm] v(T,x,q,s) = x + qs - \gamma q^2 s^2 \\[2mm] v(t,x,q,0) = x \end{cases}$$

Making transformation:

$$\tau = T - t, \quad z = \ln s - \frac{1}{2}\sigma^2\tau, \tag{4.39}$$

and denoting

$$w(\tau,x,q,z) = v(T - \tau, x, q, e^{z + \frac{1}{2}\sigma^2\tau}).$$

The HJB equation becomes

$$\begin{cases} w_\tau = \frac{1}{2}\sigma^2 w_{zz} + (\mathbb{T}w)(\tau,x,q,z) \\[2mm] w(0,x,q,z) = x + qe^z - \gamma q^2 e^{2z} \end{cases} \tag{4.40}$$

We study this equation via the corresponding integral equation

$$w(\tau,x,q,z) = e^{\tau\Delta}w(\tau,x,q,z) + \int_0^\tau e^{(\tau-r)\Delta}(\mathbb{T}w)(r,x,q,z)dr, \tag{4.41}$$

where

$$e^{\tau\Delta}f(z) = \int_{\mathbb{R}} G(\tau, z - y)f(y)dy,$$

$$G(\tau, z) = \frac{1}{\sqrt{2\pi\tau}\sigma}e^{-\frac{z^2}{2\sigma^2\tau}} \tag{4.42}$$

We collect some well known facts about the semigroup $e^{\tau\Delta}$[Weissler 1981].

Proposition 28   *1. $\|G(\tau,\cdot)\|_1 = 1$ for all $\tau > 0$.*

2. If $f \geq 0$, then $e^{\tau\Delta}f \geq 0$ and $\left\|e^{\tau\Delta}f\right\|_1 = \|f\|_1$.

3. If $1 \leq p \leq \infty$, then $\left\|e^{\tau\Delta}f\right\|_p = \|f\|_p$ for all $\tau > 0$

Theorem 29 *If $\lambda^s + \lambda^b + \lambda^d \leq M$, then equation (4.40) admits a unique solution.*

Proof: For any $w(\cdot, \cdot) \in C([0,T] \times \mathbb{R}^3)$, define operator $\mathbb{F} : C([0,T] \times \mathbb{R}^3) \rightarrow C([0,T] \times \mathbb{R}^3)$

by

$$(\mathbb{F}w)(x,q,z,\tau) = e^{\tau\Delta}w(0,x,q,z) + \int_0^\tau e^{(\tau-r)\Delta}(\mathbb{T}w)(r,x,q,z)dr \qquad (4.43)$$

The equation (4.41) becomes

$$w(\tau,x,q,z) = (\mathbb{F}w)(\tau,x,q,z)$$

We claim that, for $\delta = \frac{1}{4M}$, $\forall w_1, w_2 \in C([0,\delta] \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+)$

$$\|\mathbb{F}w_1 - \mathbb{F}w_2\|_\infty \leq \frac{1}{2}\|w_1 - w_2\|_\infty \qquad (4.44)$$

In fact, by the theorem (27) and proposition (28.2),

$$\|\mathbb{F}w_1 - \mathbb{F}w_2\|_\infty$$

$$= \sup_{(\tau,x,q,z)\in[0,\delta]\times\mathbb{R}^3} |(\mathbb{F}w_1)(\tau,x,q,z) - (\mathbb{F}w_2)(\tau,x,q,z)|$$

$$= \sup_{(\tau,x,q,z)\in[0,\delta]\times\mathbb{R}^3} |\int_0^\tau e^{(\tau-r)\Delta}((\mathbb{T}w_1)(r,x,q,z) - (\mathbb{T}w_2)(r,x,q,z))dr|$$

$$\leq \int_0^\delta \sup_{(\tau,x,q,z)\in[0,\delta]\times\mathbb{R}^3} |e^{(\tau-r)\Delta}((\mathbb{T}w_1)(r,x,q,z) - (\mathbb{T}w_2)(r,x,q,z))|dr \tag{4.45}$$

$$\leq \int_0^\delta \sup_{(\tau,x,q,z)\in[0,\delta]\times\mathbb{R}^3} |(\mathbb{T}w_1)(r,x,q,z) - (\mathbb{T}w_2)(r,x,q,z)|dr$$

$$\leq 2M \int_0^\delta \|w_1 - w_2\|_\infty \, dr$$

$$\leq 2\delta M \|w_1 - w_2\|_\infty$$

$$= \frac{1}{2} \|w_1 - w_2\|_\infty$$

Hence, by the contraction mapping principle, equation (4.38) admits a unique solution in $[0,\delta]$. Repeating this process, we obtain our conclusion.

4.6 Optimal Trading Strategy

In order to obtain the optimal trading strategy, we explore the solution to the following HJB equation:

$$\begin{cases} v_t + \frac{1}{2}\sigma^2 s^2 v_{ss} + H(t, v(t, x, q, s)) = 0, & (t, x, s, q) \in [0, T] \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+, \\ v(T, x, q, s) = x + qs - \gamma q^2 s^2, & (x, s, q) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+, \\ v(t, x, q, 0) = x, & (t, x, q) \in [0, T] \times \mathbb{R}_+ \times \mathbb{R}, \end{cases} \tag{4.46}$$

where

$$\begin{aligned} H(t, v(t, x, s, q)) = \sup_{(\delta^b, \delta^s, v^d) \in \mathbb{R}^3_+} \{ & \lambda^b(\delta^b, \delta^s) \int_\mathbb{R} [v(t, x - (s - \delta^b)y, q + y, s) - v(t, x, q, s)] F(dy) \\ & + \lambda^b(\delta^b, \delta^s) \int_\mathbb{R} [v(t, x + (s + \delta^s)y, q - y, s) - v(t, x, q, s)] F(dy) \\ & + \lambda^d [v(t, x + (s + \delta^d)v^d, q - v^d, s) - v(t, x, q, s)] \}. \end{aligned} \tag{4.47}$$

Recall that the solution to

$$\begin{cases} w_t + \frac{1}{2}\sigma^2 s^2 w_{ss} = 0, t \in [0, T] \\ w(T, q, s) = x + qs - \gamma q^2 s^2 \end{cases} \tag{4.48}$$

is given by

$$w(t, x, q, s) = x + qs - \gamma q^2 s^2 e^{\sigma^2(T-t)}.$$

Let

$$\theta(t, x, q, s) = v(t, x, q, s) - [x + qs - \gamma q^2 s^2 e^{\sigma^2(T-t)}],$$

then

$$\theta(T, x, q, s) = 0$$

We will show that $\theta(t, x, q, s)$ is actually independent of variable x. First, we find the equation which $\theta(t, x, q, s)$ satisfies. It is easy to see that

$$\theta_t + \frac{1}{2}\sigma^2 s^2 \theta_{ss} = v_t + \frac{1}{2}\sigma^2 s^2 v_{ss} = -H(t, v(t, x, q, s)) \tag{4.49}$$

Note that

$$v(t, x - (s - \delta^b)y, q + y, s) - v(t, x, q, s)$$

$$=\theta(t, x - (s - \delta^b)y, q + y, s) + [x - (s - \delta^b)y + (q + y)s - \gamma(q + y)^2 s^2 e^{\sigma^2(T-t)}]$$

$$- \theta(t, x, q, s) - [x + qs - \gamma q^2 s^2 e^{\sigma^2(T-t)}] \tag{4.50}$$

$$=\theta(t, x - (s - \delta^b)y, q + y, s) - \theta(t, x, q, s) + \delta^b y - \gamma y(2q + y)s^2 e^{\sigma^2(T-t)}$$

Also,

$$v(t, x + (s + \delta^s)y, q + y, s) - v(t, x, q, s)$$

$$=\theta(t, x + (s + \delta^s)y, q - y, s) + [x + (s + \delta^s)y + (q - y)s - \gamma(q - y)^2 s^2 e^{\sigma^2(T-t)}]$$

$$- \theta(t, x, q, s) - [x + qs - \gamma q^2 s^2 e^{\sigma^2(T-t)}] \tag{4.51}$$

$$=\theta(t, x + (s + \delta^s)y, q + y, s) - \theta(t, x, q, s) + \delta^s y + \gamma y(2q - y)s^2 e^{\sigma^2(T-t)}$$

and

$$v(t, x + (s + \delta^d)y, q - v^d, s) - v(t, x, q, s)$$

$$=\theta(t, x + (s + \delta^d)y, q - v^d, s) + [x + (s + \delta^d)v^d + (q - v^d)s - \gamma(q - v^d)^2 s^2 e^{\sigma^2(T-t)}]$$

$$- \theta(t, x, q, s) - [x + qs - \gamma q^2 s^2 e^{\sigma^2(T-t)}] \tag{4.52}$$

$$=\theta(t, x + (s + \delta^d)y, q - v^d, s) - \theta(t, x, q, s) + (p^b - s)v^d - \gamma v^d(2q - v^d)s^2 e^{\sigma^2(T-t)}$$

Hence, $\theta(t, x, q, s)$ satisfies the following:

$$
\begin{cases}
\theta_t + \frac{1}{2}\sigma^2 s^2 \theta_{ss} \\[6pt]
+ \sup_{\delta^s} \lambda^s \int_{\mathbb{R}} \{\delta^s y + \gamma y(2q - y)s^2 e^{\sigma^2(T-t)} + \theta(t, x - (s - \delta^b)y, q - y, s) - \theta(t, x, q, s)\} F(dy) \\[6pt]
+ \sup_{\delta^b} \lambda^b \int_{\mathbb{R}} \{\delta^b y - \gamma y(2q + y)s^2 e^{\sigma^2(T-t)} + \theta(t, x + (s + \delta^s)y, q - y, s) - \theta(t, x, q, s)\} F(dy) \\[6pt]
+ \sup_{v^d} \lambda^d (\delta^d v^d + \gamma v^d (2q - v^d)s^2 e^{\sigma^2(T-t)} + \theta(t, x + (s + \delta^d)v^d, q - v^d, s) - \theta(t, x, q, s)) = 0, \\[6pt]
\theta(T, x, q, s) = 0 \\[6pt]
\theta(t, x, q, 0) = v(t, x, q, 0) - x = x - x = 0,
\end{cases}
$$

(4.53)

Since the terminal condition and boundary condition do not depend on variable x, it is natural for us to introduce the following:

$$
\begin{cases}
\theta_t + \frac{1}{2}\sigma^2 s^2 \theta_{ss} + \hat{H}(t, \theta(t, q, s)) = 0, \quad (t, q, s) \in [0, T] \times \mathbb{R}^2_+ \\[6pt]
\theta(T, q, s) = 0, \quad (q, s) \in \mathbb{R} \times \mathbb{R}_+ \\[6pt]
\theta(t, q, 0) = 0, \quad (t, q) \in [0, T] \times \mathbb{R}
\end{cases}
$$

(4.54)

where

$$
\hat{H}(t, \theta(t, q, s)) = \sup_{\delta^s} \lambda^s \int_{\mathbb{R}} \{\delta^s y + \gamma y(2q - y)s^2 e^{\sigma^2(T-t)} + \theta(t, q - y, s) - \theta(t, q, s)\} F^s(dy)
$$

$$
+ \sup_{\delta^b} \lambda^b \int_{\mathbb{R}} \{\delta^b y - \gamma y(2q + y)s^2 e^{\sigma^2(T-t)} + \theta(t, q + y, s) - \theta(t, q, s)\} F^b(dy)
$$

$$
+ \sup_{v^d} \lambda^d (\delta^d v^d + \gamma v^d (2q - v^d)s^2 e^{\sigma^2(T-t)} + \theta(t, q - v^d, s) - \theta(t, q, s)),
$$

(4.55)

If we can find a solution $\theta(t, q, s)$ (independent of x) to (4.54), then

$$
v(t, x, q, s) = x + qs - \gamma q^2 s^2 e^{\sigma^2(T-t)} + \theta(t, q, s)
$$

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will be a solution to (4.46).

Now we observe that

$$\int_{\mathbb{R}} \{\delta^b y - \gamma y (2q + y) s^2 e^{\sigma^2(T-t)} + \theta(t, q+y, s) - \theta(t, q, s)\} F^b(dy)$$

$$= \int_{\mathbb{R}} \{-\gamma y (2q + y) s^2 e^{\sigma^2(T-t)} + \theta(t, q+y, s) - \theta(t, q, s)\} F^b(dy) + \delta^s \int_{\mathbb{R}} y F^b(dy)$$

$$= D_+ + \delta^b v^b$$

and   (4.56)

$$\int_{\mathbb{R}} \{\delta^s y + \gamma y (2q - y) s^2 e^{\sigma^2(T-t)} + \theta(t, q-y, s) - \theta(t, q, s)\} F^s(dy)$$

$$= \int_{\mathbb{R}} \{\gamma y (2q - y) s^2 e^{\sigma^2(T-t)} + \theta(t, q-y, s) - \theta(t, q, s)\} F^s(dy) + \delta^s \int_{\mathbb{R}} y F^s(dy)$$

$$= D_- + \delta^s v^s,$$

where $D_+ = \int_{\mathbb{R}} \{-\gamma y (2q + y) s^2 e^{\sigma^2(T-t)} + \theta(t, q+y, s) - \theta(t, q, s)\} F^b(dy) + \delta^s \int_{\mathbb{R}} y F^b(dy)$,

$D_- = \int_{\mathbb{R}} \{\gamma y (2q-y) s^2 e^{\sigma^2(T-t)} + \theta(t, q-y, s) - \theta(t, q, s)\} F^s(dy)$. From the first-order optimality

condition in equation (4.54), we obtain the optimal δ^s, δ^b and v^d. They are given by the

implicit relations

$$\delta^b = -\frac{\lambda^b(\delta^b)}{\dot{\lambda}^b(\delta^b)} - \frac{D_+}{v^b}$$

$$\delta^s = -\frac{\lambda^s(\delta^s)}{\dot{\lambda}^s(\delta^s)} - \frac{D_-}{v^s}$$ (4.57)

$$v^d = q - \frac{\theta_q - \delta^d}{2\gamma s^2 e^{\sigma^2(T-t)}}$$

Intuitively, the optimal strategy are obtained through a two-step procedure. First, we solve

the equation (4.54) in order to obtain D_+, D_- and θ_q. Second, we solve the implicit equations

(4.57) and obtain the optimal strategy.

4.6.1   A Modification of HJB Equation

We see that equation (4.54) is highly nonlinear. Thus, it is difficult to find an explicit solution.

In this section, we introduce a modified HJB equation by linearizing $\hat{H}(t, \theta(t, q, s))$.

We assume that the arrival rates

$$\lambda^s(\delta) = \lambda^b(\delta) = Ae^{-k\delta}, \tag{4.58}$$

where A, k are positive constants, and $v^b = v^s = \bar{v}$. Then,

$$\delta^b = \frac{1}{k} - \frac{D_+}{\bar{v}}$$
$$\delta^s = \frac{1}{k} - \frac{D_-}{\bar{v}} \tag{4.59}$$

Substituting (4.59) back into (4.54) yields:

$$\begin{cases} \theta_t + \frac{1}{2}\sigma^2 s^2 \theta_{ss} + \hat{H}(t, \theta(t, q, s)) = 0, & (t, q, s) \in [0, T] \times \mathbb{R} \times \mathbb{R}_+ \\ \\ \theta(T, q, s) = 0, & (q, s) \in \mathbb{R} \times \mathbb{R}_+ \\ \\ \theta(t, q, 0) = 0, & (t, q) \in [0, T] \times \mathbb{R}, \end{cases} \tag{4.60}$$

where

$$\hat{H}(t, \theta(t, q, s)) = \frac{Av}{ke}(e^{\frac{kD_+}{\bar{v}}} + e^{\frac{kD_-}{\bar{v}}})$$
$$+ \sup_{v^d} \lambda^d(\delta^d v^d + \gamma v^d(2q - v^d)s^2 e^{\sigma^2(T-t)} + \theta(t, q - v^d, s) - \theta(t, q, s)) \tag{4.61}$$

Taking a first-order approximation of the order arrival term $\frac{Av}{ke}(e^{\frac{kD_+}{v}} + e^{\frac{kD_-}{v}})$

$$\frac{Av}{ke}(e^{\frac{kD_+}{v}} + e^{\frac{kD_-}{v}}) = \frac{Av}{ke}(2 + \frac{k}{v}(D_+ + D_-) + \cdots), \tag{4.62}$$

we notice that

$$D_+ + D_- = -2\gamma s^2 e^{\sigma^2(T-t)} \int_R y^2 F(dy) + \int_R [\theta(t, q+y, s) + \theta(t, q-y, s) - 2\theta(t, q, s)] F(dy),$$

$$(4.63)$$

and the highest degree of q in $\hat{H}(t, \theta(t, q, s))$ is 2. Therefore, we now try to find $\theta(t, q, s)$ of the following form:

$$\theta(t, q, s) = \theta^0(t, s) + \theta^1(t, s)q + \theta^2(t, s)q^2, \qquad (4.64)$$

then

$$D_+ + D_- = 2(\theta^2(t, s) - \gamma s^2 e^{\sigma^2(T-t)}) \int_R y^2 F(dy), \qquad (4.65)$$

and

$$\begin{cases} \delta^s & = \frac{1}{k} - \frac{D_-}{v} \\[2mm] \delta^b & = \frac{1}{k} - \frac{D_+}{v} \\[2mm] v^d & = q + \frac{\delta^d - \theta^1(s,t)}{2(\gamma s^2 e^{\sigma^2(T-t)} - \theta^2(s,t))} \end{cases} \qquad (4.66)$$

Substituting the optimal strategy given by (4.66) and (4.65) into (4.54) and , we have

$$\begin{cases} \theta_t + \frac{1}{2}s^2\sigma^2\theta_{ss} + \frac{Av}{ke}(2 + 2\frac{k}{v}((\theta^2 - \gamma s^2 e^{\sigma^2(T-t)}) \int_R y^2 F(dy))) \\[2mm] \quad + \lambda^d(\gamma s^2 e^{\sigma^2(T-t)} - \theta^2)(q + \frac{\delta^d - \theta^1}{2(\gamma s^2 e^{\sigma^2(T-t)} - \theta^2)})^2 = 0 \\[2mm] \theta(T, q, s) = 0 \end{cases} \qquad (4.67)$$

Therefore, if we subsitute (4.64) into (4.67) and group terms of order q^0, we obtain

$$
\begin{cases}
\theta_t^0 + \frac{1}{2}s^2\sigma^2\theta_{ss}^0 + \frac{Av}{ke}(2 + 2\frac{k}{\bar{v}}((\theta^2 - \gamma s^2 e^{\sigma^2(T-t)}) \int_R y^2 F(dy))) + \lambda^d(\frac{(\delta^d - \theta^1)^2}{4(\gamma s^2 e^{\sigma^2(T-t)} - \theta^2)}) = 0 \\
\\
\theta^0(T,s) = 0
\end{cases}
$$

$$(4.68)$$

Grouping terms of order q yields

$$
\begin{cases}
\theta_t^1 + \frac{1}{2}s^2\sigma^2\theta_{ss}^1 + \lambda^d(\delta^d - \theta^1) = 0 \\
\\
\theta^1(T,s) = 0,
\end{cases}
$$

$$(4.69)$$

whose solution is $\theta^1(t,s) = \delta^d(1 - e^{-\lambda^d\ (T-t)})$.

Grouping terms of order q^2 yields

$$
\begin{cases}
\theta_t^2 + \frac{1}{2}s^2\sigma^2\theta_{ss}^2 + \lambda^d(\gamma s^2 e^{\sigma^2(T-t)} - \theta^2) = 0 \\
\\
\theta^2(T,s) = 0,
\end{cases}
$$

$$(4.70)$$

whose solution is

$$
\theta^2(s,t) = \gamma s^2(e^{\sigma^2(T-t)} - e^{(\sigma^2 - \lambda^d)(T-t)}).
$$

Now, we are at the position to compute the explicit optimal strategy δ^s, δ^b and v^d.

Theorem 30 *Under the assumptions, the explicit optimal strategy is*

$$
\begin{aligned}
\delta^s &= \max \frac{1}{k} + \delta^d(1 - e^{-\lambda^d\ (T-t)}) + \gamma s^2 e^{(\sigma^2 - \lambda^d)(T-t)}(\epsilon - 2q), 0 \\
\delta^b &= \max \frac{1}{k} - \delta^d(1 - e^{-\lambda^d\ (T-t)}) + \gamma s^2 e^{(\sigma^2 - \lambda^d)(T-t)}(\epsilon + 2q), 0 \\
v^d &= q + \frac{\delta^d}{2\gamma s^2 e^{\sigma^2(T-t)}}
\end{aligned}
$$

$$(4.71)$$

where $\epsilon = \frac{\int_R y^2 F(dy)}{\bar{v}}$

Proof: We obtain this result through substituting θ^1 and θ^2 back into (4.66). First, we compute D_- and D_+.

$$
\begin{aligned}
D_- &= \int_{\mathbb{R}} \{\gamma y(2q-y)s^2 e^{\sigma^2(T-t)} + \theta(q-y,s,t) - \theta(q,s,t)\} F(dy) \\
&= \int_{\mathbb{R}} \{\gamma y(2q-y)s^2 e^{\sigma^2(T-t)} - \theta^1(s,t)y - \theta^2(s,t)y(2q-y)\} F(dy) \\
&= (\gamma s^2 e^{\sigma^2(T-t)} - \theta^2(s,t))(2qv - \int_{\mathbb{R}} y^2 P(dy)) - \theta^1(s,t)v \\
&= \gamma s^2 e^{(\sigma^2 - \lambda^d)(T-t)}(2qv - \int_{\mathbb{R}} y^2 P(dy)) - \theta^1(s,t)v
\end{aligned}
\tag{4.72}
$$

$$
\begin{aligned}
D_+ &= \int_{\mathbb{R}} \{-\gamma y(2q+y)s^2 e^{\sigma^2(T-t)} + \theta(q+y,s,t) - \theta(q,s,t)\} F(dy) \\
&= \int_{\mathbb{R}} \{-\gamma y(2q+y)s^2 e^{\sigma^2(T-t)} + \theta^1(s,t)y + \theta^2(s,t)y(2q+y)\} F(dy) \\
&= (\theta^2(s,t) - \gamma s^2 e^{\sigma^2(T-t)})(2qv + \int_{\mathbb{R}} y^2 P(dy)) + \theta^1(s,t)v \\
&= -\gamma s^2 e^{(\sigma^2 - \lambda^d)(T-t)}(2qv + \int_{\mathbb{R}} y^2 P(dy)) + \theta^1(s,t)v
\end{aligned}
\tag{4.73}
$$

then, we substitute D_- and D_+ back into (4.66), which gives

$$
\begin{aligned}
\delta^s &= \max\{\frac{1}{k} + \delta^d(1 - e^{-\lambda^d (T-t)}) + \gamma s^2 e^{(\sigma^2 - \lambda^d)(T-t)}(\epsilon - 2q), 0\} \\
\delta^b &= \max\{\frac{1}{k} - \delta^d(1 - e^{-\lambda^d (T-t)}) + \gamma s^2 e^{(\sigma^2 - \lambda^d)(T-t)}(\epsilon + 2q), 0\}
\end{aligned}
\tag{4.74}
$$

In the third equation of (4.66), since

$$
\theta_q(t, q - v^d, s) = \theta^1(t,s) + 2\theta^2(t,s)(q - v^d),
$$

we have

$$
\begin{aligned}
v^d &= q - \frac{\theta^1(t,s) + 2\theta^2(t,s)(q - v^d) - \delta^d}{2\gamma s^2 e^{\sigma^2(T-t)}} \\
&= q - \frac{\delta^d(1 - e^{-\lambda^d\ (T-t)}) + 2\gamma s^2 (e^{\sigma^2(T-t)} - e^{(\sigma^2 - \lambda^d)(T-t)})(q - v^d) - \delta^d}{2\gamma s^2 e^{\sigma^2(T-t)}}
\end{aligned}
\tag{4.75}
$$

Solving for v^d gives

$$
v^d = q + \frac{\delta^d}{2\gamma s^2 e^{\sigma^2(T-t)}}
$$

4.6.2 Value Function

Recall that the value function is

$$
v(t, x, q, s) = x + qs - \gamma q^2 s^2 e^{\sigma^2(T-t)} + \theta(t, q, s),
$$

The solution for θ^0 is

$$
\begin{aligned}
\theta^0 =& \frac{2Av}{ke}(T - t) - \frac{2Av\gamma\epsilon}{e\lambda^d} s^2 (e^{\sigma^2(T-t)} - e^{(\sigma^2 - \lambda^d)(T-t)}) \\
&+ \frac{\lambda^d}{4\gamma(4\sigma^2 + \lambda^d)} s^{-2} (e^{3\sigma^2(T-t)} - e^{-(\sigma^2 + \lambda^d)(T-t)})),
\end{aligned}
\tag{4.76}
$$

in addition,

$$
\theta^1(t, s) = \delta^d(1 - e^{-\lambda^d\ (T-t)})
$$

and

$$
\theta^2(t, s) = \gamma s^2 (e^{\sigma^2(T-t)} - e^{(\sigma^2 - \lambda^d)(T-t)})
$$

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therefore,

$$v(t, x, q, s) = v^0(t, x, q, s) + \theta(t, q, s)$$

$$= x + \theta^0(s, t) + (\delta^d(1 - e^{-\lambda^d(T-t)}) + s)q - \gamma s^2 e^{(\sigma^2 - \lambda^d)(T-t)} q^2 \tag{4.77}$$

We define the preferred inventory level to be the inventory level at which the value function reaches the maximum. By this definition the preferred inventory $\hat{q}$ is

$$\hat{q} = \frac{\delta^d(1 - e^{-\lambda^d(T-t)}) + s}{2\gamma s^2 e^{(\sigma^2 - \lambda^d)(T-t)}} \tag{4.78}$$

4.7 A comparison with existing models

We have four quoting strategies at hand: reservation bid/ask price (δ_r^s, δ_r^b) with $y = \epsilon$, symmetric strategy $(\delta_s^s = \delta_s^b = \delta)$, without dealer trading strategy (δ_o^s, δ_o^b), and with dealer trading strategy (δ_w^s, δ_w^b).

We summarize all the strategies in the following.

$$\delta_r^s = \gamma(\epsilon - 2q)s^2 e^{\sigma^2(T-t)}, \delta_r^b = \gamma(\epsilon + 2q)s^2 e^{\sigma^2(T-t)}.$$

$$\delta_s^s = \delta_s^b = \delta = \frac{1}{k} + \gamma \epsilon s^2 e^{\sigma^2(T-t)}.$$

$$\delta_o^s = \frac{1}{k} + \gamma s^2 e^{\sigma^2(T-t)}(\epsilon - 2q), \delta_o^b = \frac{1}{k} + \gamma s^2 e^{\sigma^2(T-t)}(\epsilon + 2q). \tag{4.79}$$

$$\delta_w^s = \frac{1}{k} + \delta^d(1 - e^{-\lambda^d \ (T-t)}) + \gamma s^2 e^{(\sigma^2 - \lambda^d)(T-t)}(\epsilon - 2q),$$

$$\delta_w^b = \frac{1}{k} - \delta^d(1 - e^{-\lambda^d \ (T-t)}) + \gamma s^2 e^{(\sigma^2 - \lambda^d)(T-t)}(\epsilon + 2q)$$

It is interesting to see that

Conclusion 1

$$\delta_r^s + \delta_r^b = 2\delta - \frac{2}{k}, \quad \delta_o^s + \delta_o^b = 2\delta, \quad \delta_w^s + \delta_w^b < 2\delta \tag{4.80}$$

This conclusion tells that the market with inter-dealer trading is more liquid than the market without inter-dealer trading.

Next, we compare the value functions with these four trading strategies, listed as the following.

value function of reservation prices:

$$v^r(t, x, q, s) = x + qs - \gamma q^2 s^2 e^{\sigma^2(T-t)},$$

value function of symmetric strategy:

$$v^s(t, x, q, s) = x + qs - \gamma q^2 s^2 e^{\sigma^2(T-t)} + \frac{2Av(T-t)}{k} e^{-1 - k\gamma\epsilon s^2 e^{\sigma^2(T-t)}} \tag{4.81}$$

value function without inter-dealer trading:

$$v^o(t, x, q, s) = x + qs - \gamma q^2 s^2 e^{\sigma^2(T-t)} + \frac{2Av(T-t)}{ke}(1 - 2k\gamma\epsilon s^2 e^{\sigma^2(T-t)})$$

value function with inter-dealer trading:

$$v^w(t, x, q, s) = x + \theta^0(s, t) + (\delta^d(1 - e^{-\lambda^d(T-t)}) + s)q - \gamma s^2 e^{(\sigma^2 - \lambda^d)(T-t)}q^2$$

Conclusion 2 *If $\gamma\epsilon s^2 e^{\sigma^2(T-t)} < 1/k$ and $q \geq \frac{2Av(T-t)}{ke^2\delta^d(1-e^{-\lambda^d(T-t)})}$, then*

$$v^r < v^o < v^s < v^w \tag{4.82}$$

Proof: 1. It is obvious that $v^r < v^o$ if $\gamma\epsilon s^2 e^{\sigma^2(T-t)} < 1/k$.

2. Since $e^{-x} - 1 + x > 0$ for $x > 0$, it is easy to show that $v^o < v^s$.

3. Since $\frac{\lambda^d}{4\gamma(4\sigma^2+\lambda^d)}s^{-2}\left(e^{3\sigma^2(T-t)} - e^{-(\sigma^2+\lambda^d)(T-t)}\right) > 0$ for $0 \leq t \leq T$, $\lambda^d \geq 0$,

$$
\begin{aligned}
v^w - v^s &\geq \theta^0 - \frac{2Av(T-t)}{k}e^{-1-k\gamma\epsilon s^2 e^{\sigma^2(T-t)}} + \delta^d(1 - e^{-\lambda^d(T-t)})q \\
&\geq \frac{2Av(T-t)}{ke}(1 - e^{-k\gamma\epsilon s^2 e^{\sigma^2(T-t)}} - k\gamma\epsilon s^2 e^{\sigma^2(T-t)}) + \delta^d(1 - e^{-\lambda^d(T-t)})q \\
&\geq -\frac{2Av(T-t)}{ke^2} + \delta^d(1 - e^{-\lambda^d(T-t)})q \\
&\geq 0
\end{aligned}
\tag{4.83}
$$

Remark: The conditions for the conclusion (4.82) are not necessary. The reason to use these conditions is that it has a nice form.

4.8 The Cost of Liquidity

In practice, a market with very low transaction costs is characterized as liquid and one with high transaction costs as illiquid [M.J. Flemming 2003]. Measuring these costs is not simple, however, as they depend on the size of a trade, its timing, the trading venue, and the counterparties. Furthermore, the information needed to calculate transaction costs is often not available.

As a consequence, a variety of measures are employed to evaluate a markets liquidity. The bid-ask spread is a commonly used measure of market liquidity. It directly measures the cost of executing a small trade, with the cost typically calculated as the difference between the bid or offer price and the bid-ask midpoint (or one half of the bid-ask spread). A liquidity

measure used in the Treasury market is the liquidity spread between more and less liquid securities, often calculated as the difference between the yield of an onthe- run security and that of an off-the-run security with similar cash flow characteristics.[1] Trading volume is an indirect but widely cited measure of market liquidity. Its popularity may stem from the fact that more active markets tend to be more liquid, and from theoretical studies that link increased trading activity with improved liquidity. A closely related measure of market liquidity is trading frequency. Trading frequency equals the number of trades executed within a specified interval, without regard to trade size.

[M.J. Flemming 2003] reveals that the simple bid-ask spread is a useful measure for assessing and tracking liquidity for the U.S. Treasury market. The spread can be calculated quickly and easily with data that are widely available on a real-time basis. The bid-ask spread thus increases sharply with the equity market declines in October 1997, with the financial market turmoil in the fall of 1998, and with the market disruptions around the Treasurys quarterly refunding announcement in February 2000. By contrast, trading volume and trading frequency are weak proxies for market liquidity, as both high and low levels of trading activity are associated with periods of poor liquidity.

The spread can be difficult to interpret, however, for the reason that the spread reflects both the price of liquidity as well as differences in liquidity between securities. In addition,

---

[1]An on-the-run security is the most recently auctioned security of a given (original) maturity and an off-the-run security is an older security of a given maturity. Off-the-run securities are sometimes further classified as first off-the-run (the most recently auctioned off-the-run security of a given maturity), second off-the-run (the second most recently auctioned off-the-run security of a given maturity), and so on

factors besides liquidity can cause the differences of spread. Our results offer an interpreta-tion of the spread. We notice that the optimal trading strategy

$$
\begin{aligned}
\delta_w^s &= \frac{1}{k} + \delta^d(1 - e^{-\lambda^d\ (T-t)}) + \gamma s^2 e^{(\sigma^2 - \lambda^d)(T-t)}(\epsilon - 2q), \\
\delta_w^b &= \frac{1}{k} - \delta^d(1 - e^{-\lambda^d\ (T-t)}) + \gamma s^2 e^{(\sigma^2 - \lambda^d)(T-t)}(\epsilon + 2q)
\end{aligned}
\tag{4.84}
$$

has three components: $\frac{1}{k}$, $\pm\delta^d(1 - e^{-\lambda^d\ (T-t)})$ and $\gamma s^2 e^{(\sigma^2 - \lambda^d)(T-t)}(\epsilon \pm 2q)$. The first component is related to the sensitivity for the customers to the price changes. The more sensitive to the price changes the customers, the narrower the spread. The second component is from the inter-dealer trading, if the cost of inter-dealer trading measured by δ^d is low, the spread to the customers is also low. The last component is related to the cost for holding the bond, by comparing with the reservation strategy.

4.9   Calibration of Model

The optimal quoting bid and ask price can be computed given values for the parameters. In practice, we need to determine the parameters using observed bond prices. In this section, we will discuss how to determine the parameters involved in this model.

In terms of the meaning of the parameters, based on the statistic of bond trading data in chapter 3 , we use the following values for the simulation:

- T: the terminal time. We use 1 day or one week.

- t: the current time.

- A: the maximum order arrival rates. We let it take 40 if T is one day and 200 if T is one week.

- k: order sensitivity to the price change. We let $k = 1.5$.

- λ^d: the average inter-dealer order rates. We let it take 20 if T is one day and 100 if T is one week.

- $\bar{v}$: the average order size. We let it be 250, the median of the quantity. (In table 3.2, the median quantity is \$25000, we divide it by face value \$100.)

- ϵ: $\frac{\int_R y^2 F(dy)}{\bar{v}}$. We use a discrete distribution for F.

- σ: the variance of the log middle price of bid and ask. It is 0.01.

- s: the current middle price. It is 100.

- q: the current inventory. We let it be 0.

- γ: the risk aversion coefficient. We let it be 0.0000001. It is determined by calibrating the quoting price to a reasonable range.

4.10 Simulation

The simulation is obtained through the following procedure.

1. At time t, given the state variables, the dealer quotes

$$\delta^s = \frac{1}{k} + \delta^d(1 - e^{-\lambda^d(T-t)}) + \gamma s^2 e^{(\sigma^2 - \lambda^d)(T-t)}(\epsilon - 2q)$$

$$\delta^b = \frac{1}{k} - \delta^d(1 - e^{-\lambda^d(T-t)}) + \gamma s^2 e^{(\sigma^2 - \lambda^d)(T-t)}(\epsilon + 2q)$$

(4.85)

and sell $v^d = q + \frac{\delta^d}{2\gamma s^2 e^{\sigma^2(T-t)}}$ to dealer.

2. At time $t + dt$, the state variables are updated. Generate a random number ν with a distribution F. With probability $\lambda^s(\delta^s)dt$, the inventory variables decreases by ν and the wealth increases by $(s + \delta^s)\nu$. With probability $\lambda^b(\delta^b)dt$, the inventory variables increases by ν and the wealth decreases by $(s - \delta^b)\nu$. The ture price is updated by $S_t(1 + \alpha)$ where α is a normal distributed random number with mean 0 and variance σdt.

3. Repeat step 2 until time T.

Figure (4.1) illustrates one simulation of a trading path. The first graph shows the bid and ask quotes for one path of the bond price. The second graph shows the corresponding accumulated inventory prosition. The third graph shows the profit of the dealer. The green lines in all three graphs stand for the benchmark strategy. The red lines stand for the inventory strategy.

Notice that, at time $t = 25$, the accumulated inventory is relatively high, but the bid and ask quotes are not significantly low as we expect. This is because the dealer can trade his inventory with the other dealers. The inventory quickly returns to zero by the time $t = 26$.

Figure 4.1: Inventory strategy compare with benchmark strategy.

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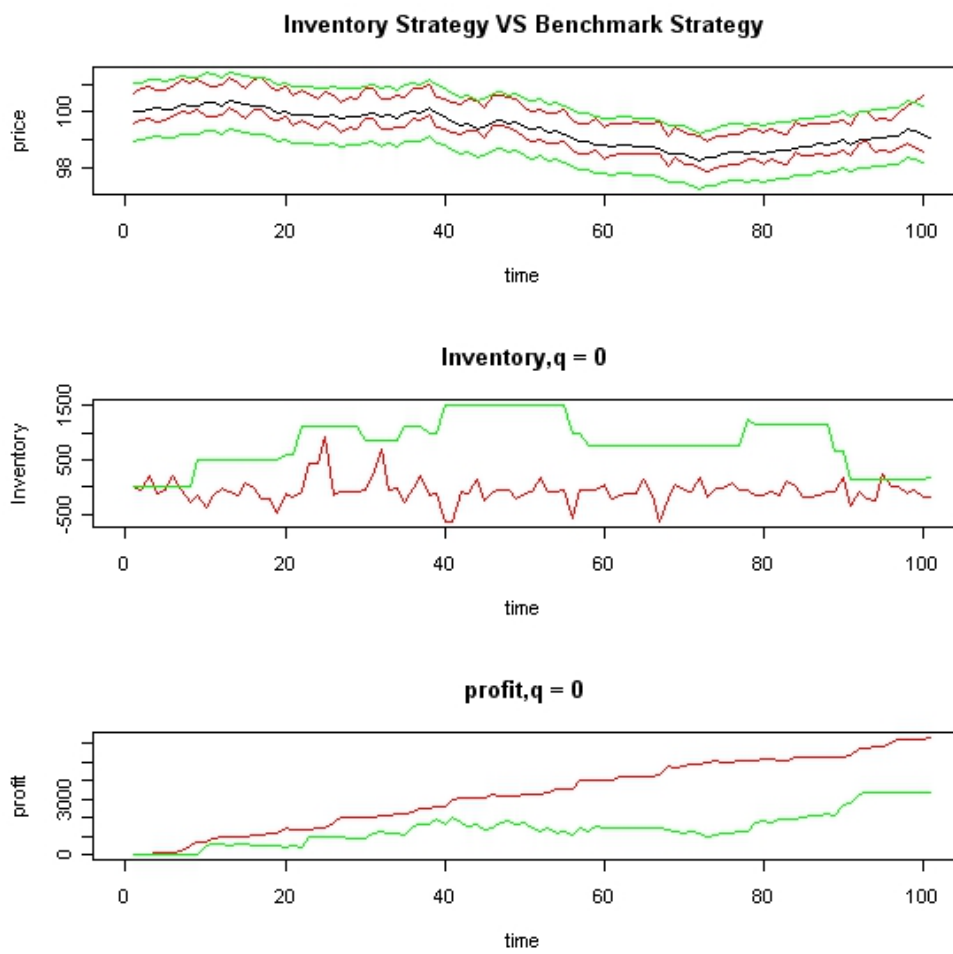
We then run 1000 simulations to compare our inventory strategy to the symmetric strategy, shown in figure (4.2). The red lines still stands for our inventory, and the green line is for the symmetric strategy. First, the spread of our inventory strategy converges to that of the symmetric strategy as the time approaches the terminal time. Indeed, when we are close to the terminal time, our inventory position is considered less risky, since the true price is less likely to move drastically. Second, the average inventory position of symmetric strategy is larger than that of our inventory strategy while the average profit of symmetric strategy is lower than that of the inventory strategy. Though our inventory strategy has a narrower spread than the symmetric strategy, it still has higher profit than the symmetric strategy since our inventory strategy involves inter-dealer trading, and therefore has a higher volume of trading than the symmetric strategy.

4.11   Conclusion

In this chapter, we introduced a continuous time trading model in the bond market with inter-dealer trading. Due to the nonliearity of the HJB equation, the explicit solution is not available even though the uniqueness and existence of the solution was approved. Therefore, we introduced an approximate method to obtain the explicit solution. This approximate solution has a better performance than the benchmark symmetric quoting strategy as following.

1. The spread of this inter-dealer trading model is smaller than that of the benchmark model.

2. The value function of this inter-dealer trading model is greater than that of the benchmark model.

The research in this chapter tells that the inter-dealer market is more liquid than the market that without inter-dealer trading. For a dealer, the risk for holding an inventory is reduced by trading with the other dealers.

Figure 4.2: The Average of 1000 Simulations
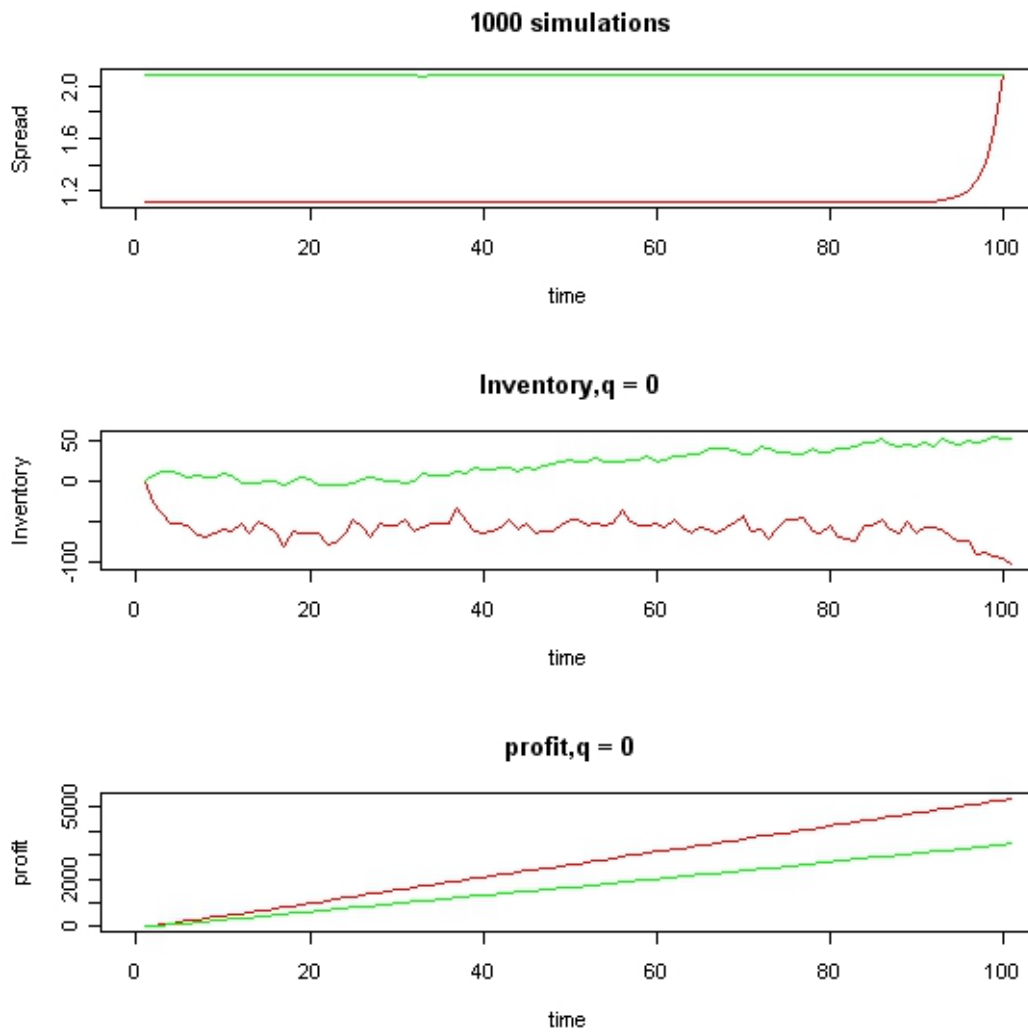
CHAPTER 5

DISCRETE TIME MODEL WITH INTER-DEALER TRADING

Since it is difficult to find the exact explicit solution in the continuous time model, we bulit a dicrete time model to explore exact explicit solution without a sacrifice for complexity of model settings. The only model setting that we have changed is that the dealer put the quotes at a fixed time instead of a continuous time. We are able to find the exact explicit solution under this discrete time model.

5.1   Model Settings

We will consider all the uncertainty on a filtered probability space $(\Omega, \mathbb{F}, \mathbb{F}_t, \mathbb{P})$.

The finite discrete time horizon. We consider a discrete time model with a trading horizon of one day. We divide a day into n periods $0 = t_0 < t_1 < \ldots < t_n = T$. At the beginning of periods i $(0 \leq i \leq n)$, the dealer sets bid and ask quotes (p_i^b, p_i^s).

The market dynamics. We assume that the true price process is exogenous, which is given by

$$dS_t = \sigma S_t dB_t, \text{ where } B_t \text{ is a Brownian Motion} \qquad (5.1)$$

since here we use discrete time model, we will use the following approximation

$$\Delta S_i = S_{i+1} - S_i$$
$$= \sigma S_i (B_{i+1} - B_i) \qquad (5.2)$$

The dealer's state variables and controls. The dynamics of the dealer's cash and inventory are described by the following eqautions:

$$C_i = C_{i-1} + V_i^s p_{i-1}^s - V_i^b p_{i-1}^b + q_i^d p_{i-1}^d,$$
$$I_i = I_{i-1} - V_i^s + V_i^b - q_i^d \qquad (5.3)$$

where C_i denotes the cash at time t_i, I_i denotes the inventory at time t_i, and V_i^s and V_i^b mean the buy orders and sell orders from customers at time t_i. The dealer decides the amount q_i^d to buy or sell to the interdealer market when he observes the interdealer price p_i^d. If p_i^d is positive (negative), it means selling to (buying from) the dealer broker. In the other words, the dealer controls p_i^s , p_i^b and q_i^d at time i.

The wealth of the dealer is

$$W_i = C_i + S_i I_i, \text{ where } S_i = S_{t_i} \qquad (5.4)$$

The change in wealth over period (t_i, t_{i+1}) will depend on (1) the arrival of transaction and (2) return on security held in inventory. Therefore, we decompose the change in wealth

into two components

$$\Delta W_i = W_{i+1} - W_i$$

$$= \Delta C_i + S_{i+1}I_{i+1} - S_i I_i$$

$$= \Delta C_i + (S_{i+1} - S_i)I_{i+1} + S_i(I_{i+1} - I_i)$$

$$= \Delta C_i + S_i \Delta I_i + \Delta S_i I_{i+1}$$

$$= V_{i+1}^s p_i^s - V_{i+1}^b p_i^b + q_i^d p_i^d + S_i \Delta I_i + \Delta S_i I_{i+1} \tag{5.5}$$

$$= V_{i+1}^s (S_i + \delta_i^s) - V_{i+1}^b (S_i - \delta_i^b) + q_i^d p_i^d + S_i \Delta I_i + \Delta S_i I_{i+1}$$

$$= V_{i+1}^s \delta_i^s + V_{i+1}^b \delta_i^b + (V_{i+1}^s - V_{i+1}^b)S_i + q_i^d p_i^d + S_i \Delta I_i + \Delta S_i I_{i+1}$$

$$= V_{i+1}^s \delta_i^s + V_{i+1}^b \delta_i^b + q_i^d(p_i^d - S_i) + \Delta X_i$$

$$= V_{i+1}^s \delta_i^s + V_{i+1}^b \delta_i^b + q_i^d \delta_i^d + \Delta X_i$$

where $V_{i+1}^s \delta_i^s + V_{i+1}^b \delta_i^b + q_i^d(p_i^d$ is the change wealth due to transaction and $\Delta X_i = \Delta S_i I_{i+1}$ is

the change in the market value of the inventory. We denote $\delta_i^d = p_i^d - S_i$.

The dealer quotes bid and ask prices around the true price,

$$p_i^s = S_i + \delta_i^s$$

$$p_i^b = S_i - \delta_i^b \tag{5.6}$$

Order dynamics. We simply assume that the order V_i^s and V_i^b are random variables

with

$$E_i(V_i^s) = \lambda^s(\delta_i^s), \quad Var_i(V_i^s) = (\sigma^s \Delta t)^2$$

$$E_i(V_i^b) = \lambda^b(\delta_i^b), \quad Var_i(V_i^b) = (\sigma^b \Delta t)^2 \tag{5.7}$$

The bid and ask quotes affect the arrival rates of orders. The further away from the ture price the dealer positions his quote, the less often he will receive buy and sell orders. In the other words, $\lambda^s(\delta^s)$ is a decreasing function of δ^s, and $\lambda^b(\delta^b)$ is a decreasing function of δ^b.

In the other way, it is equivalent to assume that

$$
\begin{aligned}
V_i^s &= \lambda^s(\delta_i^s) + \epsilon_i^s, \quad Var_i(\epsilon_i^s) = (\sigma^s \Delta t)^2 \\
V_i^b &= \lambda^b(\delta_i^b) + \epsilon_i^b, \quad Var_i(\epsilon_i^b) = (\sigma^b \Delta t)^2,
\end{aligned}
\tag{5.8}
$$

where ϵ_i^s and ϵ_i^s are the orders of uninformed customers, or noise customers. The rest of incoming order flow, $\lambda^s(\delta_i^s)$ and $\lambda^b(\delta_i^b)$, is informed. These customers react to the price change. Their demand is related to the spread.

The objective. The objective of the dealer is to maximize his profit from transaction and minimize his risk from uncertainty in the security's value. Therefore, we will use the following objective function:

$$
J(x, q, s, \mathbb{U}_0) = E(W_T) - \gamma Var(\Sigma_{j=0}^n \Delta X_j)
\tag{5.9}
$$

where $\gamma > 0$ is a fixed constant, risk aversion coefficient, $u_j = (\delta_j^s, \delta_j^b, q_j^d)$ is the control at time t_j, $\mathbb{U}_i = (u_i, u_{i+1}, \cdots, u_{T-1})$. The value function is defined by

$$
v(x, q, s) = \max_{\mathbb{U}_0 \in \mathcal{U}[0, T-1]} J(x, q, s, \mathbb{U}_0)
\tag{5.10}
$$

with $\mathcal{U}[0, T-1]$ being a set of all admissible controls $\mathbb{U}_0$.

5.2   Optimal quoting strategy for one period model

In the case of one period model, we only consider the trading period (t_i, t_{i+1}). The objective function and value function can be written as

$$J(C_i, I_i, S_i, u_i) = E_i(W_{i+1}) - \gamma Var_i((S_{i+1} - S_i)I_{i+1})$$

$$v(C_i, I_i, S_i) = \max_{u_i} J(C_i, I_i, S_i, \delta_i^s, \delta_i^b)$$

(5.11)

Lemma 31

$$E_i(W_{i+1}) = W_i + (\delta_i^s \lambda_i^s + \delta_i^b \lambda_i^b)\Delta t + q_i^d \delta_i^d,$$

$$E_i(I_{i+1}^2) = (I_i + (\lambda^b(\delta_i^b) - \lambda^s(\delta_i^s))\Delta t)^2 + ((\sigma^s)^2 + (\sigma^b)^2)(\Delta t)^2$$

(5.12)

$$Var_i((S_{i+1} - S_i)I_{i+1}) = \sigma^2 S_i^2 \Delta t((I_i + (\lambda_i^b - \lambda_i^s)\Delta t - q_i^d)^2 + ((\sigma^s)^2 + (\sigma^b)^2)(\Delta t)^2)$$

Proof:

$$E_i(W_{i+1}) = E_i(W_i + \Delta W_i)$$

$$= W_i + E_i(V_{i+1}^s \delta_i^s + V_{i+1}^b \delta_i^b + q_i^d \delta_i^d + \Delta X_i)$$

(5.13)

$$= W_i + (\delta_i^s \lambda_i^s + \delta_i^b \lambda_i^b)\Delta t + q_i^d \delta_i^d$$

$$E_i(I_{i+1}^2) = E_i(I_i + \Delta I_i)^2$$

$$= Var_i(I_i + \Delta I_i) + (E_i(I_i + \Delta I_i))^2$$

(5.14)

$$= ((\sigma^s)^2 + (\sigma^b)^2)(\Delta t)^2 + (I_i + (\lambda_i^b - \lambda_i^s)\Delta t - q_i^d)^2$$

Since $E_i(\Delta S_i I_{i+1}) = 0$, we have

$$Var_i(\Delta S_i I_{i+1}) = E_i(\Delta S_i I_{i+1})^2$$

$$= \sigma^2 S_i^2 \Delta t ((I_i + (\lambda_i^b - \lambda_i^s)\Delta t - q_i^d)^2 + ((\sigma^s)^2 + (\sigma^b)^2)(\Delta t)^2) \tag{5.15}$$

From lemma 31,

$$J(C_i, I_i, S_i, u_i) = W_i + (\delta_i^s \lambda_i^s + \delta_i^b \lambda_i^b)\Delta t + q_i^d \delta_i^d$$

$$- \gamma \sigma^2 S_i^2 \Delta t ((I_i + (\lambda_i^b - \lambda_i^s)\Delta t - q_i^d)^2 + ((\sigma^s)^2 + (\sigma^b)^2)(\Delta t)^2) \tag{5.16}$$

Lemma 32 *The first order condition of optimal $u_i = (\delta_i^s, \delta_i^b, q_i^d)$ for one-period model is*

$$\delta_i^s = -\frac{\lambda_i^s}{(\lambda_i^s)'} + \delta_i^d$$

$$\delta_i^b = -\frac{\lambda_i^b}{(\lambda_i^b)'} - \delta_i^d \tag{5.17}$$

$$q_i^d = I_i + (\lambda_i^b - \lambda_i^s)\Delta t + \frac{\delta_i^d}{2\gamma\sigma^2 S_i^2 \Delta t}$$

Proof: Based on the result of lemma 31, take the first order partial derivative of objective fuction 5.16 in terms of $\delta_i^s, \delta_i^b, q_i^d$ respectively,

$$\delta_i^s = -\frac{\lambda_i^s}{(\lambda_i^s)'} - 2\gamma\sigma^2 S_i^2 \Delta t(I_i + (\lambda_i^b - \lambda_i^s)\Delta t - q_i^d)$$

$$\delta_i^b = -\frac{\lambda_i^b}{(\lambda_i^b)'} + 2a\sigma^2 S_i^2 \Delta t^2(I_i + (\lambda_i^b - \lambda_i^s)\Delta t - q_i^d) \tag{5.18}$$

$$q_i^d = I_i + (\lambda_i^b - \lambda_i^s)\Delta t + \frac{\delta_i^d}{2a\sigma^2 S_i^2 \Delta t}.$$

Substituting q_i^d in equations of δ_i^s, δ_i^b yields

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$$\hat{\delta}_i^s = -\frac{\lambda_i^s}{(\lambda_i^s)'} + \delta_i^d$$

$$\hat{\delta}_i^b = -\frac{\lambda_i^b}{(\lambda_i^b)'} - \delta_i^d \tag{5.19}$$

$$\hat{q}_i^d = I_i + (\lambda_i^b - \lambda_i^s)\Delta t + \frac{\delta_i^d}{2a\sigma^2 S_i^2 \Delta t}$$

Since $\lambda_i^s(\lambda_i^b)$ is a function of $\delta_i^s(\delta_i^b)$. The following theorem will give the conditions that equations 5.17 exist solutions.

Theorem 33 *For decreasing function λ_i^s and λ_i^b defined on $[0, M]$, if $0 < \frac{\lambda_i^s}{(\lambda_i^s)^2}(\lambda_i^s)'' < 2$ and $0 < \frac{\lambda_i^b}{(\lambda_i^b)^2}(\lambda_i^b)'' < 2$, equations 5.17 exist an unique solution.*

Proof: We define $g(x) = -\frac{f(x)}{f'(x)} + C$. Since $g'(x) = -1 + \frac{f(x)}{(f'(x))^2}f''(x)$, if $0 < \frac{f(x)}{(f'(x))^2}f''(x) < 2$ on $[0, M]$, then $|g'(x)| < 1$. By fixed point theorem, equation $x = g(x)$ has an unique solution. Applying λ_i^s and λ_i^b as $f(x)$, q_i^d and $-q_i^d$ as C, we have the conlusion.

Theorem 34 *If $\delta_i^d \in [0, D], 0 < \frac{\lambda_i^s}{(\lambda_i^s)^2}(\lambda_i^s)'' < h < 2$, $\frac{(\lambda_i^s)'}{\lambda_i^s} > \frac{h-2}{2\delta_i^d h}$, , $(\lambda_i^s)' < 0, 0 < \frac{\lambda_i^b}{(\lambda_i^b)^2}(\lambda_i^b)'' < 2$, $(\lambda_i^b)' < 0$, then the optimal $u_i = (\delta_i^s, \delta_i^b, q_i^d)$ for one period model is*

$$\delta_i^s = -\frac{\lambda_i^s}{(\lambda_i^s)'} + \delta_i^d$$

$$\delta_i^b = -\frac{\lambda_i^b}{(\lambda_i^b)'} - \delta_i^d \tag{5.20}$$

$$q_i^d = I_i + (\lambda_i^b - \lambda_i^s)\Delta t + \frac{\delta_i^d}{2a\sigma^2 S_i^2 \Delta t}$$

Proof: We only need to prove that 5.17 is optimal. So, if $\frac{\partial^2 J}{\partial (u_i)^2}$ is negative definite, 5.17 is the optimal solution.

$$\frac{\partial^2 J}{\partial (u_i)^2} = \begin{pmatrix} \alpha^s - B((\lambda_i^s)'\Delta t)^2 + \delta_i^d(\lambda_i^s)''\Delta t & B(\lambda_i^s)'(\lambda_i^b)'(\Delta t)^2 & -B(\lambda_i^s)'\Delta t \\ B(\lambda_i^s)'(\lambda_i^b)'(\Delta t)^2 & \alpha^b - B((\lambda_i^b)'\Delta t)^2 + \delta_i^d(\lambda_i^s)''\Delta t & B(\lambda_i^b)'\Delta t \\ -B(\lambda_i^s)'\Delta t & B(\lambda_i^b)'\Delta t & -B \end{pmatrix} \cdot$$

$$\sim \begin{pmatrix} \alpha^s + \delta_i^d(\lambda_i^s)''\Delta t & 0 & 0 \\ 0 & \alpha^b + \delta_i^d(\lambda_i^s)''\Delta t & 0 \\ -B(\lambda_i^s)'\Delta t & B(\lambda_i^b)'\Delta t & -B \end{pmatrix}$$

(5.21)

where $\alpha^s = (\delta_i^s(\lambda_i^s)'' + 2(\lambda_i^s)')\Delta t$, $\alpha^b = (\delta_i^b(\lambda_i^b)'' + 2(\lambda_i^b)')\Delta t$ and $B = 2a\sigma^2 S_i^2 \Delta t$.

$$\begin{aligned} \alpha^s + \delta_i^d(\lambda_i^s)''\Delta t &= (\delta_i^s(\lambda_i^s)'' + 2(\lambda_i^s)')\Delta t + \delta_i^d(\lambda_i^s)''\Delta t \\ &= (\delta_i^s + \delta_i^d)(\lambda_i^s)''\Delta t + 2(\lambda_i^s)'\Delta t \\ &< (\delta_i^s + \delta_i^d)\Delta t \frac{h((\lambda_i^s)')^2}{\lambda_i^s} + 2(\lambda_i^s)'\Delta t \\ &= (\lambda_i^s)'\Delta t((-\frac{\lambda_i^s}{(\lambda_i^s)'} + \delta_i^d)\frac{h(\lambda_i^s)'}{\lambda_i^s} + 2) \\ &= (\lambda_i^s)'\Delta t(2 - h + 2\delta_i^d h\frac{h(\lambda_i^s)'}{\lambda_i^s}) \\ &< 0 \end{aligned}$$

(5.22)

$$\alpha^b + \delta_i^d (\lambda_i^b)'' \Delta t = (\delta_i^b (\lambda_i^b)'' + 2(\lambda_i^b)') \Delta t - \delta_i^d (\lambda_i^b)'' \Delta t$$

$$= (\delta_i^b - \delta_i^d)(\lambda_i^b)'' \Delta t + 2(\lambda_i^b)' \Delta t$$

$$< (\delta_i^b - \delta_i^d) \Delta t \frac{2((\lambda_i^b)')^2}{\lambda_i^b} + 2(\lambda_i^b)' \Delta t$$

$$= (\lambda_i^b)' \Delta t \left(\left(-\frac{\lambda_i^b}{(\lambda_i^b)'} - 2\delta_i^d \right) \frac{2(\lambda_i^b)'}{\lambda_i^b} + 2 \right)$$

$$= -4\Delta t \delta_i^d \frac{((\lambda_i^b)')^2}{\lambda_i^b}$$

$$< 0$$

(5.23)

and obviously $-B \leq 0$. Therefore, $\frac{\partial^2 J}{\partial(u_i)^2}$ is negative definite at $\hat{u}_i$. Function J attain its maximum at $\hat{u}_i$.

Example 35 *Let* $\lambda_i^s = A_1 e^{-k_1 \delta^s}$, $\lambda_i^b = A_1 e^{-k_1 \delta^b}$, A_1, A_2, k_1, k_2 *are positive. For* $\delta_i^d \in [0, D]$, *if* $k_1 < \frac{1}{2\delta_i^d}$, *the optimal solution is*

$$\delta_i^s = \frac{1}{k_1} + \delta_i^d$$

$$\delta_i^b = \frac{1}{k_2} - \delta_i^d$$

(5.24)

$$q_i^d = I_i + (\lambda_i^b - \lambda_i^s)\Delta t + \frac{\delta_i^d}{2a\sigma^2 S_i^2 \Delta t}$$

Remark 36 *1. Theorem 34 shows that the quoted bid/ask price is not directly related to dealer's inventory, but positively correlated to the price of inter-dealer. The bid/ask price to customers and the price of inter-dealer co-move up and down together, which is consistent with the finding from the empirical data analysis.*

2. The spread $\delta_i^s + \delta_i^b$ is not related to the inter-dealer price.

3. The value function is

$$V(C_i, I_i, S_i) = W_i + (\hat{\delta}_i^s \hat{\lambda}_i^s + \hat{\delta}_i^b \hat{\lambda}_i^b)\Delta t + \hat{q}_i^d \delta_i^d$$

$$- \frac{(\delta_i^d)^2}{4\gamma\sigma^2 S_i^2 \Delta t} + ((\sigma^s)^2 + (\sigma^b)^2)(\Delta t)^2) \tag{5.25}$$

$$= W_i + \hat{Y}_i + \hat{q}_i^d \delta_i^d - -\frac{(\delta_i^d)^2}{4B_i \Delta t} + A$$

5.3   Multi-period Model

In the case of multi-period model, we will consider the trading period (t_i, t_T) $0 \leq i \leq T$. The optimal control for multi-period model will refer a sequence of decisions $u_j = (\delta_j^s, \delta_j^b, q_j^d)$, $i \leq j \leq T-1$. Each decision $(\delta_j^s, \delta_j^b, q_j^d)$ only depends on the current state (C_i, I_i, S_i). We call $\mathbb{U}_i = (u_i, u_{i+1}, \cdots, u_{T-1})$ policy. We denote the optimal policy $\hat{\mathbb{U}}_i = (\hat{u}_i, \hat{u}_{i+1}, \cdots, \hat{u}_{T-1})$

We notice that

$$W_T = W_i + \Sigma_{j=i}^{T-1} \Delta W_j, \tag{5.26}$$

and

$$Var_i(\Sigma_{j=i}^n \Delta X_j) = \Sigma_{j=i}^n (Var_i \Delta X_j), \tag{5.27}$$

therefore,

$$J_i(C_i, I_i, S_i, u_i) = E_i(W_T) - \gamma Var_i(\Sigma_{j=i}^n \Delta X_j)$$

$$= W_i + \Sigma_{j=i}^n (E_i(\Delta W_j - a(\Delta X_j)^2)) \tag{5.28}$$

We denote the second term of equation $5.28 G_i(C_i, I_i, S_i, u_i) = \Sigma_{j=i}^n (E_i(\Delta W_j - a(\Delta X_j)^2))$.

Obviously, $J_i(C_i, I_i, S_i, u_i) = W_i + G(C_i, I_i, S_i, u_i)$. Since W_i is not a function of u_i, it is equivalent to maximize $G_i(C_i, I_i, S_i, \mathbb{U}_i)$. We simply denote

$$V_i(C_i, I_i, S_i) = \max_{\mathbb{U}_i} G_i(C_i, I_i, S_i, \mathbb{U}_i) \tag{5.29}$$

We summarize the multi-period model as following.

$$\max_{\mathbb{U}_i} G_i(C_i, I_i, S_i, \mathbb{U}_i) = \Sigma_{j=i}^n (E_i(\Delta W_j - a(\Delta X_j)^2))$$

where $\mathbb{U}_i$ is the control or policy

the state equations are $\tag{5.30}$

$$\Delta W_j = V_{i+1}^s \delta_j^s + V_{i+1}^b \delta_j^b + q_j^d \delta_j^d + \Delta X_j,$$

$$\Delta X_j = (S_{j+1} - Sj)(I_j - V_{j+1}^s + V_{j+1}^b - q_j^d)$$

Theorem 37 Bellman's Principle

For multi-period model 5.30, the Bellman's equation of dynamic programming is

$$V_i(C_i, I_i, S_i) = \max_{u_i} \{\delta_i^s \lambda_i^s + \delta_i^b \lambda_i^b)\Delta t + q_i^d \delta_i^d$$
$$- a\sigma^2 S_i^2 \Delta t((I_i + (\lambda_i^b - \lambda_i^s)\Delta t - q_i^d)^2 + ((\sigma^s)^2 + (\sigma^b)^2)(\Delta t)^2) \tag{5.31}$$
$$+ E_i(V_{i+1}(C_i, I_i, S_i))\}$$

We conject that the value function V_i is of form $\sum_{j=i}^n E_i(\hat{Y}_j + \hat{q}_j \delta_j^d) + C$ where $\hat{Y}_j = (\hat{\delta}_j^a \hat{\lambda}_j^a + \hat{\delta}_j^b \hat{\lambda}_j^b)\Delta t$ and C is not related to $\hat{\delta}_j^s, \hat{\delta}_j^b, \hat{q}_j^s$ $1 \le j < n$.

We assume that λ_i^s, $1 \leq i < n$ satisfies the conditions

$$
\begin{array}{ll}
1. & (\lambda_i^s)' < 0 \\[2mm]
2. & 0 < \dfrac{\lambda_i^s}{(\lambda_i^s)^2}(\lambda_i^s)'' < h < 2 \\[2mm]
3. & \dfrac{(\lambda_i^s)'}{\lambda_i^s} > \dfrac{h-2}{2\delta_i^d h}
\end{array}
\tag{5.32}
$$

λ_i^b, $1 \leq i < n$ satisfies the conditions

$$
\begin{array}{ll}
1. & (\lambda_i^b)' < 0 \\[2mm]
2. & 0 < \dfrac{\lambda_i^b}{(\lambda_i^b)^2}(\lambda_i^b)'' < 2
\end{array}
\tag{5.33}
$$

Theorem 38 The optimal policy for multi-period model

For any $1 < k < n-1$, if δ_i^d is a martingale on $(\Omega, \mathbb{F}, \mathbb{F}_t, \mathbb{P})$ and λ_i^s satisfies conditions 5.32, (λ_i^b) satisfies 5.33, then the value function induction rule is given by

$$
\begin{aligned}
\frac{\partial E_k V_{k+1}}{\partial \delta_k^s} &= -\,\delta_k^d (\lambda_k^s)' \Delta t \\[2mm]
\frac{\partial E_k V_{k+1}}{\partial \delta_k^b} &= \delta_k^d (\lambda_k^b)' \Delta t \\[2mm]
\frac{\partial E_k V_{k+1}}{\partial q_k^s} &= -\,\delta_k^d
\end{aligned}
\tag{5.34}
$$

The optimal policy for multi-period model is given by

$$
\begin{aligned}
\delta_i^s &= -\frac{\lambda_i^s}{(\lambda_i^s)'} + \delta_i^d \\[2mm]
\delta_i^b &= -\frac{\lambda_i^b}{(\lambda_i^b)'} - \delta_i^d \\[2mm]
q_i^d &= I_i + (\lambda_i^b - \lambda_i^s)\Delta t
\end{aligned}
\tag{5.35}
$$

Proof: Assume when $i = k$ the induction 5.34 holds, by Bellman's equation 5.31, the optimal $\delta_k^s, \delta_k^b, q_k^d$ will satisfy the following equations:

$$\hat{\delta}_k^s = -\frac{\lambda_k^s}{(\lambda_k^s)'} - 2\gamma\sigma^2 S_k^2 \Delta t(I_k + (\lambda_k^b - \lambda_k^s)\Delta t - q_k^d) + (n - k - 1)E_k\delta_{k+1}$$

$$\hat{\delta}_k^b = -\frac{\lambda_k^b}{(\lambda_k^b)'} + 2a\sigma^2 S_k^2 \Delta t(I_k + (\lambda_k^b - \lambda_k^s)\Delta t - q_k^d) - (n - k - 1)E_k\delta_{k+1} \qquad (5.36)$$

$$\hat{q}_k^d = I_i + (\lambda_i^b - \lambda_i^s)\Delta t.$$

Plug q_k^d back into δ_k^s and δ_k^b, then we have

$$\hat{\delta}_k^s = -\frac{\lambda_k^s}{(\lambda_k^s)'} + E_k\delta_{k+1}$$

$$\hat{\delta}_k^b = -\frac{\lambda_k^b}{(\lambda_k^b)'} - E_k\delta_{k+1} \qquad (5.37)$$

$$\hat{q}_k^d = I_i + (\lambda_i^b - \lambda_i^s)\Delta t.$$

Under the conditions 5.32, 5.33, $\frac{\partial^2 G_k}{\partial(u_k)^2}$ is negative definite, because

$$\frac{\partial^2 G_k}{\partial(u_k)^2} = \begin{pmatrix} \alpha^s - B((\lambda_k^s)'\Delta t)^2 + \delta_i^d(\lambda_k^s)''\Delta t & B(\lambda_k^s)'(\lambda_k^b)'(\Delta t)^2 & -B(\lambda_k^s)'\Delta t \\ B(\lambda_k^s)'(\lambda_k^b)'(\Delta t)^2 & \alpha^b - B((\lambda_k^b)'\Delta t)^2 + \delta_k^d(\lambda_i^s)''\Delta t & B(\lambda_k^b)'\Delta t \\ -B(\lambda_k^s)'\Delta t & B(\lambda_k^b)'\Delta t & -B \end{pmatrix} \cdot$$

$$\sim \begin{pmatrix} \alpha^s + \delta_k^d(\lambda_k^s)''\Delta t & 0 & 0 \\ 0 & \alpha^b + \delta_k^d(\lambda_k^s)''\Delta t & 0 \\ -B(\lambda_k^s)'\Delta t & B(\lambda_k^b)'\Delta t & -B \end{pmatrix}$$

$$(5.38)$$

where $\alpha^s = (\delta_k^s(\lambda_k^s)'' + 2(\lambda_k^s)')\Delta t$, $\alpha^b = (\delta_k^b(\lambda_k^b)'' + 2(\lambda_k^b)')\Delta t$ and $B = 2a\sigma^2 S_k^2 \Delta t$.

and $\alpha^s + \delta_i^d(\lambda_i^s)''\Delta t < 0, \alpha^b + \delta_k^d(\lambda_k^s)''\Delta t < 0$, then 5.37 is the optimal solution.

Now we check if it still holds when $i = k - 1$, by the conjection form of V_i,

$$
\begin{aligned}
\frac{\partial E_{k-1} V_k}{\partial \delta_{k-1}^s} &= \frac{\partial}{\partial \delta_{k-1}^s} \sum_{j=k}^{n} E_{k-1}(\hat{Y}_j + \hat{q}_j)\delta_j^d + C \\
&= \frac{\partial}{\partial \delta_{k-1}^s} \sum_{j=k}^{n} E_{k-1}\hat{q}_j \delta_k^d \\
&= -\delta_{k-1}^d (\lambda_{k-1}^s)'\Delta t
\end{aligned}
\tag{5.39}
$$

The second equality holds because $\hat{Y}_j$ is a function of δ_j^d, which is independent with δ_k^s.

Since

$$
\begin{aligned}
E_{k-1}\hat{q}_{k+1} &= E_{k-1}(E_k(\hat{q}_{k+1})) \\
&= E_{k-1}(I_k + (\hat{\lambda}_j^b - \hat{\lambda}_j^s)\Delta t - \hat{q}_k) \\
&= 0
\end{aligned}
\tag{5.40}
$$

then we have that for any $j \geq k + 1$, $E_{k-1}\hat{q}_j = 0$. Therefore, the third equality holds.

Similarly, we can prove

$$
\begin{aligned}
\frac{\partial E_{k-1} V_k}{\partial \delta_{k-1}^b} &= \delta_{k-1}^d (\lambda_{k-1}^s)'\Delta t \\
\frac{\partial E_{k-1} V_k}{\partial q_{k-1}^b} &= -\delta_{k-1}^d
\end{aligned}
\tag{5.41}
$$

Under the induction rule, 5.37 is the optimal solution.

5.4   Conclusion

The result of this trading model shows that

1. The dealer's bid and ask price is positive correlated to inter-dealer's price, which explains the finding from the data.

2. Dealer trades his whole imbalance with dealer broke or the other dealer, therefore, the price movement is not directly related to inventory movement. The price is only related to inter-dealer's price and current order flow rate. That is why data shows weak relation between trading drift and price.

3. The quoted price only reveal the cost of liquidity, not reflect the other risks such as interest rate risk and default risk. As we mention in chapter 1, dealer trade for liquidity and make profit from providing liquidity, not betting on the movement of the fair price of the bond.

APPENDIX A THE SOLUTION TO θ^0

$$\begin{cases} \theta_t^0 + \frac{1}{2}s^2\sigma^2\theta_{ss}^0 + \frac{Av}{ke}(2 - 2\gamma k\epsilon s^2 e^{(\sigma^2 - \lambda^d)(T-t)} + \frac{\lambda^d}{4\gamma s^2}e^{-(\sigma^2 + \lambda^d)(T-t)} = 0 \\ \\ \theta^0(s,T) = 0 \end{cases} \tag{.42}$$

Let

$$\tau = T - t, \quad z = lns - \frac{1}{2}\sigma^2\tau \tag{.43}$$

The equation (.42) becomes

$$\begin{cases} v_t = \frac{1}{2}\sigma^2 v_{zz} + \frac{Av}{ke}(2 - 2\gamma k e e^{2z}e^{(2\sigma^2 - \lambda^d)\tau} + \frac{\lambda^d}{4\gamma}e^{-2z}e^{-(2\sigma^2 + \lambda^d)\tau}) = 0, t \in [0,T] \\ \\ v(q,z,\tau) = 0 \end{cases} \tag{.44}$$

The solution of equation (.44) is

$$\int_0^\tau e^{(\tau - r)\Delta}\frac{Av}{ke}(2 - 2\gamma k\epsilon e^{2z}e^{(2\sigma^2 - \lambda^d)r} + \frac{\lambda^d}{4\gamma}e^{-2z}e^{-(2\sigma^2 + \lambda^d)r})dr$$

$$=\frac{2Av}{ke}\tau - \frac{2Av\gamma\epsilon}{e}\int_0^\tau e^{(2\sigma^2 - \lambda^d)r}e^{(\tau - r)\Delta}e^{2z}dr + \frac{\lambda^d}{4\gamma}\int_0^\tau e^{-(2\sigma^2 + \lambda^d)r}e^{(\tau - r)\Delta}e^{-2z}dr$$

$$=\frac{2Av}{ke}\tau - \frac{2Av\gamma\epsilon}{e}\int_0^\tau e^{(2\sigma^2 - \lambda^d)r}e^{2z + 2\sigma^2(\tau - r)}dr + \frac{\lambda^d}{4\gamma}\int_0^\tau e^{-(2\sigma^2 + \lambda^d)r}e^{-2z + 2\sigma^2(\tau - r)}dr$$

$$=\frac{2Av}{ke}\tau - \frac{2Av\gamma\epsilon}{e}e^{2z + 2\sigma^2\tau}\frac{1 - e^{-\lambda^d\tau}}{\lambda^d} + \frac{\lambda^d}{4\gamma(4\sigma^2 + \lambda^d)}e^{-2z + 2\sigma^2\tau}(1 - e^{-(4\sigma^2 + \lambda^d)\tau})$$

$$=\frac{2Av}{ke}(T - t) - \frac{2Av\gamma\epsilon}{e\lambda^d}s^2(e^{\sigma^2(T-t)} - e^{(\sigma^2 - \lambda^d)(T-t)}) + \frac{\lambda^d}{4\gamma(4\sigma^2 + \lambda^d)}s^{-2}(e^{3\sigma^2(T-t)} - e^{-(\sigma^2 + \lambda^d)(T-t)})$$

$$\tag{.45}$$

APPENDIX B BOND VALUATION

Bond valuation is to determine the fair price of a bond. The theoretical fair value of a bond is the present value of the stream of cash flows it is expected to generate. Hence, the value of a bond is obtained by discounting the bond's expected cash flows to the present using the appropriate discount rate. The discount rates in different maturity time are not linear or log linear related, but in practice refered as term structure. How to determine the term structure is usually what people say about meauring interest rate risk.

The term structure encapsulates the market's views of the future behaviour of short-term interest rates. The arrival of information leads to a revision of expectations and thus moves the yield curve. Therefore, in section 1, we derive the term structure given the spot rate of short term interest rate.

The price is not independent with the market, as bond price is contingent on interest rate. Treasury bond prices usually service as a benchmark for pricing financial products. HJM method extend this idea to use forward rate as a reference for pricing. Even though it is not perfect, it offers us a novel insight into pricing. In section 2, we give a short review of HJM method.

The parameters in single factor model change over time, which mean that it can not catch the pattern of interest-rate volatilites across maturities. Then comes the multifactor models, it shows that the parameters are persistent over time. In section 3, we introduce the idea of multifactor model by a simple case, two factor model.

In regard to default risk, in section 4, we start our analysis from a simple formular with known default risk, which is available from credit rating company. My curiosity is how the bond price of AAA company related to that of BBB company with various maturity.

.1 Interest Rate Models and Bond Pricing

In this section, we will briefly review interest rate models, under which we will price bonds without default risk. (See details in notes by Robert V. Kohn)

Basic terminology

Short rate at time t, or instantaneous interest rate, is denoted by $r(t)$.

A zero-coupon bond, maturing at time T, pays 1 at time T. Its price at time t, $B(t,T)$, is given by

$$B(t,T) = E_t[exp\{-\int_t^T r(u)du\}]$$

The instantaneous forward rate $f(t,T)$ is defined by

$$f(t,T) = -\frac{\partial log B(t,T)}{\partial T}$$

This is the instantaneous interest rate, agreed upon at time t, for money borrowed at time T. Integrating the above equation, we obtain

$$\int_t^T f(t,u)du = -\int_t^T -\frac{\partial log B(t,u)}{\partial u}du$$

$$= -log B(t,u)\|_{u=t}^{u=T} \qquad (.46)$$

$$= -log B(t,T),$$

so

$$B(t,T) = exp\{-\int_t^T f(t,u)du\}$$

You can agree at time t to recieve interest rate $f(t,u)$ at each time $u \in [t,T]$. If you invest $B(t,T) at time t and receive interest rate $f(t,u)$ at each time u between t and T, this will grow to

$$B(t,T)exp\{\int_t^T f(t,u)du\} = 1$$

at time T.

Interest Rate Models

1. Vasicek(1977). The short rate solves

$$dr(t) = (\theta - ar(t))dt + \sigma dw(t) \qquad (.47)$$

with θ, a, and σ constant and $a > 0$. The advantage of such a model is that it leads to explicit formulas. Moreover, $P(t,T)$ has lognormal statistics. The disadvantage of such a model is that it has just a few paramters. So there is no hope of calibrating it to match the entire yield curve $P(0,T)$ observed in the marketplace at time 0. For this reason Vasicek and its siblings are rarely used in practice.

2. Hull-White model. It extended Vasicek model, depending on a function of one variable.

$$dr(t) = (\theta(t) - ar(t))dt + \sigma dw(t) \tag{.48}$$

where a and σ are still constant but θ is a function of t.

3. Cox-Ingersoll-Ross (CIR) Model.

$$dr(t) = (\theta - ar(t)dt + \sigma\sqrt{r(t)}dw(t) \tag{.49}$$

4. Factor Model (Duffie & Kan 1996).

The method to derive bond price contingent on the dynamic of interest rate is:

1. Assume $B(t, T)$ of affine form

$$B(r, t, T) = A(t, T)e^{-rC(t,T)}$$

2. Noticing that $e^{-\int_t^T r(u)du}B(t, T)$ is a martingale, find $de^{-\int_t^T r(u)du}B(t, T)$ and set the dt term zero. Then we have a partial differential equation satisfying the terminal condition $B(r, T, T) = 1$.

3. Solving the partial differential equation, we obtain the formula of $B(r, t, T)$

Fortunately, under those three interest rate models, they all have closed form solution for $B(t, T)$. We list all the solutions for reference.

Bond Price under Vasicek model

Under Vasicek model, the bond price at time t is given by

$$B(t, T) = A(t, T)e^{-C(t,T)r(t)} \tag{.50}$$

where $A(t,T) = exp[(\frac{\theta}{a} - \frac{\sigma^2}{2a^2}(C(t,T) - T + t) - \frac{\sigma^2}{4a}(C^2(t,T))]$, $C(t.T) = \frac{1}{a}(1 - e^{a(T-t)})$

Bond Price under Hull-White Model Under Hull-White model, the bond price at time t is given by

$$B(t,T) = A(t,T)e^{-C(t,T)r(t)} \tag{.51}$$

where $A(t,T) = exp[-\int_t^T \theta(s)C(s,T)ds - \frac{\sigma^2}{2a^2}(C(t,T) - T + t) - \frac{\sigma^2}{4a}(C^2(t,T))]$, $C(t.T) = \frac{1}{a}(1 - e^{a(T-t)})$ We can determine θ from the term structure at time 0 to calibrate the yield curve observed from marketplace at time 0.

Bond Price under CIR Model Under CIR model, the bond price at time t is given by

$$B(t,T) = A(t,T)e^{-C(t,T)r(t)} \tag{.52}$$

where

$$
\begin{aligned}
A(t,T) =& (\frac{2\gamma e^{(\alpha+\gamma)(T-t/2)}}{(\alpha+\gamma)(e^{\gamma(T-t)} - 1) + 2\gamma})^{2\theta/\sigma^2} \\
C(t,T) =& \frac{2(e^{\gamma(T-t)} - 1)}{(\alpha+\gamma)(e^{\gamma(T-t)} - 1) + 2\gamma} \\
\gamma =& \sqrt{\alpha^2 + 2\sigma^2}
\end{aligned}
\tag{.53}
$$

.2 Heath-Jarrow-Morton Method

Rather than work in terms of a short rate, it specifies the evolution of the instantaneous forward rate $f(t,T)$ by solving an SDE in t:

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW(t)$$

The Bond prices

$$B(t,T) = exp\{-\int_t^T f(t,u)du\}$$

satisfy

$$dB(t,T)/K(t) = (B(t,T)/K(t))((1/2\sigma^2(t,T) - \int_t^T \alpha(t,u)du)dt + \sigma(t,T)dW(t))$$

To implement HJM, you specify a function $\sigma(t,T), 0 \le t \le T$.

.3 Default Risk and Bond Pricing

Since Merton (1974) to Longstaff (1995), the value of a particular issue of corporate debt is obtained through contingent-claims-based approach. In this approach, the corporate debt is contingent on if the total dynamic value of the assets of the firm falls below a given threshold value K for the firm at which financial distress occurs. Longstaff (1995) developed a simple framework for pricing risky corporate bond that incorporated both default risk and interest-rate risk. Applying the model, it derived a closed form expressions for fixed-rate and floating rate debt. However, the dynamic of the total asset value of the firm is described only by two constant parameters. It is too rough for the default risk. In addition, estimating those parameters is difficult in practice. The demand for more realistic valuation of default bonds leads to the development of an alternative, reduced form approach(Jarrow and Turnbull 1995; Duffie and Singleton 1999). The reduced form approach does not look at the structure

of firm's liabilities, but the default probability.

Credit rating companies, such as Fitch, Moody's and Standard & Poors, are professional in measuring default risk. Investors, issuers, investment banks, broker-dealers, and governments rely on their rating as reference.

In this section, we derive the bond price by using the default probability provided by the rating company.

Let $B^d(t, T)$ denote the price of a default risky discount bond at time t with maturity date T, $B(t, T)$ denote the price of a default free bond at time t with maturity date T.

Assumption 1 *1: Let r denote the short-term riskless interest rate process. The dynamics of r are given by*

$$dr(t) = u(r(t), t)dt + \sigma(r(t), t)dw_t \tag{.54}$$

Assumption 2 *2: The payoff function is expressed as*

$$1 - wI_{\gamma < T}$$

where I is an indicator function that takes value one if the firm defaults during the life of the bond, and zero otherwise, w is writedown rate, and γ, a random variable, denotes the default time. We assume that defaulting is an independent event with interest rate $r(t)$. The default risk $\alpha(t, T) = E_t(I_{\gamma < T})$ is a increasing function of maturity T, given by Moody's rating.

From the assumption 1 and 2, we are at the position to derive the formula of $B^d(t,T)$.

$$B^d(t,T) = E_t[exp\{-\int_t^T r(u)du(1-wI_{t<T})\}]$$

$$= (1 - wE_t(I_{\gamma<T}))E_t[exp\{-\int_t^T r(u)du\}] \qquad (.55)$$

$$= (1 - wP\{t < \gamma < T\})B(t,T)$$

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