2013

Partially Integrable Pt-symmetric Hierarchies Of Some Canonical Nonlinear Partial Differential Equations

Keri Pecora
University of Central Florida

PARTIALLY INTEGRABLE $\mathcal{PT}$-SYMMETRIC HIERARCHIES OF SOME CANONICAL NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

by

KERI PECORA
B.S. Stetson University, 2008
M.S. University of Central Florida, 2010

A dissertation proposal submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy
in the Department of Mathematics
in the College of Science
at the University of Central Florida
Orlando, Florida

Spring Term
2013

Major Professor: Roy Choudhury

ABSTRACT

In this dissertation, we generalize the work of Bender and co-workers to derive new partially-integrable hierarchies of various $\mathcal{PT}$-symmetric, nonlinear partial differential equations. The possible integrable members are identified employing the Painlevé Test, a necessary but not sufficient integrability condition, and are indexed by the integer n, corresponding to the negative of the order of the dominant pole in the singular part of the Painlevé expansion for the solution.

For the $\mathcal{PT}$-symmetric Korteweg-de Vries (KdV) equation, as with some other hierarchies, the first or $n = 1$ equation fails the test, the $n = 2$ member corresponds to the regular KdV equation, while the remainder form an entirely new, possibly integrable hierarchy. Integrability properties of the $n = 3$ and $n = 4$ members, typical of partially-integrable systems, including Bäcklund Transformations, a 'near-Lax Pair', and analytic solutions are derived. The solutions, or solitary waves, prove to be algebraic in form, and the extended homogeneous balance technique appears to be the most efficient in exposing the near-Lax Pair.

The $\mathcal{PT}$-symmetric Burgers' equation fails the Painlevé Test for its $n = 2$ case, but special solutions are nonetheless obtained. Also, $\mathcal{PT}$-Symmetric hierarchies of 2+1 Burgers' and Kadomtsev-Petviashvili equations, which may prove useful in applications are analyzed. Extensions of the Painlevé Test and Invariant Painlevé analysis to 2+1 dimensions are utilized, and BTs and special solutions are found for those cases that pass the Painlevé Test.

TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER 1: INTRODUCTION

Although not yet fully proven, the Painlevé tests [1] seem to provide extremely useful necessary conditions for identifying the completely integrable cases of a wide variety of families of nonlinear ordinary and partial differential equations, as well as integrodifferential equations. Originally, Ablowitz et al. [2] conjectured that a nonlinear partial differential equation is integrable if *all* its exact reductions to ordinary differential equations have the Painlevé property. This approach poses the obvious operational difficulty of finding *all* exact reductions. This difficulty was circumvented by Weiss et al. [3] by postulating that a partial differential equation has the Painlevé property if its solutions are single-valued about a movable singular manifold. In this paper, we follow this latter approach to perform the Painlevé analysis of several nonlinear evolution equations.

There is now a compelling body of evidence that if an equation possesses the Painlevé property it is likely to be integrable, i.e., the Painlevé test is a necessary test for integrability. In the cases where the criteria for the Painlevé test are met, the analysis may have failed to detect an essential singularity and further analysis would be needed to rigorously prove integrability by:

(a) constructing the full set of integrals of the motion [4], or

(b) linearizing the equations, e.g., by the inverse scattering transform [5], or

(c) reducing them to one of Painlevé transcendental equations [1, 6, 7].

The usefulness of the Painlevé approach is not limited to integrability prediction, and use of the generalized Weiss algorithm [6, 8] yields auto-Bäcklund transformations and Lax pairs for the integrable cases. Painlevé analysis also yields a systematic procedure for obtaining special solutions when the equation possesses only the conditional Painlevé property [9]-[14], when the compatibility conditions of the Painlevé analysis result in constraint equations for the movable singular manifold which is no longer completely arbitrary.

Weiss' original technique [3, 8] was extensively developed by others (see [15, 16] for instance). This

approach, which will be briefly reviewed in this chapter, involves the Weiss strategy of truncating the Painlevé singularity expansion for the solution of the system of NLPDEs at the constant term, thereby imposing a specific choice of singular manifold function and the truncated (singular part) of the Painlevé expansion are then used to semi-algorithmically derive an auto-Bäcklund transformation between two different solutions of the NLPDE(s), and also to derive the associated linear scattering problem or Lax Pair. The latter step is not completely algorithmic since it involves linearizing the overdetermined system of PDEs connecting various derivatives of the singularity manifold by employing a 'Weiss substitution' which may often involve prior, extraneous knowledge about the NLPDE(s) under consideration. References [15] and [16] also discuss the connections between Painlevé analysis and other properties of, and approaches to, integrable systems such as Lie symmetries and Hirota's method. However, the original semi-algorithmic character of the Weiss SMM persists.

A second recent approach, which has opened up a whole new sub-field, involves making the entire process of singularity analysis invariant under the homographic or Möbius transformation [17, 18]. This significantly simplifies the testing for integrability [18], the derivation of Lax Pairs [19, 20], as well as the derivation of special families of analytic solutions (see [21]-[24] for instance). Some of these special families of analytic solutions have also been employed in tandem with Melnikov theory to analytically investigate the breakdown of coherent structure solutions and the onset of chaos in NLPDEs under forcing. Note that the invariant analysis yields a fully algorithmic procedure for finding Lax pairs, but none for auto-BTs, tau functions, and multisoliton solutions.

A third approach [25, 26] involves significant extensions of the original Weiss procedure to derive the 'Weiss substitution' and the Lax Pair completely algorithmically. In addition, this technique algorithmically derives many other important features of integrable systems such as Miura Transformations, Darboux Transformations, multisoliton solutions, and Hirota's tau function. Much of this work is motivated by the connections sought to be made between the various properties of integrable systems in [15, 16]. Earlier work along these lines includes [27]. We shall develop this approach systematically.

There has also been other activity in the area in recent years, including investigations of why the Painlevé test works, and on higher-order truncations and so on. We do not refer to these at all here since they do not directly impact the topic of this dissertation. In this paper, we use the above method to find the integrable members of various recently-derived hierarchies of $\mathcal{PT}$-symmetric nonlinear wave equations [28, 29].

Many papers have been written in the past 15 years extending Hermitian quantum mechanics to non-Hermitian, but physically-meaningful, $\mathcal{PT}$-symmetric Hamiltonians [30]. In [28], Carl Bender and his co-

workers attempted to extend the ideas of $\mathcal{PT}$-symmetry to the first integrable NLPDE, the Korteweg-de Vries equation. Fring [28] subsequently derived another $\mathcal{PT}$-symmetric hierarchy.

In [28], some preliminary analysis on solitary waves and conservation laws of these $\mathcal{PT}$-symmetric hierarchies appeared to indicate that these new complex KdV hierarchies were in fact, non-integrable. In [31], Fan extended these $\mathcal{PT}$-symmetric ideas to various other nonlinear evolution equations, deriving two new hierarchies of Burgers', two families of $(2+1)$ Burgers', and four families of KP equations, among others.

We use a different approach to these equations here. In particular, the integrable members of each hierarchy are identified via the Painlevé Test, and correspond to a rational-valued, discrete, but infinite, subset of the continuous parameter that characterizes each of the original $\mathcal{PT}$-symmetric equations [28, 31]. Each integrable set thus forms a new $\mathcal{PT}$-symmetric sub-hierarchy within the original hierarchy.

In Chapter 2, the possible integrable members of the KdV hierarchy are identified employing the Painlevé Test, and are indexed by the integer n, corresponding to the negative of the order of the dominant pole in the singular part of the Painlevé expansion for the solution. As with some other hierarchies, the first or $n = 1$ equation proves non-integrable, the $n = 2$ member corresponds to the regular KdV equation, while the remainder form an entirely new hierarchy. Integrability properties of the $n = 3$ and $n = 4$ members, including auto-Bäcklund Transformations, near-Lax Pairs, and soliton solutions are derived. The solitons prove to be algebraic in form, and the extended homogeneous balance technique appears to be the most efficient in exposing the near-Lax Pair.

Chapter 3 considers one family of $\mathcal{PT}$-symmetric Burgers' equations in an analogous manner. While the possible integrable cases end up requiring a compatibility condition to be satisfied, and thus fail the Painlevé Test, some special solutions are obtained.

We proceed to a 2+1 case of the $\mathcal{PT}$-symmetric Burgers' equation in Chapter 4, utilizing regular and Invariant Painlevé Analysis, extended to three variables. The $n = 2, 3$ cases pass the Painlevé test, and are further analyzed for auto-Bäcklund Transformations and soliton solutions.

Also in 2+1, one of the Kadomtsev-Petviashvili (KP) equations is analyzed in Chapter 5. While the $n = 2$ case passes the Painlevé test, the $n = 3$ requires a compatibility condition, and thus fails. Solutions are obtained for the $n = 2$ case.

Finally, in Chapter 6, the results are summarized and potential future investigation and analysis are briefly discussed. In particular, similarity and variational methods may be worth consideration, as is the case for other partially-integrable systems similar to those considered here.

CHAPTER 2: $\mathcal{PT}$-SYMMETRIC KDV HIERARCHY

2.1 Painlevé Test and Analysis

Unlike linear differential equations that exhibit fixed singularities, nonlinear equations can have movable singularities whose location depend on initial conditions. Further, the singularities of NLPDEs are defined by movable singular manifolds. The form of the exact solutions for the differential equations admit the nature of their movable singularities, though solutions are typically not easily found, if at all. For a nonlinear differential equation to have the Painlevé Property, all movable singularities exhibited by the solution (in the complex time domain) must be ordinary poles. For a NLPDE in $z_1, \ldots, z_n$, this concept can be extended; for singular manifolds defined by

$$\phi(z_1, \ldots, z_n) = 0, \tag{2.1.1}$$

we require the analytic function $\phi(z_1, \ldots, z_n)$ to only exhibit negative integer powers in the solution for the equation to have the Painlevé Property.[6]

For the complex $\mathcal{PT}$-Symmetric KdV equation [29],

$$u_t + \lambda u u_x - i\frac{\partial^2}{\partial x^2}\left(iu_x\right)^\epsilon = u_t + \lambda u u_x + i\epsilon(\epsilon-1)\left(iu_x\right)^{\epsilon-2} u_{xx}^2 + \epsilon\left(iu_x\right)^{\epsilon-1} u_{3x} = 0, \tag{2.1.2}$$

we wish to find integrable cases, that is, values of ϵ such that (2.1.2) is integrable. The Painlevé Property provides a necessary test for integrability [32].

2.1.1 Leading Order Analysis

Since exact solutions are typically not obtainable, we must determine the nature of the movable singular manifolds by examining the local behavior of solutions around them. This is accomplished by using a leading order analysis; we make the ansatz

$$u(x,t) = u_0\phi^\alpha, \tag{2.1.3}$$

where $\alpha \in \mathbb{R}$, $u_0(x,t)$ are to be determined, and $\phi(x,t) = 0$ is the location of the singular manifold. Using this in (2.1.2), we have

$$\begin{aligned}
&\mathcal{O}(\phi^{\alpha-1}) + \lambda u_0 \phi^\alpha \left[\alpha u_0 \phi^{\alpha-1}\phi_x + \mathcal{O}(\phi^\alpha)\right] \\
&\quad + i\epsilon(\epsilon-1)\left[i\alpha u_0\phi^{\alpha-1}\phi_x + \mathcal{O}(\phi^\alpha)\right]^{\epsilon-2}\left[\alpha(\alpha-1)u_0\phi^{\alpha-2}\phi_x^2 + \mathcal{O}(\phi^{\alpha-1})\right]^2 \\
&\quad + \epsilon\left[i\alpha u_0\phi^{\alpha-1}\phi_x + \mathcal{O}(\phi^\alpha)\right]^{\epsilon-1}\left[\alpha(\alpha-1)(\alpha-2)u_0\phi^{\alpha-3}\phi_x^3 + \mathcal{O}(\phi^{\alpha-2})\right] = 0. \qquad (2.1.4)
\end{aligned}$$

The most singular terms in this expression must balance at the singularity. The smallest powers of ϕ in the last two lines are $(\alpha-1)(\epsilon-2)+2(\alpha-2)$ and $(\alpha-1)(\epsilon-1)+(\alpha-3)$, which both simplify to $\epsilon(\alpha-1)-2$. Thus, to balance the powers of all most singular terms, we require

$$\begin{aligned}
2\alpha - 1 &= \epsilon(\alpha-1) - 2, \\
\Rightarrow \epsilon &= \frac{2\alpha+1}{\alpha-1}. \qquad (2.1.5)
\end{aligned}$$

As a necessary condition for the Painlevé Property, we further impose the restriction $\alpha = -n$, where $n \in \mathbb{N}$; the equivalent of requiring singularities to be ordinary poles. Thus, the values of ϵ that we can *consider* for integrability are

$$\epsilon = \frac{2n-1}{n+1}, \qquad n = 1, 2, \ldots \qquad (2.1.6)$$

Now equating the corresponding coefficients of the most singular terms in (2.1.4), we require

$$\begin{aligned}
\lambda\alpha u_0^2\phi_x + i\epsilon(\epsilon-1)\left[i\alpha u_0\phi_x\right]^{\epsilon-2}\alpha^2(\alpha-1)^2u_0^2\phi_x^4 + \epsilon\left[i\alpha u_0\phi_x\right]^{\epsilon-1}\alpha(\alpha-1)(\alpha-2)u_0\phi_x^3 &= 0, \qquad (2.1.7) \\
\Rightarrow \lambda + \epsilon(\alpha-1)(i\alpha)^{\epsilon-1}u_0^{\epsilon-2}\phi_x^{\epsilon+1}\left[(\epsilon-1)(\alpha-1)+(\alpha-2)\right] &= 0, \\
\Rightarrow \lambda + 2n(2n-1)(-in)^{\frac{n-2}{n+1}}u_0^{\frac{-3}{n+1}}\phi_x^{\frac{3n}{n+1}} &= 0, \\
\Rightarrow u_0^{\frac{3}{n+1}} = -2n(2n-1)\lambda^{-1}(-in)^{\frac{n-2}{n+1}}\phi_x^{\frac{3n}{n+1}}, \qquad (2.1.8)
\end{aligned}$$

where we have used $\alpha = -n$ and (2.1.6). This expression for u_0 does not give immediately useful information, however it will be used to simplify subsequent calculations.

The leading order analysis only gives us the behavior of the solution at the singular manifold. Therefore, we will need to construct a local expansion. For solutions whose singularities are ordinary poles, the expansion will be a simple Laurent series; for singular manifolds, a generalized Laurent series can be used. That is, a

local expansion of the solution in the neighborhood of a singular manifold is given by

$$u(x,t) = \sum_{j=0}^{\infty} u_j(x,t)\phi^{-n+j}. \tag{2.1.9}$$

The expansion (2.1.9) is only valid if there is a full set of arbitrary functions for the order of the NLPDE.[6] Equation (2.1.2) is of order three, thus we need two arbitrary functions in addition to the arbitrariness of the movable singular manifold. These are admitted by arbitrary $u_r(x,t)$ (r to be determined) in the Laurent expansion; the number and location of which are found by a resonance analysis.

2.1.2 Resonance Analysis

We wish to find the values of r that make $u_r(x,t)$ in (2.1.9) arbitrary. To that end, we let

$$u(x,t) = u_0\phi^{-n} + p\phi^{-n+r}, \tag{2.1.10}$$

and impose conditions on r to ensure the arbitrariness of $p(x,t)$. With the exception of -1 (which indicates the arbitrariness of ϕ), we further require the resonances to be positive integers so as to correspond to locations in the Laurent expansion of the solution. If a full set of arbitrary coefficient functions is found, the NLPDE is said to have the Painlevé Property, indicating integrability. Using (2.1.10) in (2.1.2), we have

$$\begin{aligned}
&\mathcal{O}(\phi^{-n-1}) + \mathcal{O}(\phi^{-n+r-1}) \\
&\quad + \lambda\left[u_0\phi^{-n} + p\phi^{-n+r}\right]\left[-nu_0\phi^{-n-1}\phi_x + (-n+r)p\phi^{-n+r-1}\phi_x + \mathcal{O}(\phi^{-n+r}) + \mathcal{O}(\phi^{-n})\right] \\
&\quad + i\frac{2n-1}{n+1}\left[\frac{n-2}{n+1}\right]\left[i\left[-nu_0\phi^{-n-1}\phi_x + (-n+r)p\phi^{-n+r-1}\phi_x + \mathcal{O}(\phi^{-n+r}) + \mathcal{O}(\phi^{-n})\right]\right]^{\frac{-3}{n+1}} \\
&\qquad\qquad \cdot\left[u_0n(n+1)\phi^{-n-2}\phi_x^2 + p(-n+r-1)(-n+r)\phi^{-n+r-2}\phi_x^2 + \mathcal{O}(\phi^{-n+r-1}) + \mathcal{O}(\phi^{-n-1})\right]^2 \\
&\quad + \frac{2n-1}{n+1}\Big[i\left[-nu_0\phi^{-n-1}\phi_x + (-n+r)p\phi^{-n+r-1}\phi_x + \mathcal{O}(\phi^{-n+r}) + \mathcal{O}(\phi^{-n})\right]\Big]^{\frac{n-2}{n+1}} \\
&\qquad\qquad \cdot\big[-u_0n(n+1)(n+2)\phi^{-n-3}\phi_x^3 + \mathcal{O}(\phi^{-n-2+r}) + \mathcal{O}(\phi^{-n-2}) \\
&\qquad\qquad\qquad +p(-n+r-2)(-n+r-1)(-n+r)\phi^{-n-3+r}\phi_x^3\big] = 0.
\end{aligned} \tag{2.1.11}$$

Neglecting the higher-order terms, and using binomial expansions (about $\phi = 0$), we then obtain

$$\begin{aligned}
&\lambda\left[u_0\phi^{-n} + p\phi^{-n+r}\right]\left[-nu_0\phi^{-n-1}\phi_x + (-n+r)p\phi^{-n+r-1}\phi_x\right] \\
&\quad + i^{\frac{n-2}{n+1}}\frac{2n-1}{n+1}\left[\frac{n-2}{n+1}\right]\left[-nu_0\phi^{-n-1}\phi_x\right]^{\frac{-3}{n+1}}\left[1 + \frac{-3}{n+1}\cdot\frac{(-n+r)p}{-nu_0}\phi^r + \mathcal{O}(\phi^{2r})\right] \\
&\qquad\quad \cdot\left[u_0^2n^2(n+1)^2\phi^{-2n-4}\phi_x^4 + 2u_0pn(n+1)(-n+r-1)(-n+r)\phi^{-2n+r-4}\phi_x^4 + \mathcal{O}(\phi^{-2n+2r+4})\right] \\
&\quad + i^{\frac{n-2}{n+1}}\frac{2n-1}{n+1}\left[-nu_0\phi^{-n-1}\phi_x\right]^{\frac{n-2}{n+1}}\left[1 + \frac{n-2}{n+1}\cdot\frac{(-n+r)p}{-nu_0}\phi^r + \mathcal{O}(\phi^{2r})\right] \\
&\qquad\quad \cdot\left[-u_0n(n+1)(n+2)\phi^{-n-3}\phi_x^3 + p(-n+r-2)(-n+r-1)(-n+r)\phi^{-n-3+r}\phi_x^3\right] = 0. \quad (2.1.12)
\end{aligned}$$

We note that $\mathcal{O}(\phi^{-2n-1})$ terms are constant in p, and by virtue of (2.1.7), sum to zero. Turning our attention to $\mathcal{O}(\phi^{-2n+r-1})$ terms, which are linear in p, we have the following after some simplification

$$\left[\lambda(-2n+r)u_0\phi_x + i^{\frac{n-2}{n+1}}\frac{2n-1}{n+1}(-n+r)(2n-r)(2n-r-1)\,(-n)^{\frac{n-2}{n+1}}\,u_0^{\frac{n-2}{n+1}}\phi_x^{\frac{4n+1}{n+1}}\right]p = 0. \qquad (2.1.13)$$

Now, to admit an arbitrary p, we equate the bracketed expression to zero; the values of r for which this is true will yield the resonances. Making use of (2.1.8), we find

$$\begin{aligned}
\lambda(-2n+r)u_0\phi_x + i^{\frac{n-2}{n+1}}\frac{2n-1}{n+1}(-n+r)(2n-r)(2n-r-1)\,(-n)^{\frac{n-2}{n+1}}\,u_0^{\frac{n-2}{n+1}}\phi_x^{\frac{4n+1}{n+1}} &= 0, \\
\Rightarrow u_0\phi_x\left[\lambda(-2n+r) + i^{\frac{n-2}{n+1}}\frac{2n-1}{n+1}(-n+r)(2n-r)(2n-r-1)\,(-n)^{\frac{n-2}{n+1}}\,u_0^{\frac{-3}{n+1}}\phi_x^{\frac{3n}{n+1}}\right] &= 0, \\
\Rightarrow u_0\phi_x\left[\lambda(-2n+r) + \frac{1}{n+1}(-n+r)(2n-r)(2n-r-1)\frac{-\lambda}{2n}\right] &= 0, \\
\Rightarrow -\lambda u_0\phi_x(2n-r)\left[1 + \frac{(-n+r)(2n-r-1)}{2n(n+1)}\right] &= 0, \\
\Rightarrow \frac{-\lambda u_0\phi_x}{2n(n+1)}(r-2n)(r-3n)(r+1) &= 0.
\end{aligned}$$

Thus, values of r that admit an arbitrary p are $r = -1, 2n, 3n$; the (positive) resonances $r = 2n, 3n$ are integers for any positive integer n, as they should be to correspond to positions in the Laurent series (2.1.9). In addition to the movable singular manifold, this gives three arbitrary functions in the Laurent expansion (2.1.9). The case $n = 2$ of (2.1.2) yields the basic KdV equation $u_t + \lambda uu_x + u_{3x} = 0$; the resonances are known to be $r = -1, 4, 6$ which is consistent with our findings. The basic KdV equation has been thoroughly analyzed in other literature, and will not be discussed further here. The cases $n = 1, 3, 4$ are discussed in subsequent chapters.

It should be noted that for some PDEs, additional conditions may be necessary to secure the arbitrariness of the resonance coefficients. This can be verified by direct substitution of the Laurent expansion; after finding a recursion relation for the u_j, coefficients are evaluated at sequential values of j up through the resonances [6]. Due to our generalized n, deriving the recursion relation is not feasible in general, and so verification will be done only for specific n cases; the cases $n = 3, 4$ will be verified after reformulation in Chapter 2.2.

2.2 Reformulation into System

Given the form of (2.1.6), the original equation (2.1.2) will have branch points. In light of this, we make the substitution

$$v = (iu_x)^{\frac{1}{n+1}} \iff v^{n+1} = iu_x. \tag{2.2.1}$$

To allow for the above substitution, we take an x partial derivative of (2.1.2), multiply through by i, and use (2.1.6), giving us

$$iu_{xt} + i\lambda\left(uu_{xx} + u_x^2\right) + \frac{\partial^3}{\partial x^3}\left(iu_x\right)^{\frac{2n-1}{n+1}} = 0. \tag{2.2.2}$$

Now using the substitution (2.2.1), we obtain

$$(v^{n+1})_t + \lambda u(v^{n+1})_x - i\lambda v^{2n+2} + \frac{\partial^3}{\partial x^3} v^{2n-1} = 0. \tag{2.2.3}$$

Thus, the NLPDE (2.1.2) becomes the system

$$(n+1)v^n v_t + \lambda(n+1)uv^n v_x - i\lambda v^{2n+2} + \frac{\partial^3}{\partial x^3} v^{2n-1} = 0, \tag{2.2.4a}$$

$$v^{n+1} - iu_x = 0. \tag{2.2.4b}$$

Since we needed to take a derivative of our original equation, (2.2.4) is only a derivative system of (2.1.2), rather than a direct one; solutions of the reformulated system may require additional conditions in order to satisfy the original NLPDE.

To determine the behavior of $v(x,t)$ at the singular manifold $\phi = 0$, we let $v = v_0\phi^\beta$. Now, since $u = u_0\phi^{-n}$ at the singular manifold, (2.2.4b) implies

$$v_0^{n+1}\phi^{\beta(n+1)} = -inu_0\phi^{-n-1}\phi_x + \mathcal{O}(\phi^{-n}). \tag{2.2.5}$$

Equating the leading order powers of ϕ, we have $\beta(n+1) = -n-1$. Thus, for any n, $\beta = -1$; we note that

similar leading order powers from (2.2.4a) yield the same β value. This gives us

$$v = v_0\phi^{-1}. \tag{2.2.6}$$

Further, equating leading coefficients we have

$$v_0^{n+1} = -inu_0\phi_x \Longleftrightarrow u_0 = \frac{iv_0^{n+1}}{n\phi_x}. \tag{2.2.7}$$

Comparing this to the expression found for u_0 in (2.1.8), we then obtain

$$\begin{aligned} u_0^{\frac{3}{n+1}} = \left[\frac{iv_0^{n+1}}{n\phi_x}\right]^{\frac{3}{n+1}} &= -2n(2n-1)\lambda^{-1}(-in)^{\frac{n-2}{n+1}}\phi_x^{\frac{3n}{n+1}}, \\ \Rightarrow \frac{v_0^3}{(-in)^{\frac{3}{n+1}}\phi_x^{\frac{3}{n+1}}} &= -2n(2n-1)\lambda^{-1}(-in)^{\frac{n-2}{n+1}}\phi_x^{\frac{3n}{n+1}}, \\ \Rightarrow v_0^3 &= 2in^2(2n-1)\lambda^{-1}\phi_x^3, \end{aligned} \tag{2.2.8}$$

which gives us an expression for v_0 for any n; we obtain this same expression for v_0 when equating leading order coefficients in (2.2.4a) and comparing to (2.2.7). Expanding solutions u, v of (2.2.3) in a Laurent series, we seek solutions of the form

$$u(x,t) = \sum_{j=0}^{\infty} u_j\phi^{-n+j}, \tag{2.2.9a}$$

$$v(x,t) = \sum_{j=0}^{\infty} v_j\phi^{-1+j}, \tag{2.2.9b}$$

where u_0, v_0 are given by (2.2.7), (2.2.8) respectively.

2.3 Resonance Analysis for Reformulated System

Following an analogous procedure as in Chapter 2.1.2 for systems, we let

$$u(x,t) = u_0\phi^{-n} + p\phi^{-n+r}, \tag{2.3.1a}$$

$$v(x,t) = v_0\phi^{-1} + q\phi^{-1+r}. \tag{2.3.1b}$$

We want to find the values of r that make either p or q arbitrary. Since our system (2.2.4) is fourth-order, we require three arbitrary functions in addition to the arbitrary singular manifold location. Substituting (2.3.1) into (2.2.4), we balance the most singular r-powered terms from each equation, which correspond to terms linear in p and q; these are the $\mathcal{O}(\phi^{-2n-2+r})$ and $\mathcal{O}(\phi^{-n-1+r})$ terms respectively, and they yield the set of equations

$$\begin{aligned}\left[-\lambda(n+1)v_0^{n+1}\phi_x\right]p+\big[\lambda(n+1)(-n-1+r)v_0^n u_0\phi_x - i\lambda(2n+2)v_0^{2n+1}\\ +(2n-1)(-2n+1+r)(-2n+r)(-2n-1+r)v_0^{2n-2}\phi_x^3\big]\,q=0, \qquad (2.3.2a)\\ \left[-i(-n+r)\phi_x\right]p+\left[(n+1)v_0^n\right]q=0. \qquad (2.3.2b)\end{aligned}$$

For p or q to be arbitrary, we require the determinant of the above system in p, q to be zero. This gives the equation

$$\begin{aligned}-\lambda(n+1)^2v_0^{2n+1}\phi_x+i\lambda(-n+r)(n+1)(-n-1+r)u_0v_0^n\phi_x^2+\lambda(-n+r)(2n+2)v_0^{2n+1}\phi_x\\ +i(-n+r)(2n-1)(-2n+1+r)(-2n+r)(-2n-1+r)v_0^{2n-2}\phi_x^4=0. \qquad (2.3.3)\end{aligned}$$

Making use of (2.2.7) and (2.2.8), we obtain

$$\begin{aligned}&\frac{\lambda}{v_0}\big[-(n+1)^2[-inu_0\phi_x]^2\phi_x\\ &\quad+i(-n+r)(n+1)(-n-1+r)u_0[-inu_0\phi_x]\phi_x^2+(-n+r)(2n+2)[-inu_0\phi_x]^2\phi_x\\ &\quad+(-n+r)(-2n+1+r)(-2n+r)(-2n-1+r)[-inu_0\phi_x]^2\phi_x\cdot\frac{i(2n-1)\phi_x^3}{v_0^3}\Big]=0,\\ &\Rightarrow\frac{\lambda u_0^2\phi_x^3}{v_0}\big[n^2(n+1)^2+n(-n+r)(n+1)(-n-1+r)-n^2(-n+r)(2n+2)\\ &\qquad-\frac{1}{2}(-n+r)(-2n+1+r)(-2n+r)(-2n-1+r)\cdot\frac{2in^2(2n-1)\phi_x^3}{\lambda v_0^3}\Big]=0,\\ &\Rightarrow\frac{\lambda u_0^2\phi_x^3}{2v_0}\big[2n^2(n+1)^2+2n(-n+r)(n+1)(-n-1+r)-4n^2(-n+r)(n+1)\\ &\qquad-(-n+r)(-2n+1+r)(-2n+r)(-2n-1+r)\big]=0,\\ &\Rightarrow\frac{\lambda u_0^2\phi_x^3}{2v_0}\left[r^4-7nr^3+(16n^2-2n-1)r^2+(-12n^3+10n^2+5n)r+(-12n^3-6n^2)\right]=0,\\ &\qquad\Rightarrow\frac{\lambda u_0^2\phi_x^3}{2v_0}(r+1)(r-2n)(r-3n)(r-2n-1)=0. \qquad (2.3.4)\end{aligned}$$

Thus, the values $r = -1, 2n, 2n+1, 3n$ should make p or q arbitrary. This gives us three positive integer resonance values for $n \geq 2$, and therefore should yield three arbitrary coefficient functions in the Laurent series expansions of the solutions u, v.

For the $n = 1$ case, the positive resonances are $r = 2, 3$, giving us only two arbitrary function locations instead of three. Hence, we do not have the full set of arbitrary coefficients and the Laurent expansion (2.2.9) is not valid. Thus, (2.2.4) fails the Painlevé test for $n = 1$.

As previously stated, the $n = 2$ case gives the regular KdV equation, so we turn our attention to the $n = 3$ case.

2.3.1 Verification of Resonances for $n = 3$

For $n = 3$, our system (2.2.4) becomes

$$4v^3v_t + 4\lambda u v^3 v_x - i\lambda v^8 + \frac{\partial^3}{\partial x^3}v^5 = 0, \tag{2.3.5a}$$

$$v^4 - iu_x = 0. \tag{2.3.5b}$$

The positive resonances from (2.3.4) are given by $r = 6, 7, 9$. Thus, we wish to show u_j or v_j in (2.2.9) are arbitrary for $j = 6, 7, 9$. Substituting the truncated expansions

$$u(x,t) = \sum_{j=0}^{9} u_j \phi^{-3+j}, \tag{2.3.6a}$$

$$v(x,t) = \sum_{j=0}^{9} v_j \phi^{-1+j}, \tag{2.3.6b}$$

into (2.4.10), we balance coefficients in ϕ. The $\mathcal{O}(\phi^{-8})$ term from (2.3.5a) and the $\mathcal{O}(\phi^{-4})$ term from (2.3.5b) yield a system of equations in u_0, v_0. Upon solving this system, we obtain

$$u_0 = \frac{i3^{5/3}10^{4/3}\phi_x^3}{\lambda^{4/3}}, \tag{2.3.7a}$$

$$v_0 = -\frac{i3^{2/3}10^{1/3}\phi_x}{\lambda^{1/3}}. \tag{2.3.7b}$$

Using these expressions for u_0, v_0, the systems of equations for $j = 1-5$ are sequentially obtained from the $\mathcal{O}(\phi^{-7}) - \mathcal{O}(\phi^{-3})$ terms of (2.3.5a) and the $\mathcal{O}(\phi^{-3}) - \mathcal{O}(\phi^{1})$ terms from (2.3.5b), respectively. Solving these systems by recursively replacing expressions found for the u_j, v_j, the coefficient functions for $j = 1-5$ are

given in (A.1). At the next order terms, $\mathcal{O}(\phi^{-2})$ and $\mathcal{O}(\phi^2)$, we obtain the dependent set of equations for u_6, v_6

$$\left[-\frac{30^{4/3}}{4i\lambda^{5/3}\phi_x^4}\right] f_6(u_6, v_6, \phi) = 0, \tag{2.3.8a}$$

$$\left[-\frac{1}{16\cdot 3^{1/3}\lambda^{4/3}\phi_x^8}\right] f_6(u_6, v_6, \phi) = 0, \tag{2.3.8b}$$

where f_6 is given in (A.1k). Thus, we may choose u_6 to be our arbitrary coefficient, verifying the resonance $r = 6$. The coefficient v_6 in terms of the arbitrary u_6 is given in (A.1).

Using this expression for v_6, the next order terms, $\mathcal{O}(\phi^{-1})$ and $\mathcal{O}(\phi^3)$, give the linearly dependent equations in u_7, v_7

$$\left[-\frac{5^{4/3}3^{2/3}}{8\cdot 2^{1/3}\lambda^{5/3}\phi_x^6}\right] f_7(u_7, v_7, \phi) = 0, \tag{2.3.9a}$$

$$\left[-\frac{i}{144\cdot 2^{2/3}\lambda^{4/3}\phi_x^{10}}\right] f_7(u_7, v_7, \phi) = 0, \tag{2.3.9b}$$

where f_7 is given in (A.1n). Again, we may choose u_7 as our arbitrary coefficient, which verifies the resonance $r = 7$. The coefficient v_7, along with u_8, v_8 from subsequently solving the $\mathcal{O}(\phi^0)$ and $\mathcal{O}(\phi^4)$ term equations, are given in (A.1).

Using these expressions, the $\mathcal{O}(\phi^1)$ and $\mathcal{O}(\phi^5)$ terms give the linearly dependent equations in u_9, v_9

$$\left[-\frac{3^{1/3}5^{1/3}}{16\cdot 2^{1/3}\lambda^{5/3}\phi_x^{10}}\right] f_9(u_9, v_9, \phi) = 0, \tag{2.3.10a}$$

$$\left[-\frac{i}{960\cdot 3^{1/3}2^{2/3}\lambda^{4/3}\phi_x^{14}}\right] f_9(u_9, v_9, \phi) = 0, \tag{2.3.10b}$$

where f_9 is given in (A.1s). Thus, we may choose u_9 to be arbitrary, which verifies the resonance $r = 9$; v_9 is given in (A.1). We have therefore verified all resonances; the coefficients at the $j = 6, 7, 9$ positions are indeed arbitrary, giving us a full set of arbitrary functions for our system with $n = 3$.

2.3.2 Verification of Resonances for $n = 4$

For $n = 4$, our system (2.2.4) becomes

$$5v^4v_t + 5\lambda uv^4v_x - i\lambda v^{10} + \frac{\partial^3}{\partial x^3}v^7 = 0, \tag{2.3.11a}$$

$$v^5 - iu_x = 0. \tag{2.3.11b}$$

The positive resonances from (2.3.4) are given by $r = 8, 9, 12$; we wish to show u_j or v_j in (2.2.9) are arbitrary for $j = 8, 9, 12$. Following the same procedure as for the $n = 1, 3$ cases, we substitute the truncated expansions

$$u(x,t) = \sum_{j=0}^{12} u_j \phi^{-4+j}, \tag{2.3.12a}$$

$$v(x,t) = \sum_{j=0}^{12} v_j \phi^{-1+j}, \tag{2.3.12b}$$

into (2.3.11), and balance coefficients in ϕ. After recursively solving for and using the u_j, v_j for $j = 0 - 7$, the $\mathcal{O}(\phi^{-2})$ term from (2.3.11a) and $\mathcal{O}(\phi^3)$ term from (2.3.11b) yield a linearly dependent set of equations in u_8, v_8. Likewise, at the next order terms, $\mathcal{O}(\phi^{-1})$ and $\mathcal{O}(\phi^4)$ respectively, yield a linearly dependent set of equations in u_9, v_9. Choosing u_8, u_9 as the arbitrary functions, we solve for v_8, v_9, and the resulting expressions for the $j = 10, 11$ coefficients are found from subsequent ordered terms. Finally, the respective $\mathcal{O}(\phi^2)$ and $\mathcal{O}(\phi^7)$ terms yield a linearly dependent set of equations in u_{12}, v_{12}, and we may choose u_{12} to be arbitrary. All of the resonances $j = 8, 9, 12$ are verified, giving us a full set of arbitrary parameters for our $n = 4$ system (2.3.11).

2.4 Singular Manifold Method

2.4.1 Case $\lambda = 1; n = 1$

Letting $\lambda = 1$ and $n = 1$ in (2.2.4), we will be analyzing and finding solutions to the system

$$2vv_t + 2uvv_x - iv^4 + v_{3x} = 0, \tag{2.4.1a}$$

$$v^2 - iu_x = 0. \tag{2.4.1b}$$

Following the original procedure of Weiss [8], we truncate the Laurent expansion of the solution at the constant term $\mathcal{O}(\phi^0)$; that is, we assume the solutions take the form of (2.2.9), truncated at $j = n = 1$,

$$u = \frac{u_0}{\phi} + u_1, \tag{2.4.2a}$$

$$v = \frac{v_0}{\phi} + v_1. \tag{2.4.2b}$$

The expressions (2.2.7) and (2.2.8) become

$$u_0 = \frac{(-i)^2 2^{2/3}\phi_x^2}{-i\phi_x} = -i2^{2/3}\phi_x, \tag{2.4.3a}$$

$$v_0 = (2i)^{1/3}\phi_x = -i2^{1/3}\phi_x. \tag{2.4.3b}$$

Therefore, (2.4.2) is now

$$u = \frac{-i2^{2/3}\phi_x}{\phi} + u_1, \tag{2.4.4a}$$

$$v = \frac{-i2^{1/3}\phi_x}{\phi} + v_1. \tag{2.4.4b}$$

Substituting these into (2.4.1), we equate powers of ϕ; the coefficient equations are given in (A.2) and (A.3). We note the equations (A.2d), (A.3b) are simply (2.4.1) in u_1, v_1. Thus, (2.4.4) gives an auto-Bäcklund Transformation connecting two solutions (u_1, v_1) and (u, v) of (2.4.1); if (u_1, v_1) is a solution, and ϕ satisfies (A.2a)-(A.2c) and (A.3a), then (u, v) given in (2.4.4) yields a new solution.

2.4.1.1 A t-Independent Solution

It is clear the vacuum solution $(u, v) = (0, 0)$ solves (2.4.1); thus, we let $u_1 = v_1 = 0$ in (2.4.4), (A.2), and (A.3). We further require $\phi_x \neq 0$, as this leads once again to the vacuum solution. The coefficient equations

(A.2a)-(A.2c) and (A.3a) now become

$$\mathcal{O}(\phi^{-3}): \quad \phi_x^2\left[2^{1/3}\phi_t - 4i\phi_{xx}\right] = 0, \tag{2.4.5a}$$

$$\mathcal{O}(\phi^{-2}): \quad -2^{1/3}\phi_x\phi_{xt} + \frac{3}{2}\phi_{xx}^2 + 2i\phi_x\phi_{3x} = 0, \tag{2.4.5b}$$

$$\mathcal{O}(\phi^{-1}): \quad i2^{1/3}\phi_{xxxx} = 0, \tag{2.4.5c}$$

and

$$\mathcal{O}(\phi^{-1}): \quad \phi_{xx} = 0, \tag{2.4.5d}$$

Using (2.4.5d), (2.4.5a) yields $\phi_t = 0$; (2.4.5b) and (2.4.5c) are subsequently satisfied. We have then $\phi_{xx} = \phi_t = 0$, giving us

$$\phi = ax + b, \tag{2.4.6}$$

where a, b are constants, with $a \neq 0$. This yields

$$u = \frac{-ai2^{2/3}}{ax+b} = \frac{-i2^{2/3}}{x+b/a}, \tag{2.4.7a}$$

$$v = \frac{-ai2^{1/3}}{ax+b} = \frac{-i2^{1/3}}{x+b/a}. \tag{2.4.7b}$$

It can be verified that (2.4.7) satisfies our system (2.4.1).

Going back to our original equation (2.1.2), with $n = \lambda = 1$, after plugging in the solution $u(x)$ of (2.4.7) (via Mathematica), we find

$$\frac{2^{4/3}a^2\left(a + a(b/a+x)\sqrt{(b/a+x)^{-2}}\right)}{(b+ax)^3} = 0. \tag{2.4.8}$$

Thus, we further require $b/a + x < 0$ for identity. With this new restriction, we find that

$$u(x,t) = \frac{-i2^{2/3}}{x+b/a}, \qquad x + \frac{b}{a} < 0, \tag{2.4.9}$$

is a solution of (2.1.2). However, (2.4.7) is time-independent, and therefore not of much interest. To obtain a t-dependent solution, we require the use of an alternate analysis, to be covered in Chapter 2.5.

2.4.2 Case $\lambda = 1; n = 3$

Letting $\lambda = 1$ in our $n = 3$ system (2.3.5), we will be analyzing and finding solutions to the system

$$4v^3v_t + 4uv^3v_x - iv^8 + \frac{\partial^3}{\partial x^3}v^5 = 0, \tag{2.4.10a}$$

$$v^4 - iu_x = 0. \tag{2.4.10b}$$

Following the original procedure of Weiss [8], we truncate the Laurent expansion of the solution at the constant term $\mathcal{O}(\phi^0)$; that is, we assume the solutions take the form of series (2.2.9), truncated at $j = n = 3$ in (2.2.9a) and $j = 1$ in (2.2.9b), or

$$u(x,t) = \frac{u_0}{\phi^3} + \frac{u_1}{\phi^2} + \frac{u_2}{\phi} + u_3, \tag{2.4.11a}$$

$$v(x,t) = \frac{v_0}{\phi} + v_1. \tag{2.4.11b}$$

Plugging these into (2.4.10), we equate terms order by order in ϕ. Solving $\mathcal{O}(\phi^{-8})$ term from (2.4.10a) and the $\mathcal{O}(\phi^{-4})$ term from (2.4.10b) yield

$$u_0 = i3^{5/3}10^{4/3}\phi_x^3, \tag{2.4.12a}$$

$$v_0 = -i3^{2/3}10^{1/3}\phi_x. \tag{2.4.12b}$$

Using these expressions, (2.4.11) becomes

$$u(x,t) = \frac{i3^{5/3}10^{4/3}\phi_x^3}{\phi^3} + \frac{u_1}{\phi^2} + \frac{u_2}{\phi} + u_3, \tag{2.4.13a}$$

$$v(x,t) = -\frac{i3^{2/3}10^{1/3}\phi_x}{\phi} + v_1. \tag{2.4.13b}$$

The remaining coefficient equations are given in (A.4) and (A.5). We note that (A.4h), (A.5d) are simply (2.4.10) in u_3, v_1. Thus, (2.4.13) gives an auto-Bäcklund Transformation from a known solution (u_3, v_1) to a new solution (u, v), where u_1, u_2, ϕ satisfy (A.4) and (A.5).

2.4.2.1 A t-independent Solution

Using the vacuum solution of (2.4.10), we set $(u_3, v_1) = (0, 0)$ in (A.4) and (A.5); the equations in (A.4e)-(A.4h) and (A.5d) are therefore satisfied. The remaining equations become

$$3^{1/3}\phi_x u_1 + 315i10^{1/3}\phi_x^2\phi_{xx} = 0, \tag{2.4.14a}$$

$$4\cdot 3^{1/3}\phi_x^3 u_2 - 4\cdot 3^{1/3}\phi_x\phi_{xx}u_1 - 1125i10^{1/3}\phi_x^2\phi_{xx}^2 - 240i10^{1/3}\phi_x^3\phi_{3x} = 0, \tag{2.4.14b}$$

$$\begin{aligned} 4\cdot 3^{1/3}\phi_t\phi_x^3 - 4\cdot 3^{1/3}\phi_x^2\phi_{xx}u_2 + 180i10^{1/3}\phi_x\phi_{xx}^3 \\ +180i10^{1/3}\phi_x^2\phi_{xx}\phi_{3x} + 15i10^{1/3}\phi_x^3\phi_{4x} = 0, \end{aligned} \tag{2.4.14c}$$

$$\phi_x\phi_{xt} = 0, \tag{2.4.14d}$$

$$i\phi_x u_1 + 45\cdot 3^{2/3}10^{1/3}\phi_x^2\phi_{xx} = 0, \tag{2.4.14e}$$

$$\phi_x u_2 - (u_1)_x = 0, \tag{2.4.14f}$$

$$(u_2)_x = 0. \tag{2.4.14g}$$

Further requiring $\phi_x \neq 0$, the equations (2.4.14a) and (2.4.14e) are incompatible, and thus require $u_1 = \phi_{xx} = 0$. Using these, (2.4.14b) and (2.4.14f) imply $u_2 = 0$, which in turn yields $\phi_t = 0$ from (2.4.14c). Then (2.4.14d) and (2.4.14g) are subsequently satisfied. Combining our conditions on ϕ, we obtain $\phi(x,t) = ax + b, a \neq 0$. Therefore, our new solution by (2.4.13) is

$$u(x,t) = \frac{i3^{5/3}10^{4/3}}{(x+b/a)^3}, \tag{2.4.15a}$$

$$v(x,t) = -\frac{i3^{2/3}10^{1/3}}{x+b/a}, \tag{2.4.15b}$$

which is verified by Mathematica to solve (2.4.10).

2.4.3 Case $\lambda = 1; n = 4$

Letting $\lambda = 1$ in our $n = 4$ system (2.3.11), we will be analyzing and finding solutions to the system

$$5v^4v_t + 5uv^4v_x - iv^{10} + \frac{\partial^3}{\partial x^3}v^7 = 0, \tag{2.4.16a}$$

$$v^5 - iu_x = 0. \tag{2.4.16b}$$

We assume the solutions take the form of the truncated series

$$u(x,t) = \frac{u_0}{\phi^4} + \frac{u_1}{\phi^3} + \frac{u_2}{\phi^2} + \frac{u_3}{\phi} + u_4, \tag{2.4.17a}$$

$$v(x,t) = \frac{v_0}{\phi} + v_1. \tag{2.4.17b}$$

Plugging this into (2.4.16), the most singular terms of each equation yield

$$u_0 = 448 \cdot 2^{1/3} 7^{2/3} \phi_x^4, \tag{2.4.18a}$$

$$v_0 = -2i2^{2/3} 7^{1/3} \phi_x, \tag{2.4.18b}$$

and the $\mathcal{O}(\phi^0)$ terms from both (2.4.16a) and (2.4.16b) yield the same equations in u_4, v_1, thus giving us the auto-BT

$$u(x,t) = \frac{2^{19/3} 7^{5/3} \phi_x^4}{\phi^4} + \frac{u_1}{\phi^3} + \frac{u_2}{\phi^2} + \frac{u_3}{\phi} + u_4, \tag{2.4.19a}$$

$$v(x,t) = -\frac{i2^{5/3} 7^{1/3} \phi_x}{\phi} + v_1. \tag{2.4.19b}$$

The remaining $\phi-$coefficient equations yield conditions on u_1, u_2, u_3, ϕ and their derivatives.

2.4.3.1 A t-Independent Solution

Letting $(u_4, v_1) = (0, 0)$ in (2.4.19) and the $\phi-$coefficient equations, we obtain

$$5 \cdot 7^{1/3} u_1 \phi_x + 59584 \cdot 2^{1/3} \phi_x^3 \phi_{xx} = 0, \tag{2.4.20a}$$

$$5 \cdot 7^{1/3} u_1 \phi_x \phi_{xx} + 57624 \cdot 2^{1/3} \phi_x^3 \phi_{xx}^2 + 8624 \cdot 2^{1/3} \phi_x^4 \phi_{3x} - 5 \cdot 7^{1/3} u_2 \phi_x^3 = 0, \tag{2.4.20b}$$

$$5 \cdot 7^{1/3} u_3 \phi_x^3 - 5 \cdot 7^{1/3} u_2 \phi_x \phi_{xx} + 11760 \cdot 2^{1/3} \phi_x \phi_{xx}^3 + 7056 \cdot 2^{1/3} \phi_x^2 \phi_{xx} \phi_{3x} + 392 \cdot 2^{1/3} \phi_x^3 \phi_{4x} = 0, \tag{2.4.20c}$$

$$\phi_t \phi_x^2 - u_3 \phi_x \phi_{xx} = 0, \tag{2.4.20d}$$

$$\phi_x \phi_{xt} = 0, \tag{2.4.20e}$$

$$1792 \cdot 2^{1/3} 7^{2/3} \phi_x^3 \phi_{xx} - 3 u_1 \phi_x = 0, \tag{2.4.20f}$$

$$2 u_2 \phi_x - (u_1)_x = 0, \tag{2.4.20g}$$

$$u_3 \phi_x - (u_2)_x = 0, \tag{2.4.20h}$$

$$(u_3)_x = 0. \tag{2.4.20i}$$

Similar to the $n = 3$ case, these equations imply $u_1 = u_2 = u_3 = \phi_t = \phi_{xx} = 0$, so $\phi(x,t) = ax + b, a \neq 0$. Then (2.4.19) gives the new solution to (2.4.16),

$$u(x,t) = \frac{2^{19/3}7^{5/3}}{(x+b/a)^4}, \tag{2.4.21a}$$

$$v(x,t) = -\frac{i2^{5/3}7^{1/3}}{x+b/a}, \tag{2.4.21b}$$

which is verified by Mathematica.

2.5 Invariant Painlevé Analysis

All solutions thus far obtained have been time-independent and of little interest. In an attempt to find t-dependent solutions, we consider another method for analyzing the NLPDE, by way of a Ricatti-type analysis, or Invariant Painlevé formulation [34]. Here, we look at expansions of the form

$$u(x,t) = \sum_{j=0}^{\infty} u_j \chi^{-\alpha+j}, \tag{2.5.1}$$

where χ must vanish with the singular manifold $\phi - \phi_0$, and α is determined by a leading order analysis. If we choose the form of χ to be

$$\chi = \frac{\psi}{\psi_x} = \left(\frac{\phi_x}{\phi - \phi_0} - \frac{\phi_{xx}}{2\phi_x}\right)^{-1}, \tag{2.5.2a}$$

$$\psi = \frac{\phi - \phi_0}{\phi_x^{1/2}}, \tag{2.5.2b}$$

then χ satisfies the Ricatti equations

$$\chi_x = 1 + \frac{1}{2}S\chi^2, \tag{2.5.3a}$$

$$\chi_t = -C + C_x\chi - \frac{1}{2}\left(CS + C_{xx}\right)\chi^2, \tag{2.5.3b}$$

and ψ satisfies the linear equations

$$\psi_{xx} = -\frac{1}{2}S\psi, \tag{2.5.4a}$$

$$\psi_t = \frac{1}{2}C_x\psi - C\psi_x. \tag{2.5.4b}$$

The quantities $S(x,t)$ and $C(x,t)$ are defined by

$$S(x,t) = \frac{\phi_{3x}}{\phi_x} - \frac{3}{2}\left(\frac{\phi_{xx}}{\phi_x}\right)^2, \tag{2.5.5a}$$

$$C(x,t) = -\frac{\phi_t}{\phi_x}, \tag{2.5.5b}$$

and are often referred to as the Schwarzian derivative and celerity (dimension of velocity), respectively. Both quantities (2.5.5) are invariant under the Möbius transformation

$$\phi \to \frac{a\phi + b}{c\phi + d}, \quad ad - bc = 1, \tag{2.5.6}$$

and are related under the cross-derivative condition $\phi_{3xt} = \phi_{t3x}$ by

$$S_t + C_{3x} + 2C_x S + C S_x = 0. \tag{2.5.7}$$

The solution method consists of using (2.5.1) (usually truncated) in the NLPDE to be solved, recursively replacing χ derivatives using (2.5.3), and equating terms order by order in χ. Conditions on u_j, S, C, may be found, and therefore (2.5.3) or (2.5.4) may be used to solve for χ. Then (2.5.1) with u_j, χ, will give a solution to the NLPDE. [34]

The above Invariant Painlevé formulation can be generalized to systems by using expansions similar to (2.5.1) for each dependent variable. For the system (2.2.4), the analysis dictates the use of the expansions

$$u(x,t) = \sum_{j=0}^{\infty} u_j \chi^{-n+j}, \tag{2.5.8a}$$

$$v(x,t) = \sum_{j=0}^{\infty} v_j \chi^{-1+j}. \tag{2.5.8b}$$

2.5.1 Application to $n = 1$ Case

We proceed for (2.4.1) by truncating the expansions (2.5.8), with $n = 1$, at the constant term. Substituting

$$u(x,t) = \frac{u_0}{\chi} + u_1, \tag{2.5.9a}$$

$$v(x,t) = \frac{v_0}{\chi} + v_1, \tag{2.5.9b}$$

into (2.4.1), we recursively replace derivatives of χ using (2.5.3) and equate coefficients order by order in χ. The coefficient equations are given in (A.6) and (A.7). Solving the leading order coefficient equations, (A.6a) and (A.7a), for u_0, v_0, we obtain

$$u_0 = -i2^{2/3}, \tag{2.5.10a}$$

$$v_0 = -i2^{1/3}. \tag{2.5.10b}$$

Making use of (2.5.10), the coefficient equations (A.6b) and (A.7b) yield

$$u_1 = C, \tag{2.5.11a}$$

$$v_1 = 0. \tag{2.5.11b}$$

The remaining coefficient equations, with use of (2.5.10) and (2.5.11), become

$$\mathcal{O}(\chi^{-2}): \quad iS + 2^{1/3}C_x = 0, \tag{2.5.12a}$$

$$\mathcal{O}(\chi^{-1}): \quad \frac{\partial}{\partial x}\left[iS + 2^{1/3}C_x\right] = 0, \tag{2.5.12b}$$

$$\mathcal{O}(\chi^{0}): \quad S^2 + S_{xx} = 0, \tag{2.5.12c}$$

and

$$\mathcal{O}(\chi^{0}): \quad iS + 2^{1/3}C_x = 0, \tag{2.5.12d}$$

which further reduces to the system (in S, C_x),

$$S^2 + S_{xx} = 0, \tag{2.5.13a}$$

$$iS + 2^{1/3}C_x = 0. \tag{2.5.13b}$$

One solution of (2.5.13) is given by

$$S = -(-1)^{2/3}6^{1/3}\wp\left[(-6)^{-1/3}\left(x + g_1(t)\right); 0, g_2(t)\right], \tag{2.5.14a}$$

$$C_x = -i(-1)^{2/3}3^{1/3}\wp\left[(-6)^{-1/3}\left(x + g_1(t)\right); 0, g_2(t)\right], \tag{2.5.14b}$$

where $\wp$ is the Weierstrass elliptic $\wp$-function, and g_1, g_2 are arbitrary functions of t. Given the complexity of this intermediate result, we instead turn to the more obvious trivial solution of system (2.5.13)

$$S = 0, \tag{2.5.15a}$$

$$C_x = 0 \Rightarrow C = f(t), \tag{2.5.15b}$$

where $f(t)$ is an arbitrary function. This implies from (2.5.3)

$$\chi_x = 1, \tag{2.5.16a}$$

$$\chi_t = -f(t), \tag{2.5.16b}$$

$$\Rightarrow \chi = x - F(t), \tag{2.5.16c}$$

where $F(t)$ is an antiderivative of $f(t)$. We may then write $u_1 = C = f(t) = F'(t)$. Combining our results, (2.5.9) becomes

$$u(x,t) = \frac{-i2^{2/3}}{x - F(t)} + F'(t), \tag{2.5.17a}$$

$$v(x,t) = \frac{-i2^{1/3}}{x - F(t)}. \tag{2.5.17b}$$

It is verified by Mathematica that (2.5.17) solves (2.4.1).

To determine if (2.5.17a) also solves the original NLPDE (2.1.2), with $n = \lambda = 1$, from which the system (2.4.1) was derived, we substitute the expression for u in the original equation to obtain

$$2 \cdot 2^{1/3} \left(\frac{1}{|x - F(t)|^3} + \frac{1}{[x - F(t)]^3} \right) + F''(t) = 0, \tag{2.5.18}$$

Thus, we require $x - F(t) < 0$ and $F''(t) = 0$ for identity. This implies $F(t) = at + b$, where a, b are constants, which yields

$$u(x,t) = \frac{-i2^{2/3}}{x - at - b} + a, \qquad x < at + b, \tag{2.5.19}$$

as a solution of (2.1.2) with $n = 1$, verified by Mathematica. Choosing the values $a = 1,\ b = 0$ in (2.5.19), we note $\mathrm{Re}(u(x,t)) = 1$; a plot of $\mathrm{Im}(u(x,t))$ is given in Figure 1.

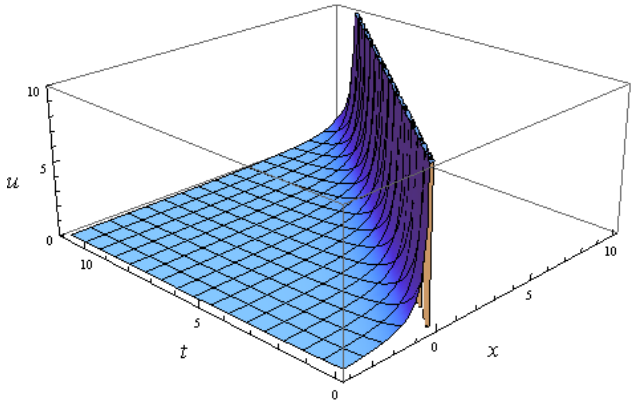


Figure 1: $\mathrm{Im}(u(x,t))$ of (2.5.19) with $a = 1,\ b = 0$

2.5.2 Application to $n = 3$ Case

Using (2.5.8) truncated at the constant term, with $n = 3$, we substitute

$$u(x,t) = \frac{u_0}{\chi^3} + \frac{u_1}{\chi^2} + \frac{u_2}{\chi} + u_3, \tag{2.5.20a}$$

$$v(x,t) = \frac{v_0}{\chi} + v_1, \tag{2.5.20b}$$

into (2.4.10) and recursively replace derivatives of χ using (2.5.3). The $\mathcal{O}(\chi^{-8})$ and $\mathcal{O}(\chi^{-4})$ terms from (2.4.10a) and (2.4.10b), respectively, yield

$$u_0 = -30(-1)^{5/6}3^{2/3}10^{1/3}, \tag{2.5.21a}$$

$$v_0 = (-1)^{5/6}3^{2/3}10^{1/3}, \tag{2.5.21b}$$

Likewise, $\mathcal{O}(\chi^{-7})$ and $\mathcal{O}(\chi^{-3})$ terms yield $u_1 = v_1 = 0$. The remaining $\chi-$coefficient equations using these values give, from (2.4.10a),

$$\mathcal{O}(\chi^{-6}): \quad 255 \cdot 10^{1/3}S - 2(-1)^{1/6}3^{1/3}u_2 = 0, \tag{2.5.22a}$$

$$\mathcal{O}(\chi^{-5}): \quad 4(-1)^{1/6}3^{1/3}C - 105 \cdot 10^{1/3}S_x - 4(-1)^{1/6}3^{1/3}u_3 = 0, \tag{2.5.22b}$$

$$\mathcal{O}(\chi^{-4}): \quad 465 \cdot 10^{1/3}S^2 - 4(-1)^{1/6}3^{1/3}Su_2 - 8(-1)^{1/6}3^{1/3}C_x + 15 \cdot 10^{1/3}S_{xx} = 0, \tag{2.5.22c}$$

$$\mathcal{O}(\chi^{-3}): \quad 2(-1)^{1/6}3^{1/3}CS + 2(-1)^{1/6}3^{1/3}C_{xx} - 2(-1)^{1/6}3^{1/3}Su_3 - 45 \cdot 10^{1/3}SS_x = 0, \tag{2.5.22d}$$

$$\mathcal{O}(\chi^{-2}): \quad S = 0, \tag{2.5.22e}$$

and from (2.4.10b),

$$\mathcal{O}(\chi^{-2}): \quad iu_2 + 45(-10)^{1/3}3^{2/3}S = 0, \tag{2.5.23a}$$

$$\mathcal{O}(\chi^{-1}): \quad (u_2)_x = 0, \tag{2.5.23b}$$

$$\mathcal{O}(\chi^{0}): \quad Su_2 - 2(u_3)_x = 0. \tag{2.5.23c}$$

By (2.5.22e), (2.5.22a) and (2.5.23a), we have $u_2 = S = 0$. Then (2.5.22b) gives $u_3 = C$, which in turn implies $(u_3)_x = C_x = 0$ from (2.5.23c). Thus, we have $C = f(t)$ and $S = 0$ as in the $n = 1$ case. Referring to our similar results in (2.5.16), we have $\chi(x,t) = x - F(t)$, where $F(t)$ is an arbitrary function and $F'(t) = f(t)$. Combining our results, (2.5.20) gives us the solution,

$$u(x,t) = -\frac{30(-1)^{5/6}3^{2/3}10^{1/3}}{[x-F(t)]^3} + F'(t), \tag{2.5.24a}$$

$$v(x,t) = \frac{(-1)^{5/6}3^{2/3}10^{1/3}}{x-F(t)}, \tag{2.5.24b}$$

which is verified by Mathematica to solve (2.4.10). Again, this is the solution to the associated system of the original NLPDE. We check our solution for u in (2.1.2) with $n = 3, \lambda = 1$, giving us the equation

$$\frac{4050i3^{1/3}10^{2/3}(i+\sqrt{3})}{[x-F(t)]^7}\left(1 + \frac{x-F(t)}{|x-F(t)|}\right) + F''(t) = 0, \tag{2.5.25}$$

which is satisfied provided $x - F(t) < 0$ and $F''(t) = 0$. Thus, as it was for the $n = 1$ case, we have $F(t) = at + b$, which gives

$$u(x,t) = -\frac{30(-1)^{5/6}3^{2/3}10^{1/3}}{[x-at-b]^3} + a, \qquad x < at + b, \tag{2.5.26}$$

as a solution to (2.1.2) with $n = 3$, verified by Mathematica. Choosing the values $a = 1,\ b = 0$ in (2.5.26), we note $\mathrm{Re}(u(x,t)) = 1$; a plot of $\mathrm{Im}(u(x,t))$ is given in Figure 2.

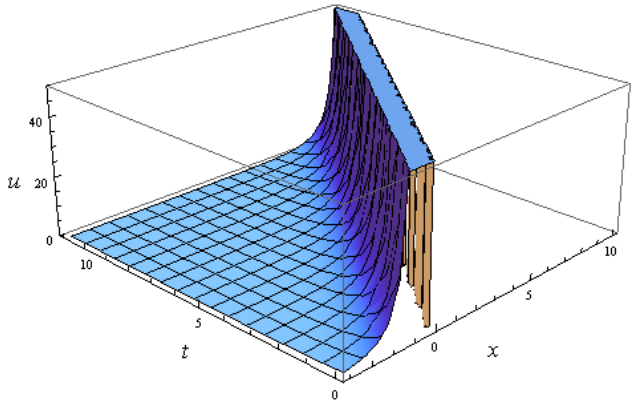


Figure 2: $\mathrm{Im}(u(x,t))$ of (2.5.26) with $a = 1,\ b = 0$

2.5.3 Application to $n = 4$ Case

Following the same procedure as for the $n = 1, 3$ cases, we truncate (2.5.8) at the constant term with $n = 4$,

$$u(x,t) = \frac{u_0}{\chi^4} + \frac{u_1}{\chi^3} + \frac{u_2}{\chi^2} + \frac{u_3}{\chi} + u_4, \tag{2.5.27a}$$

$$v(x,t) = \frac{v_0}{\chi} + v_1. \tag{2.5.27b}$$

We substitute (2.5.27) into (2.4.16) and use (2.5.3) to replace derivatives of χ. The $\mathcal{O}(\chi^{-10})$ term from (2.4.16a) and the $\mathcal{O}(\chi^{-5})$ term from (2.4.16b) gives

$$u_0 = 448(-7)^{2/3}2^{1/3}, \tag{2.5.28a}$$

$$v_0 = 2(-1)^{5/6}2^{2/3}7^{1/3}. \tag{2.5.28b}$$

Similarly, the $\mathcal{O}(\chi^{-9})$ and $\mathcal{O}(\chi^{-4})$ terms yield $u_1 = v_1 = 0$. The remaining $\chi-$coefficient equations become, from (2.4.16a)

$$\mathcal{O}(\chi^{-8}): \quad 5096(-1)^{2/3}2^{1/3}S - 7^{1/3}u_2 = 0, \tag{2.5.29a}$$

$$\mathcal{O}(\chi^{-7}): \quad 7^{1/3}u_3 + 784(-1)^{2/3}2^{1/3}S_x = 0, \tag{2.5.29b}$$

$$\mathcal{O}(\chi^{-6}): \quad 10\cdot 7^{1/3}C + 25088(-1)^{2/3}2^{1/3}S^2 + 392(-1)^{2/3}2^{1/3}S_{xx} - 5\cdot 7^{1/3}Su_2 - 10\cdot 7^{1/3}u_4 = 0, \tag{2.5.29c}$$

$$\mathcal{O}(\chi^{-5}): \quad 5\cdot 7^{1/3}Su_3 + 10\cdot 7^{1/3}C_x + 3528(-1)^{2/3}2^{1/3}SS_x = 0, \tag{2.5.29d}$$

$$\mathcal{O}(\chi^{-4}): \quad 7^{1/3}CS + 588(-1)^{2/3}2^{1/3}S^3 + 7^{1/3}C_{xx} - 7^{1/3}Su_4 = 0, \tag{2.5.29e}$$

and from (2.4.16b),

$$\mathcal{O}(\chi^{-3}): \quad 448(-7)^{2/3}2^{1/3}S + u_2 = 0, \tag{2.5.30a}$$

$$\mathcal{O}(\chi^{-2}): \quad u_3 - (u_2)_x = 0, \tag{2.5.30b}$$

$$\mathcal{O}(\chi^{-1}): \quad Su_2 - (u_3)_x = 0, \tag{2.5.30c}$$

$$\mathcal{O}(\chi^{0}): \quad Su_3 - 2(u_4)_x = 0. \tag{2.5.30d}$$

The equations (2.5.29a) and (2.5.30a) yield $u_2 = S = 0$, which gives $u_3 = 0$ from (2.5.29b). Then (2.5.29c) implies $u_4 = C$, and (2.5.29d) yields $C_x = (u_4)_x = 0$. Thus, similar to the $n = 1, 3$ cases, we have $C = f(t)$ and $\chi = x - F(t)$, where $F'(t) = f(t)$. From (2.5.27), we obtain the solution

$$u(x,t) = \frac{448(-7)^{2/3}2^{1/3}}{[x-F(t)]^4} + F'(t), \tag{2.5.31a}$$

$$v(x,t) = \frac{2(-1)^{5/6}2^{2/3}7^{1/3}}{x-F(t)}, \tag{2.5.31b}$$

which is verified by Mathematica to solve (2.4.16). Again, this is the solution to the associated system of the original NLPDE; we proceed to check our solution for u in (2.1.2) with $n = 4, \lambda = 1$, which yields the conditions $x - F(t) < 0$ and $F''(t) = 0$ for identity. Thus, as it was for the $n = 1, 3$ cases, we have $F(t) = at + b$, which gives

$$u(x,t) = \frac{448(-7)^{2/3}2^{1/3}}{[x-at-b]^4} + a, \qquad x < at + b, \tag{2.5.32}$$

as a solution to (2.1.2) with $n = 4$, verified by Mathematica. Choosing the values $a = 1,\ b = 0$ in (2.5.32), we note $\mathrm{Re}(u(x,t)) = 1$; a plot of $\mathrm{Im}(u(x,t))$ is given in Figure 3.

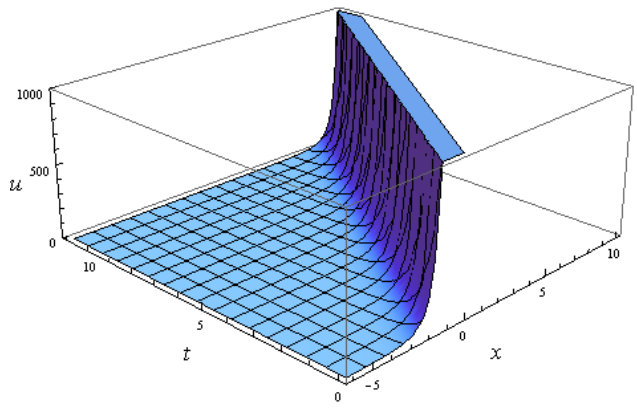


Figure 3: $\mathrm{Im}(u(x,t))$ of (2.5.32) with $a = 1,\ b = 0$

2.6 Homogeneous Balance Method

For the Homogeneous Balance method, Wang [33] proposed a special solution $u(x,t)$ of a NLPDE in x, t, could be expanded as a linear combination of various mixed derivatives of a function $f(\phi)$, where $\phi = \phi(x,t)$ is called a quasisolution if such an expansion exists. En-Gui et al. [31] generalized this method to investigate Bäcklund transformations and Lax Pairs, among other topics. They used only the highest order mixed derivative of $f(\phi)$ in the solution u, and collected all lower order derivative terms as a single function $u_1(x,t)$ to be determined. Here, we apply this technique to our NLPDE system in u, v.

Similar to a leading order analysis, to find the highest order mixed derivative term in the expansion, we assume

$$u(x,t) = \frac{\partial^{m+n} f(\phi(x,t))}{\partial t^m \partial x^n} = f^{(m+n)} \phi_t^m \phi_x^n + \mathcal{O}(\phi_t^{m-1} \phi_x^n) + \mathcal{O}(\phi_t^m \phi_x^{n-1}), \tag{2.6.1a}$$

$$v(x,y) = \frac{\partial^{p+q} g(\phi(x,t))}{\partial t^p \partial x^q} = g^{(p+q)} \phi_t^p \phi_x^q + \mathcal{O}(\phi_t^{p-1} \phi_x^q) + \mathcal{O}(\phi_t^p \phi_x^{q-1}). \tag{2.6.1b}$$

Substitution into the NLPDE system, the highest nonlinear terms are balanced in terms of the largest powers of ϕ_t and ϕ_x in each equation, which give a system of equations in m, n, p, q to be solved. Using these values and balancing the corresponding coefficients, we also get a system of nonlinear ODEs to be solved for f, g, most often resulting in logarithm functions. These solutions are used to eliminate the highest nonlinear

terms, however the remaining f, g and their derivatives are kept in symbolic form; otherwise, we would obtain an equivalent expression for u, v as we did for the regular Painlevé analysis. The nonlinear terms in f, g and their various derivatives can be replaced with higher-order derivative (linear) terms as a result of their logarithmic forms. Equating each coefficient of the now linear derivatives of f, g to zero, we obtain a system of equations in ϕ and its partial derivatives that must be satisfied, which we attempt to linearize into a Lax Pair.

2.6.1 Lax-Type Equations for $n = 1$ Case

Even though the $n = 1$ case of our $\mathcal{PT}$-symmetric KdV, (2.4.1), failed the Painlevé Test for integrability, we apply the homogeneous balance method to attempt to find a (linear) Lax Pair for the system. For this purpose, we use generalized form (2.2.4), with $n = 1$ and arbitrary λ

$$2vv_t + 2\lambda uvv_x - i\lambda v^4 + v_{3x} = 0, \tag{2.6.2a}$$

$$v^2 - iu_x = 0. \tag{2.6.2b}$$

Following the homogeneous balance method, we substitute the expressions

$$u = \frac{\partial^{m+n} f(\phi(x,t))}{\partial t^m \partial x^n} = f^{(m+n)}\phi_t^m \phi_x^n + \mathcal{O}(\phi_t^{m-1}\phi_x^n) + \mathcal{O}(\phi_t^m \phi_x^{n-1}), \tag{2.6.3a}$$

$$v = \frac{\partial^{p+q} g(\phi(x,t))}{\partial t^p \partial x^q} = g^{(p+q)}\phi_t^p \phi_x^q + \mathcal{O}(\phi_t^{p-1}\phi_x^q) + \mathcal{O}(\phi_t^p \phi_x^{q-1}). \tag{2.6.3b}$$

into (2.6.2). From (2.6.2a), this gives

$$\begin{aligned}
&2\left[g^{(p+q)}\phi_t^p\phi_x^q + \mathcal{O}(\phi_t^{p-1}\phi_x^q) + \mathcal{O}(\phi_t^p\phi_x^{q-1})\right]\left[g^{(p+q+1)}\phi_t^{p+1}\phi_x^q + \mathcal{O}(\phi_t^p\phi_x^q) + \mathcal{O}(\phi_t^{p+1}\phi_x^{q-1})\right] \\
&\quad + 2\lambda\left[f^{(m+n)}\phi_t^m\phi_x^n + \mathcal{O}(\phi_t^{m-1}\phi_x^n) + \mathcal{O}(\phi_t^m\phi_x^{n-1})\right]\left[g^{(p+q)}\phi_t^p\phi_x^q + \mathcal{O}(\phi_t^{p-1}\phi_x^q) + \mathcal{O}(\phi_t^p\phi_x^{q-1})\right] \\
&\qquad\qquad \cdot\left[g^{(p+q+1)}\phi_t^p\phi_x^{q+1} + \mathcal{O}(\phi_t^{p-1}\phi_x^{q+1}) + \mathcal{O}(\phi_t^p\phi_x^q)\right] \\
&\quad - i\lambda\left[g^{(p+q)}\phi_t^p\phi_x^q + \mathcal{O}(\phi_t^{p-1}\phi_x^q) + \mathcal{O}(\phi_t^p\phi_x^{q-1})\right]^4 \\
&\quad + \left[g^{(p+q+3)}\phi_t^p\phi_x^{q+3} + \mathcal{O}(\phi_t^{p-1}\phi_x^{q+3}) + \mathcal{O}(\phi_t^p\phi_x^{q+2})\right] = 0, \\
\Rightarrow &\mathcal{O}(\phi_t^{2p+1}\phi_x^{2q}) + 2\lambda f^{(m+n)}g^{(p+q)}g^{(p+q+1)}\phi_t^{m+2p}\phi_x^{n+2q+1} + \mathcal{O}(\phi_t^{m+2p-1}\phi_x^{n+2q+1}) + \mathcal{O}(\phi_t^{m+2p}\phi_x^{n+2q}) \\
&\quad - i\lambda\left[g^{(p+q)}\right]^4\phi_t^{4p}\phi_x^{4q} + \mathcal{O}(\phi_t^{4p-1}\phi_x^{4q}) + \mathcal{O}(\phi_t^{4p}\phi_x^{4q-1}) \\
&\quad + g^{(p+q+3)}\phi_t^p\phi_x^{q+3} + \mathcal{O}(\phi_t^{p-1}\phi_x^{q+3}) + \mathcal{O}(\phi_t^p\phi_x^{q+2}) = 0.
\end{aligned} \tag{2.6.4}$$

From (2.3.5b), we obtain

$$\left[g^{(p+q)}\phi_t^p\phi_x^q + \mathcal{O}(\phi_t^{p-1}\phi_x^q) + \mathcal{O}(\phi_t^p\phi_x^{q-1})\right]^2 - i\left[f^{(m+n+1)}\phi_t^m\phi_x^{n+1} + \mathcal{O}(\phi_t^{m-1}\phi_x^{n+1}) + \mathcal{O}(\phi_t^m\phi_x^n)\right] = 0,$$
$$\Rightarrow \left[g^{(p+q)}\right]^2\phi_t^{2p}\phi_x^{2q} + \mathcal{O}(\phi_t^{2p-1}\phi_x^{2q}) + \mathcal{O}(\phi_t^{2p}\phi_x^{2q-1}) - if^{(m+n+1)}\phi_t^m\phi_x^{n+1} + \mathcal{O}(\phi_t^{m-1}\phi_x^{n+1}) + \mathcal{O}(\phi_t^m\phi_x^n) = 0. \tag{2.6.5}$$

Next, we balance the powers of ϕ_t, ϕ_x, respectively, from the highest-order nonlinear terms. That is, those with combined powers in ϕ_t, ϕ_x of $m+n+2p+2q+1$, $4p+4q$, and $p+q+3$ from (2.6.4); $2p+2q$ and $m+n+1$ from (2.6.5). We require

$$m + 2p = 4p = p, \tag{2.6.6a}$$
$$n + 2q + 1 = 4q = q + 3, \tag{2.6.6b}$$
$$2p = m, \tag{2.6.6c}$$
$$2q = n + 1. \tag{2.6.6d}$$

Thus, $m = p = 0$ and $n = q = 1$. Then (2.6.3) becomes

$$u = \frac{\partial f(\phi(x,t))}{\partial x} = f'\phi_x, \tag{2.6.7a}$$
$$v = \frac{\partial g(\phi(x,t))}{\partial x} = g'\phi_x. \tag{2.6.7b}$$

Further, using the values of m, n, p, q (or substitution of (2.6.20) back into the NLPDE system), we equate coefficients of the highest-order ϕ-derivative terms: ϕ_x^4 from (2.6.4) and ϕ_x^2 from (2.6.5). We obtain a system of nonlinear ODEs in f, g to be solved

$$2\lambda f'g'g'' - i\lambda[g']^4 + g^{(4)} = 0, \tag{2.6.8a}$$
$$[g']^2 - if^{(2)} = 0. \tag{2.6.8b}$$

Assuming f, g are of logarithmic form, we substitute

$$f(\phi) = f_0 \ln\phi, \tag{2.6.9a}$$
$$g(\phi) = g_0 \ln\phi, \tag{2.6.9b}$$

into (2.6.8), obtaining

$$g_0\left[-2\lambda f_0 g_0 - i\lambda g_0^3 - 6\right]\phi^{-4} = 0, \tag{2.6.10a}$$

$$\left[g_0^2 + i f_0\right]\phi^{-2} = 0. \tag{2.6.10b}$$

Solving this, now algebraic, system for f_0, g_0 yields

$$f_0 = \left[\frac{2i}{\lambda}\right]^{2/3}, \tag{2.6.11a}$$

$$g_0 = \left[\frac{2i}{\lambda}\right]^{1/3}. \tag{2.6.11b}$$

Thus,

$$f(\phi) = \left[\frac{2i}{\lambda}\right]^{2/3}\ln\phi, \tag{2.6.12a}$$

$$g(\phi) = \left[\frac{2i}{\lambda}\right]^{1/3}\ln\phi. \tag{2.6.12b}$$

We note that $g = (\lambda/2i)^{1/3} f$, and so we may write

$$g^{(k)}(\phi) = \left[\frac{\lambda}{2i}\right]^{1/3} f^{(k)}(\phi), \qquad k \in \mathbb{N}, \tag{2.6.13}$$

Now, we use the full expansion of the solutions

$$u = \frac{\partial f(\phi(x,t))}{\partial x} + u_1(x,t) = f'\phi_x + u_1(x,t), \tag{2.6.14a}$$

$$v = \frac{\partial g(\phi(x,t))}{\partial x} + v_1(x,t) = \left[\frac{\lambda}{2i}\right]^{1/3} f'\phi_x + v_1(x,t). \tag{2.6.14b}$$

We note that using the form of $f = f_0 \ln\phi$, the expression for u would yield $f'\phi_x + u_1 = f_0\phi_x/\phi + u_1$, which is a similar expansion to (2.4.4) used in the regular Painlevé analysis; we keep $f^{(k)}$ symbolically throughout, so to obtain a different collection of linearly independent terms. Substitution of (2.6.14) in (2.6.2), and making use of (2.6.8) with (2.6.13) to remove the ϕ_x^4 and ϕ_x^2 terms that arise from (2.6.2a) and (2.6.2b) respectively, we obtain the equations given in (A.8) and (A.9).

Now, given the form of $f = f_0 \ln\phi$, we can replace the nonlinear f-derivative terms with linear higher-order derivatives of f. The nonlinear derivative terms from (A.8) are $(f')^2$, $f'f''$, and $(f')^3$. These can be

written as follows (keeping f_0 for brevity and generality),

$$(f')^2 = \frac{f_0^2}{\phi^2} = -f_0 \cdot \frac{-f_0}{\phi^2} = -f_0 f'', \tag{2.6.15a}$$

$$f'f'' = \frac{f_0}{\phi} \cdot \frac{-f_0}{\phi^2} = -\frac{f_0}{2} \cdot \frac{2f_0}{\phi^3} = -\frac{f_0}{2} f^{(3)}, \tag{2.6.15b}$$

$$(f')^3 = \frac{f_0^3}{\phi^3} = \frac{f_0^2}{2} \cdot \frac{2f_0}{\phi^3} = -\frac{f_0^2}{2} f^{(3)}. \tag{2.6.15c}$$

Using these substitutions and our known value of f_0 from (2.6.11a), the equations (A.8) and (A.9), become (A.10) and (A.11), respectively, which are linear in f-derivatives.

The coefficients of each $f^{(k)}$ in (A.10) and (A.11), equated to zero, give the following Lax-type equations for ϕ,

$$\begin{aligned} -2 \cdot 2^{2/3} i^{2/3} \lambda^{4/3} v_1^3 - i 2^{2/3} i^{2/3} \lambda^{1/3} (v_1)_t - i 2^{2/3} i^{2/3} \lambda^{4/3} u_1 (v_1)_x + 2\lambda v_1 (v_1)_x \\ - \frac{i 2^{2/3} i^{2/3} \lambda^{1/3} v_1 \phi_{xt}}{\phi_x} - \frac{i 2^{2/3} i^{2/3} \lambda^{4/3} u_1 v_1 \phi_{xx}}{\phi_x} - \frac{i i^{2/3} \lambda^{1/3} \phi_{4x}}{2^{1/3} \phi_x} = 0, \end{aligned} \tag{2.6.16a}$$

$$\begin{aligned} -2^{2/3} i^{1/3} \lambda^{4/3} u_1 v_1 + 6i\lambda v_1^2 - \frac{2^{2/3} i^{1/3} \lambda^{1/3} v_1 \phi_t}{\phi_x} - 2 \cdot 2^{1/3} i^{1/3} \lambda^{2/3} (v_1)_x - \frac{2\phi_{xt}}{\phi_x} \\ - \frac{2\lambda u_1 \phi_{xx}}{\phi_x} - \frac{2 \cdot 2^{1/3} i^{1/3} \lambda^{2/3} v_1 \phi_{xx}}{\phi_x} - \frac{3 i^{1/3} \lambda^{1/3} \phi_{xx}^2}{2^{1/3} \phi_x^2} - \frac{2 \cdot 2^{2/3} i^{1/3} \lambda^{1/3} \phi_{3x}}{\phi_x} = 0, \end{aligned} \tag{2.6.16b}$$

$$-\lambda u_1 - (1+2i) 2^{1/3} i^{1/3} \lambda^{2/3} v_1 - \frac{\phi_t}{\phi_x} + \frac{(1-3i) 2^{2/3} i^{2/3} \lambda^{1/3} \phi_{xx}}{\phi_x} = 0, \tag{2.6.16c}$$

$$-2^{2/3} i^{1/3} \lambda^{1/3} v_1 - \frac{i\phi_{xx}}{\phi_x} = 0. \tag{2.6.16d}$$

We note that the terms independent of f-derivatives of (A.10) and (A.11) is simply our original $n = 1$ system (2.6.2) in u_1, v_1. Thus, (2.6.14) gives an auto-BT for (2.6.2), where ϕ satisfies the Lax-type equations (2.6.16). However, since the $n = 1$ system did not pass the Painlevé Test, we do not expect these equations to behave like a Lax Pair. Indeed, attempts to reduce (2.6.16) and recover our original system have not been successful.

2.6.2 Near-Lax Pair for $n = 3$ Case

For the purposes of searching for a Lax Pair, we use the (2.3.5) form of the $n = 3$ case, with an arbitrary λ. Following the homogeneous balance method, we substitute (2.6.3) into (2.3.5). From (2.3.5a), this gives

$$
\begin{aligned}
&4\left[g^{(p+q)}\phi_t^p\phi_x^q+\mathcal{O}(\phi_t^{p-1}\phi_x^q)+\mathcal{O}(\phi_t^p\phi_x^{q-1})\right]^3\left[g^{(p+q+1)}\phi_t^{p+1}\phi_x^q+\mathcal{O}(\phi_t^p\phi_x^q)+\mathcal{O}(\phi_t^{p+1}\phi_x^{q-1})\right]\\
&\quad+4\lambda\left[f^{(m+n)}\phi_t^m\phi_x^n+\mathcal{O}(\phi_t^{m-1}\phi_x^n)+\mathcal{O}(\phi_t^m\phi_x^{n-1})\right]\left[g^{(p+q)}\phi_t^p\phi_x^q+\mathcal{O}(\phi_t^{p-1}\phi_x^q)+\mathcal{O}(\phi_t^p\phi_x^{q-1})\right]^3\\
&\qquad\cdot\left[g^{(p+q+1)}\phi_t^p\phi_x^{q+1}+\mathcal{O}(\phi_t^{p-1}\phi_x^{q+1})+\mathcal{O}(\phi_t^p\phi_x^q)\right]\\
&\quad-i\lambda\left[g^{(p+q)}\phi_t^p\phi_x^q+\mathcal{O}(\phi_t^{p-1}\phi_x^q)+\mathcal{O}(\phi_t^p\phi_x^{q-1})\right]^8\\
&\quad+\frac{\partial^3}{\partial x^3}\left[g^{(p+q)}\phi_t^p\phi_x^q+\mathcal{O}(\phi_t^{p-1}\phi_x^q)+\mathcal{O}(\phi_t^p\phi_x^{q-1})\right]^5=0,\\
\Rightarrow&\mathcal{O}(\phi_t^{4p+1}\phi_x^{4q})+4\lambda f^{(m+n)}\left[g^{(p+q)}\right]^3g^{(p+q+1)}\phi_t^{m+4p}\phi_x^{n+4q+1}+\mathcal{O}(\phi_t^{m+4p-1}\phi_x^{n+4q+1})\\
&\quad+\mathcal{O}(\phi_t^{m+4p}\phi_x^{n+4q})-i\lambda\left[g^{(p+q)}\right]^8\phi_t^{8p}\phi_x^{8q}+\mathcal{O}(\phi_t^{8p-1}\phi_x^{8q})+\mathcal{O}(\phi_t^{8p}\phi_x^{8q-1})\\
&\quad+\left[60\left[g^{(p+q)}\right]^2\left[g^{(p+q+1)}\right]^3+60\left[g^{(p+q)}\right]^3g^{(p+q+1)}g^{(p+q+2)}+5\left[g^{(p+q)}\right]^4g^{(p+q+3)}\right]\phi_t^{5p}\phi_x^{5q+3}\\
&\quad+\mathcal{O}(\phi_t^{5p-1}\phi_x^{5q+3})+\mathcal{O}(\phi_t^{5p}\phi_x^{5q+2})=0.
\end{aligned}
\tag{2.6.17}
$$

From (2.3.5b), we obtain

$$
\begin{aligned}
&\left[g^{(p+q)}\phi_t^p\phi_x^q+\mathcal{O}(\phi_t^{p-1}\phi_x^q)+\mathcal{O}(\phi_t^p\phi_x^{q-1})\right]^4\\
&\qquad-i\left[f^{(m+n+1)}\phi_t^m\phi_x^{n+1}+\mathcal{O}(\phi_t^{m-1}\phi_x^{n+1})+\mathcal{O}(\phi_t^m\phi_x^n)\right]=0,\\
\Rightarrow&\left[g^{(p+q)}\right]^4\phi_t^{4p}\phi_x^{4q}+\mathcal{O}(\phi_t^{4p-1}\phi_x^{4q})+\mathcal{O}(\phi_t^{4p}\phi_x^{4q-1})\\
&\qquad-if^{(m+n+1)}\phi_t^m\phi_x^{n+1}+\mathcal{O}(\phi_t^{m-1}\phi_x^{n+1})+\mathcal{O}(\phi_t^m\phi_x^n)=0.
\end{aligned}
\tag{2.6.18}
$$

Next, we balance the powers of ϕ_t, ϕ_x, respectively, from the highest-order nonlinear terms. That is, those with combined powers in ϕ_t, ϕ_x of $m + n + 4p + 4q + 1$, $8p + 8q$, and $5p + 5q + 3$ from (2.6.17), and

$4p+4q$ and $m+n+1$ from (2.6.18). We obtain the conditions

$$m+4p=8p=5p, \tag{2.6.19a}$$

$$n+4q+1=8q=5q+3, \tag{2.6.19b}$$

$$4p=m, \tag{2.6.19c}$$

$$4q=n+1. \tag{2.6.19d}$$

Thus, $m=p=0$, $n=3$ and $q=1$. Then (2.6.3) becomes

$$u=\frac{\partial^3 f(\phi(x,t))}{\partial x^3}=f^{(3)}\phi_x^3+3f''\phi_x\phi_{xx}+f'\phi_{3x}, \tag{2.6.20a}$$

$$v=\frac{\partial g(\phi(x,t))}{\partial x}=g'\phi_x. \tag{2.6.20b}$$

Further, using the values of m,n,p,q (or substitution of (2.6.20) back into the NLPDE system), we equate coefficients of the highest-order ϕ-derivative terms: ϕ_x^8 from (2.6.17) and ϕ_x^4 from (2.6.18). We obtain a system of nonlinear ODEs in f,g to be solved

$$\left[g'\right]^2\left[4\lambda f^{(3)}g'g''-i\lambda\left[g'\right]^6+60\left[g''\right]^3+60g'g''g^{(3)}+5\left[g'\right]^2g^{(4)}\right]=0, \tag{2.6.21a}$$

$$\left[g'\right]^4-if^{(4)}=0. \tag{2.6.21b}$$

Assuming f,g are of logarithmic form, we substitute

$$f(\phi)=f_0\ln\phi, \tag{2.6.22a}$$

$$g(\phi)=g_0\ln\phi, \tag{2.6.22b}$$

into (2.6.21), obtaining

$$g_0^4\left[8\lambda f_0+i\lambda g_0^4+210g_0\right]\phi^{-8}=0, \tag{2.6.23a}$$

$$\left[g_0^4+6if_0\right]\phi^{-4}=0. \tag{2.6.23b}$$

Solving this, now algebraic, system for f_0, g_0 yields

$$f_0 = \frac{15i3^{2/3}10^{1/3}}{\lambda^{4/3}}, \tag{2.6.24a}$$

$$g_0 = -\frac{i3^{2/3}10^{1/3}}{\lambda^{1/3}}. \tag{2.6.24b}$$

Thus,

$$f(\phi) = \frac{15i3^{2/3}10^{1/3}}{\lambda^{4/3}} \ln \phi, \tag{2.6.25a}$$

$$g(\phi) = -\frac{i3^{2/3}10^{1/3}}{\lambda^{1/3}} \ln \phi. \tag{2.6.25b}$$

We note that $g = -(\lambda/15)f$, and so we may write

$$g^{(k)}(\phi) = -\frac{\lambda}{15} f^{(k)}(\phi), \qquad k \in \mathbb{N}, \tag{2.6.26}$$

Now, we use the full expansion of the solutions

$$u = \frac{\partial^3 f(\phi(x,t))}{\partial x^3} + u_1(x,t) = f^{(3)}\phi_x^3 + 3f''\phi_x\phi_{xx} + f'\phi_{3x} + u_1(x,t), \tag{2.6.27a}$$

$$v = \frac{\partial g(\phi(x,t))}{\partial x} + v_1(x,t) = -\frac{\lambda}{15} f'\phi_x + v_1(x,t). \tag{2.6.27b}$$

Substitution of these in (2.3.5), and making use of (2.6.21) with (2.6.26) to remove the ϕ_x^8 and ϕ_x^4 terms that arise from (2.3.5a) and (2.3.5b) respectively, we obtain the equations given in (A.12) and (A.13).

Now, given the form of $f = f_0 \ln \phi$, we can replace the nonlinear f-derivative terms with linear higher-order derivatives of f, as we did for the $n = 1$ case. A few of the nonlinear derivative terms from (A.12) are $(f')^2$, $f'f''$, and $(f')^3$, which are rewritten in (2.6.15). Other nonlinear f-derivative terms can be rewritten in a similar manner. Using these substitutions and our known value of f_0 from (2.6.24a), the equations (A.12) and (A.13) become (A.14) and (A.15), respectively, which are linear in f-derivatives.

The coefficients of each $f^{(k)}$ in (A.14) and (A.15), equated to zero, give Lax-type equations for ϕ defined in (A.16). We note that the terms independent of f-derivatives of (A.14) and (A.15) is simply our original $n = 3$ system (2.3.5) in u_1, v_1. Thus, (2.6.27) gives an auto-BT for (2.3.5), where f is given by (2.6.25a) and ϕ satisfies the Lax-type equations (A.16), which we will attempt to linearize into a Lax Pair. We note that (A.16g) and (A.16j) are redundant, and so future analysis will include only one of these.

2.6.2.1 Reduction of Lax-Type Equation

Given the forms of the ϕ-derivative terms in the Lax-type equations (A.16), we set

$$\phi_t = \beta_1 \phi_x, \tag{2.6.28a}$$

$$\phi_{xt} = \beta_2 \phi_x, \tag{2.6.28b}$$

$$\phi_{xx} = \beta_3 \phi_x, \tag{2.6.28c}$$

$$\phi_{3x} = \beta_4 \phi_x, \tag{2.6.28d}$$

$$\phi_{4x} = \beta_5 \phi_x, \tag{2.6.28e}$$

$$\phi_{xx}^2 = \beta_6 \phi_x^2, \tag{2.6.28f}$$

$$\phi_{xx}^3 = \beta_7 \phi_x^3. \tag{2.6.28g}$$

While we will ultimately require the compatibility conditions

$$(\phi_{xx})^2 = \phi_{xx}^2 \quad \Rightarrow \quad \beta_6 = \beta_3^2, \tag{2.6.29a}$$

$$(\phi_{xx})^3 = \phi_{xx}^3 \quad \Rightarrow \quad \beta_7 = \beta_3^3, \tag{2.6.29b}$$

$$(\phi_{xx})_x = \phi_{3x} \quad \Rightarrow \quad (\beta_3)_x + \beta_3^2 = \beta_4, \tag{2.6.29c}$$

$$(\phi_{3x})_x = \phi_{4x} \quad \Rightarrow \quad (\beta_4)_x + \beta_3 \beta_4 = \beta_5, \tag{2.6.29d}$$

$$(\phi_t)_x = \phi_{xt} \quad \Rightarrow \quad (\beta_1)_x + \beta_1 \beta_3 = \beta_2, \tag{2.6.29e}$$

for the purpose of rewriting the Lax-type equations, we will begin by using the distinct β_j's. Making the substitution (2.6.28) into (A.16), the Lax-type equations become (A.17), now without explicit ϕ-dependence. Solving (A.17g) for β_3 and (A.17h) for β_5, we obtain

$$\beta_3 = -\frac{i 2^{2/3} \lambda^{1/3} v_1}{3^{2/3} 5^{1/3}}, \tag{2.6.30a}$$

$$\beta_5 = \frac{4}{15} i \lambda v_1^3. \tag{2.6.30b}$$

Using these, we may then solve (A.17a) for β_2, and (A.17f), (A.17i) for β_4, β_6, giving us

$$\begin{aligned}\beta_2 = &\frac{i2^{2/3}\lambda^{4/3}u_1v_1}{3^{2/3}5^{1/3}} + \frac{5}{3}i\lambda v_1^4 - \frac{3(v_1)_t}{v_1} - \frac{3\lambda u_1(v_1)_x}{v_1} \\ &+ 3i3^{1/3}10^{2/3}\lambda^{1/3}(v_1)_x^2 - \frac{30(v_1)_x^3}{v_1^2} + i3^{1/3}10^{2/3}\lambda^{1/3}v_1(v_1)_{xx} \\ &- \frac{45(v_1)_x(v_1)_{xx}}{v_1} - 5(v_1)_{3x}, \end{aligned} \tag{2.6.31a}$$

$$\beta_4 = -\frac{7\lambda^{2/3}v_1^2}{4\cdot 3^{1/3}10^{2/3}} - \frac{i3^{1/3}\lambda^{1/3}(v_1)_x}{4\cdot 10^1/3}, \tag{2.6.31b}$$

$$\beta_6 = -\frac{5^{1/3}\lambda^{2/3}v_1^2}{3\cdot 2^{2/3}3^{1/3}} + \frac{i\lambda^{1/3}(v_1)_x}{3^{2/3}10^{1/3}}. \tag{2.6.31c}$$

Next, we solve (A.17d), (A.17e) for β_1, β_7, which yields our final β_j's

$$\begin{aligned}\beta_1 = &-\lambda u_1 - \frac{115^{1/3}\lambda^{2/3}v_1^3}{2\cdot 2^{2/3}3^{1/3}} + \frac{3i3^{2/3}10^{1/3}(v_1)_t}{\lambda^{1/3}v_1^2} + \frac{3i3^{2/3}10^{1/3}\lambda^{2/3}u_1(v_1)_x}{v_1^2} \\ &+ \frac{15i3^{1/3}5^{2/3}\lambda^{1/3}v_1(v_1)_x}{2^{1/3}} + \frac{90(v_1)_x^2}{v_1} + \frac{30i3^{2/3}10^{1/3}(v_1)_x^3}{\lambda^{1/3}v_1^3} \\ &- 15(v_1)_{xx} + \frac{45i3^{2/3}10^{1/3}(v_1)_x(v_1)_{xx}}{\lambda^{1/3}v_1^2} + \frac{15i3^{2/3}5^{1/3}(v_1)_{3x}}{2\cdot 2^{2/3}\lambda^{1/3}v_1}, \end{aligned} \tag{2.6.32a}$$

$$\begin{aligned}\beta_7 = &\frac{29}{180}i\lambda v_1^3 - \frac{(v_1)_t}{5v_1^2} - \frac{\lambda u_1(v_1)_x}{5v_1^2} - \frac{\lambda^{2/3}v_1(v_1)_x}{3^{1/3}10^{2/3}} \\ &+ \frac{i3^{1/3}2^{2/3}\lambda^{1/3}(v_1)_x^2}{5^{1/3}v_1} - \frac{2(v_1)_x^3}{v_1^3} - \frac{3(v_1)_x(v_1)_{xx}}{v_1^2} - \frac{(v_1)_{3x}}{4v_1}. \end{aligned} \tag{2.6.32b}$$

The remaining two equations, (A.17b) and (A.17c), become identically the first field equation from our original system, (2.3.5a) in u_1, v_1, after utilizing the appropriate β_j's. That is,

$$4v_1(v_1)_t + 4\lambda u_1 v_1(v_1)_x - i\lambda v_1^6 + v_1^{-2}\left[v_1^5\right]_{3x} = 0, \tag{2.6.33}$$

We now return to our compatibility conditions (2.6.29). To simplify some of the following calculations, we solve (2.6.33) for u_1 to get

$$u_1 = \frac{iv_1^5}{4(v_1)_x} - \frac{(v_1)_t}{\lambda(v_1)_x} - \frac{15(v_1)_x^2}{\lambda v_1} - \frac{15(v_1)_{xx}}{\lambda} - \frac{5v_1(v_1)_{3x}}{4\lambda(v_1)_x}. \tag{2.6.34}$$

Making use of the β_j's previously found, the first compatibility condition (2.6.29a) yields

$$i3^{1/3}\lambda^{2/3}v_1^2 + 3\cdot 10^{1/3}\lambda^{1/3}(v_1)_x = 0,$$
$$\Rightarrow (v_1)_x = -\frac{i\lambda^{1/3}v_1^2}{3^{2/3}10^{1/3}}. \tag{2.6.35}$$

Thus, we have an additional requirement that will need to accompany the near-Lax Pair. We note that solving the above PDE for v_1 and letting $\lambda = 1$, we obtain

$$v_1 = -\frac{i3^{2/3}10^{1/3}}{x - F(t)} \tag{2.6.36}$$

which is the algebraic form similar to the solution for v obtained during Invariant Painlevé analysis, (2.5.24). Making use of the expression for $(v_1)_x$, and the expression for u_1 (2.6.34), the second compatibility condition (2.6.29b) is satisfied. Further substitution of $(v_1)_x$ (and $(v_1)_{xx}$) also satisfies the third and fourth equations (2.6.29c),(2.6.29d). The final compatibility equation, upon substitution of u_1 and recursive applications of $(v_1)_x$ yields

$$v_1^4 - i(u_1)_x = 0, \tag{2.6.37}$$

which is our second field equation of the original system (2.3.5b) in u_1, v_1. Thus, we have recovered our original system from our Lax-type Equations. Using $(v_1)_x$, the β_j's become

$$\beta_1 = \frac{\phi_t}{\phi_x} = 2\lambda u_1 - \frac{2\cdot 10^{1/3}\lambda^{2/3}v_1^3}{3^{1/3}} + \frac{3i3^{2/3}10^{1/3}(v_1)_t}{\lambda^{1/3}v_1^2}, \tag{2.6.38a}$$
$$\beta_2 = \frac{\phi_{xt}}{\phi_x} = \frac{i5^{2/3}\lambda^{4/3}u_1v_1}{2^{1/3}3^{2/3}} - \frac{5}{3}i\lambda v_1^4 - \frac{3(v_1)_t}{v_1}, \tag{2.6.38b}$$
$$\beta_3 = \frac{\phi_{xx}}{\phi_x} = -\frac{i2^{2/3}\lambda^{1/3}v_1}{3^{2/3}5^{1/3}}, \tag{2.6.38c}$$
$$\beta_4 = \frac{\phi_{3x}}{\phi_x} = -\frac{2^{1/3}\lambda^{2/3}v_1^2}{3^{1/3}5^{2/3}}, \tag{2.6.38d}$$
$$\beta_5 = \frac{\phi_{4x}}{\phi_x} = \frac{4}{15}i\lambda v_1^3, \tag{2.6.38e}$$
$$\beta_6 = \frac{\phi_{xx}^2}{\phi_x^2} = -\frac{2\cdot 2^{1/3}\lambda^{2/3}v_1^2}{3\cdot 3^{1/3}5^{2/3}}, \tag{2.6.38f}$$
$$\beta_7 = \frac{\phi_{xx}^3}{\phi_x^3} = \frac{i\lambda^{4/3}u_1}{5\cdot 3^{2/3}10^{1/3}} + \frac{1}{45}i\lambda v_1^3 - \frac{(v_1)_t}{5v_1^2}. \tag{2.6.38g}$$

Specifically, we consider $\beta_2 = \phi_{xt}/\phi_x$ to see if it can be integrated. Upon further substitution of the

expression for u_1, and subsequently $(v_1)_x$, we obtain the equation

$$\begin{aligned} \frac{\phi_{xt}}{\phi_x} &= \frac{2(v_1)_t}{v_1}, \\ \Rightarrow \frac{\partial}{\partial t}\ln(\phi_x) &= \frac{\partial}{\partial t} 2\ln(v_1) \\ \Rightarrow \phi_x &= \alpha(t) v_1^2, \end{aligned} \tag{2.6.39}$$

where $\alpha(t)$ is an arbitrary function. Combining this with our expression for $\phi_t = \beta_1 \phi_x$ from (2.6.38a) and removing the subscripts on u, v, our preliminary near-Lax Pair to consider is

$$\phi_x = \alpha(t) v^2, \tag{2.6.40a}$$

$$\phi_t = \left[2\lambda u - \frac{2 \cdot 10^{1/3} \lambda^{2/3} v^3}{3^{1/3}} + \frac{3i 3^{2/3} 10^{1/3} v_t}{\lambda^{1/3} v^2} \right] \phi_x, \tag{2.6.40b}$$

with

$$v_x = -\frac{i\lambda^{1/3} v^2}{3^{2/3} 10^{1/3}} \tag{2.6.40c}$$

Now checking the cross-derivative condition $(\phi_x)_t = (\phi_t)_x$, we obtain

$$v^2 \alpha'(t) - 4\alpha(t) v v_t - 2\lambda \alpha(t) v^2 u_x - 4\lambda \alpha(t) u v v_x + \frac{10 \cdot 10^{1/3} \lambda^{2/3} \alpha(t) v^4 v_x}{3^{1/3}} = 0, \tag{2.6.41}$$

where we have also utilized v_{xt} from taking the $t-$partial derivative of (2.6.40c). For the above condition to yield our original system, we further require $\alpha'(t) = 0$, thus $\alpha(t) = \alpha$. Now, the remaining terms can be algebraically manipulated to expose both field equations as compatibility conditions, although this requires some prior knowledge of the types of terms that appear in the NLPDE system. The terms are identically zero when u, v satisfy the original system (2.3.5), and after further use of v_x. Therefore, we obtain

$$\phi_x = \alpha v^2, \tag{2.6.42a}$$

$$\phi_t = \left[2\lambda u - \frac{2 \cdot 10^{1/3} \lambda^{2/3} v^3}{3^{1/3}} + \frac{3i 3^{2/3} 10^{1/3} v_t}{\lambda^{1/3} v^2} \right] \phi_x, \tag{2.6.42b}$$

as a near-Lax Pair with spectral parameter α. That is, it generates our original system, but only with the additional condition on v_x given in (2.6.40c), as well as some prior knowledge about our NLPDE system. For partially-integrable systems, i.e. systems admitting some degree of integration on analytic solutions, these 'near-Lax Pairs' have been briefly discussed for some nearly-integrable ODE and PDE systems in [35].

CHAPTER 3: $\mathcal{PT}$-SYMMETRIC BURGERS EQUATION

We now apply the methods of the previous chapter to the $\mathcal{PT}$-symmetric Burgers [29] equation

$$u_t + u^m u_x - i u_{xx} = 0. \tag{3.0.1}$$

Here, we want to find values of m for which the equation is integrable.

3.1 Leading Order and Resonance Analysis

Starting with a leading order analysis to determine the behavior of the solution at the singular manifold, we make the ansatz

$$u(x,t) = u_0 \phi^{-n}, \tag{3.1.1}$$

where we require $n \in \mathbb{N}$; n and $u_0(x,t)$ are to be determined, and $\phi(x,t) = 0$ is the location of the singular manifold. Using this in (3.0.1), we have

$$\mathcal{O}(\phi^{-n-1}) + \left[u_0\phi^{-n}\right]^m \left[-n u_0 \phi^{-n-1}\phi_x + \mathcal{O}(\phi^{-n})\right] - i\left[(-n)(-n-1)u_0\phi^{-n-2}\phi_x^2 + \mathcal{O}(\phi^{-n-1})\right] = 0. \tag{3.1.2}$$

Balancing the powers of ϕ in the most singular terms, we require $-mn - n - 1 = -n - 2$. Thus, the values of m we can consider for integrability are

$$m = \frac{1}{n}, \qquad n \in \mathbb{N}, \tag{3.1.3}$$

Now balancing the coefficients using this value for m, we also require

$$\begin{aligned} -n u_0^{1/n+1}\phi_x &= -in(n+1)u_0\phi_x^2, \\ \Rightarrow u_0^{1/n} &= -i(n+1)\phi_x. \end{aligned} \tag{3.1.4}$$

Constructing a local expansion around the singular manifold, we assume the solution to (3.0.1) takes the form of a Laurent series

$$u(x,t) = \sum_{j=0}^{\infty} u_j(x,t)\phi^{-n+j}. \tag{3.1.5}$$

As before, we require a full set of arbitrary coefficient functions for this expansion to be valid. The $\mathcal{PT}$-symmetric Burgers equation is second-order, thus we need two arbitrary functions; one should be the singular manifold itself, and the other should appear as an arbitrary u_j coefficient in (3.1.5).

Using a resonance analysis, we want to find the values of r that make $u_r(x,t)$ in (3.1.5) arbitrary. To that end, we let

$$u(x,t) = u_0\phi^{-n} + p\phi^{-n+r}, \tag{3.1.6}$$

and impose conditions on r to ensure the arbitrariness of $p(x,t)$. Using (3.1.6) in (3.0.1), we obtain

$$\begin{aligned}
&\mathcal{O}(\phi^{-n-1}) + \mathcal{O}(\phi^{-n+r-1}) \\
&\quad + \left[u_0\phi^{-n} + p\phi^{-n+r}\right]^{\frac{1}{n}} \left[-nu_0\phi^{-n-1}\phi_x + (-n+r)p\phi^{-n+r-1}\phi_x + \mathcal{O}(\phi^{-n+r}) + \mathcal{O}(\phi^{-n})\right] \\
&\quad - i\left[n(n+1)u_0\phi^{-n-2}\phi_x^2 + p(-n+r-1)(-n+r)\phi^{-n+r-2}\phi_x^2 + \mathcal{O}(\phi^{-n+r-1}) + \mathcal{O}(\phi^{-n-1})\right] = 0.
\end{aligned} \tag{3.1.7}$$

Neglecting the higher-order terms, and using binomial expansions (about $\phi = 0$), we then obtain

$$\begin{aligned}
&u_0^{1/n}\phi^{-1}\left[1 + \frac{p}{nu_0}\phi^r + \mathcal{O}(\phi^{2r})\right]\left[-nu_0\phi^{-n-1}\phi_x + (-n+r)p\phi^{-n+r-1}\phi_x\right] \\
&\quad - in(n+1)u_0\phi^{-n-2}\phi_x^2 + ip(-n+r-1)(-n+r)\phi^{-n+r-2}\phi_x^2 = 0.
\end{aligned} \tag{3.1.8}$$

We note that $\mathcal{O}(\phi^{-n-2})$ terms are constant in p, and by virtue of (3.1.4), sum to zero. Turning our attention to $\mathcal{O}(\phi^{-n+r-2})$ terms, which are linear in p, we have

$$\left[(-n+r)u_0^{1/n}\phi_x - u_0^{1/n}\phi_x - i(-n+r)(-n+r-1)\phi_x^2\right]p = 0. \tag{3.1.9}$$

For p to be arbitrary, we reqire the above expression in brackets to be identically zero. Making use of (3.1.4), this gives

$$\begin{aligned}
(-n+r-1)[-i(n+1)\phi_x]\phi_x - i(-n+r)(-n+r-1)\phi_x^2 = 0, \\
\Rightarrow -i\phi_x^2(r-n-1)(r+1) = 0.
\end{aligned} \tag{3.1.10}$$

Thus, the resonance values $r = -1, n+1$ will yield an arbitrary p. The value $r = -1$ indicates the arbitrariness of ϕ and the value $r = n + 1$ is a positive integer for any $n \in \mathbb{N}$, as needed to correspond to a location in the Laurent series expansion (3.1.5). Verification of the arbitrariness of the u_{n+1} coefficient will be done in specific n cases after the reformulation in the next section.

The case $n = 1$, in (3.0.1), yields the complex Burgers equation $u_t + uu_x - iu_{xx} = 0$, which has been analyzed in [29], as well as others, and will not be discussed further here. The case $n = 2$ yields the equation

$$u_t + u^{1/2}u_x - iu_{xx} = 0, \tag{3.1.11}$$

which will be discussed in subsequent sections.

3.2 Reformulation into System

Given the rational form of the $m = 1/n$, we make the substitution

$$v = u^{1/n} \Longleftrightarrow v^n = u, \tag{3.2.1}$$

which converts the equation (3.0.1) to the second-order system

$$u_t + vu_x - iu_{xx} = 0, \tag{3.2.2a}$$

$$v^n - u = 0. \tag{3.2.2b}$$

To determine the behavior of $v(x, t)$ at the singular manifold $\phi = 0$, we let $v = v_0\phi^{-\beta}$. Now, since $u = u_0\phi^{-n}$ at the singular manifold, (3.2.2b) implies

$$v_0^n\phi^{-\beta n} = u_0\phi^{-n}. \tag{3.2.3}$$

Balancing the powers of ϕ gives $\beta = 1$, so at leading order, $v = v_0\phi^{-1}$. Further equating coefficients, we require $v_0^n = u_0$. Then using (3.1.4), we obtain

$$v_0 = u_0^{1/n} = -i(n + 1)\phi_x. \tag{3.2.4}$$

Now expanding solutions u, v of (3.2.2) about the singular manifold, we seek solutions of the form

$$u(x,t) = \sum_{j=0}^{\infty} u_j \phi^{-n+j}, \tag{3.2.5a}$$

$$v(x,t) = \sum_{j=0}^{\infty} v_j \phi^{-1+j}, \tag{3.2.5b}$$

where u_0, v_0 are given in (3.2.4). Again, for these expansions to be valid, we require a full complement of arbitrary coefficient functions. Our system (3.2.2) is second order, so we only need one of u_j or v_j, for some $j \in \mathbb{N}$, to be arbitrary in addition to the singular manifold location.

Performing a resonance analysis on the reformulated system to determine the values of r such that u_r or v_r in (3.2.5) is arbitrary, we substitute the expressions

$$u(x,t) = u_0\phi^{-n} + p\phi^{-n+r}, \tag{3.2.6a}$$

$$v(x,t) = v_0\phi^{-1} + q\phi^{-1+r}, \tag{3.2.6b}$$

into (3.2.2) and require p, q be arbitrary. We equate the most singular r-powered terms from each equation, $\mathcal{O}(\phi^{-n-2+r})$ and $\mathcal{O}(\phi^{-n+r})$ terms respectively, which correspond to terms linear in p and q. This yields the system of equations

$$\left[(-n+r)v_0\phi_x - i(-n+r)(-n+r-1)\phi_x^2\right] p - nu_0\phi_x q = 0, \tag{3.2.7a}$$

$$-p + nv_0^{n-1} q = 0. \tag{3.2.7b}$$

For p or q to be arbitrary, we require the determinant of the above system in p, q to be zero. Making use of (3.2.4), we find

$$\begin{aligned}
nv_0^{n-1}\left[(-n+r)v_0\phi_x - i(-n+r)(-n+r-1)\phi_x^2\right] - nu_0\phi_x &= 0,\\
\Rightarrow n\left[\frac{u_0}{-i(n+1)\phi_x}\right]\left[-i(-n+r)(n+1)\phi_x^2 - i(-n+r)(-n+r-1)\phi_x^2\right] - nu_0\phi_x &= 0,\\
\Rightarrow \left[\frac{(-n+r)r}{(n+1)}\right] nu_0\phi_x - nu_0\phi_x &= 0,\\
\Rightarrow \left[r^2 - nr - (n+1)\right]\frac{nu_0\phi_x}{(n+1)} &= 0,\\
\Rightarrow (r+1)(r-(n+1))\frac{nu_0\phi_x}{(n+1)} &= 0.
\end{aligned} \tag{3.2.8}$$

Thus, the values of r that make p, q arbitrary are $r = -1, n+1$, which give the same resonance value as previously (3.1.10).

3.2.1 Verification of Resonances for $n = 2$

Letting $n = 2$ in (3.2.2), we will be further analyzing the system

$$u_t + vu_x - iu_{xx} = 0, \tag{3.2.9a}$$

$$v^2 - u = 0. \tag{3.2.9b}$$

To verify the positive resonance $r = n + 1 = 3$, we use the truncated expansions

$$u(x,t) = \sum_{j=0}^{3} u_j \phi^{-2+j} = \frac{u_0}{\phi^2} + \frac{u_1}{\phi} + u_2 + u_3\phi, \tag{3.2.10a}$$

$$v(x,t) = \sum_{j=0}^{3} v_j \phi^{-1+j} = \frac{v_0}{\phi} + v_1 + v_2\phi + v_3\phi^2, \tag{3.2.10b}$$

in (3.2.9) and want to show either u_3 or v_3 is arbitrary. Making this substitution, the $\mathcal{O}(\phi^{-4})$ term from (3.2.9a) and the $\mathcal{O}(\phi^{-2})$ term from (3.2.9b), each set equal to zero, gives

$$u_0 = -9\phi_x^2, \tag{3.2.11a}$$

$$v_0 = -3i\phi_x, \tag{3.2.11b}$$

as expected from (3.2.4) with $n = 2$. Using these expressions for u_0, v_0, we solve the equations obtained from setting the $\mathcal{O}(\phi^{-3})$ and $\mathcal{O}(\phi^{-1})$ terms, from (3.2.9a) and (3.2.9b) respectively, to zero, which yields

$$u_1 = \frac{9}{2} i \left(\phi_t - 2i\phi_{xx}\right), \tag{3.2.12a}$$

$$v_1 = -\frac{3\left(\phi_t - 2i\phi_{xx}\right)}{4\phi_x}. \tag{3.2.12b}$$

Using these, now the next ordered terms $\mathcal{O}(\phi^{-2})$ and $\mathcal{O}(\phi^0)$ respectively give

$$u_2 = \frac{15\phi_t^2}{16\phi_x^2} - \frac{9i\phi_{xt}}{2\phi_x} + \frac{9i\phi_t\phi_{xx}}{4\phi_x^2} + \frac{9\phi_{xx}^2}{4\phi_x^2} - \frac{3\phi_{3x}}{\phi_x}, \tag{3.2.13a}$$

$$v_2 = \frac{i\phi_t^2}{16\phi_x^3} + \frac{3\phi_{xt}}{4\phi_x^2} - \frac{3\phi_t\phi_{xx}}{4\phi_x^3} + \frac{3i\phi_{xx}^2}{4\phi_x^3} - \frac{i\phi_{3x}}{2\phi_x^2}. \tag{3.2.13b}$$

Finally, the $\mathcal{O}(\phi^{-1})$ and $\mathcal{O}(\phi^1)$ terms set equal to zero, is the system of equations

$$3i\phi_x^2\left[f_3(u_3,v_3)+\frac{3\phi_{tt}}{2\phi_x^2}-\frac{3\phi_t\phi_{xt}}{\phi_x^3}+\frac{3\phi_t^2\phi_{xx}}{2\phi_x^4}\right]=0, \tag{3.2.14a}$$

$$f_3(u_3,v_3)=0, \tag{3.2.14b}$$

where f_3 is given by

$$\begin{aligned} f_3(u_3,v_3) = &-u_3-6iv_3\phi_x-\frac{3i\phi_t^3}{32\phi_x^4}-\frac{9\phi_t\phi_{xt}}{8\phi_x^3}+\frac{15\phi_t^2\phi_{xx}}{16\phi_x^4} \\ &+\frac{9i\phi_{xt}\phi_{xx}}{4\phi_x^3}-\frac{27i\phi_t\phi_{xx}^2}{8\phi_x^4}-\frac{9\phi_{xx}^3}{4\phi_x^4}+\frac{3i\phi_t\phi_{3x}}{4\phi_x^3}+\frac{3\phi_{xx}\phi_{3x}}{2\phi_x^3}. \end{aligned} \tag{3.2.14c}$$

In order for one of u_3, v_3 to be arbitrary, we require (3.2.14) to be a linearly dependent set of equations in u_3, v_3. This is only true if the other terms besides f_3 in the bracket of (3.2.14a) is zero. Equivalently, we require the compatibility condition

$$\phi_x^2\phi_{tt}-2\phi_x\phi_t\phi_{xt}+\phi_t^2\phi_{xx}=0, \tag{3.2.15}$$

to insure the arbitrariness of either u_3 or v_3. Thus, ϕ is not arbitrary, making the implication of $r=-1$ corresponding to the arbitrariness of ϕ invalid, so the Laurent expansion (3.2.5) is not valid for the $n=2$ case of (3.2.2), and this system thus fails the Painlevé Test.

3.3 Special Solutions via Singular Manifold Method for $n=2$ Case

While the $n=2$ case of (3.2.2), given by (3.2.9), did not satisfy the Painlevé test for integrability, we can still find special solutions by using a truncated series expansion about the singular manifold. That is, we assume solutions of the form

$$u(x,t)=\frac{u_0}{\phi^2}+\frac{u_1}{\phi}+u_2, \tag{3.3.1a}$$

$$v(x,t)=\frac{v_0}{\phi}+v_1. \tag{3.3.1b}$$

Substituting these expressions into (3.2.9a), we obtain the coefficient equations

$$\mathcal{O}(\phi^{-4}): \quad -2u_0v_0\phi_x - 6iu_0\phi_x^2 = 0, \tag{3.3.2a}$$

$$\mathcal{O}(\phi^{-3}): \quad -2u_0\phi_t - u_1v_0\phi_x - 2u_0v_1\phi_x - 2iu_1\phi_x^2 + v_0(u_0)_x + 4i\phi_x(u_0)_x + 2iu_0\phi_{xx} = 0, \tag{3.3.2b}$$

$$\mathcal{O}(\phi^{-2}): \quad -u_1\phi_t + (u_0)_t - u_1v_1\phi_x + v_1(u_0)_x + v_0(u_1)_x + 2i\phi_x(u_1)_x + iu_1\phi_{xx} - i(u_0)_{xx} = 0, \tag{3.3.2c}$$

$$\mathcal{O}(\phi^{-1}): \quad (u_1)_t + v_1(u_1)_x + v_0(u_2)_x - i(u_1)_{xx} = 0, \tag{3.3.2d}$$

$$\mathcal{O}(\phi^{0}): \quad (u_2)_t + v_1(u_2)_x - i(u_2)_{xx} = 0, \tag{3.3.2e}$$

and from (3.2.9b),

$$\mathcal{O}(\phi^{-2}): \quad -u_0 + (v_0)^2 = 0, \tag{3.3.3a}$$

$$\mathcal{O}(\phi^{-1}): \quad -u_1 + 2v_0v_1 = 0, \tag{3.3.3b}$$

$$\mathcal{O}(\phi^{0}): \quad -u_2 + (v_1)^2 = 0. \tag{3.3.3c}$$

We note that the $\mathcal{O}(\phi^0)$ terms are simply (3.2.9) in u_2, v_1, so (3.3.1) gives an auto-BT from a known solution (u_2, v_1) to a new solution (u, v), where u_0, u_1, v_0, ϕ satisfy the above coefficient equations (3.3.2), (3.3.3).

3.3.1 First Iteration from Vacuum Solution

Starting from the vacuum solution $(u, v) = (0, 0)$ of (3.2.9), we let $u_2 = v_1 = 0$ in (3.3.1), (3.3.2), and (3.3.3). Solving (3.3.2a), (3.3.3a) for $u_0 = -9\phi_x^2$ and $v_0 = -3i\phi_x$ (or from previously solved, (3.2.11)), the remaining coefficient equations become

$$iu_1\phi_x^2 + 18\phi_t\phi_x^2 - 36i\phi_x^2\phi_{xx} = 0, \tag{3.3.4a}$$

$$-u_1\phi_t - i\phi_x(u_1)_x - 18\phi_x\phi_{xt} + iu_1\phi_{xx} + 9i\left(2\phi_{xx}^2 + 2\phi_x\phi_{3x}\right) = 0, \tag{3.3.4b}$$

$$(u_1)_t - i(u_1)_{xx} = 0, \tag{3.3.4c}$$

and

$$-u_1 = 0. \tag{3.3.4d}$$

These further reduce to the system of equations

$$[\phi_t - 2i\phi_{xx}]\,\phi_x^2 = 0, \tag{3.3.5a}$$

$$i\phi_{xx}^2 + i\phi_x\phi_{3x} - \phi_x\phi_{xt} = 0. \tag{3.3.5b}$$

Assuming $\phi_x \neq 0$ (otherwise we would get the trivial solution $u = v = 0$), we then require $\phi_t = 2i\phi_{xx}$. Differentiating once with respect to x, we obtain $\phi_{xt} = 2i\phi_{3x}$. The second equation above then becomes $\phi_{xx}^2 - \phi_x\phi_{3x} = 0$, which has the solution

$$\phi(x,t) = g_1(t)e^{xg_2(t)} + g_3(t), \tag{3.3.6}$$

where g_1, g_2, g_3 are arbitrary functions of t. Returning to the equation $\phi_t = 2i\phi_{xx}$, we substitute our expression for ϕ, yielding the equation

$$g_1'e^{xg_2} + xg_1g_2'e^{xg_2} + g_3' = 2ig_1g_2^2e^{xg_2}, \tag{3.3.7}$$

which further imposes the restrictions

$$g_3' = 0, \tag{3.3.8a}$$

$$g_1g_2' = 0, \tag{3.3.8b}$$

$$g_1' = 2ig_1g_2^2. \tag{3.3.8c}$$

Since we have already required $\phi_x \neq 0$, then $g_1 \neq 0$, leaving us with the solutions

$$g_1(t) = c_1e^{2ic_2^2t}, \tag{3.3.9a}$$

$$g_2(t) = c_2, \tag{3.3.9b}$$

$$g_3(t) = c_3, \tag{3.3.9c}$$

where c_1, c_2, c_3 are constants. Thus, we obtain

$$\phi(x,t) = c_1e^{xc_2+2ic_2^2t} + c_3. \tag{3.3.10}$$

Now, combining our results, (3.3.1) gives us

$$u(x,t) = \frac{-9\left[c_1 c_2 e^{xc_2+2ic_2^2 t}\right]^2}{\left[c_1 e^{xc_2+2ic_2^2 t}+c_3\right]^2}, \tag{3.3.11a}$$

$$v(x,t) = \frac{-3i\left[c_1 c_2 e^{xc_2+2ic_2^2 t}\right]}{\left[c_1 e^{xc_2+2ic_2^2 t}+c_3\right]}. \tag{3.3.11b}$$

These can be rewritten in terms of two arbitrary constants instead of three to correspond to the second order PDE system, and further, into the forms

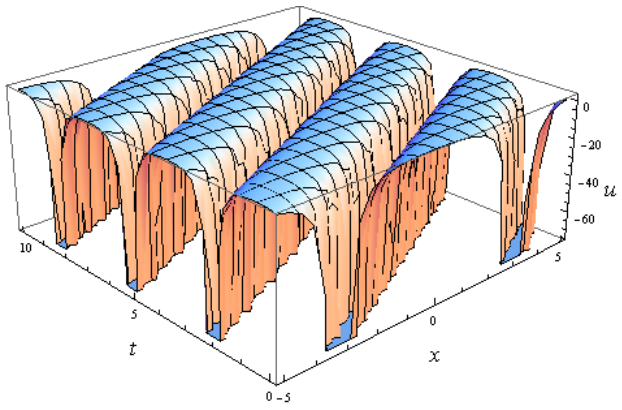
$$u(x,t) = -\frac{9}{4}k_1^2\left[\tanh\left[\frac{1}{2}\left(k_1 x+2ik_1^2 t+k_2\right)\right]+1\right]^2, \tag{3.3.12a}$$

$$v(x,t) = -\frac{3i}{2}k_1\left[\tanh\left[\frac{1}{2}\left(k_1 x+2ik_1^2 t+k_2\right)\right]+1\right], \tag{3.3.12b}$$

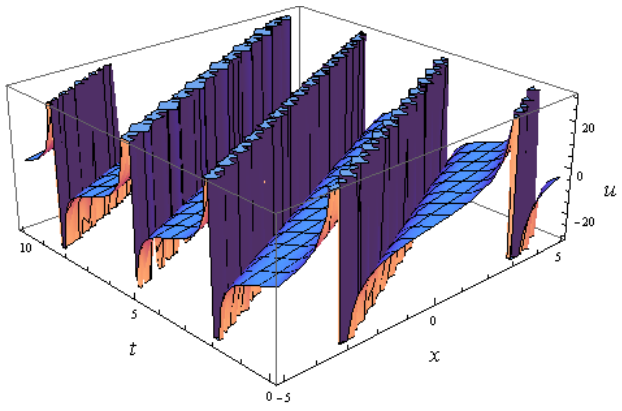
where k_1, k_2 are the arbitrary constants. These can be uniquely determined by imposing two appropriate initial/boundary conditions on u, v. It is verified by Mathematica that (3.3.12) solves (3.2.9). Next, the solution for u is checked in our original equation (3.0.1) with $n = 2$ ($m = 1/2$). We further require $k_1 = ib,\ k_2 = a,\ a, b \in \mathbb{R}$ for identity. We arrive at the solution

$$u(x,t) = \frac{9}{4}b^2\left[i\tan\left[\frac{1}{2}\left(bx-2b^2 t+ia\right)\right]+1\right]^2, \tag{3.3.13}$$

as a solution to (3.0.1) with $m = 1/2$, verified by Mathematica. Choosing the values $a = 0,\ b = 1$ in (3.3.13), plots of $\mathrm{Re}(u(x,t))$ and $\mathrm{Im}(u(x,t))$ are given in Figure 4. We note that when $a = 0$, the solution has singularities for $bx - 2b^2 t = (2j+1)\pi,\ j \in \mathbb{Z}$. Although a t-dependent solution has been obtained for the $\mathcal{PT}$-symmetric Burgers, an Invariant Analysis of the system was attempted to possibly find other solutions. However, only trivial solutions were found.



(a) $\mathrm{Re}(u(x,t))$



(b) $\mathrm{Im}(u(x,t))$

Figure 4: Solution $u(x,t)$-(3.3.13) with $a = 0,\ b = 1$

CHAPTER 4: $\mathcal{PT}$-SYMMETRIC (2+1) BURGERS EQUATION

For the complex $\mathcal{PT}$-Symmetric (2+1) Burgers equation,

$$\begin{aligned} 0 &= u_{xt} + (uu_x)_x + \frac{\partial^2}{\partial x^2}\left(iu_x\right)^\epsilon + u_{yy}, \\ &= u_{xt} + u_x^2 + uu_x + i\epsilon\left(iu_x\right)^{\epsilon-1} u_{3x} - \epsilon(\epsilon-1)\left(iu_x\right)^{\epsilon-2} u_{xx}^2 + u_{yy}, \end{aligned} \tag{4.0.1}$$

with $u = u(x, y, t)$, we wish to the values of ϵ such that (4.0.1) is integrable, and find special solutions.

4.1 Leading Order Analysis

We begin as before to determine the leading order behavior. We make the ansatz

$$u(x, y, t) = u_0\phi^{-n}, \tag{4.1.1}$$

where we require $n \in \mathbb{N}$; n and $u_0(x, y, t)$ are to be determined, and $\phi(x, y, t) = 0$ is the location of the singular manifold. Using this in (4.0.1), we have

$$\begin{aligned} &\mathcal{O}(\phi^{-n-2}) + \left[-nu_0\phi^{-n-1}\phi_x + \mathcal{O}(\phi^{-n})\right]^2 + u_0\phi^{-n}\left[-n(-n-1)u_0\phi^{-n-2}\phi_x^2 + \mathcal{O}(\phi^{-n-1})\right] \\ &\quad + i\epsilon\left[-inu_0\phi^{-n-1}\phi_x + \mathcal{O}(\phi^{-n})\right]^{\epsilon-1}\left[(-n)(-n-1)(-n-2)u_0\phi^{-n-3}\phi_x^3 + \mathcal{O}(\phi^{-n-2})\right] \\ &\quad - i\epsilon(\epsilon-1)\left[-inu_0\phi^{-n-1}\phi_x + \mathcal{O}(\phi^{-n})\right]^{\epsilon-2}\left[(-n)(-n-1)u_0\phi^{-n-2}\phi_x^2 + \mathcal{O}(\phi^{-n-1})\right]^2 = 0. \end{aligned} \tag{4.1.2}$$

Balancing the powers of ϕ in the most singular terms, we require $-2n-2 = (-n-1)(\epsilon-1) + (-n-3) = (-n-1)(\epsilon-2) + 2(-n-2)$, or equivalently $-2n-2 = -n\epsilon - \epsilon - 2$. Thus, the values of ϵ we can consider for integrability are

$$\epsilon = \frac{2n}{n+1}, \qquad n \in \mathbb{N}. \tag{4.1.3}$$

Now balancing the coefficients using this value for ϵ, we also require

$$\begin{aligned}
&n^2u_0^2\phi_x^2 + n(n+1)u_0^2\phi_x^2 - i\epsilon n(n+1)(n+2)u_0\phi_x^3\left(-inu_0\phi_x\right)^{\epsilon-1} \\
&\qquad\qquad - \epsilon(\epsilon-1)n^2(n+1)^2u_0^2\phi_x^4\left(-inu_0\phi_x\right)^{\epsilon-2} = 0, \\
&\Rightarrow n^2u_0^2\phi_x^2 + n(n+1)u_0^2\phi_x^2 - 2in^2(n+2)u_0\phi_x^3\left(-inu_0\phi_x\right)^{\frac{2n}{n+1}-1} \\
&\qquad\qquad - 2(n-1)n^3u_0^2\phi_x^4\left(-inu_0\phi_x\right)^{\frac{2n}{n+1}-2} = 0, \\
&\Rightarrow (2n+1)u_0^2 + 2(2n+1)\left(-inu_0\phi_x\right)^{\frac{2n}{n+1}} = 0, \\
&\Rightarrow u_0^{\frac{2}{n+1}} = -2\left(-in\phi_x\right)^{\frac{2n}{n+1}}. \qquad (4.1.4)
\end{aligned}$$

We will use this expression for u_0 to simplify subsequent calculations.

4.2 Reformulation into System

Given the rational form of the $\epsilon = 2n/(n+1)$, we promptly reformulate our original equation into an equivalent system before performing a resonance analysis. We make the substitution

$$v = (iu_x)^{\frac{1}{(n+1)}} \Longleftrightarrow v^{n+1} = iu_x, \qquad (4.2.1)$$

which converts the equation (4.0.1) to the third-order system

$$(n+1)v^nv_t + \left(uv^{n+1}\right)_x + i\left(v^{2n}\right)_{xx} + iu_{yy} = 0, \qquad (4.2.2a)$$

$$v^{n+1} - iu_x = 0. \qquad (4.2.2b)$$

To determine the behavior of $v(x,y,t)$ at the singular manifold $\phi = 0$, we let $v = v_0\phi^{-\beta}$. Now, since $u = u_0\phi^{-n}$ at the singular manifold, (4.2.2b) implies

$$v_0^{n+1}\phi^{-\beta(n+1)} = i\left[-nu_0\phi^{-n-1}\phi_x + \mathcal{O}(\phi^{-n})\right]. \qquad (4.2.3)$$

Balancing the powers of ϕ gives $\beta = 1$, so once again at leading order, $v = v_0\phi^{-1}$. Further, equating leading coefficients we have

$$v_0^{n+1} = -inu_0\phi_x \Longleftrightarrow u_0 = \frac{iv_0^{n+1}}{n\phi_x}. \qquad (4.2.4)$$

. Comparing this to the expression found for u_0 in (4.1.4), we then obtain

$$u_0^{\frac{2}{n+1}} = \left[\frac{iv_0^{n+1}}{n\phi_x}\right]^{\frac{2}{n+1}} = -2\left(-in\phi_x\right)^{\frac{2n}{n+1}},$$
$$\Rightarrow v_0^2 = 2n^2\phi_x^2, \tag{4.2.5}$$

which gives us an expression for v_0 for any n; we obtain this same expression for v_0 when equating leading order coefficients in (4.2.2a) and utilizing (4.2.4).

Now expanding solutions u, v of (4.2.2) about the singular manifold, we seek solutions of the form

$$u(x,y,t) = \sum_{j=0}^{\infty} u_j \phi^{-n+j}, \tag{4.2.6a}$$
$$v(x,y,t) = \sum_{j=0}^{\infty} v_j \phi^{-1+j}, \tag{4.2.6b}$$

with $u_j = u_j(x,y,t)$, where u_0, v_0 are given in (4.2.4) and (4.2.5). For these expansions to be valid, we require a full complement of arbitrary coefficient functions. Our system (4.2.2) is third order, so we need two of u_j or v_j, $j \in \mathbb{N}$, to be arbitrary in addition to the singular manifold location.

4.3 Resonance Analysis

We perform a resonance analysis on the reformulated system to determine the values of r such that u_r or v_r in (4.2.6) is arbitrary; substituting the expressions

$$u(x,y,t) = u_0\phi^{-n} + p\phi^{-n+r}, \tag{4.3.1a}$$
$$v(x,y,t) = v_0\phi^{-1} + q\phi^{-1+r}, \tag{4.3.1b}$$

into (4.2.2), we equate the coefficients of the most singular r-powered terms from each equation, $\mathcal{O}(\phi^{-2n-2+r})$ and $\mathcal{O}(\phi^{-n+r-1})$ terms respectively. These correspond to terms linear in p and q, and yields the system of equations

$$(-2n+r-1)v_0^{n+1}p + (-2n+r-1)\left[(n+1)u_0 + 2in(-2n+r)v_0^{n-1}\phi_x\right]v_0^n q = 0, \tag{4.3.2a}$$
$$-i(-n+r)\phi_x p + (n+1)v_0^n q = 0. \tag{4.3.2b}$$

For p or q to be arbitrary, we require the determinant of the above system in p, q to be zero. Making use of (4.2.4) and (4.2.5), we find

$$\begin{aligned}
&(-2n+r-1)(n+1)v_0^{2n+1} + i(-n+r)(-2n+r-1)\left[(n+1)u_0 + 2in(-2n+r)v_0^{n-1}\phi_x\right]\phi_x v_0^n = 0,\\
&\Rightarrow (-2n+r-1)\left[(n+1)v_0^{n+1} + i(-n+r)\left[(n+1)\left(\frac{iv_0^{n+1}}{n\phi_x}\right) + 2in(-2n+r)\frac{v_0^{n+1}}{v_0^2}\phi_x\right]\phi_x\right] = 0,\\
&\Rightarrow (-2n+r-1)\left[n+1+(-n+r)\left[\frac{-(n+1)}{n\phi_x} - 2n(-2n+r)\frac{1}{2n^2\phi_x^2}\phi_x\right]\phi_x\right] = 0,\\
&\Rightarrow (-2n+r-1)\left[n+1+(-n+r)\frac{n-r-1}{n}\right] = 0,\\
&\Rightarrow (-2n+r-1)(2n-r)(1+r) = 0. \quad (4.3.3)
\end{aligned}$$

Thus, the values of r that make p, q arbitrary are $r = -1, 2n, 2n+1$. With the exception of $r = -1$, these are positive integer values for all n, and thus correspond to locations in the Laurent expansion. We verify these resonances for specific values of n in the sections that follow.

4.3.1 Verification of Resonances for $n = 1$

Letting $n = 1$ in (4.2.2), we will be further analyzing the system

$$2vv_t + (uv^2)_x + i(v^2)_{xx} + iu_{yy} = 0, \tag{4.3.4a}$$

$$v^2 - iu_x = 0. \tag{4.3.4b}$$

which has the positive resonances $r = 2, 3$. To verify these, we use the truncated expansions

$$u(x,y,t) = \sum_{j=0}^{3} u_j \phi^{-1+j} = \frac{u_0}{\phi} + u_1 + u_2\phi + u_3\phi^2, \tag{4.3.5a}$$

$$v(x,y,t) = \sum_{j=0}^{3} v_j \phi^{-1+j} = \frac{v_0}{\phi} + v_1 + v_2\phi + v_3\phi^2, \tag{4.3.5b}$$

in (4.3.4) and want to show either u_2 or v_2, and u_3 or v_3 is arbitrary. Making this substitution, the $\mathcal{O}(\phi^{-4})$ term from (4.3.4a) and the $\mathcal{O}(\phi^{-2})$ term from (4.3.4b), each set equal to zero, gives

$$u_0 = 2i\phi_x, \tag{4.3.6a}$$

$$v_0 = \sqrt{2}\phi_x, \tag{4.3.6b}$$

as expected from (4.2.4) and (4.2.5) with $n = 1$. Using these expressions for u_0, v_0, we solve the equations obtained from setting the $\mathcal{O}(\phi^{-3})$ and $\mathcal{O}(\phi^{-1})$ terms, from (4.3.4a) and (4.3.4b) respectively, to zero, which yields

$$u_1 = -\frac{\phi_y^2 + \phi_t\phi_x + i\phi_x\phi_{xx}}{\phi_x^2}, \tag{4.3.7a}$$

$$v_1 = -\frac{\phi_{xx}}{\sqrt{2}\phi_x}. \tag{4.3.7b}$$

Using these, now the next ordered terms $\mathcal{O}(\phi^{-2})$ and $\mathcal{O}(\phi^0)$ respectively give the system of equations

$$f_2(u_2, v_2, \phi) + 2\phi_{yy}\phi_x = 0, \tag{4.3.8a}$$

$$f_2(u_2, v_2\phi) + 4\phi_y\phi_{xy} - \frac{2\phi_y^2\phi_{xx}}{\phi_x} = 0, \tag{4.3.8b}$$

where

$$f_2(u_2, v_2, \phi) = -2u_2\phi_x^3 - 4i\sqrt{2}v_2\phi_x^3 + 2\phi_x\phi_{xt} - 2\phi_t\phi_{xx} - \frac{2\phi_y^2\phi_{xx}}{\phi_x} - 3i\phi_{xx}^2 + 2i\phi_x\phi_{3x}.$$

In order for one of u_2, v_2 to be arbitrary, we require (4.3.8) to be a linearly dependent set of equations in u_2, v_2. This is only true if the other terms besides f_2 in (4.3.8) are equal. Equivalently, we require the compatibility condition

$$\phi_x^2\phi_{yy} - 2\phi_x\phi_y\phi_{xy} + \phi_y^2\phi_{xx} = 0, \tag{4.3.9}$$

to insure the arbitrariness of either u_2 or v_2. Thus, ϕ is not arbitrary, making the implication of $r = -1$ corresponding to the arbitrariness of ϕ invalid, so the Laurent expansion (4.2.6) is not valid for the $n = 1$ case of (4.2.2), and this system thus fails the Painlevé Test.

4.3.2 Verification of Resonances for $n = 2$

Letting $n = 2$ in (4.2.2), we will be verifying the resonances for the system

$$3v^2v_t + (uv^3)_x + i(v^4)_{xx} + iu_{yy} = 0, \tag{4.3.10a}$$

$$v^3 - iu_x = 0. \tag{4.3.10b}$$

which has the positive resonances $r = 4, 5$. To verify these, we use the truncated expansions

$$u(x, y, t) = \sum_{j=0}^{5} u_j \phi^{-2+j}, \tag{4.3.11a}$$

$$v(x, y, t) = \sum_{j=0}^{5} v_j \phi^{-1+j}, \tag{4.3.11b}$$

in (4.3.10) and want to show either u_4 or v_4, and u_5 or v_5 is arbitrary. Making this substitution, the $\mathcal{O}(\phi^{-6})$ term from (4.3.10a) and the $\mathcal{O}(\phi^{-3})$ term from (4.3.10b), each set equal to zero, gives

$$u_0 = 8i\sqrt{2}\phi_x^2, \tag{4.3.12a}$$

$$v_0 = 2\sqrt{2}\phi_x, \tag{4.3.12b}$$

as expected from (4.2.4) and (4.2.5) with $n = 2$. Using these expressions for u_0, v_0, we solve the equations obtained from setting the $\mathcal{O}(\phi^{-5})$ and $\mathcal{O}(\phi^{-2})$ terms, from (4.3.10a) and (4.3.10b) respectively, to zero, which yields

$$u_1 = -8i\sqrt{2}\phi_{xx}, \tag{4.3.13a}$$

$$v_1 = -\frac{\sqrt{2}\phi_{xx}}{\phi_x}. \tag{4.3.13b}$$

Using these, we balance the next ordered terms $\mathcal{O}(\phi^{-4})$ and $\mathcal{O}(\phi^{-1})$, and obtain

$$u_2 = -\frac{\phi_y^2}{\phi_x^2} - \frac{\phi_t}{\phi_x} - \frac{2i\sqrt{2}\phi_{xx}^2}{\phi_x^2} + \frac{8i\sqrt{2}\phi_{3x}}{3\phi_x}, \tag{4.3.14a}$$

$$v_2 = -\frac{\phi_{xx}^2}{\sqrt{2}\phi_x^3} + \frac{\sqrt{2}\phi_{3x}}{3\phi_x^2}. \tag{4.3.14b}$$

We subsequently set the $\mathcal{O}(\phi^{-3})$ and $\mathcal{O}(\phi^0)$ to zero, which gives

$$u_3 = \frac{3\phi_{yy}}{4\phi_x^2} + \frac{\phi_{xt}}{\phi_x^2} + \frac{\phi_y\phi_{xy}}{2\phi_x^3} - \frac{5\phi_y^2\phi_{xx}}{4\phi_x^4} - \frac{\phi_t\phi_{xx}}{\phi_x^3} - \frac{2i\sqrt{2}\phi_{xx}^3}{\phi_x^4} + \frac{8i\sqrt{2}\phi_{xx}\phi_{3x}}{3\phi_x^3} - \frac{2i\sqrt{2}\phi_{4x}}{3\phi_x^2}, \tag{4.3.15a}$$

$$v_3 = \frac{i\phi_{yy}}{32\phi_x^3} - \frac{i\phi_y\phi_{xy}}{16\phi_x^4} + \frac{i\phi_y^2\phi_{xx}}{32\phi_x^5} - \frac{\phi_{xx}^3}{\sqrt{2}\phi_x^5} + \frac{\phi_{xx}\phi_{3x}}{\sqrt{2}\phi_x^4} - \frac{\phi_{4x}}{6\sqrt{2}\phi_x^3}. \tag{4.3.15b}$$

Now, the next order terms, $\mathcal{O}(\phi^{-2})$ and $\mathcal{O}(\phi^1)$ respectively, give a linearly dependent system of equations in u_4 and v_4. Choosing v_4 as our arbitrary function, we solve for u_4

$$\begin{aligned} u_4 = &-12iv_4\phi_x - \frac{\phi_{xy}^2}{4\phi_x^4} - \frac{3\phi_{xyy}}{8\phi_x^3} + \frac{3\phi_{yy}\phi_{xx}}{8\phi_x^4} + \frac{3\phi_{xt}\phi_{xx}}{2\phi_x^4} + \frac{11\phi_y\phi_{xy}\phi_{xx}}{4\phi_x^5} - \frac{23\phi_y^2\phi_{xx}^2}{8\phi_x^6} \\ &- \frac{3\phi_t\phi_{xx}^2}{2\phi_x^5} - \frac{10i\sqrt{2}\phi_{xx}^4}{\phi_x^6} - \frac{\phi_{xxt}}{2\phi_x^3} - \frac{\phi_y\phi_{xxy}}{4\phi_x^4} + \frac{5\phi_y^2\phi_{3x}}{8\phi_x^5} + \frac{\phi_t\phi_{3x}}{2\phi_x^4} + \frac{14i\sqrt{2}\phi_{xx}^2\phi_{3x}}{\phi_x^5} \\ &- \frac{2i\sqrt{2}\phi_{3x}^2}{\phi_x^4} - \frac{3i\sqrt{2}\phi_{xx}\phi_{4x}}{\phi_x^4} + \frac{i\sqrt{2}\phi_{5x}}{3\phi_x^3}, \end{aligned} \tag{4.3.16a}$$

$$v_4 \quad \text{arbitrary.} \tag{4.3.16b}$$

Using this expression for u_4, the next order terms, $\mathcal{O}(\phi^{-1})$ and $\mathcal{O}(\phi^2)$, also give us a linearly dependent system of equations in u_5 and v_5. Once again, we choose v_5 as our arbitrary function, and we obtain

$$\begin{aligned} u_5 = &-8iv_5\phi_x + 4i(v_4)_x + \frac{12iv_4\phi_{xx}}{\phi_x} - \frac{5\phi_{xy}^2\phi_{xx}}{4\phi_x^6} - \frac{\phi_{xyy}\phi_{xx}}{2\phi_x^5} + \frac{7\phi_{yy}\phi_{xx}^2}{16\phi_x^6} + \frac{5\phi_{xt}\phi_{xx}^2}{2\phi_x^6} \\ &+ \frac{53\phi_y\phi_{xy}\phi_{xx}^2}{8\phi_x^7} - \frac{93\phi_y^2\phi_{xx}^3}{16\phi_x^8} - \frac{5\phi_t\phi_{xx}^3}{2\phi_x^7} - \frac{41i\phi_{xx}^5}{\sqrt{2}\phi_x^8} - \frac{\phi_{xx}\phi_{xxt}}{\phi_x^5} + \frac{\phi_{xy}\phi_{xxy}}{4\phi_x^5} \\ &- \frac{5\phi_y\phi_{xx}\phi_{xxy}}{4\phi_x^6} + \frac{\phi_{xxyy}}{8\phi_x^4} - \frac{\phi_{yy}\phi_{3x}}{24\phi_x^5} - \frac{2\phi_{xt}\phi_{3x}}{3\phi_x^5} - \frac{3\phi_y\phi_{xy}\phi_{3x}}{2\phi_x^6} + \frac{73\phi_y^2\phi_{xx}\phi_{3x}}{24\phi_x^7} \\ &+ \frac{5\phi_t\phi_{xx}\phi_{3x}}{3\phi_x^6} + \frac{115i\sqrt{2}\phi_{xx}^3\phi_{3x}}{3\phi_x^7} - \frac{118i\sqrt{2}\phi_{xx}\phi_{3x}^2}{9\phi_x^6} + \frac{\phi_{3xt}}{6\phi_x^4} + \frac{\phi_y\phi_{3xy}}{12\phi_x^5} \\ &- \frac{5\phi_y^2\phi_{4x}}{24\phi_x^6} - \frac{\phi_t\phi_{4x}}{6\phi_x^5} - \frac{53i\phi_{xx}^2\phi_{4x}}{3\sqrt{2}\phi_x^6} + \frac{23i\sqrt{2}\phi_{3x}\phi_{4x}}{9\phi_x^5} + \frac{4i\sqrt{2}\phi_{xx}\phi_{5x}}{3\phi_x^5} - \frac{i\sqrt{2}\phi_{6x}}{9\phi_x^4}, \end{aligned} \tag{4.3.17a}$$

$$v_5 \quad \text{arbitrary.} \tag{4.3.17b}$$

Thus, our resonances are verified; we have two arbitrary coefficient functions v_4 and v_5, corresponding to the resonances $r = 4, 5$. The Laurent expansion (4.2.6) is valid, and (4.2.2) passes the Painlevé test for $n = 2$

4.3.3 Verification of Resonances for $n = 3$

Letting $n = 3$ in (4.2.2), we will be verifying the resonances for the system

$$4v^3v_t + (uv^4)_x + i(v^6)_{xx} + iu_{yy} = 0, \tag{4.3.18a}$$

$$v^4 - iu_x = 0. \tag{4.3.18b}$$

which has the positive resonances $r = 6, 7$. To verify these, we use the truncated expansions

$$u(x, y, t) = \sum_{j=0}^{7} u_j \phi^{-3+j}, \tag{4.3.19a}$$

$$v(x, y, t) = \sum_{j=0}^{7} v_j \phi^{-1+j}, \tag{4.3.19b}$$

in (4.3.18) and want to show either u_6 or v_6, and u_7 or v_7 is arbitrary. Making this substitution, the $\mathcal{O}(\phi^{-8})$ term from (4.3.18a) and the $\mathcal{O}(\phi^{-4})$ term from (4.3.18b), each set equal to zero, gives

$$u_0 = 108 i \phi_x^3, \tag{4.3.20a}$$

$$v_0 = 3\sqrt{2}\phi_x, \tag{4.3.20b}$$

as expected from (4.2.4) and (4.2.5) with $n = 3$. Using these expressions for u_0, v_0, we solve the equations obtained from setting the $\mathcal{O}(\phi^{-7})$ and $\mathcal{O}(\phi^{-3})$ terms, from (4.3.18a) and (4.3.18b) respectively, to zero, which yields

$$u_1 = -162 i \phi_x \phi_{xx}, \tag{4.3.21a}$$

$$v_1 = -\frac{3\phi_{xx}}{\sqrt{2}\phi_x}. \tag{4.3.21b}$$

Continuing in this manner, we pair the $\mathcal{O}(\phi^{-6}) - \mathcal{O}(\phi^{-3})$ and $\mathcal{O}(\phi^{-2}) - \mathcal{O}(\phi^{1})$ coefficient equations, respectively, to solve for $u_j, v_j, j = 2, 3, 4, 5$. These are given in (A.18). Now, the next order terms, $\mathcal{O}(\phi^{-2})$ and $\mathcal{O}(\phi^{2})$ respectively, give a linearly dependent system of equations in u_6, v_6. Choosing v_6 as our arbitrary function, we solve for u_6, given in (A.18i). Similarly, the next ordered terms $\mathcal{O}(\phi^{-3})$ and $\mathcal{O}(\phi^{3})$ respectively, give a linearly dependent system in u_7, v_7; choosing v_7 as the arbitrary function, the expression for u_7 is given in (A.18k). Thus, our resonances are verified; we have two arbitrary coefficient functions v_6 and v_7, corresponding to the resonances $r = 6, 7$. The Laurent expansion (4.2.6) is valid, and (4.2.2) passes the Painlevé test for $n = 3$.

4.4 Singular Manifold Method

4.4.1 Case $n = 2$

We begin by analyzing the system (4.3.10). We truncate the Laurent expansion of the solution at the constant term $\mathcal{O}(\phi^0)$; that is, we assume the solutions take the form of series (4.2.6), truncated at $j = n = 2$ in (4.2.6a) and $j = 1$ in (4.2.6b), or

$$u = \frac{u_0}{\phi^2} + \frac{u_1}{\phi} + u_2, \tag{4.4.1a}$$

$$v = \frac{v_0}{\phi} + v_1. \tag{4.4.1b}$$

As found previously, we have

$$u_0 = 8i\sqrt{2}\phi_x^2, \tag{4.4.2a}$$

$$v_0 = 2\sqrt{2}\phi_x, \tag{4.4.2b}$$

$$u_1 = -8i\sqrt{2}\phi_{xx}, \tag{4.4.2c}$$

from intermediate results during verification of the resonances. Substituting (4.4.1) into (4.3.10), the order $\mathcal{O}(\phi^0)$ terms from both subequations yield

$$3(v_1)^2(v_1)_t + i(u_2)_{yy} + (v_1)^3(u_2)_x + 3u_2(v_1)^2(v_1)_x + 12i(v_1)^2(v_1)_x^2 + 4i(v_1)^3(v_1)_{xx} = 0, \tag{4.4.3a}$$

$$(v_1)^3 - i(u_2)_x = 0, \tag{4.4.3b}$$

which is our original system (4.3.10) in u_2, v_1. Thus, (4.4.1) gives an auto-Bäcklund Transformation connecting two solutions (u_2, v_1) and (u, v) of (4.3.10), provided ϕ satisfies the remaining coefficient equations. These Painlevé-Bäcklund equations are given in (A.19).

Attempts were made to derive a new solution from the vacuum solution $u_2 = v_1 = 0$ using the auto-BT (4.4.1), however only trivial results were found. We apply an invariant Painlevé analysis in a later section.

4.4.2 Case $n = 3$

We now analyze the system (4.3.18). We truncate the Laurent expansion of the solution at the constant term $\mathcal{O}(\phi^0)$; that is, we assume the solutions take the form of series (4.2.6), truncated at $j = n = 3$ in

(4.2.6a) and $j = 1$ in (4.2.6b), or

$$u = \frac{u_0}{\phi^3} + \frac{u_1}{\phi^2} + \frac{u_2}{\phi} + u_3, \tag{4.4.4a}$$

$$v = \frac{v_0}{\phi} + v_1. \tag{4.4.4b}$$

As found previously, we have

$$u_0 = 108i\phi_x^3, \tag{4.4.5a}$$

$$v_0 = 3\sqrt{2}\phi_x, \tag{4.4.5b}$$

$$u_1 = -162i\phi_x\phi_{xx}, \tag{4.4.5c}$$

$$u_2 = \frac{324i\phi_{xx}^2}{\phi_x} - 162i\phi_{3x}, \tag{4.4.5d}$$

from intermediate results during verification of the resonances. Substituting (4.4.4) into (4.3.18), the order $\mathcal{O}(\phi^0)$ terms from both sub-equations yield

$$4v_1^3(v_1)_t + i(u_3)_{yy} + v_1^4(u_3)_x + 4u_3v_1^3(v_1)_x + 30iv_1^4(v_1)_x^2 + 6iv_1^5(v_1)_{xx} = 0, \tag{4.4.6a}$$

$$v_1^4 - i(u_3)_x = 0, \tag{4.4.6b}$$

which is our original system (4.3.18) in u_3, v_1. Thus, (4.4.4) gives an auto-Bäcklund Transformation connecting two solutions (u_3, v_1) and (u, v) of (4.3.18), provided ϕ satisfies the remaining coefficient equations. These Painlevé-Bäcklund equations are given in (A.20).

Attempts were made to derive a new solution from the vacuum solution $u_3 = v_1 = 0$ using the auto-BT (4.4.4), however only trivial results were found. We apply an invariant Painlevé analysis in the next section.

4.5 Invariant Painlevé Analysis 2+1

Similar to the analysis performed on the KdV equation in Section 2.5, we apply the 2+1 formulation of Invariant Painlevé analysis to our 2+1 Burgers equation[36]. That is, we look at expansions of the form

$$u(x,y,t) = \sum_{j=0}^{\infty} u_j \chi^{-\alpha+j}, \tag{4.5.1}$$

where χ must vanish with the singular manifold $\phi - \phi_0$, and α is determined by a leading order analysis. Similar to the 1 + 1 case, if we choose the form of χ to be

$$\chi = \frac{\psi}{\psi_x} = \left(\frac{\phi_x}{\phi - \phi_0} - \frac{\phi_{xx}}{2\phi_x} \right)^{-1}, \tag{4.5.2a}$$

$$\psi = \frac{\phi - \phi_0}{\phi_x^{1/2}}, \tag{4.5.2b}$$

then χ satisfies the Ricatti equations

$$\chi_x = 1 + \frac{1}{2} S \chi^2, \tag{4.5.3a}$$

$$\chi_y = -K + K_x \chi - \frac{1}{2} \left(KS + K_{xx} \right) \chi^2, \tag{4.5.3b}$$

$$\chi_t = -C + C_x \chi - \frac{1}{2} \left(CS + C_{xx} \right) \chi^2, \tag{4.5.3c}$$

and ψ satisfies the linear equations

$$\psi_{xx} = -\frac{1}{2} S \psi, \tag{4.5.4a}$$

$$\psi_y = \frac{1}{2} K_x \psi - K \psi_x, \tag{4.5.4b}$$

$$\psi_t = \frac{1}{2} C_x \psi - C \psi_x. \tag{4.5.4c}$$

The quantities $S(x,y,t)$, $K(x,y,t)$ and $C(x,y,t)$ are defined by

$$S(x,y,t) = \frac{\phi_{3x}}{\phi_x} - \frac{3}{2} \left(\frac{\phi_{xx}}{\phi_x} \right)^2, \tag{4.5.5a}$$

$$K(x,y,t) = -\frac{\phi_y}{\phi_x}, \tag{4.5.5b}$$

$$C(x,y,t) = -\frac{\phi_t}{\phi_x}, \tag{4.5.5c}$$

and are invariant under the Möbius transformation

$$\phi \to \frac{a\phi + b}{c\phi + d}, \quad ad - bc = 1. \tag{4.5.6}$$

They are linked by the cross-derivative condition $\phi_{3xt} = \phi_{t3x}$ by

$$S_t + C_{3x} + 2C_x S + CS_x = 0, \tag{4.5.7a}$$

$$S_y + K_{3x} + 2K_x S + KS_x = 0, \tag{4.5.7b}$$

$$C_y - K_t + C_x K - CK_x = 0. \tag{4.5.7c}$$

Similar to the 1+1 case, the solution method consists of using a truncated (4.5.1) in the NLPDE to be solved, recursively replacing χ derivatives using (4.5.3), and equating terms order by order in χ. Conditions on u_j, S, K, C, may be found, and therefore (4.5.3) or (4.5.4) may be used to solve for χ. Then the expansion (4.5.1) with u_j, χ, will give a solution to the NLPDE. [36]

The above Invariant Painlevé formulation can be generalized to systems by using expansions similar to (4.5.1) for each dependent variable. For the system (4.2.2), the analysis dictates the use of the expansions

$$u(x, y, t) = \sum_{j=0}^{\infty} u_j \chi^{-n+j}, \tag{4.5.8a}$$

$$v(x, y, t) = \sum_{j=0}^{\infty} v_j \chi^{-1+j}. \tag{4.5.8b}$$

4.5.1 Application to $n = 2$ Case

We proceed for the $n = 2$ case of (4.3.10) by truncating the expansions (4.5.8) at the constant term. Substituting

$$u(x, y, t) = \frac{u_0}{\chi^2} + \frac{u_1}{\chi} + u_2, \tag{4.5.9a}$$

$$v(x, y, t) = \frac{v_0}{\chi} + v_1, \tag{4.5.9b}$$

into (4.3.10), we recursively replace derivatives of χ using (4.5.3) and equate coefficients order by order in χ. The coefficient equations are given in (A.21) and (A.22). Solving the leading order coefficient equations,

(A.21a) and (A.22a), for u_0, v_0, we obtain

$$u_0 = -8i\sqrt{2}, \tag{4.5.10a}$$

$$v_0 = -2\sqrt{2}. \tag{4.5.10b}$$

Making use of (4.5.10), the next order coefficient equations (A.21b) and (A.22b) imply

$$u_1 = 0, \tag{4.5.11a}$$

$$v_1 = 0. \tag{4.5.11b}$$

With use of (4.5.10) and (4.5.11), the coefficient equations (A.22c) and (A.22d) yield $S = 0$ and $(u_2)_x = 0$. The remaining equations from (A.21) then become

$$\mathcal{O}(\chi^{-4}): \quad u_2 = C - K^2, \tag{4.5.12a}$$

$$\mathcal{O}(\chi^{-3}): \quad K_y + 3C_x - 5KK_x = 0, \tag{4.5.12b}$$

$$\mathcal{O}(\chi^{-2}): \quad 4K_x^2 + 4KK_{xx} - 2K_{xy} - 3C_{xx} = 0, \tag{4.5.12c}$$

$$\mathcal{O}(\chi^{-1}): \quad 3K_xK_{xx} - K_{xxy} = 0, \tag{4.5.12d}$$

$$\mathcal{O}(\chi^{0}): \quad (u_2)_{yy} + 4i\sqrt{2}K_{xx}^2 = 0, \tag{4.5.12e}$$

$$\text{with} \quad (u_2)_x = 0. \tag{4.5.12f}$$

To solve the nonlinear system of PDEs above, we make a further assumption that $K(x, y, t) = F(t)$ [36], which immediately satisfies (4.5.12d). Then using (4.5.12b) yields $C_x = 0$, subsequently satisfying (4.5.12c). Further, the third cross-derivative condition of (4.5.7) is reduced to $C_y - K_t = 0$, which gives us $C = F'(t)y + g(t)$, where we assume $g(t)$ is an integrable function. Then using $u_2 = C - K^2$, the remaining equations are satisfied. Thus, from the χ-derivatives (4.5.3), we have

$$\chi_x = 1, \tag{4.5.13a}$$

$$\chi_y = -F(t), \tag{4.5.13b}$$

$$\chi_t = -F'(t)y - g(t), \tag{4.5.13c}$$

$$\Rightarrow \chi = x - F(t)y - G(t) \tag{4.5.13d}$$

where $G(t)$ is an antiderivative of $G(t)$. Combining our results, (4.5.9) becomes

$$u(x,y,t) = -\frac{8i\sqrt{2}}{[x-F(t)y-G(t)]^2} + F'(t)y + G'(t) - F(t)^2, \tag{4.5.14a}$$

$$v(x,y,t) = -\frac{2\sqrt{2}}{x-F(t)y-G(t)}. \tag{4.5.14b}$$

It is verified by Mathematica that (4.5.14) solves the $n = 2$ system (4.3.10). We further check our solution for u in the original NLPDE (4.0.1) with $n = 2$; this requires only the condition $x - F(t)y - G(t) < 0$ for identity. Thus, the solution to (4.0.1) is

$$u(x,y,t) = -\frac{8i\sqrt{2}}{[x-F(t)y-G(t)]^2} + F'(t)y + G'(t) - F(t)^2, \qquad x < F(t)y + G(t), \tag{4.5.15}$$

$$\tag{4.5.16}$$

which is verified by Mathematica. Several combinations of $F,\ G$ were used in the solution; many of those with $F'(t),\ G'(t) \neq 0$ gave a similar type of behavior, as illustrated in Figure **??**. The solution $\mathrm{Im}(u(x,y,t))$ of (4.5.15) is plotted with $F(t) = t^2,\ G(t) = e^t$ for $t = 0.5,\ 1.0,\ 1.5,\ 2.0$.

4.5.2 Application to $n = 3$ Case

We proceed for the $n = 3$ case of (4.3.18) by truncating the expansions (4.5.8) at the constant term. Substituting

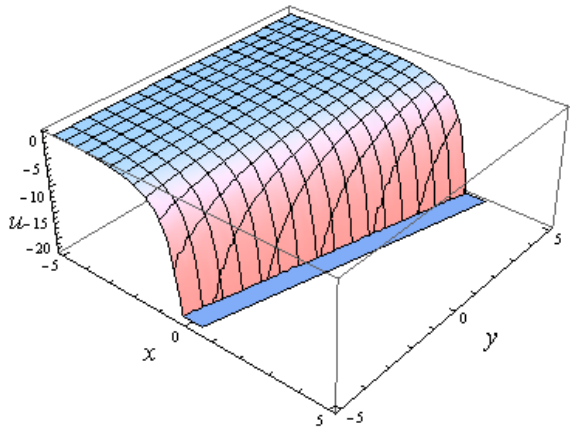
$$u(x,y,t) = \frac{u_0}{\chi^3} + \frac{u_1}{\chi^2} + \frac{u_2}{\chi} + u_3, \tag{4.5.17a}$$

$$v(x,y,t) = \frac{v_0}{\chi} + v_1, \tag{4.5.17b}$$

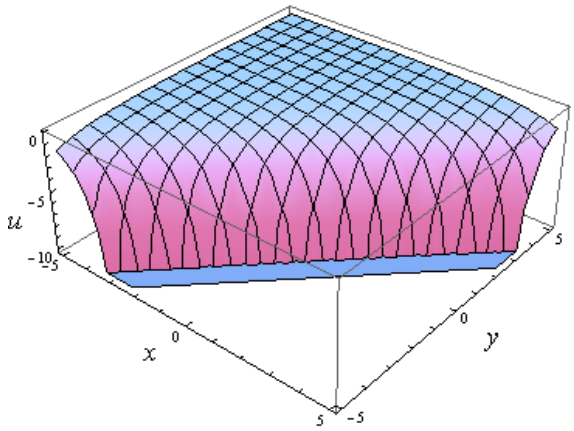
into (4.3.18), we recursively replace derivatives of χ using (4.5.3) and equate coefficients order by order in χ. The coefficient equations are given in (A.23) and (A.24). Solving the leading order coefficient equations, (A.23a) and (A.24a), for u_0, v_0, we obtain

$$u_0 = 108i, \tag{4.5.18a}$$

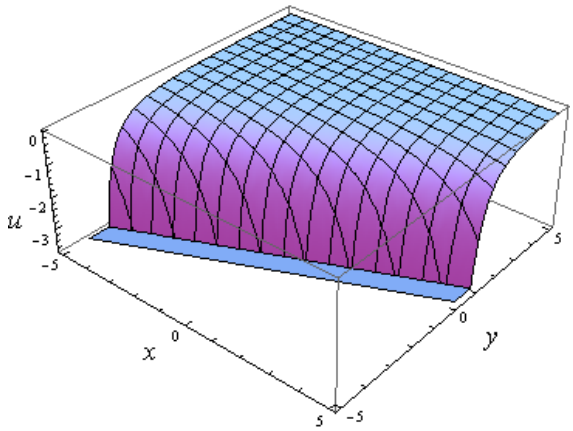
$$v_0 = 3\sqrt{2}. \tag{4.5.18b}$$



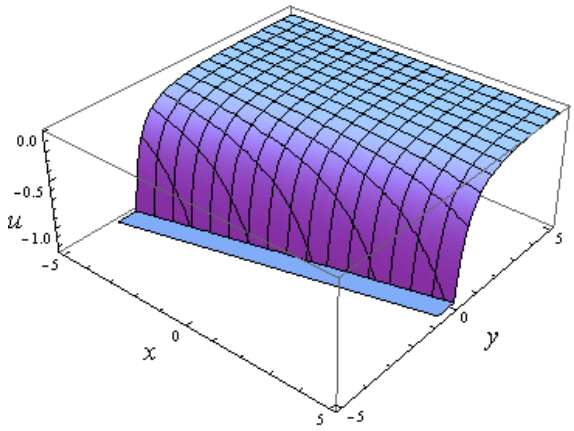
(a) $\mathrm{Im}(u(x,y,t))$ at $t = 0.5$



(b) $\mathrm{Im}(u(x,y,t))$ at $t = 1.0$



(c) $\mathrm{Im}(u(x,y,t))$ at $t = 1.5$



(d) $\mathrm{Im}(u(x,y,t))$ at $t = 2.0$

Figure 5: Solution $\mathrm{Im}(u(x,y,t))$ of (4.5.15) with $F(t) = t^2,\ G(t) = e^t$

Making use of (4.5.18), the next order coefficient equations (A.23b) and (A.24b) imply

$$u_1 = 0, \tag{4.5.19a}$$

$$v_1 = 0. \tag{4.5.19b}$$

Subsequently, we solve the next order equations (A.23c) and (A.24c) obtaining

$$S = 0, \tag{4.5.20a}$$

$$u_2 = 0, \tag{4.5.20b}$$

We also find $(u_3)_x = 0$ from (A.24e). The remaining equations from (A.23) then become

$$\mathcal{O}(\chi^{-5}): \quad u_3 = C - K^2, \tag{4.5.21a}$$

$$\mathcal{O}(\chi^{-4}): \quad K_y + 4C_x - 7KK_x = 0, \tag{4.5.21b}$$

$$\mathcal{O}(\chi^{-3}): \quad 3K_x^2 + 3KK_{xx} - K_{xy} - 2C_{xx} = 0, \tag{4.5.21c}$$

$$\mathcal{O}(\chi^{-2}): \quad 5K_xK_{xx} - K_{xxy} = 0, \tag{4.5.21d}$$

$$\mathcal{O}(\chi^{-1}): \quad K_{xx} = 0, \tag{4.5.21e}$$

$$\mathcal{O}(\chi^{0}): \quad (u_3)_{yy} = 0, \tag{4.5.21f}$$

$$\text{with} \quad (u_3)_x = 0. \tag{4.5.21g}$$

To solve the nonlinear system of PDEs above, we make a further assumption that $K(x, y, t) = F(t)$ as before, which immediately satisfies (4.5.21d) and (4.5.21e). Then we can obtain $C_x = 0$ from (4.5.21b), therefore also satisfying (4.5.21c). Further, the third cross-derivative condition of (4.5.7) is reduced to $C_y - K_t = 0$, which gives us $C = F'(t)y + g(t)$, where we assume $g(t)$ is an integrable function. Then using $u_3 = C - K^2$, the remaining equations are satisfied. Thus, from the χ-derivatives (4.5.3), we have

$$\chi_x = 1, \tag{4.5.22a}$$

$$\chi_y = -F(t), \tag{4.5.22b}$$

$$\chi_t = -F'(t)y - g(t), \tag{4.5.22c}$$

$$\Rightarrow \chi = x - F(t)y - G(t) \tag{4.5.22d}$$

where $G(t)$ is an antiderivative of $g(t)$. Combining our results, (4.5.17) becomes

$$u(x,y,t) = \frac{108i}{[x - F(t)y - G(t)]^3} + F'(t)y + G'(t) - F(t)^2, \tag{4.5.23a}$$

$$v(x,y,t) = \frac{3\sqrt{2}}{x - F(t)y - G(t)}. \tag{4.5.23b}$$

It is verified by Mathematica that (4.5.23) solves (4.3.18). We further check our solution for u in the original NLPDE (4.0.1) with $n = 3$; this requires only the condition $x - F(t)y - G(t) < 0$ for identity. Thus, the solution to (4.0.1) is

$$u(x,y,t) = \frac{108i}{[x - F(t)y - G(t)]^3} + F'(t)y + G'(t) - F(t)^2, \qquad x < F(t)y + G(t), \tag{4.5.24}$$

which is verified by Mathematica. We note that plots of the solution for various F, G mimic those obtained for the $n = 2$ case over the restricted domain $x < F(t)y + G(t)$ included in Figure 5, and so will not be repeated here.

CHAPTER 5: $\mathcal{PT}$-SYMMETRIC KPII EQUATION

We now consider one of the the complex $\mathcal{PT}$-symmetric Kadomtsev-Petviashvili (KPII) equations

$$\begin{aligned} 0 &= u_{xt} + (uu_x)_x - i\frac{\partial^3}{\partial x^3}(iu_x)^\epsilon + u_{yy}, \\ &= u_{xt} + (uu_x)_x - \epsilon(\epsilon-1)(\epsilon-2)(iu_x)^{\epsilon-3}u_{xx}^3 + 3i\epsilon(\epsilon-1)(iu_x)^{\epsilon-2}u_{xx}u_{3x} + \epsilon(iu_x)^{\epsilon-1}u_{4x} + u_{yy}, \end{aligned} \tag{5.0.1}$$

and will determine the values of ϵ for which (5.0.1) is integrable, in addition to special solutions.

5.1 Leading Order Analysis

We start with the leading order analysis; we make the ansatz

$$u(x,y,t) = u_0\phi^{-n}, \tag{5.1.1}$$

where we require $n \in \mathbb{N}$; n and $u_0(x,y,t)$ are to be determined, and $\phi(x,y,t) = 0$ is the location of the singular manifold. Using this in (5.0.1), we have

$$\begin{aligned} &\mathcal{O}(\phi^{-n-2}) + \left[-nu_0\phi^{-n-1}\phi_x + \mathcal{O}(\phi^{-n})\right]^2 + u_0\phi^{-n}\left[-n(-n-1)u_0\phi^{-n-2}\phi_x^2 + \mathcal{O}(\phi^{-n-1})\right] \\ &\quad - \epsilon(\epsilon-1)(\epsilon-2)\left[-inu_0\phi^{-n-1}\phi_x + \mathcal{O}(\phi^{-n})\right]^{\epsilon-3}\left[-n(-n-1)u_0\phi^{-n-2}\phi_x^2 + \mathcal{O}(\phi^{-n-2})\right]^3 \\ &\quad + 3i\epsilon(\epsilon-1)\left[-inu_0\phi^{-n-1}\phi_x + \mathcal{O}(\phi^{-n})\right]^{\epsilon-2}\left[-n(-n-1)u_0\phi^{-n-2}\phi_x^2 + \mathcal{O}(\phi^{-n-1})\right] \\ &\qquad\qquad \cdot\left[-n(-n-1)(-n-2)u_0\phi^{-n-3}\phi_x^3 + \mathcal{O}(\phi^{-n-2})\right] \\ &\quad + \epsilon\left[-inu_0\phi^{-n-1}\phi_x + \mathcal{O}(\phi^{-n})\right]^{\epsilon-1}\left[-n(-n-1)(-n-2)(-n-3)u_0\phi^{-n-4}\phi_x^4\right] = 0. \end{aligned} \tag{5.1.2}$$

Balancing the powers of ϕ in the most singular terms, we require

$$\begin{aligned}-2n-2 &= (-n-1)(\epsilon-3)+3(-n-2),\\ &= (-n-1)(\epsilon-2)+(-n-2)+(-n-3),\\ &= (-n-1)(\epsilon-1)+(-n-4),\end{aligned}$$

or equivalently, $-2n-2 = -n\epsilon-\epsilon-3$. Thus, the values of ϵ we can consider for integrability are

$$\epsilon = \frac{2n-1}{n+1}, \qquad n \in \mathbb{N}, \tag{5.1.3}$$

Now balancing the coefficients using this value for ϵ, we also require

$$\begin{aligned}&n^2u_0^2\phi_x^2 + n(n+1)u_0^2\phi_x^2 - \epsilon(\epsilon-1)(\epsilon-2)n^3(n+1)^3u_0^3\phi_x^6\left(-inu_0\phi_x\right)^{\epsilon-3}\\ &\qquad - 3i\epsilon(\epsilon-1)n^2(n+1)^2(n+2)u_0^2\phi_x^5\left(-inu_0\phi_x\right)^{\epsilon-2}\\ &\qquad + \epsilon n(n+1)(n+2)(n+3)u_0\phi_x^4\left(-inu_0\phi_x\right)^{\epsilon-1} = 0,\\ \Rightarrow &n(2n+1)u_0^2\phi_x^2 + 3(2n-1)(n-2)n^3u_0^3\phi_x^6\left(-inu_0\phi_x\right)^{\frac{2n-1}{n+1}-3}\\ &\qquad - 3i(2n-1)(n-2)n^2(n+2)u_0^2\phi_x^5\left(-inu_0\phi_x\right)^{\frac{2n-1}{n+1}-2}\\ &\qquad + (2n-1)n(n+2)(n+3)u_0\phi_x^4\left(-inu_0\phi_x\right)^{\frac{2n-1}{n+1}-1} = 0,\\ \Rightarrow &n(2n+1)u_0^2\phi_x^2 + i2n(2n+1)(2n-1)\left(-inu_0\phi_x\right)^{\frac{2n}{n+1}} = 0,\\ \Rightarrow &u_0^{\frac{3}{n+1}} = -2i(2n-1)\phi_x\left(-in\phi_x\right)^{\frac{2n-1}{n+1}}.\end{aligned} \tag{5.1.4}$$

We will use this expression for u_0 to simplify subsequent calculations.

5.2 Reformulation into System

Given the rational form of the $\epsilon = (2n-1)/(n+1)$, we reformulate our original equation into an equivalent system before performing a resonance analysis. We make the substitution

$$v = (iu_x)^{\frac{1}{(n+1)}} \iff v^{n+1} = iu_x, \tag{5.2.1}$$

which converts the equation (5.0.1) to the fourth-order system

$$(n+1)v^n v_t + \left(uv^{n+1}\right)_x + \left(v^{2n-1}\right)_{3x} + iu_{yy} = 0, \tag{5.2.2a}$$

$$v^{n+1} - iu_x = 0. \tag{5.2.2b}$$

As in previous cases with the same substitution, a leading order analysis yields $v = v_0\phi^{-1}$. Further, leading coefficients gives us

$$v_0^{n+1} = -inu_0\phi_x \Longleftrightarrow u_0 = \frac{iv_0^{n+1}}{n\phi_x}. \tag{5.2.3}$$

Comparing this to the expression found for u_0 in (5.1.4), we then obtain

$$\begin{aligned} u_0^{\frac{3}{n+1}} = \left[\frac{iv_0^{n+1}}{n\phi_x}\right]^{\frac{3}{n+1}} &= -2i(2n-1)\phi_x\left(-in\phi_x\right)^{\frac{2n-1}{n+1}}, \\ \Rightarrow v_0^3 &= 2in^2(2n-1)\phi_x^3, \end{aligned} \tag{5.2.4}$$

which gives us an expression for v_0 for any n; we obtain this same expression for v_0 when equating leading order coefficients in (5.2.2a) and utilizing (5.2.3).

Now expanding solutions u, v of (5.2.2) about the singular manifold, we seek solutions of the form

$$u(x,y,t) = \sum_{j=0}^{\infty} u_j\phi^{-n+j}, \tag{5.2.5a}$$

$$v(x,y,t) = \sum_{j=0}^{\infty} v_j\phi^{-1+j}, \tag{5.2.5b}$$

where u_0, v_0 are given in (5.2.3) and (5.2.4). For these expansions to be valid, we require a full complement of arbitrary coefficient functions. Our system (5.2.2) is fourth order, so we need three of u_j or v_j, $j \in \mathbb{N}$, to be arbitrary in addition to the singular manifold location.

5.3 Resonance Analysis

We perform a resonance analysis on the reformulated system to determine the values of r such that u_r or v_r in (5.2.5) is arbitrary; substituting the expressions

$$u(x,y,t) = u_0\phi^{-n} + p\phi^{-n+r}, \tag{5.3.1a}$$

$$v(x,y,t) = v_0\phi^{-1} + q\phi^{-1+r}, \tag{5.3.1b}$$

into (5.2.2), we equate the coefficients of the most singular r-powered terms from each equation, $\mathcal{O}(\phi^{-2n-2+r})$ and $\mathcal{O}(\phi^{-n+r-1})$ terms respectively. These correspond to terms linear in p and q, and yields the system of equations

$$\begin{aligned}
&(-2n+r-1)v_0^{n+1}\phi_x p \\
&\quad +(-2n+r-1)\left[(n+1)u_0 v_0^n \phi_x + (2n-1)(-2n+r+1)(2n+r)v_0^{2n-2}\phi_x^3\right] q = 0, && \text{(5.3.2a)}\\
&\quad -i(-n+r)\phi_x p + (n+1)v_0^n q = 0. && \text{(5.3.2b)}
\end{aligned}$$

Setting the determinant of the above system to zero to impose the arbitrariness of p, q, and making use of (5.2.3) and (5.2.4), we find

$$\begin{aligned}
&(-2n+r-1)(n+1)v_0^{2n+1} \\
&\quad +i(-n+r)(-2n+r-1)\left[(n+1)u_0 v_0^n \phi_x + (2n-1)(-2n+r+1)(-2n+r)v_0^{2n-2}\phi_x^3\right] = 0, \\
&\Rightarrow (-2n+r-1)\left[(n+1)v_0^{n+1} + i(-n+r)\left[(n+1)\left(\frac{iv_0^{n+1}}{n\phi_x}\right)\phi_x \right.\right. \\
&\qquad \left.\left. +(2n-1)(-2n+r+1)(-2n+r)\frac{v_0^{n+1}}{v_0^3}\phi_x^3\right]\right] = 0, \\
&\Rightarrow (-2n+r-1)\left[n+1+(-n+r)\left[\frac{-(n+1)}{n} + \frac{(-2n+r)(-2n+r+1)}{2n^2}\right]\right] = 0, \\
&\Rightarrow (-2n+r-1)(2n-r)(3n-r)(1+r) = 0. \quad \text{(5.3.3)}
\end{aligned}$$

Thus, the values of r that make p, q arbitrary are $r = -1, 2n, 2n+1, 3n$. With the exception of $r = -1$, these are positive integer values for all n, and thus correspond to locations in the Laurent expansion. Note that for $n = 1$, the resonance values $2n+1, 3n$ are redundant, and we are reduced to only two values instead of the required three positive resonances. Therefore, the KPII system does not pass the Painlevé test for the $n = 1$ case; we verify the resonances for$n = 2, 3$ in the next section.

5.3.1 Verification of Resonances for $n = 2$

Letting $n = 2$ in (5.2.2), we will be verifying the resonances for the system

$$\begin{aligned}
&3v^2 v_t + (uv^3)_x + (v^3)_{3x} + iu_{yy} = 0, && \text{(5.3.4a)}\\
&v^3 - iu_x = 0. && \text{(5.3.4b)}
\end{aligned}$$

which has the positive resonances $r = 4, 5, 6$. To verify these, we use the truncated expansions

$$u(x, y, t) = \sum_{j=0}^{6} u_j \phi^{-2+j}, \tag{5.3.5a}$$

$$v(x, y, t) = \sum_{j=0}^{6} v_j \phi^{-1+j}, \tag{5.3.5b}$$

in (5.3.4) and want to show either u_4 or v_4, u_5 or v_5, and u_6 or v_6 is arbitrary. Making this substitution, the $\mathcal{O}(\phi^{-6})$ term from (5.3.4a) and the $\mathcal{O}(\phi^{-3})$ term from (5.3.4b), each set equal to zero, gives

$$u_0 = -12\phi_x^2, \tag{5.3.6a}$$

$$v_0 = -2i3^{1/3}\phi_x, \tag{5.3.6b}$$

as expected from (5.2.3) and (5.2.4) with $n = 2$. Using these expressions for u_0, v_0, we solve the equations obtained from setting the $\mathcal{O}(\phi^{-5})$ and $\mathcal{O}(\phi^{-2})$ terms, from (5.3.4a) and (5.3.4b) respectively, to zero, which yields

$$u_1 = 12\phi_{xx}, \tag{5.3.7a}$$

$$v_1 = \frac{i3^{1/3}\phi_{xx}}{\phi_x}. \tag{5.3.7b}$$

Using these, we balance the next ordered terms $\mathcal{O}(\phi^{-4})$ and $\mathcal{O}(\phi^{-1})$, and obtain

$$u_2 = -\frac{\phi_y^2}{\phi_x^2} - \frac{\phi_t}{\phi_x} + \frac{3\phi_{xx}^2}{\phi_x^2} - \frac{4\phi_{3x}}{\phi_x}, \tag{5.3.8a}$$

$$v_2 = \frac{i3^{1/3}\phi_{xx}^2}{2\phi_x^3} - \frac{i\phi_{3x}}{3^{2/3}\phi_x^2}. \tag{5.3.8b}$$

We subsequently set the $\mathcal{O}(\phi^{-3})$ and $\mathcal{O}(\phi^0)$ to zero, which gives

$$u_3 = \frac{\phi_{yy}}{\phi_x^2} + \frac{\phi_{xt}}{\phi_x^2} - \frac{\phi_y^2\phi_{xx}}{\phi_x^4} - \frac{\phi_t\phi_{xx}}{\phi_x^3} + \frac{3\phi_{xx}^3}{\phi_x^4} - \frac{4\phi_{xx}\phi_{3x}}{\phi_x^3} + \frac{\phi_{4x}}{\phi_x^2}, \tag{5.3.9a}$$

$$v_3 = -\frac{i\phi_{yy}}{12\cdot 3^{2/3}\phi_x^3} + \frac{i\phi_y\phi_{xy}}{6\cdot 3^{2/3}\phi_x^4} - \frac{i\phi_y^2\phi_{xx}}{12\cdot 3^{2/3}\phi_x^5} + \frac{i3^{1/3}\phi_{xx}^3}{2\phi_x^5} - \frac{i3^{1/3}\phi_{xx}\phi_{3x}}{2\phi_x^4} + \frac{i\phi_{4x}}{4\cdot 3^{2/3}\phi_x^3}. \tag{5.3.9b}$$

Now, the next order terms, $\mathcal{O}(\phi^{-2})$ and $\mathcal{O}(\phi^1)$ respectively, give a linearly dependent system of equations in u_4 and v_4. Choosing v_4 as our arbitrary function, we solve for u_4 in terms of v_4; given in (A.25a). Using

this expression for u_4, the next order terms, $\mathcal{O}(\phi^{-1})$ and $\mathcal{O}(\phi^2)$, also give us a linearly dependent system of equations in u_5 and v_5. Once again, we choose v_5 as our arbitrary function, and we obtain the expression for u_5 given in (A.25c). Finally, using the expression for u_5, the $\mathcal{O}(\phi^0)$ and $\mathcal{O}(\phi^3)$ terms yield a linearly dependent system in u_6, v_6, and upon solving for u_6 with v_6 arbitrary, we obtain (A.25e).

Thus, our resonances are verified; we have three arbitrary coefficient functions v_4, v_5, and v_6, corresponding to the resonances $r = 4, 5, 6$. The Laurent expansion (5.2.5) is valid, and (5.2.2) passes the Painlevé test for $n = 2$.

5.3.2 Verification of Resonances for $n = 3$

Letting $n = 3$ in (5.2.2), we will be verifying the resonances for the system

$$4v^3v_t + (uv^4)_x + (v^5)_{3x} + iu_{yy} = 0, \tag{5.3.10a}$$

$$v^4 - iu_x = 0. \tag{5.3.10b}$$

which has the positive resonances $r = 6, 7, 9$. To verify these, we use the truncated expansions

$$u(x, y, t) = \sum_{j=0}^{9} u_j \phi^{-3+j}, \tag{5.3.11a}$$

$$v(x, y, t) = \sum_{j=0}^{9} v_j \phi^{-1+j}, \tag{5.3.11b}$$

in (5.3.10) and want to show either u_6 or v_6, u_7 or v_7, and u_9 or v_9 is arbitrary. Making this substitution, the $\mathcal{O}(\phi^{-8})$ term from (5.3.10a) and the $\mathcal{O}(\phi^{-4})$ term from (5.3.10b), each set equal to zero, gives

$$u_0 = 30i \cdot 3^{2/3} 10^{1/3} \phi_x^3, \tag{5.3.12a}$$

$$v_0 = -i3^{2/3} 10^{1/3} \phi_x, \tag{5.3.12b}$$

as expected from (5.2.3) and (5.2.4) with $n = 3$. Using these expressions for u_0, v_0, we solve the equations obtained from setting the $\mathcal{O}(\phi^{-7})$ and $\mathcal{O}(\phi^{-3})$ terms, from (5.3.10a) and (5.3.10b) respectively, to zero,

which yields

$$u_1 = -45i \cdot 3^{2/3} 10^{1/3} \phi_x \phi_{xx}, \tag{5.3.13a}$$

$$v_1 = \frac{i 3^{2/3} 5^{1/3} \phi_{xx}}{2^{2/3} \phi_x}. \tag{5.3.13b}$$

Continuing in this manner, we solve each subsequent ordered set of equations, $\mathcal{O}(\phi^{-6}) - \mathcal{O}(\phi^{-3})$ and $\mathcal{O}(\phi^{-2}) - \mathcal{O}(\phi^{1})$, respectively, for u_j, v_j, $j = 2, 3, 4, 5$. These are given in (A.26). Now, the next order terms, $\mathcal{O}(\phi^{-2})$ and $\mathcal{O}(\phi^{2})$ respectively, give a linearly dependent system of equations in u_6 and v_6. Choosing v_6 as our arbitrary function, we solve for u_6 in terms of v_6; given in (A.26i). Using this expression for u_6, the next order terms, $\mathcal{O}(\phi^{-1})$ and $\mathcal{O}(\phi^{3})$, also give us a linearly dependent system of equations in u_7 and v_7. Once again, we choose v_7 as our arbitrary function, and we obtain the expression for u_7 given in (A.26k). Now, the next ordered terms $\mathcal{O}(\phi^{0})$ and $\mathcal{O}(\phi^{4})$ give the prescribed u_8, v_8 given in (A.26). Finally, the $\mathcal{O}(\phi^{1})$ and $\mathcal{O}(\phi^{5})$ terms yield a system in u_9, v_9 of the form

$$f(u_9, v_9, \phi) + g_1(\phi) = 0, \tag{5.3.14a}$$

$$f(u_9, v_9, \phi) + g_2(\phi) = 0, \tag{5.3.14b}$$

where f, g_1, g_2 are functions of ϕ and its various mixed derivatives. Therefore, for the arbitrariness of u_9, v_9, we require $g_1 - g_2 = 0$, which gives us the condition

$$\begin{aligned}
&\phi_{4y}\phi_x^6 - 4\phi_{3y}\phi_x^5\phi_{xy} + 12\phi_{yy}\phi_x^4\phi_{xy}^2 - 24\phi_y\phi_x^3\phi_{xy}^3 - 6\phi_{yy}\phi_x^5\phi_{xyy} + 24\phi_y\phi_x^4\phi_{xy}\phi_{xyy} \\
&\quad - 4\phi_y\phi_x^5\phi_{x3y} + 3\phi_{yy}^2\phi_x^4\phi_{xx} + 4\phi_y\phi_{3y}\phi_x^4\phi_{xx} - 36\phi_y\phi_{yy}\phi_x^3\phi_{xy}\phi_{xx} + 72\phi_y^2\phi_x^2\phi_{xy}^2\phi_{xx} \\
&\quad - 18\phi_y^2\phi_x^3\phi_{xyy}\phi_{xx} + 18\phi_y^2\phi_{yy}\phi_x^2\phi_{xx}^2 - 60\phi_y^3\phi_x\phi_{xy}\phi_{xx}^2 + 15\phi_y^4\phi_{xx}^3 + 12\phi_y\phi_{yy}\phi_x^4\phi_{xxy} \\
&\quad - 36\phi_y^2\phi_x^3\phi_{xy}\phi_{xxy} + 24\phi_y^3\phi_x^2\phi_{xx}\phi_{xxy} + 6\phi_y^2\phi_x^4\phi_{xxyy} - 6\phi_y^2\phi_{yy}\phi_x^3\phi_{3x} + 16\phi_y^3\phi_x^2\phi_{xy}\phi_{3x} \\
&\quad - 10\phi_y^4\phi_x\phi_{xx}\phi_{3x} - 4\phi_y^3\phi_x^3\phi_{3xy} + \phi_y^4\phi_x^2\phi_{4x} = 0.
\end{aligned} \tag{5.3.15}$$

Thus, ϕ is not arbitrary, making the implication of $r = -1$ corresponding to the arbitrariness of ϕ invalid, so the Laurent expansion (5.2.5) is not valid, and the $n = 3$ case of (5.2.2) fails the Painlevé Test.

5.4 Singular Manifold Method for $n = 2$ Case

We will be analyzing and finding solutions to the system (5.3.4). We truncate the Laurent expansion of the solution at the constant term $\mathcal{O}(\phi^0)$; that is, we assume the solutions take the form of series (5.2.5) , truncated at $j = n = 2$ in (5.2.5a) and $j = 1$ in (5.2.5b), or

$$u = \frac{u_0}{\phi^2} + \frac{u_1}{\phi} + u_2, \tag{5.4.1a}$$

$$v = \frac{v_0}{\phi} + v_1. \tag{5.4.1b}$$

As previously found during verification of resonances, we have

$$u_0 = -12\phi_x^2, \tag{5.4.2a}$$

$$v_0 = -2i3^{1/3}\phi_x, \tag{5.4.2b}$$

$$u_1 = 12\phi_{xx}. \tag{5.4.2c}$$

Substituting this into (5.3.4), the order $\mathcal{O}(\phi^0)$ terms from both subequations yield

$$3(v_1)^2(v_1)_t + i(u_2)_{yy} + (v_1)^3(u_2)_x + 3u_2(v_1)^2(v_1)_x + 12i(v_1)^2(v_1)_x^2 + 4i(v_1)^3(v_1)_{xx} = 0, \tag{5.4.3a}$$

$$(v_1)^3 - i(u_2)_x = 0, \tag{5.4.3b}$$

which is our original equation (5.3.4) in u_2, v_1. Thus, (5.4.1) gives an auto-Bäcklund Transformation connecting two solutions (u_2, v_1) and (u, v) of (5.3.4), provided ϕ satisfies the remaining coefficient equations. These Painlevé-Bäcklund equations are given in (A.27).

Attempts were made to derive a new solution from the vacuum solution $u_2 = v_1 = 0$ using the auto-BT (4.4.1), however only trivial results were found. We apply an invariant Painlevé analysis in the next section.

5.5 Invariant Painlevé Analysis for $n = 2$ Case

We proceed with the 2+1 invariant Painlevé analysis discussed in Section 4.5 for the system (5.3.4) by truncating the expansions (4.5.8) at the constant term. Substituting

$$u(x, y, t) = \frac{u_0}{\chi^2} + \frac{u_1}{\chi} + u_2, \tag{5.5.1a}$$

$$v(x, y, t) = \frac{v_0}{\chi} + v_1, \tag{5.5.1b}$$

into (5.3.4), we recursively replace derivatives of χ using (4.5.3) and equate coefficients order by order in χ. The coefficient equations are given in (A.28) and (A.29). Solving the leading order coefficient equations, (A.28a) and (A.29a), for u_0, v_0, we obtain

$$u_0 = -12, \tag{5.5.2a}$$

$$v_0 = -2i3^{1/3}. \tag{5.5.2b}$$

Making use of (5.5.2), the next order coefficient equations (A.21b) and (A.22b) imply

$$u_1 = 0, \tag{5.5.3a}$$

$$v_1 = 0. \tag{5.5.3b}$$

With use of (5.5.2) and (5.5.3), the coefficients of $\mathcal{O}(\chi^{-1})$ and $\mathcal{O}(\chi^0)$ from the second system equation,(A.22c) and (A.22d), yield $S = 0$ and $(u_2)_x = 0$. The remaining coefficients from the first system equation, (A.21), then become

$$\mathcal{O}(\chi^{-4}): \quad u_2 = C - K^2, \tag{5.5.4a}$$

$$\mathcal{O}(\chi^{-3}): \quad K_y + 3C_x - 5KK_x = 0, \tag{5.5.4b}$$

$$\mathcal{O}(\chi^{-2}): \quad 4K_x^2 + 4KK_{xx} - 2K_{xy} - 3C_{xx} = 0, \tag{5.5.4c}$$

$$\mathcal{O}(\chi^{-1}): \quad 3K_xK_{xx} - K_{xxy} = 0, \tag{5.5.4d}$$

$$\mathcal{O}(\chi^0): \quad (u_2)_{yy} - 6K_{xx}^2 = 0, \tag{5.5.4e}$$

$$\text{with} \quad (u_2)_x = 0. \tag{5.5.4f}$$

With the exception of the K_{xx}^2 term, these conditions are the same as for the 2+1 Burgers Equation, (4.5.12). The further assumption that $K(x,y,t) = F(t)$ as before, yields an equivalent χ-solution

$$\chi_x = 1, \tag{5.5.5a}$$

$$\chi_y = -F(t), \tag{5.5.5b}$$

$$\chi_t = -F'(t)y - g(t), \tag{5.5.5c}$$

$$\Rightarrow \chi = x - F(t)y - G(t) \tag{5.5.5d}$$

where $G(t)$ is an antiderivative of $G(t)$. Combining our results, (5.5.1) becomes

$$u(x,y,t) = -\frac{12}{[x - F(t)y - G(t)]^2} + F'(t)y + G'(t) - F(t)^2, \tag{5.5.6a}$$

$$v(x,y,t) = -\frac{2i3^{1/3}}{x - F(t)y - G(t)}. \tag{5.5.6b}$$

It is verified by Mathematica that (5.5.6) solves the $n = 2$ system (5.3.4). We further check our solution for u in the original NLPDE (5.0.1) with $n = 2$, which is also identically satisfied with no further conditions.

CHAPTER 6: CONCLUSIONS AND FUTURE WORK

Generalizing the work of Bender and co-workers, as well as others, we utilized the Painlevé Test, Singular Manifold Method, Invariant Painlevé Analysis, and the Homogeneous Balance Method to analyze several hierarchies of $\mathcal{PT}$-symmetric NLPDEs. Due to the nature of the $\mathcal{PT}$-symmetric equations and their possible integrable cases, all except the (1+1) Burgers' equation were reformulated into a system with a transformation of the form $v^n = u_x$ prior to applying other analyses.

In Chapter 2, we discussed the (1+1) $\mathcal{PT}$-symmetric KdV equation (2.1.2). A leading order analysis gave the sub-hierarchy of possibly integrable members, prescribed by $\epsilon = (2n-1)/(n+1)$, with $n \in \mathbb{N}$ corresponding to the order of the singular manifold. After transforming the original NLPDE into a system in u, v, we found the positive resonances to be $r = 2n, 2n+1, 3n$, thus indicating a full set of arbitrary coefficient functions for $n \geq 2$. Though the $n = 1$ case did not pass the Painlevé test, the auto-Bäcklund Transformations for the $n = 1, 3, 4$ cases are found, though these yielded only $t-$independent solutions from the vacuum solution. Then utilizing Invariant Painlevé analysis, we derived algebraic (solitary wave) solutions the form $u = u_0[x - at - b]^{-n} + a$ for $n = 1, 3, 4$. Finally, we applied the Homogeneous Balance method to the $n = 1, 3$ cases, and derived a near-Lax Pair for the $n = 3$ case.

We briefly analyzed the (1+1) $\mathcal{PT}$-symmetric Burgers' equation (3.0.1) in Chapter 3. Leading order analysis prescribed the values $m = 1/n$ as possible integrable cases. After reformulating the NLPDE into a system, the positive resonance was found to be $r = n + 1$. However, the $n = 2$ case required a compatibility condition, and so failed the Painlevé Test. An auto-BT was found and used to derive a special solution from the vacuum solution.

Extending our analysis to (2+1) dimensions, we analyzed the (2+1) $\mathcal{PT}$-symmetric Burgers' equation (4.0.1). A leading order analysis yielded the possible integrable members $\epsilon = 2n/(n+1)$, and the equation was subsequently transformed into an equivalent system. Resonance analysis gave the positive resonances $r = 2n, 2n+1$, however, the $n = 1$ case required a compatibility condition, and thus failed the Painlevé Test. Auto-BTs were derived for the $n = 2, 3$ cases, but did not yield any new solutions from the vacuum

solution. Algebraic solutions of the form $u = u_0[x - F(t)y - G(t)]^{-n} + F'(t)y + G'(t) - F(t)^2$ were found through Invariant Painlevé analysis.

Finally, in Chapter 5, we studied the $\mathcal{PT}$-symmetric KPII equation (5.0.1). Similar to the (1+1) KdV equation, a leading order analysis gave $\epsilon = (2n-1)/(n+1)$ as possible integrable members. After reformulation into a system, we found the positive resonances $r = 2n, 2n+1, 3n$, thus requiring $n \geq 2$ for distinct resonances. While the $n = 2$ case passed the Painlevé Test, the $n = 3$ case required a compatibility condition, and thus failed the test. For the $n = 2$ case, an auto-BT was derived, however did not yield any new solutions. Invariant Painlevé Analysis yielded a similar algebraic solution $u = u_0[x-ky-F(t)]^{-n}+F'(t)y+G'(t)-F(t)^2$ to the KPII system.

As far as future work, there is still the Homogeneous Balance Method to extend and apply to the 2+1 systems and subsequently, linearization of the Lax-type equations obtained; other near-Lax Pairs could possibly be found. In addition, Yan [29] presented many other $\mathcal{PT}$-Symmetric systems, including transformations of the KP equation other than the one considered here.

A natural area of extension would be to symmetry analysis [37, 38], which usually yields additional results and insights. Also, derivations of Lagrangian and Hamiltonian formulations for the traveling wave equations [39, 40] is likely to be worthwhile to investigate other solutions and features of the $\mathcal{PT}$-symmetric NLPDEs considered here.

APPENDIX: COEFFICIENT FUNCTIONS AND EQUATION

Using (2.3.6) in (2.4.10) and equating coefficients of ϕ, the u_j, v_j are found to be

$$u_1 = -\frac{45i3^{2/3}10^{1/3}\phi_x\phi_{xx}}{\lambda^{4/3}}, \tag{A.1a}$$

$$v_1 = \frac{i3^{2/3}5^{1/3}\phi_{xx}}{2^{2/3}\lambda^{1/3}\phi_x}, \tag{A.1b}$$

$$u_2 = \frac{15i3^{2/3}10^{1/3}\phi_{3x}}{\lambda^{4/3}}, \tag{A.1c}$$

$$v_2 = \frac{i3^{2/3}5^{1/3}\phi_{xx}^2}{2\cdot 2^{2/3}\lambda^{1/3}\phi_x^3} - \frac{i5^{1/3}\phi_{3x}}{2^{2/3}3^{1/3}\lambda^{1/3}\phi_x^2} \tag{A.1d}$$

$$u_3 = -\frac{\phi_t}{\lambda\phi_x} - \frac{15i3^{2/3}5^{1/3}\phi_{xx}^3}{2\cdot 2^{2/3}\lambda^{4/3}\phi_x^3} + \frac{15i3^{2/3}5^{1/3}\phi_{xx}\phi_{3x}}{2^{2/3}\lambda^{4/3}\phi_x^2} - \frac{15i3^{2/3}5^{1/3}\phi_{4x}}{2\cdot 2^{2/3}\lambda^{4/3}\phi_x}, \tag{A.1e}$$

$$v_3 = \frac{i3^{2/3}5^{1/3}\phi_{xx}^3}{2\cdot 2^{2/3}\lambda^{1/3}\phi_x^5} - \frac{i3^{2/3}5^{1/3}\phi_{xx}\phi_{3x}}{2\cdot 2^{2/3}\lambda^{1/3}\phi_x^4} + \frac{i5^{1/3}\phi_{4x}}{4\cdot 2^{2/3}3^{1/3}\lambda^{1/3}\phi_x^3}, \tag{A.1f}$$

$$\begin{aligned} u_4 = {} & \frac{\phi_{xt}}{\lambda\phi_x^2} - \frac{\phi_t\phi_{xx}}{\lambda\phi_x^3} - \frac{45i3^{2/3}5^{1/3}\phi_{xx}^4}{4\cdot 2^{2/3}\lambda^{4/3}\phi_x^5} + \frac{45i3^{2/3}5^{1/3}\phi_{xx}^2\phi_{3x}}{2\cdot 2^{2/3}\lambda^{4/3}\phi_x^4} \\ & - \frac{5i3^{2/3}5^{1/3}\phi_{3x}^2}{2^{2/3}\lambda^{4/3}\phi_x^3} - \frac{15i3^{2/3}5^{1/3}\phi_{xx}\phi_{4x}}{2\cdot 2^{2/3}\lambda^{4/3}\phi_x^3} + \frac{3i3^{2/3}5^{1/3}\phi_{5x}}{2\cdot 2^{2/3}\lambda^{4/3}\phi_x^2}, \end{aligned} \tag{A.1g}$$

$$\begin{aligned} v_4 = {} & \frac{5i3^{2/3}5^{1/3}\phi_{xx}^4}{8\cdot 2^{2/3}\lambda^{1/3}\phi_x^7} - \frac{5i5^{1/3}\phi_{xx}^2\phi_{3x}}{2\cdot 2^{2/3}3^{1/3}\lambda^{1/3}\phi_x^6} + \frac{i5^{1/3}\phi_{3x}^2}{3\cdot 2^{2/3}3^{1/3}\lambda^{1/3}\phi_x^5} \\ & + \frac{i5^{1/3}\phi_{xx}\phi_{4x}}{2\cdot 2^{2/3}3^{1/3}\lambda^{1/3}\phi_x^5} - \frac{i\phi_{5x}}{4\cdot 3^{1/3}10^{2/3}\lambda^{1/3}\phi_x^4}, \end{aligned} \tag{A.1h}$$

$$\begin{aligned} u_5 = {} & \frac{3\phi_{xt}\phi_{xx}}{2\lambda\phi_x^4} - \frac{3\phi_t\phi_{xx}^2}{2\lambda\phi_x^5} - \frac{135i3^{2/3}5^{1/3}\phi_{xx}^5}{8\cdot 2^{2/3}\lambda^{4/3}\phi_x^7} - \frac{\phi_{xxt}}{2\lambda\phi_x^3} + \frac{\phi_t\phi_{3x}}{2\lambda\phi_x^4} \\ & + \frac{75i3^{2/3}5^{1/3}\phi_{xx}^3\phi_{3x}}{2\cdot 2^{2/3}\lambda^{4/3}\phi_x^6} - \frac{15i3^{2/3}5^{1/3}\phi_{xx}\phi_{3x}^2}{2^{2/3}\lambda^{4/3}\phi_x^5} - \frac{45i3^{2/3}5^{1/3}\phi_{xx}^2\phi_{4x}}{4\cdot 2^{2/3}\lambda^{4/3}\phi_x^5} \\ & + \frac{15i3^{2/3}5^{1/3}\phi_{3x}\phi_{4x}}{4\cdot 2^{2/3}\lambda^{4/3}\phi_x^4} + \frac{9i3^{2/3}5^{1/3}\phi_{xx}\phi_{5x}}{4\cdot 2^{2/3}\lambda^{4/3}\phi_x^4} - \frac{i3^{2/3}5^{1/3}\phi_{6x}}{4\cdot 2^{2/3}\lambda^{4/3}\phi_x^3}, \end{aligned} \tag{A.1i}$$

$$\begin{aligned} v_5 = {} & \frac{7i3^{2/3}5^{1/3}\phi_{xx}^5}{8\cdot 2^{2/3}\lambda^{1/3}\phi_x^9} + \frac{5i5^{1/3}\phi_{xx}\phi_{3x}^2}{4\cdot 2^{2/3}3^{1/3}\lambda^{1/3}\phi_x^7} - \frac{i5^{1/3}\phi_{xx}\phi_{5x}}{8\cdot 2^{2/3}3^{1/3}\lambda^{1/3}\phi_x^6} + \frac{5i3^{2/3}5^{1/3}\phi_{xx}^2\phi_{4x}}{16\cdot 2^{2/3}\lambda^{1/3}\phi_x^7} \\ & - \frac{5i5^{1/3}\phi_{3x}\phi_{4x}}{24\cdot 2^{2/3}3^{1/3}\lambda^{1/3}\phi_x^6} - \frac{35i5^{1/3}\phi_{xx}^3\phi_{3x}}{8\cdot 2^{2/3}3^{1/3}\lambda^{1/3}\phi_x^8} + \frac{i\phi_{6x}}{24\cdot 3^{1/3}10^{2/3}\lambda^{1/3}\phi_x^5}, \end{aligned} \tag{A.1j}$$

$$\begin{aligned} f_6(u_6, v_6, \phi) = {} & [48i3^{1/3}\lambda^{4/3}\phi_x^9]u_6 - [5760i3^{1/3}\lambda^{1/3}\phi_x^{11}]v_6 \\ & - 8i3^{1/3}\lambda^{1/3}\left[15\phi_{xt}\phi_x^3\phi_{xx}^2 - 15\phi_t\phi_x^2\phi_{xx}^3 - 4\phi_{xt}\phi_{3x}\phi_x^4 + \phi_{3xt}\phi_x^5\right. \\ & \qquad \left. + 10\phi_t\phi_{xx}\phi_{3x}\phi_x^3 - 6\phi_{xx}\phi_{xxt}\phi_x^4 - \phi_t\phi_{4x}\phi_x^4\right] \\ & - 10^{1/3}\left[13230\phi_{xx}^6 - 27405\phi_x\phi_{3x}\phi_{xx}^4 - 6\phi_{7x}\phi_x^5 - 760\phi_x^3\phi_{3x}^3\right. \\ & \qquad + 6390\phi_x^2\phi_{xx}^3\phi_{4x} + 12780\phi_x^2\phi_{xx}^2\phi_{3x}^2 - 3420\phi_x^3\phi_{xx}\phi_{3x}\phi_{4x} \\ & \qquad \left. - 1026\phi_{5x}\phi_x^3\phi_{xx}^2 + 216\phi_{3x}\phi_{5x}\phi_x^4 + 108\phi_{xx}\phi_{6x}\phi_x^4 + 135\phi_x^4\phi_{4x}^2\right], \end{aligned} \tag{A.1k}$$

u_6 arbitrary (A.1l)

$$v_6 = \frac{\lambda}{120\phi_x^2}u_6 - \frac{\phi_{xt}\phi_{xx}^2}{48\phi_x^8} + \frac{\phi_t\phi_{xx}^3}{48\phi_x^9} + \frac{49i3^{2/3}5^{1/3}\phi_{xx}^6}{32\cdot 2^{2/3}\lambda^{1/3}\phi_x^{11}} + \frac{\phi_{xx}\phi_{xxt}}{120\phi_x^7} + \frac{\phi_{xt}\phi_{3x}}{180\phi_x^7} - \frac{\phi_t\phi_{xx}\phi_{3x}}{72\phi_x^8}$$
$$- \frac{203i3^{2/3}5^{1/3}\phi_{xx}^4\phi_{3x}}{64\cdot 2^{2/3}\lambda^{1/3}\phi_x^{10}} + \frac{71i5^{1/3}\phi_{xx}^2\phi_{3x}^2}{16\cdot 2^{2/3}3^{1/3}\lambda^{1/3}\phi_x^9} - \frac{19i5^{1/3}\phi_{3x}^3}{72\cdot 2^{2/3}3^{1/3}\lambda^{1/3}\phi_x^8} - \frac{\phi_{3xt}}{720\phi_x^6} + \frac{\phi_t\phi_{4x}}{720\phi_x^7}$$
$$+ \frac{71i5^{1/3}\phi_{xx}^3\phi_{4x}}{32\cdot 2^{2/3}3^{1/3}\lambda^{1/3}\phi_x^9} - \frac{19i5^{1/3}\phi_{xx}\phi_{3x}\phi_{4x}}{16\cdot 2^{2/3}3^{1/3}\lambda^{1/3}\phi_x^8} + \frac{i3^{2/3}5^{1/3}\phi_{4x}^2}{64\cdot 2^{2/3}\lambda^{1/3}\phi_x^7} - \frac{19i3^{2/3}\phi_{xx}^2\phi_{5x}}{32\cdot 10^{2/3}\lambda^{1/3}\phi_x^8}$$
$$+ \frac{i3^{2/3}\phi_{3x}\phi_{5x}}{8\cdot 10^{2/3}\lambda^{1/3}\phi_x^7} + \frac{i3^{2/3}\phi_{xx}\phi_{6x}}{16\cdot 10^{2/3}\lambda^{1/3}\phi_x^7} - \frac{i\phi_{7x}}{96\cdot 3^{1/3}10^{2/3}\lambda^{1/3}\phi_x^6} \tag{A.1m}$$

$$f_7(u_7,v_7,\phi) = \left[576\cdot 2^{2/3}\lambda^{4/3}\phi_x^{11}\right]u_7 - \left[51840\cdot 2^{2/3}\lambda^{1/3}\phi_x^{13}\right]v_7 + 72\cdot 2^{2/3}\lambda^{4/3}\phi_x^9\left[2\phi_x(u_6)_x + 9\phi_{xx}u_6\right]$$
$$+ 108\cdot 2^{2/3}\lambda^{1/3}\phi_x^2\phi_{xx}\left[-15\phi_x\phi_{xt}\phi_{xx}^2 + 15\phi_t\phi_{xx}^3 + 6\phi_x^2\phi_{xx}\phi_{xxt} + 4\phi_x^2\phi_{xt}\phi_{3x}\right.$$
$$\left. -10\phi_t\phi_x\phi_{xx}\phi_{3x} - \phi_x^3\phi_{3xt} + \phi_t\phi_x^2\phi_{4x}\right]$$
$$+ 3i3^{2/3}5^{1/3}\left[38070\phi_{xx}^7 - 87255\phi_x\phi_{xx}^5\phi_{3x} + 50100\phi_x^2\phi_{xx}^3\phi_{3x}^2 - 5720\phi_x^3\phi_{xx}\phi_{3x}^3\right.$$
$$+19800\phi_x^2\phi_{xx}^4\phi_{4x} - 14220\phi_x^3\phi_{xx}^2\phi_{3x}\phi_{4x} + 720\phi_x^4\phi_{3x}^2\phi_{4x}$$
$$+675\phi_x^4\phi_{xx}\phi_{4x}^2 - 3114\phi_x^3\phi_{xx}^3\phi_{5x} + 1080\phi_x^4\phi_{xx}\phi_{3x}\phi_{5x}$$
$$\left. -36\phi_x^5\phi_{4x}\phi_{5x} + 324\phi_x^4\phi_{xx}^2\phi_{6x} - 24\phi_x^5\phi_{3x}\phi_{6x} - 18\phi_x^5\phi_{xx}\phi_{7x}\right], \tag{A.1n}$$

u_7 arbitrary (A.1o)

$$v_7 = \frac{\lambda}{90\phi_x^2}u_7 + \frac{\lambda}{360\phi_x^3}(u_6)_x + \frac{\lambda\phi_{xx}}{80\phi_x^4}u_6 - \frac{\phi_{xt}\phi_{xx}^3}{32\phi_x^{10}} + \frac{\phi_t\phi_{xx}^4}{32\phi_x^{11}} + \frac{141i3^{2/3}5^{1/3}\phi_{xx}^7}{64\cdot 2^{2/3}\lambda^{1/3}\phi_x^{13}} + \frac{\phi_{xx}^2\phi_{xxt}}{80\phi_x^9}$$
$$+ \frac{\phi_{xt}\phi_{xx}\phi_{3x}}{120\phi_x^9} - \frac{\phi_t\phi_{xx}^2\phi_{3x}}{48\phi_x^{10}} - \frac{1939i5^{1/3}\phi_{xx}^5\phi_{3x}}{128\cdot 2^{2/3}3^{1/3}\lambda^{1/3}\phi_x^{12}} + \frac{835i5^{1/3}\phi_{xx}^3\phi_{3x}^2}{96\cdot 2^{2/3}3^{1/3}\lambda^{1/3}\phi_x^{11}} - \frac{143i5^{1/3}\phi_{xx}\phi_{3x}^3}{144\cdot 2^{2/3}3^{1/3}\lambda^{1/3}\phi_x^{10}}$$
$$- \frac{\phi_{xx}\phi_{3xt}}{480\phi_x^8} + \frac{\phi_t\phi_{xx}\phi_{4x}}{480\phi_x^9} + \frac{55i5^{1/3}\phi_{xx}^4\phi_{4x}}{16\cdot 2^{2/3}3^{1/3}\lambda^{1/3}\phi_x^{11}} - \frac{79i5^{1/3}\phi_{xx}^2\phi_{3x}\phi_{4x}}{32\cdot 2^{2/3}3^{1/3}\lambda^{1/3}\phi_x^{10}} + \frac{i5^{1/3}\phi_{3x}^2\phi_{4x}}{8\cdot 2^{2/3}3^{1/3}\lambda^{1/3}\phi_x^9}$$
$$+ \frac{5i3^{2/3}5^{1/3}\phi_{xx}\phi_{4x}^2}{128\cdot 2^{2/3}\lambda^{1/3}\phi_x^9} - \frac{173i\phi_{xx}^3\phi_{5x}}{64\cdot 3^{1/3}10^{2/3}\lambda^{1/3}\phi_x^{10}} + \frac{i3^{2/3}5^{1/3}\phi_{xx}\phi_{3x}\phi_{5x}}{16\cdot 2^{2/3}\lambda^{1/3}\phi_x^9} - \frac{i\phi_{4x}\phi_{5x}}{32\cdot 3^{1/3}10^{2/3}\lambda^{1/3}\phi_x^8}$$
$$+ \frac{3i3^{2/3}\phi_{xx}^2\phi_{6x}}{32\cdot 10^{2/3}\lambda^{1/3}\phi_x^9} - \frac{i\phi_{3x}\phi_{6x}}{48\cdot 3^{1/3}10^{2/3}\lambda^{1/3}\phi_x^8} - \frac{i\phi_{xx}\phi_{7x}}{64\cdot 3^{1/3}10^{2/3}\lambda^{1/3}\phi_x^8} \tag{A.1p}$$

$$
\begin{aligned}
u_8 = &-\frac{1}{\phi_x}(u_7)_x - \frac{2\phi_{xx}}{\phi_x^2}u_7 - \frac{1}{2\phi_x^2}(u_6)_{xx} - \frac{3\phi_{xx}}{2\phi_x^3}(u_6)_x - \left[\frac{3\phi_{xx}^2}{4\phi_x^4} + \frac{\phi_{3x}}{2\phi_x^3}\right]u_6 + \frac{183\phi_{xt}\phi_{xx}^4}{8\lambda\phi_x^{10}} - \frac{183\phi_t\phi_{xx}^5}{8\lambda\phi_x^{11}} \\
&- \frac{34335i3^{2/3}5^{1/3}\phi_{xx}^8}{64\cdot 2^{2/3}\lambda^{4/3}\phi_x^{13}} - \frac{21\phi_{xx}^3\phi_{xxt}}{2\lambda\phi_x^9} - \frac{105\phi_{xt}\phi_{xx}^2\phi_{3x}}{4\lambda\phi_x^9} + \frac{147\phi_t\phi_{xx}^3\phi_{3x}}{4\lambda\phi_x^{10}} \\
&+ \frac{60375i3^{2/3}5^{1/3}\phi_{xx}^6\phi_{3x}}{32\cdot 2^{2/3}\lambda^{4/3}\phi_x^{12}} + \frac{15\phi_{xx}\phi_{xxt}\phi_{3x}}{2\lambda\phi_x^8} + \frac{19\phi_{xt}\phi_{3x}^2}{6\lambda\phi_x^8} - \frac{32\phi_t\phi_{xx}\phi_{3x}^2}{3\lambda\phi_x^9} - \frac{32125i3^{2/3}5^{1/3}\phi_{xx}^4\phi_{3x}^2}{16\cdot 2^{2/3}\lambda^{4/3}\phi_x^{11}} \\
&+ \frac{985i10^{1/3}\phi_{xx}^2\phi_{3x}^3}{3^{1/3}\lambda^{4/3}\phi_x^{10}} - \frac{1145i5^{1/3}\phi_{3x}^4}{12\cdot 2^{2/3}3^{1/3}\lambda^{4/3}\phi_x^9} + \frac{23\phi_{xx}^2\phi_{3xt}}{8\lambda\phi_x^8} - \frac{11\phi_{3x}\phi_{3xt}}{12\lambda\phi_x^7} + \frac{4\phi_{xt}\phi_{xx}\phi_{4x}}{\lambda\phi_x^8} \\
&- \frac{55\phi_t\phi_{xx}^2\phi_{4x}}{8\lambda\phi_x^9} - \frac{2235i3^{2/3}5^{1/3}\phi_{xx}^5\phi_{4x}}{4\cdot 2^{2/3}\lambda^{4/3}\phi_x^{11}} - \frac{3\phi_{xxt}\phi_{4x}}{4\lambda\phi_x^7} + \frac{5\phi_t\phi_{3x}\phi_{4x}}{3\lambda\phi_x^8} + \frac{7255i3^{2/3}5^{1/3}\phi_{xx}^3\phi_{3x}\phi_{4x}}{8\cdot 2^{2/3}\lambda^{4/3}\phi_x^{10}} \\
&- \frac{1045i3^{2/3}5^{1/3}\phi_{xx}\phi_{3x}^2\phi_{4x}}{4\cdot 2^{2/3}\lambda^{4/3}\phi_x^9} - \frac{1425i3^{2/3}5^{1/3}\phi_{xx}^2\phi_{4x}^2}{16\cdot 2^{2/3}\lambda^{4/3}\phi_x^9} + \frac{125i5^{1/3}\phi_{3x}\phi_{4x}^2}{2\cdot 2^{2/3}3^{1/3}\lambda^{4/3}\phi_x^8} - \frac{\phi_{xx}\phi_{4xt}}{2\lambda\phi_x^7} \\
&- \frac{3\phi_{xt}\phi_{5x}}{10\lambda\phi_x^7} + \frac{4\phi_t\phi_{xx}\phi_{5x}}{5\lambda\phi_x^8} + \frac{987i3^{2/3}5^{1/3}\phi_{xx}^4\phi_{5x}}{8\cdot 2^{2/3}\lambda^{4/3}\phi_x^{10}} - \frac{1125i3^{2/3}5^{1/3}\phi_{xx}^2\phi_{3x}\phi_{5x}}{8\cdot 2^{2/3}\lambda^{4/3}\phi_x^9} \\
&+ \frac{197i5^{1/3}\phi_{3x}^2\phi_{5x}}{4\cdot 2^{2/3}3^{1/3}\lambda^{4/3}\phi_x^8} + \frac{11i3^{2/3}10^{1/3}\phi_{xx}\phi_{4x}\phi_{5x}}{\lambda^{4/3}\phi_x^8} - \frac{17i3^{2/3}\phi_{5x}^2}{4\cdot 10^{2/3}\lambda^{4/3}\phi_x^7} + \frac{\phi_{5xt}}{20\lambda\phi_x^6} - \frac{\phi_t\phi_{6x}}{20\lambda\phi_x^7} \\
&- \frac{167i3^{2/3}5^{1/3}\phi_{xx}^3\phi_{6x}}{8\cdot 2^{2/3}\lambda^{4/3}\phi_x^9} + \frac{173i5^{1/3}\phi_{xx}\phi_{3x}\phi_{6x}}{4\cdot 2^{2/3}3^{1/3}\lambda^{4/3}\phi_x^8} - \frac{17i5^{1/3}\phi_{4x}\phi_{6x}}{4\cdot 2^{2/3}3^{1/3}\lambda^{4/3}\phi_x^7} + \frac{43i3^{2/3}5^{1/3}\phi_{xx}^2\phi_{7x}}{16\cdot 2^{2/3}\lambda^{4/3}\phi_x^8} \\
&- \frac{19i5^{1/3}\phi_{3x}\phi_{7x}}{8\cdot 2^{2/3}3^{1/3}\lambda^{4/3}\phi_x^7} - \frac{i3^{2/3}5^{1/3}\phi_{xx}\phi_{8x}}{4\cdot 2^{2/3}\lambda^{4/3}\phi_x^7} + \frac{i5^{1/3}\phi_{9x}}{24\cdot 2^{2/3}3^{1/3}\lambda^{4/3}\phi_x^6},
\end{aligned}
\tag{A.1q}
$$

$$
\begin{aligned}
v_8 = &-\frac{\lambda}{90\phi_x^3}(u_7)_x - \frac{\lambda\phi_{xx}}{90\phi_x^4}u_7 - \frac{\lambda}{144\phi_x^4}(u_6)_{xx} - \frac{\lambda\phi_{xx}}{60\phi_x^5}(u_6)_x + \left[\frac{\lambda\phi_{xx}^2}{120\phi_x^6} - \frac{\lambda\phi_{3x}}{90\phi_x^5}\right]u_6 + \frac{13\phi_{xt}\phi_{xx}^4}{48\phi_x^{12}} \\
&- \frac{13\phi_t\phi_{xx}^5}{48\phi_x^{13}} - \frac{539i3^{2/3}5^{1/3}\phi_{xx}^8}{128\cdot 2^{2/3}\lambda^{1/3}\phi_x^{15}} - \frac{61\phi_{xx}^3\phi_{xxt}}{480\phi_x^{11}} - \frac{41\phi_{xt}\phi_{xx}^2\phi_{3x}}{120\phi_x^{11}} + \frac{15\phi_t\phi_{xx}^3\phi_{3x}}{32\phi_x^{12}} \\
&+ \frac{859i5^{1/3}\phi_{xx}^6\phi_{3x}}{16\cdot 2^{2/3}3^{1/3}\lambda^{1/3}\phi_x^{14}} + \frac{\phi_{xx}\phi_{xxt}\phi_{3x}}{10\phi_x^{10}} + \frac{89\phi_{xt}\phi_{3x}^2}{2160\phi_x^{10}} - \frac{61\phi_t\phi_{xx}\phi_{3x}^2}{432\phi_x^{11}} - \frac{6295i5^{1/3}\phi_{xx}^4\phi_{3x}^2}{96\cdot 2^{2/3}3^{1/3}\lambda^{1/3}\phi_x^{13}} \\
&+ \frac{565i5^{1/3}\phi_{xx}^2\phi_{3x}^3}{24\cdot 2^{2/3}3^{1/3}\lambda^{1/3}\phi_x^{12}} - \frac{515i5^{1/3}\phi_{3x}^4}{432\cdot 2^{2/3}3^{1/3}\lambda^{1/3}\phi_x^{11}} + \frac{53\phi_{xx}^2\phi_{3xt}}{1440\phi_x^{10}} - \frac{13\phi_{3x}\phi_{3xt}}{1080\phi_x^9} + \frac{\phi_{xt}\phi_{xx}\phi_{4x}}{18\phi_x^{10}} \\
&- \frac{133\phi_t\phi_{xx}^2\phi_{4x}}{1440\phi_x^{11}} - \frac{571i5^{1/3}\phi_{xx}^5\phi_{4x}}{32\cdot 2^{2/3}3^{1/3}\lambda^{1/3}\phi_x^{13}} - \frac{\phi_{xxt}\phi_{4x}}{96\phi_x^9} + \frac{97\phi_t\phi_{3x}\phi_{4x}}{4320\phi_x^{10}} + \frac{6215i5^{1/3}\phi_{xx}^3\phi_{3x}\phi_{4x}}{192\cdot 2^{2/3}3^{1/3}\lambda^{1/3}\phi_x^{12}} \\
&- \frac{955i5^{1/3}\phi_{xx}\phi_{3x}^2\phi_{4x}}{96\cdot 2^{2/3}3^{1/3}\lambda^{1/3}\phi_x^{11}} - \frac{73i3^{2/3}5^{1/3}\phi_{xx}^2\phi_{4x}^2}{64\cdot 2^{2/3}\lambda^{1/3}\phi_x^{11}} + \frac{949i5^{1/3}\phi_{3x}\phi_{4x}^2}{1152\cdot 2^{2/3}3^{1/3}\lambda^{1/3}\phi_x^{10}} - \frac{\phi_{xx}\phi_{4xt}}{144\phi_x^9} - \frac{\phi_{xt}\phi_{5x}}{240\phi_x^9} \\
&+ \frac{\phi_t\phi_{xx}\phi_{5x}}{90\phi_x^{10}} + \frac{551i5^{1/3}\phi_{xx}^4\phi_{5x}}{128\cdot 2^{2/3}3^{1/3}\lambda^{1/3}\phi_x^{12}} - \frac{107i\phi_{xx}^2\phi_{3x}\phi_{5x}}{4\cdot 3^{1/3}10^{2/3}\lambda^{1/3}\phi_x^{11}} + \frac{185i5^{1/3}\phi_{3x}^2\phi_{5x}}{288\cdot 2^{2/3}3^{1/3}\lambda^{1/3}\phi_x^{10}} \\
&+ \frac{85i5^{1/3}\phi_{xx}\phi_{4x}\phi_{5x}}{96\cdot 2^{2/3}3^{1/3}\lambda^{1/3}\phi_x^{10}} - \frac{167i\phi_{5x}^2}{960\cdot 3^{1/3}10^{2/3}\lambda^{1/3}\phi_x^9} + \frac{\phi_{5xt}}{1440\phi_x^8} - \frac{\phi_t\phi_{6x}}{1440\phi_x^9} - \frac{251i\phi_{xx}^3\phi_{6x}}{64\cdot 3^{1/3}10^{2/3}\lambda^{1/3}\phi_x^{11}} \\
&+ \frac{413i\phi_{xx}\phi_{3x}\phi_{6x}}{144\cdot 3^{1/3}10^{2/3}\lambda^{1/3}\phi_x^{10}} - \frac{167i\phi_{4x}\phi_{6x}}{576\cdot 3^{1/3}10^{2/3}\lambda^{1/3}\phi_x^9} + \frac{103i\phi_{xx}^2\phi_{7x}}{192\cdot 3^{1/3}10^{2/3}\lambda^{1/3}\phi_x^{10}} \\
&- \frac{23i\phi_{3x}\phi_{7x}}{144\cdot 3^{1/3}10^{2/3}\lambda^{1/3}\phi_x^9} - \frac{i5^{1/3}\phi_{xx}\phi_{8x}}{96\cdot 2^{2/3}3^{1/3}\lambda^{1/3}\phi_x^9} + \frac{i5^{1/3}\phi_{9x}}{1728\cdot 2^{2/3}3^{1/3}\lambda^{1/3}\phi_x^8}
\end{aligned}
\tag{A.1r}
$$

$$
\begin{aligned}
f_9(u_9,v_9,\phi) = &\left[5760\cdot 2^{2/3}3^{1/3}\lambda^{4/3}\phi_x^{15}\right]u_9 - \left[345600\cdot 2^{2/3}3^{1/3}\lambda^{1/3}\phi_x^{17}\right]v_9 \\
&+120\cdot 2^{2/3}3^{1/3}\lambda^{4/3}\phi_x^9\left[-8\phi_x^4(u_7)_{xx}-56\phi_x^3\phi_{xx}(u_7)_x-\left(16\phi_x^2\phi_{xx}^2+32\phi_x^3\phi_{3x}\right)u_7\right. \\
&\qquad -4\phi_x^3(u_6)_{3x}-34\phi_x^2\phi_{xx}(u_6)_{xx}-\left(20\phi_x^2\phi_{3x}+42\phi_x\phi_{xx}^2\right)(u_6)_x \\
&\qquad \left.+\left(63\phi_{xx}^3-72\phi_x\phi_{xx}\phi_{3x}-\phi_x^2\phi_{4x}\right)u_6\right] \\
&+4\cdot 2^{2/3}3^{1/3}\lambda^{1/3}\phi_x^2\left[-25515\phi_x\phi_{xt}\phi_{xx}^5+25515\phi_t\phi_{xx}^6+11790\phi_x^2\phi_{xx}^4\phi_{xxt}+45060\phi_x^2\phi_{xt}\phi_{xx}^3\phi_{3x}\right. \\
&\qquad -56850\phi_t\phi_x\phi_{xx}^4\phi_{3x}-16020\phi_x^3\phi_{xx}^2\phi_{xxt}\phi_{3x}-16380\phi_x^3\phi_{xt}\phi_{xx}\phi_{3x}^2 \\
&\qquad +32400\phi_t\phi_x^2\phi_{xx}^2\phi_{3x}^2+2560\phi_x^4\phi_{xxt}\phi_{3x}^2-2560\phi_t\phi_x^3\phi_{3x}^3-3285\phi_x^3\phi_{xx}^3\phi_{3xt} \\
&\qquad +3280\phi_x^4\phi_{xx}\phi_{3x}\phi_{3xt}-8655\phi_x^3\phi_{xt}\phi_{xx}^2\phi_{4x}+11940\phi_t\phi_x^2\phi_{xx}^3\phi_{4x} \\
&\qquad +2760\phi_x^4\phi_{xx}\phi_{xxt}\phi_{4x}+2940\phi_x^4\phi_{xt}\phi_{3x}\phi_{4x}-8980\phi_t\phi_x^3\phi_{xx}\phi_{3x}\phi_{4x} \\
&\qquad -415\phi_x^5\phi_{3xt}\phi_{4x}+415\phi_t\phi_x^4\phi_{4x}^2+630\phi_x^4\phi_{xx}^2\phi_{4xt}-340\phi_x^5\phi_{3x}\phi_{4xt} \\
&\qquad +1116\phi_x^4\phi_{xt}\phi_{xx}\phi_{5x}-1746\phi_t\phi_x^3\phi_{xx}^2\phi_{5x}-252\phi_x^5\phi_{xxt}\phi_{5x} \\
&\qquad +592\phi_t\phi_x^4\phi_{3x}\phi_{5x}-102\phi_x^5\phi_{xx}\phi_{5xt}-84\phi_x^5\phi_{xt}\phi_{6x}+186\phi_t\phi_x^4\phi_{xx}\phi_{6x} \\
&\qquad \left.+12\phi_x^6\phi_{6xt}-12\phi_t\phi_x^5\phi_{7x}\right] \\
&+i5^{1/3}\left[13565475\phi_{xx}^9-51270975\phi_x\phi_{xx}^7\phi_{3x}+65551500\phi_x^2\phi_{xx}^5\phi_{3x}^2-31677000\phi_x^3\phi_{xx}^3\phi_{3x}^3\right. \\
&\qquad +4188000\phi_x^4\phi_{xx}\phi_{3x}^4+14156100\phi_x^2\phi_{xx}^6\phi_{4x}-30193875\phi_x^3\phi_{xx}^4\phi_{3x}\phi_{4x} \\
&\qquad +15844500\phi_x^4\phi_{xx}^2\phi_{3x}^2\phi_{4x}-1153400\phi_x^5\phi_{3x}^3\phi_{4x}+3256425\phi_x^4\phi_{xx}^3\phi_{4x}^2 \\
&\qquad -2146350\phi_x^5\phi_{xx}\phi_{3x}\phi_{4x}^2+61875\phi_x^6\phi_{4x}^3-2959200\phi_x^3\phi_{xx}^5\phi_{5x}+5059080\phi_x^4\phi_{xx}^3\phi_{3x}\phi_{5x} \\
&\qquad -1654440\phi_x^5\phi_{xx}\phi_{3x}^2\phi_{5x}-989910\phi_x^5\phi_{xx}^2\phi_{4x}\phi_{5x}+283320\phi_x^6\phi_{3x}\phi_{4x}\phi_{5x} \\
&\qquad +63324\phi_x^6\phi_{xx}\phi_{5x}^2+491940\phi_x^4\phi_{xx}^4\phi_{6x}-634200\phi_x^5\phi_{xx}^2\phi_{3x}\phi_{6x}+89760\phi_x^6\phi_{3x}^2\phi_{6x} \\
&\qquad +105360\phi_x^6\phi_{xx}\phi_{4x}\phi_{6x}-9048\phi_x^7\phi_{5x}\phi_{6x}-66510\phi_x^5\phi_{xx}^3\phi_{7x}+57120\phi_x^6\phi_{xx}\phi_{3x}\phi_{7x} \\
&\qquad \left.-6450\phi_x^7\phi_{4x}\phi_{7x}+7380\phi_x^6\phi_{xx}^2\phi_{8x}-3000\phi_x^7\phi_{3x}\phi_{8x}-660\phi_x^7\phi_{xx}\phi_{9x}+40\phi_x^8\phi_{10x}\right],
\end{aligned}
\tag{A.1s}
$$

u_9 arbitrary (A.1t)

$$
\begin{aligned}
v_9 = {} & \frac{\lambda}{60\phi_x^2}u_9 - \frac{\lambda}{360\phi_x^4}(u_7)_{xx} - \frac{7\lambda\phi_{xx}}{360\phi_x^5}(u_7)_x - \left[\frac{\lambda\phi_{xx}^2}{180\phi_x^6} + \frac{\lambda\phi_{3x}}{90\phi_x^5}\right]u_7 - \frac{\lambda}{720\phi_x^5}(u_6)_{3x} - \frac{17\lambda\phi_{xx}}{1440\phi_x^6}(u_6)_{xx} \\
& - \left[\frac{7\lambda\phi_{xx}^2}{480\phi_x^7} + \frac{\lambda\phi_{3x}}{144\phi_x^6}\right](u_6)_x + \left[\frac{7\lambda\phi_{xx}^3}{320\phi_x^8} - \frac{\lambda\phi_{xx}\phi_{3x}}{40\phi_x^7} - \frac{\lambda\phi_{4x}}{2880\phi_x^6}\right]u_6 - \frac{189\phi_{xt}\phi_{xx}^5}{640\phi_x^{14}} + \frac{189\phi_t\phi_{xx}^6}{640\phi_x^{15}} \\
& + \frac{6699i3^{2/3}5^{1/3}\phi_{xx}^9}{512\cdot 2^{2/3}\lambda^{1/3}\phi_x^{17}} + \frac{131\phi_{xx}^4\phi_{xxt}}{960\phi_x^{13}} + \frac{751\phi_{xt}\phi_{xx}^3\phi_{3x}}{1440\phi_x^{13}} - \frac{379\phi_t\phi_{xx}^4\phi_{3x}}{576\phi_x^{14}} - \frac{4\phi_t\phi_{3x}^3}{135\phi_x^{12}} \\
& - \frac{25319i3^{2/3}5^{1/3}\phi_{xx}^7\phi_{3x}}{512\cdot 2^{2/3}\lambda^{1/3}\phi_x^{16}} - \frac{89\phi_{xx}^2\phi_{xxt}\phi_{3x}}{480\phi_x^{12}} - \frac{91\phi_{xt}\phi_{xx}\phi_{3x}^2}{480\phi_x^{12}} + \frac{3\phi_t\phi_{xx}^2\phi_{3x}^2}{8\phi_x^{13}} + \frac{4\phi_{xxt}\phi_{3x}^2}{135\phi_x^{11}} \\
& + \frac{72835i5^{1/3}\phi_{xx}^5\phi_{3x}^2}{384\cdot 2^{2/3}3^{1/3}\lambda^{1/3}\phi_x^{15}} - \frac{52795i5^{1/3}\phi_{xx}^3\phi_{3x}^3}{576\cdot 2^{2/3}3^{1/3}\lambda^{1/3}\phi_x^{14}} + \frac{1745i5^{1/3}\phi_{xx}\phi_{3x}^4}{144\cdot 2^{2/3}3^{1/3}\lambda^{1/3}\phi_x^{13}} - \frac{73\phi_{xx}^3\phi_{3xt}}{1920\phi_x^{12}} \\
& + \frac{41\phi_{xx}\phi_{3x}\phi_{3xt}}{1080\phi_x^{11}} - \frac{577\phi_{xt}\phi_{xx}^2\phi_{4x}}{5760\phi_x^{12}} + \frac{199\phi_t\phi_{xx}^3\phi_{4x}}{1440\phi_x^{13}} + \frac{5243i5^{1/3}\phi_{xx}^6\phi_{4x}}{128\cdot 2^{2/3}3^{1/3}\lambda^{1/3}\phi_x^{15}} + \frac{23\phi_{xx}\phi_{xxt}\phi_{4x}}{720\phi_x^{11}} \\
& + \frac{49\phi_{xt}\phi_{3x}\phi_{4x}}{1440\phi_x^{11}} - \frac{449\phi_t\phi_{xx}\phi_{3x}\phi_{4x}}{4320\phi_x^{12}} - \frac{134195i5^{1/3}\phi_{xx}^4\phi_{3x}\phi_{4x}}{1536\cdot 2^{2/3}3^{1/3}\lambda^{1/3}\phi_x^{14}} + \frac{17605i5^{1/3}\phi_{xx}^2\phi_{3x}^2\phi_{4x}}{384\cdot 2^{2/3}3^{1/3}\lambda^{1/3}\phi_x^{13}} \\
& - \frac{5767i5^{1/3}\phi_{3x}^3\phi_{4x}}{1728\cdot 2^{2/3}3^{1/3}\lambda^{1/3}\phi_x^{12}} - \frac{83\phi_{3xt}\phi_{4x}}{17280\phi_x^{10}} + \frac{83\phi_t\phi_{4x}^2}{17280\phi_x^{11}} + \frac{14473i5^{1/3}\phi_{xx}^3\phi_{4x}^2}{1536\cdot 2^{2/3}3^{1/3}\lambda^{1/3}\phi_x^{13}} + \frac{7\phi_{xx}^2\phi_{4xt}}{960\phi_x^{11}} \\
& - \frac{14309i5^{1/3}\phi_{xx}\phi_{3x}\phi_{4x}^2}{2304\cdot 2^{2/3}3^{1/3}\lambda^{1/3}\phi_x^{12}} + \frac{275i5^{1/3}\phi_{4x}^3}{1536\cdot 2^{2/3}3^{1/3}\lambda^{1/3}\phi_x^{11}} - \frac{17\phi_{3x}\phi_{4xt}}{4320\phi_x^{10}} + \frac{31\phi_{xt}\phi_{xx}\phi_{5x}}{2400\phi_x^{11}} \\
& - \frac{137i5^{1/3}\phi_{xx}^5\phi_{5x}}{16\cdot 2^{2/3}3^{1/3}\lambda^{1/3}\phi_x^{14}} - \frac{7\phi_{xxt}\phi_{5x}}{2400\phi_x^{10}} + \frac{37\phi_t\phi_{3x}\phi_{5x}}{5400\phi_x^{11}} + \frac{14053i\phi_{xx}^3\phi_{3x}\phi_{5x}}{192\cdot 3^{1/3}10^{2/3}\lambda^{1/3}\phi_x^{13}} - \frac{17\phi_{xx}\phi_{5xt}}{14400\phi_x^{10}} \\
& - \frac{13787i\phi_{xx}\phi_{3x}^2\phi_{5x}}{576\cdot 3^{1/3}10^{2/3}\lambda^{1/3}\phi_x^{12}} - \frac{10999i\phi_{xx}^2\phi_{4x}\phi_{5x}}{768\cdot 3^{1/3}10^{2/3}\lambda^{1/3}\phi_x^{12}} + \frac{787i\phi_{3x}\phi_{4x}\phi_{5x}}{192\cdot 3^{1/3}10^{2/3}\lambda^{1/3}\phi_x^{11}} - \frac{7\phi_{xt}\phi_{6x}}{7200\phi_x^{10}} \\
& + \frac{1759i\phi_{xx}\phi_{5x}^2}{1920\cdot 3^{1/3}10^{2/3}\lambda^{1/3}\phi_x^{11}} + \frac{31\phi_t\phi_{xx}\phi_{6x}}{14400\phi_x^{11}} + \frac{911i\phi_{xx}^4\phi_{6x}}{128\cdot 3^{1/3}10^{2/3}\lambda^{1/3}\phi_x^{13}} - \frac{1057i5^{1/3}\phi_{xx}^2\phi_{3x}\phi_{6x}}{576\cdot 2^{2/3}3^{1/3}\lambda^{1/3}\phi_x^{12}} \\
& + \frac{187i\phi_{3x}^2\phi_{6x}}{144\cdot 3^{1/3}10^{2/3}\lambda^{1/3}\phi_x^{11}} + \frac{439i\phi_{xx}\phi_{4x}\phi_{6x}}{288\cdot 3^{1/3}10^{2/3}\lambda^{1/3}\phi_x^{11}} - \frac{377i\phi_{5x}\phi_{6x}}{2880\cdot 3^{1/3}10^{2/3}\lambda^{1/3}\phi_x^{10}} + \frac{\phi_{6xt}}{7200\phi_x^9} \\
& - \frac{739i\phi_{xx}^3\phi_{7x}}{768\cdot 3^{1/3}10^{2/3}\lambda^{1/3}\phi_x^{12}} + \frac{119i\phi_{xx}\phi_{3x}\phi_{7x}}{144\cdot 3^{1/3}10^{2/3}\lambda^{1/3}\phi_x^{11}} - \frac{43i5^{1/3}\phi_{4x}\phi_{7x}}{2304\cdot 2^{2/3}3^{1/3}\lambda^{1/3}\phi_x^{10}} \\
& - \frac{5i5^{1/3}\phi_{3x}\phi_{8x}}{576\cdot 2^{2/3}3^{1/3}\lambda^{1/3}\phi_x^{10}} - \frac{\phi_t\phi_{7x}}{7200\phi_x^{10}} + \frac{41i\phi_{xx}^2\phi_{8x}}{384\cdot 3^{1/3}10^{2/3}\lambda^{1/3}\phi_x^{11}} - \frac{11i\phi_{xx}\phi_{9x}}{1152\cdot 3^{1/3}10^{2/3}\lambda^{1/3}\phi_x^{10}} \\
& + \frac{i\phi_{10x}}{1728\cdot 3^{1/3}10^{2/3}\lambda^{1/3}\phi_x^9} - \frac{97\phi_t\phi_{xx}^2\phi_{5x}}{4800\phi_x^{12}}
\end{aligned}
\tag{A.1u}
$$

Using (2.4.4) in (2.4.1a) under the Singular Manifold Method, the coefficients in powers of ϕ are

$$\mathcal{O}(\phi^{-3}): \quad \phi_x^2\left[2^{2/3}\phi_t + 2^{2/3}u_1\phi_x + 6v_1\phi_x - 4i2^{1/3}\phi_{xx}\right] = 0, \tag{A.2a}$$

$$
\begin{aligned}
\mathcal{O}(\phi^{-2}): \quad & iv_1\phi_t\phi_x + iu_1v_1\phi_x^2 + 3i2^{1/3}v_1^2\phi_x^2 - 2^{2/3}\phi_x^2(v_1)_x \\
& - 2^{1/3}\phi_x\phi_{xt} - 2^{1/3}u_1\phi_x\phi_{xx} - 2^{2/3}v_1\phi_x\phi_{xx} + \frac{3}{2}\phi_{xx}^2 + 2i\phi_x\phi_{3x} = 0,
\end{aligned}
\tag{A.2b}
$$

$$
\begin{aligned}
\mathcal{O}(\phi^{-1}): \quad & 4v_1^3\phi_x + 2i(v_1)_t\phi_x + 2iu_1(v_1)_x\phi_x \\
& + 2i2^{1/3}v_1(v_1)_x\phi_x + 2iv_1\phi_{xt} + 2iu_1v_1\phi_{xx} + i\phi_{xxxx} = 0,
\end{aligned}
\tag{A.2c}
$$

$$\mathcal{O}(\phi^0): \quad 2v_1(v_1)_t + 2u_1v_1(v_1)_x - iv_1^4 + (v_1)_{3x} = 0. \tag{A.2d}$$

And from (2.4.1b),

$$\mathcal{O}(\phi^{-1}): \quad 2iv_1\phi_x + 2^{1/3}\phi_{xx} = 0, \tag{A.3a}$$

$$\mathcal{O}(\phi^{0}): \quad v_1^2 - i(u_1)_x = 0. \tag{A.3b}$$

Using (2.4.13) in (2.4.10a), we obtain the ϕ-coefficient equations

$$\mathcal{O}(\phi^{-7}): \quad 3^{1/3}\phi_x u_1 - 120\cdot 3^{1/3}\phi_x^3 v_1 + 315i10^{1/3}\phi_x^2\phi_{xx} = 0, \tag{A.4a}$$

$$\begin{aligned}\mathcal{O}(\phi^{-6}): \quad & -12i\phi_x^2 u_1 v_1 - 4\cdot 3^{2/3}10^{1/3}\phi_x^3 u_2 + 2280i\phi_x^4 v_1^2 + 180\cdot 3^{2/3}10^{1/3}\phi_x^4(v_1)_x \\ & + 4\cdot 3^{2/3}10^{1/3}\phi_x\phi_{xx}u_1 + 1140\cdot 3^{2/3}10^{1/3}\phi_x^3\phi_{xx}v_1 + 1125i3^{1/3}10^{2/3}\phi_x^2\phi_{xx}^2 \\ & + 240i3^{1/3}10^{2/3}\phi_x^3\phi_{3x} = 0,\end{aligned} \tag{A.4b}$$

$$\begin{aligned}\mathcal{O}(\phi^{-5}): \quad & -2\cdot 3^{1/3}10^{2/3}\phi_x^2 u_1 v_1^2 + 60i\phi_x^3 u_2 v_1 + 20\cdot 3^{2/3}10^{1/3}\phi_t\phi_x^3 + 20\cdot 3^{2/3}10^{1/3}\phi_x^4 u_3 \\ & + 820\cdot 3^{1/3}10^{2/3}\phi_x^4 v_1^3 - 20i\phi_x^2 u_1(v_1)_x - 1800i\phi_x^4 v_1(v_1)_x - 60i\phi_x\phi_{xx}u_1v_1 \\ & - 20\cdot 3^{2/3}10^{1/3}\phi_x^2\phi_{xx}u_2 - 5400i\phi_x^3\phi_{xx}v_1^2 + 2700\cdot 3^{2/3}10^{1/3}\phi_x^3\phi_{xx}(v_1)_x \\ & + 4800\cdot 3^{2/3}10^{1/3}\phi_x^2\phi_{xx}^2 v_1 + 900i3^{1/3}10^{2/3}\phi_x\phi_{xx}^3 + 300\cdot 3^{2/3}10^{1/3}\phi_x^4(v_1)_{xx} \\ & + 1300\cdot 3^{2/3}10^{1/3}\phi_x^3\phi_{3x}v_1 + 900i3^{1/3}10^{2/3}\phi_x^2\phi_{xx}\phi_{3x} + 75i3^{1/3}10^{2/3}\phi_x^3\phi_{4x} = 0,\end{aligned} \tag{A.4c}$$

$$\begin{aligned}\mathcal{O}(\phi^{-4}): \quad & 2i3^{2/3}10^{1/3}\phi_x^2 u_1 v_1^3 + 18\cdot 3^{1/3}10^{2/3}\phi_x^3 u_2 v_1^2 - 540i\phi_t\phi_x^3 v_1 - 540i\phi_x^4 u_3 v_1 \\ & - 3135i3^{2/3}10^{1/3}\phi_x^4 v_1^4 - 18\cdot 3^{1/3}10^{2/3}\phi_x^2 u_1 v_1(v_1)_x + 180i\phi_x^3 u_2(v_1)_x \\ & - 270\cdot 3^{1/3}10^{2/3}\phi_x^4 v_1^2(v_1)_x - 8100i\phi_x^4(v_1)_x^2 + 180\cdot 3^{2/3}10^{1/3}\phi_x^3\phi_{xt} \\ & - 18\cdot 3^{1/3}10^{2/3}\phi_x\phi_{xx}u_1v_1^2 + 540i\phi_x^2\phi_{xx}u_2v_1 + 180\cdot 3^{2/3}10^{1/3}\phi_x^3\phi_{xx}u_3 \\ & - 630\cdot 3^{1/3}10^{2/3}v_1^3\phi_x^3\phi_{xx} - 56700i\phi_x^3 v_1(v_1)_x\phi_{xx} + 900\cdot 3^{2/3}10^{1/3}\phi_x^3\phi_{4x}v_1 \\ & + 8100\cdot 3^{2/3}10^{1/3}\phi_x^2\phi_{xx}^2(v_1)_x + 5400\cdot 3^{2/3}10^{1/3}\phi_x\phi_{xx}^3 v_1 - 8100i\phi_x^4 v_1(v_1)_{xx} \\ & + 2700\cdot 3^{2/3}10^{1/3}\phi_x^3\phi_{xx}(v_1)_{xx} - 13500i\phi_x^3\phi_{3x}v_1^2 + 2700\cdot 3^{2/3}10^{1/3}\phi_x^3\phi_{3x}(v_1)_x \\ & + 8100\cdot 3^{2/3}10^{1/3}\phi_x^2\phi_{xx}\phi_{3x}v_1 + 225\cdot 3^{2/3}10^{1/3}\phi_x^4(v_1)_{3x} - 36450i\phi_x^2\phi_{xx}^2 v_1^2 = 0,\end{aligned} \tag{A.4d}$$

$$\begin{aligned}\mathcal{O}(\phi^{-3}): \quad & i3^{2/3}10^{1/3}\phi_x^2 u_2 v_1^3 + 9\cdot 3^{1/3}10^{2/3}\phi_t\phi_x^2 v_1^2 + 9\cdot 3^{1/3}10^{2/3}\phi_x^3 u_3 v_1^2 + 1260\phi_x^3 v_1^5 \\ & + 90i(v_1)_t\phi_x^3 - 3i3^{2/3}10^{1/3}\phi_x u_1 v_1^2(v_1)_x - 9\cdot 3^{1/3}10^{2/3}\phi_x^2 u_2 v_1(v_1)_x + 90i\phi_x^3 u_3(v_1)_x \\ & + 270\cdot 3^{1/3}10^{2/3}\phi_x^3 v_1(v_1)_x^2 + 270i\phi_x^2\phi_{xt}v_1 - i3^{2/3}10^{1/3}\phi_{xx}u_1v_1^3 \\ & - 9\cdot 3^{1/3}10^{2/3}\phi_x\phi_{xx}u_2v_1^2 + 270i\phi_x^2\phi_{xx}u_3v_1 - 15i3^{2/3}10^{1/3}\phi_x^2\phi_{xx}v_1^4 \\ & + 675\cdot 3^{1/3}10^{2/3}\phi_x^2 v_1^2(v_1)_x\phi_{xx} + 4050i\phi_x^2\phi_{xx}(v_1)_x^2 + 180\cdot 3^{1/3}10^{2/3}\phi_x\phi_{xx}^2 v_1^3 \\ & + 8100i\phi_x v_1(v_1)_x\phi_{xx}^2 + 1350i\phi_{xx}^3 v_1^2 + 135\cdot 3^{1/3}10^{2/3}\phi_x^3 v_1^2(v_1)_{xx} \\ & + 1350i\phi_x^3(v_1)_x(v_1)_{xx} + 4050i\phi_x^2\phi_{xx}v_1(v_1)_{xx} + 105\cdot 3^{1/3}10^{2/3}\phi_x^2\phi_{3x}v_1^3 \\ & + 4050i\phi_x^2 v_1(v_1)_x\phi_{3x} + 4050i\phi_x\phi_{xx}\phi_{3x}v_1^2 + 450i\phi_x^3 v_1(v_1)_{3x} + 675i\phi_x^2\phi_{4x}v_1^2 = 0,\end{aligned} \tag{A.4e}$$

$$\begin{aligned}
\mathcal{O}(\phi^{-2}):\quad & 4i3^{2/3}10^{1/3}\phi_t\phi_x v_1^3 + 4i3^{2/3}10^{1/3}\phi_x^2 u_3 v_1^3 + 84i3^{1/3}10^{2/3}\phi_x^2 v_1^6 + 4u_1 v_1^3 (v_1)_x \\
& - 4i3^{2/3}10^{1/3}\phi_{xx} u_2 v_1^3 + 180i3^{2/3}10^{1/3}\phi_x v_1^3 (v_1)_x \phi_{xx} + 60i3^{2/3}10^{1/3}\phi_x^2 v_1^3 (v_1)_{xx} \\
& + 20i3^{2/3}10^{1/3}\phi_x\phi_{3x} v_1^4 - 180\cdot 3^{1/3}10^{2/3}\phi_{xx}\phi_{3x} v_1^3 - 60\cdot 3^{1/3}10^{2/3}\phi_x\phi_{4x} v_1^3 \\
& - 36\cdot 3^{1/3}10^{2/3}\phi_x^2 v_1 (v_1)_t - 12i3^{2/3}10^{1/3}\phi_x u_2 v_1^2 (v_1)_x - 36\cdot 3^{1/3}10^{2/3}\phi_x^2 u_3 v_1 (v_1)_x \\
& + 180i3^{2/3}10^{1/3}\phi_x^2 v_1^2 (v_1)_x^2 - 180\cdot 3^{1/3}10^{2/3}\phi_x^2 (v_1)_x^3 - 36\cdot 3^{1/3}10^{2/3}\phi_x\phi_{xt} v_1^2 \\
& - 36\cdot 3^{1/3}10^{2/3}\phi_x\phi_{xx} u_3 v_1^2 - 1080\cdot 3^{1/3}10^{2/3}\phi_x\phi_{xx} v_1 (v_1)_x^2 - 540\cdot 3^{1/3}10^{2/3}\phi_{xx}^2 v_1^2 (v_1)_x \\
& - 540\cdot 3^{1/3}10^{2/3}\phi_x^2 v_1 (v_1)_x (v_1)_{xx} - 15i3^{2/3}10^{1/3}\phi_{xx}^2 v_1^4 - 540\cdot 3^{1/3}10^{2/3}\phi_x\phi_{xx} v_1^2 (v_1)_{xx} \\
& - 540\cdot 3^{1/3}10^{2/3}\phi_x\phi_{3x} v_1^2 (v_1)_x - 90\cdot 3^{1/3}10^{2/3}\phi_x^2 v_1^2 (v_1)_{3x} = 0, \qquad \text{(A.4f)} \\
\mathcal{O}(\phi^{-1}):\quad & 8\cdot 3^{2/3}10^{1/3}\phi_x v_1^7 + 12i3^{2/3}10^{1/3}\phi_x v_1^2 (v_1)_t - 4u_2 v_1^3 (v_1)_x + 12i3^{2/3}10^{1/3}\phi_x u_3 v_1^2 (v_1)_x \\
& + 120i3^{2/3}10^{1/3}\phi_x v_1 (v_1)_x^3 + 4i3^{2/3}10^{1/3}\phi_{xt} v_1^3 + 4i3^{2/3}10^{1/3}\phi_{xx} u_3 v_1^3 \\
& + 180i3^{2/3}10^{1/3}\phi_{xx} v_1^2 (v_1)_x^2 + 180i3^{2/3}10^{1/3}\phi_x v_1^2 (v_1)_x (v_1)_{xx} + 5i3^{2/3}10^{1/3}\phi_{4x} v_1^4 \\
& + 60i3^{2/3}10^{1/3}\phi_{xx} v_1^3 (v_1)_{xx} + 60i3^{2/3}10^{1/3}\phi_{3x} v_1^3 (v_1)_x + 20i3^{2/3}10^{1/3}\phi_x v_1^3 (v_1)_{3x} = 0, \qquad \text{(A.4g)} \\
\mathcal{O}(\phi^{0}):\quad & -iv_1^7 + 4v_1^2 (v_1)_t + 4u_3 v_1^2 (v_1)_x + 60 v_1 (v_1)_x^3 + 60 v_1^2 (v_1)_x (v_1)_{xx} + 5 v_1^3 (v_1)_{3x} = 0. \qquad \text{(A.4h)}
\end{aligned}$$

And from (2.4.10b)

$$\begin{aligned}
\mathcal{O}(\phi^{-3}):\quad & i\phi_x u_1 + 180i\phi_x^3 v_1 + 45\cdot 3^{2/3}10^{1/3}\phi_x^2\phi_{xx} = 0, \qquad \text{(A.5a)} \\
\mathcal{O}(\phi^{-2}):\quad & \phi_x u_2 + 18i3^{1/3}10^{2/3}\phi_x^2 v_1^2 - (u_1)_x = 0, \qquad \text{(A.5b)} \\
\mathcal{O}(\phi^{-1}):\quad & 4\cdot 3^{2/3}10^{1/3}\phi_x v_1^3 + (u_2)_x = 0, \qquad \text{(A.5c)} \\
\mathcal{O}(\phi^{0}):\quad & v_1^4 - i(u_3)_x = 0. \qquad \text{(A.5d)}
\end{aligned}$$

Under the Invariant Painlevé Analysis, (2.4.1a) gives the χ-coefficient equations

$$\begin{aligned}
\mathcal{O}(\chi^{-4}):\quad & v_0(-6i - 2iu_0 v_0 + v_0^3) = 0, \qquad \text{(A.6a)} \\
\mathcal{O}(\chi^{-3}):\quad & Cv_0^2 - 2iv_0^3 v_1 + 3(v_0)_x + u_0 v_0 (v_0)_x - u_1 v_0^2 - u_0 v_0 v_1 = 0, \qquad \text{(A.6b)} \\
\mathcal{O}(\chi^{-2}):\quad & 2Cv_0 v_1 - 6iv_0^2 v_1^2 + 2v_0 (v_0)_t + 2u_1 v_0 (v_0)_x + 2u_0 v_1 (v_0)_x (v_1)_x \\
& + 2u_0 v_0 - 4Sv_0 - Su_0 v_0^2 - 2u_1 v_0 v_1 - 2v_0^2 C_x - 3(v_0)_{xx} = 0, \qquad \text{(A.6c)} \\
\mathcal{O}(\chi^{-1}):\quad & CSv_0^2 + 2v_1 (v_0)_t + 2u_1 v_1 (v_0)_x + 2u_0 v_1 (v_1)_x - 4iv_0 v_1^3 + 2v_0 (v_1)_t + v_0 S_x \\
& + 2v_0 u_1 (v_1)_x + v_0^2 C_{xx} + (v_0)_{3x} - 2v_0 v_1 C_x - Su_1 v_0^2 - Su_0 v_0 v_1 + 3S(v_0)_x = 0, \qquad \text{(A.6d)} \\
\mathcal{O}(\chi^{0}):\quad & S^2 v_0 - 2CSv_0 v_1 + 2Su_1 v_0 v_1 + 2iv_1^4 + 3S_x (v_0)_x + v_0 S_{xx} \\
& + 3S(v_0)_{xx} - 4v_1 (v_1)_t - 4u_1 v_1 (v_1)_x - 2v_0 v_1 C_{xx} - 2(v_1)_{3x} = 0, \qquad \text{(A.6e)}
\end{aligned}$$

and (2.4.1b) gives

$$\begin{aligned}
\mathcal{O}(\chi^{-2}):\quad & iu_0 + v_0^2 = 0, \qquad \text{(A.7a)} \\
\mathcal{O}(\chi^{-1}):\quad & 2v_0 v_1 - i(u_0)_x = 0, \qquad \text{(A.7b)} \\
\mathcal{O}(\chi^{0}):\quad & iSu_0 + 2v_1^2 - 2i(u_1)_x = 0, \qquad \text{(A.7c)}
\end{aligned}$$

Under the homogeneous balance method, using (2.6.14) in (2.6.2a) gives

$$
\begin{aligned}
&- i\lambda v_1^4 + 2v_1(v_1)_t - 2\cdot 2^{2/3} i^{2/3}\lambda^{4/3} v_1^3 f'\phi_x - i2^{2/3} i^{2/3}\lambda^{1/3} v_1 f''\phi_t\phi_x - i2^{2/3} i^{2/3}\lambda^{1/3} f'(v_1)_t\phi_x \\
&\quad - 3\cdot 2^{1/3} i^{1/3}\lambda^{5/3} v_1^2 f'^2\phi_x^2 - i2^{2/3} i^{2/3}\lambda^{4/3} u_1 v_1 f''\phi_x^2 - i2^{1/3} i^{1/3}\lambda^{2/3} f' f''\phi_t\phi_x^2 \\
&\quad - 2\lambda^2 v_1 (f')^3\phi_x^3 - i2^{1/3} i^{1/3}\lambda^{5/3} u_1 f' f''\phi_x^3 - i2^{2/3} i^{2/3}\lambda^{4/3} v_1 f' f''\phi_x^3 + 2\lambda u_1 v_1 (v_1)_x \\
&\quad - i2^{2/3} i^{2/3}\lambda^{4/3} u_1 f'\phi_x (v_1)_x + 2\lambda v_1 f'\phi_x (v_1)_x - i2^{2/3} i^{2/3}\lambda^{4/3} f'^2\phi_x^2 (v_1)_x + (v_1)_{3x} \\
&\quad - i2^{2/3} i^{2/3}\lambda^{1/3} v_1 f'\phi_{xt} - i2^{1/3} i^{1/3}\lambda^{2/3} (f')^2\phi_x\phi_{xt} - i2^{2/3} i^{2/3}\lambda^{4/3} u_1 v_1 f'\phi_{xx} \\
&\quad - i2^{1/3} i^{1/3}\lambda^{5/3} u_1 (f')^2\phi_x\phi_{xx} - i2^{2/3} i^{2/3}\lambda^{4/3} v_1 (f')^2\phi_x\phi_{xx} - i2^{1/3} i^{1/3}\lambda^{5/3} (f')^3\phi_x^2\phi_{xx} \\
&\quad - 3i2^{2/3} i^{2/3}\lambda^{1/3} f^{(3)}\phi_x^2\phi_{xx} - \frac{3i^{1/3}\lambda^{1/3} f''\phi_{xx}^2}{2^{1/3}} - 2i2^{2/3} i^{2/3}\lambda^{1/3} f''\phi_x\phi_{3x} - \frac{i^{1/3}\lambda^{1/3} f'\phi_{4x}}{2^{1/3}} = 0
\end{aligned}
\tag{A.8}
$$

and (2.6.2b) yields

$$
v_1^2 - 2^{2/3} i^{1/3}\lambda^{1/3} v_1 f'\phi_x - i(u_1)_x - if'\phi_{xx} = 0 \tag{A.9}
$$

After substitution of (2.6.15) and similar f-derivative terms, (A.8) becomes

$$
\begin{aligned}
&2v_1(v_1)_t + 2\lambda u_1 v_1 (v_1)_x - i\lambda v_1^4 + (v_1)_{3x} \\
&\quad + f'\Big[-2\cdot 2^{2/3} i^{2/3}\lambda^{4/3} v_1^3\phi_x - i2^{2/3} i^{2/3}\lambda^{1/3} (v_1)_t\phi_x - i2^{2/3} i^{2/3}\lambda^{4/3} u_1\phi_x (v_1)_x \\
&\qquad\quad + 2\lambda v_1\phi_x (v_1)_x - i2^{2/3} i^{2/3}\lambda^{1/3} v_1\phi_{xt} - i2^{2/3} i^{2/3}\lambda^{4/3} u_1 v_1\phi_{xx} - \frac{i^{1/3}\lambda^{1/3}\phi_{4x}}{2^{1/3}}\Big] \\
&\quad + f''\Big[-i2^{2/3} i^{2/3}\lambda^{1/3} v_1\phi_t\phi_x - i2^{2/3} i^{2/3}\lambda^{4/3} u_1 v_1\phi_x^2 + 6i\lambda v_1^2\phi_x^2 \\
&\qquad\quad - 2\cdot 2^{1/3} i^{1/3}\lambda^{2/3}\phi_x^2 (v_1)_x - 2\phi_x\phi_{xt} - 2\lambda u_1\phi_x\phi_{xx} \\
&\qquad\quad - 2\cdot 2^{1/3} i^{1/3}\lambda^{2/3} v_1\phi_x\phi_{xx} - \frac{3i^{1/3}\lambda^{1/3}\phi_{xx}^2}{2^{1/3}} - 2i2^{2/3} i^{2/3}\lambda^{1/3}\phi_x\phi_{3x}\Big] \\
&\quad + f^{(3)}\Big[-\phi_t\phi_x^2 - \lambda u_1\phi_x^3 - (1+2i)2^{1/3} i^{1/3}\lambda^{2/3} v_1\phi_x^3 + (1-3i)2^{2/3} i^{2/3}\lambda^{1/3}\phi_x^2\phi_{xx}\Big] = 0
\end{aligned}
\tag{A.10}
$$

and since (A.9) is already linear in f-derivatives, it is simply regrouped as

$$
v_1^2 - i(u_1)_x + f'\Big[-i2^{2/3} i^{2/3}\lambda^{1/3} v_1\phi_x - i\phi_{xx}\Big] \tag{A.11}
$$

Under the homogeneous balance method, using (2.6.27) in (2.3.5a) gives

$$
\begin{aligned}
&-i\lambda v_1^8+4v_1^3(v_1)_t+\frac{8}{15}i\lambda^2v_1^7f'\phi_x-\frac{4}{15}\lambda v_1^3f''\phi_t\phi_x-\frac{4}{5}\lambda v_1^2f'(v_1)_t\phi_x-\frac{28}{225}i\lambda^3v_1^6(f')^2\phi_x^2\\
&-\frac{4}{15}\lambda^2u_1v_1^3f''\phi_x^2+\frac{4}{75}\lambda^2v_1^2f'f''\phi_t\phi_x^2+\frac{4}{75}\lambda^2v_1(f')^2(v_1)_t\phi_x^2+\frac{56i\lambda^4v_1^5(f')^3\phi_x^3}{3375}+\frac{4}{75}\lambda^3u_1v_1^2f'f''\phi_x^3\\
&-\frac{4\lambda^3v_1(f')^2f''\phi_t\phi_x^3}{1125}-\frac{4\lambda^3(f')^3(v_1)_t\phi_x^3}{3375}-\frac{14i\lambda^5v_1^4(f')^4\phi_x^4}{10125}-\frac{4\lambda^4u_1v_1(f')^2f''\phi_x^4}{1125}-\frac{1}{3}\lambda v_1^4f^{(4)}\phi_x^4\\
&+\frac{4\lambda^4(f')^3f''\phi_t\phi_x^4}{50625}+\frac{56i\lambda^6v_1^3(f')^5\phi_x^5}{759375}+\frac{4\lambda^5u_1(f')^3f''\phi_x^5}{50625}+\frac{4}{45}\lambda^2v_1^3f'f^{(4)}\phi_x^5-\frac{28i\lambda^7v_1^2(f')^6\phi_x^6}{11390625}\\
&-\frac{4}{225}\lambda^3v_1^2(f'')^3\phi_x^6-\frac{2}{225}\lambda^3v_1^2(f')^2f^{(4)}\phi_x^6+\frac{8i\lambda^8v_1(f')^7\phi_x^7}{170859375}+\frac{8\lambda^4v_1f'(f'')^3\phi_x^7}{3375}+\frac{4\lambda^4v_1(f')^3f^{(4)}\phi_x^7}{10125}\\
&+4\lambda u_1v_1^3(v_1)_x-\frac{4}{5}\lambda^2u_1v_1^2f'\phi_x(v_1)_x+\frac{4}{75}\lambda^3u_1v_1(f')^2\phi_x^2(v_1)_x-\frac{4\lambda^4u_1(f')^3\phi_x^3(v_1)_x}{3375}\\
&+\frac{4}{5}\lambda^2v_1^2(f'')^2\phi_x^4(v_1)_x-\frac{8}{75}\lambda^3v_1f'(f'')^2\phi_x^5(v_1)_x+\frac{4\lambda^4(f')^2(f'')^2\phi_x^6(v_1)_x}{1125}-12\lambda v_1^2f''\phi_x^2(v_1)_x^2\\
&+\frac{8}{5}\lambda^2v_1f'f''\phi_x^3(v_1)_x^2-\frac{4}{75}\lambda^3(f')^2f''\phi_x^4(v_1)_x^2+60v_1^2(v_1)_x^3-8\lambda v_1f'\phi_x(v_1)_x^3+\frac{4}{15}\lambda^2(f')^2\phi_x^2(v_1)_x^3\\
&-\frac{4}{15}\lambda v_1^3f'\phi_{xt}+\frac{4}{75}\lambda^2v_1^2(f')^2\phi_x\phi_{xt}-\frac{4\lambda^3v_1(f')^3\phi_x^2\phi_{xt}}{1125}+\frac{4\lambda^4(f')^4\phi_x^3\phi_{xt}}{50625}-\frac{4}{15}\lambda^2u_1v_1^3f'\phi_{xx}\\
&+\frac{4}{75}\lambda^3u_1v_1^2(f')^2\phi_x\phi_{xx}-\frac{4\lambda^4u_1v_1(f')^3\phi_x^2\phi_{xx}}{1125}-2\lambda v_1^4f^{(3)}\phi_x^2\phi_{xx}+\frac{4\lambda^5u_1(f')^4\phi_x^3\phi_{xx}}{50625}\\
&+\frac{8}{15}\lambda^2v_1^3f'f^{(3)}\phi_x^3\phi_{xx}-\frac{4}{75}\lambda^3v_1^2f'(f'')^2\phi_x^4\phi_{xx}-\frac{4}{75}\lambda^3v_1^2(f')^2f^{(3)}\phi_x^4\phi_{xx}+\frac{8\lambda^4v_1(f')^2(f'')^2\phi_x^5\phi_{xx}}{1125}\\
&+\frac{8\lambda^4v_1(f')^3f^{(3)}\phi_x^5\phi_{xx}}{3375}-\frac{4\lambda^5(f')^3(f'')^2\phi_x^6\phi_{xx}}{16875}-\frac{2\lambda^5(f')^4f^{(3)}\phi_x^6\phi_{xx}}{50625}+\frac{8}{5}\lambda^2v_1^2f'f''\phi_x^2(v_1)_x\phi_{xx}\\
&-\frac{16}{75}\lambda^3v_1(f')^2f''\phi_x^3(v_1)_x\phi_{xx}+\frac{8\lambda^4(f')^3f''\phi_x^4(v_1)_x\phi_{xx}}{1125}-12\lambda v_1^2f'(v_1)_x^2\phi_{xx}\\
&+\frac{8}{5}\lambda^2v_1(f')^2\phi_x(v_1)_x^2\phi_{xx}-\frac{4}{75}\lambda^3(f')^3\phi_x^2(v_1)_x^2\phi_{xx}-\lambda v_1^4f''\phi_{xx}^2+\frac{4}{15}\lambda^2v_1^3f'f''\phi_x\phi_{xx}^2\\
&-\frac{2}{25}\lambda^3v_1^2(f')^2f''\phi_x^2\phi_{xx}^2+\frac{28\lambda^4v_1(f')^3f''\phi_x^3\phi_{xx}^2}{3375}-\frac{13\lambda^5(f')^4f''\phi_x^4\phi_{xx}^2}{50625}+\frac{4}{5}\lambda^2v_1^2(f')^2(v_1)_x\phi_{xx}^2\\
&-\frac{8}{75}\lambda^3v_1(f')^3\phi_x(v_1)_x\phi_{xx}^2+\frac{4\lambda^4(f')^4\phi_x^2(v_1)_x\phi_{xx}^2}{1125}-\frac{4}{225}\lambda^3v_1^2(f')^3\phi_{xx}^3+\frac{8\lambda^4v_1(f')^4\phi_x\phi_{xx}^3}{3375}\\
&-\frac{4\lambda^5(f')^5\phi_x^2\phi_{xx}^3}{50625}-4\lambda v_1^3f''\phi_x^2(v_1)_{xx}+\frac{4}{5}\lambda^2v_1^2f'f''\phi_x^3(v_1)_{xx}-\frac{4}{75}\lambda^3v_1(f')^2f''\phi_x^4(v_1)_{xx}\\
&+\frac{4\lambda^4(f')^3f''\phi_x^5(v_1)_{xx}}{3375}+60v_1^3(v_1)_x(v_1)_{xx}-12\lambda v_1^2f'\phi_x(v_1)_x(v_1)_{xx}+\frac{4}{5}\lambda^2v_1(f')^2\phi_x^2(v_1)_x(v_1)_{xx}\\
&-\frac{4}{225}\lambda^3(f')^3\phi_x^3(v_1)_x(v_1)_{xx}-4\lambda v_1^3f'\phi_{xx}(v_1)_{xx}+\frac{4}{5}\lambda^2v_1^2(f')^2\phi_x\phi_{xx}(v_1)_{xx}-\frac{1}{3}\lambda v_1^4f'\phi_{4x}\\
&-\frac{4}{75}\lambda^3v_1(f')^3\phi_x^2\phi_{xx}(v_1)_{xx}+\frac{4\lambda^4(f')^4\phi_x^3\phi_{xx}(v_1)_{xx}}{3375}-\frac{4}{3}\lambda v_1^4f''\phi_x\phi_{3x}+\frac{16}{45}\lambda^2v_1^3f'f''\phi_x^2\phi_{3x}\\
&-\frac{8}{225}\lambda^3v_1^2(f')^2f''\phi_x^3\phi_{3x}+\frac{16\lambda^4v_1(f')^3f''\phi_x^4\phi_{3x}}{10125}-\frac{4\lambda^5(f')^4f''\phi_x^5\phi_{3x}}{151875}+5v_1^4(v_1)_{3x}\\
&-\frac{4}{3}\lambda v_1^3f'\phi_x(v_1)_{3x}+\frac{2}{15}\lambda^2v_1^2(f')^2\phi_x^2(v_1)_{3x}-\frac{4}{675}\lambda^3v_1(f')^3\phi_x^3(v_1)_{3x}+\frac{\lambda^4(f')^4\phi_x^4(v_1)_{3x}}{10125}\\
&+\frac{4}{45}\lambda^2v_1^3(f')^2\phi_x\phi_{4x}-\frac{2}{225}\lambda^3v_1^2(f')^3\phi_x^2\phi_{4x}+\frac{4\lambda^4v_1(f')^4\phi_x^3\phi_{4x}}{10125}-\frac{\lambda^5(f')^5\phi_x^4\phi_{4x}}{151875}=0,
\end{aligned}
\tag{A.12}
$$

and (2.3.5b) yields

$$v_1^4 - \frac{4}{15}\lambda v_1^3 f'\phi_x + \frac{2}{75}\lambda^2 v_1^2 (f')^2\phi_x^2 - \frac{4\lambda^3 v_1 (f')^3\phi_x^3}{3375} - i(u_1)_x$$
$$- 6if^{(3)}\phi_x^2\phi_{xx} - 3if''\phi_{xx}^2 - 4if''\phi_x\phi_{3x} - if'\phi_{4x} = 0, \tag{A.13}$$

After substitution of (2.6.15) and similar f-derivative terms, (A.12) becomes

$$\begin{aligned}
&4v_1^3(v_1)_t + 4\lambda u_1 v_1^3(v_1)_x - i\lambda v_1^8 + 60v_1^2(v_1)_x^3 + 60v_1^3(v_1)_x(v_1)_{xx} + 5v_1^4(v_1)_{3x} \\
&+ f'\Bigg[\frac{8}{15}i\lambda^2 v_1^7\phi_x - \frac{4}{5}\lambda v_1^2(v_1)_t\phi_x - \frac{4}{5}\lambda^2 u_1 v_1^2\phi_x(v_1)_x - 8\lambda v_1\phi_x(v_1)_x^3 - \frac{4}{15}\lambda v_1^3\phi_{xt} - \frac{1}{3}\lambda v_1^4\phi_{3x} \\
&\quad -\frac{4}{15}\lambda^2 u_1 v_1^3\phi_{xx} - 12\lambda v_1^2(v_1)_x^2\phi_{xx} - 12\lambda v_1^2\phi_x(v_1)_x(v_1)_{xx} - 4\lambda v_1^3\phi_{xx}(v_1)_{xx} - \frac{4}{3}\lambda v_1^3\phi_x(v_1)_{3x}\Bigg] \\
&+ f''\Bigg[-\frac{4}{15}\lambda v_1^3\phi_t\phi_x - \frac{4}{15}\lambda^2 u_1 v_1^3\phi_x^2 - \frac{28\cdot 2^{1/3}\lambda^{5/3}v_1^6\phi_x^2}{3^{1/3}5^{2/3}} - \frac{4i2^{1/3}3^{2/3}\lambda^{2/3}v_1(v_1)_t\phi_x^2}{5^{2/3}} \\
&\quad -\frac{4i2^{1/3}3^{2/3}\lambda^{5/3}u_1 v_1\phi_x^2(v_1)_x}{5^{2/3}} - 12\lambda v_1^2\phi_x^2(v_1)_x^2 - 4i3^{2/3}10^{1/3}\lambda^{2/3}\phi_x^2(v_1)_x^3 - \frac{4}{3}\lambda v_1^4\phi_x\phi_{3x} \\
&\quad -\frac{4i2^{1/3}3^{2/3}\lambda^{2/3}v_1^2\phi_x\phi_{xt}}{5^{2/3}} - \frac{4i2^{1/3}3^{2/3}\lambda^{5/3}u_1 v_1^2\phi_x\phi_{xx}}{5^{2/3}} - 24i3^{2/3}10^{1/3}\lambda^{2/3}v_1\phi_x(v_1)_x^2\phi_{xx} \\
&\quad -\lambda v_1^4\phi_{xx}^2 - 12i3^{2/3}10^{1/3}\lambda^{2/3}v_1^2(v_1)_x\phi_{xx}^2 - 4\lambda v_1^3\phi_x^2(v_1)_{xx} - 12i3^{2/3}10^{1/3}\lambda^{2/3}v_1\phi_x^2(v_1)_x(v_1)_{xx} \\
&\quad -12i3^{2/3}10^{1/3}\lambda^{2/3}v_1^2\phi_x\phi_{xx}(v_1)_{xx} - 2i3^{2/3}10^{1/3}\lambda^{2/3}v_1^2\phi_x^2(v_1)_{3x} - \frac{4i10^{1/3}\lambda^{2/3}v_1^3\phi_x\phi_{3x}}{3^{1/3}}\Bigg] \\
&+ f^{(3)}\Bigg[-\frac{2i2^{1/3}3^{2/3}\lambda^{2/3}v_1^2\phi_t\phi_x^2}{5^{2/3}} - \frac{2i2^{1/3}3^{2/3}\lambda^{5/3}u_1 v_1^2\phi_x^3}{5^{2/3}} - \frac{28i3^{1/3}2^{2/3}\lambda^{4/3}v_1^5\phi_x^3}{5^{1/3}} \\
&\quad +\frac{2\cdot 3^{1/3}2^{2/3}\lambda^{1/3}(v_1)_t\phi_x^3}{5^{1/3}} + \frac{2\cdot 3^{1/3}2^{2/3}\lambda^{4/3}u_1\phi_x^3(v_1)_x}{5^{1/3}} - 12i3^{2/3}10^{1/3}\lambda^{2/3}v_1\phi_x^3(v_1)_x^2 \\
&\quad +\frac{6\cdot 3^{1/3}2^{2/3}\lambda^{1/3}v_1\phi_x^2\phi_{xt}}{5^{1/3}} + \frac{6\cdot 3^{1/3}2^{2/3}\lambda^{4/3}u_1 v_1\phi_x^2\phi_{xx}}{5^{1/3}} - 12i3^{2/3}10^{1/3}\lambda^{2/3}v_1^2\phi_x^2(v_1)_x\phi_{xx} \\
&\quad +18\cdot 3^{1/3}10^{2/3}\lambda^{1/3}\phi_x^2(v_1)_x^2\phi_{xx} - 2i3^{2/3}10^{1/3}\lambda^{2/3}v_1^3\phi_x\phi_{xx}^2 + 36\cdot 3^{1/3}10^{2/3}\lambda^{1/3}v_1\phi_x(v_1)_x\phi_{xx}^2 \\
&\quad +6\cdot 3^{1/3}10^{2/3}\lambda^{1/3}v_1^2\phi_{xx}^3 - 6i3^{2/3}10^{1/3}\lambda^{2/3}v_1^2\phi_x^3(v_1)_{xx} + 6\cdot 3^{1/3}10^{2/3}\lambda^{1/3}\phi_x^3(v_1)_x(v_1)_{xx} \\
&\quad +18\cdot 3^{1/3}10^{2/3}\lambda^{1/3}v_1\phi_x^2\phi_{xx}(v_1)_{xx} - \frac{8i10^{1/3}\lambda^{2/3}v_1^3\phi_x^2\phi_{3x}}{3^{1/3}} - 2\lambda v_1^4\phi_x^2\phi_{xx} \\
&\quad +2\cdot 3^{1/3}10^{2/3}\lambda^{1/3}v_1\phi_x^3(v_1)_{3x} + 3\cdot 3^{1/3}10^{2/3}\lambda^{1/3}v_1^2\phi_x^2\phi_{3x}\Bigg] \\
&+ f^{(4)}\Bigg[\frac{2\cdot 3^{1/3}2^{2/3}\lambda^{1/3}v_1\phi_t\phi_x^3}{5^{1/3}} + \frac{2\cdot 3^{1/3}2^{2/3}\lambda^{4/3}u_1 v_1\phi_x^4}{5^{1/3}} + \frac{209}{3}\lambda v_1^4\phi_x^4 - 2i3^{2/3}10^{1/3}\lambda^{2/3}v_1^2\phi_x^4(v_1)_x \\
&\quad +6\cdot 3^{1/3}10^{2/3}\lambda^{1/3}\phi_x^4(v_1)_x^2 + 4i\phi_x^3\phi_{xt} + 4i\lambda u_1\phi_x^3\phi_{xx} - \frac{8i10^{1/3}\lambda^{2/3}v_1^3\phi_x^3\phi_{xx}}{3^{1/3}} \\
&\quad +24\cdot 3^{1/3}10^{2/3}\lambda^{1/3}v_1\phi_x^3(v_1)_x\phi_{xx} + 9\cdot 3^{1/3}10^{2/3}\lambda^{1/3}v_1^2\phi_x^2\phi_{xx}^2 + 180i\phi_x^2(v_1)_x\phi_{xx}^2 \\
&\quad +120iv_1\phi_x\phi_{xx}^3 + 6\cdot 3^{1/3}10^{2/3}\lambda^{1/3}v_1\phi_x^4(v_1)_{xx} + 60i\phi_x^3\phi_{xx}(v_1)_{xx} \\
&\quad +4\cdot 3^{1/3}10^{2/3}\lambda^{1/3}v_1^2\phi_x^3\phi_{3x} + 5i\phi_x^4(v_1)_{3x} + 20iv_1\phi_x^3\phi_{3x}\Bigg]
\end{aligned}$$

$$
\begin{aligned}
&+ f^{(5)}\left[i\phi_t\phi_x^4 + i\lambda u_1\phi_x^5 + \frac{41i10^{1/3}\lambda^{2/3}v_1^3\phi_x^5}{3^{1/3}} + 3\cdot 3^{1/3}10^{2/3}\lambda^{1/3}v_1\phi_x^5(v_1)_x \right.\\
&\qquad + \frac{9\cdot 3^{1/3}5^{2/3}\lambda^{1/3}v_1^2\phi_x^4\phi_{xx}}{2^{1/3}} + 90i\phi_x^4(v_1)_x\phi_{xx} + 105iv_1\phi_x^3\phi_{xx}^2\\
&\qquad \left. - \frac{15\cdot 3^{2/3}10^{1/3}\phi_x^2\phi_{xx}^3}{\lambda^{1/3}} + 15i\phi_x^5(v_1)_{xx} + 20iv_1\phi_x^4\phi_{3x} - \frac{5\cdot 3^{2/3}5^{1/3}\phi_x^4\phi_{3x}}{2\cdot 2^{2/3}\lambda^{1/3}} \right]\\
&+ f^{(6)}\left[\frac{-193^{1/3}2^{2/3}\lambda^{1/3}v_1^2\phi_x^6}{5^{1/3}} + 9i\phi_x^6(v_1)_x + 30iv_1\phi_x^5\phi_{xx} - \frac{39\cdot 3^{2/3}5^{1/3}\phi_x^4\phi_{xx}^2}{2\cdot 2^{2/3}\lambda^{1/3}} - \frac{3^{2/3}10^{1/3}\phi_x^5\phi_{3x}}{\lambda^{1/3}} \right]\\
&+ f^{(7)}\left[-4iv_1\phi_x^7 - \frac{2\cdot 3^{2/3}10^{1/3}\phi_x^6\phi_{xx}}{\lambda^{1/3}} \right] = 0, \qquad \text{(A.14)}
\end{aligned}
$$

and (A.13) becomes

$$
\begin{aligned}
&v_1^4 - i(u_1)_x + f'\left(-\frac{4}{15}\lambda v_1^3\phi_x - i\phi_{3x} \right) + f''\left(\frac{-2i3^{2/3}2^{1/3}\lambda^{2/3}v_1^2\phi_x^2}{5^{2/3}} - 3i\phi_{xx}^2 - 4i\phi_x\phi_{3x} \right)\\
&\qquad + f^{(3)}\left(\frac{2\cdot 3^{1/3}2^{2/3}\lambda^{1/3}v_1\phi_x^3}{5^{1/3}} - 6i\phi_x^2\phi_{xx} \right) = 0. \qquad \text{(A.15)}
\end{aligned}
$$

The coefficients of the f-derivative terms give the Lax-type Equations

$$
\begin{aligned}
&\frac{8}{15}i\lambda v_1^5 - \frac{4}{5}(v_1)_t - \frac{4}{5}\lambda u_1(v_1)_x - \frac{8(v_1)_x^3}{v_1} - \frac{4v_1\phi_{xt}}{15\phi_x} - \frac{4\lambda u_1v_1\phi_{xx}}{15\phi_x} - \frac{12(v_1)_x^2\phi_{xx}}{\phi_x}\\
&\qquad - 12(v_1)_x(v_1)_{xx} - \frac{4v_1\phi_{xx}(v_1)_{xx}}{\phi_x} - \frac{4}{3}v_1(v_1)_{3x} - \frac{v_1^2\phi_{4x}}{3\phi_x} = 0, \qquad \text{(A.16a)}\\
&-\frac{4}{15}\lambda^2u_1v_1^3 - \frac{28\cdot 2^{1/3}\lambda^{5/3}v_1^6}{3^{1/3}5^{2/3}} - \frac{4i2^{1/3}3^{2/3}\lambda^{2/3}v_1(v_1)_t}{5^{2/3}} - \frac{4\lambda v_1^3\phi_t}{15\phi_x} - \frac{4i3^{2/3}2^{1/3}\lambda^{5/3}u_1v_1(v_1)_x}{5^{2/3}}\\
&\qquad - 12\lambda v_1^2(v_1)_x^2 - 4i3^{2/3}10^{1/3}\lambda^{2/3}(v_1)_x^3 - \frac{4i3^{2/3}2^{1/3}\lambda^{2/3}v_1^2\phi_{xt}}{5^{2/3}\phi_x} - \frac{4i3^{2/3}2^{1/3}\lambda^{5/3}u_1v_1^2\phi_{xx}}{5^{2/3}\phi_x}\\
&\qquad - \frac{24i3^{2/3}10^{1/3}\lambda^{2/3}v_1(v_1)_x^2\phi_{xx}}{\phi_x} - \frac{\lambda v_1^4\phi_{xx}^2}{\phi_x^2} - \frac{12i3^{2/3}10^{1/3}\lambda^{2/3}v_1^2(v_1)_x\phi_{xx}^2}{\phi_x^2}\\
&\qquad - 4\lambda v_1^3(v_1)_{xx} - 12i3^{2/3}10^{1/3}\lambda^{2/3}v_1(v_1)_x(v_1)_{xx} - \frac{12i3^{2/3}10^{1/3}\lambda^{2/3}v_1^2\phi_{xx}(v_1)_{xx}}{\phi_x}\\
&\qquad - \frac{4\lambda v_1^4\phi_{3x}}{3\phi_x} - 2i3^{2/3}10^{1/3}\lambda^{2/3}v_1^2(v_1)_{3x} - \frac{4i10^{1/3}\lambda^{2/3}v_1^3\phi_{4x}}{3^{1/3}\phi_x} = 0, \qquad \text{(A.16b)}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{2i2^{1/3}3^{2/3}\lambda^{5/3}u_1v_1^2}{5^{2/3}} - \frac{28i3^{1/3}2^{2/3}\lambda^{4/3}v_1^5}{5^{1/3}} + \frac{2\cdot 3^{1/3}2^{2/3}\lambda^{1/3}(v_1)_t}{5^{1/3}} - \frac{2i3^{2/3}2^{1/3}\lambda^{2/3}v_1^2\phi_t}{5^{2/3}\phi_x} \\
&\quad + \frac{2\cdot 3^{1/3}2^{2/3}\lambda^{4/3}u_1(v_1)_x}{5^{1/3}} - 12i3^{2/3}10^{1/3}\lambda^{2/3}v_1(v_1)_x^2 + \frac{6\cdot 3^{1/3}2^{2/3}\lambda^{1/3}v_1\phi_{xt}}{5^{1/3}\phi_x} - \frac{2\lambda v_1^4\phi_{xx}}{\phi_x} \\
&\quad + \frac{6\cdot 3^{1/3}2^{2/3}\lambda^{4/3}u_1v_1\phi_{xx}}{5^{1/3}\phi_x} - \frac{12i3^{2/3}10^{1/3}\lambda^{2/3}v_1^2(v_1)_x\phi_{xx}}{\phi_x} + \frac{3\cdot 3^{1/3}10^{2/3}\lambda^{1/3}v_1^2\phi_{4x}}{\phi_x} \\
&\quad + \frac{18\cdot 3^{1/3}10^{2/3}\lambda^{1/3}(v_1)_x^2\phi_{xx}}{\phi_x} - \frac{2i3^{2/3}10^{1/3}\lambda^{2/3}v_1^3\phi_{xx}^2}{\phi_x^2} + \frac{36\cdot 3^{1/3}10^{2/3}\lambda^{1/3}v_1(v_1)_x\phi_{xx}^2}{\phi_x^2} \\
&\quad + \frac{6\cdot 3^{1/3}10^{2/3}\lambda^{1/3}v_1^2\phi_{xx}^3}{\phi_x^3} - 6i3^{2/3}10^{1/3}\lambda^{2/3}v_1^2(v_1)_{xx} + 6\cdot 3^{1/3}10^{2/3}\lambda^{1/3}(v_1)_x(v_1)_{xx} \\
&\quad + \frac{18\cdot 3^{1/3}10^{2/3}\lambda^{1/3}v_1\phi_{xx}(v_1)_{xx}}{\phi_x} - \frac{8i10^{1/3}\lambda^{2/3}v_1^3\phi_{3x}}{3^{1/3}\phi_x} + 2\cdot 3^{1/3}10^{2/3}\lambda^{1/3}v_1(v_1)_{3x} = 0, \qquad \text{(A.16c)} \\
&\frac{2\cdot 3^{1/3}2^{2/3}\lambda^{4/3}u_1v_1}{5^{1/3}} + \frac{209}{3}\lambda v_1^4 + \frac{2\cdot 3^{1/3}2^{2/3}\lambda^{1/3}v_1\phi_t}{5^{1/3}\phi_x} - 2i3^{2/3}10^{1/3}\lambda^{2/3}v_1^2(v_1)_x \\
&\quad + 6\cdot 3^{1/3}10^{2/3}\lambda^{1/3}(v_1)_x^2 + \frac{4i\phi_{xt}}{\phi_x} + \frac{4i\lambda u_1\phi_{xx}}{\phi_x} - \frac{8i10^{1/3}\lambda^{2/3}v_1^3\phi_{xx}}{3^{1/3}\phi_x} + \frac{20iv_1\phi_{4x}}{\phi_x} \\
&\quad + \frac{24\cdot 3^{1/3}10^{2/3}\lambda^{1/3}v_1(v_1)_x\phi_{xx}}{\phi_x} + \frac{9\cdot 3^{1/3}10^{2/3}\lambda^{1/3}v_1^2\phi_{xx}^2}{\phi_x^2} + \frac{180i(v_1)_x\phi_{xx}^2}{\phi_x^2} + \frac{120iv_1\phi_{xx}^3}{\phi_x^3} \\
&\quad + 6\cdot 3^{1/3}10^{2/3}\lambda^{1/3}v_1(v_1)_{xx} + \frac{60i\phi_{xx}(v_1)_{xx}}{\phi_x} + \frac{4\cdot 3^{1/3}10^{2/3}\lambda^{1/3}v_1^2\phi_{3x}}{\phi_x} + 5i(v_1)_{3x} = 0, \qquad \text{(A.16d)} \\
&i\lambda u_1 + \frac{41i10^{1/3}\lambda^{2/3}v_1^3}{3^{1/3}} + \frac{i\phi_t}{\phi_x} + 3\cdot 3^{1/3}10^{2/3}\lambda^{1/3}v_1(v_1)_x + \frac{9\cdot 3^{1/3}5^{2/3}\lambda^{1/3}v_1^2\phi_{xx}}{2^{1/3}\phi_x} + 15i(v_1)_{xx} \\
&\quad + \frac{90i(v_1)_x\phi_{xx}}{\phi_x} + \frac{105iv_1\phi_{xx}^2}{\phi_x^2} - \frac{15\cdot 3^{2/3}10^{1/3}\phi_{xx}^3}{\lambda^{1/3}\phi_x^3} + \frac{20iv_1\phi_{3x}}{\phi_x} - \frac{5\cdot 3^{2/3}5^{1/3}\phi_{4x}}{2\cdot 2^{2/3}\lambda^{1/3}\phi_x} = 0, \qquad \text{(A.16e)} \\
&- \frac{19\cdot 3^{1/3}2^{2/3}\lambda^{1/3}v_1^2}{5^{1/3}} + 9i(v_1)_x + \frac{30iv_1\phi_{xx}}{\phi_x} - \frac{39\cdot 3^{2/3}5^{1/3}\phi_{xx}^2}{2\cdot 2^{2/3}\lambda^{1/3}\phi_x^2} - \frac{3^{2/3}10^{1/3}\phi_{3x}}{\lambda^{1/3}\phi_x} = 0, \qquad \text{(A.16f)} \\
&- 4iv_1 - \frac{2\cdot 3^{2/3}10^{1/3}\phi_{xx}}{\lambda^{1/3}\phi_x} = 0, \qquad \text{(A.16g)} \\
&- \frac{4}{15}\lambda v_1^3 - \frac{i\phi_{4x}}{\phi_x} = 0, \qquad \text{(A.16h)} \\
&- \frac{2i3^{2/3}2^{1/3}\lambda^{2/3}v_1^2}{5^{2/3}} - \frac{3i\phi_{xx}^2}{\phi_x^2} - \frac{4i\phi_{3x}}{\phi_x} = 0, \qquad \text{(A.16i)} \\
&\frac{2\cdot 3^{1/3}2^{2/3}\lambda^{1/3}v_1}{5^{1/3}} - \frac{6i\phi_{xx}}{\phi_x} = 0. \qquad \text{(A.16j)}
\end{aligned}
$$

Making the β_j-form substitutions (2.6.28), the above Lax-type equations become

$$4\lambda\beta_2 v_1^3 + 4\lambda^2\beta_3 u_1 v_1^3 + 5\lambda\beta_5 v_1^4 - 8i\lambda^2 v_1^7 + 12\lambda v_1^2 (v_1)_t + 12\lambda^2 u_1 v_1^2 (v_1)_x$$
$$+ 180\lambda\beta_3 v_1^2 (v_1)_x^2 + 120\lambda v_1 (v_1)_x^3 + 60\lambda\beta_3 v_1^3 (v_1)_{xx} + 180\lambda v_1^2 (v_1)_x (v_1)_{xx} + 20\lambda v_1^3 (v_1)_{3x} = 0, \quad \text{(A.17a)}$$

$$12i3^{2/3}10^{1/3}\lambda^{1/3}\beta_2 v_1^2 + 12i3^{2/3}10^{1/3}\lambda^{4/3}\beta_3 u_1 v_1^2 + 4\lambda^{2/3}\beta_1 v_1^3 + 20i3^{2/3}10^{1/3}\lambda^{1/3}\beta_5 v_1^3 + 4\lambda^{5/3}u_1 v_1^3$$
$$+ 20\lambda^{2/3}\beta_4 v_1^4 + 15\lambda^{2/3}\beta_6 v_1^4 + 28\cdot 3^{2/3}10^{1/3}\lambda^{4/3}v_1^6 + 12i3^{2/3}10^{1/3}\lambda^{1/3}v_1 (v_1)_t$$
$$+ 12i3^{2/3}10^{1/3}\lambda^{4/3}u_1 v_1 (v_1)_x + 180i3^{2/3}10^{1/3}\lambda^{1/3}\beta_6 v_1^2 (v_1)_x + 360i3^{2/3}10^{1/3}\lambda^{1/3}\beta_3 v_1 (v_1)_x^2$$
$$+ 180\lambda^{2/3}v_1^2 (v_1)_x^2 + 60i3^{2/3}10^{1/3}\lambda^{1/3}(v_1)_x^3 + 180i3^{2/3}10^{1/3}\lambda^{1/3}\beta_3 v_1^2 (v_1)_{xx}$$
$$+ 60\lambda^{2/3}v_1^3 (v_1)_{xx} + 180i3^{2/3}10^{1/3}\lambda^{1/3}v_1 (v_1)_x (v_1)_{xx} + 30i3^{2/3}10^{1/3}\lambda^{1/3}v_1^2 (v_1)_{3x} = 0, \quad \text{(A.17b)}$$

$$18\cdot 3^{1/3}10^{2/3}\lambda^{1/3}\beta_2 v_1 + 18\cdot 3^{1/3}10^{2/3}\lambda^{4/3}\beta_3 u_1 v_1 - 6i3^{2/3}10^{1/3}\lambda^{2/3}\beta_1 v_1^2 + 45\cdot 3^{1/3}10^{2/3}\lambda^{1/3}\beta_5 v_1^2$$
$$+ 90\cdot 3^{1/3}10^{2/3}\lambda^{1/3}\beta_7 v_1^2 - 6i3^{2/3}10^{1/3}\lambda^{5/3}u_1 v_1^2 + 90\cdot 3^{1/3}10^{2/3}\lambda^{1/3}(v_1)_x (v_1)_{xx}$$
$$- 30i3^{2/3}10^{1/3}\lambda^{2/3}\beta_6 v_1^3 - 30\lambda\beta_3 v_1^4 - 84i3^{1/3}10^{2/3}\lambda^{4/3}v_1^5 + 6\cdot 3^{1/3}10^{2/3}\lambda^{1/3}(v_1)_t$$
$$+ 6\cdot 3^{1/3}10^{2/3}\lambda^{4/3}u_1 (v_1)_x + 540\cdot 3^{1/3}10^{2/3}\lambda^{1/3}\beta_6 v_1 (v_1)_x - 180i3^{2/3}10^{1/3}\lambda^{2/3}\beta_3 v_1^2 (v_1)_x$$
$$+ 270\cdot 3^{1/3}10^{2/3}\lambda^{1/3}\beta_3 (v_1)_x^2 - 180i3^{2/3}10^{1/3}\lambda^{2/3}v_1 (v_1)_x^2 + 270\cdot 3^{1/3}10^{2/3}\lambda^{1/3}\beta_3 v_1 (v_1)_{xx}$$
$$- 90i3^{2/3}10^{1/3}\lambda^{2/3}v_1^2 (v_1)_{xx} + 30\cdot 3^{1/3}10^{2/3}\lambda^{1/3}v_1 (v_1)_{3x} - 40i3^{2/3}10^{1/3}\lambda^{2/3}\beta_4 v_1^3 = 0, \quad \text{(A.17c)}$$

$$4i\beta_2 + 4i\lambda\beta_3 u_1 + \frac{2\cdot 3^{1/3}2^{2/3}\lambda^{1/3}\beta_1 v_1}{5^{1/3}} + 20i\beta_5 v_1 + 120i\beta_7 v_1 + \frac{2\cdot 3^{1/3}2^{2/3}\lambda^{4/3}u_1 v_1}{5^{1/3}}$$
$$+ 4\cdot 3^{1/3}10^{2/3}\lambda^{1/3}\beta_4 v_1^2 + 9\cdot 3^{1/3}10^{2/3}\lambda^{1/3}\beta_6 v_1^2 - \frac{8i10^{1/3}\lambda^{2/3}\beta_3 v_1^3}{3^{1/3}} + \frac{209}{3}\lambda v_1^4$$
$$+ 180i\beta_6 (v_1)_x + 24\cdot 3^{1/3}10^{2/3}\lambda^{1/3}\beta_3 v_1 (v_1)_x - 2i3^{2/3}10^{1/3}\lambda^{2/3}v_1^2 (v_1)_x$$
$$+ 6\cdot 3^{1/3}10^{2/3}\lambda^{1/3}(v_1)_x^2 + 60i\beta_3 (v_1)_{xx} + 6\cdot 3^{1/3}10^{2/3}\lambda^{1/3}v_1 (v_1)_{xx} + 5i(v_1)_{3x} = 0, \quad \text{(A.17d)}$$

$$i\beta_1 + i\lambda u_1 + 20i\beta_4 v_1 + 105i\beta_6 v_1 + \frac{9\cdot 3^{1/3}5^{2/3}\lambda^{1/3}\beta_3 v_1^2}{2^{1/3}} + \frac{41i10^{1/3}\lambda^{2/3}v_1^3}{3^{1/3}} + 90i\beta_3 (v_1)_x$$
$$+ 3\cdot 3^{1/3}10^{2/3}\lambda^{1/3}v_1 (v_1)_x + 15i(v_1)_{xx} - \frac{5\cdot 3^{2/3}5^{1/3}\beta_5}{2\cdot 2^{2/3}\lambda^{1/3}} - \frac{15\cdot 3^{2/3}10^{1/3}\beta_7}{\lambda^{1/3}} = 0, \quad \text{(A.17e)}$$

$$\frac{3^{2/3}10^{1/3}\beta_4}{\lambda^{1/3}} + \frac{39\cdot 3^{2/3}5^{1/3}\beta_6}{2\cdot 2^{2/3}\lambda^{1/3}} + \frac{19\cdot 3^{1/3}2^{2/3}\lambda^{1/3}v_1^2}{5^{1/3}} - 30i\beta_3 v_1 - 9i(v_1)_x = 0, \quad \text{(A.17f)}$$

$$4iv_1 + \frac{2\cdot 3^{2/3}10^{1/3}\beta_3}{\lambda^{1/3}} = 0, \quad \text{(A.17g)}$$

$$15\beta_5 - 4i\lambda v_1^3 = 0, \quad \text{(A.17h)}$$

$$20\beta_4 + 15\beta_6 + 2\cdot 3^{2/3}10^{1/3}\lambda^{2/3}v_1^2 = 0. \quad \text{(A.17i)}$$

The coefficient equations from the resonance analysis of the $n = 3$ KPII system (5.3.10) gives

$$u_2 = 54i\phi_{3x}, \tag{A.18a}$$

$$v_2 = -\frac{3\phi_{xx}^2}{2\sqrt{2}\phi_x^3} + \frac{\phi_{3x}}{\sqrt{2}\phi_x^2}, \tag{A.18b}$$

$$u_3 = -\frac{\phi_y^2}{\phi_x^2} - \frac{\phi_t}{\phi_x} - \frac{27i\phi_{xx}^3}{2\phi_x^3} + \frac{27i\phi_{xx}\phi_{3x}}{\phi_x^2} - \frac{27i\phi_{4x}}{2\phi_x}, \tag{A.18c}$$

$$v_3 = -\frac{3\phi_{xx}^3}{2\sqrt{2}\phi_x^5} + \frac{3\phi_{xx}\phi_{3x}}{2\sqrt{2}\phi_x^4} - \frac{\phi_{4x}}{4\sqrt{2}\phi_x^3}, \tag{A.18d}$$

$$\begin{aligned} u_4 =& \frac{2\phi_{yy}}{5\phi_x^2} + \frac{\phi_{xt}}{\phi_x^2} + \frac{6\phi_y\phi_{xy}}{5\phi_x^3} - \frac{8\phi_y^2\phi_{xx}}{5\phi_x^4} - \frac{\phi_t\phi_{xx}}{\phi_x^3} \\ & - \frac{81i\phi_{xx}^4}{4\phi_x^5} + \frac{81i\phi_{xx}^2\phi_{3x}}{2\phi_x^4} - \frac{9i\phi_{3x}^2}{\phi_x^3} - \frac{27i\phi_{xx}\phi_{4x}}{2\phi_x^3} + \frac{27i\phi_{5x}}{10\phi_x^2}, \end{aligned} \tag{A.18e}$$

$$v_4 = \frac{i\phi_{yy}}{540\sqrt{2}\phi_x^4} - \frac{i\phi_y\phi_{xy}}{270\sqrt{2}\phi_x^5} + \frac{i\phi_y^2\phi_{xx}}{540\sqrt{2}\phi_x^6} - \frac{15\phi_{xx}^4}{8\sqrt{2}\phi_x^7} + \frac{5\phi_{xx}^2\phi_{3x}}{2\sqrt{2}\phi_x^6} - \frac{\phi_{3x}^2}{3\sqrt{2}\phi_x^5} - \frac{\phi_{xx}\phi_{4x}}{2\sqrt{2}\phi_x^5} + \frac{\phi_{5x}}{20\sqrt{2}\phi_x^4}, \tag{A.18f}$$

$$\begin{aligned} u_5 =& -\frac{\phi_{xy}^2}{5\phi_x^4} - \frac{2\phi_{xyy}}{5\phi_x^3} + \frac{3\phi_{yy}\phi_{xx}}{5\phi_x^4} + \frac{3\phi_{xt}\phi_{xx}}{2\phi_x^4} + \frac{11\phi_y\phi_{xy}\phi_{xx}}{5\phi_x^5} - \frac{13\phi_y^2\phi_{xx}^2}{5\phi_x^6} \\ & - \frac{3\phi_t\phi_{xx}^2}{2\phi_x^5} - \frac{243i\phi_{xx}^5}{8\phi_x^7} - \frac{\phi_{xxt}}{2\phi_x^3} - \frac{\phi_y\phi_{xxy}}{5\phi_x^4} + \frac{3\phi_y^2\phi_{3x}}{5\phi_x^5} + \frac{\phi_t\phi_{3x}}{2\phi_x^4} \\ & + \frac{135i\phi_{xx}^3\phi_{3x}}{2\phi_x^6} - \frac{27i\phi_{xx}\phi_{3x}^2}{\phi_x^5} - \frac{81i\phi_{xx}^2\phi_{4x}}{4\phi_x^5} + \frac{27i\phi_{3x}\phi_{4x}}{4\phi_x^4} + \frac{81i\phi_{xx}\phi_{5x}}{20\phi_x^4} - \frac{9i\phi_{6x}}{20\phi_x^3}, \end{aligned} \tag{A.18g}$$

$$\begin{aligned} v_5 =& \frac{i\phi_{xy}^2}{270\sqrt{2}\phi_x^6} - \frac{i\phi_{xyy}}{540\sqrt{2}\phi_x^5} + \frac{i\phi_{yy}\phi_{xx}}{216\sqrt{2}\phi_x^6} - \frac{i\phi_y\phi_{xy}\phi_{xx}}{60\sqrt{2}\phi_x^7} + \frac{i\phi_y^2\phi_{xx}^2}{120\sqrt{2}\phi_x^8} - \frac{21\phi_{xx}^5}{8\sqrt{2}\phi_x^9} + \frac{i\phi_y\phi_{xxy}}{270\sqrt{2}\phi_x^6} \\ & - \frac{i\phi_y^2\phi_{3x}}{540\sqrt{2}\phi_x^7} + \frac{35\phi_{xx}^3\phi_{3x}}{8\sqrt{2}\phi_x^8} - \frac{5\phi_{xx}\phi_{3x}^2}{4\sqrt{2}\phi_x^7} - \frac{15\phi_{xx}^2\phi_{4x}}{16\sqrt{2}\phi_x^7} + \frac{5\phi_{3x}\phi_{4x}}{24\sqrt{2}\phi_x^6} + \frac{\phi_{xx}\phi_{5x}}{8\sqrt{2}\phi_x^6} - \frac{\phi_{6x}}{120\sqrt{2}\phi_x^5}, \end{aligned} \tag{A.18h}$$

$$\begin{aligned} u_6 =& -72i\sqrt{2}v_6\phi_x^2 - \frac{7\phi_{xy}^2\phi_{xx}}{5\phi_x^6} - \frac{2\phi_{xyy}\phi_{xx}}{5\phi_x^5} + \frac{3\phi_{yy}\phi_{xx}^2}{10\phi_x^6} + \frac{5\phi_{xt}\phi_{xx}^2}{2\phi_x^6} + \frac{36\phi_y\phi_{xy}\phi_{xx}^2}{5\phi_x^7} \\ & - \frac{61\phi_y^2\phi_{xx}^3}{10\phi_x^8} - \frac{5\phi_t\phi_{xx}^3}{2\phi_x^7} - \frac{1323i\phi_{xx}^6}{4\phi_x^9} - \frac{\phi_{xx}\phi_{xxt}}{\phi_x^5} + \frac{\phi_{xy}\phi_{xxy}}{5\phi_x^5} - \frac{7\phi_y\phi_{xx}\phi_{xxy}}{5\phi_x^6} \\ & + \frac{2\phi_{xxyy}}{15\phi_x^4} - \frac{2\phi_{yy}\phi_{3x}}{15\phi_x^5} - \frac{2\phi_{xt}\phi_{3x}}{3\phi_x^5} - \frac{19\phi_y\phi_{xy}\phi_{3x}}{15\phi_x^6} + \frac{3\phi_y^2\phi_{xx}\phi_{3x}}{\phi_x^7} + \frac{5\phi_t\phi_{xx}\phi_{3x}}{3\phi_x^6} \\ & + \frac{5481i\phi_{xx}^4\phi_{3x}}{8\phi_x^8} - \frac{639i\phi_{xx}^2\phi_{3x}^2}{2\phi_x^7} + \frac{19i\phi_{3x}^3}{\phi_x^6} + \frac{\phi_{3xt}}{6\phi_x^4} + \frac{\phi_y\phi_{3xy}}{15\phi_x^5} - \frac{\phi_y^2\phi_{4x}}{5\phi_x^6} - \frac{\phi_t\phi_{4x}}{6\phi_x^5} + \frac{3i\phi_{7x}}{20\phi_x^4} \\ & - \frac{639i\phi_{xx}^3\phi_{4x}}{4\phi_x^7} + \frac{171i\phi_{xx}\phi_{3x}\phi_{4x}}{2\phi_x^6} - \frac{27i\phi_{4x}^2}{8\phi_x^5} + \frac{513i\phi_{xx}^2\phi_{5x}}{20\phi_x^6} - \frac{27i\phi_{3x}\phi_{5x}}{5\phi_x^5} - \frac{27i\phi_{xx}\phi_{6x}}{10\phi_x^5}, \end{aligned} \tag{A.18i}$$

$$v_6 \quad \text{arbitrary}, \tag{A.18j}$$

$$
\begin{aligned}
u_7 = &-54i\sqrt{2}v_7\phi_x^2 + 18i\sqrt{2}\phi_x(v_6)_x + 117i\sqrt{2}v_6\phi_{xx} - \frac{39\phi_{xy}^2\phi_{xx}^2}{10\phi_x^8} - \frac{23\phi_{xyy}\phi_{xx}^2}{40\phi_x^7} + \frac{7\phi_{yy}\phi_{xx}^3}{16\phi_x^8} \\
&+ \frac{35\phi_{xt}\phi_{xx}^3}{8\phi_x^8} + \frac{627\phi_y\phi_{xy}\phi_{xx}^3}{40\phi_x^9} - \frac{977\phi_y^2\phi_{xx}^4}{80\phi_x^{10}} - \frac{35\phi_t\phi_{xx}^4}{8\phi_x^9} - \frac{729i\phi_{xx}^7}{\phi_x^{11}} - \frac{15\phi_{xx}^2\phi_{xxt}}{8\phi_x^7} \\
&+ \frac{13\phi_{xy}\phi_{xx}\phi_{xxy}}{10\phi_x^7} - \frac{39\phi_y\phi_{xx}^2\phi_{xxy}}{10\phi_x^8} - \frac{\phi_{xxy}^2}{20\phi_x^6} + \frac{7\phi_{xx}\phi_{xxyy}}{30\phi_x^6} + \frac{23\phi_{xy}^2\phi_{3x}}{30\phi_x^7} + \frac{\phi_{xyy}\phi_{3x}}{12\phi_x^6} \\
&- \frac{\phi_{yy}\phi_{xx}\phi_{3x}}{6\phi_x^7} - \frac{5\phi_{xt}\phi_{xx}\phi_{3x}}{2\phi_x^7} - \frac{15\phi_y\phi_{xy}\phi_{xx}\phi_{3x}}{2\phi_x^8} + \frac{403\phi_y^2\phi_{xx}^2\phi_{3x}}{40\phi_x^9} + \frac{35\phi_t\phi_{xx}^2\phi_{3x}}{8\phi_x^8} \\
&+ \frac{15309i\phi_{xx}^5\phi_{3x}}{8\phi_x^{10}} + \frac{5\phi_{xxt}\phi_{3x}}{12\phi_x^6} + \frac{23\phi_y\phi_{xxy}\phi_{3x}}{30\phi_x^7} - \frac{4\phi_y^2\phi_{3x}^2}{5\phi_x^8} - \frac{5\phi_t\phi_{3x}^2}{12\phi_x^7} - \frac{2709i\phi_{xx}^3\phi_{3x}^2}{2\phi_x^9} \\
&+ \frac{441i\phi_{xx}\phi_{3x}^3}{2\phi_x^8} + \frac{5\phi_{xx}\phi_{3xt}}{12\phi_x^6} - \frac{\phi_{xy}\phi_{3xy}}{15\phi_x^6} + \frac{13\phi_y\phi_{xx}\phi_{3xy}}{30\phi_x^7} - \frac{\phi_{3xyy}}{30\phi_x^5} + \frac{\phi_{yy}\phi_{4x}}{48\phi_x^6} + \frac{5\phi_{xt}\phi_{4x}}{24\phi_x^6} \\
&+ \frac{53\phi_y\phi_{xy}\phi_{4x}}{120\phi_x^7} - \frac{17\phi_y^2\phi_{xx}\phi_{4x}}{16\phi_x^8} - \frac{5\phi_t\phi_{xx}\phi_{4x}}{8\phi_x^7} - \frac{1827i\phi_{xx}^4\phi_{4x}}{4\phi_x^9} + \frac{7119i\phi_{xx}^2\phi_{3x}\phi_{4x}}{16\phi_x^8} \\
&- \frac{339i\phi_{3x}^2\phi_{4x}}{8\phi_x^7} - \frac{225i\phi_{xx}\phi_{4x}^2}{8\phi_x^7} - \frac{\phi_{4xt}}{24\phi_x^5} - \frac{\phi_y\phi_{4xy}}{60\phi_x^6} + \frac{\phi_y^2\phi_{5x}}{20\phi_x^7} + \frac{\phi_t\phi_{5x}}{24\phi_x^6} + \frac{315i\phi_{xx}^3\phi_{5x}}{4\phi_x^8} \\
&- \frac{45i\phi_{xx}\phi_{3x}\phi_{5x}}{\phi_x^7} + \frac{27i\phi_{4x}\phi_{5x}}{8\phi_x^6} - \frac{783i\phi_{xx}^2\phi_{6x}}{80\phi_x^7} + \frac{9i\phi_{3x}\phi_{6x}}{4\phi_x^6} + \frac{33i\phi_{xx}\phi_{7x}}{40\phi_x^6} - \frac{3i\phi_{8x}}{80\phi_x^5}, \qquad \text{(A.18k)}
\end{aligned}
$$

v_7 arbitrary. (A.18l)

The Painlevé-Bäcklund equations from substitution of (4.4.1) into (4.3.10) are

$$
\begin{aligned}
\phi_y^2\phi_x + \phi_t\phi_x^2 + u_2\phi_x^3 + 4i\phi_x^3(v_1)_x = 0, \qquad \text{(A.19a)} \\
-3v_1\phi_t\phi_x^2 - 3u_2v_1\phi_x^3 - 12iv_1\phi_x^3(v_1)_x + \sqrt{2}\phi_{yy}\phi_x^2 + \sqrt{2}\phi_x^3(u_2)_x + 3\sqrt{2}\phi_x^2\phi_{xt} + \sqrt{2}\phi_y^2\phi_{xx} \\
+4\sqrt{2}\phi_y\phi_x\phi_{xy} + 3\sqrt{2}u_2\phi_x^2\phi_{xx} + 12i\sqrt{2}\phi_x^2(v_1)_x\phi_{xx} + 4i\sqrt{2}\phi_x^3(v_1)_{xx} = 0, \qquad \text{(A.19b)} \\
3\sqrt{2}(v_1)^2\phi_t\phi_x + 3\sqrt{2}u_2(v_1)^2\phi_x^2 - 12(v_1)_t\phi_x^2 - 12u_2\phi_x^2(v_1)_x + 12i\sqrt{2}(v_1)^2\phi_x^2(v_1)_x + 8\sqrt{2}\phi_{xy}^2 \\
-48i\phi_x^2(v_1)_x^2 + 8\sqrt{2}\phi_x\phi_{xyy} + 4\sqrt{2}\phi_{yy}\phi_{xx} + 8\sqrt{2}\phi_y\phi_{xxy} - 12v_1\phi_x^2(u_2)_x \\
-24v_1\phi_x\phi_{xt} - 24u_2v_1\phi_x\phi_{xx} - 96iv_1\phi_x(v_1)_x\phi_{xx} - 48iv_1\phi_x^2(v_1)_{xx} = 0, \qquad \text{(A.19c)} \\
6v_1(v_1)_t\phi_x + 3(v_1)^2\phi_x(u_2)_x + 6u_2v_1\phi_x(v_1)_x + 24iv_1\phi_x(v_1)_x^2 + 3(v_1)^2\phi_{xt} \\
+3u_2(v_1)^2\phi_{xx} + 12i(v_1)^2(v_1)_x\phi_{xx} + 12i(v_1)^2\phi_x(v_1)_{xx} + 4\phi_{xxyy} = 0, \qquad \text{(A.19d)}
\end{aligned}
$$

and

$$
\begin{aligned}
v_1\phi_x^2 + \sqrt{2}\phi_x\phi_{xx} = 0, \qquad \text{(A.19e)} \\
3(v_1)^2\phi_x - 4\phi_{3x} = 0. \qquad \text{(A.19f)}
\end{aligned}
$$

The Painlevé-Bäcklund equations from substitution of (4.4.4) into (4.3.18) are

$$
\begin{aligned}
\sqrt{2}v_1\phi_x^2 + 3\phi_x\phi_{xx} = 0, \qquad \text{(A.20a)} \\
20(v_1)^2\phi_x^3 - 18\sqrt{2}\phi_x^3(v_1)_x - 3\sqrt{2}v_1\phi_x^2\phi_{xx} - 315\phi_x\phi_{xx}^2 + 126\phi_x^2\phi_{3x} = 0, \qquad \text{(A.20b)} \\
2\phi_y^2\phi_x^2 + 2\phi_t\phi_x^3 + 2u_3\phi_x^4 - 12i\sqrt{2}(v_1)^3\phi_x^4 + 84iv_1\phi_x^4(v_1)_x + 72i(v_1)^2\phi_x^3\phi_{xx} \\
-36i\sqrt{2}\phi_x^3(v_1)_x\phi_{xx} + 468i\sqrt{2}v_1\phi_x^2\phi_{xx}^2 - 486i\phi_x\phi_{xx}^3 - 9i\sqrt{2}\phi_x^4(v_1)_{xx} \\
-207i\sqrt{2}v_1\phi_x^3\phi_{3x} + 81i\phi_x^3\phi_{4x} = 0, \qquad \text{(A.20c)}
\end{aligned}
$$

$$-2\sqrt{2}v_1\phi_t\phi_x^3+\phi_{yy}\phi_x^3-2\sqrt{2}u_3v_1\phi_x^4+4i(v_1)^4\phi_x^4+\phi_x^4(u_3)_x-48i\sqrt{2}(v_1)^2\phi_x^4(v_1)_x$$
$$+30i\phi_x^4(v_1)_x^2+4\phi_x^3\phi_{xt}+6\phi_y\phi_x^2\phi_{xy}+3\phi_y^2\phi_x\phi_{xx}+4u_3\phi_x^3\phi_{xx}-36i\sqrt{2}(v_1)^3\phi_x^3\phi_{xx}$$
$$+132iv_1\phi_x^3(v_1)_x\phi_{xx}-306i(v_1)^2\phi_x^2\phi_{xx}^2+216i\sqrt{2}\phi_x^2(v_1)_x\phi_{xx}^2+432i\sqrt{2}v_1\phi_x\phi_{xx}^3$$
$$+30iv_1\phi_x^4(v_1)_{xx}+168i(v_1)^2\phi_x^3\phi_{3x}-108i\sqrt{2}\phi_x^3(v_1)_x\phi_{3x}+108i\sqrt{2}v_1\phi_x^2\phi_{xx}\phi_{3x}$$
$$-108i\sqrt{2}v_1\phi_x^3\phi_{4x}=0, \tag{A.20d}$$
$$-6(v_1)^2\phi_t\phi_x^2-6u_3(v_1)^2\phi_x^3+i\sqrt{2}(v_1)^5\phi_x^3+6\sqrt{2}(v_1)_t\phi_x^3+6\sqrt{2}v_1\phi_x^3(u_3)_x$$
$$+6\sqrt{2}u_3\phi_x^3(v_1)_x-108i(v_1)^3\phi_x^3(v_1)_x+180i\sqrt{2}v_1\phi_x^3(v_1)_x^2+18\sqrt{2}v_1\phi_x^2\phi_{xt}$$
$$-18\phi_x\phi_{xy}^2-9\phi_x^2\phi_{xyy}-9\phi_{yy}\phi_x\phi_{xx}+18\sqrt{2}u_3v_1\phi_x^2\phi_{xx}-57i(v_1)^4\phi_x^2\phi_{xx}$$
$$+378i\sqrt{2}(v_1)^2\phi_x^2(v_1)_x\phi_{xx}-18\phi_y\phi_{xy}\phi_{xx}-\frac{18\phi_y^2\phi_{xx}^2}{\phi_x}-144i\sqrt{2}(v_1)^3\phi_x\phi_{xx}^2$$
$$-18\phi_y\phi_x\phi_{xxy}+1944iv_1\phi_x(v_1)_x\phi_{xx}^2+972i(v_1)^2\phi_{xx}^3+90i\sqrt{2}(v_1)^2\phi_x^3(v_1)_{xx}$$
$$+9\phi_y^2\phi_{3x}+144i\sqrt{2}(v_1)^3\phi_x^2\phi_{3x}-972iv_1\phi_x^2(v_1)_x\phi_{3x}$$
$$+972i(v_1)^2\phi_x\phi_{xx}\phi_{3x}-486i(v_1)^2\phi_x^2\phi_{4x}=0, \tag{A.20e}$$
$$-2\sqrt{2}(v_1)^3\phi_t\phi_x^2-2\sqrt{2}u_3(v_1)^3\phi_x^3+36v_1(v_1)_t\phi_x^3+18(v_1)^2\phi_x^3(u_3)_x$$
$$+36u_3v_1\phi_x^3(v_1)_x-30i\sqrt{2}(v_1)^4\phi_x^3(v_1)_x+540i(v_1)^2\phi_x^3(v_1)_x^2+36(v_1)^2\phi_x^2\phi_{xt}$$
$$-\frac{108\phi_y\phi_{xy}\phi_{xx}^2}{\phi_x}+36u_3(v_1)^2\phi_x^2\phi_{xx}-9i\sqrt{2}(v_1)^5\phi_x^2\phi_{xx}+612i(v_1)^3\phi_x^2(v_1)_x\phi_{xx}$$
$$+27\phi_x\phi_{xyy}\phi_{xx}+54\phi_{yy}\phi_{xx}^2+9i(v_1)^4\phi_x\phi_{xx}^2+1944i\sqrt{2}(v_1)^2\phi_x(v_1)_x\phi_{xx}^2$$
$$+180i(v_1)^3\phi_x^3(v_1)_{xx}+54\phi_x\phi_{xy}\phi_{xxy}+216\phi_y\phi_{xx}\phi_{xxy}+27\phi_x^2\phi_{xxyy}$$
$$-27\phi_{yy}\phi_x\phi_{3x}+90i(v_1)^4\phi_x^2\phi_{3x}-972i\sqrt{2}(v_1)^2\phi_x^2(v_1)_x\phi_{3x}$$
$$+972i\sqrt{2}(v_1)^3\phi_x\phi_{xx}\phi_{3x}-54\phi_y\phi_x\phi_{3xy}-324i\sqrt{2}(v_1)^3\phi_x^2\phi_{4x}=0, \tag{A.20f}$$
$$6\sqrt{2}(v_1)^2(v_1)_t\phi_x^3+2\sqrt{2}(v_1)^3\phi_x^3(u_3)_x+6\sqrt{2}u_3(v_1)^2\phi_x^3(v_1)_x+60i\sqrt{2}(v_1)^3\phi_x^3(v_1)_x^2$$
$$-108\phi_x\phi_{xxy}^2+2\sqrt{2}u_3(v_1)^3\phi_x^2\phi_{xx}+30i\sqrt{2}(v_1)^4\phi_x^2(v_1)_x\phi_{xx}+216i(v_1)^3\phi_x(v_1)_x\phi_{xx}^2$$
$$+2\sqrt{2}(v_1)^3\phi_x^2\phi_{xt}+54\phi_{xyy}\phi_{xx}^2-54i(v_1)^4\phi_{xx}^3+15i\sqrt{2}(v_1)^4\phi_x^3(v_1)_{xx}$$
$$-\frac{108\phi_{xy}^2\phi_{xx}^2}{\phi_x}+216\phi_{xy}\phi_{xx}\phi_{xxy}-108\phi_x\phi_{xx}\phi_{xxyy}+3i\sqrt{2}(v_1)^5\phi_x^2\phi_{3x}$$
$$-108i(v_1)^3\phi_x^2(v_1)_x\phi_{3x}+108i(v_1)^4\phi_x\phi_{xx}\phi_{3x}+27\phi_x^2\phi_{3xyy}-27i(v_1)^4\phi_x^2\phi_{4x}=0, \tag{A.20g}$$

and

$$\sqrt{2}v_1\phi_x^2+3\phi_x\phi_{xx}=0, \tag{A.20h}$$
$$2(v_1)^2\phi_x^2-9\phi_{xx}^2=0, \tag{A.20i}$$
$$2\sqrt{2}(v_1)^3\phi_x^2-\frac{54\phi_{xx}^3}{\phi_x}+108\phi_{xx}\phi_{3x}-27\phi_x\phi_{4x}=0. \tag{A.20j}$$

The substitution of (4.5.9) to the 2+1 Burgers Equation (4.3.10a) yields the coefficient equations

$$\mathcal{O}(\chi^{-6}): \quad u_0v_0 - 4iv_0^2 = 0, \tag{A.21a}$$

$$\mathcal{O}(\chi^{-5}): \quad -4u_1v_0^3 - 12u_0v_0^2v_1 + 48iv_0^3v_1 + v_0^3(u_0)_x + 3u_0v_0^2(v_0)_x - 32iv_0^3(v_0)_x = 0, \tag{A.21b}$$

$$\begin{aligned}\mathcal{O}(\chi^{-4}): \quad & 12iK^2u_0 + 6Cv_0^3 - 5Su_0v_0^3 - 6u_2v_0^3 + 32iSv_0^4 - 18u_1v_0^2v_1 - 18u_0v_0v_1^2 \\ & + 72iv_0^2v_1^2 + 6v_0^2v_1(u_0)_x + 2v_0^3(u_1)_x + 6u_1v_0^2(v_0)_x + 12u_0v_0v_1(v_0)_x \\ & - 144iv_0^2v_1(v_0)_x + 24iv_0^2(v_0)_x^2 + 6u_0v_0^2(v_1)_x - 48iv_0^3(v_1)_x + 8iv_0^3(v_0)_{xx} = 0,\end{aligned} \tag{A.21c}$$

$$\begin{aligned}\mathcal{O}(\chi^{-3}): \quad & 2iK^2u_1 + 6Cv_0^2v_1 + 8iv_0v_1^3 + 3v_0^2(v_0)_t + 2iu_0K_y + 4iK(u_0)_y - 10iKu_0K_x \\ & - 2iv_0^4S_x + 3v_0v_1^2(u_0)_x + 3v_0^2v_1(u_1)_x + v_0^3(u_2)_x + 3u_2v_0^2(v_0)_x + 6u_1v_0v_1(v_0)_x \\ & + 3u_0v_1^2(v_0)_x - 48iv_0v_1^2(v_0)_x + 24iv_0v_1(v_0)_x^2 + 3u_1v_0^2(v_1)_x + 6u_0v_0v_1(v_1)_x \\ & - 48iv_0^2v_1(v_1)_x + 24iv_0^2(v_0)_x(v_1)_x + 12iv_0^2v_1(v_0)_{xx} + 4iv_0^3(v_1)_{xx} - 2Su_1v_0^3 \\ & - 6Su_0v_0^2v_1 - 6u_2v_0^2v_1 + 36iSv_0^3v_1 - 6u_1v_0v_1^2 - 2u_0v_1^3 - 3v_0^3C_x - 16iSv_0^3(v_0)_x = 0,\end{aligned} \tag{A.21d}$$

$$\begin{aligned}\mathcal{O}(\chi^{-2}): \quad & 4iK^2Su_0 + \frac{3}{2}CSv_0^3 + 3iS^2v_0^4 + 3Cv_0v_1^2 + 24iSv_0^2v_1^2 + 6v_0v_1(v_0)_t + 3v_0^2(v_1)_t + iu_1K_y \\ & + 2iK(u_1)_y + i(u_0)_{yy} - 3iKu_1K_x - 4i(u_0)_yK_x + 4iu_0K_x^2 - 6iv_0^3v_1S_x + v_1^3(u_0)_x - u_1v_1^3 \\ & + 3v_0v_1^2(u_1)_x + 3v_0^2v_1(u_2)_x + 6u_2v_0v_1(v_0)_x - 36iSv_0^2v_1(v_0)_x + 3u_1v_1^2(v_0)_x - 8iv_1^3(v_0)_x \\ & + 12iv_1^2(v_0)_x^2 + 3u_2v_0^2(v_1)_x - 12iSv_0^3(v_1)_x + 6u_1v_0v_1(v_1)_x + 3u_0v_1^2(v_1)_x - 24iv_0v_1^2(v_1)_x \\ & + 48iv_0v_1(v_0)_x(v_1)_x + 12iv_0^2(v_1)_x^2 - 2iu_0K_{xy} + \frac{3}{2}v_0^3C_{xx} + 4iKu_0K_{xx} + 12iv_0v_1^2(v_0)_{xx} \\ & + 12iv_0^2v_1(v_1)_{xx} - \frac{3}{2}Su_2v_0^3 - \frac{9}{2}Su_1v_0^2v_1 - \frac{9}{2}Su_0v_0v_1^2 - 3u_2v_0v_1^2 - 6v_0^2v_1C_x = 0,\end{aligned} \tag{A.21e}$$

$$\begin{aligned}\mathcal{O}(\chi^{-1}): \quad & iK^2Su_1 + 3CSv_0^2v_1 - 3Su_2v_0^2v_1 + 6iS^2v_0^3v_1 - 3Su_1v_0v_1^2 - Su_0v_1^3 + 4iSv_0v_1^3 \\ & + 3v_1^2(v_0)_t + 6v_0v_1(v_1)_t + iSu_0K_y + iKu_0S_y + 2iKS(u_0)_y + i(u_1)_{yy} - 3v_0v_1^2C_x \\ & - 3iKSu_0K_x - 2i(u_1)_yK_x + iu_1K_x^2 - 6iv_0^2v_1^2S_x + v_1^3(u_1)_x + 3v_0v_1^2(u_2)_x \\ & + 3u_2v_1^2(v_0)_x - 24iSv_0v_1^2(v_0)_x + 6u_2v_0v_1(v_1)_x - 24iSv_0^2v_1(v_1)_x + 3u_1v_1^2(v_1)_x \\ & + 24iv_1^2(v_0)_x(v_1)_x + 24iv_0v_1(v_1)_x^2 - iu_1K_{xy} + 3v_0^2v_1C_{xx} + iKu_1K_{xx} \\ & + 2i(u_0)_yK_{xx} - 3iu_0K_xK_{xx} + 4iv_1^3(v_0)_{xx} + 12iv_0v_1^2(v_1)_{xx} + iu_0K_{xxy} = 0,\end{aligned} \tag{A.21f}$$

$$\begin{aligned}\mathcal{O}(\chi^{0}): \quad & iK^2S^2u_0 + 3CSv_0v_1^2 - 3Su_2v_0v_1^2 + 6iS^2v_0^2v_1^2 - Su_1v_1^3 + 6v_1^2(v_1)_t + iSu_1K_y \\ & + iKu_1S_y + 2iKS(u_1)_y + 2i(u_2)_{yy} - iKSu_1K_x - 4iv_0v_1^3S_x + 2v_1^3(u_2)_x \\ & - 8iSv_1^3(v_0)_x + 6u_2v_1^2(v_1)_x - 24iSv_0v_1^2(v_1)_x + 24iv_1^2(v_1)_x^2 + 3v_0v_1^2C_{xx} \\ & + 2iKSu_0K_{xx} + 2i(u_1)_yK_{xx} - iu_1K_xK_{xx} + iu_0K_{xx}^2 + 8iv_1^3(v_1)_{xx} + iu_1K_{xxy} = 0,\end{aligned} \tag{A.21g}$$

and from (4.3.10b), we have

$$\mathcal{O}(\chi^{-3}): \quad 2iu_0 + v_0^3 = 0, \tag{A.22a}$$

$$\mathcal{O}(\chi^{-2}): \quad iu_1 + 3v_0^2v_1 - i(u_0)_x = 0, \tag{A.22b}$$

$$\mathcal{O}(\chi^{-1}): \quad iSu_0 + 3v_0v_1^2 - i(u_1)_x = 0, \tag{A.22c}$$

$$\mathcal{O}(\chi^{0}): \quad iSu_1 + 2v_1^3 - 2i(u_2)_x = 0. \tag{A.22d}$$

The substitution of (4.5.17) to the 2+1 Burgers Equation (4.3.18a) yields the coefficient equations

$$\mathcal{O}(\chi^{-8}): \quad u_0v_0 - 6i(v_0)^3 = 0, \tag{A.23a}$$

$$\mathcal{O}(\chi^{-7}): \quad -6u_1(v_0)^2 - 24u_0v_0v_1 + 180i(v_0)^3v_1 + (v_0)^2(u_0)_x + 4u_0v_0(v_0)_x - 72i(v_0)^3(v_0)_x = 0, \tag{A.23b}$$

$$\begin{aligned}\mathcal{O}(\chi^{-6}): \quad & -7Su_0(v_0)^3 - 10u_2(v_0)^3 + 72iS(v_0)^5 - 40u_1(v_0)^2v_1 - 60u_0v_0(v_1)^2 + 600i(v_0)^3(v_1)^2 \\ & + 8(v_0)^2v_1(u_0)_x + 2(v_0)^3(u_1)_x + 8u_1(v_0)^2(v_0)_x + 24u_0v_0v_1(v_0)_x - 600i(v_0)^3v_1(v_0)_x \\ & + 60i(v_0)^3(v_0)_x^2 + 8u_0(v_0)^2(v_1)_x - 120i(v_0)^4(v_1)_x + 12i(v_0)^4(v_0)_{xx} = 0,\end{aligned} \tag{A.23c}$$

$$\begin{aligned}\mathcal{O}(\chi^{-5}): \quad & 12iK^2u_0 + 4C(v_0)^4 - 3Su_1(v_0)^4 - 4u_3(v_0)^4 - 12Su_0(v_0)^3v_1 - 16u_2(v_0)^3v_1 \\ & + 150iS(v_0)^5v_1 - 24u_1(v_0)^2(v_1)^2 - 16u_0v_0(v_1)^3 + 240i(v_0)^3(v_1)^3 - 3i(v_0)^6S_x \\ & + 6(v_0)^2(v_1)^2(u_0)_x + 4(v_0)^3v_1(u_1)_x + (v_0)^4(u_2)_x + 4u_2(v_0)^3(v_0)_x - 36iS(v_0)^5(v_0)_x \\ & + 12u_1(v_0)^2v_1(v_0)_x + 12u_0v_0(v_1)^2(v_0)_x - 480i(v_0)^3(v_1)^2(v_0)_x + 120i(v_0)^3v_1(v_0)_x^2 \\ & + 4u_1(v_0)^3(v_1)_x + 12u_0(v_0)^2v_1(v_1)_x - 240i(v_0)^4v_1(v_1)_x + 60i(v_0)^4(v_0)_x(v_1)_x \\ & + 30i(v_0)^4v_1(v_0)_{xx} + 6i(v_0)^5(v_1)_{xx} = 0,\end{aligned} \tag{A.23d}$$

$$\begin{aligned}\mathcal{O}(\chi^{-4}): \quad & 12iK^2u_1 + 15iS^2(v_0)^6 + 24C(v_0)^3v_1 - 24u_3(v_0)^3v_1 - 36u_2(v_0)^2(v_1)^2 - 24u_1v_0(v_1)^3 \\ & - 6u_0(v_1)^4 + 180i(v_0)^2(v_1)^4 + 8(v_0)^3(v_0)_t + 6iu_0K_y + 12iK(u_0)_y - 8(v_0)^4C_x \\ & - 42iKu_0K_x - 30i(v_0)^5v_1S_x + 8v_0(v_1)^3(u_0)_x + 12(v_0)^2(v_1)^2(u_1)_x + 8(v_0)^3v_1(u_2)_x \\ & + 2(v_0)^4(u_3)_x + 8u_3(v_0)^3(v_0)_x + 24u_2(v_0)^2v_1(v_0)_x + 24u_1v_0(v_1)^2(v_0)_x + 8u_0(v_1)^3(v_0)_x \\ & - 720i(v_0)^2(v_1)^3(v_0)_x + 360i(v_0)^2(v_1)^2(v_0)_x^2 + 8u_2(v_0)^3(v_1)_x + 24u_1(v_0)^2v_1(v_1)_x \\ & + 24u_0v_0(v_1)^2(v_1)_x - 720i(v_0)^3(v_1)^2(v_1)_x + 480i(v_0)^3v_1(v_0)_x(v_1)_x + 60i(v_0)^4(v_1)_x^2 \\ & + 120i(v_0)^3(v_1)^2(v_0)_{xx} + 60i(v_0)^4v_1(v_1)_{xx} - 5Su_2(v_0)^4 - 20Su_1(v_0)^3v_1 \\ & - 30Su_0(v_0)^2(v_1)^2 + 480iS(v_0)^4(v_1)^2 - 300iS(v_0)^4v_1(v_0)_x - 60iS(v_0)^5(v_1)_x = 0,\end{aligned} \tag{A.23e}$$

$$\begin{aligned}\mathcal{O}(\chi^{-3}): \quad & 9iK^2Su_0 + 2iK^2u_2 + 2CS(v_0)^4 + 30iS^2(v_0)^5v_1 + 12C(v_0)^2(v_1)^2 + 180iS(v_0)^3(v_1)^3 \\ & + 12iv_0(v_1)^5 + 12(v_0)^2v_1(v_0)_t + 4(v_0)^3(v_1)_t + 2iu_1K_y + 4iK(u_1)_y + i(u_0)_{yy} \\ & - 10iKu_1K_x - 6i(u_0)_yK_x + 9iu_0K_x^2 - 30i(v_0)^4(v_1)^2S_x + (v_1)^4(u_0)_x + 4v_0(v_1)^3(u_1)_x \\ & + 6(v_0)^2(v_1)^2(u_2)_x + 4(v_0)^3v_1(u_3)_x + 12u_3(v_0)^2v_1(v_0)_x + 12u_2v_0(v_1)^2(v_0)_x \\ & - 240iS(v_0)^3(v_1)^2(v_0)_x + 4u_1(v_1)^3(v_0)_x - 120iv_0(v_1)^4(v_0)_x + 120iv_0(v_1)^3(v_0)_x^2 \\ & + 4u_3(v_0)^3(v_1)_x + 12u_2(v_0)^2v_1(v_1)_x - 120iS(v_0)^4v_1(v_1)_x + 12u_1v_0(v_1)^2(v_1)_x \\ & + 4u_0(v_1)^3(v_1)_x - 240i(v_0)^2(v_1)^3(v_1)_x + 360i(v_0)^2(v_1)^2(v_0)_x(v_1)_x + 120i(v_0)^3v_1(v_1)_x^2 \\ & - 3iu_0K_{xy} + 2(v_0)^4C_{xx} + 9iKu_0K_{xx} + 60i(v_0)^2(v_1)^3(v_0)_{xx} + 60i(v_0)^3(v_1)^2(v_1)_{xx} \\ & - 2Su_3(v_0)^4 - 8Su_2(v_0)^3v_1 - 12Su_1(v_0)^2(v_1)^2 - 12u_3(v_0)^2(v_1)^2 - 8Su_0v_0(v_1)^3 \\ & - 8u_2v_0(v_1)^3 - 2u_1(v_1)^4 - 12(v_0)^3v_1C_x = 0,\end{aligned} \tag{A.23f}$$

$$\begin{aligned}
\mathcal{O}(\chi^{-2}):\quad & 8iK^2Su_1+12CS(v_0)^3v_1-12Su_3(v_0)^3v_1-18Su_2(v_0)^2(v_1)^2+90iS^2(v_0)^4(v_1)^2\\
&+8Cv_0(v_1)^3-12Su_1v_0(v_1)^3-8u_3v_0(v_1)^3-3Su_0(v_1)^4-2u_2(v_1)^4+120iS(v_0)^2(v_1)^4\\
&+24v_0(v_1)^2(v_0)_t+24(v_0)^2v_1(v_1)_t+3iSu_0K_y+2iu_2K_y+3iKu_0S_y+6iKS(u_0)_y\\
&+4iK(u_2)_y+2i(u_1)_{yy}-24(v_0)^2(v_1)^2C_x-15iKSu_0K_x-6iKu_2K_x-8i(u_1)_yK_x\\
&+8iu_1K_x^2-60i(v_0)^3(v_1)^3S_x+2(v_1)^4(u_1)_x+8v_0(v_1)^3(u_2)_x+12(v_0)^2(v_1)^2(u_3)_x\\
&+24u_3v_0(v_1)^2(v_0)_x+8u_2(v_1)^3(v_0)_x-360iS(v_0)^2(v_1)^3(v_0)_x-24i(v_1)^5(v_0)_x\\
&+60i(v_1)^4(v_0)_x^2+24u_3(v_0)^2v_1(v_1)_x+24u_2v_0(v_1)^2(v_1)_x-360iS(v_0)^3(v_1)^2(v_1)_x\\
&+8u_1(v_1)^3(v_1)_x-120iv_0(v_1)^4(v_1)_x+480iv_0(v_1)^3(v_0)_x(v_1)_x+360i(v_0)^2(v_1)^2(v_1)_x^2\\
&-4iu_1K_{xy}+12(v_0)^3v_1C_{xx}+8iKu_1K_{xx}+6i(u_0)_yK_{xx}-15iu_0K_xK_{xx}\\
&+60iv_0(v_1)^4(v_0)_{xx}+120i(v_0)^2(v_1)^3(v_1)_{xx}+3iu_0K_{xxy}=0, \qquad \text{(A.23g)}\\
\mathcal{O}(\chi^{-1}):\quad & 3iK^2S^2u_0+2iK^2Su_2+12CS(v_0)^2(v_1)^2-12Su_3(v_0)^2(v_1)^2-8Su_2v_0(v_1)^3\\
&+60iS^2(v_0)^3(v_1)^3-2Su_1(v_1)^4+12iSv_0(v_1)^5+8(v_1)^3(v_0)_t+24v_0(v_1)^2(v_1)_t\\
&+2iSu_1K_y+2iKu_1S_y+4iKS(u_1)_y+2i(u_2)_{yy}-8v_0(v_1)^3C_x-6iKSu_1K_x\\
&-4i(u_2)_yK_x+2iu_2K_x^2-30i(v_0)^2(v_1)^4S_x+2(v_1)^4(u_2)_x+8v_0(v_1)^3(u_3)_x\\
&+8u_3(v_1)^3(v_0)_x-120iSv_0(v_1)^4(v_0)_x+24u_3v_0(v_1)^2(v_1)_x+8u_2(v_1)^3(v_1)_x\\
&-240iS(v_0)^2(v_1)^3(v_1)_x+120i(v_1)^4(v_0)_x(v_1)_x+240iv_0(v_1)^3(v_1)_x^2-2iu_2K_{xy}\\
&+12(v_0)^2(v_1)^2C_{xx}+6iKSu_0K_{xx}+2iKu_2K_{xx}+4i(u_1)_yK_{xx}-6iu_1K_xK_{xx}\\
&+3iu_0K_{xx}^2+12i(v_1)^5(v_0)_{xx}+60iv_0(v_1)^4(v_1)_{xx}+2iu_1K_{xxy}=0, \qquad \text{(A.23h)}\\
\mathcal{O}(\chi^{0}):\quad & iK^2S^2u_1+4CSv_0(v_1)^3-4Su_3v_0(v_1)^3-Su_2(v_1)^4+15iS^2(v_0)^2(v_1)^4+8(v_1)^3(v_1)_t\\
&+iSu_2K_y+iKu_2S_y+2iKS(u_2)_y+2i(u_3)_{yy}-iKSu_2K_x-6iv_0(v_1)^5S_x+2(v_1)^4(u_3)_x\\
&-12iS(v_1)^5(v_0)_x+8u_3(v_1)^3(v_1)_x-60iSv_0(v_1)^4(v_1)_x+60i(v_1)^4(v_1)_x^2+4v_0(v_1)^3C_{xx}\\
&+2iKSu_1K_{xx}+2i(u_2)_yK_{xx}-iu_2K_xK_{xx}+iu_1K_{xx}^2+12i(v_1)^5(v_1)_{xx}+iu_2K_{xxy}=0, \qquad \text{(A.23i)}
\end{aligned}$$

and from (4.3.18b), we have

$$\begin{aligned}
\mathcal{O}(\chi^{-4}):\quad & 3iu_0+(v_0)^4=0, \qquad \text{(A.24a)}\\
\mathcal{O}(\chi^{-3}):\quad & 2iu_1+4(v_0)^3v_1-i(u_0)_x=0, \qquad \text{(A.24b)}\\
\mathcal{O}(\chi^{-2}):\quad & 3iSu_0+2iu_2+12(v_0)^2(v_1)^2-2i(u_1)_x=0, \qquad \text{(A.24c)}\\
\mathcal{O}(\chi^{-1}):\quad & iSu_1+4v_0(v_1)^3-i(u_2)_x=0, \qquad \text{(A.24d)}\\
\mathcal{O}(\chi^{0}):\quad & iSu_2+2(v_1)^4-2i(u_3)_x=0. \qquad \text{(A.24e)}
\end{aligned}$$

Solving the linearly dependent systems for u_j, v_j to verify the resonances of the $n=2$ KPII equation (5.3.4), we obtain

$$\begin{aligned}
u_4=\ & 6i3^{2/3}v_4\phi_x-\frac{\phi_{xyy}}{2\phi_x^3}+\frac{\phi_{yy}\phi_{xx}}{2\phi_x^4}+\frac{3\phi_{xt}\phi_{xx}}{2\phi_x^4}+\frac{2\phi_y\phi_{xy}\phi_{xx}}{\phi_x^5}-\frac{5\phi_y^2\phi_{xx}^2}{2\phi_x^6}-\frac{3\phi_t\phi_{xx}^2}{2\phi_x^5}\\
&+\frac{15\phi_{xx}^4}{\phi_x^6}-\frac{\phi_{xxt}}{2\phi_x^3}+\frac{\phi_y^2\phi_{3x}}{2\phi_x^5}+\frac{\phi_t\phi_{3x}}{2\phi_x^4}-\frac{21\phi_{xx}^2\phi_{3x}}{\phi_x^5}+\frac{3\phi_{3x}^2}{\phi_x^4}+\frac{9\phi_{xx}\phi_{4x}}{2\phi_x^4}-\frac{\phi_{5x}}{2\phi_x^3}, \qquad \text{(A.25a)}\\
v_4\ & \text{ arbitrary}, \qquad \text{(A.25b)}
\end{aligned}$$

$$u_5 = 4i3^{2/3}v_5\phi_x - 2i3^{2/3}(v_4)_x - \frac{6i3^{2/3}v_4\phi_{xx}}{\phi_x} - \frac{2\phi_{xy}^2\phi_{xx}}{3\phi_x^6} - \frac{2\phi_{xyy}\phi_{xx}}{3\phi_x^5} + \frac{7\phi_{yy}\phi_{xx}^2}{12\phi_x^6} + \frac{5\phi_{xt}\phi_{xx}^2}{2\phi_x^6}$$
$$+ \frac{31\phi_y\phi_{xy}\phi_{xx}^2}{6\phi_x^7} - \frac{61\phi_y^2\phi_{xx}^3}{12\phi_x^8} - \frac{5\phi_t\phi_{xx}^3}{2\phi_x^7} + \frac{123\phi_{xx}^5}{4\phi_x^8} - \frac{\phi_{xx}\phi_{xxt}}{\phi_x^5} - \frac{2\phi_y\phi_{xx}\phi_{xxy}}{3\phi_x^6} + \frac{\phi_{xxyy}}{6\phi_x^4}$$
$$- \frac{\phi_{yy}\phi_{3x}}{18\phi_x^5} - \frac{2\phi_{xt}\phi_{3x}}{3\phi_x^5} - \frac{11\phi_y\phi_{xy}\phi_{3x}}{9\phi_x^6} + \frac{47\phi_y^2\phi_{xx}\phi_{3x}}{18\phi_x^7} + \frac{5\phi_t\phi_{xx}\phi_{3x}}{3\phi_x^6} - \frac{115\phi_{xx}^3\phi_{3x}}{2\phi_x^7}$$
$$+ \frac{59\phi_{xx}\phi_{3x}^2}{3\phi_x^6} + \frac{\phi_{3xt}}{6\phi_x^4} - \frac{\phi_y^2\phi_{4x}}{6\phi_x^6} - \frac{\phi_t\phi_{4x}}{6\phi_x^5} + \frac{53\phi_{xx}^2\phi_{4x}}{4\phi_x^6} - \frac{23\phi_{3x}\phi_{4x}}{6\phi_x^5} - \frac{2\phi_{xx}\phi_{5x}}{\phi_x^5} + \frac{\phi_{6x}}{6\phi_x^4}, \tag{A.25c}$$

v_5 arbitrary, (A.25d)

$$u_6 = \frac{\phi_{yy}^2}{288\phi_x^6} + 3i3^{2/3}v_6\phi_x - i3^{2/3}(v_5)_x - \frac{\phi_y\phi_{yy}\phi_{xy}}{72\phi_x^7} + \frac{\phi_y^2\phi_{xy}^2}{72\phi_x^8} + \frac{\phi_y^2\phi_{yy}\phi_{xx}}{144\phi_x^8} - \frac{4i3^{2/3}v_5\phi_{xx}}{\phi_x}$$
$$+ \frac{3i3^{2/3}(v_4)_x\phi_{xx}}{2\phi_x^2} - \frac{\phi_y^3\phi_{xy}\phi_{xx}}{72\phi_x^9} + \frac{\phi_y^4\phi_{xx}^2}{288\phi_x^{10}} - \frac{9i3^{2/3}v_4\phi_{xx}^2}{4\phi_x^3} - \frac{55\phi_{xy}^2\phi_{xx}^2}{24\phi_x^8} - \frac{47\phi_{xyy}\phi_{xx}^2}{48\phi_x^7}$$
$$+ \frac{13\phi_{yy}\phi_{xx}^3}{16\phi_x^8} + \frac{35\phi_{xt}\phi_{xx}^3}{8\phi_x^8} + \frac{281\phi_y\phi_{xy}\phi_{xx}^3}{24\phi_x^9} - \frac{491\phi_y^2\phi_{xx}^4}{48\phi_x^{10}} - \frac{35\phi_t\phi_{xx}^4}{8\phi_x^9} + \frac{1965\phi_{xx}^6}{32\phi_x^{10}}$$
$$+ \frac{i3^{2/3}(v_4)_{xx}}{2\phi_x} - \frac{15\phi_{xx}^2\phi_{xxt}}{8\phi_x^7} + \frac{\phi_{xy}\phi_{xx}\phi_{xxy}}{2\phi_x^7} - \frac{55\phi_y\phi_{xx}^2\phi_{xxy}}{24\phi_x^8} + \frac{\phi_{xx}\phi_{xxyy}}{3\phi_x^6}$$
$$+ \frac{5i3^{2/3}v_4\phi_{3x}}{2\phi_x^2} + \frac{17\phi_{xy}^2\phi_{3x}}{36\phi_x^7} + \frac{13\phi_{xyy}\phi_{3x}}{72\phi_x^6} - \frac{5\phi_{yy}\phi_{xx}\phi_{3x}}{18\phi_x^7} - \frac{5\phi_{xt}\phi_{xx}\phi_{3x}}{2\phi_x^7}$$
$$- \frac{53\phi_y\phi_{xy}\phi_{xx}\phi_{3x}}{9\phi_x^8} + \frac{1219\phi_y^2\phi_{xx}^2\phi_{3x}}{144\phi_x^9} + \frac{35\phi_t\phi_{xx}^2\phi_{3x}}{8\phi_x^8} - \frac{557\phi_{xx}^4\phi_{3x}}{4\phi_x^9} + \frac{5\phi_{xxt}\phi_{3x}}{12\phi_x^6}$$
$$+ \frac{17\phi_y\phi_{xxy}\phi_{3x}}{36\phi_x^7} - \frac{47\phi_y^2\phi_{3x}^2}{72\phi_x^8} - \frac{5\phi_t\phi_{3x}^2}{12\phi_x^7} + \frac{583\phi_{xx}^2\phi_{3x}^2}{8\phi_x^8} - \frac{44\phi_{3x}^3}{9\phi_x^7} + \frac{5\phi_{xx}\phi_{3xt}}{12\phi_x^6}$$
$$+ \frac{\phi_y\phi_{xx}\phi_{3xy}}{6\phi_x^7} - \frac{\phi_{3x2y}}{24\phi_x^5} - \frac{\phi_{yy}\phi_{4x}}{144\phi_x^6} + \frac{5\phi_{xt}\phi_{4x}}{24\phi_x^6} + \frac{31\phi_y\phi_{xy}\phi_{4x}}{72\phi_x^7} - \frac{133\phi_y^2\phi_{xx}\phi_{4x}}{144\phi_x^8}$$
$$- \frac{5\phi_t\phi_{xx}\phi_{4x}}{8\phi_x^7} + \frac{551\phi_{xx}^3\phi_{4x}}{16\phi_x^8} - \frac{43\phi_{xx}\phi_{3x}\phi_{4x}}{2\phi_x^7} + \frac{95\phi_{4x}^2}{96\phi_x^6} - \frac{\phi_{4xt}}{24\phi_x^5} + \frac{\phi_y^2\phi_{5x}}{24\phi_x^7} + \frac{\phi_t\phi_{5x}}{24\phi_x^6}$$
$$- \frac{93\phi_{xx}^2\phi_{5x}}{16\phi_x^7} + \frac{35\phi_{3x}\phi_{5x}}{24\phi_x^6} + \frac{2\phi_{xx}\phi_{6x}}{3\phi_x^6} - \frac{\phi_{7x}}{24\phi_x^5}, \tag{A.25e}$$

v_6 arbitrary. (A.25f)

The coefficient equations from the resonance analysis of the $n = 3$ KPII system (5.3.10) gives

$$u_2 = 15i3^{2/3}10^{1/3}\phi_{3x}, \tag{A.26a}$$

$$v_2 = \frac{i3^{2/3}5^{1/3}\phi_{xx}^2}{2\cdot 2^{2/3}\phi_x^3} - \frac{i5^{1/3}\phi_{3x}}{3^{1/3}2^{2/3}\phi_x^2}, \tag{A.26b}$$

$$u_3 = -\frac{\phi_y^2}{\phi_x^2} - \frac{\phi_t}{\phi_x} - \frac{15i3^{2/3}5^{1/3}\phi_{xx}^3}{2\cdot 2^{2/3}\phi_x^3} + \frac{15i3^{2/3}5^{1/3}\phi_{xx}\phi_{3x}}{2^{2/3}\phi_x^2} - \frac{15i3^{2/3}5^{1/3}\phi_{4x}}{2\cdot 2^{2/3}\phi_x}, \tag{A.26c}$$

$$v_3 = \frac{i3^{2/3}5^{1/3}\phi_{xx}^3}{2\cdot 2^{2/3}\phi_x^5} - \frac{i3^{2/3}5^{1/3}\phi_{xx}\phi_{3x}}{2\cdot 2^{2/3}\phi_x^4} + \frac{i5^{1/3}\phi_{4x}}{4\cdot 3^{1/3}2^{2/3}\phi_x^3}, \tag{A.26d}$$

$$\begin{aligned} u_4 =& \frac{12\phi_{yy}}{25\phi_x^2} + \frac{\phi_{xt}}{\phi_x^2} + \frac{26\phi_y\phi_{xy}}{25\phi_x^3} - \frac{38\phi_y^2\phi_{xx}}{25\phi_x^4} - \frac{\phi_t\phi_{xx}}{\phi_x^3} - \frac{45i3^{2/3}5^{1/3}\phi_{xx}^4}{4\cdot 2^{2/3}\phi_x^5} \\ &+ \frac{45i3^{2/3}5^{1/3}\phi_{xx}^2\phi_{3x}}{2\cdot 2^{2/3}\phi_x^4} - \frac{5i3^{2/3}5^{1/3}\phi_{3x}^2}{2^{2/3}\phi_x^3} - \frac{15i3^{2/3}5^{1/3}\phi_{xx}\phi_{4x}}{2\cdot 2^{2/3}\phi_x^3} + \frac{3i3^{2/3}5^{1/3}\phi_{5x}}{2\cdot 2^{2/3}\phi_x^2}, \end{aligned} \tag{A.26e}$$

$$\begin{aligned} v_4 =& \frac{\phi_{yy}}{750\phi_x^4} - \frac{\phi_y\phi_{xy}}{375\phi_x^5} + \frac{\phi_y^2\phi_{xx}}{750\phi_x^6} + \frac{5i3^{2/3}5^{1/3}\phi_{xx}^4}{8\cdot 2^{2/3}\phi_x^7} \\ &- \frac{5i5^{1/3}\phi_{xx}^2\phi_{3x}}{2\cdot 3^{1/3}2^{2/3}\phi_x^6} + \frac{i5^{1/3}\phi_{3x}^2}{3\cdot 3^{1/3}2^{2/3}\phi_x^5} + \frac{i5^{1/3}\phi_{xx}\phi_{4x}}{2\cdot 3^{1/3}2^{2/3}\phi_x^5} - \frac{i\phi_{5x}}{4\cdot 3^{1/3}10^{2/3}\phi_x^4}, \end{aligned} \tag{A.26f}$$

$$\begin{aligned} u_5 =& -\frac{\phi_{xy}^2}{25\phi_x^4} - \frac{12\phi_{xyy}}{25\phi_x^3} + \frac{18\phi_{yy}\phi_{xx}}{25\phi_x^4} + \frac{3\phi_{xt}\phi_{xx}}{2\phi_x^4} + \frac{41\phi_y\phi_{xy}\phi_{xx}}{25\phi_x^5} - \frac{58\phi_y^2\phi_{xx}^2}{25\phi_x^6} - \frac{3\phi_t\phi_{xx}^2}{2\phi_x^5} - \frac{\phi_{xxt}}{2\phi_x^3} \\ &- \frac{135i3^{2/3}5^{1/3}\phi_{xx}^5}{8\cdot 2^{2/3}\phi_x^7} - \frac{\phi_y\phi_{xxy}}{25\phi_x^4} + \frac{13\phi_y^2\phi_{3x}}{25\phi_x^5} + \frac{\phi_t\phi_{3x}}{2\phi_x^4} + \frac{75i3^{2/3}5^{1/3}\phi_{xx}^3\phi_{3x}}{2\cdot 2^{2/3}\phi_x^6} - \frac{i3^{2/3}5^{1/3}\phi_{6x}}{4\cdot 2^{2/3}\phi_x^3} \\ &- \frac{15i3^{2/3}5^{1/3}\phi_{xx}\phi_{3x}^2}{2^{2/3}\phi_x^5} - \frac{45i3^{2/3}5^{1/3}\phi_{xx}^2\phi_{4x}}{4\cdot 2^{2/3}\phi_x^5} + \frac{15i3^{2/3}5^{1/3}\phi_{3x}\phi_{4x}}{4\cdot 2^{2/3}\phi_x^4} + \frac{9i3^{2/3}5^{1/3}\phi_{xx}\phi_{5x}}{4\cdot 2^{2/3}\phi_x^4}, \end{aligned} \tag{A.26g}$$

$$\begin{aligned} v_5 =& \frac{\phi_{xy}^2}{375\phi_x^6} - \frac{\phi_{xyy}}{750\phi_x^5} + \frac{\phi_{yy}\phi_{xx}}{300\phi_x^6} - \frac{3\phi_y\phi_{xy}\phi_{xx}}{250\phi_x^7} + \frac{7i3^{2/3}5^{1/3}\phi_{xx}^5}{8\cdot 2^{2/3}\phi_x^9} \\ &+ \frac{3\phi_y^2\phi_{xx}^2}{500\phi_x^8} + \frac{\phi_y\phi_{xxy}}{375\phi_x^6} - \frac{\phi_y^2\phi_{3x}}{750\phi_x^7} - \frac{35i5^{1/3}\phi_{xx}^3\phi_{3x}}{8\cdot 3^{1/3}2^{2/3}\phi_x^8} + \frac{5i5^{1/3}\phi_{xx}\phi_{3x}^2}{4\cdot 3^{1/3}2^{2/3}\phi_x^7} \\ &+ \frac{5i3^{2/3}5^{1/3}\phi_{xx}^2\phi_{4x}}{16\cdot 2^{2/3}\phi_x^7} - \frac{5i5^{1/3}\phi_{3x}\phi_{4x}}{24\cdot 3^{1/3}2^{2/3}\phi_x^6} - \frac{i5^{1/3}\phi_{xx}\phi_{5x}}{8\cdot 3^{1/3}2^{2/3}\phi_x^6} + \frac{i\phi_{6x}}{24\cdot 3^{1/3}10^{2/3}\phi_x^5}, \end{aligned} \tag{A.26h}$$

$$\begin{aligned} u_6 =& 120v_6\phi_x^2 - \frac{27\phi_{xy}^2\phi_{xx}}{25\phi_x^6} - \frac{12\phi_{xyy}\phi_{xx}}{25\phi_x^5} + \frac{9\phi_{yy}\phi_{xx}^2}{25\phi_x^6} + \frac{5\phi_{xt}\phi_{xx}^2}{2\phi_x^6} + \frac{161\phi_y\phi_{xy}\phi_{xx}^2}{25\phi_x^7} - \frac{143\phi_y^2\phi_{xx}^3}{25\phi_x^8} \\ &- \frac{5\phi_t\phi_{xx}^3}{2\phi_x^7} - \frac{735i3^{2/3}5^{1/3}\phi_{xx}^6}{4\cdot 2^{2/3}\phi_x^9} - \frac{\phi_{xx}\phi_{xxt}}{\phi_x^5} + \frac{\phi_{xy}\phi_{xxy}}{25\phi_x^5} - \frac{27\phi_y\phi_{xx}\phi_{xxy}}{25\phi_x^6} + \frac{4\phi_{xxyy}}{25\phi_x^4} \\ &- \frac{4\phi_{yy}\phi_{3x}}{25\phi_x^5} - \frac{2\phi_{xt}\phi_{3x}}{3\phi_x^5} - \frac{79\phi_y\phi_{xy}\phi_{3x}}{75\phi_x^6} + \frac{41\phi_y^2\phi_{xx}\phi_{3x}}{15\phi_x^7} + \frac{5\phi_t\phi_{xx}\phi_{3x}}{3\phi_x^6} - \frac{\phi_t\phi_{4x}}{6\phi_x^5} \\ &+ \frac{3045i3^{2/3}5^{1/3}\phi_{xx}^4\phi_{3x}}{8\cdot 2^{2/3}\phi_x^8} - \frac{355i3^{2/3}5^{1/3}\phi_{xx}^2\phi_{3x}^2}{2\cdot 2^{2/3}\phi_x^7} + \frac{95i5^{1/3}\phi_{3x}^3}{3\cdot 3^{1/3}2^{2/3}\phi_x^6} + \frac{\phi_{3xt}}{6\phi_x^4} + \frac{\phi_y\phi_{3xy}}{75\phi_x^5} \\ &- \frac{13\phi_y^2\phi_{4x}}{75\phi_x^6} - \frac{355i3^{2/3}5^{1/3}\phi_{xx}^3\phi_{4x}}{4\cdot 2^{2/3}\phi_x^7} + \frac{95i3^{2/3}5^{1/3}\phi_{xx}\phi_{3x}\phi_{4x}}{2\cdot 2^{2/3}\phi_x^6} - \frac{15i3^{2/3}5^{1/3}\phi_{4x}^2}{8\cdot 2^{2/3}\phi_x^5} \\ &+ \frac{57i3^{2/3}5^{1/3}\phi_{xx}^2\phi_{5x}}{4\cdot 2^{2/3}\phi_x^6} - \frac{3i3^{2/3}5^{1/3}\phi_{3x}\phi_{5x}}{2^{2/3}\phi_x^5} - \frac{3i3^{2/3}5^{1/3}\phi_{xx}\phi_{6x}}{2\cdot 2^{2/3}\phi_x^5} + \frac{i5^{1/3}\phi_{7x}}{4\cdot 3^{1/3}2^{2/3}\phi_x^4}, \end{aligned} \tag{A.26i}$$

v_6 arbitrary, (A.26j)

$$
\begin{aligned}
u_7 =& 90v_7\phi_x^2 - 30\phi_x(v_6)_x - 195v_6\phi_{xx} - \frac{323\phi_{xy}^2\phi_{xx}^2}{100\phi_x^8} - \frac{69\phi_{xyy}\phi_{xx}^2}{100\phi_x^7} + \frac{21\phi_{yy}\phi_{xx}^3}{40\phi_x^8} + \frac{35\phi_{xt}\phi_{xx}^3}{8\phi_x^8} \\
&+ \frac{354\phi_y\phi_{xy}\phi_{xx}^3}{25\phi_x^9} - \frac{2291\phi_y^2\phi_{xx}^4}{200\phi_x^{10}} - \frac{35\phi_t\phi_{xx}^4}{8\phi_x^9} - \frac{405i3^{2/3}5^{1/3}\phi_{xx}^7}{2^{2/3}\phi_x^{11}} - \frac{15\phi_{xx}^2\phi_{xxt}}{8\phi_x^7} \\
&+ \frac{43\phi_{xy}\phi_{xx}\phi_{xxy}}{50\phi_x^7} - \frac{323\phi_y\phi_{xx}^2\phi_{xxy}}{100\phi_x^8} - \frac{\phi_{xxy}^2}{100\phi_x^6} + \frac{7\phi_{xx}\phi_{xxyy}}{25\phi_x^6} + \frac{49\phi_{xy}^2\phi_{3x}}{75\phi_x^7} + \frac{\phi_{xyy}\phi_{3x}}{10\phi_x^6} \\
&- \frac{\phi_{yy}\phi_{xx}\phi_{3x}}{5\phi_x^7} - \frac{5\phi_{xt}\phi_{xx}\phi_{3x}}{2\phi_x^7} - \frac{203\phi_y\phi_{xy}\phi_{xx}\phi_{3x}}{30\phi_x^8} + \frac{703\phi_y^2\phi_{xx}^2\phi_{3x}}{75\phi_x^9} + \frac{35\phi_t\phi_{xx}^2\phi_{3x}}{8\phi_x^8} \\
&+ \frac{8505i3^{2/3}5^{1/3}\phi_{xx}^5\phi_{3x}}{8\cdot 2^{2/3}\phi_x^{10}} + \frac{5\phi_{xxt}\phi_{3x}}{12\phi_x^6} + \frac{49\phi_y\phi_{xxy}\phi_{3x}}{75\phi_x^7} - \frac{223\phi_y^2\phi_{3x}^2}{300\phi_x^8} - \frac{5\phi_t\phi_{3x}^2}{12\phi_x^7} + \frac{\phi_t\phi_{5x}}{24\phi_x^6} \\
&- \frac{1505i3^{2/3}5^{1/3}\phi_{xx}^3\phi_{3x}^2}{2\cdot 2^{2/3}\phi_x^9} + \frac{245i3^{2/3}5^{1/3}\phi_{xx}\phi_{3x}^3}{2\cdot 2^{2/3}\phi_x^8} + \frac{5\phi_{xx}\phi_{3xt}}{12\phi_x^6} - \frac{\phi_{xy}\phi_{3xy}}{75\phi_x^6} + \frac{43\phi_y\phi_{xx}\phi_{3xy}}{150\phi_x^7} \\
&- \frac{\phi_{3xyy}}{25\phi_x^5} + \frac{\phi_{yy}\phi_{4x}}{40\phi_x^6} + \frac{5\phi_{xt}\phi_{4x}}{24\phi_x^6} + \frac{19\phi_y\phi_{xy}\phi_{4x}}{50\phi_x^7} - \frac{23\phi_y^2\phi_{xx}\phi_{4x}}{24\phi_x^8} - \frac{5\phi_t\phi_{xx}\phi_{4x}}{8\phi_x^7} + \frac{13\phi_y^2\phi_{5x}}{300\phi_x^7} \\
&- \frac{1015i3^{2/3}5^{1/3}\phi_{xx}^4\phi_{4x}}{4\cdot 2^{2/3}\phi_x^9} + \frac{3955i3^{2/3}5^{1/3}\phi_{xx}^2\phi_{3x}\phi_{4x}}{16\cdot 2^{2/3}\phi_x^8} - \frac{565i5^{1/3}\phi_{3x}^2\phi_{4x}}{8\cdot 3^{1/3}2^{2/3}\phi_x^7} - \frac{\phi_{4xt}}{24\phi_x^5} - \frac{\phi_y\phi_{4xy}}{300\phi_x^6} \\
&- \frac{125i3^{2/3}5^{1/3}\phi_{xx}\phi_{4x}^2}{8\cdot 2^{2/3}\phi_x^7} + \frac{175i3^{2/3}5^{1/3}\phi_{xx}^3\phi_{5x}}{4\cdot 2^{2/3}\phi_x^8} + \frac{11i5^{1/3}\phi_{xx}\phi_{7x}}{8\cdot 3^{1/3}2^{2/3}\phi_x^6} + \frac{5i3^{2/3}5^{1/3}\phi_{3x}\phi_{6x}}{4\cdot 2^{2/3}\phi_x^6} \\
&- \frac{25i3^{2/3}5^{1/3}\phi_{xx}\phi_{3x}\phi_{5x}}{2^{2/3}\phi_x^7} + \frac{15i3^{2/3}5^{1/3}\phi_{4x}\phi_{5x}}{8\cdot 2^{2/3}\phi_x^6} - \frac{87i3^{2/3}5^{1/3}\phi_{xx}^2\phi_{6x}}{16\cdot 2^{2/3}\phi_x^7} - \frac{i5^{1/3}\phi_{8x}}{16\cdot 3^{1/3}2^{2/3}\phi_x^5}, \qquad \text{(A.26k)}
\end{aligned}
$$

v_7 arbitrary, (A.26l)

$$
\begin{aligned}
u_8 =& -\frac{1609i\phi_{yy}^2}{28125\cdot 3^{2/3}10^{1/3}\phi_x^7} - \frac{26i2^{2/3}\phi_y\phi_{3y}}{1125\cdot 3^{2/3}5^{1/3}\phi_x^7} - \frac{13i2^{2/3}\phi_{yyt}}{1125\cdot 3^{2/3}5^{1/3}\phi_x^6} + \frac{13i2^{2/3}\phi_{yy}\phi_{xt}}{1125\cdot 3^{2/3}5^{1/3}\phi_x^7} \\
&+ \frac{5168i2^{2/3}\phi_y\phi_{yy}\phi_{xy}}{28125\cdot 3^{2/3}5^{1/3}\phi_x^8} + \frac{26i2^{2/3}\phi_{yt}\phi_{xy}}{1125\cdot 3^{2/3}5^{1/3}\phi_x^7} - \frac{52i2^{2/3}\phi_y\phi_{xt}\phi_{xy}}{1125\cdot 3^{2/3}5^{1/3}\phi_x^8} - \frac{7118i2^{2/3}\phi_y^2\phi_{xy}^2}{28125\cdot 3^{2/3}5^{1/3}\phi_x^9} \\
&- \frac{26i2^{2/3}\phi_t\phi_{xy}^2}{1125\cdot 3^{2/3}5^{1/3}\phi_x^8} + \frac{26i2^{2/3}\phi_y\phi_{xyt}}{1125\cdot 3^{2/3}5^{1/3}\phi_x^7} + \frac{26i2^{2/3}\phi_y^2\phi_{xyy}}{375\cdot 3^{2/3}5^{1/3}\phi_x^8} + \frac{13i2^{2/3}\phi_t\phi_{xyy}}{1125\cdot 3^{2/3}5^{1/3}\phi_x^7} \\
&- \frac{3559i2^{2/3}\phi_y^2\phi_{yy}\phi_{xx}}{28125\cdot 3^{2/3}5^{1/3}\phi_x^9} - \frac{26i2^{2/3}\phi_y\phi_{yt}\phi_{xx}}{1125\cdot 3^{2/3}5^{1/3}\phi_x^8} - \frac{13i2^{2/3}\phi_t\phi_{yy}\phi_{xx}}{1125\cdot 3^{2/3}5^{1/3}\phi_x^8} - \frac{135(v_6)_x\phi_{xx}}{\phi_x} - 90\phi_x(v_7)_x \\
&+ \frac{13i2^{2/3}\phi_y^2\phi_{xt}\phi_{xx}}{375\cdot 3^{2/3}5^{1/3}\phi_x^9} + \frac{9068i2^{2/3}\phi_y^3\phi_{xy}\phi_{xx}}{28125\cdot 3^{2/3}5^{1/3}\phi_x^{10}} + \frac{26i2^{2/3}\phi_t\phi_y\phi_{xy}\phi_{xx}}{375\cdot 3^{2/3}5^{1/3}\phi_x^9} - \frac{5509i\phi_y^4\phi_{xx}^2}{28125\cdot 3^{2/3}10^{1/3}\phi_x^{11}} \\
&- \frac{13i2^{2/3}\phi_t\phi_y^2\phi_{xx}^2}{375\cdot 3^{2/3}5^{1/3}\phi_x^{10}} - \frac{180v_6\phi_{xx}^2}{\phi_x^2} + \frac{199\phi_{xy}^2\phi_{xx}^3}{25\phi_x^{10}} - \frac{159\phi_{xyy}\phi_{xx}^3}{40\phi_x^9} + \frac{126\phi_{yy}\phi_{xx}^4}{25\phi_x^{10}} + \frac{63\phi_{xt}\phi_{xx}^4}{8\phi_x^{10}} \\
&- \frac{41\phi_y\phi_{xy}\phi_{xx}^4}{4\phi_x^{11}} - \frac{11\phi_y^2\phi_{xx}^5}{4\phi_x^{12}} - \frac{63\phi_t\phi_{xx}^5}{8\phi_x^{11}} + \frac{111645i3^{2/3}5^{1/3}\phi_{xx}^8}{64\cdot 2^{2/3}\phi_x^{13}} - 30(v_6)_{xx} - 360v_7\phi_{xx} \\
&- \frac{13i2^{2/3}\phi_y^2\phi_{xxt}}{1125\cdot 3^{2/3}5^{1/3}\phi_x^8} - \frac{7\phi_{xx}^3\phi_{xxt}}{2\phi_x^9} - \frac{26i2^{2/3}\phi_y^3\phi_{xxy}}{375\cdot 3^{2/3}5^{1/3}\phi_x^9} - \frac{26i2^{2/3}\phi_t\phi_y\phi_{xxy}}{1125\cdot 3^{2/3}5^{1/3}\phi_x^8} - \frac{701\phi_{xy}\phi_{xx}^2\phi_{xxy}}{100\phi_x^9} \\
&+ \frac{199\phi_y\phi_{xx}^3\phi_{xxy}}{25\phi_x^{10}} + \frac{26\phi_{xx}\phi_{xxy}^2}{25\phi_x^8} + \frac{27\phi_{xx}^2\phi_{xxyy}}{20\phi_x^8} + \frac{26i2^{2/3}\phi_y^4\phi_{3x}}{1125\cdot 3^{2/3}5^{1/3}\phi_x^{10}} + \frac{13i2^{2/3}\phi_t\phi_y^2\phi_{3x}}{1125\cdot 3^{2/3}5^{1/3}\phi_x^9}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{15v_6\phi_{3x}}{\phi_x}-\frac{109\phi_{xy}^2\phi_{xx}\phi_{3x}}{25\phi_x^9}+\frac{107\phi_{xyy}\phi_{xx}\phi_{3x}}{50\phi_x^8}-\frac{923\phi_{yy}\phi_{xx}^2\phi_{3x}}{200\phi_x^9}-\frac{7\phi_{xt}\phi_{xx}^2\phi_{3x}}{\phi_x^9}\\
&+\frac{274\phi_y\phi_{xy}\phi_{xx}^2\phi_{3x}}{25\phi_x^{10}}+\frac{26\phi_y^2\phi_{xx}^3\phi_{3x}}{25\phi_x^{11}}+\frac{21\phi_t\phi_{xx}^3\phi_{3x}}{2\phi_x^{10}}-\frac{10395i3^{2/3}5^{1/3}\phi_{xx}^6\phi_{3x}}{2\cdot 2^{2/3}\phi_x^{12}}\\
&+\frac{7\phi_{xx}\phi_{xxt}\phi_{3x}}{4\phi_x^8}+\frac{3\phi_{xy}\phi_{xxy}\phi_{3x}}{2\phi_x^8}-\frac{109\phi_y\phi_{xx}\phi_{xxy}\phi_{3x}}{25\phi_x^9}-\frac{3\phi_{xxyy}\phi_{3x}}{10\phi_x^7}+\frac{11\phi_{yy}\phi_{3x}^2}{25\phi_x^8}\\
&+\frac{7\phi_{xt}\phi_{3x}^2}{12\phi_x^8}-\frac{91\phi_y\phi_{xy}\phi_{3x}^2}{75\phi_x^9}+\frac{34\phi_y^2\phi_{xx}\phi_{3x}^2}{75\phi_x^{10}}-\frac{7\phi_t\phi_{xx}\phi_{3x}^2}{3\phi_x^9}+\frac{37065i3^{2/3}5^{1/3}\phi_{xx}^4\phi_{3x}^2}{8\cdot 2^{2/3}\phi_x^{11}}\\
&-\frac{7525i5^{1/3}\phi_{xx}^2\phi_{3x}^3}{2\cdot 3^{1/3}2^{2/3}\phi_x^{10}}+\frac{595i5^{1/3}\phi_{3x}^4}{4\cdot 3^{1/3}2^{2/3}\phi_x^9}+\frac{7\phi_{xx}^2\phi_{3xt}}{8\phi_x^8}-\frac{\phi_{3x}\phi_{3xt}}{6\phi_x^7}+\frac{104\phi_{xy}\phi_{xx}\phi_{3xy}}{75\phi_x^8}\\
&-\frac{701\phi_y\phi_{xx}^2\phi_{3xy}}{300\phi_x^9}-\frac{7\phi_{xxy}\phi_{3xy}}{30\phi_x^7}+\frac{\phi_y\phi_{3x}\phi_{3xy}}{2\phi_x^8}-\frac{6\phi_{xx}\phi_{3xyy}}{25\phi_x^7}+\frac{41\phi_{xy}^2\phi_{4x}}{150\phi_x^8}-\frac{29\phi_{xyy}\phi_{4x}}{200\phi_x^7}\\
&+\frac{13\phi_{yy}\phi_{xx}\phi_{4x}}{25\phi_x^8}+\frac{7\phi_{xt}\phi_{xx}\phi_{4x}}{8\phi_x^8}-\frac{367\phi_y\phi_{xy}\phi_{xx}\phi_{4x}}{300\phi_x^9}+\frac{3\phi_y^2\phi_{xx}^2\phi_{4x}}{100\phi_x^{10}}-\frac{7\phi_t\phi_{xx}^2\phi_{4x}}{4\phi_x^9}\\
&+\frac{9975i3^{2/3}5^{1/3}\phi_{xx}^5\phi_{4x}}{8\cdot 2^{2/3}\phi_x^{11}}-\frac{\phi_{xxt}\phi_{4x}}{8\phi_x^7}+\frac{41\phi_y\phi_{xxy}\phi_{4x}}{150\phi_x^8}-\frac{19\phi_y^2\phi_{3x}\phi_{4x}}{200\phi_x^9}+\frac{7\phi_t\phi_{3x}\phi_{4x}}{24\phi_x^8}\\
&-\frac{3325i3^{2/3}5^{1/3}\phi_{xx}^3\phi_{3x}\phi_{4x}}{2\cdot 2^{2/3}\phi_x^{10}}+\frac{4655i5^{1/3}\phi_{xx}\phi_{3x}^2\phi_{4x}}{4\cdot 3^{1/3}2^{2/3}\phi_x^9}+\frac{4015i3^{2/3}5^{1/3}\phi_{xx}^2\phi_{4x}^2}{32\cdot 2^{2/3}\phi_x^9}-\frac{\phi_{xx}\phi_{4xt}}{8\phi_x^7}\\
&-\frac{185i3^{2/3}5^{1/3}\phi_{3x}\phi_{4x}^2}{8\cdot 2^{2/3}\phi_x^8}-\frac{7\phi_{xy}\phi_{4xy}}{60\phi_x^7}+\frac{26\phi_y\phi_{xx}\phi_{4xy}}{75\phi_x^8}+\frac{\phi_{4x2y}}{50\phi_x^6}-\frac{29\phi_{yy}\phi_{5x}}{1000\phi_x^7}+\frac{7\phi_t\phi_{xx}\phi_{5x}}{40\phi_x^8}\\
&-\frac{\phi_{xt}\phi_{5x}}{20\phi_x^7}+\frac{28\phi_y\phi_{xy}\phi_{5x}}{375\phi_x^8}-\frac{107\phi_y^2\phi_{xx}\phi_{5x}}{3000\phi_x^9}-\frac{441i3^{2/3}5^{1/3}\phi_{xx}^4\phi_{5x}}{2\cdot 2^{2/3}\phi_x^{10}}+\frac{\phi_{5xt}}{120\phi_x^6}-\frac{7\phi_y\phi_{5xy}}{300\phi_x^7}\\
&+\frac{3233i3^{2/3}5^{1/3}\phi_{xx}^2\phi_{3x}\phi_{5x}}{16\cdot 2^{2/3}\phi_x^9}-\frac{149i3^{2/3}5^{1/3}\phi_{3x}^2\phi_{5x}}{8\cdot 2^{2/3}\phi_x^8}-\frac{187i3^{2/3}5^{1/3}\phi_{xx}\phi_{4x}\phi_{5x}}{8\cdot 2^{2/3}\phi_x^8}-\frac{\phi_t\phi_{6x}}{120\phi_x^7}\\
&+\frac{13i3^{2/3}\phi_{5x}^2}{4\cdot 10^{2/3}\phi_x^7}+\frac{\phi_y^2\phi_{6x}}{300\phi_x^8}+\frac{465i3^{2/3}5^{1/3}\phi_{xx}^3\phi_{6x}}{16\cdot 2^{2/3}\phi_x^9}-\frac{377i5^{1/3}\phi_{xx}\phi_{3x}\phi_{6x}}{8\cdot 3^{1/3}2^{2/3}\phi_x^8}+\frac{13i5^{1/3}\phi_{4x}\phi_{6x}}{4\cdot 3^{1/3}2^{2/3}\phi_x^7}\\
&-\frac{45i3^{2/3}5^{1/3}\phi_{xx}^2\phi_{7x}}{16\cdot 2^{2/3}\phi_x^8}+\frac{5i3^{2/3}5^{1/3}\phi_{3x}\phi_{7x}}{8\cdot 2^{2/3}\phi_x^7}+\frac{3i3^{2/3}5^{1/3}\phi_{xx}\phi_{8x}}{16\cdot 2^{2/3}\phi_x^7}-\frac{i5^{1/3}\phi_{9x}}{48\cdot 3^{1/3}2^{2/3}\phi_x^6},
\end{aligned}
\tag{A.26m}
$$

$$
\begin{aligned}
v_8=&-\frac{1009i\phi_{yy}^2}{1265625\cdot 3^{2/3}10^{1/3}\phi_x^9}-\frac{13i\phi_y\phi_{3y}}{20250\cdot 3^{2/3}10^{1/3}\phi_x^9}-\frac{13i\phi_{yyt}}{40500\cdot 3^{2/3}10^{1/3}\phi_x^8}-\frac{(v_7)_x}{\phi_x}+\frac{111\phi_{xy}^2\phi_{xx}^3}{500\phi_x^{12}}\\
&+\frac{13i\phi_{yy}\phi_{xt}}{40500\cdot 3^{2/3}10^{1/3}\phi_x^9}+\frac{12947i\phi_y\phi_{yy}\phi_{xy}}{2531250\cdot 3^{2/3}10^{1/3}\phi_x^{10}}+\frac{13i\phi_{yt}\phi_{xy}}{20250\cdot 3^{2/3}10^{1/3}\phi_x^9}-\frac{81\phi_{xyy}\phi_{xx}^3}{2000\phi_x^{11}}\\
&-\frac{13i\phi_y\phi_{xt}\phi_{xy}}{10125\cdot 3^{2/3}10^{1/3}\phi_x^{10}}-\frac{8911i\phi_y^2\phi_{xy}^2}{1265625\cdot 3^{2/3}10^{1/3}\phi_x^{11}}-\frac{13i\phi_t\phi_{xy}^2}{20250\cdot 3^{2/3}10^{1/3}\phi_x^{10}}+\frac{59\phi_{yy}\phi_{xx}^4}{1000\phi_x^{12}}\\
&+\frac{13i\phi_y\phi_{xyt}}{20250\cdot 3^{2/3}10^{1/3}\phi_x^9}+\frac{13i\phi_y^2\phi_{xyy}}{6750\cdot 3^{2/3}10^{1/3}\phi_x^{10}}+\frac{13i\phi_t\phi_{xyy}}{40500\cdot 3^{2/3}10^{1/3}\phi_x^9}-\frac{281\phi_y\phi_{xy}\phi_{xx}^4}{500\phi_x^{13}}\\
&-\frac{8911i\phi_y^2\phi_{yy}\phi_{xx}}{2531250\cdot 3^{2/3}10^{1/3}\phi_x^{11}}-\frac{13i\phi_y\phi_{yt}\phi_{xx}}{20250\cdot 3^{2/3}10^{1/3}\phi_x^{10}}-\frac{13i\phi_t\phi_{yy}\phi_{xx}}{40500\cdot 3^{2/3}10^{1/3}\phi_x^{10}}-\frac{3v_7\phi_{xx}}{\phi_x^2}\\
&-\frac{5(v_6)_x\phi_{xx}}{2\phi_x^3}+\frac{13i\phi_y^2\phi_{xt}\phi_{xx}}{13500\cdot 3^{2/3}10^{1/3}\phi_x^{11}}+\frac{22697i\phi_y^3\phi_{xy}\phi_{xx}}{2531250\cdot 3^{2/3}10^{1/3}\phi_x^{12}}+\frac{13i\phi_t\phi_y\phi_{xy}\phi_{xx}}{6750\cdot 3^{2/3}10^{1/3}\phi_x^{11}}\\
&-\frac{6893i\phi_y^4\phi_{xx}^2}{2531250\cdot 3^{2/3}10^{1/3}\phi_x^{13}}-\frac{13i\phi_t\phi_y^2\phi_{xx}^2}{13500\cdot 3^{2/3}10^{1/3}\phi_x^{12}}-\frac{5v_6\phi_{xx}^2}{2\phi_x^4}+\frac{281\phi_y^2\phi_{xx}^5}{1000\phi_x^{14}}-\frac{(v_6)_{xx}}{2\phi_x^2}
\end{aligned}
$$

$$+\frac{4677i3^{2/3}5^{1/3}\phi_{xx}^{8}}{128\cdot 2^{2/3}\phi_x^{15}}-\frac{13i\phi_y^2\phi_{xxt}}{40500\cdot 3^{2/3}10^{1/3}\phi_x^{10}}-\frac{13i\phi_y^3\phi_{xxy}}{6750\cdot 3^{2/3}10^{1/3}\phi_x^{11}}+\frac{17\phi_{xx}\phi_{xxy}^2}{1000\phi_x^{10}}$$
$$-\frac{13i\phi_t\phi_y\phi_{xxy}}{20250\cdot 3^{2/3}10^{1/3}\phi_x^{10}}-\frac{141\phi_{xy}\phi_{xx}^2\phi_{xxy}}{1000\phi_x^{11}}+\frac{111\phi_y\phi_{xx}^3\phi_{xxy}}{500\phi_x^{12}}+\frac{73\phi_{xx}^2\phi_{xxyy}}{6000\phi_x^{10}}$$
$$+\frac{13i\phi_y^4\phi_{3x}}{20250\cdot 3^{2/3}10^{1/3}\phi_x^{12}}+\frac{13i\phi_t\phi_y^2\phi_{3x}}{40500\cdot 3^{2/3}10^{1/3}\phi_x^{11}}-\frac{5v_6\phi_{3x}}{6\phi_x^3}-\frac{83\phi_{xy}^2\phi_{xx}\phi_{3x}}{750\phi_x^{11}}+\frac{3\phi_{xyy}\phi_{xx}\phi_{3x}}{125\phi_x^{10}}$$
$$-\frac{343\phi_{yy}\phi_{xx}^2\phi_{3x}}{6000\phi_x^{11}}+\frac{143\phi_y\phi_{xy}\phi_{xx}^2\phi_{3x}}{300\phi_x^{12}}-\frac{1753i3^{2/3}5^{1/3}\phi_{xx}^6\phi_{3x}}{16\cdot 2^{2/3}\phi_x^{14}}-\frac{i\phi_{9x}}{432\cdot 3^{1/3}10^{2/3}\phi_x^8}$$
$$+\frac{43\phi_{xy}\phi_{xxy}\phi_{3x}}{1500\phi_x^{10}}-\frac{83\phi_y\phi_{xx}\phi_{xxy}\phi_{3x}}{750\phi_x^{11}}-\frac{7\phi_{xxyy}\phi_{3x}}{2250\phi_x^9}+\frac{17\phi_{yy}\phi_{3x}^2}{3000\phi_x^{10}}+\frac{13i\phi_{4x}\phi_{6x}}{36\cdot 3^{1/3}10^{2/3}\phi_x^9}$$
$$+\frac{113\phi_y^2\phi_{xx}\phi_{3x}^2}{1500\phi_x^{12}}+\frac{4705i5^{1/3}\phi_{xx}^4\phi_{3x}^2}{16\cdot 3^{1/3}2^{2/3}\phi_x^{13}}-\frac{5713i5^{1/3}\phi_{xx}^2\phi_{3x}^3}{72\cdot 3^{1/3}2^{2/3}\phi_x^{12}}+\frac{667i5^{1/3}\phi_{3x}^4}{216\cdot 3^{1/3}2^{2/3}\phi_x^{11}}+\frac{\phi_y^2\phi_{6x}}{6000\phi_x^{10}}$$
$$+\frac{17\phi_{xy}\phi_{xx}\phi_{3xy}}{750\phi_x^{10}}-\frac{47\phi_y\phi_{xx}^2\phi_{3xy}}{1000\phi_x^{11}}-\frac{\phi_{xxy}\phi_{3xy}}{300\phi_x^9}+\frac{43\phi_y\phi_{3x}\phi_{3xy}}{4500\phi_x^{10}}-\frac{\phi_{xx}\phi_{3xyy}}{500\phi_x^9}+\frac{7\phi_{xy}^2\phi_{4x}}{1000\phi_x^{10}}$$
$$-\frac{11\phi_{xyy}\phi_{4x}}{6000\phi_x^9}+\frac{41\phi_{yy}\phi_{xx}\phi_{4x}}{6000\phi_x^{10}}-\frac{151\phi_y\phi_{xy}\phi_{xx}\phi_{4x}}{3000\phi_x^{11}}+\frac{73\phi_y^2\phi_{xx}^2\phi_{4x}}{1500\phi_x^{12}}+\frac{641i5^{1/3}\phi_{xx}^5\phi_{4x}}{8\cdot 3^{1/3}2^{2/3}\phi_x^{13}}$$
$$+\frac{7\phi_y\phi_{xxy}\phi_{4x}}{1000\phi_x^{10}}-\frac{149\phi_y^2\phi_{3x}\phi_{4x}}{18000\phi_x^{11}}-\frac{5153i5^{1/3}\phi_{xx}^3\phi_{3x}\phi_{4x}}{48\cdot 3^{1/3}2^{2/3}\phi_x^{12}}+\frac{199i5^{1/3}\phi_{xx}\phi_{3x}^2\phi_{4x}}{8\cdot 3^{1/3}2^{2/3}\phi_x^{11}}-\frac{\phi_y\phi_{5xy}}{3000\phi_x^9}$$
$$+\frac{133i5^{1/3}\phi_{xx}^2\phi_{4x}^2}{16\cdot 3^{1/3}2^{2/3}\phi_x^{11}}-\frac{217i5^{1/3}\phi_{3x}\phi_{4x}^2}{144\cdot 3^{1/3}2^{2/3}\phi_x^{10}}-\frac{\phi_{xy}\phi_{4xy}}{600\phi_x^9}+\frac{17\phi_y\phi_{xx}\phi_{4xy}}{3000\phi_x^{10}}+\frac{\phi_{4x2y}}{6000\phi_x^8}-\frac{11\phi_{yy}\phi_{5x}}{30000\phi_x^9}$$
$$+\frac{3\phi_y\phi_{xy}\phi_{5x}}{1250\phi_x^{10}}-\frac{121\phi_y^2\phi_{xx}\phi_{5x}}{30000\phi_x^{11}}-\frac{2279i\phi_{xx}^4\phi_{5x}}{32\cdot 3^{1/3}10^{2/3}\phi_x^{12}}+\frac{2093i\phi_{xx}^2\phi_{3x}\phi_{5x}}{32\cdot 3^{1/3}10^{2/3}\phi_x^{11}}-\frac{853i\phi_{3x}^2\phi_{5x}}{144\cdot 3^{1/3}10^{2/3}\phi_x^{10}}$$
$$-\frac{187i\phi_{xx}\phi_{4x}\phi_{5x}}{24\cdot 3^{1/3}10^{2/3}\phi_x^{10}}+\frac{13i\phi_{5x}^2}{60\cdot 3^{1/3}10^{2/3}\phi_x^9}+\frac{101i3^{2/3}\phi_{xx}^3\phi_{6x}}{32\cdot 10^{2/3}\phi_x^{11}}-\frac{733i\phi_{xx}\phi_{3x}\phi_{6x}}{144\cdot 3^{1/3}10^{2/3}\phi_x^{10}}$$
$$-\frac{89i\phi_{xx}^2\phi_{7x}}{96\cdot 3^{1/3}10^{2/3}\phi_x^{10}}+\frac{29i\phi_{3x}\phi_{7x}}{144\cdot 3^{1/3}10^{2/3}\phi_x^9}+\frac{i\phi_{xx}\phi_{8x}}{16\cdot 3^{1/3}10^{2/3}\phi_x^9}-\frac{\phi_y\phi_{xy}\phi_{3x}^2}{25\phi_x^{11}}-\frac{131\phi_y^2\phi_{xx}^3\phi_{3x}}{375\phi_x^{13}} \tag{A.26n}$$

The Painlevé-Bäcklund equations from substitution of (5.4.1) into (5.3.4) are

$$3^{2/3}v_1\phi_x^2 - 3i\phi_x\phi_{xx} = 0, \quad \text{(A.27a)}$$

$$\begin{aligned}&-2i\phi_y^2\phi_x - 2i\phi_t\phi_x^2 - 2iu_2\phi_x^3 - 5i3^{1/3}(v_1)^2\phi_x^3 - 2\cdot 3^{2/3}\phi_x^3(v_1)_x\\ &+10\cdot 3^{2/3}v_1\phi_x^2\phi_{xx} - 30i\phi_x\phi_{xx}^2 - 12i\phi_x^2\phi_{3x} = 0,\end{aligned} \quad \text{(A.27b)}$$

$$\begin{aligned}&3^{2/3}v_1\phi_t\phi_x^2 + i\phi_{yy}\phi_x^2 + 3^{2/3}u_2v_1\phi_x^3 + (v_1)^3\phi_x^3 + i\phi_x^3(u_2)_x + 3i3^{1/3}v_1\phi_x^3(v_1)_x + 3i\phi_x^2\phi_{xt}\\ &+4i\phi_y\phi_x\phi_{xy} + i\phi_y^2\phi_{xx} + 3iu_2\phi_x^2\phi_{xx} + 12i3^{1/3}(v_1)^2\phi_x^2\phi_{xx} + 9\cdot 3^{2/3}\phi_x^2(v_1)_x\phi_{xx}\\ &+6i\phi_{xx}^3 + 3\cdot 3^{2/3}\phi_x^3(v_1)_{xx} + 3^{2/3}v_1\phi_x^2\phi_{3x} + 18i\phi_x\phi_{xx}\phi_{3x} + 3i\phi_x^2\phi_{4x} = 0,\end{aligned} \quad \text{(A.27c)}$$

$$\begin{aligned}&4\cdot 3^{2/3}u_2v_1\phi_x\phi_{xx} + 6(v_1)^3\phi_x\phi_{xx} + 6i3^{1/3}v_1\phi_x(v_1)_x\phi_{xx} + 9i3^{1/3}(v_1)^2\phi_{xx}^2 + 12\cdot 3^{2/3}(v_1)_x\phi_{xx}^2\\ &-i3^{1/3}(v_1)^2\phi_t\phi_x - i3^{1/3}u_2(v_1)^2\phi_x^2 + 2\cdot 3^{2/3}(v_1)_t\phi_x^2 + 2\cdot 3^{2/3}v_1\phi_x^2(u_2)_x + 2\cdot 3^{2/3}u_2\phi_x^2(v_1)_x\\ &+6(v_1)^2\phi_x^2(v_1)_x - 6i3^{1/3}\phi_x^2(v_1)_x^2 + 4\cdot 3^{2/3}v_1\phi_x\phi_{xt} + 4i\phi_{xy}^2 + 4i\phi_x\phi_{xyy} + 2i\phi_{yy}\phi_{xx}\\ &-6i3^{1/3}v_1\phi_x^2(v_1)_{xx} + 12\cdot 3^{2/3}\phi_x\phi_{xx}(v_1)_{xx} + 4i\phi_y\phi_{xxy} + 8i3^{1/3}(v_1)^2\phi_x\phi_{3x}\\ &+12\cdot 3^{2/3}\phi_x(v_1)_x\phi_{3x} + 12\cdot 3^{2/3}v_1\phi_{xx}\phi_{3x} + 2\cdot 3^{2/3}\phi_x^2(v_1)_{3x} + 4\cdot 3^{2/3}v_1\phi_x\phi_{4x} = 0,\end{aligned} \quad \text{(A.27d)}$$

$$\begin{aligned}&-2i3^{1/3}v_1(v_1)_t\phi_x - i3^{1/3}(v_1)^2\phi_x(u_2)_x - 2i3^{1/3}u_2v_1\phi_x(v_1)_x - i3^{1/3}(v_1)^2\phi_{xt} - i3^{1/3}u_2(v_1)^2\phi_{xx}\\ &+6(v_1)^2(v_1)_x\phi_{xx} - 6i3^{1/3}(v_1)_x^2\phi_{xx} - 6i3^{1/3}\phi_x(v_1)_x(v_1)_{xx} - 6i3^{1/3}v_1\phi_{xx}(v_1)_{xx}\\ &+2i\phi_{xxyy} + 2(v_1)^3\phi_{3x} - 6i3^{1/3}v_1(v_1)_x\phi_{3x} - 2i3^{1/3}v_1\phi_x(v_1)_{3x} - i3^{1/3}(v_1)^2\phi_{4x} = 0,\end{aligned} \quad \text{(A.27e)}$$

and

$$3^{2/3}v_1\phi_x^2 - 3i\phi_x\phi_{xx} = 0, \quad \text{(A.27f)}$$

$$3^{1/3}(v_1)^2\phi_x + 2\phi_{3x} = 0. \quad \text{(A.27g)}$$

Substituting (5.5.1) into (5.3.4) and replacing χ-derivatives by (4.5.3), the coefficient equations are

$$\mathcal{O}(\chi^{-6}): \quad 12v_0 + u_0v_0 = 0, \quad \text{(A.28a)}$$

$$\mathcal{O}(\chi^{-5}): \quad 4u_1v_0^2 + 72v_0v_1 + 12u_0v_0v_1 - v_0^2(u_0)_x - 108v_0(v_0)_x - 3u_0v_0(v_0)_x = 0, \quad \text{(A.28b)}$$

$$\begin{aligned}\mathcal{O}(\chi^{-4}): \quad &12iK^2u_0 + 6Cv_0^3 - 114Sv_0^3 - 5Su_0v_0^3 - 6u_2v_0^3 - 18u_1v_0^2v_1 - 36v_0v_1^2\\ &- 18u_0v_0v_1^2 + 6v_0^2v_1(u_0)_x + 2v_0^3(u_1)_x + 6u_1v_0^2(v_0)_x + 216v_0v_1(v_0)_x\\ &+ 12u_0v_0v_1(v_0)_x - 108v_0(v_0)_x^2 + 108v_0^2(v_1)_x + 6u_0v_0^2(v_1)_x - 54v_0^2(v_0)_{xx} = 0,\end{aligned} \quad \text{(A.28c)}$$

$$\begin{aligned}\mathcal{O}(\chi^{-3}): \quad &2iK^2u_1 + 6Cv_0^2v_1 + 3v_0^2(v_0)_t + 2iu_0K_y + 4iK(u_0)_y - 10iKu_0K_x + 12v_0^3S_x\\ &+ 3v_0v_1^2(u_0)_x + 3v_0^2v_1(u_1)_x + v_0^3(u_2)_x + 3u_2v_0^2(v_0)_x + 6u_1v_0v_1(v_0)_x + 18v_1^2(v_0)_x\\ &+ 3u_0v_1^2(v_0)_x + 6(v_0)_x^3 + 3u_1v_0^2(v_1)_x + 36v_0v_1(v_1)_x + 6u_0v_0v_1(v_1)_x + 18v_0(v_0)_x(v_0)_{xx}\\ &+ 3v_0^2(v_0)_{3x} - 2Su_1v_0^3 - 60Sv_0^2v_1 - 6Su_0v_0^2v_1 - 6u_2v_0^2v_1 - 6u_1v_0v_1^2 - 2u_0v_1^3 - 3v_0^3C_x\\ &+ 81Sv_0^2(v_0)_x - 36v_1(v_0)_x^2 - 72v_0(v_0)_x(v_1)_x - 36v_0v_1(v_0)_{xx} - 18v_0^2(v_1)_{xx} = 0,\end{aligned} \quad \text{(A.28d)}$$

$$
\begin{aligned}
\mathcal{O}(\chi^{-2}):\quad & 8iK^2Su_0 + 3CSv_0^3 + 6Cv_0v_1^2 + 12v_0v_1(v_0)_t + 6v_0^2(v_1)_t + 2iu_1K_y + 4iK(u_1)_y + 2i(u_0)_{yy} \\
& - 6iKu_1K_x - 8i(u_0)_yK_x + 8iu_0K_x^2 + 30v_0^2v_1S_x + 2v_1^3(u_0)_x + 6v_0v_1^2(u_1)_x \\
& + 144Sv_0v_1(v_0)_x + 12u_2v_0v_1(v_0)_x + 6u_1v_1^2(v_0)_x + 72Sv_0^2(v_1)_x + 6u_2v_0^2(v_1)_x \\
& + 12u_1v_0v_1(v_1)_x + 6u_0v_1^2(v_1)_x + 36(v_0)_x^2(v_1)_x - 4iu_0K_{xy} + 3v_0^3C_{xx} + 8iKu_0K_{xx} \\
& + 36v_1(v_0)_x(v_0)_{xx} + 36v_0(v_1)_x(v_0)_{xx} + 36v_0(v_0)_x(v_1)_{xx} + 12v_0v_1(v_0)_{3x} + 6v_0^2(v_1)_{3x} \\
& - 30S^2v_0^3 - 3Su_2v_0^3 - 9Su_1v_0^2v_1 - 24Sv_0v_1^2 - 9Su_0v_0v_1^2 - 6u_2v_0v_1^2 + 2u_1v_1^3 \\
& - 12v_0^2v_1C_x - 27v_0^2S_x(v_0)_x - 54Sv_0(v_0)_x^2 - 72v_1(v_0)_x(v_1)_x - 36v_0(v_1)_x^2 - 3v_0^3S_{xx} \\
& - 27Sv_0^2(v_0)_{xx} - 18v_1^2(v_0)_{xx} - 36v_0v_1(v_1)_{xx} + 6v_0^2v_1(u_2)_x = 0, \qquad \text{(A.28e)} \\
\mathcal{O}(\chi^{-1}):\quad & iK^2Su_1 + 3CSv_0^2v_1 + 3v_1^2(v_0)_t + 6v_0v_1(v_1)_t + iSu_0K_y + iKu_0S_y + 2iKS(u_0)_y \\
& + i(u_1)_{yy} - 3iKSu_0K_x - 2i(u_1)_yK_x + iu_1K_x^2 + \frac{9}{2}Sv_0^3S_x + 3v_0v_1^2S_x + v_1^3(u_1)_x \\
& + 3v_0v_1^2(u_2)_x + \frac{27}{2}S^2v_0^2(v_0)_x + 9Sv_1^2(v_0)_x + 3u_2v_1^2(v_0)_x + 18Sv_0v_1(v_1)_x \\
& + 6u_2v_0v_1(v_1)_x + 3u_1v_1^2(v_1)_x + 18(v_0)_x(v_1)_x^2 - iu_1K_{xy} + 3v_0^2v_1C_{xx} + iKu_1K_{xx} \\
& + 2i(u_0)_yK_{xx} - 3iu_0K_xK_{xx} + 18v_1(v_1)_x(v_0)_{xx} + 18v_1(v_0)_x(v_1)_{xx} \\
& + iu_0K_{xxy} + 3v_1^2(v_0)_{3x} + 6v_0v_1(v_1)_{3x} - 12S^2v_0^2v_1 - 3Su_2v_0^2v_1 - 3Su_1v_0v_1^2 \\
& - 3v_0v_1^2C_x - 18v_0v_1S_x(v_0)_x - 18Sv_1(v_0)_x^2 - 9v_0^2S_x(v_1)_x - 36Sv_0(v_0)_x(v_1)_x \\
& - 3v_0^2v_1S_{xx} - 18Sv_0v_1(v_0)_{xx} - 9Sv_0^2(v_1)_{xx} + 18v_0(v_1)_x(v_1)_{xx} - Su_0v_1^3 = 0, \qquad \text{(A.28f)} \\
\mathcal{O}(\chi^{0}):\quad & 2iK^2S^2u_0 + 6CSv_0v_1^2 - 6S^2v_0v_1^2 - 6Su_2v_0v_1^2 - 2Su_1v_1^3 + 12v_1^2(v_1)_t + 2iSu_1K_y \\
& + 2iKu_1S_y + 4iKS(u_1)_y + 4i(u_2)_{yy} - 2iKSu_1K_x + 18Sv_0^2v_1S_x + 4v_1^3(u_2)_x \\
& + 36S^2v_0v_1(v_0)_x - 18v_1^2S_x(v_0)_x + 18S^2v_0^2(v_1)_x + 12u_2v_1^2(v_1)_x - 36v_0v_1S_x(v_1)_x \\
& - 72Sv_1(v_0)_x(v_1)_x - 36Sv_0(v_1)_x^2 + 24(v_1)_x^3 + 6v_0v_1^2C_{xx} + 4iKSu_0K_{xx} \\
& - 2iu_1K_xK_{xx} + 2iu_0K_{xx}^2 - 6v_0v_1^2S_{xx} - 18Sv_1^2(v_0)_{xx} - 36Sv_0v_1(v_1)_{xx} \\
& + 72v_1(v_1)_x(v_1)_{xx} + 2iu_1K_{xxy} + 12v_1^2(v_1)_{3x} - 3S^3v_0^3 + 4i(u_1)_yK_{xx} = 0, \qquad \text{(A.28g)}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{O}(\chi^{-3}):\quad & 2iu_0 + v_0^3 = 0, \qquad \text{(A.29a)} \\
\mathcal{O}(\chi^{-2}):\quad & iu_1 + 3v_0^2v_1 - i(u_0)_x = 0, \qquad \text{(A.29b)} \\
\mathcal{O}(\chi^{-1}):\quad & iSu_0 + 3v_0v_1^2 - i(u_1)_x = 0, \qquad \text{(A.29c)} \\
\mathcal{O}(\chi^{0}):\quad & iSu_1 + 2v_1^3 - 2i(u_2)_x = 0. \qquad \text{(A.29d)}
\end{aligned}
$$

REFERENCES

[1] A. Ramani, B. Grammaticos and T. Bountis, The Painlevé property and singularity sanalysis of integrable and non-integrable systems, *Phys. Rep.* **180** (1989), 160.

[2] M.J. Ablowitz, A. Ramani and H. Segur, A connection between nonlinear evolution equations and ODEs of P-type: I and II, *J. Math. Phys.* **21** (1980), 715, 1006.

[3] J. Weiss, M. Tabor and G. Carnevale, The Painlevé property for partial differential equations, *J. Math. Phys.* **24** (1983), 522.

[4] J. Hietarinta, Direct methods for the search of the second invariant, *Phys. Rep.* **147** (1987), 87

[5] M.J. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Transform*, SIAM, Philadelphia, 1981; R.K. Dodd, J.C. Eilbec, J.D. Gibbon and H.C. Morris, *Solitons and Nonlinear Wave Equations*, Academic Press, New York, 1982.

[6] M. Tabor, *Chaos and integrability in nonlinear dynamics: an introduction*, Wiley, New York, 1989

[7] E.L. Ince, *Ordinary Differential Equations*, Dover, New York, 1956

[8] J. Weiss, The Painlevé property for partial differential equations. II, *J. Math. Phys.* **24** (1983), 1405; **25** (1984), 13, 2226; **26** (1985), 258, 2174; **27** (1986), 1293, 2647; **28** (1987), 2025;

[9] N.A. Kudryashov, Exact solution of the generalized KS equation, *Phys. Lett. A* **147** (1990), 287

[10] R. Conte and M. Musette, Painlevé analysis and BT in the Duramoto-Sivashinsky equation, *J. Phys. A* **22** (1989), 169.

[11] F. Cariello and M. Tabor, Painlevé expansions for nonintegrable evolution equations, *Physica D* **39** (1989), 77

[12] S. Roy Choudhury, Painlevé analysis and special solutions of two families of reaction-diffusion equations, *Phys. Lett. A* **159** (1991), 311.

[13] B. Yu Guo and Z. Xiong, Analytic solutions of the Fisher equation, *J. Phys. A* **24** (1991), 645

[14] S. Roy Choudhury, BTs, truncated Painlevé expansions and special solutions of nonintegrable long-wave evolution equations, *Can. J. Phys.* **70** (1992), 595; Painlevé analysis and partial integrability of a class of reaction-diffusion equations, *Nonlin. Anal: Theory, Meth. & Appl.* **18** (1992), 445;

[15] A.C. Newell, M. Tabor and Y.B. Zeng, A unified approach to Painlevé expansions, *Physica D* **29** (1987), 1.

[16] H. Flaschka, A.C. Newell and M. Tabor, Monodromy- and spectrum-preserving deformations, in *What is Integrability*, V.E. Zakharov (Ed.), Springer, Berlin, 1991.

[17] E. Hille, *ODEs in the Complex Domain*, Wiley, New York, 1976.

[18] R. Conte, Invariant Painlevé analysis of PDEs, *Phys. Lett. A* **140** (1989), 383.

[19] M. Musette and R. Conte, Algorithmic method for deriving Lax pairs from the invariant Painlevé analysis of NLPDEs, *J. Math. Phys.* **32** (1991), 1450.

[20] M.Musette and R. Conte, The two-singular-manifold method: I. MKdV and sine-Gordon equations, *J. Phys. A: Math. Gen* **27** (1994), 3895.

[21] S. Roy Choudhury, Invariant Painlevé analysis and coherent structures of two families of reaction-diffusion equations, *J. Math. Phys.* **40** (1999), 3643.

[22] S. Roy Choudhury, One and 2D coherent structures of the Zakharov-Kuznetsov equations, *Problems of Nonlin. Anal.* **6** (2000), 1.

[23] N. Isldore and W. Malfliet, New special solutions of the 'Brusselator' reaction model, *J. Phys. A: Math. Gen* **30** (1997), 5151.

[24] R. Conte and M. Musette, Linearity inside nonlinearity: Exact solutions of the 1D Quintic CGL equation, *Physica D* **69** (1993), 1.

[25] P.G. Estevez, E. Conde and P.R. Gordoa, Unified approach to Miura, Bäcklund and Darboux transformations for NLPDEs, *J. Math. Phys.* **5** (1998), 82.

[26] P.G. Estevez and P.R. Gordoa, Darboux transformations via Painleé analysis, *Inverse Problems* **13** (1997), 939.

[27] M. Wadati, H. Sanuki, and K. Konno, Relationships among the inverse method, BTs, and an infinite number of conservation laws, *Prog. Theor. Phys.* **53** (1975), 419.

[28] A. Fring, $\mathcal{PT}$-symmetric deformations of the Korteweg-de-Vries equation, *J. Phys. A: Math. Theor.* **40** (2007) 4215-4224; C.M. Bender, D.C. Brody, J. Chen and E. Furlan, $\mathcal{PT}$-symmetric Extension of the Korteweg-de Vries Equation, arXiv:math-ph/0610003v1, 2 Oct 2006.

[29] Z. Yan, Complex $\mathcal{PT}$-symmetric extensions of the non-$\mathcal{PT}$-symmetric Burgers equation, *Physica Scripta.* **77** (2008) 025006.

[30] C.M. Bender, Making Sense of non-Hermitian Hamiltonians, *Reports on Progress in Physics* **70** (2007) 947 - 1018.

[31] F. En-Gui, Z. Hong-Qing, and L. Gang, Bäcklund Transformation, Lax Pairs, Symmetries and Exact Solutions for Variable Coefficient KdV Equation, *Acta Phys. Sin. (Overseas Edn.)* **7** (1998) 649.

[32] S. Roy Choudhury, A Unified Approach to Integrable Systems via Painlevé Analysis,*Contemporary Mathematics* **301** (2002), 139. pg 322-357.

[33] M. Wang, Exact solutions for a compound KdV-Burgers equation, *Phys. Lett. A* **213** (1996) 279-287.

[34] S. Roy Choudhury, Invariant Painlevé Analysis and Coherent Structures of Long-wave Equations, *Physica Scripta.* **62** (2000), 156.

[35] R. Conte, Exact Solutions of NLPDEs by Singularity Analysis, *Lec. Notes in Physics* **632** (2003), 1-83.

[36] U. Tanriver and S. Roy Choudhury, One and Two Dimensional Coherent Structures of the Zakharov-Kuznetsov Equation via Invariant Painlevé Analysis, *Problems of Nonlinear Analysis of Engineering Systems* **6** (2000).

[37] H. Stephani, *Differential Equations - Their Solutions Using Symmetries*, ed M. MacCullum, Cambridge Univ. Press, Cambridge, 1989.

[38] P.E. Hydon, *Symmetry Methods for Differential Equations*, Cambridge Univ. Press, Cambridge, 1999.

[39] M.E. Fels, The inverse problem of the calculus of variations for scalar fourth-order ordinary differential equations, *Trans. Amer. Math. Soc.* **348** (1996), 5007-5029.

[40] M.C. Nucci and A.M. Arthurs, On the inverse problem of calculus for fourth-order equations, *Proc. R. Soc. A* **466** (2010), doi: 10.1098/rspa.2009.0618