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Impulse Formulations Of The Euler Equations For Incompressible And Compressible Fluids

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IMPULSE FORMULATIONS OF THE EULER EQUATIONS FOR INCOMPRESSIBLE AND COMPRESSIBLE FLUIDS

by

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ABSTRACT

The purpose of this paper is to consider the impulse formulations of the Euler equations for incompressible and compressible fluids. Different gauges are considered. In particular, the Kuz'min gauge provides an interesting case as it allows the fluid impulse velocity to describe the evolution of material surface elements. This result affords interesting physical interpretations of the Kuz'min invariant. Some exact solutions in the impulse formulation are studied. Finally, generalizations to compressible fluids are considered as an extension of these results. The arrangement of the paper is as follows: in the first chapter we will give a brief explanation on the importance of the study of fluid impulse. In chapters two and three we will derive the Kuz'min, E & Liu, Maddocks & Pego and the Zero gauges for the evolution equation of the impulse density, as well as their properties. The first three of these gauges have been named after their authors. Chapter four will study two exact solutions in the impulse formulation. Physical interpretations are examined in chapter five. In chapter six, we will begin with the generalization to the compressible case for the Kuz'min gauge, based on Shivamoggi et al. (2007), and we will derive similar results for the remaining gauges. In Chapter seven we will examine physical interpretations for the compressible case.

This thesis is dedicated to my sons, Tyler and Matthew, my greatest gifts. Their selfless love and tacit support kept me moving forward throughout the years; to Shonna, master interpreter, with whom I endured difficulties and shared successes. Her support was always steadfast and, without her loving encouragement, this work would not have existed; to my parents, Ana and Carlos, and the outstanding job they performed in raising a man of discipline and direction; to Erika Blanken, the greatest colleague and friend anyone could ever have. Her support, companionship and teamwork made this journey endurable; to Staff Sergeant Félix García, USMC, the prototype of Honor, Courage and Commitment, whose lessons are forever engraved in my heart; to Gabi Booth, whose friendship I value enormously; to William Lai, best friend and brother; to Sabrina Massey, my dearly loved Texan sister; to Suzy McDowell and Barry Gibson, who believed in a young man's potential to advance as a professional; to Michelle McCraney, Paul Grau and the Advanced Technology College staff; to Frank Lombardo, Marc Campbell, and the entire Daytona Beach College family, who provided me the opportunity to pursue and complete this degree; to my students, past and present; to the Marines and Sailors of 4^{th} Medical Bn., Bravo Co., East Coast 1, 4^{th} Marine Logistics Group, Fleet Marine Force, with whom I have been honored to serve. *Semper Fidelis!*

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TABLE OF CONTENTS

LIST OF IDENTITIES AND CONVENTIONS viii

CHAPTER ONE: INTRODUCTION 1

CHAPTER TWO: IMPULSE FORMULATIONS 2

2.1 Kuz'min (Geometric) Gauge 4

2.2 E & Liu Gauge 5

2.3 Zero Gauge 5

2.4 Maddocks & Pego (Impetus) Gauge 7

CHAPTER THREE: GAUGE PROPERTIES 8

3.1 L^2 Norm 8

3.2 Total Impulse 9

3.3 Kelvin-Helmholtz Theorem 11

3.4 Kuz'min Invariant 12

3.5 Invariance of Impulse 14

CHAPTER FOUR: EXACT SOLUTIONS 16

4.1 Exact Solution 16

4.2 2π-periodic Solution 18

CHAPTER FIVE: PHYSICAL INTERPRETATIONS 20

5.1 Evolution of Line Elements 20

5.2 Evolution of Surface Elements 20

5.3 Conservation of Volume Elements 21

CHAPTER SIX: GENERALIZATIONS 22

6.1 General Results for the Kuz'min Gauge 22

6.2 General Results for the Zero Gauge 26

6.3 General Results for the E & Liu Gauge 26

6.4 General Results for the Maddocks & Pego Gauge 27

CHAPTER SEVEN: PHYSICAL INTERPRETATIONS FOR COMPRESSIBLE FLUIDS 30

7.1 Evolution of Line Elements 30

7.2 Evolution of Surface Elements 30

7.3 Conservation of Mass Elements 31

CHAPTER EIGHT: SUMMARY 32

LIST OF REFERENCES 33

LIST OF IDENTITIES AND CONVENTIONS

Identities

[1] $\nabla \times (\nabla \psi) = 0$

[2] $\nabla \cdot (\nabla \times \mathbf{A}) = 0$

[3] $\nabla \cdot (\nabla \psi) = \nabla^2 \psi$

[4] $\nabla \times \left(\nabla \times \mathbf{A}\right) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$

[5] $\nabla \cdot (\mathbf{A} + \mathbf{B}) = \nabla \cdot \mathbf{A} + \nabla \cdot \mathbf{B}$

[6] $\nabla \times (\mathbf{A} + \mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B}$

[7] $\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A})$

[8] $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B}$

[9] $\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}$

[10] $\nabla \cdot (\psi \mathbf{A}) = \mathbf{A} \cdot \nabla \psi + \psi \nabla \cdot \mathbf{A}$

[11] $\nabla \times (\psi \mathbf{A}) = \psi \nabla \times \mathbf{A} - \mathbf{A} \times \nabla \psi$

[12] $\nabla(\psi \phi) = \phi \nabla \psi + \psi \nabla \phi$

[13] $\frac{1}{2} \nabla A^2 = \mathbf{A} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{A}$

[14] $\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} = \sum_{i=1}^{n} \vec{e}_i \frac{\partial}{\partial x_i}$

[15] $\int_V \left(\nabla \cdot \mathbf{F}\right) dV = \int_{\partial V} \mathbf{F} \cdot \mathbf{n} dS$ (Divergence Theorem)

[16] $\int_V \left(\nabla \times \mathbf{F}\right) \cdot d\Omega = \int_{\partial V} \mathbf{F} \cdot d\mathbf{r}$ (Kelvin-Stokes Theorem)

Conventions

$\mathbf{u}$ Velocity at a specified time and position in space

$\mathbf{r}$ Position vector

$\mathbf{p} = \mathbf{u} + \nabla \phi$ Impulse density

$\Delta = \nabla \cdot \mathbf{u}$ Rate of expansion

$\boldsymbol{\omega} = \nabla \times \mathbf{u} = \nabla \times \mathbf{p}$ Vorticity

ρ Fluid density

$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla$ Operator of the material derivative

CHAPTER ONE: INTRODUCTION

The study of fluid dynamics of the Euler equations in terms of a momentum variable is of great interest. The reason behind such interest lies in the fact that the total momentum of a fluid is not well defined, especially if the velocity field does not vanish at infinity. It is therefore necessary to utilize a more suitable, physically significant measure which we call the *impulse* of the fluid. It is this quantity that has the property of acting as the fluid's momentum, in the form of a convergent integral. The Euler equations for incompressible fluid can be written in terms of a vector field **p**, commonly known as the *impulse density*. The relationship between **p** and the fluid velocity, **u**, is that the latter is the divergenceless projection of **p**. Different impulse formulations are possible, contingent on using different gauge conditions imposed on the impulse density. Given our choice of gauge, an evolution equation for **p** will describe the motion of the fluid. The Kuz'min gauge is of special importance because it describes the evolution of material surface elements in the fluid. Not surprisingly, this gauge has been widely studied. In the following two chapters, we will explore the fluid impulse formulations by using the most common gauges and then examine some of the properties of these gauges. We will discuss some exact solutions of these formulations in chapter four. We will explore the existence of some local invariants in the impulse formulations and discuss the physical interpretations in chapter five. The interest in local invariants, as opposed to standard (global) invariants, arises due to the lack of usefulness of the latter to keep track of changes in vortex line topology, which is local in nature. Local invariants are able to provide the track of such changes in vortex line topology. These local invariants are provided in the fluid impulse formulations. In this paper we will provide the generalizations of the impulse formulation for the Zero, E & Liu, and Maddocks & Pego gauges (see Chapter Six). We will finally extend these formulations in the compressible case in chapter seven.

CHAPTER TWO: IMPULSE FORMULATIONS

Fluid impulse P is the integral of dispersed force impulse that would produce motion of a given fluid, when applied at any moment to its volume Ω (or at its bounding surface). For an ideal fluid (unbounded and incompressible) motionless at infinity, with bounded vorticity distribution, the fluid impulse is:

$$P = \frac{1}{2}\rho\int_{\Omega}\mathbf{r}\times\boldsymbol{\omega}d\mathbf{v}$$

To describe the evolution equations, we consider the Euler equations describing the motion of an incompressible fluid in $\mathbb{R}^n$ (n = 2, 3):

$$(2.1) \qquad \nabla\cdot\mathbf{u} = 0$$

$$(2.2) \qquad \frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla\left(\frac{\mathrm{P}}{\rho}\right)$$

where **u** is the *fluid velocity*, ρ is the *density*, and P is the *pressure*. Introduce the *impulse density* as:

$$(2.3) \qquad \mathbf{p} = \mathbf{u} + \nabla\phi$$

The vector field **p** is the divergenceless projection of the fluid velocity **u** ($\nabla\cdot\mathbf{u} = 0$). Both have the same vorticity, or local rotation ($\nabla\times\mathbf{u} = \nabla\times\mathbf{p} = \boldsymbol{\omega}$), and differ only by the gradient of the scalar field ϕ. By making use of identity [13] and setting ρ = 1 we can rewrite equation (2.2),

$$\frac{\partial\mathbf{u}}{\partial t} + \frac{1}{2}\nabla u^2 - \mathbf{u}\times(\nabla\times\mathbf{u}) = -\nabla\mathrm{P}$$

This is,

$$\frac{\partial\mathbf{u}}{\partial t} - \mathbf{u}\times(\nabla\times\mathbf{u}) = -\nabla\left(\mathrm{P} + \frac{u^2}{2}\right)$$

By replacing **u** with **p** through manipulation,

$$\frac{\partial}{\partial t}(\mathbf{p}-\nabla\phi)-\mathbf{u}\times\left[\nabla\times(\mathbf{p}-\nabla\phi)\right]=-\nabla\left(\mathrm{P}+\frac{u^2}{2}\right)$$

Distributing the differential operator and expanding the curl inside brackets,

$$\frac{\partial\mathbf{p}}{\partial t}-\frac{\partial\nabla\phi}{\partial t}-\mathbf{u}\times\left[(\nabla\times\mathbf{p})-(\nabla\times\nabla\phi)\right]=-\nabla\left(\mathrm{P}+\frac{u^2}{2}\right)$$

In utilizing identity [1],

$$\frac{\partial\mathbf{p}}{\partial t}-\mathbf{u}\times(\nabla\times\mathbf{p})=-\nabla\left(\mathrm{P}+\frac{u^2}{2}\right)+\frac{\partial\nabla\phi}{\partial t}$$

Factoring the *Del* operator, we obtain,

$$\frac{\partial\mathbf{p}}{\partial t}-\mathbf{u}\times(\nabla\times\mathbf{p})=-\nabla\left(\mathrm{P}+\frac{u^2}{2}-\frac{\partial\phi}{\partial t}\right)$$

Define now a scalar field $\Lambda=\mathrm{P}+\frac{u^2}{2}-\frac{\partial\phi}{\partial t}$. We obtain

$$(2.4)\qquad \frac{\partial\mathbf{p}}{\partial t}-\mathbf{u}\times(\nabla\times\mathbf{p})=-\nabla\Lambda$$

In tensor notation, (2.4) is,

$$\frac{\partial p_i}{\partial t}-\varepsilon_{ijk}u_j\varepsilon_{klm}\frac{\partial}{\partial x_l}p_m=-\frac{\partial}{\partial x_i}\Lambda$$

Λ is called the *gauge* and its various forms will dictate the behavior of the evolution equation (2.4).

■

In the following pages, we will now discuss the most commonly used gauges in the fluid impulse formulations beginning with the customarily used Kuz'min gauge. We shall use tensor notation for simplicity purposes.

2.1 <u>Kuz'min (Geometric) Gauge</u>

Let $\Lambda = u_j p_j$, then (2.4) becomes

$$\frac{\partial p_i}{\partial t} - \varepsilon_{kij} u_j \varepsilon_{klm} \frac{\partial}{\partial x_l} p_m = -\frac{\partial}{\partial x_i} u_j p_j$$

We first develop the left-hand side (LHS),

$$\frac{\partial p_i}{\partial t} - \varepsilon_{kij} u_j \varepsilon_{klm} \frac{\partial}{\partial x_l} p_m = \frac{\partial p_i}{\partial t} - \varepsilon_{kij} \varepsilon_{klm} u_j \frac{\partial}{\partial x_l} p_m$$

Rewriting the *Levi-Civita* operator into *Kronecker* delta operator, we obtain

$$\frac{\partial p_i}{\partial t} - \varepsilon_{kij} \varepsilon_{klm} u_j \frac{\partial}{\partial x_l} p_m = \frac{\partial p_i}{\partial t} - \left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right) u_j \frac{\partial}{\partial x_l} p_m$$

Substituting back, we find

$$(2.5) \qquad \frac{\partial p_i}{\partial t} - u_j \frac{\partial p_j}{\partial x_i} + u_j \frac{\partial p_i}{\partial x_j} = -\frac{\partial}{\partial x_i} u_j p_j$$

And the right-hand side (RHS) is

$$-\frac{\partial}{\partial x_i} u_j p_j = -u_j \frac{\partial}{\partial x_i} p_j - p_j \frac{\partial}{\partial x_i} u_j$$

Substituting back, we obtain,

$$\frac{\partial p_i}{\partial t} - u_j \frac{\partial p_j}{\partial x_i} + u_j \frac{\partial p_i}{\partial x_j} = -u_j \frac{\partial p_j}{\partial x_i} - p_j \frac{\partial u_j}{\partial x_i}$$

This yields,

$$(2.6) \qquad \frac{\partial p_i}{\partial t} + u_j \frac{\partial p_i}{\partial x_j} = -p_j \frac{\partial u_j}{\partial x_i}. \qquad \blacksquare$$

2.2 E & Liu Gauge

Let $\Lambda = \frac{1}{2} u_j u_j$, then (2.5) becomes

$$\frac{\partial p_i}{\partial t} - u_j \frac{\partial p_j}{\partial x_i} + u_j \frac{\partial p_i}{\partial x_j} = -\frac{1}{2} \frac{\partial}{\partial x_i} u_j u_j$$

Expanding the RHS,

$$\frac{\partial p_i}{\partial t} - u_j \frac{\partial p_j}{\partial x_i} + u_j \frac{\partial p_i}{\partial x_j} = -u_j \frac{\partial}{\partial x_i} u_j$$

Rearranging,

$$\frac{\partial p_i}{\partial t} + u_j \frac{\partial p_i}{\partial x_j} = -u_j \frac{\partial u_j}{\partial x_i} + u_j \frac{\partial p_j}{\partial x_i}$$

Factoring the partial differential and using equation (2.3),

$$\frac{\partial p_i}{\partial t} + u_j \frac{\partial p_i}{\partial x_j} = u_j \frac{\partial}{\partial x_i} \left(u_j + \frac{\partial \phi}{\partial x_j} - u_j \right)$$

This is,

$$\frac{\partial p_i}{\partial t} + u_j \frac{\partial p_i}{\partial x_j} = u_j \frac{\partial}{\partial x_i} \frac{\partial \phi}{\partial x_j}$$

so

$$\text{(2.7)} \qquad \frac{\partial p_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = 0. \qquad \blacksquare$$

2.3 Zero Gauge

Let $\Lambda = 0$, so that we immediately obtain

$$\text{(2.8)} \qquad \frac{\partial p_i}{\partial t} - \varepsilon_{kij} u_j \varepsilon_{klm} \frac{\partial}{\partial x_l} p_m = 0$$

or equivalently

$$\frac{\partial p_i}{\partial t} + \left(\delta_{im}\delta_{jl} - \delta_{il}\delta_{jm}\right) u_j \frac{\partial}{\partial x_l} p_m = 0$$

And applying the *Kronecker* delta on u_j only,

$$\frac{\partial p_i}{\partial t} + \left(\delta_{im} u_l - \delta_{il} u_m\right) \frac{\partial}{\partial x_l} p_m = 0$$

$$\frac{\partial p_i}{\partial t} + A_{mi}^l \frac{\partial}{\partial x_l} p_m = 0$$

where $A_{mi}^l \equiv \delta_{im} u_l - \delta_{il} u_m$. The significance of this partial differential equation (PDE) is that it will be hyperbolic or degenerate hyperbolic depending on the eigenvalues of the matrix

$$A_{mi}^l n^l = \begin{pmatrix} u_2 n_2 + u_3 n_3 & -u_2 n_1 & -u_3 n_1 \\ -u_1 n_2 & u_1 n_1 + u_3 n_3 & -u_3 n_2 \\ -u_1 n_3 & -u_2 n_3 & u_1 n_1 + u_2 n_2 \end{pmatrix}$$

When we calculate $\det\left(A_{mi}^l n^l - \lambda I\right) = 0$, we obtain the eigenvalues

$$\lambda = \left\{0, u_j n_j, u_j n_j\right\}$$

By substituting $\lambda = 0$, we can solve the system

$$\begin{pmatrix} u_2 n_2 + u_3 n_3 - \lambda & -u_2 n_1 & -u_3 n_1 \\ -u_1 n_2 & u_1 n_1 + u_3 n_3 - \lambda & -u_3 n_2 \\ -u_1 n_3 & -u_2 n_3 & u_1 n_1 + u_2 n_2 - \lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

This leads to the conclusion that for $\lambda = 0$, the corresponding eigenvector is the normal vector **n**.

Since if $(x_1, x_2, x_3)^T = (n_1, n_2, n_3)^T$ then we can verify that

$$\begin{aligned} &u_2 n_2 n_1 + u_3 n_3 n_1 - u_2 n_1 n_2 - u_3 n_1 n_3 = 0 \\ &-u_1 n_2 n_1 + u_1 n_1 n_2 + u_3 n_3 n_2 - u_3 n_2 n_3 = 0 \\ &-u_1 n_3 n_1 - u_2 n_3 n_2 + u_1 n_1 n_3 - u_2 n_2 n_3 = 0 \end{aligned}$$

For the case where $\lambda = \mathbf{u}\cdot\mathbf{n} = 0$, the system does not have independent eigenvectors and thus it is degenerate hyperbolic. ■

2.4 Maddocks & Pego (Impetus) Gauge

Let $\Lambda = u_j p_j - \frac{1}{2}u_j u_j$. By replacing on (2.5),

$$\frac{\partial p_i}{\partial t} - u_j \frac{\partial p_j}{\partial x_i} + u_j \frac{\partial p_i}{\partial x_j} = -\frac{\partial}{\partial x_i}\left(u_j p_j - \frac{1}{2}u_j u_j\right)$$

which is

$$\frac{\partial p_i}{\partial t} - u_j \frac{\partial p_j}{\partial x_i} + u_j \frac{\partial p_i}{\partial x_j} = -u_j \frac{\partial p_j}{\partial x_i} - p_j \frac{\partial u_j}{\partial x_i} + u_j \frac{\partial u_j}{\partial x_i}$$

Further reduction yields

$$\frac{\partial p_i}{\partial t} + u_j \frac{\partial p_i}{\partial x_j} = \left(u_j - p_j\right)\frac{\partial u_j}{\partial x_i}$$

And in using (2.3) we readily see that

$$\text{(2.9)} \qquad \frac{\partial p_i}{\partial t} + u_j \frac{\partial p_i}{\partial x_j} = -\frac{\partial \phi}{\partial x_j}\frac{\partial u_j}{\partial x_i}. \qquad ■$$

CHAPTER THREE: GAUGE PROPERTIES

3.1 L^2 Norm

Theorem (1). For a given flow on a fixed domain Ω, the value of **p** that has the minimal L^2 norm is obtained for **p** = **u**.

Proof. By definition the L^p norm is:

$$\| f \|_{\Omega}^{p} = \left(\int_{\Omega} | f(x) |^{p} dx \right)^{\frac{1}{p}}$$

Substituting **p**,

$$\|\mathbf{p}\|^2 = \int_{\Omega} |\mathbf{u} + \nabla \phi|^2 dx$$

Expanding the square,

$$\int_{\Omega} |\mathbf{u} + \nabla \phi|^2 dx = \int_{\Omega} \left(|\mathbf{u}|^2 + 2(\mathbf{u} \cdot \nabla \phi) + |\nabla \phi|^2 \right) dx$$

By making use of the divergence theorem,

$$\int_{\Omega} |\mathbf{u} + \nabla \phi|^2 dx = \int_{\Omega} \left(|\mathbf{u}|^2 + |\nabla \phi|^2 \right) dx + \int_{\partial\Omega} 2(\mathbf{u} \cdot \mathbf{n}) \phi dS$$

But $\mathbf{u} \cdot \mathbf{n} = 0$ everywhere on the surface $\partial\Omega$, thus

$$\|\mathbf{p}\|^2 = \int_{\Omega} \left(|\mathbf{u}|^2 + |\nabla \phi|^2 \right) dx. \quad \blacksquare \tag{3.1}$$

Proof for the Kuz'min gauge. By taking the norm of the gauge,

$$\|\mathbf{u} \cdot \mathbf{p}\|^2 = \int_{\Omega} |\mathbf{u} \cdot (\mathbf{u} + \nabla \phi)|^2 dx$$

Distributing the dot product,

$$\|\mathbf{u}\cdot\mathbf{p}\|^2 = \int_\Omega \left|\mathbf{u}^2 + (\mathbf{u}\cdot\nabla\phi)\right|^2 dx$$

This is equivalent to

$$\|\mathbf{u}\cdot\mathbf{p}\|^2 = \int_\Omega \left[\left(\mathbf{u}^2\right)^2 + 2\mathbf{u}^2(\mathbf{u}\cdot\nabla\phi) + (\mathbf{u}\cdot\nabla\phi)^2\right] dx$$

As above, using the divergence theorem and $\mathbf{u}\cdot\mathbf{n} = 0$ on $\partial\Omega$:

$$\|\mathbf{u}\cdot\mathbf{p}\|^2 = \int_\Omega \left[\left(\mathbf{u}^2\right)^2 + (\mathbf{u}\cdot\nabla\phi)^2 - \phi(\mathbf{u}\cdot\nabla)2u^2\right] dx$$

In the case where **p** = **u**, we obtain

$$\|\mathbf{u}\cdot\mathbf{u}\|^2 = \int_\Omega \left(\mathbf{u}^2\right)^2 dx$$

Thus, the minimal L^2 norm is given when **p** = **u**. ■

3.2 Total Impulse

Theorem (2). Total fluid impulse is

$$P = \int_\Omega \mathbf{p}\,dV + \frac{1}{2}\int_{\partial\Omega} \left[(\mathbf{r}\cdot\mathbf{p})\mathbf{n} - (\mathbf{r}\cdot\mathbf{n})\mathbf{p}\right] dS$$

where P is the fluid impulse.

Proof. Recall from Batchelor (1967) that for unit density,

$$(3.2) \qquad P = \frac{1}{2}\int_\Omega \mathbf{r}\times\boldsymbol{\omega}\,d\mathbf{v}$$

Further, recall that $\boldsymbol{\omega} = \nabla\times\mathbf{u} = \nabla\times\mathbf{p}$, thus

$$P = \frac{1}{2}\int_\Omega \mathbf{r}\times(\nabla\times\mathbf{p})\,d\mathbf{v}$$

Change to tensors to ease in computation,

$$P=\frac{1}{2}\int_{\Omega}\varepsilon_{ijk}r_j\varepsilon_{klm}\frac{\partial p_m}{\partial x_l}dV$$

On rearranging the *Levi-Civita* operator,

$$P=\frac{1}{2}\int_{\Omega}\varepsilon_{kij}\varepsilon_{klm}r_j\frac{\partial p_m}{\partial x_l}dV$$

which is equivalent to

$$P=\frac{1}{2}\int_{\Omega}\left(\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl}\right)r_j\frac{\partial p_m}{\partial x_l}dV$$

And so,

$$\text{(3.3)} \qquad P=\frac{1}{2}\left(\int_{\Omega}r_j\frac{\partial p_j}{\partial x_i}dV-\int_{\Omega}r_j\frac{\partial p_i}{\partial x_j}dV\right)$$

Consider the multivariate integration by parts,

$$\int_{\Omega}\frac{\partial u}{\partial x_i}v\,dx=\int_{\partial\Omega}uvn_i\,d\sigma-\int_{\Omega}u\frac{\partial v}{\partial x_i}dx$$

where n_i is the component form of the normal $\mathbf{n}$. We can apply this to (3.3),

$$P=\frac{1}{2}\left(\int_{\partial\Omega}p_jr_jn_i\,dS-\int_{\Omega}p_j\frac{\partial r_j}{\partial x_i}dV-\int_{\partial\Omega}p_ir_jn_j\,dS+\int_{\Omega}p_i\frac{\partial r_j}{\partial x_j}dV\right)$$

Rearranging,

$$P=\frac{1}{2}\left[\int_{\partial\Omega}\left(p_jr_jn_i-p_ir_jn_j\right)dS+\int_{\Omega}\left(-p_j\frac{\partial r_j}{\partial x_i}+p_i\frac{\partial r_j}{\partial x_j}\right)dV\right]$$

where the partial differentials are

$$\frac{\partial r_j}{\partial x_i}=\delta_{ji}=1 \quad \text{and} \quad \frac{\partial r_j}{\partial x_j}=\delta_{jj}=3$$

Thus

$$(3.4) \qquad P = \frac{1}{2}\int_{\partial\Omega}\left(p_j r_j n_i - p_i r_j n_j\right)dS + \int_{\Omega} p_i dV$$

or

$$(3.5) \qquad P = \int_{\Omega}\mathbf{p}dV + \frac{1}{2}\int_{\partial\Omega}\left[(\mathbf{r}\cdot\mathbf{p})\mathbf{n} - (\mathbf{r}\cdot\mathbf{n})\mathbf{p}\right]dS$$

If we now choose appropriate boundary conditions to eliminate the surface integral, then

$$P = \int_{\Omega}\mathbf{p}dV$$

and hence, **p** is the fluid impulse density. ■

3.3 <u>Kelvin-Helmholtz Theorem</u>

Theorem (3). Given (2.3), then the Kelvin-Helmholtz Theorem holds for **p**, such that

$$\frac{d}{dt}\oint_{\Omega}\mathbf{p}\cdot dl = 0$$

where Ω is a closed material curve in the fluid.

Proof. The total derivative of the closed material curve integral of **u** can be written as

$$\frac{d}{dt}\oint_{\Omega}\mathbf{u}\cdot d\mathbf{l} = \oint_{\Omega}\frac{d\mathbf{u}}{dt}\cdot d\mathbf{l} + \oint_{\Omega}\mathbf{u}\cdot\frac{d\left(d\mathbf{l}\right)}{dt}$$

Substituting (2.3) into the equation, we see that the first term in the RHS is,

$$\oint_{\Omega}\frac{d\mathbf{u}}{dt}\cdot d\mathbf{l} = \oint_{\Omega}\frac{d\left(\mathbf{p} - \nabla\phi\right)}{dt}\cdot d\mathbf{l}$$

or

$$\oint_{\Omega} \frac{d\mathbf{u}}{dt} \cdot d\mathbf{l} = \oint_{\Omega} \frac{d\mathbf{p}}{dt} \cdot d\mathbf{l}$$

On examination of the second term in the RHS,

$$\oint_{\Omega} \mathbf{u} \cdot \frac{d(d\mathbf{l})}{dt} = \oint_{\Omega} \mathbf{u} \cdot d\mathbf{u} = \oint_{\Omega} u_1 du_1 + u_2 du_2 + u_3 du_3$$

This can be written as,

$$\oint_{\Omega} u_1 du_1 + u_2 du_2 + u_3 du_3 = \oint_{\Omega} d\left(u_1^2 + u_2^2 + u_3^2\right) = \oint_{\Omega} d\mathbf{u}^2 = 0$$

Ultimately, we can see that

$$\frac{d}{dt} \oint_{\Omega} \mathbf{u} \cdot d\mathbf{l} = \frac{d}{dt} \oint_{\Omega} \mathbf{p} \cdot d\mathbf{l}$$

And by the Kelvin-Helmholtz Theorem,

$$\frac{d}{dt} \oint_{\Omega} \mathbf{u} \cdot d\mathbf{l} = 0$$

This implies that

$$\text{(3.6)} \qquad \frac{d}{dt} \oint_{\Omega} \mathbf{p} \cdot d\mathbf{l} = 0$$

or

$$\text{(3.7)} \qquad \oint_{\Omega} \mathbf{p} \cdot d\mathbf{l} = k \quad (k \text{ constant}). \qquad \blacksquare$$

3.4 Kuz'min Invariant

Theorem (4). In the Kuz'min gauge, there exists a local invariant

$$\mathbf{p} \cdot \boldsymbol{\omega} = \textit{constant.}$$

Proof. Consider the curl of equation (2.4) and identity [4] such that

$$\nabla\times\left(\frac{\partial\mathbf{p}}{\partial t}-\mathbf{u}\times(\nabla\times\mathbf{p})\right)=\nabla\times(-\nabla\Lambda)=0$$

Given that $\nabla \times \mathbf{p} = \boldsymbol{\omega}$,

$$\nabla\times\left(\frac{\partial\mathbf{p}}{\partial t}-\mathbf{u}\times\boldsymbol{\omega}\right)=0$$

On distributing the curl,

$$\text{(3.8)}\qquad \frac{\partial(\nabla\times\mathbf{p})}{\partial t}=\nabla\times(\mathbf{u}\times\boldsymbol{\omega})$$

In applying identity [9],

$$\frac{\partial\boldsymbol{\omega}}{\partial t}=\mathbf{u}(\nabla\cdot\boldsymbol{\omega})-\boldsymbol{\omega}(\nabla\cdot\mathbf{u})+(\boldsymbol{\omega}\cdot\nabla)\mathbf{u}-(\mathbf{u}\cdot\nabla)\boldsymbol{\omega}$$

As a consequence of identity [2] and due to incompressibility (in 3D),

$$\text{(3.9)}\qquad \frac{\partial\boldsymbol{\omega}}{\partial t}+(\mathbf{u}\cdot\nabla)\boldsymbol{\omega}=(\boldsymbol{\omega}\cdot\nabla)\mathbf{u}$$

This can be written as a material derivative,

$$\text{(3.10)}\qquad \frac{D\boldsymbol{\omega}}{Dt}=(\boldsymbol{\omega}\cdot\nabla)\mathbf{u}$$

Now, consider the Kuz'min gauge, equation (2.6), in vector notation,

$$\text{(3.11)}\qquad \frac{\partial\mathbf{p}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{p}=-(\nabla\mathbf{u})^{T}\,\mathbf{p}$$

If we now use (3.9) and (3.11), we can readily see that

$$\text{(3.12)}\qquad \frac{D(\mathbf{p}\cdot\boldsymbol{\omega})}{Dt}=0$$

since equating the two RHS gives

$$(\mathbf{p}\cdot\boldsymbol{\omega})\cdot\nabla\mathbf{u}=-(\nabla\mathbf{u})^{T}(\mathbf{p}\cdot\boldsymbol{\omega})=0$$

An implication of (3.12) is that

$$\mathbf{p} \cdot \boldsymbol{\omega} = constant$$

This result leads to the physical implication that the volume of the fluid element remains unchanged. This is, of course, conservation of mass since the fluid is incompressible (see Chapter Five). ■

3.5 Invariance of Impulse

Theorem (5). The impulse *P* required to generate motion of the fluid at rest is independent of time such that

$$\frac{dP}{dt} = 0$$

Proof. If we consider the time derivative of equation (3.2), then

$$\frac{dP}{dt} = \frac{1}{2}\rho \int_{\Omega} \mathbf{r} \times \frac{\partial \boldsymbol{\omega}}{\partial t} d\mathbf{v}$$

and utilizing (3.8), we obtain

$$\frac{dP}{dt} = \frac{1}{2}\rho \int_{\Omega} \mathbf{r} \times \left[\nabla \times (\mathbf{u} \times \boldsymbol{\omega})\right] d\mathbf{v}$$

If we let $\alpha = \mathbf{u} \times \boldsymbol{\omega}$, then in changing to tensor notation, the integrand becomes

$$\mathbf{r} \times \left[\nabla \times (\mathbf{u} \times \boldsymbol{\omega})\right] = \varepsilon_{ijk} r_j \varepsilon_{klm} \frac{\partial}{\partial x_l} \alpha_m = \varepsilon_{kij} \varepsilon_{klm} r_j \frac{\partial}{\partial x_l} \alpha_m$$

On changing to *Kronecker* delta

$$\varepsilon_{kij} \varepsilon_{klm} r_j \frac{\partial}{\partial x_l} \alpha_m = \left(\delta_{il}\delta_{jm} - \delta_{lm}\delta_{jl}\right) r_j \frac{\partial}{\partial x_l} \alpha_m$$

If we make use of the identity $\delta_{ij} x_j = x_i$, then

$$\left(\delta_{il}\delta_{jm} - \delta_{lm}\delta_{jl}\right) r_j \frac{\partial}{\partial x_l} \alpha_m = \frac{\partial}{\partial x_i}\left(r_j \alpha_j\right) - \alpha_i \left(r_j \frac{\partial}{\partial x_j}\right)$$

If we now change to vector notation, we see that

$$(3.13)\quad \frac{dP}{dt} = \frac{1}{2}\rho\int_{\Omega} 2\alpha_i dV = \rho\int_{\Omega} \mathbf{u}\times\boldsymbol{\omega} d\mathbf{v}$$

since the surface integrals resulting by application of the divergence theorem vanish at infinity.

Rewriting (3.13) as

$$\frac{dP}{dt} = \rho\int_{\Omega} \mathbf{u}\times(\nabla\times\mathbf{u}) d\mathbf{v}$$

and using identity [13] yields

$$\frac{dP}{dt} = \rho\int_{\Omega}\left(\frac{1}{2}\nabla\mathbf{u}^2 - (\mathbf{u}\cdot\nabla)\mathbf{u}\right)d\mathbf{v}$$

On transforming to surface integral by means of the divergence theorem,

$$\frac{dP}{dt} = \rho\int_{\partial\Omega}\left(\frac{1}{2}\mathbf{n}\mathbf{u}^2 - (\mathbf{u}\cdot\mathbf{n})\mathbf{u}\right)\cdot d\mathbf{S}$$

where the velocity **u** vanishes at infinity. Thus,

$$(3.14)\quad \frac{dP}{dt} = 0. \quad \blacksquare$$

CHAPTER FOUR: EXACT SOLUTIONS

4.1 Exact Solution

Let's consider the following exact solution for the two-dimensional Euler equations, as studied by Russo & Smereka (1999):

$$\omega = f(r) \quad \text{where} \quad r = \sqrt{x^2 + y^2},$$

$$\mathbf{u} = U(r)\hat{\mathbf{u}}_\theta = <0, U(r)> \quad \text{where} \quad U(r) = \frac{1}{r}\int_0^r f(s)s\,ds,$$

Recall equation (2.6),

$$\frac{\partial p_i}{\partial t} + u_j \frac{\partial p_i}{\partial x_j} = -p_j \frac{\partial u_j}{\partial x_i}$$

and

$$\nabla \cdot \mathbf{u} = \frac{1}{r}\frac{\partial (r u_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} = 0$$

The r component is

$$\frac{\partial p_r}{\partial t} + u_r \frac{\partial p_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial p_r}{\partial \theta} - \frac{u_\theta p_\theta}{r} = -p_\theta\left(\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\right) - p_r\left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r}\right)$$

which reduces to

$$\frac{\partial p_r}{\partial t} + u_r \frac{\partial p_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial p_r}{\partial \theta} = -p_\theta \frac{\partial u_\theta}{\partial r} - p_r\left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r}\right)$$

But since the r and θ components of **u** are $u_r = 0$, $u_\theta = U(r)$ then,

$$\frac{\partial p_r}{\partial t} + \frac{U(r)}{r}\frac{\partial p_r}{\partial \theta} = -p_\theta \frac{\partial U(r)}{\partial r}$$

Since the p_r component is independent of θ,

$$\frac{\partial p_r}{\partial t} = -p_\theta U'(r)$$

And integrating with respect to *t*,

$$p_r = -p_\theta U'(r) t \tag{4.1}$$

Next, the θ component is

$$\frac{\partial}{\partial t}(rp_\theta) + \frac{u_\theta}{r}\frac{\partial}{\partial \theta}(rp_\theta) = 0$$

which implies that the total derivative,

$$\frac{D}{Dt}(rp_\theta) = 0$$

Thus, the term rp_θ is a function of r,

$$rp_\theta = F(r)$$

We obtain:

$$p_\theta = \frac{1}{r}F(r)$$

And since $\mathbf{p} = \mathbf{u} + \nabla\phi$, then the θ component of $\mathbf{p}$ is,

$$p_\theta = u_\theta + \frac{1}{r}\frac{\partial \phi}{\partial \theta}$$

And we observe that

$$\frac{1}{r}\frac{\partial \phi}{\partial \theta} = 0$$

This gives us,

$$p_\theta = u_\theta = U(r)$$

If we now replace the above result in,

$$p_r = -p_\theta U'(r)t$$

Then,

$$p_r = -U(r)U'(r)t$$

And given,

$$\mathbf{p} = p_r\hat{\mathbf{r}} + p_\theta\hat{\boldsymbol{\theta}}$$

We obtain the solution

$$(4.2) \qquad \mathbf{p} = -U(r)U'(r)t\hat{\mathbf{r}} + U(r)\boldsymbol{\theta}. \quad \blacksquare$$

4.2 <u>2π-periodic Solution</u>

Now consider $\mathbf{u} = \mathbf{u}_s = (u,v)$ where

$$u(x,y) = \sin y \cos x \qquad , \qquad v(x,y) = -\sin x \cos y$$

With initial condition

$$\mathbf{p}(x,y,0) = \mathbf{u}_s(x,y) + \nabla\psi \;\text{ where }\; \psi = \sin x \cos y + \sin y \cos x$$

If we write in vector notation, we see that

$$\mathbf{u} = (\sin y \cos x, -\sin x \cos y)$$

And

$$\nabla\psi(x,y) = (\cos y \cos x - \sin y \sin x, -\sin x \sin y + \cos x \cos y)$$

This yield

$$\mathbf{p}(x,y,t) = (\sin y \cos x + \cos y \cos x - \sin y \sin x, -\sin x \cos y - \sin x \sin y + \cos x \cos y)$$

For the case of $x = y = \frac{\pi}{2}$,

$$\mathbf{p}\left(\tfrac{\pi}{2},\tfrac{\pi}{2},t\right) = \left(\sin\tfrac{\pi}{2}\cos\tfrac{\pi}{2} + \cos\tfrac{\pi}{2}\cos\tfrac{\pi}{2} - \sin\tfrac{\pi}{2}\sin\tfrac{\pi}{2}, -\sin\tfrac{\pi}{2}\cos\tfrac{\pi}{2} - \sin\tfrac{\pi}{2}\sin\tfrac{\pi}{2} + \cos\tfrac{\pi}{2}\cos\tfrac{\pi}{2}\right)$$

This reduces to,

$$\mathbf{p}\left(\tfrac{\pi}{2},\tfrac{\pi}{2},t\right)=\left(-\sin\tfrac{\pi}{2}\sin\tfrac{\pi}{2},-\sin\tfrac{\pi}{2}\sin\tfrac{\pi}{2}\right)=\left(-1,-1\right)$$

We see that the components of **p** are:

$$\begin{aligned} p_x &= \sin y\cos x+\cos y\cos x-\sin y\sin x \\ p_y &= -\sin x\cos y-\sin x\sin y+\cos x\cos y \end{aligned}$$

Then

$$\begin{aligned} \frac{\partial p_x}{\partial t} &= -\sin y\sin x-2\left(\sin x\cos y+\sin y\cos x\right)+\cos y\cos x \\ \frac{\partial p_y}{\partial t} &= \sin x\sin y-2\left(\sin x\cos y+\sin y\cos x\right)-\cos x\cos y \end{aligned}$$

We can easily see that

$$\frac{\partial p_x\left(\tfrac{\pi}{2},\tfrac{\pi}{2},t\right)}{\partial t}=-\sin\tfrac{\pi}{2}\sin\tfrac{\pi}{2}=-1 \quad , \quad \frac{\partial p_y\left(\tfrac{\pi}{2},\tfrac{\pi}{2},t\right)}{\partial t}=\sin\tfrac{\pi}{2}\sin\tfrac{\pi}{2}=1$$

and thus

$$\frac{\partial p_x\left(\tfrac{\pi}{2},\tfrac{\pi}{2},t\right)}{\partial t}=p_x\left(\tfrac{\pi}{2},\tfrac{\pi}{2},t\right) \quad , \quad \frac{\partial p_y\left(\tfrac{\pi}{2},\tfrac{\pi}{2},t\right)}{\partial t}=-p_y\left(\tfrac{\pi}{2},\tfrac{\pi}{2},t\right)$$

By solving the differential equations,

$$\begin{pmatrix} \left(p_x\right)_t = p_x \\ \left(p_y\right)_t = -p_y \end{pmatrix} \rightarrow \begin{pmatrix} \ln p_x = t+C_1 \\ \ln p_y = -t+C_2 \end{pmatrix} \rightarrow \begin{pmatrix} p_x\left(\tfrac{\pi}{2},\tfrac{\pi}{2},t\right)=k_1e^t \\ p_y\left(\tfrac{\pi}{2},\tfrac{\pi}{2},t\right)=k_2e^{-t} \end{pmatrix}$$

Utilizing the initial condition we find

$$k_1=k_2=-1$$

$$\begin{aligned} p_x\left(\tfrac{\pi}{2},\tfrac{\pi}{2},t\right) &= -e^t \\ p_y\left(\tfrac{\pi}{2},\tfrac{\pi}{2},t\right) &= -e^{-t} \end{aligned}$$

■

CHAPTER FIVE: PHYSICAL INTERPRETATIONS

The results found above have interesting physical implications.

5.1 Evolution of Line Elements

Consider, for example, the presence of an infinitesimal material element $\boldsymbol{\delta}\mathbf{l}$ in the volume of the fluid. If we now examine the material derivative of this element, we see that

$$(5.1) \qquad \frac{D\boldsymbol{\delta}\mathbf{l}}{Dt} = (\boldsymbol{\delta}\mathbf{l}\cdot\nabla)\mathbf{u}$$

On expansion of the operator we obtain

$$(5.2) \qquad \frac{\partial\boldsymbol{\delta}\mathbf{l}}{\partial t} + (\mathbf{u}\cdot\nabla)\boldsymbol{\delta}\mathbf{l} = (\boldsymbol{\delta}\mathbf{l}\cdot\nabla)\mathbf{u}$$

which, on comparing with (3.9), implies that the material element $\boldsymbol{\delta}\mathbf{l}$ evolves like the vorticity $\boldsymbol{\omega}$. In other words, any vortex line can be considered a material line element. ■

5.2 Evolution of Surface Elements

Now, consider a material surface element, represented by the vector field $\boldsymbol{\delta\sigma}$. If we now consider the evolution of this vector field in $\mathbf{u}$ (i.e. $\mathbf{u}\cdot\boldsymbol{\delta\sigma}$), then by substituting in (2.4),

$$\frac{\partial\boldsymbol{\delta\sigma}}{\partial t} - \mathbf{u}\times(\nabla\times\boldsymbol{\delta\sigma}) = -\nabla(\mathbf{u}\cdot\boldsymbol{\delta\sigma})$$

which is the same as the equation for the impulse density in the Kuz'min gauge. By making use of identity [7], we obtain

$$\frac{\partial\boldsymbol{\delta\sigma}}{\partial t} - \nabla(\mathbf{u}\cdot\boldsymbol{\delta\sigma}) + \mathbf{u}\cdot\nabla\boldsymbol{\delta\sigma} + \boldsymbol{\delta\sigma}\times(\nabla\times\mathbf{u}) + \boldsymbol{\delta\sigma}\cdot\nabla\mathbf{u} = -\nabla(\mathbf{u}\cdot\boldsymbol{\delta\sigma})$$

Canceling the gradient terms and further manipulation yields,

$$\frac{\partial \boldsymbol{\delta\sigma}}{\partial t} + \mathbf{u}\cdot\nabla\boldsymbol{\delta\sigma} = -\left[\boldsymbol{\delta\sigma}\times(\nabla\times\mathbf{u}) + \boldsymbol{\delta\sigma}\cdot\nabla\mathbf{u}\right]$$

Using identity [4],

$$\frac{\partial \boldsymbol{\delta\sigma}}{\partial t} + \mathbf{u}\cdot\nabla\boldsymbol{\delta\sigma} = -\left[\nabla(\boldsymbol{\delta\sigma}\cdot\mathbf{u}) - (\boldsymbol{\delta\sigma}\cdot\nabla)\mathbf{u} + (\boldsymbol{\delta\sigma}\cdot\nabla)\mathbf{u}\right]$$

Cancelation gives

$$\frac{\partial \boldsymbol{\delta\sigma}}{\partial t} + \mathbf{u}\cdot\nabla\boldsymbol{\delta\sigma} = -\nabla(\mathbf{u}\cdot\boldsymbol{\delta\sigma})$$

By the properties of the gradient operator, this yields,

$$(5.3) \qquad \frac{\partial \boldsymbol{\delta\sigma}}{\partial t} + (\mathbf{u}\cdot\nabla)\boldsymbol{\delta\sigma} = -(\nabla\mathbf{u})^T\,\boldsymbol{\delta\sigma}$$

In comparison with (2.6), we see that they are identical. The physical interpretation of this is that the surface element given by the vector field **δσ** evolves as the fluid impulse density **p**! ■

5.3 Conservation of Volume Elements

Consider a fluid element in the form of a cylinder surface with its base **δσ** and generator **δl**; its volume is given by

$$\delta v = \boldsymbol{\delta}\mathbf{l}\cdot\boldsymbol{\delta\sigma}$$

Consider the rate of change of the volume of this fluid element. In light of (3.9) and (3.11) we can arrive at

$$(5.4) \qquad \frac{D(\boldsymbol{\delta}\mathbf{l}\cdot\boldsymbol{\delta\sigma})}{Dt} = 0$$

This result has the physical interpretation that the fluid element preserves its volume. This establishes conservation of mass given incompressibility of the fluid, according to equation (2.1). ■

CHAPTER SIX: GENERALIZATIONS

A natural extension of the results found above for incompressible fluids is to that for compressible fluids. We shall now examine the generalization of the previous chapters to the four gauges to such fluids. We begin by establishing the Euler equations for compressible fluids:

$$(6.1) \qquad \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

and

$$(6.2) \qquad \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho} \nabla \mathrm{P}$$

We know from Chapter Two that equation (6.2) can be written as

$$(6.3) \qquad \frac{\partial \mathbf{u}}{\partial t} - \mathbf{u} \times (\nabla \times \mathbf{u}) = -\nabla \left(\mathscr{P} + \frac{u^2}{2} \right)$$

where, assuming the fluid to be barotropic, we have

$$(6.4) \qquad \mathscr{P}(\rho) = \int \frac{d\mathrm{P}}{\rho}$$

If we now consider the Helmholtz decomposition (2.3), then for arbitrary scalar ϕ, **p** evolves according to (6.3), such that

$$(6.5) \qquad \frac{\partial \mathbf{p}}{\partial t} - \mathbf{u} \times (\nabla \times \mathbf{p}) = -\nabla \left(\mathscr{P} + \frac{u^2}{2} - \frac{\partial \phi}{\partial t} \right) \equiv -\nabla \Lambda$$

6.1 General Results for the Kuz'min Gauge

Theorem (6). In a compressible barotropic fluid, **p** evolves, in the Kuz'min gauge, according to the equation

$$\frac{\partial \mathbf{p}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{p} = -(\nabla \mathbf{u})^T \mathbf{p}$$

which is the same as that for the incompressible case, namely, equation (2.6).

Proof. If we let $\Lambda = \mathbf{u} \cdot \mathbf{p}$, we see that, in light of (6.5), ϕ satisfies the PDE

$$\frac{\partial \phi}{\partial t} + (\mathbf{u} \cdot \nabla)\phi = \mathscr{P} - \frac{u^2}{2} \tag{6.6}$$

In substituting (6.6) into (6.5) we clearly see that by making use of identity [7], we obtain

$$\frac{\partial \mathbf{p}}{\partial t} - \nabla(\mathbf{u} \cdot \mathbf{p}) + \mathbf{u} \cdot \nabla \mathbf{p} + \mathbf{p} \times (\nabla \times \mathbf{u}) + \mathbf{p} \cdot \nabla \mathbf{u} = -\nabla(\mathbf{u} \cdot \mathbf{p})$$

Canceling the gradient terms and further manipulation yields,

$$\frac{\partial \mathbf{p}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{p} = -\left[\mathbf{p} \times (\nabla \times \mathbf{u}) + \mathbf{p} \cdot \nabla \mathbf{u}\right]$$

Using identity [4],

$$\frac{\partial \mathbf{p}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{p} = -\left[\nabla(\mathbf{p} \cdot \mathbf{u}) - (\mathbf{p} \cdot \nabla)\mathbf{u} + (\mathbf{p} \cdot \nabla)\mathbf{u}\right]$$

Cancelation gives

$$\frac{\partial \mathbf{p}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{p} = -\nabla(\mathbf{u} \cdot \mathbf{p})$$

Rewriting in matrix notation,

$$\frac{\partial \mathbf{p}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{p} = -\nabla\left(\mathbf{u}^T \mathbf{p}\right)$$

By the properties of the gradient operator, this yields,

$$\frac{\partial \mathbf{p}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{p} = -(\nabla \mathbf{u})^T \mathbf{p} \tag{6.7}$$

which is the same as that for the incompressible case, namely, equation (2.6)! ■

Theorem (7). The evolution equation of the potential vorticity $\frac{\boldsymbol{\omega}}{\rho}$ is

$$\frac{D}{Dt}\left(\frac{\boldsymbol{\omega}}{\rho}\right) = \left(\frac{\boldsymbol{\omega}}{\rho}\cdot\nabla\right)\mathbf{u}$$

Proof. In the case of compressible barotropic fluid, the density is a function of only pressure. Thus, we can write the RHS of equation (6.2) as

$$-\frac{1}{\rho}\nabla \mathrm{P} = -\nabla\left(\int \frac{1}{\rho} d\mathrm{P}\right) \equiv -\nabla \mathscr{P}$$

and on writing the LHS of (6.2) as a material derivative, we have

$$\text{(6.8)} \qquad \frac{D\mathbf{u}}{Dt} = -\nabla \mathscr{P}$$

If we now take the curl of (6.8), we see that

$$\nabla \times \left(\frac{D\mathbf{u}}{Dt}\right) = \nabla \times \left(-\nabla \mathscr{P}\right) = 0$$

On distributing the curl and using identity [13], this leads to the vorticity evolution equation

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{u}\cdot\nabla)\boldsymbol{\omega} + \boldsymbol{\omega}(\nabla\cdot\mathbf{u}) = (\boldsymbol{\omega}\cdot\nabla)\mathbf{u}$$

Combining this equation with the mass conservation equation (6.1), we obtain for the evolution of the potential vorticity $\frac{\boldsymbol{\omega}}{\rho}$

$$\text{(6.9)} \qquad \frac{\partial}{\partial t}\left(\frac{\boldsymbol{\omega}}{\rho}\right) + (\mathbf{u}\cdot\nabla)\left(\frac{\boldsymbol{\omega}}{\rho}\right) = \left(\frac{\boldsymbol{\omega}}{\rho}\cdot\nabla\right)\mathbf{u}$$

or

$$\text{(6.10)} \qquad \frac{D}{Dt}\left(\frac{\boldsymbol{\omega}}{\rho}\right) = \left(\frac{\boldsymbol{\omega}}{\rho}\cdot\nabla\right)\mathbf{u}$$

This result indicates that changes in the potential vortex lines evolve the same way as the material line elements. Further, this implies that vortex lines move with the fluid. ■

Theorem (8). The impulse density **p**, in a compressible fluid, satisfies the Kelvin-Helmholtz circulation theorem (3.6):

$$\frac{d}{dt}\oint_{\Omega}\mathbf{p}\cdot d\mathbf{l}=0$$

Proof. The proof of this theorem is analogous to the proof exhibited in Chapter Three. ■

Theorem (9). For a compressible barotropic fluid, there exists, in the Kuz'min gauge, a local invariant

$$\frac{\mathbf{p}\cdot\boldsymbol{\omega}}{\rho}=constant$$

Proof. Equation (6.7) along with the potential vorticity evolution equation (6.9), in a manner analogous to that discussed in Chapter Three, gives

$$\text{(6.11)}\quad \frac{D}{Dt}\left(\frac{\mathbf{p}\cdot\boldsymbol{\omega}}{\rho}\right)=0$$

Equation (6.11) leads to the local invariant

$$\frac{\mathbf{p}\cdot\boldsymbol{\omega}}{\rho}=constant$$

This result leads to the physical implication that the mass of the fluid element is invariant. This is, of course, conservation of the mass condition for a compressible fluid (see Chapter Seven). ■

6.2 General Results for the Zero Gauge

Theorem (10). In a compressible barotropic fluid, **p** evolves, in the Zero gauge, according to the equation

$$\frac{\partial \mathbf{p}}{\partial t} = \mathbf{u} \times \frac{\boldsymbol{\omega}}{\rho}$$

which is the same as that for the incompressible case, namely, equation (2.8).

Proof. If we let $\Lambda = 0$, ϕ satisfies the PDE

$$(6.12) \quad \frac{\partial \phi}{\partial t} = \mathscr{P} + \frac{u^2}{2}$$

which immediately yields

$$(6.13) \quad \frac{\partial \mathbf{p}}{\partial t} - \mathbf{u} \times \left(\nabla \times \mathbf{p} \right) = 0$$

And by the definition of vorticity,

$$(6.14) \quad \frac{\partial \mathbf{p}}{\partial t} = \mathbf{u} \times \frac{\boldsymbol{\omega}}{\rho}$$

which is the same as that of the incompressible case, namely, equation (2.8)! The physical interpretation of (6.14) is that **p** is invariant all along vortex lines **l** and families of such vortex lines are overlapped by material surfaces ρ**S**. ■

6.3 General Results for the E & Liu Gauge

Theorem (11). In a compressible barotropic fluid, **p** evolves, in the E & Liu gauge, according to the equation

$$\frac{\partial \mathbf{p}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = 0$$

which is the same as that for the incompressible case, namely, equation (2.7).

Proof. If we now let $\Lambda = \dfrac{u^2}{2}$, ϕ satisfies the PDE

$$(6.15) \quad \frac{\partial \phi}{\partial t} = \mathscr{P}.$$

Then, (6.5) becomes

$$\frac{\partial \mathbf{p}}{\partial t} - \mathbf{u} \times (\nabla \times \mathbf{p}) = -\frac{1}{2} \nabla \mathbf{u} \cdot \mathbf{u}$$

Replacing **p** with (2.3)

$$\frac{\partial \mathbf{p}}{\partial t} - \mathbf{u} \times \left[\nabla \times (\mathbf{u} + \nabla \phi)\right] = -\mathbf{u} \times (\nabla \times \mathbf{u}) - (\mathbf{u} \cdot \nabla)\mathbf{u}$$

Distributing the curl, we have

$$\frac{\partial \mathbf{p}}{\partial t} - \mathbf{u} \times (\nabla \times \mathbf{u}) + (\nabla \times \nabla \phi) = -\mathbf{u} \times (\nabla \times \mathbf{u}) - (\mathbf{u} \cdot \nabla)\mathbf{u}$$

Finally, by identity [2]

$$(6.16) \quad \frac{\partial \mathbf{p}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = 0$$

which is the same as that for the incompressible case, namely, equation (2.7)! ■

6.4 General Results for the Maddocks & Pego Gauge

Theorem (12). In a compressible barotropic fluid, **p** evolves, in the Maddocks & Pego, according to the equation

$$\frac{\partial \mathbf{p}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{p} = -(\nabla \mathbf{u})^T \nabla \phi.$$

which is the same as that for the incompressible case, namely, equation (2.9).

Proof. If we now let $\Lambda = \mathbf{u}\cdot\mathbf{p} - \frac{u^2}{2}$, ϕ satisfies the PDE

$$(6.17) \quad \frac{\partial \phi}{\partial t} + \mathbf{u}\cdot\nabla\phi = \mathscr{P}.$$

Then, (6.5) becomes, with the use of [7],

$$\frac{\partial \mathbf{p}}{\partial t} - \nabla(\mathbf{u}\cdot\mathbf{p}) + \mathbf{u}\cdot\nabla\mathbf{p} + \mathbf{p}\times(\nabla\times\mathbf{u}) + \mathbf{p}\cdot\nabla\mathbf{u} = -\nabla(\mathbf{u}\cdot\mathbf{p}) + \frac{1}{2}\nabla\mathbf{u}\cdot\mathbf{u}$$

Immediate cancelation and rewriting gives

$$\frac{\partial \mathbf{p}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{p} = -\mathbf{p}\times(\nabla\times\mathbf{u}) - \mathbf{p}\cdot\nabla\mathbf{u} + \frac{1}{2}\nabla\mathbf{u}\cdot\mathbf{u}$$

Making use of identity [13],

$$\frac{\partial \mathbf{p}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{p} = -\mathbf{p}\times(\nabla\times\mathbf{u}) - \mathbf{p}\cdot\nabla\mathbf{u} + \mathbf{u}\times(\nabla\times\mathbf{u}) + (\mathbf{u}\cdot\nabla)\mathbf{u}$$

Using (2.3),

$$\frac{\partial \mathbf{p}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{p} = -(\mathbf{u}+\nabla\phi)\times(\nabla\times\mathbf{u}) - (\mathbf{u}+\nabla\phi)\cdot\nabla\mathbf{u} + \mathbf{u}\times(\nabla\times\mathbf{u}) + (\mathbf{u}\cdot\nabla)\mathbf{u}$$

Distribution of the curl yields,

$$\frac{\partial \mathbf{p}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{p} = -\mathbf{u}\times(\nabla\times\mathbf{u}) - \nabla\phi\times(\nabla\times\mathbf{u}) - (\mathbf{u}+\nabla\phi)\cdot\nabla\mathbf{u} + \mathbf{u}\times(\nabla\times\mathbf{u}) + (\mathbf{u}\cdot\nabla)\mathbf{u}$$

Cancelation and distribution of the dot product gives us,

$$\frac{\partial \mathbf{p}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{p} = -\nabla\phi\times(\nabla\times\mathbf{u}) - (\mathbf{u}\cdot\nabla)\mathbf{u} - (\nabla\phi\cdot\nabla)\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}$$

or

$$\frac{\partial \mathbf{p}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{p} = -[\nabla\phi\times(\nabla\times\mathbf{u})] - (\nabla\phi\cdot\nabla)\mathbf{u}$$

which is the same as,

$$(6.18) \quad \frac{\partial \mathbf{p}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{p} = -(\nabla \mathbf{u})^T \nabla \phi.$$

which is the same as that for the incompressible case, namely, equation (2.9)! ■

CHAPTER SEVEN: PHYSICAL INTERPRETATIONS FOR COMPRESSIBLE FLUIDS

We are now ready to recognize the physical interpretations for the compressible, barotropic fluid.

7.1 Evolution of Line Elements

Consider a vector field linked to an infinitesimal material element **l** in the volume of the fluid. If we now examine the material derivative of this element, we see that

$$\frac{D\mathbf{l}}{Dt} = (\mathbf{l}\cdot\nabla)\mathbf{u} \tag{7.1}$$

On expansion of the operator we obtain

$$\frac{\partial \mathbf{l}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{l} = (\mathbf{l}\cdot\nabla)\mathbf{u} \tag{7.2}$$

which is identical to (6.9). This implies that the material element **l** evolves as does the potential vorticity $\frac{\boldsymbol{\omega}}{\rho}$. This is equivalent to saying that the potential vortex line can be considered a material line element. ■

7.2 Evolution of Surface Elements

If now $\rho\mathbf{S}$ represents a vector field linked to a material surface element, then by substituting in (6.5),

$$\frac{\partial(\rho\mathbf{S})}{\partial t} - \mathbf{u}\times(\nabla\times\rho\mathbf{S}) = -\nabla(\mathbf{u}\cdot\rho\mathbf{S})$$

which is the same as the equation for the impulse density in the Kuz'min gauge. By making use of identity [7], we obtain

$$\frac{\partial(\rho\mathbf{S})}{\partial t} - \nabla(\mathbf{u}\cdot\rho\mathbf{S}) + \mathbf{u}\cdot\nabla(\rho\mathbf{S}) + (\rho\mathbf{S})\times(\nabla\times\mathbf{u}) + (\rho\mathbf{S})\cdot\nabla\mathbf{u} = -\nabla(\mathbf{u}\cdot\rho\mathbf{S})$$

Canceling the gradient terms and further manipulation yields,

$$\frac{\partial(\rho\mathbf{S})}{\partial t}+\mathbf{u}\cdot\nabla(\rho\mathbf{S})=-\left[(\rho\mathbf{S})\times(\nabla\times\mathbf{u})+(\rho\mathbf{S})\cdot\nabla\mathbf{u}\right]$$

Using identity [4],

$$\frac{\partial(\rho\mathbf{S})}{\partial t}+\mathbf{u}\cdot\nabla(\rho\mathbf{S})=-\left[\nabla(\rho\mathbf{S}\cdot\mathbf{u})-(\rho\mathbf{S}\cdot\nabla)\mathbf{u}+(\rho\mathbf{S}\cdot\nabla)\mathbf{u}\right]$$

Cancelation gives

$$\frac{\partial(\rho\mathbf{S})}{\partial t}+\mathbf{u}\cdot\nabla(\rho\mathbf{S})=-\nabla(\mathbf{u}\cdot\rho\mathbf{S})$$

By the properties of the gradient operator, this yields,

$$(7.3)\qquad \frac{\partial(\rho\mathbf{S})}{\partial t}+(\mathbf{u}\cdot\nabla)(\rho\mathbf{S})=-(\nabla\mathbf{u})^{T}(\rho\mathbf{S})$$

On comparison with (6.7), we see that they are identical. The physical interpretation is that the surface element given by the vector field $\rho\mathbf{S}$ evolves as the fluid impulse density **p,** as in the incompressible case! ■

7.3 Conservation of Mass Elements

Consider a fluid element in the form of a cylinder surface with its base $\rho\mathbf{S}$ and generator $\mathbf{l}$. We see that its mass δm is given by

$$\delta m=\mathbf{l}\cdot\rho\mathbf{S}$$

Consider now the rate of change of the volume of this fluid element. In light of (6.7) and (6.9) we see

$$(7.4)\qquad \frac{D(\mathbf{l}\cdot\rho\mathbf{S})}{Dt}=0$$

This result has the physical interpretation that the fluid element preserves its mass. This establishes conservation of mass in the case of compressible fluid. ■

CHAPTER EIGHT: SUMMARY

In this paper, we have considered impulse formulations of the Euler equations for both incompressible and compressible fluids. Different gauge conditions are considered. The geometric gauge provided a remarkable physical interpretation. In this gauge, the impulse density evolves the same way as material surfaces: its direction is orthogonal to the material surface element, and its length is proportional to the area of the surface element. The impulse density has a local invariant associated with it which has the physical implication of conservation of volume of fluid elements. It is interesting that, in the compressible barotropic case, the results turn out to be similar to those for the incompressible case.

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