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## Modified Pal Interpolation And Sampling Bilevel Signals With Finite Rate Of Innovation

Gayatri Ramesh  
*University of Central Florida*



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# **MODIFIED PÁL INTERPOLATION AND SAMPLING BILEVEL SIGNALS WITH FINITE RATE OF INNOVATION**

GAYATRI RAMESH

B.S., University of Tennessee at Martin, 2006

M.S., University of Central Florida, 2008

A dissertation submitted in partial fulfillment of the requirements  
for the degree of Doctor of Philosophy  
in the Department of Mathematics  
in the College of Sciences  
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Major Professors: Qiyu Sun and Ram Mohapatra

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## **ABSTRACT**

Sampling and interpolation are two important topics in signal processing. Signal processing is a vast field of study that deals with analysis and operations of signals such as sounds, images, sensor data, telecommunications and so on. It also utilizes many mathematical theories such as approximation theory, analysis and wavelets. This dissertation is divided into two chapters: Modified Pál Interpolation and Sampling Bilevel Signals with Finite Rate of Innovation. In the first chapter, we introduce a new interpolation process, the modified Pál interpolation, based on papers by Pál, József and Szabó, and we establish the existence and uniqueness of interpolation polynomials of modified Pál type.

The paradigm to recover signals with finite rate of innovation from their samples is a fairly recent field of study. In the second chapter, we show that causal bilevel signals with finite rate of innovation can be stably recovered from their samples provided that the sampling period is at or above the maximal local rate of innovation, and that the sampling kernel is causal and positive on the first sampling period. Numerical simulations are presented to discuss the recovery of bilevel causal signals in the presence of noise.

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## CHAPTER 1: Modified Pál Interpolation

A purpose of interpolation is to approximate a function (signal) from its limited information, such as sampling data or a discrete set. Interpolation has shown to be important in signal processing, approximation theory, computer aided design etc. There are many classical methods available such as polynomial interpolation, trigonometric interpolation, fractal interpolation, spline interpolation and wavelet interpolation.

In the first section of this chapter, we recall the interpolation of Pál type. In the second section, we introduce the modified Pál interpolation. In third and fourth sections, we modify Szabó and Joó's generalized Pál interpolation.

### 1.1 Pál interpolation

In this section, we recall Hermite-Fejer interpolation discussed in [14], which is now known as Pál interpolation.

Let  $X := \{x_1, \dots, x_n\}$  contain  $n$  distinct nodes  $x_1 < x_2 < \dots < x_n$  on the real line, and define

$$\omega_X(x) = \prod_{k=1}^n (x - x_k). \quad (1.1.1)$$

Between two neighbouring roots of  $\omega_X(x)$  there is one and only one root for its derivative  $\omega'_X(x)$ .

In other words, the roots  $x_1, \dots, x_n$  of the polynomial  $\omega_X(x)$  and the roots  $x_1^*, \dots, x_{n-1}^*$  of its derivative

$$\omega'_X(x) = n \prod_{l=1}^{n-1} (x - x_l^*) \quad (1.1.2)$$

has the following interlacing property:

$$x_1 < x_1^* < x_2 < x_2^* < \dots < x_{n-1}^* < x_n.$$

In [14], Pál considered the following interpolation problem: Find a polynomial  $P(x)$  of lowest degree such that

$$P(x_k) = y_k \text{ for all } 1 \leq k \leq n \text{ and } P'(x_l^*) = y_l^* \text{ for all } 1 \leq l \leq n-1 \quad (1.1.3)$$

for any given interpolation data  $\{y_k\}_{k=1}^n$  and  $\{y_l^*\}_{l=1}^{n-1}$ .

**Theorem 1.1.1.** ([14]) *Let  $X := \{x_1, \dots, x_n\}$  contain  $n$  distinct nodes on the real line, and  $X^* := \{x_1^*, \dots, x_{n-1}^*\}$  be the set of roots of the polynomial  $\omega'_X(x)$ , the derivative of the polynomial  $\omega_X(x) = \prod_{k=1}^n (x - x_k)$ . Then given any interpolation data  $\{y_k\}_{k=1}^n$  and  $\{y_l^*\}_{l=1}^{n-1}$  there exists a polynomial  $P(x)$  of degree  $2n - 1$  that satisfies (1.1.3). Moreover,*

$$\begin{aligned} P(x) = & - \sum_{k=1}^n y_k \frac{\omega_X(x)}{(\omega'_X(x_k))^2 \prod_{i \neq k} (x_k - x_i)} \int \frac{\omega'_X(x)}{(x - x_k)^2} (\omega'_X(x_k) - \omega''_X(x_k)(x - x_k)) dx \\ & + \sum_{l=1}^{n-1} y_l^* \frac{\omega_X(x)}{\omega_X(x_l^*)} \int \frac{\prod_{j \neq l} (x - x_j^*)}{\prod_{j \neq l} (x_l^* - x_j^*)} dx. \end{aligned} \quad (1.1.4)$$

Observe that for any polynomial  $P$  satisfying (1.1.3),  $P(x) + C\omega_X(x)$  has the same interpolation property (1.1.3) for any constants  $C$ . The uniqueness of polynomials satisfying (1.1.3) is

established when an additional interpolation condition is imposed.

**Theorem 1.1.2.** ([14]) *Let  $\{x_k\}_{k=1}^n$ ,  $\{x_l^*\}_{l=1}^{n-1}$ ,  $\{y_k\}_{k=1}^n$  and  $\{y_l^*\}_{l=1}^{n-1}$  be as in Theorem 1.1.1, and let  $a \neq x_k$  for all  $k = 1, 2, \dots, n$ . Then the polynomial  $R(x)$  defined by*

$$\begin{aligned} R(x) = & - \sum_{k=1}^n y_k \frac{\omega_X(x)}{(\omega'_X(x_k))^2 \prod_{i \neq k} (x_k - x_i)} \int_a^x \frac{\omega'_X(t)}{(t - x_k)^2} (\omega'_X(x_k) - \omega''_X(x_k)(t - x_k)) dt \\ & + \sum_{l=1}^{n-1} y_l^* \frac{\omega_X(x)}{\omega_X(x_l^*)} \int_a^x \frac{\prod_{j \neq l} (t - x_j^*)}{\prod_{j \neq l} (x_l^* - x_j^*)} dt \quad \text{for } x \in (a - \delta, a + \delta), \end{aligned} \quad (1.1.5)$$

is the unique polynomial of degree at most  $2n - 1$  that satisfies (1.1.3) and  $R(a) = 0$ , where  $\delta = \min_{1 \leq k \leq n} |x_k - a|$ .

## 1.2 Modified Pál interpolation I

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In this section, we consider finding a polynomial  $P(x)$  of lowest degree for any given interpolation data  $\{y_k\}_{k=1}^n$  and  $\{y_l^*\}_{l=1}^{n-1}$  such that

$$P(x_l^*) = y_l^* \text{ for all } 1 \leq l \leq n - 1, \text{ and } \int_{x_k}^{x_{k+1}} P(x) dx = y_{k+1} \text{ for all } 1 \leq k \leq n - 1.$$

**Theorem 1.2.1.** *Let  $X := \{x_1, \dots, x_n\}$  contain  $n$  distinct nodes on the real line ordered by  $x_1 < x_2 < \dots < x_n$ , and let  $X^* = \{x_l^*\}_{l=1}^{n-1}$  denote the set of the roots of the polynomial  $\omega'_X$  (the derivative of the polynomial  $\omega_X$  in (1.1.1)). Given interpolation data  $\{y_k\}_{k=2}^n$  and  $\{y_l^*\}_{l=1}^{n-1}$ , the*

polynomial  $P(x)$  of degree  $2n - 2$  defined by

$$P(x) := \frac{d}{dx} \left[ - \sum_{k=2}^n z_k \frac{\omega_X(x)}{(\omega'_X(x_k))^2 \prod_{i \neq k} (x_k - x_i)} \int \frac{\omega'_X(x)}{(x - x_k)^2} (\omega'_X(x_k) - \omega''_X(x_k)(x - x_k)) dx \right. \\ \left. + \sum_{l=1}^{n-1} y_l^* \frac{\omega_X(x)}{\omega_X(x_l^*)} \int \frac{\prod_{j \neq l} (x - x_j^*)}{\prod_{j \neq l} (x_l^* - x_j^*)} dx \right], \quad (1.2.1)$$

satisfies

$$P(x_l^*) = y_l^*, \quad 1 \leq l \leq n - 1, \quad (1.2.2)$$

and

$$\int_{x_k}^{x_{k+1}} P(x) dx = y_{k+1}, \quad 1 \leq k \leq n - 1, \quad (1.2.3)$$

where  $z_k = \sum_{q=2}^k y_q$ ,  $2 \leq k \leq n$ .

*Proof.* Let us show that a polynomial  $P$  of the following form

$$P(x) = \frac{d}{dx} \left[ \sum_{k=2}^n z_k A_k(x) + \sum_{l=1}^{n-1} y_l^* B_l(x) \right] \quad (1.2.4)$$

has the interpolation properties (1.2.2) and (1.2.3), where polynomials  $\{A_k(x)\}_{k=2}^n$  and  $\{B_l(x)\}_{l=1}^{n-1}$  of degree at most  $2n - 1$  satisfy

$$\begin{cases} \text{(a)} & A_k(x_i) = \delta_{ki} \text{ for all } 2 \leq k \leq n \text{ and } 1 \leq i \leq n \\ \text{(b)} & A'_k(x_j^*) = 0 \text{ for all } 2 \leq k \leq n \text{ and } 1 \leq j \leq n - 1, \end{cases} \quad (1.2.5)$$

and

$$\begin{cases} \text{(c)} & B_l(x_i) = 0 \text{ for all } 1 \leq l \leq n-1 \text{ and } 1 \leq i \leq n \\ \text{(d)} & B'_l(x_j^*) = \delta_{lj} \text{ for all } 1 \leq l \leq n-1 \text{ and } 1 \leq j \leq n-1. \end{cases} \quad (1.2.6)$$

Here  $\delta_{ij}$  stands for the Kronecker symbol defined by  $\delta_{ij} = 1$  if  $i = j$  and  $\delta_{ij} = 0$  otherwise.

Set  $z_1 = 0$ . By (1.2.4)–(1.2.6),

$$\int_{x_i}^{x_{i+1}} P(x) dx = \left( \sum_{k=2}^n z_k A_k(x) + \sum_{l=1}^{n-1} y_l^* B_l(x) \right) \Big|_{x_i}^{x_{i+1}} = z_{i+1} - z_i = y_{i+1}$$

for all  $1 \leq i \leq n-1$ , and

$$P(x_j^*) = \sum_{k=2}^n z_k A'_k(x_j^*) + \sum_{l=1}^{n-1} y_l^* B'_l(x_j^*) = y_j^*$$

for all  $1 \leq j \leq n-1$ . This proves that a polynomial  $P$  of the form (1.2.4) satisfies the interpolation requirements (1.2.2) and (1.2.3).

Now it remains to construct polynomials  $A_k(x)$ ,  $2 \leq k \leq n$ , and  $B_l(x)$ ,  $1 \leq l \leq n-1$ , of degree at most  $2n-1$  satisfying (1.2.5) and (1.2.6) respectively. First we construct polynomials  $B_l(x)$ ,  $1 \leq l \leq n-1$ , of degree at most  $2n-1$  satisfying (1.2.6). Take  $1 \leq l \leq n-1$ . From the requirement (c) in (1.2.6),

$$B_l(x) = \omega_X(x) V_l(x) \quad (1.2.7)$$

for some polynomial  $V_l(x)$  of degree at most  $n-1$ . Consequently,

$$B'_l(x) = \omega'_X(x) V_l(x) + \omega_X(x) V'_l(x) = \frac{\omega'_X(x)}{(x-x_l^*)} W_l(x) \quad (1.2.8)$$

for some polynomial  $W_l(x)$  of degree at most  $n$ , where the last equality follows from the requirement (d) in (1.2.6). Multiplying  $x - x_l^*$  at both sides of the above equation leads to

$$[\omega'_X(x)V_l(x) + \omega_X(x)V'_l(x)](x - x_l^*) = \omega'_X(x)W_l(x).$$

Rearranging above equation yields

$$\omega'_X(x)((x - x_l^*)V_l(x) - W_l(x)) = -(x - x_l^*)\omega_X(x)V'_l(x). \quad (1.2.9)$$

Recall that  $\omega_X$  and its derivative  $\omega'_X$  do not have common roots. Then it follows from (1.2.9) that

$$(x - x_l^*)V'_l(x) = \omega'_X(x)M_l(x) \quad (1.2.10)$$

for some polynomial  $M_l(x)$ . Comparing the degree of both sides of the equation (1.2.10) shows that  $M_l(x)$  has degree zero, i.e.  $M_l(x) = M$  for some constant  $M$ .

Evaluating (1.2.8) at  $x = x_l^*$  and recalling the requirement (d) in (1.2.6) gives

$$1 = \omega'_X(x_l^*)V_l(x_l^*) + \omega_X(x_l^*)V'_l(x_l^*) = \omega_X(x_l^*)V'_l(x_l^*), \quad (1.2.11)$$

and hence

$$V'_l(x_l^*) = (\omega_X(x_l^*))^{-1}.$$

Substituting this in (1.2.10) and recalling that  $M_l$  is a constant function, we obtain

$$V'_l(x) = \frac{1}{\omega_X(x_l^*)} \frac{\prod_{j \neq l} (x - x_j^*)}{\prod_{j \neq l} (x_l^* - x_j^*)}. \quad (1.2.12)$$

Therefore

$$V_l(x) = \int \frac{1}{\omega_X(x_l^*)} \frac{\prod_{j \neq l}(x - x_j^*)}{\prod_{j \neq l}(x_l^* - x_j^*)} dx. \quad (1.2.13)$$

Substituting the above expression about  $V_l(x)$  into (1.2.7) yields

$$B_l(x) = \frac{\omega_X(x)}{\omega_X(x_l^*)} \int \frac{\prod_{j \neq l}(x - x_j^*)}{\prod_{j \neq l}(x_l^* - x_j^*)} dx, \quad 1 \leq l \leq n - 1.$$

The polynomials  $B_l, 1 \leq l \leq n - 1$ , just defined have degree at most  $2n - 1$ . It satisfies the requirement (c) in (1.2.6), and also the requirement (d) in (1.2.6) since

$$B_l'(x) = \frac{\omega_X'(x)}{\omega_X(x_l^*)} \int \frac{\prod_{j \neq l}(x - x_j^*)}{\prod_{j \neq l}(x_l^* - x_j^*)} dx + \frac{\omega_X(x)}{\omega_X(x_l^*)} \frac{\prod_{j \neq l}(x - x_j^*)}{\prod_{j \neq l}(x_l^* - x_j^*)}$$

and hence

$$B_l'(x_{j'}^*) = \frac{\omega_X(x_{j'}^*)}{\omega_X(x_l^*)} \frac{\prod_{j \neq l}(x_{j'}^* - x_j^*)}{\prod_{j \neq l}(x_l^* - x_j^*)} = \begin{cases} 1 & \text{if } j' = l \\ 0 & \text{if } j' \neq l. \end{cases}$$

This completes the construction of polynomials  $B_l, 1 \leq l \leq n - 1$ , of degree at most  $2n - 1$  satisfying (1.2.6).

Now we start to find  $A_k, 2 \leq k \leq n$ , of degree at most  $2n - 1$  that satisfies (1.2.5). From the requirement (a) in (1.2.5) it follows that

$$A_k(x) = \frac{\omega_X(x)}{x - x_k} S_k(x), \quad 2 \leq k \leq n \quad (1.2.14)$$

for some polynomial  $S_k(x)$  of degree at most  $n$  that satisfies

$$S_k(x_k) \neq 0. \quad (1.2.15)$$

Taking derivative of both sides of (1.2.14) and applying the requirement (b) in (1.2.5), we have

$$A'_k(x) = \left( \frac{\omega'_X(x)}{(x-x_k)} - \frac{\omega_X(x)}{(x-x_k)^2} \right) S_k(x) + \frac{\omega_X(x)}{(x-x_k)} S'_k(x) = \omega'_X(x) T_k(x)$$

for some polynomial  $T_k(x)$  of degree at most  $n-1$ . Thus

$$\omega'_X(x)(x-x_k)(S_k(x) - T_k(x)(x-x_k)) = \omega_X(x)(S_k(x) - (x-x_k)S'_k(x)). \quad (1.2.16)$$

Again, recall that  $\omega_X(x)$  and  $\omega'_X(x)$  do not share any roots. Then

$$S_k(x) - (x-x_k)S'_k(x) = \omega'_X(x)U_k(x) \quad (1.2.17)$$

and

$$S_k(x) - (x-x_k)T_k(x) = \frac{\omega_X(x)}{x-x_k}U_k(x) \quad (1.2.18)$$

for some polynomial  $U_k(x)$  of degree at most one. Substituting  $x$  by  $x_k$  in (1.2.18) and recalling that  $A_k(x_k) = 1$  by the requirement (a) in (1.2.5), we obtain

$$U_k(x_k) = \frac{1}{\omega'_X(x_k) \prod_{i \neq k} (x_k - x_i)}. \quad (1.2.19)$$

Taking derivative of both sides of (1.2.17) yields

$$(\omega'_X(x)U_k(x))' = -(x-x_k)S''_k(x),$$



which implies that

$$\omega_X''(x_k)U_k(x_k) + \omega_X'(x_k)U_k'(x_k) = 0. \quad (1.2.20)$$

Thus  $\omega_X'U_k$  has the following Taylor expansion at  $x = x_k$ :

$$\omega_X'(x)U_k(x) = \omega_X'(x_k)U_k(x_k) + c_2(x - x_k)^2 + c_3(x - x_k)^3 + \dots + c_n(x - x_k)^N \quad (1.2.21)$$

for some  $c_i$ ,  $2 \leq i \leq N = n - 1 + \deg U_k$ . Dividing both sides of (1.2.17) by  $(x - x_k)^2$  gives

$$\frac{\omega'(x)U_k(x)}{(x - x_k)^2} = \frac{S_k(x)}{(x - x_k)^2} - \frac{S_k'(x)}{x - x_k} = - \left( \frac{S_k(x)}{x - x_k} \right)'$$

This together with (1.2.21) implies that

$$\frac{S_k(x)}{x - x_k} = - \int \frac{\omega_X'(x)U_k(x)}{(x - x_k)^2} dx.$$

Hence

$$A_k(x) = -\omega_X(x) \int \frac{\omega_X'(x)U_k(x)}{(x - x_k)^2} dx. \quad (1.2.22)$$

Now it remains to figure out the polynomial  $U_k$  of degree at most one. Write

$$U_k(x) = r_0 + r_1(x - x_k). \quad (1.2.23)$$

Then

$$r_0 = U_k(x_k) = \frac{1}{\omega_X'(x_k) \prod_{i \neq k} (x_k - x_i)} \quad (1.2.24)$$

by (1.2.19). From (1.2.20) and 1.2.23 it follows that

$$r_1 = -\frac{\omega_X''(x_k)}{(\omega_X'(x_k))^2 \prod_{i \neq k} (x_k - x_i)}. \quad (1.2.25)$$

Therefore

$$U_k(x) = \frac{1}{(\omega_X'(x_k))^2 \prod_{i \neq k} (x_k - x_i)} (\omega_X'(x_k) - \omega_X''(x_k)(x - x_k)).$$

Substituting this into (1.2.22), we obtain

$$A_k(x) = -\frac{\omega_X(x)}{(\omega_X'(x_k))^2 \prod_{i \neq k} (x_k - x_i)} \int \frac{\omega_X'(x)}{(x - x_k)^2} (\omega_X'(x_k) - \omega_X''(x_k)(x - x_k)) dx, \quad 2 \leq k \leq n. \quad (1.2.26)$$

Finally let us verify that the functions  $A_k, 2 \leq k \leq n$ , satisfy (1.2.5). Notice that

$$\begin{aligned} A_k'(x) &= -\frac{\omega_X'(x)}{\prod_{i \neq k} (x_k - x_i)} \int \frac{\omega_X'(x)}{(x - x_k)^2} \left(1 - \frac{\omega_X''(x_k)}{\omega_X'(x_k)}(x - x_k)\right) dx \\ &\quad - \frac{\omega_X(x)}{\prod_{i \neq k} (x_k - x_i)} \frac{\omega_X'(x)}{(x - x_k)^2} \left(1 - \frac{\omega_X''(x_k)}{\omega_X'(x_k)}(x - x_k)\right), \end{aligned}$$

which implies that  $A_k'(x_l^*) = 0$  for all  $1 \leq l \leq n - 1$ . On the other hand,  $A_k(x_{k'}) = 0$  for all  $k' \neq k$  as  $\omega_X(x_{k'}) = 0$ , and

$$\begin{aligned} A_k(x_k) &= -\lim_{x \rightarrow x_k} \frac{\omega_X(x)}{(\omega_X'(x_k))^2 \prod_{i \neq k} (x_k - x_i)} \int \frac{1}{(x - x_k)^2} ((\omega_X'(x_k))^2 + Q(x - x_k)) dx \\ &= \lim_{x \rightarrow x_k} \frac{\omega_X(x)}{\prod_{i \neq k} (x_k - x_i)(x - x_k)} = 1 \end{aligned}$$

where  $Q$  is a polynomial such that  $Q(0) = 0$ . This proves that polynomials  $A_k, 2 \leq k \leq n$ , in (1.2.26) satisfies (1.2.5).  $\square$

We remark that there are many polynomials that satisfies (1.2.2) and (1.2.3). Consider a polynomial  $\tilde{P}$  of the following form:

$$\tilde{P}(x) = P(x) + \omega'_X(x)(\alpha + \beta\omega_X(x)) \quad (1.2.27)$$

where  $\alpha, \beta \in \mathbf{R}$ . Then

$$\begin{aligned} \int_{x_k}^{x_{k+1}} \tilde{P}(x) dx &= \int_{x_k}^{x_{k+1}} P(x) dx + \int_{x_k}^{x_{k+1}} \omega'_X(x)(\alpha + \beta\omega_X(x)) dx \\ &= y_{k+1} + \left( \alpha\omega_X(x) + (\beta/2)(\omega_X(x))^2 \right) \Big|_{x_k}^{x_{k+1}} = y_{k+1}, 1 \leq k \leq n-1. \end{aligned}$$

Also, observe that

$$\tilde{P}(x_l^*) = P(x_l^*) + \omega'_X(x_l^*)(\alpha + \beta\omega_X(x_l^*)) = y_l^*, 1 \leq l \leq n-1.$$

Therefore a polynomial  $P$  of the form of (1.2.27) satisfies (1.2.2) and (1.2.3). On the other hand, If  $Q$  is a polynomial of degree at most  $2n-1$  that satisfies (1.2.2) and (1.2.3), then  $R(x) := Q(x) - P(x)$  satisfies

$$R(x_l^*) = 0 \text{ for all } 1 \leq l \leq n-1, \text{ and } \int_{x_k}^{x_{k+1}} R(x) dx = 0 \text{ for all } 1 \leq k \leq n-1. \quad (1.2.28)$$

From the above requirement, the antiderivative of the polynomial

$$\int R(x) dx = c + \omega_X(x)S(x) \quad (1.2.29)$$

for some polynomial  $S$  of degree at most  $n$ , and

$$R(x) = \omega'_X(x)M(x) \tag{1.2.30}$$

for some polynomial  $M$  of degree at most  $n$ . Therefore

$$\omega'_X(x)S(x) + \omega_X(x)S'(x) = \omega'_X(x)M(x). \tag{1.2.31}$$

Rearranging the above equation gives

$$\omega'_X(x)(S(x) - M(x)) = -\omega_X(x)S'(x). \tag{1.2.32}$$

Recall that  $\omega_X(x)$  and  $\omega'_X(x)$  do not have common roots, and that  $S'(x)$  has degree at most  $n - 1$ . Therefore  $S'(x) = \frac{\beta}{2}\omega'_X(x)$  for some constant  $\beta$ . This implies that

$$M(x) = \alpha + \beta\omega_X(x),$$

or equivalently

$$R(x) = P(x) + \omega'_X(x)(\alpha + \beta\omega_X(x)) \tag{1.2.33}$$

for some constant  $\alpha, \beta$ . This leads to the following theorem.

**Theorem 1.2.2.** *Let  $X := \{x_1, \dots, x_n\}$  contain  $n$  distinct nodes on the real line ordered by  $x_1 < x_2 < \dots < x_n$ , and let  $X^* = \{x_l^*\}_{l=1}^{n-1}$  denote the set of the roots of the polynomial  $\omega'_X$  (the derivative of the polynomial  $\omega_X$  in (1.1.1)). Given data  $\{y_k\}_{k=1}^n$  and  $\{y_l^*\}_{l=1}^{n-1}$ , define a polynomial  $P(x)$  of degree  $2n - 2$  as in (1.2.1). Then a polynomial  $R$  of degree at most  $2n - 1$  that satisfies*

(1.2.2) and (1.2.3) if and only if  $R(x) = P(x) + \omega'_X(x)(\alpha + \beta\omega_X(x))$  for some constants  $\alpha, \beta$ .

### 1.3 Modified Pál interpolation II

Let  $a, b$ , and  $c$  be real numbers, and let  $x_k^*, k = 1, 2, \dots, n^*$ , be the real roots of

$$\tilde{\omega}_X(x) := a\omega_X(x) + (bx + c)\omega'_X(x).$$

Szabó and Joó [21] [22] [23] [24] generalized the Pál interpolation problem to the following:

*Let  $a, b, c$  be real numbers, and let  $x_l^*, l = 1, 2, \dots, n^*$  be the real roots of  $\tilde{\omega}_X(x) := a\omega_X(x) + (bx + c)\omega'_X(x)$ . Determine a polynomial  $R(x)$  of the lowest possible degree that has the properties  $R(x_k) = y_k, 1 \leq k \leq n$ , and  $R'(x_l^*) = y_l^*, 1 \leq l \leq n^*$ .*

They found general polynomials for the following cases: (1)  $b = 0$  and (2)  $a < 0, b = 1$ . If  $a = b = 0$  and  $c = 1$ , the above interpolation become Pál interpolation.

In this section, we modify the work done by Szabó and Joó [21] to fit the following conditions:

$$R(x_l^*) = y_l^*, \quad 1 \leq l \leq n,$$

and

$$\int_{x_k}^{x_{k+1}} R(x) dx = y_{k+1}, \quad 1 \leq k \leq n-1,$$

under the assumption that  $a \neq 0$  and  $b = 0$ . In this case  $n^* = n$ . Moreover, roots of  $\omega_X(x)$  and  $\tilde{\omega}_X(x)$  have the following interlacing property:

$$x_1 < x_1^* < x_2 < \dots < x_n < x_n^*$$

if  $a/c < 0$ ; and

$$x_1^* < x_1 < x_2^* < \cdots < x_n^* < x_n$$

if  $a/c > 0$ .

**Theorem 1.3.1.** *Let  $a, c \neq 0$ ,  $X := \{x_1, \dots, x_n\}$  contain  $n$  distinct nodes on the real line ordered by  $x_1 < x_2 < \dots < x_n$ , denote by  $X^* := \{x_1^*, x_2^*, \dots, x_n^*\}$  the set of real roots of the polynomial  $\tilde{\omega}_X(x) := a\omega_X(x) + c\omega'_X(x)$ , and let*

$$\Omega(x_l^*) = \frac{a\omega_X(x) + c\omega'_X(x)}{x - x_l^*} \Big|_{x=x_l^*}. \quad (1.3.1)$$

Given the interpolation data  $\{y_k\}_{k=2}^n$  and  $\{y_l^*\}_{l=1}^n$ , set  $z_k = \sum_{q=2}^k y_q$ ,  $2 \leq k \leq n$ , and define the polynomial  $R(x)$  of degree  $2n - 2$  by

$$\begin{aligned} R(x) &:= \frac{d}{dx} \left[ \sum_{k=2}^n z_k \frac{\omega_X(x)}{c\omega'_X(x_k) \prod_{i \neq k} (x_k - x_i)} e^{\frac{a}{c}x} \right. \\ &\quad \times \int_x^\infty \frac{a\omega_X(t) + c\omega'_X(t)}{(t - x_k)^2} \left( 1 - \frac{\omega''_X(x_k)}{\omega'_X(x_k)} (t - x_k) \right) e^{-\frac{a}{c}t} dt \\ &\quad \left. - \sum_{l=1}^n y_l^* \frac{\omega_X(x) e^{\frac{a}{c}x}}{\omega_X(x_l^*) \Omega(x_l^*)} \int_x^\infty \frac{a\omega_X(t) + c\omega'_X(t)}{(t - x_l^*)} e^{-\frac{a}{c}t} dt \right] \quad \text{for } x > x_n \end{aligned} \quad (1.3.2)$$

if  $\frac{a}{c} > 0$ , and

$$\begin{aligned}
R(x) &:= \frac{d}{dx} \left[ - \sum_{k=2}^n z_k \frac{\omega_X(x)}{c\omega'_X(x_k) \prod_{i \neq k} (x_k - x_i)} e^{\frac{a}{c}x} \right. \\
&\quad \times \int_{-\infty}^x \frac{a\omega_X(t) + c\omega'_X(t)}{(t - x_k)^2} \left( 1 - \frac{\omega''_X(x_k)}{\omega'_X(x_k)} (t - x_k) \right) e^{-\frac{a}{c}t} dt \\
&\quad \left. + \sum_{l=1}^{n-1} y_l^* \frac{\omega_X(x) e^{\frac{a}{c}x}}{\omega_X(x_l^*) \Omega(x_l^*)} \int_{-\infty}^x \frac{a\omega_X(t) + c\omega'_X(t)}{(t - x_l^*)} e^{-\frac{a}{c}t} dt \right] \quad \text{for } x < x_1
\end{aligned} \tag{1.3.3}$$

if  $\frac{a}{c} < 0$ . Then  $R(x)$  satisfies

$$R(x_l^*) = y_l^*, \quad 1 \leq l \leq n, \tag{1.3.4}$$

and

$$\int_{x_k}^{x_{k+1}} R(x) dx = y_{k+1}, \quad 1 \leq k \leq n-1. \tag{1.3.5}$$

*Proof.* We start by decomposing  $R(x)$  into a sum of two functions, as in the previous section,

$$R(x) = \frac{d}{dx} \left[ \sum_{k=2}^n z_k A_k(x) + \sum_{l=1}^n y_l^* B_l(x) \right], \tag{1.3.6}$$

where polynomials  $\{A_k(x)\}_{k=2}^n$  and  $\{B_l(x)\}_{l=1}^n$  of degree at most  $2n-1$  satisfy

$$\begin{cases}
\text{(a)} & A_k(x_i) = \delta_{ki} \quad \text{for all } 2 \leq k \leq n \text{ and } 1 \leq i \leq n \\
\text{(b)} & A'_k(x_j^*) = 0 \quad \text{for all } 2 \leq k \leq n \text{ and } 1 \leq j \leq n,
\end{cases} \tag{1.3.7}$$

and

$$\begin{cases} \text{(c)} & B_l(x_i) = 0 \text{ for all } 1 \leq l \leq n \text{ and } 1 \leq i \leq n \\ \text{(d)} & B'_l(x_j^*) = \delta_{lj} \text{ for all } 1 \leq l \leq n \text{ and } 1 \leq j \leq n. \end{cases} \quad (1.3.8)$$

Similar to the previous section let's first construct the polynomials  $B_l(x)$ ,  $1 \leq l \leq n$ . From the requirement (c) in (1.3.8) we know that

$$B_l(x) = \omega_X(x)V_l(x) \quad (1.3.9)$$

for a polynomial  $V_l(x)$  of degree at most  $n - 1$ . Recall that roots of  $a\omega_X(x) + c\omega'_X(x)$  are real and have a multiplicity of one. Consequently,

$$B'_l(x) = \omega'_X(x)V_l(x) + \omega_X(x)V'_l(x) = \frac{a\omega_X(x) + c\omega'_X(x)}{(x - x_l^*)}W_l(x) \quad (1.3.10)$$

for some polynomial  $W_l(x)$  of degree at most  $n - 1$ , where the last equality follows from the requirement (d) in (1.3.8). Multiplying  $x - x_l^*$  at both sides of the above equation leads to

$$\omega'_X(x)[(x - x_l^*)V_l(x) - cW_l(x)] = \omega_X(x)[-(x - x_l^*)V'_l(x) + aW_l(x)].$$

Recall that  $\omega_X$  and its derivative  $\omega'_X$  do not have common roots. Then

$$M\omega'_X(x) = -(x - x_l^*)V'_l(x) + aW_l(x) \quad (1.3.11)$$

and

$$M\omega_X(x) = (x - x_l^*)V_l(x) - cW_l(x) \quad (1.3.12)$$



for a constant  $M$ . Multiplying (1.3.11) with  $c$  and (1.3.12) with  $a$ , and then adding them together, we obtain

$$(x - x_l^*)[aV_l(x) - cV_l'(x)] = M[a\omega_X(x) + c\omega_X'(x)].$$

Multiplying both sides by  $-\frac{e^{-\frac{a}{c}x}}{c(x-x_l^*)}$  gives

$$\frac{d}{dx} \left( e^{-\frac{a}{c}x} V_l(x) \right) = -\frac{M e^{-\frac{a}{c}x}}{c} \frac{a\omega_X(x) + c\omega_X'(x)}{(x - x_l^*)}. \quad (1.3.13)$$

Integrating both sides leads to

$$V_l(x) = \frac{M e^{\frac{a}{c}x}}{c} \int_x^{\infty} \frac{a\omega_X(t) + c\omega_X'(t)}{(t - x_l^*)} e^{-\frac{a}{c}t} dt, \quad x > x_l^*, \quad \text{if } \frac{a}{c} > 0, \quad (1.3.14)$$

and

$$V_l(x) = -\frac{M e^{\frac{a}{c}x}}{c} \int_{-\infty}^x \frac{a\omega_X(t) + c\omega_X'(t)}{(t - x_l^*)} e^{-\frac{a}{c}t} dt, \quad x < x_l^*, \quad \text{if } \frac{a}{c} < 0. \quad (1.3.15)$$

The next step is to determine the constant  $M$ . Note from (1.3.10) and the condition (d) in (1.3.8) that

$$B_l'(x_l^*) = \omega_X(x_l^*)V_l'(x_l^*) + V_l(x_l^*)\omega_X'(x_l^*) = 1. \quad (1.3.16)$$

Multiplying both sides of (1.3.13) by  $-c$  and replacing  $x$  with  $x_l^*$  gives

$$-cV_l'(x_l^*) + aV_l(x_l^*) = \left. \frac{a\omega_X(x) + c\omega_X'(x)}{x - x_l^*} \right|_{x=x_l^*} M. \quad (1.3.17)$$

Note that the right hand side of the above equation is nonzero because  $x_l^*$  is a simple root of the polynomial  $a\omega_X(x) + c\omega_X'(x)$ . Multiplying both sides of (1.3.16) by  $-\frac{c}{\omega_X(x_l^*)}$  and recalling that

$a\omega_X(x_l^*) + c\omega_X'(x_l^*) = 0$ , we get

$$-\frac{c}{\omega_X(x_l^*)} = -cV_l'(x_l^*) + aV_l(x_l^*). \quad (1.3.18)$$

Let  $\Omega(x_l^*) = \frac{a\omega_X(x) + c\omega_X'(x)}{x - x_l^*} \Big|_{x=x_l^*}$ . Thus combining (1.3.17) and (1.3.18) determines the constant

$$M = \frac{-c}{\omega_X(x_l^*)\Omega(x_l^*)}. \quad (1.3.19)$$

Therefore,

$$B_l(x) = -\frac{\omega_X(x)e^{\frac{a}{c}x}}{\omega_X(x_l^*)\Omega(x_l^*)} \int_x^\infty \frac{a\omega_X(t) + c\omega_X'(t)}{(t - x_l^*)} e^{-\frac{a}{c}t} dx \quad \text{if } \frac{a}{c} > 0 \quad (1.3.20)$$

and

$$B_l(x) = \frac{\omega_X(x)e^{\frac{a}{c}x}}{\omega_X(x_l^*)\Omega(x_l^*)} \int_{-\infty}^x \frac{a\omega_X(t) + c\omega_X'(t)}{(t - x_l^*)} e^{-\frac{a}{c}t} dt \quad \text{if } \frac{a}{c} < 0. \quad (1.3.21)$$

The polynomials  $B_l, 1 \leq l \leq n$ , just defined have degree at most  $2n - 1$ . They satisfy the requirement (c) in (1.3.8) as they have the factor  $\omega_X$ , and also the requirement (d) in (1.3.8) as

$$\begin{aligned} B_l'(x) &= \frac{\omega_X(x)e^{\frac{a}{c}x}}{\omega_X(x_l^*)\Omega(x_l^*)} \frac{a\omega_X(x) + c\omega_X'(x)}{x - x_l^*} e^{-\frac{a}{c}x} - \\ &\quad \frac{e^{\frac{a}{c}x}(a\omega_X(x) + c\omega_X'(x))}{c\omega_X(x_l^*)\Omega(x_l^*)} \int_x^\infty \frac{a\omega_X(t) + c\omega_X'(t)}{(t - x_l^*)} e^{-\frac{a}{c}t} dt \quad \text{if } \frac{a}{c} > 0, \end{aligned}$$

and

$$B'_l(x) = \frac{\omega_X(x)e^{\frac{a}{c}x}}{\omega_X(x_l^*)\Omega(x_l^*)} \frac{a\omega_X(x) + c\omega'_X(x)}{x - x_l^*} e^{-\frac{a}{c}x} + \frac{e^{\frac{a}{c}x}(a\omega_X(x) + c\omega'_X(x))}{c\omega_X(x_l^*)\Omega(x_l^*)} \int_{-\infty}^x \frac{a\omega_X(t) + c\omega'_X(t)}{(t - x_l^*)} e^{-\frac{a}{c}t} dt \text{ if } \frac{a}{c} < 0.$$

Thus

$$B'_l(x_{j'}) = \frac{\omega_X(x_{j'}^*) \Omega(x_{j'}^*)}{\omega_X(x_l^*) \Omega(x_l^*)} = \begin{cases} 1 & \text{if } j' = l \\ 0 & \text{if } j' \neq l. \end{cases}$$

This completes the construction of polynomials  $B_l$ ,  $1 \leq l \leq n$ , of degree at most  $2n - 1$  satisfying (1.3.8).

Polynomial  $A_k(x)$ ,  $2 \leq k \leq n$ , can be constructed in a similar way. Condition (a) implies that

$$A_k(x) = \frac{\omega_X(x)}{x - x_k} S(x), \quad (1.3.22)$$

where  $S(x)$  is a nonzero polynomial of degree at most  $n$ . Taking derivative on both sides of (1.3.22) gives

$$A'_k(x) = \frac{\omega_X(x)}{x - x_k} S'(x) + \left[ \frac{\omega'_X(x)}{x - x_k} - \frac{\omega_X(x)}{(x - x_k)^2} \right] S(x). \quad (1.3.23)$$

Recall that  $a\omega_X(x) + c\omega'_X(x)$  has all roots being real and simple, we obtained from Condition (b) that  $A'_k(x) = (a\omega_X(x) + c\omega'_X(x))T(x)$  for some polynomial  $T$  of degree at most  $n - 2$ . Thus

$$(a\omega_X(x) + c\omega'_X(x))T(x) = \frac{\omega_X(x)}{x - x_k} S'(x) + \left[ \frac{\omega'_X(x)}{x - x_k} - \frac{\omega_X(x)}{(x - x_k)^2} \right] S(x). \quad (1.3.24)$$

Multiplying both sides by  $(x - x_k)^2$  and then moving all terms with the factor  $\omega_X(x)$  to the right

hand side, we obtain

$$\omega'_X(x)(x-x_k)[S(x)-c(x-x_k)T(x)] = \omega_X(x)[S(x)-(x-x_k)S'(x)+a(x-x_k)^2T(x)]. \quad (1.3.25)$$

Since  $\omega_X$  and  $\omega'_X$  are relatively prime (i.e. they do not have any zeros in common),

$$\omega'_X(x)U_k(x) = S(x) - (x-x_k)S'(x) + a(x-x_k)^2T(x) \quad (1.3.26)$$

and

$$\frac{\omega_X(x)}{x-x_k}U_k(x) = S(x) - c(x-x_k)T(x) \quad (1.3.27)$$

for a polynomial  $U_k(x)$  of degree at most 1. From (1.3.26) and (1.3.27),

$$\frac{\omega'_X(x)U_k(x)}{(x-x_k)^2} = \frac{S(x)}{(x-x_k)^2} - \frac{S'(x)}{x-x_k} + aT(x) = -\left(\frac{S(x)}{x-x_k}\right)' + aT(x). \quad (1.3.28)$$

and

$$\frac{\omega_X(x)U_k(x)}{(x-x_k)^2} = \frac{S(x)}{x-x_k} - cT(x). \quad (1.3.29)$$

Multiplying (1.3.29) with  $a/c$ , and then adding it with (1.3.28) yields

$$-\left(\frac{S(x)}{x-x_k}\right)' + \frac{a}{c} \frac{S(x)}{x-x_k} = \frac{1}{c} \frac{a\omega_X(x) + c\omega'_X(x)}{(x-x_k)^2} U_k(x).$$

Multiplying both sides with  $e^{-\frac{a}{c}x}$  gives

$$\frac{d}{dx} \left( e^{-\frac{a}{c}x} \frac{S(x)}{x-x_k} \right) = -\frac{e^{-\frac{a}{c}x}}{c} \frac{a\omega_X(x) + c\omega'_X(x)}{(x-x_k)^2} U_k(x).$$

Integrating both sides leads to

$$\frac{S(x)}{x - x_k} = -\frac{e^{\frac{a}{c}x}}{c} \int \frac{a\omega_X(x) + c\omega'_X(x)}{(x - x_k)^2} U_k(x) e^{-\frac{a}{c}x} dx. \quad (1.3.30)$$

The above equation combined with equation (1.3.22) gives

$$A_k(x) = -\frac{1}{c} e^{\frac{a}{c}x} \omega_X(x) \int \frac{a\omega_X(x) + c\omega'_X(x)}{(x - x_k)^2} U_k(x) e^{-\frac{a}{c}x} dx. \quad (1.3.31)$$

Now it remains to figure out the polynomial  $U_k$  of degree at most one. Write

$$U_k(x) = r_0 + r_1(x - x_k). \quad (1.3.32)$$

Then by (1.2.19)

$$r_0 = U_k(x_k) = \frac{1}{\omega'_X(x_k) \prod_{i \neq k} (x_k - x_i)}, \quad (1.3.33)$$

and by (1.3.26),

$$\omega''_X(x_k) U_k(x_k) + \omega'_X(x_k) U'_k(x_k) = 0. \quad (1.3.34)$$

The above equation implies that

$$r_1 = -\frac{\omega''_X(x_k)}{(\omega'_X(x_k))^2 \prod_{i \neq k} (x_k - x_i)}. \quad (1.3.35)$$

Therefore

$$U_k(x) = \frac{1}{(\omega'_X(x_k))^2 \prod_{i \neq k} (x_k - x_i)} (\omega'_X(x_k) - \omega''_X(x_k)(x - x_k)).$$

Substituting this into (1.3.31), we obtain that

$$A_k(x) = \frac{1}{c\omega'_X(x_k) \prod_{i \neq k} (x_k - x_i)} e^{\frac{a}{c}x} \omega_X(x) \times \int_x^\infty \frac{a\omega_X(t) + c\omega'_X(t)}{(t - x_k)^2} \left(1 - \frac{\omega''_X(x_k)}{\omega'_X(x_k)}(t - x_k)\right) e^{-\frac{a}{c}t} dt \quad \text{for } x > x_k, \quad (1.3.36)$$

if  $\frac{a}{c} > 0$ , and

$$A_k(x) = -\frac{1}{c\omega'_X(x_k) \prod_{i \neq k} (x_k - x_i)} e^{\frac{a}{c}x} \omega_X(x) \int_{-\infty}^x \frac{a\omega_X(t) + c\omega'_X(t)}{(t - x_k)^2} \left(1 - \frac{\omega''_X(x_k)}{\omega'_X(x_k)}(t - x_k)\right) e^{-\frac{a}{c}t} dt \quad \text{for } x < x_k, \quad (1.3.37)$$

if  $\frac{a}{c} < 0$ . Note that  $A_k(x)$  satisfies condition (b) because

$$A'_k(x) = -\frac{e^{\frac{a}{c}x} \omega_X(x)}{c\omega'_X(x_k) \prod_{i \neq k} (x_k - x_i)} \frac{a\omega_X(x) + c\omega'_X(x)}{(x - x_k)^2} \left(1 - \frac{\omega''_X(x_k)}{\omega'_X(x_k)}(x - x_k)\right) e^{-\frac{a}{c}x} - \frac{1}{c^2} e^{\frac{a}{c}x} (a\omega_X(x) + c\omega'_X(x)) \times \int \frac{a\omega_X(x) + c\omega'_X(x)}{\omega'_X(x_k)(x - x_k)^2 \prod_{i \neq k} (x_k - x_i)} \left(1 - \frac{\omega''_X(x_k)}{\omega'_X(x_k)}(x - x_k)\right) e^{-\frac{a}{c}x} dx.$$

Thus  $A'_k(x_j^*) = 0$  because both terms have the factor  $a\omega_X(x) + c\omega'_X(x)$  which is zero when  $x$  is

replaced by  $x_j^*$ . Note that

$$\begin{aligned}
& (a\omega_X(x) + c\omega'_X(x))U_k(x) \\
= & (a\omega_X(x_k) + c\omega'_X(x_k))U_k(x_k) + ((a\omega_X(x) + c\omega'_X(x))U_k(x))'|_{x=x_k}(x - x_k) \\
& + c_2(x - x_k)^2 + \cdots + c_N(x - x_k)^N \\
= & \frac{c}{\prod_{i \neq k} (x_k - x_i)} - \frac{a}{\prod_{i \neq k} (x_k - x_i)}(x - x_k) + c_2(x - x_k)^2 + \cdots + c_N(x - x_k)^N
\end{aligned}$$

by (1.3.33) and (1.3.35). Therefore,

$$\int \frac{a\omega_X(x) + c\omega'_X(x)}{(x - x_k)^2} U_k(x) e^{-\frac{a}{c}x} dx = -\frac{ce^{-\frac{a}{c}x}}{\prod_{i \neq j} (x_k - x_j)} \frac{1}{x - x_k} + Q(x)e^{-\frac{a}{c}x}$$

for some polynomial  $Q$  of degree at most  $n$ . Therefore  $A_k$  is a polynomial of degree at most  $2n - 1$  satisfying

$$A_k(x_j) = 0 \quad \text{for all } j \neq k$$

and

$$A_k(x_k) = -\frac{1}{c} e^{\frac{a}{c}x_k} \lim_{x \rightarrow x_k} \frac{-ce^{-\frac{a}{c}x}}{\prod_{i \neq j} (x_k - x_j)} \frac{\omega_X(x)}{x - x_k} = 1.$$

□

#### 1.4 Modified Pál interpolation III

In this section, we consider the modified Pál interpolation associated with  $\tilde{\omega}_X(x) := a\omega_X(x) + (bx + c)\omega'_X(x)$  with  $a < 0$  and  $b = 1$ .

**Theorem 1.4.1.** Let  $X := \{x_1, \dots, x_n\}$  contain  $n$  distinct nodes on the real line ordered by  $x_1 < x_2 < \dots < x_n$ ,  $0 > a \notin \{-1, -n\}$  and  $0 \neq c \notin -X$ . Assume that the polynomial  $\tilde{\omega}_X(x) := a\omega_X(x) + (x+c)\omega'_X(x)$  has  $n$  simple roots, which is denoted by  $X^* := \{x_1^*, x_2^*, \dots, x_n^*\}$ . Then the polynomial  $R(x)$  of degree  $2n - 2$  defined by

$$R(x) := \frac{d}{dx} \left[ - \sum_{k=2}^n z_k \frac{\omega_X(x)}{|x+c|^{-a}} \int_{-c}^x \frac{|t+c|^{-a}}{t+c} \frac{\tilde{\omega}_X(t)}{\omega'_X(x_k)(t-x_k)^2 \prod_{i \neq k} (x_k - x_i)} (1 - \alpha_k(t)) dt \right. \\ \left. + \sum_{l=1}^{n-1} y_l^* \frac{\beta_l}{\Omega(x_l^*)} \omega_X(x) \int_{-c}^x \left| \frac{t+c}{x+c} \right|^{-a} \frac{1}{t+c} \frac{\tilde{\omega}_X(t)}{t-x_l^*} dt \right]$$

satisfies the following conditions

$$R(x_l^*) = y_l^*, \quad 1 \leq l \leq n, \quad (1.4.1)$$

and

$$\int_{x_k}^{x_{k+1}} R(x) dx = y_{k+1}, \quad 1 \leq k \leq n-1, \quad (1.4.2)$$

where

$$z_k = \sum_{q=2}^k y_q, \quad \alpha_k(t) = \frac{\omega''_X(x_k)}{\omega'_X(x_k)}(t-x_k), \quad 2 \leq k \leq n,$$

and

$$\beta_l = \frac{x_l^* + c}{\omega_X(x_l^*)}, \quad \Omega(x_l^*) = \frac{\tilde{\omega}_X(x)}{x-x_l^*} \Big|_{x=x_l^*}, \quad 1 \leq l \leq n-1.$$

*Proof.* We begin the proof the same way as the previous sections, by decomposing  $R(x)$  into a



sum of polynomials  $A_k(x)$  and  $B_l(x)$ , c.f.(1.2.4) and (1.3.6), which satisfy the conditions:

$$\begin{cases} \text{(a)} & A_k(x_i) = \delta_{ki} \text{ for all } 2 \leq k \leq n \text{ and } 1 \leq i \leq n \\ \text{(b)} & A'_k(x_j^*) = 0 \text{ for all } 2 \leq k \leq n \text{ and } 1 \leq j \leq n, \end{cases} \quad (1.4.3)$$

and

$$\begin{cases} \text{(c)} & B_l(x_i) = 0 \text{ for all } 1 \leq l \leq n \text{ and } 1 \leq i \leq n \\ \text{(d)} & B'_l(x_j^*) = \delta_{lj} \text{ for all } 1 \leq l \leq n \text{ and } 1 \leq j \leq n. \end{cases} \quad (1.4.4)$$

Again, we start by obtaining the polynomial  $B_l(x)$ ,  $1 \leq l \leq n$ . By (1.4.4),

$$B_l(x) = \omega_X(x)V_l(x) \quad (1.4.5)$$

where  $V_l(x)$  is a polynomial of degree at most  $n - 1$ . Taking derivative of the above equality leads to

$$B'_l(x) = \omega_X(x)V'_l(x) + \omega'_X(x)V_l(x) = \frac{a\omega_X(x) + (x+c)\omega'_X(x)}{x-x_l^*}W_l(x) \quad (1.4.6)$$

for a polynomial  $W_l(x)$  of degree at most  $n - 1$ , where the last equality holds by (1.4.4) and the assumption that all roots of  $a\omega_X(x) + (x+c)\omega'_X(x)$  are real and simple. Recall that  $\omega_X$  and  $\omega'_X$  have no common roots. Hence we obtain from (1.4.6) that

$$-(x-x_l^*)V'_l(x) + aW_l(x) = M\omega'_X(x) \quad (1.4.7)$$

and

$$(x-x_l^*)V_l(x) - (x+c)W_l(x) = M\omega_X(x) \quad (1.4.8)$$

for some constant  $M$ . Dividing (1.4.7) by  $x - x_l^*$  and (1.4.8) by  $(x + c)(x - x_l^*)/a$ , and then taking their sum, we have

$$V_l'(x) - \frac{a}{x+c}V_l(x) = -\frac{M}{x+c} \frac{a\omega_X(x) + (x+c)\omega_X'(x)}{x-x_l^*}. \quad (1.4.9)$$

Multiplying by  $|x+c|^{-a}$  and then integrating at both sides yields

$$V_l(x) = -M|x+c|^a \int_{-c}^x |t+c|^{-a} \frac{a\omega_X(t) + (t+c)\omega_X'(t)}{(t-x_l^*)(t+c)} dt. \quad (1.4.10)$$

To find the constant  $M$ , we note that from condition (d) and (1.4.6)

$$B_l'(x_k^*) = 1 = \omega_X(x_l^*)V_l'(x_l^*) + \omega_X'(x_l^*)V_l(x_l^*). \quad (1.4.11)$$

Replacing  $x$  with  $x_l^*$  in (1.4.9) we get

$$-(x_l^*+c)V_l'(x_l^*) + aV_l(x_l^*) = M \frac{a\omega_X(x) + (x+c)\omega_X'(x)}{x-x_l^*} \Big|_{x=x_l^*} := M\Omega(x_l^*). \quad (1.4.12)$$

We remark that  $\Omega(x_l^*)$  is nonzero because roots of  $a\omega_X(x) + (x+c)\omega_X'(x)$  are simple. Multiplying both sides of (1.4.11) with  $x_l^* + c$  gives

$$x_l^* + c = (x_l^* + c)\omega_X'(x_l^*)V_l(x_l^*) + (x_l^* + c)V_l'(x_l^*)\omega_X(x_l^*). \quad (1.4.13)$$

Recall that  $x_l^*$  is a root of the polynomial  $a\omega_X(x) + (x+c)\omega'_X(x)$ , i.e.,

$$a\omega_X(x_l^*) + (x_l^* + c)\omega'_X(x_l^*) = 0. \quad (1.4.14)$$

This together with (1.4.11) implies that

$$x_l^* + c = -a\omega_X(x_l^*)V_l(x_l^*) + (x_l^* + c)\omega_X(x_l^*)V_l'(x_l^*). \quad (1.4.15)$$

Observe that  $\omega_X(x_l^*) \neq 0$ , as otherwise  $(x_l^* + c)\omega'_X(x_l^*) = 0$ , which contradicts to the assumptions on  $c$  and the simple root property for  $\omega_X(x)$ . Therefore,

$$-\frac{x_l^* + c}{\omega(x_l^*)} = -V_l'(x_l^*)(x_l^* + c) + aV_l(x_l^*). \quad (1.4.16)$$

Thus

$$M = -\frac{x_l^* + c}{\omega_X(x_l^*)\Omega(x_l^*)} \quad (1.4.17)$$

by (1.4.12) and (1.4.16). From (1.4.10) and (1.4.17) we conclude that

$$B_l(x) = \frac{x_l^* + c}{\omega_X(x_l^*)\Omega(x_l^*)}\omega_X(x) \int_{-c}^x \left| \frac{t+c}{x+c} \right|^{-a} \frac{1}{t+c} \frac{a\omega_X(t) + (t+c)\omega'_X(t)}{t-x_l^*} dt. \quad (1.4.18)$$

The polynomials  $B_l$ ,  $1 \leq l \leq n-1$ , satisfy the requirement (c) in (1.4.4) as they have the factor

$\omega_X$  by (1.4.5), and also the requirement (d) in (1.4.4) as

$$B'_l(x) = M\omega_X(x) \frac{1}{x+c} \frac{a\omega_X(x) + (x+c)\omega'_X(x)}{x-x_l^*} \\ + M(\omega_X(x)(x+c)^a)' \int_{-c}^x |t+c|^{-a} \frac{a\omega_X(t) + (t+c)\omega'_X(t)}{(t+c)(t-x_l^*)} dt$$

and hence

$$B'_l(x_{j'}^*) = \begin{cases} 1 & \text{if } j' = l \\ 0 & \text{if } j' \neq l. \end{cases}$$

This completes the construction of polynomials  $B_l, 1 \leq l \leq n-1$ .

We finish this section by the construction of polynomials  $A_k, 2 \leq k \leq n$ . Condition (a) in (1.4.3) implies that

$$A_k(x) = \frac{\omega_X(x)}{x-x_k} S(x) \tag{1.4.19}$$

where  $S(x)$  is a nonzero polynomial of degree at most  $n$ . The above equation (1.4.19) together with condition (b) in (1.4.3) implies that

$$A'_k(x) = \left( \frac{(x-x_k)\omega'_X(x) - \omega_X(x)}{(x-x_k)^2} \right) S(x) + \frac{\omega_X(x)}{x-x_k} S'(x) \\ = (a\omega_X(x) + (x+c)\omega'_X(x))T(x) \tag{1.4.20}$$

for a polynomial  $T(x)$  of degree at most  $n-2$ . Multiplying (1.4.20) with  $(x-x_k)^2$  and rearranging the equation yields

$$\omega'_X(x)(x-x_k)[S(x) - (x+c)(x-x_k)T(x)] \\ = \omega_X(x)[S(x) - (x-x_k)S'(x) + a(x-x_k)T(x)].$$

Recalling that  $\omega_X(x)$  and  $\omega'_X(x)$  have no roots in common, we have

$$\omega'_X(x)U_k(x) = S(x) - (x - x_k)S'(x) + a(x - x_k)^2T(x) \quad (1.4.21)$$

and

$$\frac{\omega_X(x)}{x - x_k}U_k(x) = S(x) - (x + c)(x - x_k)T(x) \quad (1.4.22)$$

for some polynomial  $U_k$  of degree at most one. Rearranging equations (1.4.21) and (1.4.22) yields

$$-\left(\frac{S(x)}{x - x_k}\right)' + aT(x) = \frac{\omega'_X(x)U_k(x)}{(x - x_k)^2} \quad (1.4.23)$$

and

$$\frac{S(x)}{x - x_k} - (x + c)T(x) = \frac{\omega_X(x)U_k(x)}{(x - x_k)^2}. \quad (1.4.24)$$

Multiplying (1.4.24) by  $a/(x + c)$  and adding it to (1.4.23) gives

$$-\left(\frac{S(x)}{x - x_k}\right)' + \frac{a}{x + c}\left(\frac{S(x)}{x - x_k}\right) = -\frac{U_k(x)}{x + c} \frac{a\omega_X(x) + (x + c)\omega'_X(x)}{(x - x_k)^2}. \quad (1.4.25)$$

Multiplying both sides of the above equation by  $|x + c|^{-a}$  leads to

$$\frac{d}{dx} \left( |x + c|^{-a} \frac{S(x)}{x - x_k} \right) = -\frac{U_k(x)}{x + c} |x + c|^{-a} (a\omega_X(x) + (x + c)\omega'_X(x)). \quad (1.4.26)$$

Hence

$$\frac{S(x)}{x - x_k} = -|x + c|^a \int \frac{|x + c|^{-a} a\omega_X(x) + (x + c)\omega'_X(x)}{x + c} U_k(x) dx. \quad (1.4.27)$$

Comparing (1.4.27) to (1.4.19) yields

$$A_k(x) = -\omega_X(x)|x+c|^a \int \frac{|x+c|^{-a} a\omega_X(x) + (x+c)\omega'_X(x)}{x+c} \frac{U_k(x)}{(x-x_k)^2} dx. \quad (1.4.28)$$

From (1.4.21)

$$S(x_k) = \omega'_X(x_k)U_k(x_k). \quad (1.4.29)$$

From (1.4.19) and condition (a) it follows that

$$A_k(x_k) = \omega'_X(x_k)S(x_k) = 1.$$

Therefore,

$$S(x_k) = \frac{1}{\omega'_X(x_k)} \quad (1.4.30)$$

and

$$U_k(x_k) = \frac{S(x_k)}{\omega'_X(x_k)} = \frac{1}{(\omega'_X(x_k))^2}. \quad (1.4.31)$$

Multiplying (1.4.21) by  $x+c$  and (1.4.22) by  $a$  and adding the two gives

$$(a\omega_X(x) + (x+c)\omega'_X(x))U_k(x) = S(x)[(x+c) + a(x-x_k)] - (x+c)(x-x_k)S'(x). \quad (1.4.32)$$

Replacing  $x$  with  $x_k$  yields

$$\tilde{\omega}(x)U_k(x)|_{x=x_k} = (x+c)S(x)|_{x=x_k} = \frac{x+c}{\omega'_X(x)} \Big|_{x=x_k}. \quad (1.4.33)$$

Now taking the derivative on both sides of (1.4.32) yields

$$\left( (a\omega_X(x) + (x+c)\omega'_X(x))U_k(x) \right)' = -S''(x)(x+c)(x-x_k) + S'(x)(x-x_k)(a-1) + (a+1)S(x). \quad (1.4.34)$$

Replacing  $x$  with  $x_k$  in the above equation and then applying (1.4.30) gives

$$\left( (a\omega_X(x) + (x+c)\omega'_X(x))U_k(x) \right)' \Big|_{x=x_k} = \frac{a+1}{\omega'_X(x_k)}. \quad (1.4.35)$$

Therefore, the Taylor series expansion of  $\tilde{\omega}(x)U_k(x)$  about the point  $x_k$  is

$$\begin{aligned} (a\omega_X(x) + (x+c)\omega'_X(x))U_k(x) &= \frac{x_k+c}{\omega'_X(x_k)} + \frac{a+1}{\omega'_X(x_k)}(x-x_k) \\ &\quad + c_2(x-x_k)^2 + \cdots + c_N(x-x_k)^N \end{aligned} \quad (1.4.36)$$

for some constants  $c_i, 2 \leq i \leq N$ , where  $N = n + \deg U_k \leq n + 1$ . Hence by (1.4.28)

$$\begin{aligned} A_k(x) &= -\frac{\omega_X(x)}{|x+c|^{-a}} \int \frac{|x+c|^{-a}}{x+c} \left( \frac{x_k+c}{\omega'_X(x_k)} \cdot \frac{1}{(x-x_k)^2} \right. \\ &\quad \left. + \frac{a+1}{\omega'_X(x_k)} \cdot \frac{1}{x-x_k} + c_2 + \cdots + c_N(x-x_k)^{N-2} \right) dx. \end{aligned} \quad (1.4.37)$$

Note that

$$\int \frac{(x+c)^{-a-1}}{(x-x_k)^2} dx = -\frac{(x+c)^{-a-1}}{x-x_k} - (a+1) \int \frac{(x+c)^{-a-2}}{x-x_k} dx. \quad (1.4.38)$$

Therefore for  $x > c$

$$\begin{aligned}
& \int \frac{|x+c|^{-a}}{x+c} \left( \frac{x_k+c}{\omega'_X(x_k)} \cdot \frac{1}{(x-x_k)^2} + \frac{a+1}{\omega'_X(x_k)} \cdot \frac{1}{x-x_k} \right) dx \\
= & \frac{x_k+c}{\omega'_X(x_k)} \frac{(x+c)^{-a-1}}{x-x_k} - \frac{(a+1)(x_k+c)}{\omega'_X(x_k)} \int \frac{(x+c)^{-a-2}}{x-x_k} dx \\
& + \frac{a+1}{\omega'_X(x_k)} \int \frac{(x+c)^{-a-1}}{x-x_k} dx \\
= & \frac{x_k+c}{\omega'_X(x_k)} \frac{(x+c)^{-a-1}}{x-x_k} - \frac{1}{\omega'_X(x_k)} (x+c)^{-a-1} + C \\
= & -\frac{(x+c)^{-a}}{\omega'_X(x_k)(x-x_k)} + C. \tag{1.4.39}
\end{aligned}$$

By (1.4.28), (1.4.37) and (1.4.39), we then obtain

$$\begin{aligned}
A_k(x) &= \frac{\omega_X(x)}{\omega'_X(x_k)(x-x_k)} \\
&\quad -\omega_X(x)|x+c|^a \int (x+c)^{-a-1} (c_2 + \dots + c_N(x-x_k)^{N-2}) dx
\end{aligned}$$

which implies that  $A_k(x)$  is a polynomial and

$$A_k(x) = -\omega_X(x)|x+c|^a \int_{-c}^x \frac{|t+c|^{-a}}{t+c} \frac{\tilde{\omega}(x)}{\omega'_X(x_k)(t-x_k)^2 \prod_{i \neq k} (x_k-x_i)} U_k(x) dx. \tag{1.4.40}$$

Recall that  $U_k(x)$  is a linear function. Then we may write

$$U_k(x) = r_0 + r_1(x-x_k). \tag{1.4.41}$$



From (1.4.35) and (1.4.31),

$$\begin{aligned}
((a\omega_X(x) + (x+c)\omega'_X(x)U_k(x))'|_{x=x_k} &= \frac{a+1}{\omega'_X(x_k)} + \frac{(x_k+c)\omega''_X(x_k)}{(\omega'_X(x_k))^2} + (x_k+c)r_1\omega'_X(x_k) \\
&= \frac{a+1}{\omega'_X(x_k)}. \tag{1.4.42}
\end{aligned}$$

This together with (1.4.31) implies that

$$r_0 = \frac{1}{(\omega'_X(x_k))^2} \quad \text{and} \quad r_1 = -\frac{\omega''(x_k)}{(\omega'_X(x_k))^3}. \tag{1.4.43}$$

Finally,

$$A_k(x) = -\frac{\omega_X(x)}{|x+c|^{-a}} \int_{-c}^x \frac{|t+c|^{-a}}{t+c} \frac{a\omega_X(t) + (t+c)\omega'_X(t)}{\omega'_X(x_k)(t-x_k)^2 \prod_{i \neq k} (x_k - x_i)} \left(1 - \frac{\omega''_X(x_k)}{\omega'_X(x_k)}(t-x_k)\right) dt. \tag{1.4.44}$$

Using (1.4.37) and (1.4.39), we can verify that  $A_k, 2 \leq k \leq n$ , just defined satisfy the requirement (a) and (b) in (1.4.3).

□

## CHAPTER 2: Sampling Bilevel Signals With Finite Rate of Innovation

Sampling plays a very important role in signal processing. Many sampling techniques were developed in past last sixty years and they have been used extensively in engineering and life sciences, especially in processing audio signals, images and in communication channels. Sampling also brings into light some mathematical tools from Fourier analysis and approximation theory.

Bilevel signals are one of the simplest of signals but they have many important applications such as coding image, audio, seismic and ECG data. Some examples of bilevel signals are documents (where black indicates text and white indicates background), line art, hand-written signatures, bar codes, and vehicle license plates. All of which are frequently handled by machine vision systems for automatic recognition and identification. Vetterli, Marziliano and Blu show in [30] that a bilevel signal  $x$  can be reconstructed from its samples  $x * h(nT)$ ,  $n \geq 0$ , when the sampling kernel  $h$  is the box spline  $\chi_{[0,T)}$  (or the hat spline  $(1/T - |t|)\chi_{[-1/T,1/T)}$ ) and the sampling rate  $T$  is at (or above) the maximal local rate of innovation  $R$  of the signal  $x$ . In this section, we show that bilevel causal signals  $x$  are uniquely determined from their samples  $x * h(n/T)$ ,  $n \geq 1$ , if the causal sampling kernel  $h$  is positive on  $(0, T)$  and the sample rate  $T$  is at (or above) the maximal local rate of innovation  $R$  (see Theorem 1). Our numerical simulations indicate that the bilevel signal recovery procedure from noisy samples  $x * h(n/T) + \epsilon_n$ ,  $n \geq 1$ , is stable when the number of transition positions are not too large. This chapter is based on [15].

## 2.1 Bilevel signals with finite rate of innovation

---

A time signal is said to have *finite rate of innovation* if it can be determined by finitely many freedoms (free variables) per unit of time [31]. Prototype examples of signals with *finite rate of innovation* include bandlimited signals, time signals in shift-invariant spaces, delta pulses, signals in ultra-wide band communication, bilevel signals, and mass spectrometry data in medical diagnosis. There are several ways to define the rate of innovation [3, 19, 20, 29, 30, 31].

A bilevel time signal is a continuous time-signal that takes only two values 0 and 1. Denote by  $\chi_E$  the characteristic function on a set  $E$ . In this chapter, we consider causal bilevel signal  $x$  of the following type:

$$x(t) := \sum_{i=1}^N \chi_{[t_{2i-1}, t_{2i}]}(t), \quad (2.1.1)$$

where  $N \geq 0$ , and transition values (positions)  $t_i$ ,  $1 \leq i \leq 2N$ , satisfy

$$t_i < t_{i+1}, \quad 1 \leq i \leq 2N. \quad (2.1.2)$$

A bilevel signal  $x$  is said to have finite rate of innovation if its maximal local rate of innovation is finite. For the bilevel causal signal  $x$  in (2.1.1), define its *maximal local rate of innovation*  $R$  by reciprocal of the maximal positive number  $\tau_0$  such that there is at most one transition position  $t_i$ ,  $1 \leq i \leq 2N$ , in any time interval  $[t, t + \tau_0)$ ,  $t \geq 0$ , that is,

$$R = \sup_{1 \leq i < 2N} \frac{1}{t_{i+1} - t_i}. \quad (2.1.3)$$

## 2.2 Recovery of bilevel causal signals

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In this section, we provide a necessary condition on the sampling kernel  $h$  such that bilevel signals  $x$  can be uniquely determined from their samples. We also propose an algorithm for the stable recovery of bilevel signals.

**Theorem 2.2.1.** *If  $h$  is a causal sampling kernel with  $h(t) > 0$  on  $(0, 1/T)$ , then any bilevel causal signal  $x$  in (2.1.1) with the maximal local rate of innovation  $R$  less than or equal to the sampling rate  $T$  can be recovered from its samples  $x * h(n/T)$ ,  $n \geq 1$ .*

*Proof.* Let

$$H(t) = \int_0^t h(s) ds, \quad 0 \leq t \leq 1/T. \quad (2.2.1)$$

Then  $H(0) = 0$  and  $H$  is a strictly increasing function on  $[0, 1/T)$  as  $h$  is strictly positive on  $(0, 1/T)$ . Denote its inverse function on  $[0, T]$  by  $H^{-1} : [0, H(T)] \rightarrow [0, T]$ .

Let  $x$  be a bilevel causal signal in (2.1.1) with transition positions  $t_i$ ,  $1 \leq i \leq 2N$ , satisfying (2.1.2). Then its first sample  $y_1 = x * h(1/T)$  is given by

$$\begin{aligned} y_1 &= \int_0^{\infty} x(t) h\left(\frac{1}{T} - t\right) dt = \int_0^{1/T} x(t) h\left(\frac{1}{T} - t\right) dt \\ &= \int_0^{1/T} \chi_{[t_1, t_2)}(t) h\left(\frac{1}{T} - t\right) dt. \end{aligned}$$

Here the first two equalities hold due to the causality of the signal  $x$  and the sampling kernel  $h$ ,

and the third equality follows from (2.1.1) and the observation that

$$t_i \geq t_2 = (t_2 - t_1) + t_1 \geq \frac{1}{R} + 0 \geq \frac{1}{T}, \quad i \geq 2.$$

Thus,

$$y_1 = \int_0^{1/T} \chi_{[t_1, 1/T)}(t) h\left(\frac{1}{T} - t\right) dt = H(\max\{T^{-1} - t_1, 0\}). \quad (2.2.2)$$

Recall that  $H$  is strictly increasing on  $[0, 1/T)$ . Then there exists a transition position in the time interval  $[0, 1/T)$  if and only if  $y_1 = x * h(1/T) > 0$ . Moreover, if it exists, we can solve

$$H\left(\frac{1}{T} - t_1\right) = y_1$$

to obtain

$$t_1 = 1/T - H^{-1}(y_1). \quad (2.2.3)$$

Thus for a bilevel causal signal, we may determine from its first sample  $x * h(1/T)$  the existence (or nonexistence) of its transition position on the time period  $[0, 1/T)$ , and determine the transition in that time period if it does exist.

Inductively, we may assume that all transition positions of the bilevel signal  $x$  on the time interval  $[0, n/T)$  has been determined from its samples  $y_k = x * h(k/T)$ ,  $1 \leq k \leq n$ . We examine four cases to determine its transition location (if it exists) on the time period  $[n/T, (n+1)/T)$  from sample  $y_{n+1} = x * h((n+1)/T)$ .

**Case 1:** There is no transition position on the time interval  $[0, n/T)$ .

In this case, following above argument to determine transition positions in the interval  $[0, 1/T)$ ,

we have that there exists a transition position on  $[n/T, (n+1)/T)$  if and only if  $y_{n+1} > 0$ . If it exists, the transition position is the first one,  $t_1$ , of the bilevel signal  $x$  and

$$t_1 = \frac{n+1}{T} - H^{-1}(y_{n+1}). \quad (2.2.4)$$

**Case 2:** The last transition position on the time interval  $[0, n/T)$  is  $t_{2i_0-1}$  for some  $i_0 \geq 1$ .

In this case,  $t_{2i_0} \geq n/T$  and  $t_i \geq (n+1)/T$  for all  $i > 2i_0$ . Thus

$$\begin{aligned} y_{n+1} &= x * h\left(\frac{n+1}{T}\right) = \int_0^{\frac{n+1}{T}} x(t)h\left(\frac{n+1}{T} - t\right)dt \\ &= \int_0^{\frac{n+1}{T}} \left( \sum_{1 \leq i < i_0} \chi_{[t_{2i-1}, t_{2i}]}(t) + \chi_{[t_{2i_0-1}, \frac{n+1}{T}]}(t) \right) h\left(\frac{n+1}{T} - t\right)dt \\ &\quad - \int_{\frac{n}{T}}^{\frac{n+1}{T}} \chi_{[\min(t_{2i_0}, \frac{n+1}{T}), \frac{n+1}{T}]}(t)h\left(\frac{n+1}{T} - t\right)dt. \end{aligned}$$

Hence there exists a transition position  $t_{2i_0}$  in the time interval  $[n/T, (n+1)/T)$  if and only if

$$\tilde{y}_{n+1} := \int_0^{\frac{n+1}{T}} \left( \sum_{1 \leq i < i_0} \chi_{[t_{2i-1}, t_{2i}]}(t) + \chi_{[t_{2i_0-1}, \frac{n+1}{T}]}(t) \right) h\left(\frac{n+1}{T} - t\right)dt - y_{n+1} \quad (2.2.5)$$

is positive. Moreover, if  $\tilde{y}_{n+1} > 0$ , the transition position  $t_{2i_0}$  in the time interval  $[n/T, (n+1)/T)$  is determined by

$$t_{2i_0} = \frac{n+1}{T} - H^{-1}(\tilde{y}_{k+1}). \quad (2.2.6)$$

**Case 3:** The last transition position on the time interval  $[0, n/T)$  is  $t_{2i_0}$  for some  $1 \leq i_0 < N$ .

In this case, the  $(n + 1)$ -th sample  $y_{n+1} = x * h((n + 1)/T)$  is given by

$$\begin{aligned}
y_{n+1} &= \int_0^{n/T} \left( \sum_{i=1}^{i_0} \chi_{[t_{2i-1}, t_{2i}]}(t) \right) h\left(\frac{n+1}{T} - t\right) dt \\
&\quad + \int_{\min\{t_{2i_0+1}, \frac{n+1}{T}\}}^{\frac{n+1}{T}} h\left(\frac{n+1}{T} - t\right) dt.
\end{aligned} \tag{2.2.7}$$

Therefore, there exists a transition value  $t_{2i_0+1} \in [n/T, (n + 1)/T)$  if and only if

$$\tilde{y}_{n+1} := y_{n+1} - \int_0^{n/T} \left( \sum_{i=1}^{i_0} \chi_{[t_{2i-1}, t_{2i}]}(t) \right) h\left(\frac{n+1}{T} - t\right) dt \tag{2.2.8}$$

is positive. Also, we see that if  $\tilde{y}_{n+1} > 0$  then the transition value  $t_{2i_0} + 1$  can be obtained by

$$t_{2i_0+1} = \frac{n+1}{T} - H^{-1}(\tilde{y}_{n+1}). \tag{2.2.9}$$

**Case 4:** The last transition position on the time range  $[0, n/T)$  is  $t_{2N}$ . In this case, all transition positions of the bilevel signal,  $x$ , have been recovered already. Hence the bilevel signal,  $x$ , is fully recovered.

This completes the inductive proof. □

From the above argument of Theorem 1, we can use the following algorithm to recover a bilevel causal signal  $x$  in (2.1.1) from its samples  $x * h(n/T)$ ,  $1 \leq n \leq K$ , in the noiseless environment, where  $K > t_{2N}T$ .

**Bilevel Signal Recovery Algorithm:**

*Step 1:* If all samples  $y_n = x * h(n/T)$ ,  $n \geq 1$ , are zero, then the bilevel signal  $x$  is the zero signal, else find the first nonzero sample, say  $y_{n_0} > 0$ , the first transition location of the bilevel signal,  $x$ , is located at  $t_1 := n_0/T - H^{-1}(y_{n_0})$ , and set  $n = n_0$ .

*Step 2:* Do Step 2a if the last transition position on the time interval  $[0, n/T)$  is  $t_{2i_0-1}$  for some  $i_0 \geq 1$ , Step 2b else if the last transition location on the interval  $[0, n/T)$  is  $t_{2i_0}$  for some  $1 \leq i_0 < N$ , or else Step 4.

- *Step 2a:* Let  $\tilde{y}_{n+1}$  as in (2.2.5). Define  $t_{2i_0}$  as in (2.2.6) and set  $n = n + 1$  if  $\tilde{y}_{n+1} > 0$ .
- *Step 2b:* Let  $\tilde{y}_{n+1}$  as in (2.2.8). Define  $t_{2i_0+1}$  as in (2.2.9) and set  $n = n + 1$  if  $\tilde{y}_{n+1} > 0$ .

*Step 3:* Set  $n = n + 1$ . Do Step 2 if  $n < K$ , and Step 4 if  $n = K$ .

*Step 4:* Stop as all transition positions  $t_i$ ,  $1 \leq i \leq 2N$ , of bilevel signal  $x$  have been recovered.

### **2.3 Stable recovery of bilevel causal signals from noisy samples**

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In this section, we consider the stable recovery of a bilevel signal  $x$  in (2.1.1) from its noisy samples  $\{x * h(n/T) + \epsilon_n\}$  where  $\epsilon_n$ ,  $n \geq 1$ , are bounded noises.

First, we note that the sampling procedure from bilevel signals  $x$  to their samples are stable in bounded norm.

**Theorem 2.3.1.** *Let  $T > 0$ ,  $h$  be a bounded filter supported in  $[0, M/T)$ ,  $x(t) = \sum_{i=1}^N \chi_{[t_{2i-1}, t_{2i})}(t)$  be a bilevel causal signal with maximal local innovation rate  $R \leq T$ , and*

$$\tilde{x}(t) = \sum_{i=N}^N \chi_{[t_{2i-1} + \delta_{2i-1}, t_{2i} + \delta_{2i})}(t) \tag{2.3.1}$$



be a perturbation of the bilevel signal  $x$  with perturbed transition positions  $\{t_i + \delta_i\}_{i=1}^{2N}$  satisfying

$$\delta := \sup_{1 \leq i \leq 2N} |\tilde{t}_i - t_i| < \frac{1}{2R}. \quad (2.3.2)$$

Then the sample error between  $x * h(n/T)$  and  $\tilde{x} * h(n/T)$ ,  $n \geq 1$ , are dominated by  $(\lfloor \frac{MR}{T} \rfloor + 2)\|h\|_\infty \delta$ , i.e.,

$$\left| x * h\left(\frac{n}{T}\right) - \tilde{x} * h\left(\frac{n}{T}\right) \right| \leq \left( \lfloor \frac{MR}{T} \rfloor + 2 \right) \|h\|_\infty \delta, \quad n \geq 1, \quad (2.3.3)$$

where  $\|h\|_\infty$  is the  $L^\infty$  norm of the sampling kernel  $h$ .

*Proof.* By the assumption on maximal local innovation rate  $R$  of the bilevel signal  $x$  and the maximal transition position perturbation  $\delta$  between bilevel signals  $x$  and  $\tilde{x}$ , we have that

$$|x(t) - \tilde{x}(t)| = \sum_{i=1}^{2N} \chi_{t_i + [\min\{\delta_i, 0\}, \max\{\delta_i, 0\}]}(t). \quad (2.3.4)$$

Therefore,

$$\begin{aligned} & |x * h(n/T) - \tilde{x} * h(n/T)| \\ &= \left| \int_0^{n/T} (x(t) - \tilde{x}(t)) h(n/T - t) dt \right| \\ &\leq \delta \|h\|_\infty \int_{(n-M)/T}^{n/T} \sum_{i=1}^{2N} \chi_{t_i + [\min\{\delta_i, 0\}, \max\{\delta_i, 0\}]}(t) dt. \end{aligned} \quad (2.3.5)$$

Therefore,

$$\begin{aligned}
& |x * h(n/T) - \tilde{x} * h(n/T)| \\
& \leq \delta \|h\|_\infty \#\{t_i : t_i \in [(n-M)/T - \delta, n/T + \delta]\} \\
& \leq \delta \|h\|_\infty (\lfloor (M/T + 2\delta)/(1/R) \rfloor + 1) \\
& \leq \delta \|h\|_\infty (\lfloor MR/T \rfloor + 2),
\end{aligned}$$

where the first inequality holds as  $t_i \in [(n-m)/T - \delta, n/T + \delta]$  if the intersection of  $t_i + [\min(\delta_i - 0), \max(\delta_i, 0)]$  and  $[(n-M)/T, n/T]$  is nonempty, the second inequality holds true as  $t_{i+1} - t_i \geq 1/R$  for all  $1 \leq i < 2N$ , and the last inequality follows from the assumptions that  $\delta < 1/(2R)$  and  $R \leq T$ . This proves that the sampling error estimate (2.3.3) between the bilevel causal signals  $x$  and  $\tilde{x}$ .  $\square$

Now we consider recovering a bilevel signal  $x$  from its noisy samples  $\{x * h(n/T) + \epsilon_n\}$ , where  $\epsilon_n, n \geq 1$ , are bounded noises. Let us start this nonlinear problem by looking at two examples.

*Example 1:* Take

$$x_1(t) = \sum_{i=1}^{\infty} \chi_{[2i-1, 2i)}(t)$$

as the original bilevel signal and

$$h_1(t) = \chi_{[0, 2)}(t)$$

as the sampling kernel. For sufficiently small  $\epsilon > 0$ , define

$$x_{1,\epsilon} = \sum_{i=1}^{\infty} \chi_{[(1+\epsilon)(2i-1), 2(1+\epsilon)i)}(t). \quad (2.3.6)$$

Then for every,  $i \geq 1$ , the  $i$ -th innovation positions of bilevel signals  $x_1$  and  $x_{1,\epsilon}$  are  $i$  and  $i(1 + \epsilon)$  respectively (hence their difference is  $i\epsilon$  which could be arbitrarily large for sufficiently large  $i$ ). On the other hand, maximal sampling errors for those two bilevel signals  $x_1$  and  $x_{1,\epsilon}$  are bounded by  $\epsilon$  as

$$|x_{1,\epsilon} * h_1(n) - x_1 * h_1(n)| = |x_{1,\epsilon} * h_1(n) - 1| \leq \epsilon, \quad n \geq 1.$$

This leads to the instability of the recovery procedure from samples  $\{x_1 * h_1(n/T) + \epsilon_n\}$  to the bilevel signal  $x_1$  in the presence of bounded noises  $\{\epsilon_n\}$ .

*Example 2:* Take  $x_1$  and  $h_1$  in Example 1 as the original bilevel signal and the sampling kernel respectively. Define

$$x_{2,\epsilon} = \sum_{i=1}^{\infty} \chi_{[2i-1+\epsilon, 2i+\epsilon)}(t)$$

for sufficiently small  $\epsilon > 0$ . Then the difference between the  $i$ -th transition positions of bilevel signals  $x_1$  and  $x_{2,\epsilon}$  is always  $\epsilon_0$  for every  $i \geq 1$ , and there is no difference between their  $n$ -th samples except for  $n = 1$ . This suggests that the recovery procedure from samples  $\{x_1 * h_1(n/T) + \epsilon_n\}$  to the bilevel signal  $x_1$  is not locally-behaved and the reconstruction error on transition positions could disseminate.

From the above two examples, we see that the recovery procedure from samples  $\{x * h(n)\}$  to bilevel signals  $x$  is unstable in the presence of bounded noises and that it is globally-behaved in general. In the following, we present some numerical simulations with small number of transition positions and sampling rate over maximal local rate of innovation of bilevel signals.

Take a sampling kernel  $h_0(t) = (t + 1)/2\chi_{[0,1)}(t) + (2t - 1)\chi_{[1,2)}$ , and a bilevel signal

$$\begin{aligned}
 x_0(t) &= \chi_{[0.3791,1.9885)}(t) + \chi_{[3.1306,4.3440)}(t) \\
 &+ \chi_{[5.7552,7.1820)}(t) + \chi_{[8.7423,10.1052)}(t) \\
 &+ \chi_{[11.4200,12.6884)}(t)
 \end{aligned} \tag{2.3.7}$$

containing 10 transition positions, see Figure 2.1. Here the transition positions  $t_i^0$ ,  $1 \leq i \leq 10$ ,

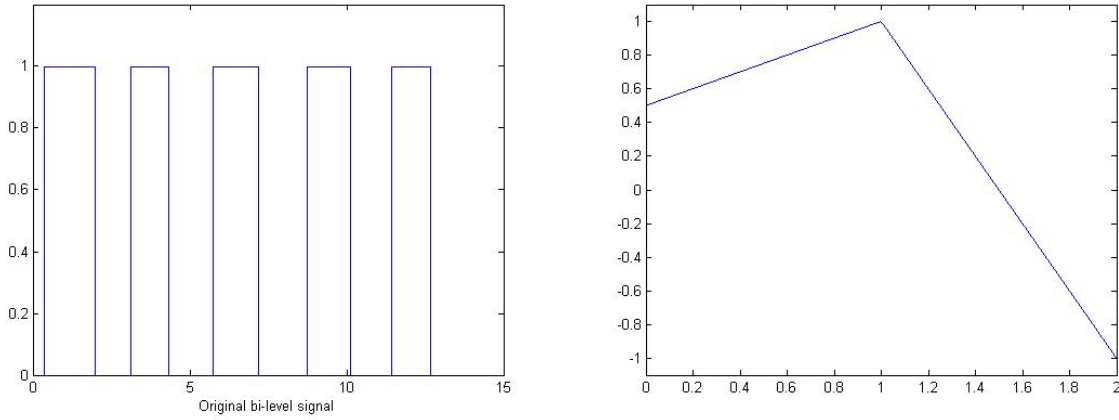


Figure 2.1: Bilevel signal  $x_0$  (left) and sampling kernel  $h_0$  (right)

of the bilevel signal  $x_0$  are randomly selected so that  $t_i^0 - t_{i-1}^0 \in [1.1, 1.9]$ ,  $2 \leq i \leq 10$ . The bilevel signal  $x_0$  in (2.3.7) has 0.8756 as its maximal local rate of innovation. We sample the convolution  $x_0 * h_1$  between  $x_0$  and  $h_1$  every second, which gives the sampling vectors  $Y_0 = (x_0 * h(1), \dots, x_0 * h(14))$ , and then add bounded random noise to the sampling vectors

$$Y_\delta = Y_0 + \delta(\epsilon_1, \dots, \epsilon_{14}) \tag{2.3.8}$$

where  $\delta \geq 0$  and  $\epsilon_i \in [-1, 1]$ ,  $1 \leq i \leq 14$ , are random noises. We apply the bilevel signal recovery

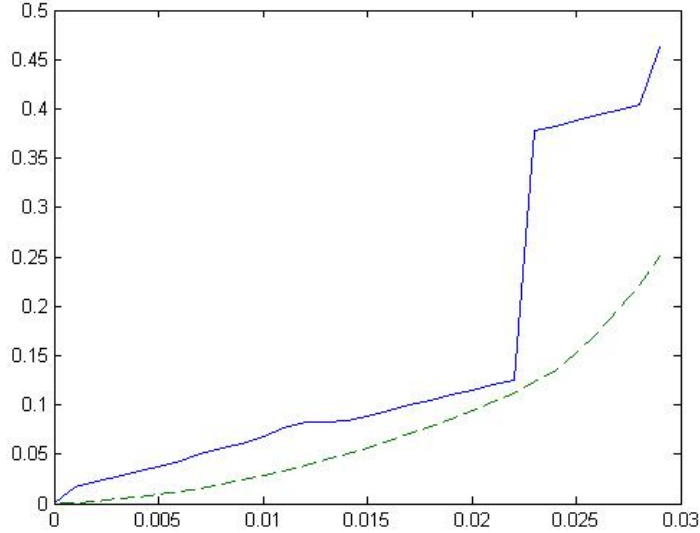


Figure 2.2: Maximal transition position error

algorithm in Section 2.2 and denote the reconstructed bilevel signal  $x_\delta$  with the first 10 transition positions being  $t_{1,\delta}, \dots, t_{10,\delta}$ . Define the maximal error of transition positions by

$$P(\delta) = \max_{1 \leq i \leq 10} |t_{i,\delta} - t_i^0|,$$

where  $t_1^0, \dots, t_{10}^0$  are transition positions of the bilevel signal  $x_0$ . We perform the recovery algorithm 50 times for every noise level  $\delta \in [0, 0.03]$ . The maximal value of  $P(\delta)$  after performing 50 times is plotted in Figure 2.2 with a solid line, while the average value of  $P(\delta)$  plotted with dashed line. Notice that  $\max_{1 \leq n \leq 14} |x_0 * h_1(n)| = 0.9796$ . So this numerical simulation indicates that our algorithm to recover the bilevel signal  $x_0$  from its noisy samples  $x_0 * h_0(n/T) + \epsilon_n$ ,  $n \geq 1$ , is reliable when the noise level  $\epsilon + \max_{n \geq 1} |\epsilon_n|$  is at (or below) 2% of the maximal sample value

$$\max_{n \geq 1} |x_0 * h_0(n/T)|.$$

In conclusion, we show that bilevel causal signals  $x$  could be reconstructed from their samples  $x * h(n/T)$ ,  $n \geq 1$ , if the sampling kernel  $h$  is causal and positive on  $(0, 1/T)$  and if the sample rate  $T$  is at (or above) the maximal local rate of innovation  $R$ . We also propose a stable bilevel signal recovery algorithm in the presence of bounded noise if the number of transition positions of bilevel signals is not large.

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