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Optimal Dual Frames For Erasures And Discrete Gabor Frames

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by

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Abstract

Since their discovery in the early 1950's, frames have emerged as an important tool in areas such as signal processing, image processing, data compression and sampling theory, just to name a few. Our purpose of this dissertation is to investigate dual frames and the ability to find dual frames which are optimal when coping with the problem of erasures in data transmission. In addition, we study a special class of frames which exhibit algebraic structure, discrete Gabor frames. Much work has been done in the study of discrete Gabor frames in $\mathbb{R}^n$, but very little is known about the $\ell^2(\mathbb{Z})$ case or the $\ell^2(\mathbb{Z}^d)$ case. We establish some basic Gabor frame theory for $\ell^2(\mathbb{Z})$ and then generalize to the $\ell^2(\mathbb{Z}^d)$ case.

For my family...

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CHAPTER 1

INTRODUCTION

The concept of an orthonormal basis is fundamental in the study of inner product spaces, and Hilbert spaces in particular. Results for orthonormal bases make it easier to study such topics as dimension, projections, separability of Hilbert spaces, and countless others. However, their most fundamental use is in representing any vector as a linear combination of the orthnormal basis vectors, and the ease with which the coefficients of that linear combination can be found.

For example, if $\{e_i\}_{i=1}^n$ is an orthonormal basis for a Hilbert space, H, and x is any vector of the space with $\sum_{i=1}^n c_i e_i$ its linear combination, then by taking the inner product with e_j we see that

$$\begin{aligned} x &= \sum_{i=1}^{n} c_i e_i \\ \langle x, e_j \rangle &= \left\langle \sum_{i=1}^{n} c_i e_i, e_j \right\rangle \\ \langle x, e_j \rangle &= \sum_{i=1}^{n} c_i \langle e_i, e_j \rangle \\ \langle x, e_j \rangle &= c_j \end{aligned}$$

Therefore, it follows that for any x in H

$$x = \sum_{i=1}^{n} \langle x, e_i \rangle e_i \tag{1.1}$$

which gives a nice (and unique) representation for the vector x in that orthonormal basis.

As a consequence of Equation 1.1, by taking the inner product with x, we have a very useful identity known as the Parseval identity

$$\|x\|^2 = \sum_{i=1}^{n} |\langle x, e_i \rangle|^2 \tag{1.2}$$

which holds for all x in H.

So then, given these advantages, what are some of the disadvantages of using orthonormal bases?

Firstly, they are sensitive to data loss. For example, in the context of signal transmission, a signal can be thought of as a vector, represented by a linear combination in a particular orthonormal basis. The coefficients of the linear combination are the data that is transmitted to a receiver. If even one of the coefficients is lost in transmission, the signal cannot be reconstructed again.

Another shortcoming of orthonormal bases is evident when we wish to choose basis vectors that satisfy some other conditions, such as a group structure, and it may be impossible to find an orthonormal basis which satisfies the additional conditions.

One solution to these issues is the notion of a *frame*.

In 1952, while working on problems in nonharmonic Fourier series, Duffin and Schaeffer introduced frames for a Hilbert space, although their work was not continued until the 1980s, when Morlet, Grossmann, and others brought about the "wavelet era", and with it a renewed interest in overcomplete systems.

Frames generalize the concept of a basis by sacrificing the uniqueness of a vector's orthonormal basis representation, which is often unnecessary in applications, in exchange for redundancy which makes the frame more robust for applications such as data transmission.

Moreover, a special subset of frames known as Parseval frames, continue to satisfy Equations 1.1 and 1.2, offering even more of the benefits of orthonormal bases, such as the ability to easily compute the coefficients of a representation using the inner product.

Because of these advantages, the last few years have seen a tremendous growth in the research area of frames. They appear in the fields of signal processing, image processing, quantum mechanics, harmonic analysis, and many others. They are also interesting from a purely mathematical standpoint, which will be our primary focus.

The rest of the chapters are laid out as follows. Chapter 2 begins with a brief introduction to frames, including some of the basic results for general frames. In addition, the idea of using dual pairs of frames for the trace of an operator is introduced in Section 2.7.

Chapter 3 continues the overview of frames by focusing on a class of frames which exhibit algebraic structure, in particular group structure. This will help lay the foundation for some of the later work in finding optimal dual frames for group representation frames, as well as for studying the discrete Gabor frames.

The main results of this work are presented in Chapters 4 and 5.

Chapter 4 begins with a simple introduction to using frames for signal transmission. As mentioned above, frames have proven to be useful in such applications, since their redundant nature makes them more robust when dealing with *erasures*, a loss of some of the transmitted data. The error of such a loss, that is, a measure of the difference between the reconstructed signal and the original signal, can be made smaller by choosing an appropriate frame to use for encoding the signal. Finding frames which are "optimal" in this sense has been studied, see for example [23]. This method, however, will naturally add some constraints on which frames can be used for coding.

We take a slightly different approach. Rather than minimizing the error at the outset, consider coding a vector using a frame already chosen, and then, if there are erasures, reconstructing the signal using a dual frame which minimizes the error. Finding such an *optimal dual frame* for a given frame is the problem which is studied in Chapter 4.

We first prove the existence of optimal dual frames for any number of erasures. Then we go on to show that for many important classes of frames, the canonical dual frame is an optimal dual frame, and, moreover, it is the unique optimal dual frame. We show this result for both uniform tight frames and group representation frames, and then go on to generalize this result to any frame where $\|S^{-1}x_i\| \cdot \|x_i\|$ is a constant for all i.

We then give some examples, one of which shows that it is possible for a frame to have a unique optimal dual frame which is not the canonical dual. Another example shows that a frame can have infinitely many optimal dual frames for one erasure.

In Chapter 5, we change gears and begin studying another class of structured frame, the Gabor (or Weyl-Heisenberg) frame. Gabor frames are the result of taking a base function, known as a *Gabor atom*, and applying time translations and frequency modulations to generate a sequence of functions which form a frame.

Much of the work in this area has involved the infinite-dimensional function space $L^2(\mathbb{R}^d)$ and the finite-dimensional signal space $\mathbb{R}^d$ (or $\mathbb{C}^d$). However, very little is known about the infinite-dimensional discrete signal space $\ell^2(\mathbb{Z}^d)$, especially when $d > 1$. Studying the fundamental aspects of discrete Gabor frames in $\ell^2(\mathbb{Z}^d)$ is the focus of Chapter 5.

We begin by reviewing some of the definitions and properties for frames, with special attention to those things which are different in the infinite-dimensional setting. We define the Gabor family, and give some basic properties of Gabor frames.

Then we show some results for the $\ell^2(\mathbb{Z})$ case which are analogous to a few fundamental theorems about Gabor frames which are well known in $L^2(\mathbb{R}^d)$. These include the density theorems for frames and super-frames, the characterizations of dual frame pairs and tight frames, and the characterization of orthogonal (strongly disjoint) frames. We also give the existence theorem for the tight dual frame of the Gabor type in the $\ell^2(\mathbb{Z}^d)$ case.

Next, the characterizations and density theorems are generalized to $\ell^2(\mathbb{Z}^d)$. There are some technical difficulties in doing this because of the complexity involved with the higher dimension lattices in $\mathbb{Z}^d$. In particular, the density theorem for Gabor super-frames requires the generalization of an existence theorem for common subgroup coset representatives.

Finally, Chapter 6 concludes with some ideas for further work in frames.

CHAPTER 2

PRELIMINARIES

2.1 Frames in Hilbert Space

A *frame*, in the simplest sense, is a generalization of a *basis* for a vector space. For a finite-dimensional vector space, this generalization can be characterized quite simply. While a basis is a set of linearly independent vectors which span the space, a frame is *any* set of vectors which span the space. In other words, the vectors of a frame may be linearly dependent.

Allowing a spanning set to be linearly dependent offers several benefits, including:

- Redundancy
- Relaxed conditions, making it easier to find a spanning set with additional properties (e.g. group structure).

For an infinite-dimensional space, the situation is slightly more complicated. Instead of spanning sets there are complete sequences, but not every complete sequence is a frame [18].

Fortunately, there is a definition which is valid for both the finite- and infinite-dimensional cases.

Definition 2.1 *Let H be a Hilbert space and $\{v_i\}_{i\in\mathcal{I}} \subseteq H$. If there exist constants $A, B > 0$ such that, for every $x \in H$*

$$A\|x\|^2 \leq \sum_i |\langle x, v_i\rangle|^2 \leq B\|x\|^2 \tag{2.1}$$

*then the sequence $\{v_i\}$ is called a **frame**. The constant A which is maximal is called the **lower frame bound** and the constant B which is minimal the **upper frame bound**.*

If $A = B$ the frame is called a **tight frame**. If $A = B = 1$, Equation 2.1 becomes the Parseval identity (Equation 1.2) and so the frame is called a **Parseval tight frame**, or Parseval frame. A **uniform** (or **equal-norm**) **frame** is a frame in which all vectors have equal norm.

As mentioned, the above definition is valid for both finite-dimensional and infinite-dimensional spaces. However, for a finite-dimensional space, the condition that the frame spans the space is sometimes more convenient to use than the frame bounds. This leads to an alternate definition for a finite frame

Definition 2.2 *Let H be a finite-dimensional Hilbert space and $\{v_i\}_{i=1}^k \subseteq H$ such that span$\{v_i\} = H$. The sequence $\{v_i\}$ is called a **frame**.*

It can be shown that this is equivalent to Definition 2.1. The following proof is adapted from Proposition 3.18 of [18].

Proof: First, suppose that $\{v_i\}_{i=1}^k$ does not span H. Then there exists a nonzero vector x such that x is in the orthogonal complement of span$\{v_i\}_{i=1}^k$. Thus, for all i,

$\langle x, v_i \rangle = 0$. But then

$$\sum_{i=1}^{k} |\langle x, v_i \rangle|^2 = 0$$

Therefore, $A = 0$ in Equation 2.1. In other words, there is no lower frame bound, and so $\{v_i\}_{i=1}^{k}$ is not a frame.

Conversely, suppose that $\{v_i\}_{i=1}^{k}$ violates the lower frame bound condition of Definition 2.1 (the upper condition always holds for a finite sequence). Then, for every $m \in \mathbb{N}$, there exists $y_m \in H$ such that $\|y_m\| = 1$ and

$$\sum_{i=1}^{k} |\langle y_m, v_i \rangle|^2 < \frac{1}{m}$$

Since $\{y_m\}_{m=1}^{\infty}$ is a bounded sequence, it must have a convergent subsequence $\{y_{m_j}\}$ with limit vector y. Thus

$$\begin{aligned} 0 &= \lim_{j\to\infty} \sum_{i=1}^{k} |\langle y_{m_j}, v_i \rangle|^2 \\ &= \sum_{i=1}^{k} |\langle y, v_i \rangle|^2 \end{aligned}$$

and y is orthogonal to every v_i. So either $y = 0$ or $\{v_i\}_{i=1}^{k}$ does not span H, but $\|y\| = 1$ since every $\|y_{m_j}\| = 1$. Therefore, $\text{span}\{v_i\}_{i=1}^{k} \neq H$.

■

It follows from this definition that every basis is also a frame.

Next, we look at a few simple examples of frames.

Example 2.1 *The vectors* $\{x_i\}_{i=1}^3$ *given by*

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

is a frame for $H = \mathbb{R}^2$*.*

Notice, in particular, that it is acceptable to repeat a vector multiple times in a frame. Consequently, the idea of a frame as a *set* of vectors, while convenient in casual discussion, is actually not the best description, which is why we define a frame as a *sequence* of vectors. However, this can also have its problems in some applications. For example, we may wish to treat two frames as equal to each other if they contain the same vectors in a different sequence ordering. See Section 2.6 for more details.

The next example is slightly more interesting. It will be revisited in Chapter 3.

Example 2.2 (Mercedes-Benz Frame) *The vectors* $\{x_i\}_{i=1}^3$ *given by*

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{bmatrix} \right\}$$

is a frame for $H = \mathbb{R}^2$*.*

Another example shows a simple frame in the infinite-dimensional case.

Example 2.3 *Let* $\{e_i\}_{i=1}^\infty$ *be an orthonormal basis for the Hilbert space* $H = \ell^2(\mathbb{N})$*. Then by repeating each element of* $\{e_i\}_{i=1}^\infty$ *twice we have*

$$\{x_i\}_{i=1}^\infty = \{e_1, e_1, e_2, e_2, \ldots\}$$

which is a tight frame for H with $A = B = 2$.

Chapter 5 will continue exploring frames in the infinite-dimensional setting.

2.2 Analysis Operator and Frame Operator

We begin the study of frames by defining some operators that are associated with an arbitrary sequence of vectors. Then we study the properties of these operators, and show how they relate back to sequences of vectors which are frames.

Definition 2.3 *Let H, K be Hilbert spaces, with K of dimension k. Let $\{e_i\}_{i=1}^k$ be an orthonormal basis for K, and $\{v_i\}_{i=1}^k \subseteq H$. The* ***analysis operator*** *is the linear operator $\Theta : H \to K$ such that*

$$\Theta x = \sum_{i=1}^{k} \langle x, v_i \rangle e_i$$

If $K = \mathbb{C}^k$, then this is equivalent to

$$\Theta x = \begin{bmatrix} \langle x, v_1 \rangle \\ \langle x, v_2 \rangle \\ \vdots \\ \langle x, v_k \rangle \end{bmatrix}$$

When dealing with more than one set of frame vectors, it will often be convenient to use a subscript notation to differentiate between their respective analysis operators.

For example, if $\{v_i\}_{i=1}^k \subseteq H$ and $\{w_i\}_{i=1}^k \subseteq H$, then

$$\Theta_v x = \sum_{i=1}^{k} \langle x, v_i \rangle e_i \quad \text{and} \quad \Theta_w x = \sum_{i=1}^{k} \langle x, w_i \rangle e_i$$

Definition 2.4 *The* ***synthesis operator*** *is the adjoint of the analysis operator.*

This is equivalent to

$$\Theta^* x = \sum_{i=1}^{k} \langle x, e_i \rangle v_i$$

for $x \in K$, which can be derived from the definition of the analysis operator. Alternatively, the synthesis operator can be characterized by

$$\Theta^* e_i = v_i$$

and this can be derived from the previous equation and the properties of orthonormal basis, by plugging e_j in for x.

Definition 2.5 *The* ***frame operator*** *is the operator* $\Theta^*\Theta$.

From $\Theta^* e_i = v_i$ it follows that

$$\Theta^*\Theta x = \sum_{i=1}^{k} \langle x, v_i \rangle v_i \tag{2.2}$$

The frame operator is often denoted by S.

Definition 2.6 *The* ***Grammian operator*** *is the operator* $\Theta\Theta^*$.

If $K = \mathbb{C}^k$, then from the above definitions

$$\begin{aligned}
\Theta\Theta^* x &= \Theta(\Theta^* x) \\
&= \sum_{i=1}^{k} \langle \Theta^* x, v_i \rangle e_i \\
&= \sum_{i=1}^{k} \langle x, \Theta v_i \rangle e_i \\
&= \sum_{i=1}^{k} \left\langle x, \sum_{j=1}^{k} \langle v_i, v_j \rangle e_j \right\rangle e_i \\
&= \sum_{i=1}^{k} \sum_{j=1}^{k} \langle v_j, v_i \rangle \langle x, e_j \rangle e_i \\
&= Ax
\end{aligned}$$

where $A = (\langle v_j, v_i \rangle)$. That is, $\Theta\Theta^*$ is the matrix

$$\Theta\Theta^* = \begin{bmatrix} \langle v_1, v_1 \rangle & \langle v_2, v_1 \rangle & \dots & \langle v_k, v_1 \rangle \\ \langle v_1, v_2 \rangle & \langle v_2, v_2 \rangle & \dots & \langle v_k, v_2 \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle v_1, v_k \rangle & \langle v_2, v_k \rangle & \dots & \langle v_k, v_k \rangle \end{bmatrix}$$

Note in particular that the diagonal elements of the Grammian are $\|v_i\|^2$. See Property 2.9 for more details.

In addition to these standard operators, it will also be useful to take compositions of operators associated with different sets of vectors. That is, to create operators of the form

$$\Theta_w^* \Theta_v x = \sum_{i=1}^{k} \langle x, v_i \rangle w_i$$

We now review some of the basic properties of these operators, beginning with the analysis operator.

Property 2.1 *The analysis operator is injective if, and only if,* $\{v_i\}_{i=1}^k$ *is a frame.*

Proof: First, suppose $\{v_i\}_{i=1}^k$ is a frame. If $\Theta x = 0$, then

$$\sum_{i=1}^{k} \langle x, v_i \rangle e_i = 0$$

and since e_i is a basis, $\langle x, v_i \rangle = 0$ for all i. Since $\{v_i\}_{i=1}^k$ is a spanning set for H, there exists α_i such that $x = \sum_{i=1}^k \alpha_i v_i$, which gives

$$\begin{aligned} \langle x, x \rangle &= \langle x, \sum_{i=1}^{k} \alpha_i v_i \rangle \\ &= \sum_{i=1}^{k} \overline{\alpha_i} \langle x, v_i \rangle \\ &= 0 \end{aligned}$$

Therefore, $x = 0$, and the kernel of Θ is trivial, so the analysis operator is injective.

Now, suppose instead that the analysis operator is injective. Suppose, by way of contradiction, that the span of $\{v_i\}_{i=1}^k$ is not the entire space H. Then pick y such that $y \perp \operatorname{span}\{v_i\}_{i=1}^k$ and $y \neq 0$. Thus, $\langle y, v_i \rangle = 0$ for all i, and so $\Theta y = 0$. But then the analysis operator is not injective, which is a contradiction. Therefore, $\{v_i\}_{i=1}^k$ spans the entire space, and so is a frame.

■

Property 2.2 *If* $\{v_i\}_{i=1}^k$ *is a Parseval frame, then the analysis operator is an isometry.*

Proof:

$$\begin{aligned}
\|\Theta_v x\|^2 &= \langle \Theta_v x, \Theta_v x \rangle \\
&= \langle \Theta_v^* \Theta_v x, x \rangle \\
&= \left\langle \sum_{i=1}^{k} \langle x, v_i \rangle v_i, x \right\rangle \\
&= \sum_{i=1}^{k} \langle x, v_i \rangle \langle v_i, x \rangle \\
&= \sum_{i=1}^{k} |\langle x, v_i \rangle|^2 \\
&= \|x\|^2
\end{aligned}$$

Where the last equality follows from Equation 2.1, with $A = B = 1$, since $\{v_i\}_{i=1}^k$ is a Parseval frame. Therefore, the analysis operator is an isometry.

■

Property 2.3 *Let $T : H \to H$ be a linear operator so that the set of vectors $\{Tv_i\}_{i=1}^k$ has analysis operator Θ_{Tv}. Then $\Theta_{Tv} x = \Theta_v T^* x$.*

Proof:

$$\begin{aligned}
\Theta_{Tv} x &= \sum_{i=1}^{k} \langle x, Tv_i \rangle e_i \\
&= \sum_{i=1}^{k} \langle T^* x, v_i \rangle e_i \\
&= \Theta_v T^* x
\end{aligned}$$

■

Property 2.4 *Let α be a scalar so that the set of vectors $\{\alpha v_i\}_{i=1}^k$ has analysis operator $\Theta_{\alpha v}$. Then $\Theta_{\alpha v} = \overline{\alpha}\Theta_v$.*

Proof: This can be shown from the definition, or simply by using Property 2.3 with $T = \alpha I$.

■

Next, we give some of the properties of the frame operator.

Property 2.5 *The frame operator is invertible if, and only if, $\{v_i\}_{i=1}^k$ is a frame.*

Proof: If S^{-1} exists, then for all $x \in H$

$$x = \sum_{i=1}^{k} \langle x, S^{-1}v_i \rangle v_i$$

by Proposition 2.2. Thus $\{v_i\}_{i=1}^k$ spans H, and so is a frame.

Conversely, suppose $\{v_i\}_{i=1}^k$ is a frame with analysis operator Θ. If $x \neq 0 \in H$, then $\Theta x \neq 0 \in \Theta(H)$, by Property 2.1. Since $K = \ker \Theta^* \oplus \text{Range } \Theta$, if $y \neq 0 \in \Theta(H)$, then $\Theta^* y \neq 0$. Thus, $\Theta^*(\Theta x) = \Theta^*\Theta x \neq 0$. Therefore, $\Theta^*\Theta$ is invertible.

■

Property 2.6 *The frame operator is self-adjoint.*

From the definition, $S^* = (\Theta^*\Theta)^* = \Theta^*(\Theta^*)^* = \Theta^*\Theta = S$.

Property 2.7 *The frame operator is the identity operator I if, and only if, $\{v_i\}_{i=1}^k$ is a Parseval frame.*

This follows from the reconstruction formula. See Section 2.3 for more details.

Property 2.8 *The frame operator is a scalar multiple of the identity operator, λI, if, and only if $\{v_i\}_{i=1}^k$ is a tight frame with frame bound λ.*

Proof: Let $\{v_i\}_{i=1}^k$ be a tight frame with frame bound $\lambda > 0$, so that for all x

$$\begin{aligned}
\lambda\|x\|^2 &= \sum_{i=1}^{k} |\langle x, v_i\rangle|^2 \\
\|x\|^2 &= \lambda^{-1} \sum_{i=1}^{k} |\langle x, v_i\rangle|^2 \\
&= \sum_{i=1}^{k} |\lambda^{-1/2}\langle x, v_i\rangle|^2 \\
&= \sum_{i=1}^{k} |\langle x, \lambda^{-1/2} v_i\rangle|^2
\end{aligned}$$

Thus the set of vectors $\{\lambda^{-1/2}v_i\}_{i=1}^k$ is a Parseval frame. From Property 2.7, this frame has frame operator I, and so by Property 2.4

$$\begin{aligned}
I &= \Theta^*_{\lambda^{-1/2}v}\Theta_{\lambda^{-1/2}v} \\
&= \lambda^{-1/2}\Theta^*_v\Theta_v\lambda^{-1/2} \\
&= \lambda^{-1}\Theta^*_v\Theta_v \\
\lambda I &= \Theta^*_v\Theta_v
\end{aligned}$$

Therefore the frame operator is a scalar multiple of the identity operator.

■

Finally, we show a useful property of the Grammian operator.

Property 2.9 *For a frame* $\{v_i\}_{i=1}^k$ *with* Θ *its analysis operator,* $tr(\Theta\Theta^*) = \sum_{i=1}^k \|v_i\|^2$.

Proof: Let $\Theta : H \to K$ be the analysis operator for the frame $\{v_i\}_{i=1}^k$. Then the Grammian operator $\Theta\Theta^*$ is an operator from K to K. So if $\{e_i\}_{i=1}^k$ is an orthonormal basis for K , then by the definition of the synthesis operator

$$\begin{aligned}\operatorname{tr}(\Theta\Theta^*) &= \sum_{i=1}^k \langle \Theta\Theta^* e_i, e_i \rangle \\ &= \sum_{i=1}^k \langle \Theta^* e_i, \Theta^* e_i \rangle \\ &= \sum_{i=1}^k \langle v_i, v_i \rangle \\ &= \sum_{i=1}^k \|v_i\|^2\end{aligned}$$

■

2.3 Parseval Frames

One of the most important properties of an orthonormal basis for a vector space is the ability to represent any vector x in the space as a linear combination of the basis vectors, where the coefficients are unique in that basis. Indeed, if $\{e_i\}_{i=1}^n$ is an orthonormal basis, then

$$x = \sum_{i=1}^n \langle x, e_i \rangle e_i$$

It turns out that there are sets of vectors other than orthonormal bases which exhibit this extremely useful reconstruction property.

Definition 2.7 *Let $\{v_i\}_{i=1}^k$ be a set of vectors in H, and $x \in H$. Then the* ***reconstruction formula*** *is*

$$x = \sum_{i=1}^{k} \langle x, v_i \rangle v_i \tag{2.3}$$

This is equivalent to the equation $x = \Theta_v^* \Theta_v x$. In other words, the frame operator is the identity operator. This leads to the following theorem

Theorem 2.1 *A set of vectors $\{v_i\}_{i=1}^k \subseteq H$ is a Parseval frame if, and only if, it satisfies the reconstruction formula (Equation 2.3).*

Proof: Suppose $\{v_i\}_{i=1}^k$ satisfies the reconstruction formula. Then

$$\begin{aligned}
\|x\|^2 &= \langle x, x \rangle \\
&= \left\langle \sum_{i=1}^{k} \langle x, v_i \rangle v_i, x \right\rangle \\
&= \sum_{i=1}^{k} \langle \langle x, v_i \rangle v_i, x \rangle \\
&= \sum_{i=1}^{k} \langle x, v_i \rangle \langle v_i, x \rangle \\
&= \sum_{i=1}^{k} |\langle x, v_i \rangle|^2
\end{aligned}$$

Therefore, the Parseval identity is satisfied for all $x \in H$, and so $\{v_i\}_{i=1}^k$ is a Parseval frame.

Conversely, suppose $\{v_i\}_{i=1}^k$ is a Parseval frame. By Property 2.2, Θ_v is an isometry, and so it also preserves inner products. Let $\{u_i\}_{i=1}^n$ be an orthonormal basis for

H and $\{e_i\}_{i=1}^k$ be an orthonormal basis for K. Thus

$$\begin{aligned}
x &= \sum_{i=1}^{n} \langle x, u_i \rangle u_i \\
&= \sum_{i=1}^{n} \langle \Theta_v x, \Theta_v u_i \rangle u_i \\
&= \sum_{i=1}^{n} \left\langle \sum_{j=1}^{k} \langle x, v_j \rangle e_j, \sum_{m=1}^{k} \langle u_i, v_m \rangle e_m \right\rangle u_i \\
&= \sum_{i=1}^{n} \sum_{j=1}^{k} \sum_{m=1}^{k} \langle \langle x, v_j \rangle e_j, \langle u_i, v_m \rangle e_m \rangle \, u_i \\
&= \sum_{i=1}^{n} \sum_{j=1}^{k} \sum_{m=1}^{k} \langle x, v_j \rangle \langle v_m, u_i \rangle \langle e_j, e_m \rangle u_i \\
&= \sum_{i=1}^{n} \sum_{j=1}^{k} \langle x, v_j \rangle \langle v_j, u_i \rangle u_i \\
&= \sum_{j=1}^{k} \langle x, v_j \rangle \sum_{i=1}^{n} \langle v_j, u_i \rangle u_i \\
&= \sum_{j=1}^{k} \langle x, v_j \rangle v_j
\end{aligned}$$

Therefore, the reconstruction formula is satisfied, as required.

■

Parseval frames are an important class of frames with many useful results. As was just shown in Theorem 2.1, Parseval frames satisfy the same reconstruction formula as orthonormal bases,

$$x = \sum_{i=1}^{k} \langle x, v_i \rangle v_i \qquad \forall x \in H$$

This allows for easy computation of the linear combination coefficients using the inner product. However, unlike for orthonormal bases, a Parseval frame representation for a vector is not necessarily unique.

Parseval frames make up for this with the added advantage of redundancy. Theorem 2.4 is one example of that redundancy where, under certain conditions, removing one vector of a Parseval frame leaves a collection of vectors which still form a frame (i.e., they still span the space).

What follows are some of the basic results for Parseval frames, beginning with a very important theorem which shows that every frame has a Parseval frame associated with it.

Theorem 2.2 *For any frame* $\{v_i\}_{i=1}^k$*, the set* $\{S^{-1/2}v_i\}_{i=1}^k$ *is a Parseval frame.*

Proof: It is enough to show that $\{S^{-1/2}v_i\}_{i=1}^k$ satisfies the reconstruction formula

$$\begin{aligned} x &= (S^{-1/2}SS^{-1/2})x \\ &= S^{-1/2}S(S^{-1/2}x) \\ &= S^{-1/2}\sum_{i=1}^{k}\langle S^{-1/2}x, v_i\rangle v_i \\ &= \sum_{i=1}^{k}\langle x, S^{-1/2}v_i\rangle S^{-1/2}v_i \end{aligned}$$

since S, and thus $S^{-1/2}$, is self-adjoint. Therefore, $\{S^{-1/2}v_i\}_{i=1}^k$ satisfies the reconstruction formula and is a Parseval frame.

■

Proposition 2.1 *If* $\{v_i\}_{i=1}^k$ *is a Parseval frame, then for all* i, $\|v_i\| \leq 1$, *with equality if, and only if,* v_i *is orthogonal to all* v_j *where* $j \neq i$.

Proof: From the Parseval identity,

$$\|v_j\|^2 = \sum_{i=1}^{k} |\langle v_j, v_i\rangle|^2 \geq |\langle v_j, v_j\rangle|^2 = \|v_j\|^4$$

Thus, $\|v_j\| \leq 1$ for all j.

For equality, suppose $\|v_j\| = 1$. Then

$$1 = \|v_j\|^2 = \sum_{i=1}^{k} |\langle v_j, v_i\rangle|^2 = |\langle v_j, v_j\rangle|^2 + \sum_{i \neq j} |\langle v_j, v_i\rangle|^2$$

which implies that $\langle v_j, v_i\rangle = 0$ for all $i \neq j$.

Conversely, suppose v_j is orthogonal to all $j \neq i$. Then

$$\|v_j\|^2 = \sum_{i=1}^{k} |\langle v_j, v_i\rangle|^2 = |\langle v_j, v_j\rangle|^2 = \|v_j\|^4$$

so that $\|v_j\| = 1$ if $v_j \neq 0$.

■

This leads to the following Corollary.

Corollary 2.1 *Let* $\{v_i\}_{i=1}^k$ *be a Parseval frame. Then* $\{v_i\}_{i=1}^k$ *is an orthonormal basis if, and only if, every* v_i *is a unit vector.*

In fact, a more general proposition proved identically to Proposition 2.1 gives $\|v_i\|^2 \leq A$ for a tight frame with frame bound A, with equality only for v_i which are orthogonal to all v_j such that $j \neq i$.

For a general frame with upper frame bound B

$$B\|v_j\|^2 \geq \sum_{i=1}^{k} |\langle v_j, v_i\rangle|^2 \geq |\langle v_j, v_j\rangle|^2 = \|v_j\|^4$$

so that $\|v_i\|^2 \leq B$ for all $v_i \neq 0$.

Lemma 2.1 *Let H be a Hilbert space of dimension $n \leq k$. If $\{v_i\}_{i=1}^k$ is a uniform Parseval frame, then $\|v_i\|^2 = \frac{n}{k}$ for all i.*

Proof: Since the frame is Parseval, $\Theta^*\Theta = I_n$ by Property 2.7. By the definition of uniform, $\|v_i\| = \|v_j\|$ for all i, j. Combining this with Property 2.9 gives, for any j

$$\begin{aligned}
\|v_j\|^2 &= \frac{1}{k}\sum_{i=1}^{k} \|v_i\|^2 \\
&= \frac{1}{k}\operatorname{tr}(\Theta\Theta^*) \\
&= \frac{1}{k}\operatorname{tr}(\Theta^*\Theta) \\
&= \frac{1}{k}\operatorname{tr}(I_n) \\
&= \frac{n}{k}
\end{aligned}$$

■

Theorem 2.3 *Let H be a Hilbert space of dimension $n < k$. If $\{v_i\}_{i=1}^k \subseteq H$ is a uniform Parseval frame, then for any index j, $\{v_i\}_{i\neq j}$ is still a frame.*

Proof: From Equation 2.1, $\forall x \in H$

$$\begin{aligned}
\|x\|^2 &= \sum_{i=1}^{k} |\langle x, v_i \rangle|^2 \\
&= |\langle x, v_j \rangle|^2 + \sum_{\substack{i=1 \\ i \neq j}}^{k} |\langle x, v_i \rangle|^2 \\
&\leq \|x\|^2 \|v_j\|^2 + \sum_{\substack{i=1 \\ i \neq j}}^{k} |\langle x, v_i \rangle|^2
\end{aligned}$$

by Cauchy-Schwarz. Thus

$$\|x\|^2 (1 - \|v_j\|^2) \leq \sum_{\substack{i=1 \\ i \neq j}}^{k} |\langle x, v_i \rangle|^2$$

Therefore, provided $(1 - \|v_j\|^2) > 0$, there is a lower frame bound and so the remaining set of vectors is a frame. But by Lemma 2.1, $\|v_j\|^2 = \frac{n}{k}$ for all j, and since

$$\begin{aligned}
k &> n \\
1 &> \frac{n}{k} \\
1 - \frac{n}{k} &> 0
\end{aligned}$$

the lower frame bound exists, and the set of vectors is a frame.

■

In fact this is a special case of a more general theorem which makes use of Proposition 2.1.

Theorem 2.4 *Let H be a Hilbert space of dimension $n \leq k$. If $\{v_i\}_{i=1}^{k} \subseteq H$ is a*

Parseval frame, then for any index j where v_j is not orthogonal to every v_i such that $i \neq j$, $\{v_i\}_{i \neq j}$ is still a frame.

Proof: The proof is similar to the proof of Theorem 2.3, where Proposition 2.1 implies that $\|v_j\| < 1$ since v_j is not orthogonal to every v_i with $i \neq j$. Thus $(1 - \|v_j\|^2) > 0$, so there is a lower frame bound for the remaining vectors, as required.

■

Note that for an orthonormal basis, there is no such v_j not orthogonal to every other v_i, which corresponds to the fact that removing one vector from an orthonormal basis leaves a set which does not span the space.

2.4 General Reconstruction Formula and Dual Frames

For general frames, there is also a reconstruction formula similar to Equation 2.3. Let $\{v_i\}_{i=1}^k$ be a frame for H. Then, there exists a frame $\{w_i\}_{i=1}^k$ such that every $x \in H$ can be reconstructed with the formula

$$x = \sum_{i=1}^{k} \langle x, w_i \rangle v_i = \sum_{i=1}^{k} \langle x, v_i \rangle w_i \tag{2.4}$$

Definition 2.8 *Let $\{v_i\}_{i=1}^k$ be a frame for H. Then any frame $\{w_i\}_{i=1}^k$ which satisfies Equation 2.4 is called a **dual frame** for $\{v_i\}_{i=1}^k$.*

The frame $\{S^{-1}v_i\}_{i=1}^k$ always satisfies this reconstruction formula and is called the **canonical** (or **standard**) **dual frame**. Any other frame which satisfies the equation is called an **alternate dual frame**.

Proposition 2.2 *The set of vectors* $\{S^{-1}v_i\}_{i=1}^k$ *is a dual frame for* $\{v_i\}_{i=1}^k$.

Proof: Substituting $S^{-1}x$ for x in Equation 2.2 gives

$$\begin{aligned} Sx &= \sum_{i=1}^{k} \langle x, v_i \rangle v_i \\ S(S^{-1}x) &= \sum_{i=1}^{k} \langle S^{-1}x, v_i \rangle v_i \\ x &= \sum_{i=1}^{k} \langle x, S^{-1}v_i \rangle v_i \end{aligned}$$

since S, and thus S^{-1}, is self-adjoint. Similarly, applying S^{-1} to both sides of Equation 2.2 gives

$$\begin{aligned} Sx &= \sum_{i=1}^{k} \langle x, v_i \rangle v_i \\ S^{-1}(Sx) &= S^{-1}\left(\sum_{i=1}^{k} \langle x, v_i \rangle v_i\right) \\ x &= \sum_{i=1}^{k} \langle x, v_i \rangle S^{-1}v_i \end{aligned}$$

Therefore

$$x = \sum_{i=1}^{k} \langle x, S^{-1}v_i \rangle v_i = \sum_{i=1}^{k} \langle x, v_i \rangle S^{-1}v_i \tag{2.5}$$

as required.

■

The next result gives a characterization for all of the alternate dual frames of a given frame.

Proposition 2.3 *Every dual frame of $\{v_i\}$ is of the form $w_i = S^{-1}v_i + h_i$, where*

$$\sum_{i=1}^{k}\langle x, v_i\rangle h_i = \sum_{i=1}^{k}\langle x, h_i\rangle v_i = 0, \quad \forall x \in H$$

Proof: Let $\{w_i\}$ be a dual frame for $\{v_i\}$, and define $h_i = w_i - S^{-1}v_i$

$$\begin{aligned}\sum_{i=1}^{k}\langle x, v_i\rangle h_i &= \sum_{i=1}^{k}\langle x, v_i\rangle (w_i - S^{-1}v_i)\\ &= \sum_{i=1}^{k}\langle x, v_i\rangle w_i - \sum_{i=1}^{k}\langle x, v_i\rangle S^{-1}v_i\\ &= x - x\\ &= 0\end{aligned}$$

Conversely, suppose $\{w_i\}_{i=1}^{k}$ is a set of vectors such that $w_i = S^{-1}v_i + h_i$, where $\sum_{i=1}^{k}\langle x, v_i\rangle h_i = 0$. Then

$$\begin{aligned}\sum_{i=1}^{k}\langle x, v_i\rangle w_i &= \sum_{i=1}^{k}\langle x, v_i\rangle (S^{-1}v_i + h_i)\\ &= \sum_{i=1}^{k}\langle x, v_i\rangle S^{-1}v_i + \sum_{i=1}^{k}\langle x, v_i\rangle h_i\\ &= x + 0\\ &= x\end{aligned}$$

Thus $\{w_i\}$ satisfies the dual reconstruction formula, and so it is a dual frame for $\{v_i\}$. The proof for $\sum_{i=1}^{k}\langle x, h_i\rangle v_i = 0$ is similar.

■

The condition that $\sum_{i=1}^{k}\langle x, v_i\rangle h_i = 0$, or in operator notation $\Theta_h^*\Theta_v = 0$, is related to the concept of orthogonal frames. See Section 2.5 for more details.

Next, we review some properties of the canonical dual frame.

Let $\{v_i\}_{i=1}^k$ be a frame with frame operator S and canonical dual $\{S^{-1}v_i\}_{i=1}^k$.

Property 2.10 *If $\{v_i\}_{i=1}^k$ is a Parseval frame, then it is its own canonical dual.*

This follows from Property 2.7, that the frame operator of a Parseval frame is the identity operator $S = I$, so that $\{S^{-1}v_i\}_{i=1}^k = \{v_i\}_{i=1}^k$.

Property 2.11 *If $\{v_i\}_{i=1}^k$ is a tight frame with frame bound λ, then $\{\lambda^{-1}v_i\}_{i=1}^k$ is its canonical dual.*

This follows from Property 2.8, that the frame operator of a λ-tight frame is $S = \lambda I$, so that $\{S^{-1}v_i\}_{i=1}^k = \{\lambda^{-1}v_i\}_{i=1}^k$.

The next two properties together show that a frame and its canonical dual frame are actually canonical duals of each other.

Property 2.12 *The frame operator for $\{S^{-1}v_i\}_{i=1}^k$ is S^{-1}.*

Proof: Let $\Theta_{S^{-1}v}$ be the analysis operator for the canonical dual, so that $T = \Theta_{S^{-1}v}^*\Theta_{S^{-1}v}$ is its frame operator. Then by Property 2.3

$$\begin{aligned}
T &= \Theta_{S^{-1}v}^*\Theta_{S^{-1}v} \\
&= (S^{-1}\Theta_v^*)(\Theta_v S^{-1}) \\
&= S^{-1}(\Theta_v^*\Theta_v)S^{-1} \\
&= S^{-1}(S)S^{-1} \\
&= S^{-1}
\end{aligned}$$

■

Property 2.13 *The canonical dual of* $\{S^{-1}v_i\}_{i=1}^k$ *is* $\{v_i\}_{i=1}^k$.

That $\{S^{-1}v_i\}_{i=1}^k$ and $\{v_i\}_{i=1}^k$ are dual frames of each other is readily apparent from the definition of dual frames and Equation 2.5. That $\{S^{-1}v_i\}_{i=1}^k$ is the canonical dual of $\{v_i\}_{i=1}^k$ follows from Property 2.12, since $T^{-1} = S$.

2.5 Orthogonal Frames

Let $\{v_i\}_{i=1}^k \subset H$ and $\{w_i\}_{i=1}^k \subset K$ with $\Theta_v : H \to \mathbb{C}^k$ and $\Theta_w : K \to \mathbb{C}^k$ their respective analysis operators. Then $\{v_i\}$ and $\{w_i\}$ are *orthogonal sequences* if the range space of Θ_v is orthogonal to the range space of Θ_w, that is, $\Theta_v(H) \perp \Theta_w(K)$. In addition, if $\{v_i\}$ and $\{w_i\}$ are frames for their respective spaces, they are called *orthogonal frames.*

Orthogonal frames are useful in applications such as multiplexing of data and will be revisited in Chapter 5. In addition, the following result shows that orthogonal frames are related to the characterization of all possible alternate dual frames as given in Proposition 2.3.

Proposition 2.4 $\Theta_v(H) \perp \Theta_w(K)$ *if and only if* $\Theta_v^*\Theta_w = 0$

Proof: First, suppose $\Theta_v^*\Theta_w = 0$. Then for all $a \in H$, $b \in K$

$$\begin{aligned} \langle \Theta_v a, \Theta_w b \rangle &= \langle a, \Theta_v^*\Theta_w b \rangle \\ &= \langle a, 0 \rangle \\ &= 0 \end{aligned}$$

So that $\Theta_v(H) \perp \Theta_w(K)$.

Now, suppose that $\Theta_v(H) \perp \Theta_w(K)$. Then for all $a \in H$, $b \in K$

$$0 = \langle \Theta_v a, \Theta_w b \rangle = \langle a, \Theta_v^* \Theta_w b \rangle$$

and since this must hold for all $a \in H$, $\Theta_v^* \Theta_w b = 0$. And since this must hold for all $b \in K$, $\Theta_v^* \Theta_w = 0$.

■

2.6 Similar Frames

Definition 2.9 *Let $\{v_i\}_{i=1}^k$ and $\{w_i\}_{i=1}^k$ be frames. Then these frames are said to be **similar** if there exists an invertible operator T such that $Tv_i = w_i$ for $i = 1, \ldots, k$. If T is a unitary operator, the frames are said to be **unitarily equivalent**.*

Similar frames are denoted as $\{v_i\}_{i=1}^k \sim \{w_i\}_{i=1}^k$. Clearly every frame is similar to its canonical dual frame. More importantly, from Theorem 2.2 every frame is similar to a Parseval frame.

Property 2.14 *Similarity (and consequently unitary equivalence) is an equivalence relation.*

Proof: Let $\{v_i\}, \{w_i\}$, and $\{z_i\}$ be frames. Let T, S be invertible operators.

- Reflexive: For all i, $Iv_i = v_i$, so that $\{v_i\} \sim \{v_i\}$.

- Symmetric: For all i, if $Tv_i = w_i$, then $v_i = T^{-1} w_i$.

- Transitive: For all i, if $Tv_i = w_i$ and $Sw_i = z_i$, then $z_i = Sw_i = S(Tv_i) = (ST)v_i$, with (ST) invertible.

■

Proposition 2.5 *If $\{v_i\}$ and $\{w_i\}$ are similar, Range(Θ_v) = Range(Θ_w).*

Proof: Suppose $Av_i = w_i$, for all i, where A is invertible. Let $y \in \text{Range}(\Theta_v)$. Then there exists x such that

$$y = \Theta_v x = \Theta_{A^{-1}w} x = \Theta_w (A^{-1})^* x$$

so that $y \in \text{Range}(\Theta_w)$. Thus $\text{Range}(\Theta_v) \subseteq \text{Range}(\Theta_w)$.

Similarly, if $y \in \text{Range}(\Theta_w)$, then there exists x such that

$$y = \Theta_w x = \Theta_{Av} x = \Theta_v A^* x$$

so that $y \in \text{Range}(\Theta_v)$. Thus $\text{Range}(\Theta_w) \subseteq \text{Range}(\Theta_v)$.

Therefore, $\text{Range}(\Theta_v) = \text{Range}(\Theta_w)$.

■

Note that if $\{v_i\}$ and $\{w_i\}$ are unitarily equivalent, this becomes

$$\Theta_v x = \Theta_w A x \quad \text{and} \quad \Theta_w x = \Theta_v A^* x$$

2.7 Operator Trace Using Dual Frames

Let T be a linear operator on a Hilbert space H having dimension n. Then the trace of T is defined as $\text{tr}(T) = \sum_{i=1}^{n} \langle Te_i, e_i \rangle$, for any orthonormal basis $\{e_i\}_{i=1}^{n}$ of H.

The standard proof of the independence of the choice of the orthonormal basis for calculating the trace appears to use only the reconstruction property of orthonormal bases. So consider the following generalizations using dual frames.

Theorem 2.5 *Let T be a linear operator on H having dimension n. Let $k \geq n$ and $\ell \geq n$. If $\{e_i\}_{i=1}^{k}$ and $\{f_i\}_{i=1}^{\ell}$ are frames, with dual frames $\{v_i\}_{i=1}^{k}$ and $\{w_i\}_{i=1}^{\ell}$ respectively, then*

$$\sum_{i=1}^{k} \langle Te_i, v_i \rangle = \sum_{i=1}^{\ell} \langle Tf_i, w_i \rangle$$

Proof:

$$\begin{aligned}
\sum_{i=1}^{k} \langle Te_i, v_i \rangle &= \sum_{i=1}^{k} \langle \left(\sum_{j=1}^{\ell} \langle Te_i, w_j \rangle f_j \right), v_i \rangle \\
&= \sum_{i=1}^{k} \sum_{j=1}^{\ell} \langle Te_i, w_j \rangle \langle f_j, v_i \rangle \\
&= \sum_{j=1}^{\ell} \sum_{i=1}^{k} \langle f_j, v_i \rangle \langle e_i, T^* w_j \rangle \\
&= \sum_{j=1}^{\ell} \langle \left(\sum_{i=1}^{k} \langle f_j, v_i \rangle e_i \right), T^* w_j \rangle \\
&= \sum_{j=1}^{\ell} \langle f_j, T^* w_j \rangle \\
&= \sum_{j=1}^{\ell} \langle Tf_j, w_j \rangle
\end{aligned}$$

■

Corollary 2.2 *For any Parseval frame* $\{f_i\}_{i=1}^k$, $tr(T) = \sum_{i=1}^k \langle Tf_i, f_i\rangle$.

Proof: For any orthonormal basis $\{e_i\}_{i=1}^n$ of H, using $\{e_i\}_{i=1}^n = \{v_i\}_{i=1}^n$ and $\{f_i\}_{i=1}^k = \{w_i\}_{i=1}^k$ in the above theorem gives

$$\begin{aligned}\mathrm{tr}(T) &= \sum_{i=1}^n \langle Te_i, e_i\rangle \\ &= \sum_{i=1}^k \langle Tf_i, f_i\rangle\end{aligned}$$

■

Using $T = S = I_n$ in the above Corollary gives an alternate proof that the sum of the frame vector norms is the dimension of H, without using the Grammian matrix.

Corollary 2.3 *For any Parseval frame,* $\{f_i\}_{i=1}^k$, *on a Hilbert space* H *of dimension* n, $\sum_{i=1}^k \|f_i\|^2 = n$.

Proof:

$$\begin{aligned}n &= \mathrm{tr}(I_n) \\ &= \sum_{i=1}^k \langle I_n f_i, f_i\rangle \\ &= \sum_{i=1}^k \|f_i\|^2\end{aligned}$$

■

The following Corollaries for general frames mirror the ones for Parseval frames

Corollary 2.4 *For any frame* $\{f_i\}_{i=1}^k$ *with dual frame* $\{w_i\}_{i=1}^k$, *tr*$(T) = \sum_{i=1}^k \langle Tf_i, w_i\rangle$.

Proof: For any orthonormal basis $\{e_i\}_{i=1}^n$ of H, using $\{e_i\}_{i=1}^n = \{v_i\}_{i=1}^n$ in the above theorem gives

$$\begin{aligned} \mathrm{tr}(T) &= \sum_{i=1}^n \langle Te_i, e_i\rangle \\ &= \sum_{i=1}^k \langle Tf_i, w_i\rangle \end{aligned}$$

■

Using $T = I_n$ in the above Corollary

Corollary 2.5 *For any frame,* $\{f_i\}_{i=1}^k$ *with dual frame* $\{w_i\}_{i=1}^k$, *on a Hilbert space* H *of dimension* n, $\sum_{i=1}^k \langle f_i, w_i\rangle = n$.

Proof:

$$\begin{aligned} n &= \mathrm{tr}(I_n) \\ &= \sum_{i=1}^k \langle I_n f_i, w_i\rangle \\ &= \sum_{i=1}^k \langle f_i, w_i\rangle \end{aligned}$$

■

In addition, if the dual used is the canonical dual

Corollary 2.6 *For any frame,* $\{f_i\}_{i=1}^k$ *with frame operator* S *and canonical dual frame* $\{S^{-1}f_i\}_{i=1}^k$, *tr*$(S) = \sum_{i=1}^k \|f_i\|^2$ *and tr*$(S^{-1}) = \sum_{i=1}^k \|S^{-1}f_i\|^2$.

Proof:

$$\begin{aligned}
\mathrm{tr}(S) &= \sum_{i=1}^{k} \langle S(S^{-1}f_i), f_i \rangle \\
&= \sum_{i=1}^{k} \langle f_i, f_i \rangle \\
&= \sum_{i=1}^{k} \|f_i\|^2 \\
\mathrm{tr}(S^{-1}) &= \sum_{i=1}^{k} \langle S^{-1}f_i, S^{-1}f_i \rangle \\
&= \sum_{i=1}^{k} \|S^{-1}f_i\|^2
\end{aligned}$$

■

Finally, the independence of the choice of Parseval frame in the trace shows that no Parseval frame can have a Parseval frame as a proper subset (with non-zero vectors omitted).

Corollary 2.7 *If $\{f_i\}_{i=1}^{\ell}$ is a Parseval frame for a Hilbert space H of dimension n, then there is no Parseval frame $\{f_i\}_{i=1}^{k}$, with $n \leq k < \ell$, with some $f_j \neq 0$ for $k < j \leq \ell$.*

Proof: The proof is by contradiction. Let $\{f_i\}_{i=1}^{k} \subset \{f_i\}_{i=1}^{\ell}$ both be Parseval frames, with $f_j \neq 0$ for some $k < j \leq \ell$. Then by the trace,

$$\sum_{i=1}^{\ell} \|f_i\|^2 - \sum_{i=1}^{k} \|f_i\|^2 = n - n = 0$$

$$\sum_{i=k+1}^{\ell} \|f_i\|^2 + \sum_{i=1}^{k} \|f_i\|^2 - \sum_{i=1}^{k} \|f_i\|^2 = 0$$

$$\sum_{i=k+1}^{\ell} \|f_i\|^2 = 0$$

But this is impossible if $f_j \neq 0$ for some $k < j \leq \ell$. Therefore, there is no such Parseval frame with a Parseval frame (or orthonormal basis) as a proper subset.

■

CHAPTER 3

GROUP REPRESENTATION FRAMES

Several special classes of finite frames exist by allowing for some structure in the set of frame vectors, rather than just an arbitrary collection. One such structure is the group structure. Specific types of groups, such as cyclic and abelian, will demonstrate different properties in their associated frames.

3.1 Unitary Representations

Let G be a group, and H a Hilbert space. Let $B(H)$ denote the set of bounded, linear operators from H to H. An operator $T \in B(H)$ is called unitary if $T^* = T^{-1}$, that is, if the adjoint operator of T is the inverse operator of T.

Definition 3.1 *The set of all unitary operators of H form a group, and a group homomorphism π from G into this group of unitary operators is called a* ***unitary representation****.*

Definition 3.2 *For any unitary representation π, the* ***commutant*** *is the set*

$$\pi(G)' = \{T \in B(H) \mid T\pi(g) = \pi(g)T, \forall g \in G\}$$

The following properties hold for any unitary representation.

Property 3.1 *For any unitary representation π, $\pi(e) = I$.*

Proof: Since π is a homomorphism

$$\begin{aligned}\pi(e) &= \pi(gg^{-1})\\ &= \pi(g)\pi(g^{-1})\\ &= \pi(g)\pi(g)^{-1}\\ &= I\end{aligned}$$

■

Property 3.2 *For any unitary representation π, $\pi(g)^* = \pi(g^{-1})$.*

Since $\pi(g)$ is unitary, $\pi(g)^* = \pi(g)^{-1} = \pi(g^{-1})$

Property 3.3 *For any unitary representation π, $\|\pi(g)\xi\| = \|\xi\|$ for all $g \in G$.*

Proof: Since π(g) is unitary

$$\begin{aligned}\|\pi(g)\xi\|^2 &= \langle \pi(g)\xi, \pi(g)\xi\rangle\\ &= \langle \xi, \pi(g)^*\pi(g)\xi\rangle\\ &= \langle \xi, \xi\rangle\\ &= \|\xi\|^2\end{aligned}$$

■

3.2 Frame Representations

Definition 3.3 *A unitary representation on a Hilbert space H is called a **frame representation** if $\exists\xi \in H$ such that $\{\pi(g)\xi\}_{g\in G}$ is a frame for H. In this case ξ is* *called a **frame vector** for π, and π is said to admit a frame.*

For frame representations, most definitions follow directly from those of a general frame. Let $\{\pi(g)\xi\}_{g\in G}$ be a collection of vectors in H, and $\{\chi_g\}$ an orthonormal basis for $K = \ell^2(G)$.

Definition 3.4 *The **analysis operator** is the operator $\Theta : H \to K$ defined by*

$$\Theta x = \sum_{g\in G} \langle x, \pi(g)\xi\rangle \chi_g$$

A given frame representation can admit multiple frame vectors, and so the subscripted notation will be useful

$$\Theta_\xi = \sum_{g\in G} \langle x, \pi(g)\xi\rangle \chi_g \quad \text{and } \Theta_\eta = \sum_{g\in G} \langle x, \pi(g)\eta\rangle \chi_g$$

Note that this notation is inconsistent with the notation given in Chapter 2 for general frames, since here the subscript denotes the frame vector of the representation, rather than the vectors themselves. For example, when applying a linear operator, as in Property 2.3, the notation $\Theta_{T\xi}$ means

$$\Theta_{T\xi} = \sum_{g\in G} \langle x, \pi(g)T\xi\rangle \chi_g$$

rather than

$$\sum_{g\in G} \langle x, T\pi(g)\xi\rangle \chi_g$$

In general, these two are different unless $T \in \pi(G)'$.

Definition 3.5 *The **synthesis operator** is the adjoint of the analysis operator.*

$$\Theta^* x = \sum_{g \in G} \langle x, \chi_g \rangle \pi(g)\xi$$

or

$$\Theta^* \chi_g = \pi(g)\xi$$

Definition 3.6 *The* ***frame operator*** *is the operator* $\Theta^*\Theta$*.*

$$\Theta^*\Theta x = \sum_{g \in G} \langle x, \pi(g)\xi \rangle \pi(g)\xi$$

Definition 3.7 *The* ***Grammian operator*** *is the operator* $\Theta\Theta^*$*.*

In addition, it will also be useful to take compositions of operators associated with different frame vectors for a given representation. That is, to create operators of the form

$$\Theta_\eta^* \Theta_\xi = \sum_{g \in G} \langle x, \pi(g)\xi \rangle \pi(g)\eta$$

The first result is a basic property of frame representations.

Proposition 3.1 *If* $\pi(g)$ *admits a frame, it is a uniform frame.*

Proof: This is an immediate consequence of Property 3.3, since every vector in the frame has the same norm, the norm of the frame vector ξ.

∎

In addition to the properties for the analysis and frame operators given in Chapter 2, we have the following additional property for the frame operator of a group representation frame.

Property 3.4 *The frame operator is in the commutant of* $\pi(g)$.

Proof: This follows from $S = \Theta_\xi^* \Theta_\xi$ and Lemma 3.3. See Section 3.3 for more details.

■

Consequently, S^{-1} is also in the commutant of $\pi(g)$. This leads to the following

Proposition 3.2 *If* ξ *is a frame vector so that* $\{\pi(g)\xi\}_{g\in G}$ *is a frame for* H*, then* $S^{-1}\xi$ *generates a dual frame for* $\{\pi(g)\xi\}_{g\in G}$.

Proof: From Proposition 2.2, $\{S^{-1}\pi(g)\xi\}_{g\in G}$ is a dual frame for $\{\pi(g)\xi\}_{g\in G}$, and since S^{-1} commutes with $\pi(g)$, this becomes $\{\pi(g)S^{-1}\xi\}_{g\in G}$.

■

We say that $(\xi, S^{-1}\xi)$ form a *dual pair*.

3.3 Commutant of Group Representation Frames

Lemma 3.1 *If* $T \in \pi(G)'$*, then* $\Theta_{T\xi} = \Theta_\xi T^*$

Proof: For all $x \in H$

$$\begin{aligned}
\Theta_{T\xi} x &= \sum_{g\in G} \langle x, \pi(g) T\xi \rangle \chi_g \\
&= \sum_{g\in G} \langle x, T\pi(g)\xi \rangle \chi_g \\
&= \sum_{g\in G} \langle T^* x, \pi(g)\xi \rangle \chi_g \\
&= \Theta_\xi T^* x
\end{aligned}$$

■

Lemma 3.2 *If $\pi(g)$ admits a Parseval frame vector ξ, then* $\pi(G)' \subseteq \{\Theta_y^*\Theta_z \mid y, z \in H\}$

Proof: Let $T \in \pi(G)'$. Then from Lemma 3.1

$$\begin{aligned} T &= I \cdot T \\ &= \Theta_\xi^*\Theta_\xi \cdot T \\ &= \Theta_\xi^*\Theta_{T^*\xi} \end{aligned}$$

so that $T \in \{\Theta_y^*\Theta_z \mid y, z \in H\}$. Therefore, $\pi(G)' \subseteq \{\Theta_y^*\Theta_z \mid y, z \in H\}$.

■

In addition

Lemma 3.3 $\{\Theta_y^*\Theta_z \mid y, z \in H\} \subseteq \pi(G)'$

Proof: If $T = \Theta_y^*\Theta_z$ for some $y, z \in H$, then for any $x \in H$

$$
\begin{aligned}
T\pi(h)x &= \Theta_y^*\Theta_z\pi(h)x \\
&= \sum_{g\in G}\langle \pi(h)x, \pi(g)z\rangle \pi(g)y \\
&= \sum_{g\in G}\langle x, \pi(h)^*\pi(g)z\rangle \pi(g)y \\
&= \sum_{g\in G}\langle x, \pi(h^{-1}g)z\rangle \pi(g)y \\
&= \pi(h)\pi(h^{-1})\sum_{g\in G}\langle x, \pi(h^{-1}g)z\rangle \pi(g)y \\
&= \pi(h)\sum_{g\in G}\langle x, \pi(h^{-1}g)z\rangle \pi(h^{-1})\pi(g)y \\
&= \pi(h)\sum_{g\in G}\langle x, \pi(h^{-1}g)z\rangle \pi(h^{-1}g)y \\
&= \pi(h)\sum_{\widetilde{g}\in G}\langle x, \pi(\widetilde{g})z\rangle \pi(\widetilde{g})y \\
&= \pi(h)\Theta_y^*\Theta_z x \\
&= \pi(h)Tx
\end{aligned}
$$

so that, $T \in \pi(G)'$. Therefore, $\{\Theta_y^*\Theta_z \mid y, z \in H\} \subseteq \pi(G)'$.

■

Theorem 3.1 *If $\pi(g)$ admits a Parseval frame vector ξ, then $\pi(G)' = \{\Theta_\xi^*\Theta_\eta \mid \xi, \eta \in H\}$*

Proof: Follows from Lemma 3.2 and Lemma 3.3.

■

In fact, a more general result is given by the following.

Theorem 3.2 $\pi(G)' = span\{\Theta_\xi^*\Theta_\eta \mid \xi, \eta \in H\}$

Theorem 3.3 *Range* Θ_ξ = *Range* $\Theta_\eta \iff \xi \sim \eta$ *i.e.* $\exists$ *invertible* $A \in \pi(G)'$ *s.t.* $A\xi = \eta \iff A\pi(g)\xi = \pi(g)\eta$

3.4 Unitary Equivalence

Recall that if two frames are unitarily equivalent, denoted $\{\pi(g)\xi\}_{g\in G} \sim \{\sigma(g)\eta\}_{g\in G}$, then there exists a unitary operator U such that for all $g \in G$

$$U\pi(g)\xi = \sigma(g)\eta$$

In addition, if two unitary representations are unitarily equivalent, denoted $\pi \sim \sigma$, then there exists a unitary operator U such that for all $g \in G$

$$U\pi(g) = \sigma(g)U$$

Lemma 3.4 *If* $\{\pi(g)\xi\} \sim \{\sigma(g)\eta\}$ *as frames, then* $U\xi = \eta$.

Proof: Let $\{\pi(g)\xi\} \sim \{\sigma(g)\eta\}$ as frames. Then, by definition, there exists a unitary operator U such that $U\pi(g)\xi = \sigma(g)\eta$, $\forall g \in G$. Thus, by Property 3.1

$$U\pi(e)\xi = \sigma(e)\eta$$

$$UI\xi = I\eta$$

$$U\xi = \eta$$

■

Lemma 3.5 *If $\{\pi(g)\xi\} \sim \{\sigma(g)\eta\}$ as frames, then $\pi \sim \sigma$.*

Proof: Let $\{\pi(g)\xi\} \sim \{\sigma(g)\eta\}$ as frames. Then, by definition, there exists a unitary operator U such that $U\pi(g)\xi = \sigma(g)\eta$, $\forall g \in G$. Thus, by Lemma 3.4

$$\begin{aligned} U\pi(g)\xi &= \sigma(g)\eta \\ &= \sigma(g)U\xi \\ U\pi(g) &= \sigma(g)U \end{aligned}$$

Therefore, $\pi \sim \sigma$ by definition.

■

3.5 Abelian and Cyclic Groups

An abelian group is one where the group operation is commutative.

A cyclic group $G = \langle a \rangle$ is a group of the form $\{a^i \mid i \in \mathbb{Z}\}$, where a^0 is the identity, and $a^{i+j} = a^i a^j$. An immediate consequence is that every cyclic group is abelian.

Frame representations induced by abelian and cyclic groups have additional properties.

One example of a group frame where the group is cyclic is the Mercedes-Benz frame from Example 2.2. This frame is equivalent to the 3rd roots of unity, and each vector in the frame comes from a group operation of "rotation by 120 degrees".

Example 3.1 (Mercedes-Benz Frame) *The vectors $\{x_i\}_{i=1}^3$ given by*

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{bmatrix} \right\}$$

is a frame for $H = \mathbb{R}^2$*.*

For one particular cyclic group, the K-th roots of unity, the following lemma will be useful.

Lemma 3.6 *If* $e^{2\pi i \frac{A}{K}}$ *is a* K*-th root of unity that is not equal to 1, then*

$$\sum_{n=0}^{K-1} e^{2\pi i \frac{A}{K} n} = 0$$

Proof:

$$\begin{aligned}
e^{2\pi i \frac{A}{K}} \sum_{n=0}^{K-1} e^{2\pi i \frac{A}{K} n} &= \sum_{n=1}^{K} e^{2\pi i \frac{A}{K} n} \\
&= \left(\sum_{n=1}^{K-1} e^{2\pi i \frac{A}{K} n} \right) + e^{2\pi i \frac{A}{K} K} \\
&= \left(\sum_{n=1}^{K-1} e^{2\pi i \frac{A}{K} n} \right) + e^{2\pi i A} \\
&= \left(\sum_{n=1}^{K-1} e^{2\pi i \frac{A}{K} n} \right) + 1 \\
&= \left(\sum_{n=1}^{K-1} e^{2\pi i \frac{A}{K} n} \right) + e^{2\pi i 0} \\
&= \sum_{n=0}^{K-1} e^{2\pi i \frac{A}{K} n}
\end{aligned}$$

Therefore, $e^{2\pi i \frac{A}{K}} = 1$, or $\sum_{n=0}^{K-1} e^{2\pi i \frac{A}{K} n} = 0$.

■

The following results are summarized without proof from Han and Larson. The first is Proposition 1.14 in [19], which says that distinct alternate duals for a given frame are never similar.

Lemma 3.7 *Suppose that* $\{x_n\}$ *is a frame and* $\{y_n\}$ *is an alternate dual for* $\{x_n\}$*. If* $T \in B(H)$ *is an invertible operator such that* $\{Ty_n\}$ *is also an alternate dual for* $\{x_n\}$*, then* $T = I$*.*

The next result is Corollary 3.14 in [19], adapted for our notation.

Lemma 3.8 *Let* G *be an abelian group and let* π *be a representation of* G *on a Hilbert space* H*. Suppose that* ξ *is a Parseval frame vector of* π *for* H*. Then for every frame vector* η *for* H*, there is a (unique) invertible operator* $V \in \pi(G)'$ *such that* $\eta = V\xi$*.*

These two results combine together to prove the following.

Proposition 3.3 *Let* $\{\pi(g)\xi\}_{g\in G}$ *be a frame, with* G *an abelian group. Then the only dual frame for* $\{\pi(g)\xi\}_{g\in G}$ *with the same group structure is the canonical dual frame.*

3.6 Orthogonal Group Frames and Super-Frames

As discussed in Section 2.5, we say that two sequences are orthogonal if the range spaces of their respective analysis operators are orthogonal. This was shown to be equivalent to the condition $\Theta_2^*\Theta_1 = 0$. If the sequences are also frames we call them *orthogonal frames* or *strongly disjoint*.

In terms of group representation frames, suppose that ξ and η are frame vectors for π. Then $\{\pi(g)\xi\}_{g\in G}$ and $\{\pi(g)\eta\}_{g\in G}$ are orthogonal frames if $\Theta_\eta^*\Theta_\xi = 0$.

Proposition 3.4 *If* η *and* ζ *generate two dual frames for* $\{\pi(g)\xi\}_{g\in G}$*, then* $u = \eta - \zeta$ *generates a sequence which is orthogonal to* $\{\pi(g)\xi\}_{g\in G}$*.*

Proof: For every $x \in H$

$$\begin{aligned}\sum_{g\in G}\langle x, \pi(g)\xi\rangle\pi(g)u &= \sum_{g\in G}\langle x, \pi(g)\xi\rangle\pi(g)(\eta-\zeta)\\ &= \sum_{g\in G}\langle x, \pi(g)\xi\rangle\pi(g)\eta - \sum_{g\in G}\langle x, \pi(g)\xi\rangle\pi(g)\zeta\\ &= x - x\\ &= 0\end{aligned}$$

■

We say that (ξ, u) form an *orthogonal pair.*

Orthogonal frames have applications in multiplexing, and we briefly mention some of the ideas used later. Let $\{\phi_j^{(\ell)}\}_{j\in\mathcal{J}}$ be Parseval frames for Hilbert spaces H_ℓ, $\ell = 1, \ldots, k$. We say that $\left(\{\phi_j^{(1)}\}, \{\phi_j^{(2)}\}, \ldots, \{\phi_j^{(k)}\}\right)$ is a *disjoint k-tuple* if $\{\phi_j^{(1)} \oplus \ldots \oplus \phi_j^{(k)}\}$ is a frame for the orthogonal direct sum space $H_1 \oplus \ldots \oplus H_k$, and is a *strongly disjoint k-tuple* if it is a Parseval frame for the direct sum space. A strongly disjoint k-tuple is also called a *superframe* of length k [1, 16, 19].

Lemma 3.9 (Han-Larson) *$\{\phi_j^{(1)} \oplus \ldots \oplus \phi_j^{(k)}\}_{j\in\mathcal{J}}$ is a superframe for $H \oplus \ldots \oplus H$ if and only if all of the following hold*

(i) Each $\{\phi_j^{(\ell)}\}$ is a Parseval frame for H.

(ii) $\{\phi_j^{(m)}\}$ and $\{\phi_j^{(n)}\}$ are orthogonal when $m \neq n$

CHAPTER 4

ERASURES

4.1 Introduction

Suppose we wish to use frames to add redundancy to transmitted data. What follows is a simple example. Let $H = \mathbb{C}^2$ and $K = \mathbb{C}^3$. Suppose $\{v_i\}_{i=1}^3$ is a uniform Parseval frame with analysis operator Θ_v. A vector $x = (x_1, x_2)'$, which is the information to be transmitted, is encoded by computing $\Theta_v x$

$$\Theta_v x = \sum_{i=1} \langle x, v_i \rangle e_i = \begin{bmatrix} \langle x, v_1 \rangle \\ \langle x, v_2 \rangle \\ \langle x, v_3 \end{bmatrix}$$

The coefficients of $\Theta_v x$ can then be transmitted to a receiver, who recovers x by computing $\Theta_v^*(\Theta_v x)$

$$\Theta_v^*(\Theta_v x) = (\Theta_v^* \Theta_v)x = Ix = x$$

Suppose, however, that one of the coefficients is lost in tranmission, and the receiver only receives

$$\begin{bmatrix} \langle x, v_1 \rangle \\ 0 \\ \langle x, v_3 \rangle \end{bmatrix}$$

The receiver could still potentially recover the transmitted data. By Theorem 2.3, a uniform Parseval frame which loses one vector is still a frame, and so v_1 and v_3 span the space. Thus there exists α and β such that $v_2 = \alpha v_1 + \beta v_3$, and so

$$\begin{aligned} x &= \Theta_v^* \Theta_v x \\ &= \langle x, v_1 \rangle v_1 + \langle x, v_2 \rangle v_2 + \langle x, v_3 \rangle v_3 \\ &= \langle x, v_1 \rangle v_1 + \langle x, \alpha v_1 + \beta v_3 \rangle v_2 + \langle x, v_3 \rangle v_3 \\ &= \langle x, v_1 \rangle v_1 + \left(\overline{\alpha} \langle x, v_1 \rangle + \overline{\beta} \langle x, v_3 \rangle \right) v_2 + \langle x, v_3 \rangle v_3 \end{aligned}$$

where all of the inner products in the last equality were received in transmission. Thus the receiver can reconstruct x exactly.

This example, while demonstrating the redudant nature of frames, is greatly simplified. In fact, this type of reconstruction requires exact knowledge of which coefficient was lost in transmission, which is not available in most applications. In addition, as the number of vectors in the frame increases, the computations needed to recover x in this way increase in complexity, becoming unfeasable even when possible.

Fortunately, in many applications exact reconstruction is not always necessary, and so the study of erasures focuses on achieving an optimal estimation of x, given a loss during transmission.

4.2 Optimal Frames for Erasures

What follows is an overview of the typical method of dealing with erasures, as from [23], et. al.

Let H be a Hilbert space of dimension n, and $\{v_i\}_{i=1}^k$ a Parseval frame with analysis operator Θ. The original vector, x, can be coded as Θx and then transmitted to a receiver, who decodes the data by applying the synthesis operator

$$\Theta^*(\Theta x) = (\Theta^*\Theta)x = Ix = x$$

Suppose, however, that some number, say m, of the components of the vector Θx are lost, garbled, or delayed in transmission. The received vector can be represented as $E\Theta x$, where E is a diagonal matrix of m 0's and $k - m$ 1's, corresponding to the entries of Θx that are lost and received, respectively. The 0's in E can be thought of as the coordinates of Θx that have been "erased".

One option to recover the original data is to attempt to compute a left inverse for $E\Theta$. An alternative would be to continue to use Θ^* to reconstruct, accepting the fact that the recovered data is only an approximation of the original x. The error of the reconstruction is then given by

$$x - \Theta^* E\Theta x = \Theta^*(I - E)\Theta x = \Theta^* D\Theta x$$

where D is the diagonal matrix with m 1's, corresponding to the lost coordinates of Θx, and $k - m$ 0's, corresponding to the received coordinates.

The goal, then, is to find the "best" frames in this circumstance. That is, to find a frame for which the norm of this error operator is minimized, independently of the coordinates which are erased. Such frames would then be considered *optimal frames for m-erasures.*

To achieve independence of the erased coordinates, we must assign to each analysis operator (and, thus, to each Parseval frame), a number representing the worst-case for the norm of the error operator given m erasures. Thus, let $\mathcal{D}_m$ be the set of all $k \times k$ diagonal matrices with m 1's and $k - m$ 0's. Then

$$d_m(\Theta) = \max\{\|\Theta^* D\Theta\| \mid D \in \mathcal{D}_m\}$$

Now, minimizing $d_m(\Theta)$ over all possible Θ, would in some sense, be optimal. However, it would be preferable if a frame which is optimal for m erasures is, in fact, optimal for m or less erasures. Thus, we create the decreasing family of frames

$$\mathcal{E}_1(k,n) = \min_{\Theta \in \mathcal{F}(k,n)} d_1(\Theta)$$
$$\mathcal{E}_m(k,n) = \min_{\Theta \in \mathcal{E}_{m-1}(k,n)} d_m(\Theta)$$

where $\mathcal{F}(k,n)$ is the compact set of all Parseval frames of k vectors in $\mathbb{F}^n$. Thus, $\Theta \in \mathcal{E}_m$ implies that $\Theta \in \mathcal{E}_i$, for $1 \leq i \leq m$. Therefore, the optimal frames for m-erasures are the ones whose analysis operator is in $\mathcal{E}_m$, and they are sometimes referred to as *m-erasure frames*

There are several results for optimal frames. It was shown that uniform Parseval frames are optimal for one erasure (1-erasure frames). In addition equiangular, uniform Parseval frames are optimal for two erasures (2-erasure frames). These 2-erasure frames are also known as Grassmannian frames [30].

4.3 Optimal Dual Frames for Erasures

Optimal frames for erasures have some limitations. Firstly, a particular application may require that vectors be coded using frames with certain specific properties. For example, grouping higher concentrations of frame vectors together where more data is expected to occur would not allow the vectors to be equiangular. Or perhaps it is desirable for the coding frame and decoding frame to be different, rather than using a self-dual Parseval frame.

This leads to a slightly different scenario: consider coding a vector using a (not necessarily tight) frame already chosen, and then, if there are missing coordinates, reconstructing the vector using a dual frame that minimizes the error of the reconstruction. Such a dual frame will be referred to as an optimal dual with respect to erasures.

To make this precise, we adapt the notation from [23]. Let $\mathcal{D}_m$ be the set of all $k \times k$ diagonal matrices with m $1's$ and $k-m$ $0's$. For any dual frame pair (X, Y) with $X = \{x_i\}_{i=1}^k$ and $Y = \{y_i\}_{i=1}^k$, we define

$$d_m(X, Y) = \max\{||\Theta_Y^* D \Theta_X|| : D \in \mathcal{D}_m\},$$

where Θ_X and Θ_Y are the analysis operators for X and Y, respectively. If $J = \{i_1, \ldots, i_m\}$ are indices where 1 appears in D, then, when approximating x by $\bar{x} = \sum_{j \neq i_1, \ldots, i_m} \langle x, x_j \rangle y_j$, the *error operator* E_J with the given m erasures is

$$\begin{aligned} E_J x &= (\Theta_Y^* D \Theta_X)(x) \\ &= x - \sum_{j \neq i_1, \ldots, i_m} \langle x, x_j \rangle y_j \\ &= \sum_{i=1}^n \langle x, x_i \rangle y_i - \sum_{j \neq i_1, \ldots, i_m} \langle x, x_j \rangle y_j \\ &= \sum_{j = i_1, \ldots, i_m} \langle x, x_j \rangle y_j. \end{aligned}$$

So the measure of the error operator $\Theta_Y^* D \Theta_X$ tells us the accuracy of the approximation. Again, we wish to minimize this error operator so that a dual frame pair is optimal for m or less erasures. Thus an optimal dual frame pair can be defined inductively as follow: An (n,k)-dual frame pair $(\tilde{X}, \tilde{Y})$ is called *optimal for m-erasures* if it is optimal for $(m-1)$ erasures and $d_m(\tilde{X}, \tilde{Y})$ minimizes $d_m(X,Y)$ for all (n,k)-dual frame pairs. When restricted to the class of all the (n,k)-Parseval frames, with $Y = X$, this reduces to the case for optimal frames described in Section 4.2.

In what follows, we begin by proving the existence of optimal dual frames and then go on to demonstrate some further results about optimal dual frames. In particular, we give examples to show that, in general, the canonical dual is not necessarily optimal. Then we show that there exists a large class of frames for which the canonical dual is, in fact, always optimal for any number of erasures.

4.3.1 Existence of Optimal Dual Frames

Let $X = \{x_i\}_{i=1}^n$ be an (n,k)-frame for H, with S the frame operator for X. We say that a dual frame, Y, for X is *an optimal dual frame of X for 1-erasure* if

$$d_1(X,Y) = \min\{d_1(X,Z) : Z \text{ is a dual frame for } X\},$$

and Y is called an *optimal dual frame of X for m-erasures* if it is optimal for $(m-1)$-erasures and

$$d_m(X,Y) = \min\{d_m(X,Z) : Z \text{ is a dual frame for } X\}.$$

From Proposition 2.3, $Y = \{y_i\}_{i=1}^n$ is a dual frame for X if and only if $Y =$

$S^{-1}X + U$ for some $U = \{u_i\}_{i=1}^n$ such that

$$\sum_{i=1}^{n} \langle x, x_i \rangle u_i = 0$$

for all $x \in H$, that is, $\Theta_U^* \Theta_X = 0$. Let $\mathcal{N}_X$ be the set of all U such that $\Theta_U^* \Theta_X = 0$. Then an optimal dual frame of X for m-erasures will be one which minimizes

$$\begin{aligned} \min\{d_m(X, S^{-1}X + U) : U \in \mathcal{N}_X\} \\ &= \min\{\max\{\|(\Theta_{S^{-1}X+U}^* D\Theta_X\| : D \in \mathcal{D}_m\} : U \in \mathcal{N}_X\} \\ &= \min\{\max\{||S^{-1}\Theta_X^* D\Theta_X + \Theta_U^* D\Theta_X|| : D \in \mathcal{D}_m\} : U \in \mathcal{N}_X\} \end{aligned}$$

We first prove the existence of an optimal dual frame for one erasure.

Let $x, y \in H$. We will use $x \otimes y$ to denote the rank-one operator defined by $(x \otimes y)(v) = \langle v, y \rangle x$ for all $v \in H$.

Note that if $D \in \mathcal{D}_1$ and $Y = \{S^{-1}x_i + u_i\}_{i=1}^n$ with $U = \{u_i\}_{i=1}^n \in \mathcal{N}_X$, then we have

$$||\Theta_Y^* D\Theta_X|| = ||(S^{-1}x_i + u_i) \otimes x_i|| = ||(S^{-1}x_i + u_i)|| \cdot ||x_i||$$

for some $1 \leq i \leq n$. Therefore when we consider 1-erasure optimal dual frames, it is reasonable to assume that $x_i \neq 0$ for all $1 \leq i \leq n$. Thus the function defined by

$$F(U) = d_1(X, S^{-1}X + U) = \max\{||(S^{-1}x_i + u_i)|| \cdot ||x_i|| : 1 \leq i \leq n\}$$

will be a continuous function of U with the property that $F(U) \to \infty$ when $||U|| \to \infty$, where we view U as a vector in the orthogonal direct sum Hilbert space $H^{(n)} :=$

$H \oplus \ldots \oplus H$. Therefore, by restricting to a bounded subset of $\mathcal{N}_X$, the minimum of F is attained.

This leads to the following:

Lemma 4.1 *Let $X = \{x_i\}_{i=1}^n$ be a frame for H with $x_i \neq 0$ for all i. Then optimal dual frames of X exist for 1-erasure. Moreover, the set of all the optimal dual frames of X for m-erasures form a convex, closed and bounded subset of $H^{(n)}$.*

Proof: We only need to show the convexity of the set. Let $Y^{(1)}$ and $Y^{(2)}$ be two optimal dual frames of X for m-erasure. Then we have

$$d_m(X, Y^{(1)}) = d_m(X, Y^{(2)}) = \min\{d_m(X, Z) : Z \text{ is a dual frame for } X\}.$$

Let $Y = \lambda Y^{(1)} + (1-\lambda)Y^{(2)}$ for $\lambda \in [0,1]$. Clearly, Y is a dual of X. It remains to show that $d_m(X, Y) = d_m(X, Y^{(1)}) = d_m(X, Y^{(2)})$. In fact, for any $D \in \mathcal{D}_m$ we have

$$\begin{aligned} ||\Theta_Y^* D \Theta_X|| &= ||\lambda \Theta_{Y^{(1)}}^* D \Theta_X + (1-\lambda)\Theta_{Y^{(2)}}^* D \Theta_X|| \\ &\leq \lambda ||\Theta_{Y^{(1)}}^* D \Theta_X|| + (1-\lambda)||\Theta_{Y^{(2)}}^* D \Theta_X|| \\ &\leq \lambda d_m(X, Y^{(1)}) + (1-\lambda) d_m(X, Y^{(2)}) \\ &= d_m(X, Y^{(1)})(= d_m(X, Y^{(2)}) \end{aligned}$$

Thus

$$d_m(X, Y) \leq d_m(X, Y^{(1)}) = d_m(X, Y^{(2)})$$

and so the equality holds.

■

Following from the above lemma and using the induction argument we obtain:

Corollary 4.1 *Let $X = \{x_i\}_{i=1}^n$ be a frame for H with $x_i \neq 0$ for all i. Then optimal dual frames of X exist for any m-erasures. Moreover, the set of all the optimal dual frames of X for m-erasures form a convex and closed subset of $H^{(n)}$.*

The next two sections show that for some cases, the canonical dual frame is the unique optimal dual frame.

4.3.2 Optimal Dual Frame for a Uniform, Tight Frame

Proposition 4.1 *For one erasure, the unique optimal dual frame for a tight frame with uniform length is the canonical dual frame.*

Proof: Let $\{x_i\}_{i=1}^n$ be a tight frame with equal norms, $\|x_i\| = \sqrt{\frac{\lambda k}{n}}, \forall i$. Then $S = \lambda I$ for some $\lambda \neq 0$, and so $S^{-1} = \frac{1}{\lambda}I$. Thus, the canonical dual frame is $\{\frac{1}{\lambda}x_i\}$. Suppose $\{y_i\}$ is a dual of $\{x_i\}$. We need to show

$$\begin{aligned}
\max_{1\leq i\leq n}\{\|y_i\| \cdot \|x_i\|\} &\geq \max_{1\leq i\leq n}\{\|\frac{1}{\lambda}x_i\| \cdot \|x_i\|\} \\
\max_{1\leq i\leq n}\{\|y_i\|\} &\geq \max_{1\leq i\leq n}\{\|\frac{1}{\lambda}x_i\|\} \\
\max_{1\leq i\leq n}\{\|y_i\|\} &\geq \sqrt{\frac{k}{\lambda n}}
\end{aligned}$$

Now, by property of the trace

$$\begin{aligned}\sum_{i=1}^{n}\langle x_i, y_i\rangle &= \operatorname{tr}(\Theta_x\Theta_y^*)\\ &= \operatorname{tr}(\Theta_y^*\Theta_x)\\ &= \operatorname{tr}(I)\\ &= k\end{aligned}$$

So, by Cauchy-Schwarz

$$\begin{aligned}k &= \sum_{i=1}^{n}\langle x_i, y_i\rangle\\ &\leq \sum_{i=1}^{n}\|x_i\|\cdot\|y_i\|\\ &\leq \sum_{i=1}^{n}\sqrt{\frac{\lambda k}{n}}\cdot\|y_i\|\end{aligned}$$

Thus, $\sum_{i=1}^{n}\|y_i\| \geq \sqrt{\frac{nk}{\lambda}}$. Now, suppose that

$$\max_{1\leq i\leq n}\{\|y_i\|\} < \sqrt{\frac{k}{\lambda n}}$$

Then

$$\begin{aligned}\sum_{i=1}^{n} \|y_i\| &< \sum_{i=1}^{n} \sqrt{\frac{k}{\lambda n}} \\ &< n\sqrt{\frac{k}{\lambda n}} \\ &< \sqrt{\frac{nk}{\lambda}}\end{aligned}$$

which is a contradiction. Therefore, the canonical dual is an optimal dual for a uniform tight frame.

In fact, it is the unique optimal dual. Suppose $\{z_i\}$ is an optimal dual frame. Then

$$\max_{1\leq i\leq n}\{\|z_i\|\} = \sqrt{\frac{k}{\lambda n}}$$

so $\|z_i\| \leq \sqrt{\frac{k}{\lambda n}}, \forall i$. If $\|z_j\| < \sqrt{\frac{k}{\lambda n}}$ for some j, then

$$\sum_{i=1}^{n} \|z_i\| < \sqrt{\frac{nk}{\lambda}}$$

which is the same contradiction as above. So, if $\{z_i\}$ is optimal, $\|z_i\| = \sqrt{\frac{k}{\lambda n}}$ for all i.

So

$$\begin{aligned}
\frac{k}{\lambda} &= \sum_{i=1}^{n} \|z_i\|^2 \\
&= \sum_{i=1}^{n} \langle \frac{1}{\lambda} x_i + h_i, \frac{1}{\lambda} x_i + h_i \rangle \\
&= \sum_{i=1}^{n} \|\frac{1}{\lambda} x_i\|^2 + \sum_{i=1}^{n} \|h_i\|^2 + 2 \sum_{i=1} \frac{1}{\lambda} \langle x_i, h_i \rangle \\
&= \frac{k}{\lambda} + \sum_{i=1}^{n} \|h_i\|^2 + 0
\end{aligned}$$

Thus $\sum_{i=1}^{n} \|h_i\|^2 = 0$, and so $h_i = 0$ for all i. Therefore, the canonical dual is the unique optimal dual.

4.3.3 Optimal Dual Frame for a Group Representation Frame

Let G be a group and H a Hilbert space, with $\{\pi(g)\xi : g \in G\}$ a frame for H.

First, note that for an abelian group, $\{\pi(g)S^{-1}\xi\}$ is the only dual with the same group structure, so the problem is only interesting for non-abelian groups.

Proposition 4.2 *For one erasure, the optimal dual frame with the same group structure is the canonical dual frame.*

Proof: Let $\pi(g)\eta = \pi(g)S^{-1}\xi + \pi(g)h$, where

$$\sum_{g \in G} \langle x, \pi(g)\xi \rangle \pi(g)h = 0$$

It follows, by setting $x = S^{-3/2}\xi$ and taking the inner product with $S^{-1/2}\xi$, that

$$\begin{aligned}
\langle \sum_{g\in G} \langle S^{-3/2}\xi, \pi(g)\xi\rangle \pi(g)h, S^{-1/2}\xi\rangle &= 0 \\
\sum_{g\in G} \langle S^{-1/2}\xi, \pi(g)S^{-1}\xi\rangle \langle \pi(g)h, S^{-1/2}\xi\rangle &= 0 \\
\sum_{g\in G} \langle \pi(g^{-1})S^{-1/2}\xi, S^{-1}\xi\rangle \langle h, \pi(g^{-1})S^{-1/2}\xi\rangle &= 0 \\
\langle h, \sum_{g\in G} \langle \pi(g^{-1})S^{-1/2}\xi, S^{-1}\xi\rangle \pi(g^{-1})S^{-1/2}\xi\rangle &= 0 \\
\langle h, S^{-1}\xi\rangle &= 0
\end{aligned}$$

Then, since S^{-1} is in the commutant of $\pi(G)$ and $\pi(g)$ is an isometry

$$\begin{aligned}
\min_h \max_{g\in G} \|S^{-1}\pi(g)\xi + \pi(g)h\| &= \min_h \max_{g\in G} \|\pi(g)\left(S^{-1}\xi + h\right)\| \\
&= \min_h \max_{g\in G} \|S^{-1}\xi + h\| \\
&= \min_h \|S^{-1}\xi + h\|
\end{aligned}$$

But $\|S^{-1}\xi + h\|^2 = \|S^{-1}\xi\|^2 + \|h\|^2$, since $S^{-1}\xi$ and h are orthogonal. Thus, the minimum occurs when $\|h\|^2 = 0$, so that $h = 0$. Therefore, the canonical dual is the optimal dual with the same group structure for one erasure.

Proposition 4.3 *For one erasure, the canonical dual frame is the unique optimal dual for a group representation frame.*

Proof: See Corollary 4.3 below.

4.3.4 Standard Dual Frame as the Unique Optimal Dual Frame

The results of the two previous sections can actually be shown to be corollaries of a more general result. We require the following lemma.

Lemma 4.2 *If, for all $x \in H$,*

$$\sum_{i=1}^{n} \langle x, x_i \rangle h_i = 0$$

then $\sum_{i=1}^{n} \langle S^{-1} x_i, h_i \rangle = 0$.

Proof: In operator notation, $\Theta_h^* \Theta_{S^{-1}x} = 0$, since for any x in H

$$\begin{aligned} \sum_{i=1}^{n} \langle x, S^{-1} x_i \rangle h_i &= \sum_{i=1}^{n} \langle S^{-1} x, x_i \rangle h_i \\ &= 0 \end{aligned}$$

Therefore, by the property of the trace

$$\begin{aligned} \sum_{i=1}^{n} \langle S^{-1} x_i, h_i \rangle &= \mathrm{tr}(\Theta_{S^{-1}x} \Theta_h^*) \\ &= \mathrm{tr}(\Theta_h^* \Theta_{S^{-1}x}) \\ &= 0 \end{aligned}$$

as required.

■

Theorem 4.1 *For one erasure, the canonical dual frame is the unique optimal dual* *frame for any frame where*

$$\|S^{-1}x_i\| \cdot \|x_i\|$$

is a constant for all i.

Let $\{x_i\}$ be a frame with $\|S^{-1}x_i\| \cdot \|x_i\| = c$, a constant for all i. Let $\{y_i\} = \{S^{-1}x_i + h_i\}$ be an optimal dual frame. Then

$$\begin{aligned}
\max_i \|S^{-1}x_i + h_i\| \cdot \|x_i\| &\leq \max_i \|S^{-1}x_i\| \cdot \|x_i\| \\
&\leq \max_i \ c \\
&\leq c
\end{aligned}$$

Thus

$$\begin{aligned}
\max_i \|y_i\| \cdot \|x_i\| &\leq \|S^{-1}x_j\| \cdot \|x_j\|, \ \forall j \\
\|y_i\| \cdot \|x_i\| &\leq \|S^{-1}x_j\| \cdot \|x_j\|, \ \forall i, j \\
\|y_j\| \cdot \|x_j\| &\leq \|S^{-1}x_j\| \cdot \|x_j\|, \ \forall j \\
\|y_j\| &\leq \|S^{-1}x_j\|, \ \forall j
\end{aligned}$$

Now, for all i

$$\begin{aligned}
\|y_i\|^2 &= \|S^{-1}x_i + h_i\|^2 \\
&= \|S^{-1}x_i\|^2 + \|h_i\|^2 + 2\mathrm{Re}\langle S^{-1}x_i, h_i\rangle \\
\|h_i\|^2 + 2\mathrm{Re}\langle S^{-1}x_i, h_i\rangle &= \|y_i\|^2 - \|S^{-1}x_i\|^2 \\
\|h_i\|^2 + 2\mathrm{Re}\langle S^{-1}x_i, h_i\rangle &\leq 0
\end{aligned}$$

Thus, by the lemma,

$$\begin{aligned} \sum_{i=1}^{n} \left(\|h_i\|^2 + 2\mathrm{Re}\langle S^{-1}x_i, h_i\rangle \right) &\leq 0 \\ \sum_{i=1}^{n} \|h_i\|^2 + 2\mathrm{Re} \sum_{i=1}^{n} \langle S^{-1}x_i, h_i\rangle &\leq 0 \\ \sum_{i=1}^{n} \|h_i\|^2 &\leq 0 \end{aligned}$$

and so $h_i = 0$ for all i. Therefore, $\{y_i\}$ is the canonical dual frame, and so the optimal dual frame is unique.

Corollary 4.2 *For one erasure, the canonical dual frame is the unique optimal dual frame for a tight frame with uniform length.*

This follows with $\|S^{-1}x_i\| \cdot \|x_i\| = \sqrt{\frac{k}{\lambda n}}$.

Corollary 4.3 *For one erasure, the canonical dual frame is the unique optimal dual frame for a group representation frame.*

This follows since $\pi(g)$ is an isometry for all g, with $\|\pi(g)S^{-1}\xi\| \cdot \|\pi(g)\xi\| = \|S^{-1}\xi\| \cdot \|\xi\|$.

In fact, since our definition of an optimal dual frame is inductive, that is, a dual frame which is optimal for m-erasures must be optimal for $(m-1)$-erasures, a dual frame which is the unique optimal dual for 1-erasure must be optimal for any number of erasures, since there are no other choices. Thus, we can restate the above result as:

Theorem 4.2 *Let $\{x_i\}_{i=1}^n$ be a frame for a k-dimensional Hilbert space H and S be its frame operator. If $\|S^{-1}x_i\| \cdot \|x_i\| = c$ is a constant for all i, then the canonical dual frame is the unique optimal dual frame for any m-erasures.*

In viewing Theorem 4.2, we would wonder whether $\{y_i\}_{i=1}^n$ is an optimal dual of $\{x_i\}_{i=1}^n$ if $\|y_i\| \cdot \|x_i\|$ is a constant. With the help of Corollary 4.3 we point out that the answer to this question is negative.

Proposition 4.4 *There exists a group frame $\{\pi(g)\varphi\}_{g\in G}$ such that it admits a dual frame of the form $\{\pi(g)\eta\}_{g\in G}$ that is not the canonical dual, and consequently $\{\pi(g)\eta\}_{g\in G}$ is not optimal.*

Proof: Let π be a group representation on H and $\{\pi(g)\varphi\}_{g\in G}$ is a Parseval frame for H with the property that there exists $g_1, g_2 \in G$ such that

$$\langle \pi(g_1)\varphi, \pi(g_2)\varphi\rangle \neq \langle \pi(g_2)\varphi, \pi(g_1)\varphi\rangle.$$

Then by the main result on the uniqueness of dual frame generators in [11], there exists $\eta \in H$ such that $\eta \neq S^{-1}\varphi$ and $\{\pi(g)\eta\}_{g\in G}$ is a dual frame of $\{\pi(g)\varphi\}_{g\in G}$, where S is the frame operator for $\{\pi(g)\varphi\}_{g\in G}$. Since π is an isometry, $\|\pi(g)\eta\| \cdot \|\pi(g)\xi\|$ is a constant for all $g \in G$. However, by Corollary 4.3, $\{\pi(g)\eta\}_{g\in G}$ is not optimal.

■

One further generalization of Theorem 4.2 requires an additional definition. Let $X = \{x_i\}_{i=1}^n$ be a sequence. We say that a decomposition $\bigcup_{j=1}^m I_j = \{1, 2, \ldots, n\}$ is *X-linearly independent* if $M_1 + \ldots + M_m$ is a direct sum, where $M_j = \text{span}\{x_i | i \in I_j\}$.

Theorem 4.3 *Let $\{I_1, \ldots, I_m\}$ be an X-linearly independent decomposition of $\{1, \ldots, n\}$. Suppose that*

$$\|S^{-1}x_i\| \cdot \|x_i\| = c_j$$

for all $i \in I_j$. Then

(i) $\{S^{-1}x_i\}_{i=1}^n$ is 1-optimal

(ii) Assume $c_m = c_{m-1} = \ldots = c_k > c_{k-1} \geq c_{k-2} \geq \ldots \geq c_1$. Then $\{x_i\}_{i=1}^n$ has a unique 1-optimal dual if and only if $\{x_i\}_{i \in \bigcup_{j=1}^{k-1} I_j}$ is linearly independent.

Proof: Let $\{y_i\}_{i=1}^n$ be 1-optimal, where $y_i = S^{-1}x_i + u_i$, and

$$\sum_{i=1}^{n} \langle x, u_i \rangle x_i = 0$$

$$\sum_{i \in I_1} \langle x, u_i \rangle x_i + \ldots + \sum_{i \in I_m} \langle x, u_i \rangle x_i = 0$$

Thus, by the X-linearly independent decomposition, for every $1 \leq j \leq m$

$$\sum_{i \in I_j} \langle x, u_i \rangle x_i = 0$$

Let $c_{j_0} = \max\{c_j : 1 \leq j \leq m\}$. Then, for all $1 \leq i \leq n$

$$\|y_i\| \cdot \|x_i\| \leq c_{j_0}$$

In particular, for all $i \in I_{j_0}$

$$\|y_i\| \cdot \|x_i\| \leq c_{j_0} = \|S^{-1}x_i\| \cdot \|x_i\|$$
$$\|y_i\| \leq \|S^{-1}x_i\|$$

Thus, just as in Theorem 4.1, $u_i = 0$ for all $i \in I_{j_0}$. Therefore, $y_i = S^{-1}x_i$ for all $i \in I_{j_0}$. Consequently

$$\max\{\|S^{-1}x_i\| \cdot \|x_i\| : 1 \leq i \leq n\} = c_{j_0} = \max\{\|y_i\| \cdot \|x_i\| : 1 \leq i \leq n\}$$

Therefore $\{S^{-1}x_i\}_{i=1}^n$ is optimal for 1-erasure as claimed.

For part (ii), assume that $\{x_i\}_{i\in\bigcup_{j=1}^{k-1} I_j}$ is linearly independent, and let $\{y_i\}_{i=1}^n$ be a 1-optimal dual for $\{x_i\}_{i=1}^n$, with $y_i = S^{-1}x_i + u_i$ and

$$\sum_{i=1}^{n} \langle x, u_i\rangle x_i = 0$$

for all $x \in H$. From part (i), we already know that $y_i = S^{-1}x_i$ (that is, $u_i = 0$) for all $i \in I_k \cup I_{k+1} \cup \ldots \cup I_m$. It remains to check that $u_i = 0$ for $i < k$. This is immediate, however, by the linear independence, since

$$0 = \sum_{i=1}^{n} \langle x, u_i\rangle x_i = \sum_{i\in\bigcup_{j=1}^{k-1} I_j} \langle x, u_i\rangle x_i$$

implies that $\langle x, u_i\rangle = 0$ for all x. Therefore, $u_i = 0$ for all i and the optimal dual frame is unique.

Conversely, suppose that $\{x_i\}_{i\in\bigcup_{j=1}^{k-1} I_j}$ is linearly dependent. Then there exists u_i not all 0 such that

$$\sum_{i\in\bigcup_{j=1}^{k-1} I_j} \langle x, u_i\rangle x_i = 0$$

for all $x \in H$. Let $u_i = 0$ when $i \in \bigcup_{j=k}^{m} I_j$. Then $\{u_i\}_{i=1}^{n}$ is a non-zero finite sequence such that

$$\sum_{i=1}^{n} \langle x, tu_i\rangle x_i = 0$$

holds for every $x \in H$, where $t \neq 0$ is any constant. Thus $Y_t = \{S^{-1}x_i + tu_i\}_{i=1}^{n}$ is a dual frame for $\{x_i\}_{i=1}^{n}$. Since $\|S^{-1}x_i\| \cdot \|x_i\| < c_m = \max\{\|S^{-1}x_i\| \cdot \|x_i\| : 1 \leq i \leq n\}$ for all $i \in i \in \bigcup_{j=1}^{k-1} I_j$, there exists $\delta > 0$ such that if $|t| \leq \delta$, then

$$\|S^{-1}x_i + tu_i\| \cdot \|x_i\| < c_m$$

holds for every $i \in \bigcup_{j=1}^{k-1} I_j$. Thus Y_t is 1-erasure optimal whenever $|t| \leq \delta$. This implies that $\{x_i\}_{i=1}^{n}$ has infinitely many 1-erasure optimal duals, since $\{u_i\}_{i=1}^{n}$ is a non-zero finite sequence.

■

4.3.5 Examples

One example of a frame where the canonical dual is optimal is a Mercedes-Benz frame.

Example 4.1 *Let $H = \mathbb{R}^2$, and consider the frame $X = \{x_i\}_{i=1}^3$ given by*

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{bmatrix} \right\}$$

In fact this frame is a group frame, and the standard dual is the unique optimal dual, see Corollary 4.3.

All of the results for optimal dual frames presented so far involve the canonical dual frame and it is natural to wonder if the canonical dual is always optimal, or, moreover, to wonder if it is always the *unique* optimal dual.

Next, we give two examples showing that a frame may have infinitely many optimal duals, and that the canonical dual is not necessarily optimal even if the optimal dual is unique.

Example 4.2 (*Frame with a unique, non-canonical optimal dual*)
Let $H = \mathbb{R}^2$, and consider the frame $\{x_i\}_{i=1}^3$ given by

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \right\}$$

This is a uniform length, non-Parseval frame, where $\|x_i\| = 1$, for all i. This frame has a unique 1-erasure (and hence m-erasure) optimal dual frame which is not the canonical dual.

First, note that

$$S = \frac{1}{2} \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}, \quad S^{-1} = \frac{1}{4} \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$$

and so the standard dual, $\{S^{-1}x_i\}$, is given by

$$\left\{ \begin{bmatrix} \frac{3}{4} \\ -\frac{1}{4} \end{bmatrix}, \begin{bmatrix} -\frac{1}{4} \\ \frac{3}{4} \end{bmatrix}, \begin{bmatrix} \frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} \end{bmatrix} \right\}$$

Therefore

$$\begin{aligned} \max_i \|S^{-1}x_i\| \cdot \|x_i\| &= \max \left\{ \left\| \begin{bmatrix} \frac{3}{4} \\ -\frac{1}{4} \end{bmatrix} \right\| \cdot 1, \left\| \begin{bmatrix} -\frac{1}{4} \\ \frac{3}{4} \end{bmatrix} \right\| \cdot 1, \left\| \begin{bmatrix} \frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} \end{bmatrix} \right\| \cdot 1 \right\} \\ &= \max \left\{ \sqrt{\frac{5}{8}}, \sqrt{\frac{5}{8}}, \frac{1}{2} \right\} \\ &= \sqrt{\frac{5}{8}} \end{aligned}$$

Now, consider the alternate dual frame $\{S^{-1}x_i + h_i\}$, where $\{h_i\}$ is given by

$$\left\{ \begin{bmatrix} \frac{3}{4} - \frac{\sqrt{3}}{2} \\ \frac{3}{4} - \frac{\sqrt{3}}{2} \end{bmatrix}, \begin{bmatrix} \frac{3}{4} - \frac{\sqrt{3}}{2} \\ \frac{3}{4} - \frac{\sqrt{3}}{2} \end{bmatrix}, \begin{bmatrix} \frac{2\sqrt{3}-3}{2\sqrt{2}} \\ \frac{2\sqrt{3}-3}{2\sqrt{2}} \end{bmatrix} \right\}$$

so that $\{S^{-1}x_i + h_i\}$ is

$$\left\{ \begin{bmatrix} \frac{3-\sqrt{3}}{2} \\ \frac{1-\sqrt{3}}{2} \end{bmatrix}, \begin{bmatrix} \frac{1-\sqrt{3}}{2} \\ \frac{3-\sqrt{3}}{2} \end{bmatrix}, \begin{bmatrix} \frac{\sqrt{3}-1}{\sqrt{2}} \\ \frac{\sqrt{3}-1}{\sqrt{2}} \end{bmatrix} \right\}$$

Thus,

$$
\begin{aligned}
\max_i \|S^{-1}x_i + h_i\| \cdot \|x_i\| &= \max\left\{ \left\| \begin{bmatrix} \frac{3-\sqrt{3}}{2} \\ \frac{1-\sqrt{3}}{2} \end{bmatrix} \right\| \cdot 1, \left\| \begin{bmatrix} \frac{1-\sqrt{3}}{2} \\ \frac{3-\sqrt{3}}{2} \end{bmatrix} \right\| \cdot 1, \left\| \begin{bmatrix} \frac{\sqrt{3}-1}{\sqrt{2}} \\ \frac{\sqrt{3}-1}{\sqrt{2}} \end{bmatrix} \right\| \cdot 1 \right\} \\
&= \max\left\{ \sqrt{3}-1, \sqrt{3}-1, \sqrt{3}-1 \right\} \\
&= \sqrt{3}-1 < \sqrt{\frac{5}{8}}
\end{aligned}
$$

Therefore,

$$\max_i \|S^{-1}x_i + h_i\| < \max_i \|S^{-1}x_i\|$$

and so the standard dual is not an optimal dual. In fact, this alternate dual is the unique optimal dual. The computations showing this are lengthy, and given in the following proof.

Proof: An optimal dual frame is the sequence $\{S^{-1}x_i + u_i\}_{i=1}^{3}$ such that

$$\max_i \{\|S^{-1}x_i + u_i\|\}$$

is minimal for all $\{u_i\}_{i=1}^{3}$ where $\sum_{i=1}^{3} \langle x, x_i \rangle u_i = 0$ for all $x \in H$, and $\{S^{-1}x_i\}_{i=1}^{3}$ is the canonical dual given by

$$\left\{ \begin{bmatrix} \frac{3}{4} \\ -\frac{1}{4} \end{bmatrix}, \begin{bmatrix} -\frac{1}{4} \\ \frac{3}{4} \end{bmatrix}, \begin{bmatrix} \frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} \end{bmatrix} \right\}.$$

By letting $x = e_1$ and $x = e_2$ we get

$$\begin{aligned} 1u_1 + 0u_2 + \frac{1}{\sqrt{2}}u_3 &= 0 \\ 0u_1 + 1u_2 + \frac{1}{\sqrt{2}}u_3 &= 0 \end{aligned}$$

and so all such $\{u_i\}$ must be of the form

$$u_1 = u_2 = \begin{bmatrix} a \\ b \end{bmatrix}, \text{ and } u_3 = \begin{bmatrix} -\sqrt{2}a \\ -\sqrt{2}b \end{bmatrix}$$

and so the function that needs to be minimized is

$$F(u) := \max\left\{ \left\|u + S^{-1}x_1\right\|, \left\|u + S^{-1}x_2\right\|, \left\|-\sqrt{2}u + S^{-1}x_3\right\| \right\},$$

where $u = \begin{bmatrix} a \\ b \end{bmatrix}$.

To simplify the calculations, we first point out that there is an optimal dual with $a = b$. This can be proved if we can show that $F(\tilde{u}) \leq F(u)$, where $u = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\tilde{u} = \begin{bmatrix} \frac{a+b}{2} \\ \frac{a+b}{2} \end{bmatrix}$.

Let $\dagger : H \to H$ be the involution $\dagger : \begin{bmatrix} a \\ b \end{bmatrix} \mapsto \begin{bmatrix} b \\ a \end{bmatrix}$. Note that $(S^{-1}x_1)^\dagger = S^{-1}x_2$ and $(S^{-1}x_3)^\dagger = S^{-1}x_3$. Therefore we have

$$\begin{aligned}
F(\tilde{u}) &= \max\left\{\left\|\frac{u+u^\dagger}{2} + S^{-1}x_1\right\|, \left\|\frac{u+u^\dagger}{2} + S^{-1}x_2\right\|, \left\|\frac{-\sqrt{2}(u+u^\dagger)}{2} + S^{-1}x_3\right\|\right\} \\
&= \max\left\{\frac{1}{2}\left\|u+u^\dagger+2S^{-1}x_1\right\|, \frac{1}{2}\left\|u+u^\dagger+2S^{-1}x_2\right\|, \frac{\sqrt{2}}{2}\left\|u+u^\dagger - \frac{2}{\sqrt{2}}S^{-1}x_3\right\|\right\} \\
&= \max\left\{\frac{1}{2}\left\|(u+S^{-1}x_1)+(u^\dagger+S^{-1}x_1)\right\|, \frac{1}{2}\left\|(u+S^{-1}x_2)+(u^\dagger+S^{-1}x_2)\right\|,\right. \\
&\qquad \left.\frac{\sqrt{2}}{2}\left\|\left(u-\frac{1}{\sqrt{2}}S^{-1}x_3\right)+\left(u^\dagger\frac{1}{\sqrt{2}}S^{-1}x_3\right)\right\|\right\} \\
&\le \max\left\{\frac{1}{2}\left(\left\|u+S^{-1}x_1\right\|+\left\|u^\dagger+S^{-1}x_1\right\|\right), \frac{1}{2}\left(\left\|u+S^{-1}x_2\right\|+\left\|u^\dagger+S^{-1}x_2\right\|\right),\right. \\
&\qquad \left.\frac{\sqrt{2}}{2}\left(\left\|u-\frac{1}{\sqrt{2}}S^{-1}x_3\right\|+\left\|u^\dagger-\frac{1}{\sqrt{2}}S^{-1}x_3\right\|\right)\right\} \\
&= \max\left\{\frac{1}{2}\left(\left\|u+S^{-1}x_1\right\|+\left\|(u+S^{-1}x_2)^\dagger\right\|\right), \frac{1}{2}\left(\left\|u+S^{-1}x_2\right\|+\left\|(u+S^{-1}x_1)^\dagger\right\|\right),\right. \\
&\qquad \left.\frac{\sqrt{2}}{2}\left(\left\|u-\frac{1}{\sqrt{2}}S^{-1}x_3\right\|+\left\|(u-\frac{1}{\sqrt{2}}S^{-1}x_3)^\dagger\right\|\right)\right\} \\
&= \max\left\{\frac{1}{2}\left(\left\|u+S^{-1}x_1\right\|+\left\|u+S^{-1}x_2\right\|\right), \frac{1}{2}\left(\left\|u+S^{-1}x_2\right\|+\left\|u+S^{-1}x_1\right\|\right),\right. \\
&\qquad \left.\sqrt{2}\left\|u-\frac{1}{\sqrt{2}}S^{-1}x_3\right\|\right\} \\
&\le \max\left\{\left\|u+S^{-1}x_1\right\|, \left\|u+S^{-1}x_2\right\|, \left\|-\sqrt{2}u+S^{-1}x_3\right\|\right\} \\
&= F(u),
\end{aligned}$$

where the last line follows since $\frac{x+y}{2} \le \max\{x, y\}$ for $x, y \ge 0$.

Now taking $b = a$ and squaring the norms, we wish to find the a that minimizes

$$f(a) := \max\left\{2a^2 + a + \frac{5}{8}, 2a^2 + a + \frac{5}{8}, 4a^2 - 2a + \frac{1}{4}\right\}$$

We show that for $a = \frac{3-2\sqrt{3}}{4}$,

$$\max\left\{(\sqrt{3}-1)^2, (\sqrt{3}-1)^2, (\sqrt{3}-1)^2\right\} = (\sqrt{3}-1)^2$$

is minimal, and thus

$$\left\{ \begin{bmatrix} \frac{3-\sqrt{3}}{2} \\ \frac{1-\sqrt{3}}{2} \end{bmatrix}, \begin{bmatrix} \frac{1-\sqrt{3}}{2} \\ \frac{3-\sqrt{3}}{2} \end{bmatrix}, \begin{bmatrix} \frac{\sqrt{3}-1}{\sqrt{2}} \\ \frac{\sqrt{3}-1}{\sqrt{2}} \end{bmatrix} \right\}$$

is an optimal dual for $\{x_i\}_{i=1}^3$. In fact, letting $a = \frac{3-2\sqrt{3}}{4} + \epsilon$, the quadratics in $f(a)$ become

$$\begin{cases} (\sqrt{3}-1)^2 + 4\epsilon - 2\sqrt{3}\epsilon + 2\epsilon^2 \\ (\sqrt{3}-1)^2 + 4\epsilon - 4\sqrt{3}\epsilon + 4\epsilon^2 \end{cases}$$

In order for the maximum to be less than $(\sqrt{3}-1)^2$, both $4\epsilon - 2\sqrt{3}\epsilon + 2\epsilon^2$ and $4\epsilon - 4\sqrt{3}\epsilon + 4\epsilon^2$ must be negative simultaneously. But $4\epsilon - 2\sqrt{3}\epsilon + 2\epsilon^2$ is only negative from $\epsilon = 0$ to $\epsilon = 2 - \sqrt{3}$, and $4\epsilon - 4\sqrt{3}\epsilon + 4\epsilon^2$ is only negative from $\epsilon = 1 - \sqrt{3}$ to $\epsilon = 0$, and so the equations are never simultaneously negative. Therefore

$$\min\max\left\{2a^2 + a + \frac{5}{8}, 2a^2 + a + \frac{5}{8}, 4a^2 - 2a + \frac{1}{4}\right\} = (\sqrt{3}-1)^2.$$

The above argument implies that there is a only one optimal dual with the property that $a = b$. By using the fact that $|\langle x, y\rangle| = \|x\| \cdot \|y\|$ if and only if x and y are linearly independent, we can easily derive that when $a \neq b$, we always have

$$\|(u + S^{-1}x_1) + (u^\dagger + S^{-1}x_1)\| < \|u + S^{-1}x_1\| + \|u^\dagger + S^{-1}x_1\|,$$

$$\|(u + S^{-1}x_2) + (u^\dagger + S^{-1}x_2)\| < \|u + S^{-1}x_2\| + \|u^\dagger + S^{-1}x_2\|$$

and

$$\|(u - \frac{1}{\sqrt{2}}S^{-1}x_3) + (u^\dagger - \frac{1}{\sqrt{2}}S^{-1}x_3)\| < \|u - \frac{1}{\sqrt{2}}S^{-1}x_3\| + \|u^\dagger - \frac{1}{\sqrt{2}}S^{-1}x_3\|$$

Thus the first inequality in the proof of "$F(\tilde{u}) \leq F(u)$" becomes a strict inequality when $a \neq b$. Hence the optimal dual happens only when $a = b$, and therefore the optimal dual is unique.

■

Example 4.3 (Frame with infinitely many optimal duals) *Let $H = \mathbb{R}^2$, and consider the frame $\{x_i\}_{i=1}^3$ given by*

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \frac{1}{2}\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \frac{1}{2}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

This frame has infinitely many 1-erasure optimal dual frames.

First, note that

$$S = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \quad S^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

and so the standard dual, $\{S^{-1}x_i\}$, is given by

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

Therefore

$$\begin{aligned} \max_i \|S^{-1}x_i\| \cdot \|x_i\| &= \max \left\{ \left\| \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\| \cdot 1, \left\| \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\| \cdot \frac{1}{2}, \left\| \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\| \cdot \frac{1}{2} \right\} \\ &= \max \left\{ 1, \frac{1}{2}, \frac{1}{2} \right\} \\ &= 1 \end{aligned}$$

Now, consider the alternate dual frame $\{S^{-1}x_i + h_i\}$, where $\{h_i\}$ is given by

$$\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} -a \\ -b \end{bmatrix} \right\}$$

for arbitrary a, b. Thus $\|S^{-1}x_1 + h_1\| \cdot \|x_1\| = \|S^{-1}x_1\| = 1$, and so all choices of a, b such that

$$\|S^{-1}x_2 + h_2\| \cdot \|x_2\| \leq 1 \text{ and } \|S^{-1}x_3 + h_3\| \cdot \|x_3\| \leq 1$$

give optimal dual frames. So

$$\begin{aligned} \|S^{-1}x_2 + h_2\| \cdot \|x_2\| &= \left\| \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix} \right\| \cdot \frac{1}{2} \\ &= \frac{1}{2}\sqrt{a^2 + (1+b)^2} \end{aligned}$$

and similarly

$$\|S^{-1}x_3 + h_3\| \cdot \|x_3\| = \frac{1}{2}\sqrt{a^2 + (1-b)^2}$$

Therefore, all a, b such that

$$a^2 + (1+b)^2 \leq 4 \text{ and } a^2 + (1-b)^2 \leq 4$$

satisfy the condition, and so there are infinitely many optimal dual frames.

Since this example does not have a unique optimal dual for 1-erasure, it gives us the opportunity to study a more interesting scenario for the 2-erasure case. We find, for example, that there are infinitely many optimal dual frames even for 2-erasures.

Example Continued - Two Erasures

Let

$$V = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix}$$

and consider $(V^\dagger + Z)DV$, with $ZV = 0$, and $V^\dagger$ the least square inverse. Then

$$\mathcal{E}_1 = (V^\dagger + Z)E_{1,2}V = \begin{bmatrix} 1 & \frac{1}{2}a \\ 0 & \frac{1}{2}(1+b) \end{bmatrix}$$

$$\mathcal{E}_2 = (V^\dagger + Z)E_{1,3}V = \begin{bmatrix} 1 & -\frac{1}{2}a \\ 0 & \frac{1}{2}(1-b) \end{bmatrix}$$

$$\mathcal{E}_3 = (V^\dagger + Z)E_{2,3}V = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

The norm is the square root of the max eigenvalue of $\mathcal{E}\mathcal{E}^*$, and so $\|\mathcal{E}_3\| = 1$. Thus the max over all 2-erasures is greater than or equal to 1. Choose $a = b = 0$ and then $\|\mathcal{E}_1\| = \|\mathcal{E}_2\| = 1$ as well.

From 1-erasure

$$\frac{1}{2}\sqrt{a^2 + (1+b)^2} < 1$$
$$\frac{1}{2}\sqrt{a^2 + (1-b)^2} < 1$$

so if $a = 0$ and b is small enough, $\|\mathcal{E}_1\|$ and $\|\mathcal{E}_2\| = 1$. Therefore there are infinite optimal duals even for 2-erasure.

We conclude the examples with pseudocode and a sample Mathematica code for finding the optimal dual frame.

Pseudocode

(i) Calculate the frame operator, S, as a matrix.

(ii) Calculate S^{-1}.

(iii) Calculate the standard dual, $\{S^{-1}x_i\}$.

(iv) Solve the linear system of equations

$$\sum_{i=1}^{n} \langle e_j, x_i \rangle u_i = 0$$

with $j = 1, \ldots, k$ for the u_i in terms of paramters.

(v) Substitute the u_i back into the norm expressions

$$\|S^{-1}x_i + u_i\|$$

(vi) Determine the parameter(s) which give

$$\min_u \max_i \|S^{-1}x_i + u_i\|$$

Example *Mathematica* Code

The following code is an example of using *Mathematica* to find the u_i which give the minimum error. This particular example uses the uniform, non-Parseval frame from above.

```
Minimize[
  {Max[Norm[{3/4, -1/4} + {u11, u12}],
    Norm[{-1/4, 3/4} + {u21, u22}],
    Norm[{1/(2*Sqrt[2]), 1/(2*Sqrt[2])} + {u31, u32}]],
```

```
 {(3/4)*{u11, u12} + (-1/4)*{u21, u22} +
    (1/(2*Sqrt[2]))*{u31, u32} == {0, 0},
  (-1/4)*{u11, u12} + (3/4)*{u21, u22} +
    (1/(2*Sqrt[2]))*{u31, u32} == {0, 0}}},
{u11, u12, u21, u22, u31, u32}]
```

CHAPTER 5

DISCRETE GABOR FRAMES

5.1 Introduction

In 1946, D. Gabor proposed the short-time frequency analysis to expand a signal in $L^2(\mathbb{R})$ with the building blocks $\{g_{m,n}\}$, where

$$g_{m,n}(x) = e^{2\pi imbx} g(x - na), \quad m, n \in \mathbb{Z}$$

for fixed $a, b \in \mathbb{R}$. In many applications, such as signal processing, we require this Gabor family to be either an orthonormal basis or a frame for $L^2(\mathbb{R})$, to provide for decomposition and reconstruction of signals.

Although most of the research in this area focuses on the function space $L^2(\mathbb{R})$, there are practical reasons for studying the discrete version of Gabor analysis on $\mathbb{R}^n$. Much work has been done for the $\mathbb{R}^n$ case, and some work has been done for the $\ell^2(\mathbb{Z})$ case. See, for example, [9, 27]. However, very little is known about the $\ell^2(\mathbb{Z}^d)$ case.

Since this chapter considers infinite-dimensional Hilbert spaces, Section 5.2 begins by reviewing the basic definitions and ideas about frames, noting in particular those things which are specific to infinite dimensions. In Section 5.3, some results for the $\ell^2(\mathbb{Z})$ case are given. Then, in Section 5.4, the results are generalized to the $\ell^2(\mathbb{Z}^d)$ case, which has its own unique set of difficulties due to the complexity of the higher dimension indices.

5.2 Preliminaries

We begin by first recalling the basic definitions and notations about frames for infinite-dimensional Hilbert spaces.

A *frame* for a separable (real or complex) Hilbert space H is a sequence $\{x_j\}_{j\in\mathcal{J}}$ of H such that there exist two positive constants $A, B > 0$ with the property that

$$A||x||^2 \leq \sum_{j\in\mathcal{J}} |\langle x, x_j\rangle|^2 \leq B||x||^2$$

holds for every $x \in H$. The optimal constants (maximal for A and minimal for B) are called *frame bounds*. A *tight frame* is a frame with equal frame bounds ($A = B$). It is called a *Parseval frame* if $A = B = 1$. A *uniform* frame is a frame where all the elements in the frame sequence have the same norm.

Unlike the finite-dimensional case, in the infinite-dimensional setting it is possible for the right inequality not to hold. If the right inequality does hold for a sequence, we call the sequence a *Bessel sequence*.

For a frame $\{x_j\}_{j\in\mathcal{J}}$ of H, the associated *analysis operator* is the linear mapping $\Theta : H \to \ell^2(\mathcal{J})$ defined by:

$$\Theta(x) = \sum_{j\in\mathcal{J}} \langle x, x_j\rangle e_j,$$

where $\{e_j\}$ is the standard orthonormal basis for $\ell^2(\mathcal{J})$. The adjoint operator Θ^* of Θ is given by

$$\Theta^*(\sum_{j\in\mathcal{J}} c_j e_j) = \sum_{j\in\mathcal{J}} c_j x_j.$$

If we let $S = \Theta^*\Theta$, then we have

$$Sx = \sum_{j\in\mathcal{J}} \langle x, x_j \rangle x_j, \qquad x \in H.$$

Thus S is a positive invertible bounded linear operator on H, which is called the *frame operator* for $\{x_j\}_{j\in\mathcal{J}}$. A direct calculation yields

$$\begin{aligned} x &= \sum_{j\in\mathcal{J}} \langle x, S^{-1/2}x_j \rangle S^{-1/2}x_j \\ &= \sum_{j\in\mathcal{J}} \langle x, S^{-1}x_j \rangle x_j \\ &= \sum_{j\in\mathcal{J}} \langle x, x_j \rangle S^{-1}x_j \qquad (x \in H). \end{aligned}$$

This tells us that $\{S^{-1/2}x_j\}_{j\in J}$ is a Parseval frame, and $\{S^{-1}x_j\}_{j\in\mathcal{J}}$ is also a frame for H. The frame $\{S^{-1}x_j\}_{j\in\mathcal{J}}$ is called the *canonical (or standard) dual* of $\{x_j\}_{j\in\mathcal{J}}$.

Besides the canonical dual, there can also exist many (in fact, infinitely many) frames $\{y_j\}_{j\in\mathcal{J}}$ for H that yields a reconstruction formula for H:

$$x = \sum_{j\in\mathcal{J}} \langle x, x_j \rangle y_j, \quad x \in H.$$

A frame $\{y_j\}_{j\in\mathcal{J}}$ satisfying the above reconstruction formula is called an *alternate dual frame* or just simply called a *dual frame* for $\{x_j\}_{j\in\mathcal{J}}$. The connection between the canonical dual and the alternate duals is given by the following: $\{y_j\}_{j\in\mathcal{J}}$ is an alternate dual for $\{x_j\}_{j\in\mathcal{J}}$ if and only if $y_j = S^{-1}x_j + h_j$ for $j \in \mathcal{J}$, where $\{h_j\}_{j\in\mathcal{J}}$ satisfies the condition

$$\sum_{j\in\mathcal{J}} \langle x, x_j \rangle h_j = 0 \quad (\forall x \in H).$$

and we say that $\{x_j\}_{j\in\mathcal{J}}$ and $\{h_j\}_{j\in\mathcal{J}}$ are *orthogonal sequences.*

Orthogonal frames have applications in multiplexing, and we recall some of the ideas used later. Let $\{\phi_j^{(\ell)}\}_{j\in\mathcal{J}}$ be Parseval frames for Hilbert spaces H_ℓ, $\ell = 1,\ldots,k$. We say that $\left(\{\phi_j^{(1)}\},\{\phi_j^{(2)}\},\ldots,\{\phi_j^{(k)}\}\right)$ is a *disjoint k-tuple* if $\{\phi_j^{(1)}\oplus\ldots\oplus\phi_j^{(k)}\}$ is a frame for the orthogonal direct sum space $H_1\oplus\ldots\oplus H_k$, and is a *strongly disjoint k-tuple* if it is a Parseval frame for the direct sum space. A strongly disjoint k-tuple is also called a *superframe* of length k [1, 16, 19].

Lemma 5.1 (Han-Larson) *$\{\phi_j^{(1)}\oplus\ldots\oplus\phi_j^{(k)}\}_{j\in\mathcal{J}}$ is a superframe for $H\oplus\ldots\oplus H$ if and only if all of the following hold*

(i) Each $\{\phi_j^{(\ell)}\}$ is a Parseval frame for H.

(ii) $\{\phi_j^{(m)}\}$ and $\{\phi_j^{(n)}\}$ are orthogonal when $m\neq n$

5.3 Discrete Gabor Frames in $\ell^2(\mathbb{Z})$

Next, we introduce some terms and ideas specific to Gabor frames on $\ell^2(\mathbb{Z})$.

Let $H=\ell^2(\mathbb{Z})$ with the inner product

$$\langle f,g\rangle=\sum_{x=-\infty}^{\infty} f(x)\overline{g(x)}$$

For fixed $N,K\in\mathbb{N}$, an element $g\in\ell^2(\mathbb{Z})$ generates a sequence of elements $\{g_{k,m}: 0\leq k\leq K-1, m\in\mathbb{Z}\}$, with $k\in\mathbb{N}$, via time translations and frequency modulations given by

$$g_{k,m}(n)=e^{2\pi i\frac{k}{K}n}g(n-mN),\quad n\in\mathbb{Z}$$

If $\{g_{k,m}\}$ is a frame, it is called a *Gabor* (or *Weyl-Heisenberg*) frame. The element g is referred to as the *Gabor atom* or *Gabor (mother) wavelet*. Then for any $f \in \ell^2(\mathbb{Z})$, there is a Gabor expansion

$$f = \sum_{k=0}^{K-1} \sum_{m \in \mathbb{Z}} c_{k,m} g_{k,m}$$

where $c_{k,m}$ are the *Gabor coefficients*.

We can consider an analysis operator $\Theta : \ell^2(\mathbb{Z}) \to \ell^2(G)$, where G is the group $\mathbb{Z}_k \otimes \mathbb{Z}$. Let $\{e_{k,m}\}_{(k,m)\in G}$ be the standard orthonormal basis for $\ell^2(\mathbb{Z}_k \otimes \mathbb{Z})$. Then for a given $g \in \ell^2(\mathbb{Z})$, the analysis operator for the Gabor family is given by

$$\Theta_g f = \sum_{(k,m)\in G} \langle f, g_{k,m} \rangle e_{k,m}$$

As for general frames, the synthesis operator is the adjoint of the anaysis operator. Thus, the frame operator is their composition, $S = \Theta^* \Theta$, and so it is given by

$$Sf = \sum_{m \in \mathbb{Z}} \sum_{k=0}^{K-1} \langle f, g_{k,m} \rangle g_{k,m}$$

If two Gabor atoms, g and h, generate dual frames, $\{g_{k,m}\}$ and $\{h_{k,m}\}$, we will say that (g, h) is a *dual frame pair* or simply *dual pair*.

For convenience, the translation and modulation operators are written, respectively, as

$$T_a f(n) = f(n - a) \quad \text{and} \quad E_b f(n) = e^{2\pi i b n} f(n)$$

where $a \in \mathbb{Z}$, $b \in \mathbb{R}$.

These operators are linear, since for fixed a, b

$$\begin{aligned} T_a(\alpha g + \beta f)(n) &= (\alpha g + \beta f)(n-a) \\ &= \alpha g(n-a) + \beta f(n-a) \\ &= \alpha T_a g(n) + \beta T_a f(n) \end{aligned}$$

and

$$\begin{aligned} E_b(\alpha g + \beta f)(n) &= e^{2\pi ibn}(\alpha g + \beta f)(n) \\ &= \alpha e^{2\pi ibn} g(n) + \beta e^{2\pi ibn} f(n) \\ &= \alpha E_b g(n) + \beta E_b f(n) \end{aligned}$$

and they are bounded (in fact, isometries), since

$$\begin{aligned} \|T_a f\|^2 &= \sum_{n\in\mathbb{Z}} |T_a f(n)|^2 \\ &= \sum_{n\in\mathbb{Z}} |f(n-a)|^2 \\ &= \sum_{\tilde{n}\in\mathbb{Z}} |f(\tilde{n})|^2 \\ &= \|f\|^2 \end{aligned}$$

and

$$\begin{aligned}
\|E_b f\|^2 &= \sum_{n\in\mathbb{Z}} |E_b f(n)|^2 \\
&= \sum_{n\in\mathbb{Z}} |e^{2\pi ibn} f(n)|^2 \\
&= \sum_{n\in\mathbb{Z}} |f(n)|^2 \\
&= \|f\|^2
\end{aligned}$$

Moreover, both operators are unitary, since by reindexing

$$\begin{aligned}
\langle T_a f, g\rangle &= \sum_{n\in\mathbb{Z}} f(n-a)\overline{g(n)} \\
&= \sum_{n\in\mathbb{Z}} f(n)\overline{g(n+a)} \\
&= \langle f, T_{-a} g\rangle
\end{aligned}$$

but also $T_{-a}T_a f(n) = T_{-a} f(n-a) = f(n-a+a) = f(n)$, so that $T_a^* = T_{-a} = T_a^{-1}$. Similarly, for the modulation operator

$$\begin{aligned}
\langle E_b f, g\rangle &= \sum_{n\in\mathbb{Z}} e^{2\pi ibn} f(n)\overline{g(n)} \\
&= \sum_{n\in\mathbb{Z}} f(n)\overline{e^{-2\pi ibn} g(n)} \\
&= \langle f, E_{-b} g\rangle
\end{aligned}$$

so that $E_b^* = E_{-b} = E_b^{-1}$.

The following properties are immediate.

Property 5.1

$$T_aT_bg(n) = T_{a+b}g(n)$$

Property 5.2

$$E_aE_bg(n) = E_{a+b}g(n)$$

Property 5.3

$$E_bT_ag(n) = e^{2\pi iab}T_aE_bg(n)$$

Proof:

$$\begin{aligned}
E_bT_ag(n) &= E_bg(n-a) \\
&= e^{2\pi ibn}g(n-a) \\
&= \left(e^{2\pi iab}e^{-2\pi iab}\right)e^{2\pi ibn}g(n-a) \\
&= e^{2\pi iab}e^{2\pi ib(n-a)}g(n-a) \\
&= e^{2\pi iab}T_a\left(e^{2\pi ibn}g(n)\right) \\
&= e^{2\pi iab}T_aE_bg(n)
\end{aligned}$$

■

From these properties, we see that the discrete Gabor frame is a type of "group-like" frame, where the group is given by $\mathbb{Z}/K\mathbb{Z} \times N\mathbb{Z}$. It is an example of what is sometimes called a *projective unitary representation*.

Using these operators, the Gabor family can now be written as

$$\begin{aligned}\{g_{k,m} : 0 \leq k \leq K-1, m \in \mathbb{Z}\} &= \{e^{2\pi i \frac{k}{K} n} g(n - mN) : 0 \leq k \leq K-1, m \in \mathbb{Z}\} \\ &= \{E_{\frac{k}{K}} T_{mN} g(n) : 0 \leq k \leq K-1, m \in \mathbb{Z}\}\end{aligned}$$

where $k \in \mathbb{N}$.

In addition, the frame operator can now be written as

$$\begin{aligned}Sf &= \sum_{m \in \mathbb{Z}} \sum_{k=0}^{K-1} \langle f, g_{k,m} \rangle g_{k,m} \\ &= \sum_{m \in \mathbb{Z}} \sum_{k=0}^{K-1} \langle f, E_{\frac{k}{K}} T_{mN} g \rangle E_{\frac{k}{K}} T_{mN} g\end{aligned}$$

It is clear from the operator notation that every Gabor family is uniform, since the operators are isometries, and so every vector has the same norm as the Gabor atom. That is, $\|g_{k,m}\|^2 = \|g\|^2$.

Proposition 5.1 *For a Gabor frame $\{g_{k,m}\}$ with fixed N, K, the frame operator S commutes with the translation operators of the form T_{aN}, where $a \in \mathbb{Z}$.*

Proof: For the translation operator

$$
\begin{aligned}
ST_{aN}f &= \sum_{m\in\mathbb{Z}}\sum_{k=0}^{K-1}\langle T_{aN}f, g_{k,m}\rangle g_{k,m} \\
&= \sum_{m\in\mathbb{Z}}\sum_{k=0}^{K-1}\langle f, T_{-aN}g_{k,m}\rangle g_{k,m} \\
&= T_{aN}T_{-aN}\sum_{m\in\mathbb{Z}}\sum_{k=0}^{K-1}\langle f, T_{-aN}g_{k,m}\rangle g_{k,m} \\
&= T_{aN}\sum_{m\in\mathbb{Z}}\sum_{k=0}^{K-1}\langle f, T_{-aN}g_{k,m}\rangle T_{-aN}g_{k,m} \\
&= T_{aN}\sum_{m\in\mathbb{Z}}\sum_{k=0}^{K-1}\langle f, T_{-aN}E_{\frac{k}{K}}T_{mN}g\rangle T_{-aN}E_{\frac{k}{K}}T_{mN}g \\
&= T_{aN}\sum_{m\in\mathbb{Z}}\sum_{k=0}^{K-1}\langle f, e^{-2\pi i\frac{k}{K}(-aN)}E_{\frac{k}{K}}T_{-aN}T_{mN}g\rangle e^{-2\pi i\frac{k}{K}(-aN)}E_{\frac{k}{K}}T_{-aN}T_{mN}g \\
&= T_{aN}\sum_{m\in\mathbb{Z}}\sum_{k=0}^{K-1} e^{-2\pi i\frac{k}{K}(aN)}\langle f, E_{\frac{k}{K}}T_{mN-aN}g\rangle e^{2\pi i\frac{k}{K}(aN)}E_{\frac{k}{K}}T_{mN-aN}g \\
&= T_{aN}\sum_{m\in\mathbb{Z}}\sum_{k=0}^{K-1}\langle f, E_{\frac{k}{K}}T_{(m-a)N}g\rangle E_{\frac{k}{K}}T_{(m-a)N}g \\
&= T_{aN}\sum_{\tilde{m}\in\mathbb{Z}}\sum_{k=0}^{K-1}\langle f, E_{\frac{k}{K}}T_{\tilde{m}N}g\rangle E_{\frac{k}{K}}T_{\tilde{m}N}g \\
&= T_{aN}\sum_{\tilde{m}\in\mathbb{Z}}\sum_{k=0}^{K-1}\langle f, g_{k,\tilde{m}}\rangle g_{k,\tilde{m}} \\
&= T_{aN}Sf
\end{aligned}
$$

■

The following is an example of a Parseval frame.

Example 5.1 *Let* $\{e_i\}_{i\in\mathbb{Z}}$ *be the standard orthonormal basis for* $\ell^2(\mathbb{Z})$*. The family*

$\{g_{k,m} : 0 \leq k \leq K-1, m \in \mathbb{Z}\}$, *with* $g = \frac{1}{\sqrt{K}}(e_0 + \ldots + e_{N-1})$ *is a Parseval frame if* $K \geq N$.

Proof: The g vector is of the form

$$(\ldots, 0, 0, \underbrace{\frac{1}{\sqrt{K}}, \frac{1}{\sqrt{K}}, \ldots, \frac{1}{\sqrt{K}}}_{N \text{ coordinates}}, 0, 0, \ldots)$$

First, note that since the vector g is of length N, translations by integer multiples of N do not overlap. That is, $\langle g_{k,m}, g_{k,j} \rangle = 0$ for all $j \neq m$. So consider the spaces

$$M_m = \text{span}\{T_{mN} e_i\}_{i=0}^{N-1}$$

Then $\ell^2(\mathbb{Z}) = \bigoplus_{m \in \mathbb{Z}} M_m$. Therefore, it is enough to show that for any fixed m, $\{g_{k,m}\}_{k=0}^{K-1}$ is a Parseval frame for M_m. So consider M_0, with $\{g_{k,0}\} = \{e^{2\pi i \frac{k}{K} n} g(n)\}$. This space is isomorphic to $\mathbb{C}^N$, and $\{g_{k,0}\}$ is the Parseval frame generated by the K-th roots of unity provided that $K \geq N$.

■

The following is well known, the so-called density condition.

Proposition 5.2 *There exists an element* $g \in \ell^2(\mathbb{Z})$ *such that* $\{g_{k,m} : 0 \leq k \leq K-1, m \in \mathbb{Z}\}$ *is a frame for* $\ell^2(\mathbb{Z})$ *if and only if* $\frac{N}{K} \leq 1$, *with equality only for a basis.*

We require the following lemma

Lemma 5.2 *If* $\{g_{k,m}\}$ *and* $\{h_{k,m}\}$ *are Parseval frames for* $\ell^2(\mathbb{Z})$, *then* $\|g\|^2 = \|h\|^2$.

Proof:

$$\begin{aligned}
\|g\|^2 &= \sum_{m\in\mathbb{Z}}\sum_{k=0}^{K-1} |\langle g, h_{k,m}\rangle|^2 \\
&= \sum_{m\in\mathbb{Z}}\sum_{k=0}^{K-1} |\langle g, E_{\frac{k}{K}} T_{mN} h\rangle|^2 \\
&= \sum_{m\in\mathbb{Z}}\sum_{k=0}^{K-1} |\langle e^{-2\pi i \frac{k}{K} mN} E_{-\frac{k}{K}} T_{-mN} g, h\rangle|^2 \\
&= \sum_{m\in\mathbb{Z}}\sum_{k=0}^{K-1} |\langle g_{-k,-m}, h\rangle|^2 \\
&= \|h\|^2
\end{aligned}$$

■

Remark: Since $\{g_{k,m}\}$ with $g = \frac{1}{\sqrt{K}}(e_0 + \ldots + e_{N-1})$ was shown to be a Parseval frame, it follows that every Parseval frame $\{h_{k,m}\}$ for $\ell^2(\mathbb{Z})$ has $\|h\|^2 = \frac{N}{K}$.

5.3.1 Characterization of Tight Gabor Frames and Dual Frames

Theorem 5.1 *Let $g, h \in \ell^2(\mathbb{Z})$. Then (g,h) is a dual pair if and only if*

$$\sum_{m\in\mathbb{Z}} g(n-mN)\overline{h(n-mN-jK)} = \frac{1}{K}\delta_{0,j}$$

for $j \in \mathbb{Z}$ and $n = 0, 1, \ldots, N-1$.

Proof: Let $\xi, \eta \in \ell^2(\mathbb{Z})$ be of finite length. Then

$$\begin{aligned}
\sum_{m\in\mathbb{Z}}\sum_{k=0}^{K-1}\langle \xi, g_{k,m}\rangle\langle h_{k,m},\eta\rangle &= \sum_{m\in\mathbb{Z}}\sum_{k=0}^{K-1}\langle \xi, E_{\frac{k}{K}}T_{mN}g\rangle\langle E_{\frac{k}{K}}T_{mN}h,\eta\rangle \\
&= \sum_{m\in\mathbb{Z}}\sum_{k=0}^{K-1}\left(\sum_{n\in\mathbb{Z}}\xi(n)\overline{e^{2\pi i\frac{k}{K}n}g(n-mN)}\right)\left(\sum_{j\in\mathbb{Z}}e^{2\pi i\frac{k}{K}j}h(j-mN)\overline{\eta(j)}\right) \\
&= \sum_{m\in\mathbb{Z}}\sum_{k=0}^{K-1}\sum_{n,j\in\mathbb{Z}}\xi(n)\overline{\eta(j)}e^{2\pi i\frac{k}{K}(j-n)}\overline{g(n-mN)}h(j-mN) \\
&= \sum_{n,j\in\mathbb{Z}}\xi(n)\overline{\eta(j)}\sum_{m\in\mathbb{Z}}\left(\sum_{k=0}^{K-1}e^{2\pi i\frac{k}{K}(j-n)}\right)\overline{g(n-mN)}h(j-mN)
\end{aligned}$$

where changing the order of summation is justified by ξ, η of finite length. Now, (g,h) is a dual pair if and only if this sum equals

$$\langle \xi, \eta\rangle = \sum_{n\in\mathbb{Z}}\xi(n)\overline{\eta(n)}$$

In other words, if and only if

$$\sum_{m\in\mathbb{Z}}\left(\sum_{k=0}^{K-1}e^{2\pi i\frac{k}{K}(j-n)}\right)\overline{g(n-mN)}h(j-mN) = \delta_{j,n}$$

Note that $\sum_{k=0}^{K-1}e^{2\pi i\frac{k}{K}(j-n)} = K$ if $j-n\in K\mathbb{Z}$ and 0 otherwise, so this holds if and only if

$$\sum_{m\in\mathbb{Z}}\overline{g(n-mN)}h(n+\ell K-mN) = \frac{1}{K}\delta_{0,\ell}$$

holds for all $\ell\in\mathbb{Z}$, as required.

■

Corollary 5.1 *Let $g \in \ell^2(\mathbb{Z})$. Then $\{g_{k,m}\}$ is an A-tight frame for $\ell^2(\mathbb{Z})$ if and only* *if*

$$\sum_{m\in\mathbb{Z}} g(n-mN)\overline{g(n-mN-jK)} = \frac{A}{K}\delta_{0,j}$$

for $j \in \mathbb{Z}$ *and* $n = 0, 1, \ldots, N-1$.

Proof: This follows from Theorem 5.1 and the fact that $\{g_{k,m}\}$ is an A-tight frame for $\ell^2(\mathbb{Z})$ if and only if $(g, \frac{1}{A}g)$ is a dual pair.

■

Let G be a subgroup of $\mathbb{Z}$. We say that a set $\mathcal{D}$ tiles $\mathbb{Z}$ by G if $\{G+m : m \in \mathcal{D}\}$ is a disjoint partition of $\mathbb{Z}$ (in this case $\mathcal{D}$ is also called a complete digital set for $\mathbb{Z}/G$). If $(G+m)\cap(G+n) = \emptyset$ for $m, n \in \mathcal{D}, m \neq n$, then we say that $\mathcal{D}$ packs $\mathbb{Z}$ by G.

Corollary 5.2 *Let* $\Lambda = \{i_1, \ldots i_L\}$ *be an index set, and* $g = \frac{1}{\sqrt{K}}(e_{i_1} + \ldots + e_{i_L})$. *Then* $\{g_{k,m}\}$ *is a Parseval frame for* $\ell^2(\mathbb{Z})$ *if and only if* Λ *tiles* $\mathbb{Z}$ *by* $N\mathbb{Z}$ *and packs by* $K\mathbb{Z}$. *In particular, if* $K \in N\mathbb{Z}(N \leq K)$, *then* $\{g_{k,m}\}$ *is a Parseval frame for* $\ell^2(\mathbb{Z})$ *if and only if* Λ *tiles* $\mathbb{Z}$ *by* $N\mathbb{Z}$.

Proof: From Corollary 5.1, $\{g_{k,m}\}$ is a Parseval frame for $\ell^2(\mathbb{Z})$ if and only if

$$\begin{aligned}\frac{1}{K}\delta_{0,j} &= \sum_{m\in\mathbb{Z}} g(n-mN)\overline{g(n-mN-jK)} \\ &= \frac{1}{K}\sum_{m\in\mathbb{Z}}\left(\sum_{s=1}^{L} e_0(n-mN-i_s)\right)\left(\sum_{s=1}^{L} e_0(n-mN-i_s-jK)\right)\end{aligned}$$

For $j = 0$

$$\frac{1}{K} = \frac{1}{K} \sum_{m \in \mathbb{Z}} \left(\sum_{s=1}^{L} e_0(n - mN - i_s) \right)^2$$
$$1 = \sum_{m \in \mathbb{Z}} \left(\sum_{s=1}^{L} e_0(n - mN - i_s) \right)^2$$

which holds if and only if Λ tiles $\mathbb{Z}$ by $N\mathbb{Z}$.

For $j \neq 0$

$$0 = \frac{1}{K} \sum_{m \in \mathbb{Z}} \left(\sum_{s=1}^{L} e_0(n - mN - i_s) \right) \left(\sum_{s=1}^{L} e_0(n - mN - i_s - jK) \right)$$

which holds if and only if Λ packs $\mathbb{Z}$ by $K\mathbb{Z}$.

■

Corollary 5.2 gives an alternate proof for Example 5.1.

5.3.2 Orthogonal Gabor Frames and Gabor Super-Frames

We say that two Bessel sequences are orthogonal if the range spaces of their respective analysis operators are orthogonal. This can be shown to be equivalent to the condition $\Theta_2^* \Theta_1 = 0$. If the sequences are also frames we call them *orthogonal frames*.

Note that if Gabor atoms h and v generate two dual frames for g, then $u = h - v$ generates a Gabor sequence which is orthogonal (strongly disjoint) with $\{g_{k,m}\}$. The following characterizes the orthogonality of Gabor Bessel sequences.

Proposition 5.3 *Let $\{g_{k,m}\}$ and $\{u_{k,m}\}$ be Bessel sequences. Then they are orthog-*

onal if and only if

$$\sum_{m\in\mathbb{Z}} g(n-mN)\overline{u(n-mN-jK)} = 0$$

for $j \in \mathbb{Z}$ and $n = 0, 1, \ldots, N-1$.

Proof: Let $\xi, \eta \in \ell^2(\mathbb{Z})$ be of finite length. Then, as in the proof of Theorem 5.1

$$\sum_{m\in\mathbb{Z}}\sum_{k=0}^{K-1}\langle \xi, g_{k,m}\rangle\langle u_{k,m}, \eta\rangle = \sum_{n,j\in\mathbb{Z}} \xi(n)\overline{\eta(j)} \sum_{m\in\mathbb{Z}} \left(\sum_{k=0}^{K-1} e^{2\pi i \frac{k}{K}(j-n)}\right) \overline{g(n-mN)}u(j-mN)$$

Now, (g, u) is an orthogonal pair if and only if

$$\sum_{m\in\mathbb{Z}} \left(\sum_{k=0}^{K-1} e^{2\pi i \frac{k}{K}(j-n)}\right) \overline{g(n-mN)}u(j-mN) = 0$$

Note that $\sum_{k=0}^{K-1} e^{2\pi i \frac{k}{K}(j-n)} = K$ if $j - n \in K\mathbb{Z}$ and 0 otherwise, so this holds if and only if

$$\sum_{m\in\mathbb{Z}} \overline{g(n-mN)}u(n+\ell K-mN) = 0$$

holds for all $\ell \in \mathbb{Z}$, as required.

■

Remark: If this holds for all j, then in particular it holds for $j = 0$. And so, summing over n gives the proof of the following corollary.

Corollary 5.3 *If $\{g_{k,m}\}$ and $\{u_{k,m}\}$ are orthogonal Bessel sequences, then $\langle g, u\rangle = 0$.*

Note that in general $\langle g, u\rangle = 0$ does not imply that (g, u) is an orthogonal pair. For example

Example 5.2 *Consider the standard orthonormal basis vectors e_0 and e_K. Clearly*

$\langle e_0, e_K \rangle = 0$, but for $n = 0$, $j = -1$

$$\sum_{m\in\mathbb{Z}} e_0(n-mN)\overline{e_K(n-mN-jK)} = \sum_{m\in\mathbb{Z}} e_0(-mN)\overline{e_K(-mN+K)}$$
$$= 1$$

Since it equals 1 when $m = 0$ and 0 otherwise. Therefore the condition of Proposition 5.3 is not satisfied, and so (e_0, e_K) do not form an orthogonal pair.

Alternate Proof of Proposition 5.3 for Frames

In the case when $\{g_{k,m}\}$ and $\{u_{k,m}\}$ are frames (as opposed to just Bessel sequences), the following variation of Proposition 5.3 can be proven using Theorem 5.1.

Proposition 5.4 *The pair (g, u) generate orthogonal frames if and only if*

$$\sum_{m\in\mathbb{Z}} g(n-mN)\overline{u(n-mN-jK)} = 0$$

for $j \in \mathbb{Z}$ and $n = 0, 1, \ldots, N-1$.

Proof: From the characterization of dual pairs, if $(g, S^{-1}g + u)$ is a dual pair

$$\begin{aligned}
\frac{1}{K}\delta_{0,j} &= \sum_{m\in\mathbb{Z}} g(n-mN)\overline{(S^{-1}g+u)(n-mN-jK)} \\
&= \sum_{m\in\mathbb{Z}} g(n-mN)\overline{[S^{-1}g(n-mN-jK)+u(n-mN-jK)]} \\
&= \sum_{m\in\mathbb{Z}} g(n-mN)\overline{S^{-1}g(n-mN-jK)} + \sum_{m\in\mathbb{Z}} g(n-mN)\overline{u(n-mN-jK)} \\
&= \frac{1}{K}\delta_{0,j} + \sum_{m\in\mathbb{Z}} g(n-mN)\overline{u(n-mN-jK)}
\end{aligned}$$

This implies that

$$\sum_{m\in\mathbb{Z}} g(n-mN)\overline{u(n-mN-jK)} = 0$$

for $j \in \mathbb{Z}$ and $n = 0, 1, \ldots, N-1$.

■

The following result gives a useful method for applying the orthogonality characterization.

Corollary 5.4 *Let* $g = \chi_{\Lambda_1}$ *and* $h = \chi_{\Lambda_2}$ *with* $\Lambda_1, \Lambda_2 \subset \mathbb{Z}$. *If* Λ_1 *and* Λ_2 *are* $K\mathbb{Z}$-*translation disjoint, then* $\{g_{k,m}\}$ *and* $\{h_{k,m}\}$ *are orthogonal.*

Proof: Applying Proposition 5.3, (g, h) are an orthogonal pair if and only if

$$\sum_{m\in\mathbb{Z}} \chi_{\Lambda_1}(n-mN)\chi_{\Lambda_2}(n-mN-jK) = 0$$

for $j \in \mathbb{Z}$ and $n = 0, 1, \ldots, N-1$. The left side can only be nonzero if $n - mN \in \Lambda_1$ and $n - mN - jK \in \Lambda_2$ simultaneously. In other words, for some j, $n - mN \in \Lambda_1 \cap (\Lambda_2 + jK)$. But if Λ_1 and Λ_2 are $K\mathbb{Z}$-translation disjoint, $\Lambda_1 \cap (\Lambda_2 + jK) = \emptyset$ for all j, and so the left side is always 0 and the equality holds.

■

The existence of Gabor super-frames is established in the following

Theorem 5.2 *The following are equivalent*

(i) There exists a Gabor super-frame of length L

(ii) $\frac{N}{K} \leq \frac{1}{L}$

Proof: For $(i) \implies (ii)$, let $h = g_1 \oplus \ldots \oplus g_L$ be a Gabor super-frame of length L. Then $\|h\|^2 \leq 1$, and since each $\{(g_i)_{k,m}\}$ is a Parseval frame for $\ell^2(\mathbb{Z})$, $\|g_i\|^2 = \frac{N}{K}$ for all i, by Lemma 5.2. Thus

$$\|h\|^2 = \sum_{i=1}^{L} \|g_i\|^2 = L \cdot \frac{N}{K} \leq 1$$

Therefore, $\frac{N}{K} \leq \frac{1}{L}$.

For $(ii) \implies (i)$, let $\frac{N}{K} \leq \frac{1}{L}$, so that $NL \leq K$.

$$\begin{aligned} \Lambda_1 &= \{0, \ldots, N-1\} \\ \Lambda_2 &= \{N, \ldots, 2N-1\} \\ &\vdots \\ \Lambda_L &= \{(L-1)N, \ldots, NL\} \end{aligned}$$

Let $g_i = \chi_{\Lambda_i}$. Since Λ_i and Λ_j are $K\mathbb{Z}$-translation disjoint for $i \neq j$, $\{(g_i)_{k,m}\}$ and $\{(g_j)_{k,m}\}$ are orthogonal by Corollary 5.4. In addition, each Λ_i tiles by $N\mathbb{Z}$ and packs by $K\mathbb{Z}$, so that $\{(g_i)_{k,m}\}$ is a Parseval frame for $\ell^2(\mathbb{Z})$. Therefore, by Lemma 5.1, $g_1 \oplus \ldots \oplus g_L$ is a Gabor super-frame of length L.

Moreover, $g_1 \oplus \ldots \oplus g_L$ is an orthonormal Gabor super-frame only if equality holds, $NL = K$ or $\frac{NL}{K} = 1$.

■

5.4 Discrete Gabor Frames in $\ell^2(\mathbb{Z}^d)$

We now consider the Hilbert space $\ell^2(\mathbb{Z}^d)$, the space of square-summable sequences indexed by integer vectors of length d, with inner product

$$\langle f, g\rangle = \sum_{\mathbf{n}\in\mathbb{Z}^d} f(\mathbf{n})\overline{g(\mathbf{n})}$$

Let G be a subgroup of $\mathbb{Z}^d$. We say that a set $\mathcal{D}$ tiles $\mathbb{Z}^d$ by G if $\{G+\mathbf{m} : \mathbf{m}\in\mathcal{D}\}$ is a disjoint partition of $\mathbb{Z}^d$ (in this case $\mathcal{D}$ is also called a complete digit set for $\mathbb{Z}^d/G$). If $(G+\mathbf{m})\cap(G+\mathbf{n}) = \emptyset$ for $\mathbf{m},\mathbf{n}\in\mathcal{D}, \mathbf{m}\neq\mathbf{n}$, then we say that $\mathcal{D}$ packs $\mathbb{Z}^d$ by G.

Given fixed integer matrices $A, B \in M_{d\times d}(\mathbb{Z})$ with B invertible, let $\Omega = \{\mathbf{k}_1, \mathbf{k}_2, \ldots, \mathbf{k}_L\}$ be a complete digit set of $B^*\mathbb{Z}^d$ in $\mathbb{Z}^d$. For a Gabor atom $g \in \ell^2(\mathbb{Z}^d)$, the Gabor sequence $\{g_{\mathbf{k},\mathbf{m}} : \mathbf{k}\in\Omega, \mathbf{m}\in\mathbb{Z}^d\}$ is given by

$$g_{\mathbf{k},\mathbf{m}}(\mathbf{n}) = e^{2\pi i\langle \mathbf{k}, B^{-1}\mathbf{n}\rangle} g(\mathbf{n} - A\mathbf{m}), \quad \mathbf{n}\in\mathbb{Z}^d$$

We begin with a characterization for dual frames, as in Theorem 5.1, but we require some basic lemmas on the nature of modulation in higher dimensions.

The first lemma shows that modulation values only depend on the $B^*\mathbb{Z}^d$-tile.

Lemma 5.3 *Modulation is well-defined for the quotient group $\mathbb{Z}^d/B^*\mathbb{Z}^d$. That is if $\mathbf{x},\mathbf{y}\in\mathbf{k}_i + B^*\mathbb{Z}^d$ for some $\mathbf{k}_i\in\Omega$ a $B^*\mathbb{Z}^d$-tile, then $e^{2\pi i\langle\mathbf{x},B^{-1}\mathbf{n}\rangle} = e^{2\pi i\langle\mathbf{y},B^{-1}\mathbf{n}\rangle}$. In fact, they both equal $e^{2\pi i\langle\mathbf{k}_i,B^{-1}\mathbf{n}\rangle}$.*

Proof: Let $\mathbf{x} = \mathbf{k}_i + B^*\mathbf{v}$ and $\mathbf{y} = \mathbf{k}_i + B^*\mathbf{w}$ for some $\mathbf{k}_i \in \Omega$. Then

$$\begin{aligned}
e^{2\pi i\langle \mathbf{x}, B^{-1}\mathbf{n}\rangle} &= e^{2\pi i\langle \mathbf{k}_i + B^*\mathbf{v}, B^{-1}\mathbf{n}\rangle} \\
&= e^{2\pi i\langle \mathbf{k}_i, B^{-1}\mathbf{n}\rangle} \cdot e^{2\pi i\langle B^*\mathbf{v}, B^{-1}\mathbf{n}\rangle} \\
&= e^{2\pi i\langle \mathbf{k}_i, B^{-1}\mathbf{n}\rangle} \cdot e^{2\pi i\langle \mathbf{v}, \mathbf{n}\rangle} \\
&= e^{2\pi i\langle \mathbf{k}_i, B^{-1}\mathbf{n}\rangle} \cdot (1) \\
&= e^{2\pi i\langle \mathbf{k}_i, B^{-1}\mathbf{n}\rangle} \cdot e^{2\pi i\langle \mathbf{w}, \mathbf{n}\rangle} \\
&= e^{2\pi i\langle \mathbf{k}_i + B^*\mathbf{w}, B^{-1}\mathbf{n}\rangle} \\
&= e^{2\pi i\langle \mathbf{y}, B^{-1}\mathbf{n}\rangle}
\end{aligned}$$

■

Also, we require the following lemma which generalizes the behavior of the roots of unity.

Lemma 5.4 *If* Ω *is a* $B^*\mathbb{Z}^d$*-tile of* $\mathbb{Z}^d$

$$\sum_{\mathbf{k}\in\Omega} e^{2\pi i\langle \mathbf{k}, B^{-1}\mathbf{n}\rangle} = \begin{cases} |\Omega| & \textit{if } \mathbf{n} \in B\mathbb{Z}^d \\ 0 & \textit{otherwise} \end{cases}$$

Proof: Let $\mathbf{k}_i \in \Omega$ be arbitrary. Then

$$\begin{aligned}
e^{2\pi i\langle \mathbf{k}_i, B^{-1}\mathbf{n}\rangle}\left(\sum_{\mathbf{k}\in\Omega} e^{2\pi i\langle \mathbf{k}, B^{-1}\mathbf{n}\rangle}\right) &= \sum_{\mathbf{k}\in\Omega} e^{2\pi i\langle \mathbf{k}_i + \mathbf{k}, B^{-1}\mathbf{n}\rangle} \\
&= \sum_{\widetilde{\mathbf{k}}\in\Omega} e^{2\pi i\langle \widetilde{\mathbf{k}}, B^{-1}\mathbf{n}\rangle}
\end{aligned}$$

So either $\sum_{\mathbf{k}\in\Omega} e^{2\pi i\langle \mathbf{k}, B^{-1}\mathbf{n}\rangle} = 0$ or $e^{2\pi i\langle \mathbf{k}_i, B^{-1}\mathbf{n}\rangle} = 1$. But $e^{2\pi i\langle \mathbf{k}_i, B^{-1}\mathbf{n}\rangle} = 1$ if and only if $\langle \mathbf{k}_i, B^{-1}\mathbf{n}\rangle \in \mathbb{Z}$. And since $\mathbf{k}_i$ was arbitrary, we have $B^{-1}\mathbf{n} \in \mathbb{Z}^d$, or equivalently, $\mathbf{n} \in B\mathbb{Z}^d$.

■

Now we are ready to prove the dual frame characterization for $\ell^2(\mathbb{Z}^d)$.

Theorem 5.3 *Let $g, h \in \ell^2(\mathbb{Z}^d)$. Then (g, h) is a dual pair if and only if*

$$\sum_{\mathbf{m}\in\mathbb{Z}^d} g(\mathbf{n} - A\mathbf{m})\overline{h(\mathbf{n} - A\mathbf{m} - B\mathbf{j})} = \frac{1}{|\Omega|}\delta_{\mathbf{0},\mathbf{j}}$$

for $\mathbf{j} \in \mathbb{Z}^d$ and $\mathbf{n} \in \mathbb{Z}^d$ (in fact, $\mathbf{n}$ in any $A\mathbb{Z}^d$-tile is enough).

Proof: Let $\xi, \eta \in \ell^2(\mathbb{Z}^d)$ be of finite length. Then

$$\begin{aligned}
&\sum_{\mathbf{m}\in\mathbb{Z}^d}\sum_{\mathbf{k}\in\Omega}\langle \xi, g_{\mathbf{k},\mathbf{m}}\rangle\langle h_{\mathbf{k},\mathbf{m}}, \eta\rangle \\
&\qquad = \sum_{\mathbf{m}\in\mathbb{Z}^d}\sum_{\mathbf{k}\in\Omega}\left(\sum_{\mathbf{n}\in\mathbb{Z}^d}\xi(\mathbf{n})\overline{e^{2\pi i\langle \mathbf{k}, B^{-1}\mathbf{n}\rangle}g(\mathbf{n} - A\mathbf{m})}\right)\left(\sum_{\mathbf{j}\in\mathbb{Z}^d} e^{2\pi i\langle \mathbf{k}, B^{-1}\mathbf{j}\rangle}h(\mathbf{j} - A\mathbf{m})\overline{\eta(\mathbf{j})}\right) \\
&\qquad = \sum_{\mathbf{m}\in\mathbb{Z}^d}\sum_{\mathbf{k}\in\Omega}\sum_{\mathbf{n},\mathbf{j}\in\mathbb{Z}^d}\xi(\mathbf{n})\overline{\eta(\mathbf{j})}e^{2\pi i\langle \mathbf{k}, B^{-1}(\mathbf{j}-\mathbf{n})\rangle}\overline{g(\mathbf{n} - A\mathbf{m})}h(\mathbf{j} - A\mathbf{m}) \\
&\qquad = \sum_{\mathbf{n},\mathbf{j}\in\mathbb{Z}^d}\xi(\mathbf{n})\overline{\eta(\mathbf{j})}\sum_{\mathbf{m}\in\mathbb{Z}^d}\left(\sum_{\mathbf{k}\in\Omega} e^{2\pi i\langle \mathbf{k}, B^{-1}(\mathbf{j}-\mathbf{n})\rangle}\right)\overline{g(\mathbf{n} - A\mathbf{m})}h(\mathbf{j} - A\mathbf{m})
\end{aligned}$$

where changing the order of summation is justified by ξ, η of finite length. Now, (g, h) is a dual pair if and only if this sum equals

$$\langle \xi, \eta\rangle = \sum_{\mathbf{n}\in\mathbb{Z}^d}\xi(\mathbf{n})\overline{\eta(\mathbf{n})}$$

In other words, if and only if

$$\sum_{\mathbf{m}\in\mathbb{Z}^d}\left(\sum_{\mathbf{k}\in\Omega} e^{2\pi i\langle \mathbf{k}, B^{-1}(\mathbf{j}-\mathbf{n})\rangle}\right)\overline{g(\mathbf{n}-A\mathbf{m})}h(\mathbf{j}-A\mathbf{m}) = \delta_{\mathbf{j},\mathbf{n}}$$

From Lemma 5.4, $\sum_{\mathbf{k}\in\Omega} e^{2\pi i\langle \mathbf{k}, B^{-1}(\mathbf{j}-\mathbf{n})\rangle} = |\Omega|$ if $\mathbf{j}-\mathbf{n}\in B\mathbb{Z}^d$ and 0 otherwise, so this holds if and only if

$$\sum_{\mathbf{m}\in\mathbb{Z}^d}\overline{g(\mathbf{n}-A\mathbf{m})}h(\mathbf{n}+B\ell-A\mathbf{m}) = \frac{1}{|\Omega|}\delta_{\mathbf{0},\ell}$$

holds for all $\ell\in\mathbb{Z}^d$, as required. .

■

The characterization of tight frames follows immediately

Corollary 5.5 *Let $g\in\ell^2(\mathbb{Z}^d)$. Then $\{g_{\mathbf{k},\mathbf{m}}\}$ is a λ-tight frame for $\ell^2(\mathbb{Z}^d)$ if and only if*

$$\sum_{\mathbf{m}\in\mathbb{Z}^d} g(\mathbf{n}-A\mathbf{m})\overline{g(\mathbf{n}-A\mathbf{m}-B\mathbf{j})} = \frac{\lambda}{|\Omega|}\delta_{\mathbf{0},\mathbf{j}}$$

for $\mathbf{j}\in\mathbb{Z}^d$ and $\mathbf{n}\in\mathbb{Z}^d$.

Proof: This follows from Theorem 5.3 and the fact that $\{g_{\mathbf{k},\mathbf{m}}\}$ is a λ-tight frame for $\ell^2(\mathbb{Z}^d)$ if and only if $(g, \frac{1}{\lambda}g)$ is a dual pair.

■

The following corollary is one application of this characterization formula.

Corollary 5.6 *Let $\Lambda = \{\mathbf{i}_1, \ldots \mathbf{i}_L\}$ be an index set, and $g = \frac{1}{\sqrt{|\Omega|}}(e_{\mathbf{i}_1}+\ldots+e_{\mathbf{i}_L})$. Then $\{g_{\mathbf{k},\mathbf{m}}\}$ is a Parseval frame for $\ell^2(\mathbb{Z}^d)$ if and only if Λ tiles $\mathbb{Z}^d$ by $A\mathbb{Z}^d$ and packs by $B\mathbb{Z}^d$.*

Proof: From Corollary 5.5, $\{g_{\mathbf{k},\mathbf{m}}\}$ is a Parseval frame for $\ell^2(\mathbb{Z}^d)$ if and only if

$$\begin{aligned}\frac{1}{|\Omega|}\delta_{\mathbf{0},\mathbf{j}} &= \sum_{\mathbf{m}\in\mathbb{Z}^d} g(\mathbf{n}-A\mathbf{m})\overline{g(\mathbf{n}-A\mathbf{m}-B\mathbf{j})} \\ &= \frac{1}{|\Omega|}\sum_{\mathbf{m}\in\mathbb{Z}^d}\left(\sum_{\mathbf{i}_s\in\Lambda} e_{\mathbf{0}}(\mathbf{n}-A\mathbf{m}-\mathbf{i}_s)\right)\left(\sum_{\mathbf{i}_s\in\Lambda} e_{\mathbf{0}}(\mathbf{n}-A\mathbf{m}-\mathbf{i}_s-B\mathbf{j})\right)\end{aligned}$$

For $\mathbf{j}=\mathbf{0}$

$$\begin{aligned}\frac{1}{|\Omega|} &= \frac{1}{|\Omega|}\sum_{\mathbf{m}\in\mathbb{Z}^d}\left(\sum_{\mathbf{i}_s\in\Lambda} e_{\mathbf{0}}(\mathbf{n}-A\mathbf{m}-\mathbf{i}_s)\right)^2 \\ 1 &= \sum_{\mathbf{m}\in\mathbb{Z}^d}\left(\sum_{\mathbf{i}_s\in\Lambda} e_{\mathbf{0}}(\mathbf{n}-A\mathbf{m}-\mathbf{i}_s)\right)^2\end{aligned}$$

which holds if and only if Λ tiles $\mathbb{Z}^d$ by $A\mathbb{Z}^d$.

For $\mathbf{j}\neq\mathbf{0}$

$$0 = \frac{1}{|\Omega|}\sum_{\mathbf{m}\in\mathbb{Z}^d}\left(\sum_{\mathbf{i}_s\in\Lambda} e_{\mathbf{0}}(\mathbf{n}-A\mathbf{m}-\mathbf{i}_s)\right)\left(\sum_{\mathbf{i}_s\in\Lambda} e_{\mathbf{0}}(\mathbf{n}-A\mathbf{m}-\mathbf{i}_s-B\mathbf{j})\right)$$

which holds if and only if Λ packs $\mathbb{Z}^d$ by $B\mathbb{Z}^d$.

■

Next, we prove the corresponding characterization for orthogonal Bessel sequences in $\ell^2(\mathbb{Z}^d)$.

Proposition 5.5 *Let $\{g_{\mathbf{k},\mathbf{m}}\}$ and $\{u_{\mathbf{k},\mathbf{m}}\}$ be Bessel sequences. Then they are orthogonal if and only if*

$$\sum_{\mathbf{m}\in\mathbb{Z}^d} g(\mathbf{n}-A\mathbf{m})\overline{u(\mathbf{n}-A\mathbf{m}-B\mathbf{j})} = 0$$

for $\mathbf{j} \in \mathbb{Z}^d$ *and* $\mathbf{n} \in \mathbb{Z}^d$.

Proof: Let $\xi, \eta \in \ell^2(\mathbb{Z}^d)$ be of finite length. Then, as in the proof of Theorem 5.3

$$\sum_{\mathbf{m}\in\mathbb{Z}^d} \sum_{\mathbf{k}\in\Omega} \langle \xi, g_{\mathbf{k},\mathbf{m}} \rangle \langle h_{\mathbf{k},\mathbf{m}}, \eta \rangle$$
$$= \sum_{\mathbf{n},\mathbf{j}\in\mathbb{Z}^d} \xi(\mathbf{n})\overline{\eta(\mathbf{j})} \sum_{\mathbf{m}\in\mathbb{Z}^d} \left(\sum_{\mathbf{k}\in\Omega} e^{2\pi i \langle \mathbf{k}, B^{-1}(\mathbf{j}-\mathbf{n})\rangle} \right) \overline{g(\mathbf{n} - A\mathbf{m})} u(\mathbf{j} - A\mathbf{m})$$

Now, (g, u) is an orthogonal pair if and only if

$$\sum_{\mathbf{n},\mathbf{j}\in\mathbb{Z}^d} \sum_{\mathbf{m}\in\mathbb{Z}^d} \left(\sum_{\mathbf{k}\in\Omega} e^{2\pi i \langle \mathbf{k}, B^{-1}(\mathbf{j}-\mathbf{n})\rangle} \right) \overline{g(\mathbf{n} - A\mathbf{m})} u(\mathbf{j} - A\mathbf{m}) = 0$$

From Lemma 5.4, $\sum_{\mathbf{k}\in\Omega} e^{2\pi i \langle \mathbf{k}, B^{-1}(\mathbf{j}-\mathbf{n})\rangle} = |\Omega|$ if $\mathbf{j} - \mathbf{n} \in B\mathbb{Z}^d$ and 0 otherwise, so this holds if and only if

$$\sum_{\mathbf{m}\in\mathbb{Z}^d} \overline{g(\mathbf{n} - A\mathbf{m})} u(\mathbf{n} + B\ell - A\mathbf{m}) = 0$$

holds for all $\ell \in \mathbb{Z}^d$, as required.

■

Another corollary, which will be useful in proving Theorem 5.6, relates an orthogonal pair of Gabor sequences generated by characteristic functions to the tiling properties of their index sets.

Corollary 5.7 *Let* $g = \chi_{\Lambda_1}$ *and* $h = \chi_{\Lambda_2}$ *with* $\Lambda_1, \Lambda_2 \subset \mathbb{Z}^d$. *If* Λ_1 *and* Λ_2 *are* $B\mathbb{Z}^d$-*translation disjoint, then* $\{g_{\mathbf{k},\mathbf{m}}\}$ *and* $\{h_{\mathbf{k},\mathbf{m}}\}$ *are orthogonal.*

Proof: Applying Proposition 5.5, (g,h) are an orthogonal pair if and only if

$$\sum_{\mathbf{m}\in\mathbb{Z}^d} \chi_{\Lambda_1}(\mathbf{n}-A\mathbf{m})\chi_{\Lambda_2}(\mathbf{n}-A\mathbf{m}-B\mathbf{j}) = 0$$

for $\mathbf{j} \in \mathbb{Z}^d$ and $\mathbf{n} \in \mathbb{Z}^d$. The left side can only be nonzero if $\mathbf{n} - A\mathbf{m} \in \Lambda_1$ and $\mathbf{n}-A\mathbf{m}-B\mathbf{j} \in \Lambda_2$ simultaneously. In other words, for some $\mathbf{j}$, $\mathbf{n}-A\mathbf{m} \in \Lambda_1 \cap (\Lambda_2+B\mathbf{j})$. But if Λ_1 and Λ_2 are $B\mathbb{Z}^d$-translation disjoint, $\Lambda_1 \cap (\Lambda_2 + B\mathbf{j}) = \emptyset$ for all $\mathbf{j}$, and so the left side is always 0 and the equality holds.

■

We require the following lemma, a generalization of Lemma 5.2

Lemma 5.5 *If* $\{g_1, \ldots, g_L\}$ *and* $\{h_1, \ldots, h_N\}$ *both generate Parseval frames for* $\ell^2(\mathbb{Z}^d)$*, then*

$$\sum_{i=1}^{L} \|g_i\|^2 = \sum_{j=1}^{N} \|h_j\|^2$$

Proof:

$$\begin{aligned}
\sum_{i=1}^{L} \|g_i\|^2 &= \sum_{i=1}^{L}\sum_{j=1}^{N}\sum_{\mathbf{k}\in\Omega}\sum_{\mathbf{m}\in\mathbb{Z}^d} |\langle g_i, (h_j)_{\mathbf{k},\mathbf{m}}\rangle|^2 \\
&= \sum_{j=1}^{N}\sum_{i=1}^{L}\sum_{\mathbf{k}\in\Omega}\sum_{\mathbf{m}\in\mathbb{Z}^d} |\langle (g_i)_{-\mathbf{k},-\mathbf{m}}, h_j\rangle|^2 \\
&= \sum_{j=1}^{N} \|h_j\|^2
\end{aligned}$$

■

We now turn to Theorem 5.6, the density condition for Gabor super-frames in $\ell^2(\mathbb{Z}^d)$. This theorem provides for the existence of Gabor super-frames and Parseval frames based on the determinants of the integer matrices A and B.

The density condition will follow from the tiling and packing results above. For one dimension, the tiling of $\mathbb{Z}$ by $a\mathbb{Z}$ and $b\mathbb{Z}$ is not very complicated. In higher dimensions, however, a bit more work is required and before we can prove this theorem, we need to generalize some results concerning common representatives for cosets and prove Theorem 5.4.

It is well known that if an abelian group G has two subgroups of finite index, H, K, then they have a common set of representatives for their cosets if and only if $|G/H| = |G/K|$. See, for example, [28]. We require something more general, for the case when $|G/H| \geq |G/K|$.

Consider subgroup $K + H$ of G, and let $|G/(K + H)| = N$. Then

$$G = \bigcup_{i=1}^{N} (d_i + K + H)$$

Lemma 5.6 *$\forall i, j \leq N$, the number of cosets of K contained in $d_i + K + H$ and $d_j + K + H$ are the same.*

Proof: Let $\{a_m + K \mid 1 \leq m \leq t\}$ be all of the cosets of K contained in $d_i + K + H$. Then, for every m

$$\begin{aligned}(d_j - d_i) + (a_m + K) &\subseteq d_j - d_i + d_i + K + H \\ &= d_j + K + H\end{aligned}$$

Note that each pair of cosets $(d_j - d_i) + (a_m + K)$ and $(d_j - d_i) + (a_n + K)$ are disjoint for $a_m \neq a_n$, since if $x \in [(d_j - d_i) + (a_m + K)] \cap [(d_j - d_i) + (a_n + K)]$

$$x = d_i - d_j + a_m + k_1 = d_i - d_j + a_n + k_2$$

$$a_m - a_n = k_2 - k_1$$

$$a_m - a_n \in K$$

Thus the number of cosets of K contained in $d_j + K + H$ is greater than or equal to the number of cosets of K contained in $d_i + K + H$. Applying the same argument with d_j and d_i reversed shows that the number of cosets of K contained in $d_i + K + H$ and $d_j + K + H$ are equal.

■

The same argument with H instead of K gives

Lemma 5.7 *$\forall i, j \leq N$, the number of cosets of H contained in $d_i + K + H$ and $d_j + K + H$ are the same.*

Lemma 5.8 *Any coset $g_1 + K$ contained in $d_i + H + K$ has non-empty intersection with any coset $g_2 + H$ contained in $d_i + H + K$.*

Proof: For some h_0, $g_1 + K = d_i + h_0 + K$, and for some k_0, $g_2 + H = d_i + k_0 + H$. The element $d_i + k_0 + h_0$ is contained in both cosets.

■

Theorem 5.4 *Assume $n = |G/K| \geq L|G/H| = Lm$. Then there exists*

$$g_{11}, g_{12}, \ldots, g_{1m}$$
$$g_{21}, g_{22}, \ldots, g_{2m}$$
$$\vdots$$
$$g_{L1}, g_{L2}, \ldots, g_{Lm}$$

such that $\{g_{i1}, \ldots, g_{im}\}$ tiles G by H and $\{g_{11}, \ldots, g_{1m}, g_{21}, \ldots, g_{2m}, \ldots, g_{L1}, \ldots, g_{Lm}\}$ packs by K.

Proof: Since there are n cosets of K, and each $d_i + K + H$ contains the same number of them for $1 \leq i \leq N$, then each $d_i + K + H$ contains $\frac{n}{N}$ cosets of K. Similarly, each $d_i + K + H$ contains $\frac{m}{N}$ cosets of H. Since $n \geq Lm$, $\frac{n}{N} \geq L\frac{m}{N}$. Let $K_{i1}, \ldots, K_{i\frac{n}{N}}$ be the cosets of K contained in $d_i + K + H$ and $H_{i1}, \ldots, H_{i\frac{m}{N}}$ be the cosets of H contained in $d_i + K + H$. By Lemma 5.8, any coset of K and coset of H contained in $d_i + K + H$ have non-empty intersection, so for all $1 \leq j \leq \frac{m}{N}$ choose

$$a_{ij}^{(1)} \in K_{ij} \cap H_{ij}$$
$$a_{ij}^{(2)} \in K_{i,j+\frac{m}{N}} \cap H_{ij}$$
$$\vdots$$
$$a_{ij}^{(L)} \in K_{i,j+(L-1)\frac{m}{N}} \cap H_{ij}$$

Now, relabel the representatives as

$$\{g_{11}, g_{12}, \ldots, g_{1m}\} = \bigcup_{i=1}^{N} \left\{a_{i1}^{(1)}, \ldots, a_{i\frac{m}{N}}^{(1)}\right\}$$
$$\{g_{21}, g_{22}, \ldots, g_{2m}\} = \bigcup_{i=1}^{N} \left\{a_{i1}^{(2)}, \ldots, a_{i\frac{m}{N}}^{(2)}\right\}$$
$$\vdots$$
$$\{g_{L1}, g_{L2}, \ldots, g_{Lm}\} = \bigcup_{i=1}^{N} \left\{a_{i1}^{(L)}, \ldots, a_{i\frac{m}{N}}^{(L)}\right\}$$

It remains to show that each $\{g_{i1}, \ldots, g_{im}\}$ tiles G by H and $\{g_{ij}\}$ packs by K. But the $a_{ij}^{(p)}$ each represent one of Lm of the n different cosets of K, so they pack by K (and tile if equality holds). In addition, for each fixed $1 \leq p \leq L$, every $a_{ij}^{(p)}$ represents one of the m different cosets of H, and so they tile by H.

■

We are now ready to prove the density condition.

Theorem 5.5 *The following are equivalent:*

(i) There exists a Gabor frame $\{g_{\mathbf{k},\mathbf{m}}\}$ *for fixed* A, B

(ii) $|\det(AB^{-1})| \leq 1$

Proof: For $(i) \implies (ii)$, let $\Lambda = \{\mathbf{i}_1, \ldots, \mathbf{i}_L\}$ be a fixed $A\mathbb{Z}^d$-tile, where $L = |\det A|$.

Define

$$g_1 = \frac{1}{\sqrt{|\det B|}} e_{\mathbf{i}_1}$$
$$\vdots$$
$$g_L = \frac{1}{\sqrt{|\det B|}} e_{\mathbf{i}_L}$$

Since Λ tiles $\mathbb{Z}^d$ by $A\mathbb{Z}^d$,

$$\mathbb{Z}^d = \biguplus_{n=1}^{L} (\mathbf{i}_n + A\mathbb{Z}^d)$$

Let $H_j = \{\xi \in \ell^2(\mathbb{Z}^d) \mid \text{supp}(\xi) \subseteq (\mathbf{i}_j + A\mathbb{Z}^d)\}$. Then

$$\ell^2(\mathbb{Z}^d) = H_1 \oplus H_2 \oplus \ldots \oplus H_L$$

Now, each $\{(g_j)_{\mathbf{k},\mathbf{m}}\}$ is a Parseval frame for H_j, since for any $\xi \in H_j$

$$\begin{aligned}
\sum_{\mathbf{k}\in\Omega} \sum_{\mathbf{m}\in\mathbb{Z}^d} |\langle \xi, (g_j)_{\mathbf{k},\mathbf{m}} \rangle|^2 &= \sum_{\mathbf{k}\in\Omega} \sum_{\mathbf{m}\in\mathbb{Z}^d} \left| \sum_{\mathbf{n}\in\mathbb{Z}^d} \xi(\mathbf{n}) \overline{\frac{1}{\sqrt{|\det B|}} e^{2\pi i \langle \mathbf{k}, B^{-1}\mathbf{n}\rangle} e_{\mathbf{i}_j + A\mathbf{m}}(\mathbf{n})} \right|^2 \\
&= \sum_{\mathbf{k}\in\Omega} \sum_{\mathbf{m}\in\mathbb{Z}^d} \left| \frac{1}{\sqrt{|\det B|}} \xi(\mathbf{i}_j + A\mathbf{m}) \right|^2 \\
&= \sum_{\mathbf{k}\in\Omega} \frac{1}{|\det B|} \|\xi\|^2 \\
&= \|\xi\|^2
\end{aligned}$$

Thus, $\{g_1, \ldots, g_L\}$ generates a Parseval frame for $\ell^2(\mathbb{Z}^d)$. Since $\{g_{\mathbf{k},\mathbf{m}}\}$ is a Parseval frame, $\|g\|^2 \leq 1$, and so applying Lemma 5.5

$$\begin{aligned}\|g\|^2 &= \sum_{i=1}^{|\det A|} \|g_i\|^2 \\ &= \sum_{i=1}^{|\det A|} \frac{1}{|\det B|} \\ &= |\det(AB^{-1})|\end{aligned}$$

Therefore, $|\det(AB^{-1})| \leq 1$.

For $(ii) \implies (i)$, suppose $|\det(AB^{-1})| \leq 1$. Then $|\det(A)| \leq |\det(B)|$. Thus $|\mathbb{Z}^d/A\mathbb{Z}^d| \leq |\mathbb{Z}^d/B\mathbb{Z}^d|$. By Theorem 5.4, there exists a set of representatives Λ which tiles $\mathbb{Z}^d$ by $A\mathbb{Z}^d$ and packs by $B\mathbb{Z}^d$. Therefore, by Corollary 5.6, there exists a Gabor frame.

■

In fact, the above theorem is a special case of the following more general density condition for Gabor super-frames, by letting $L = 1$.

Theorem 5.6 *The following are equivalent*

(i) There exists a Gabor super-frame of length L *for* $\ell^2(\mathbb{Z}^d)$

(ii) $|\det(AB^{-1})| \leq \frac{1}{L}$

Proof: For $(i) \implies (ii)$, suppose $\{(g_1)_{\mathbf{k},\mathbf{m}} \oplus \ldots \oplus (g_L)_{\mathbf{k},\mathbf{m}} \mid \mathbf{k} \in \Omega, \mathbf{m} \in A\mathbb{Z}^d\}$ is a

Parseval frame for $\ell^2(\mathbb{Z}^d) \oplus \ldots \oplus \ell^2(\mathbb{Z}^d)$. Then

$$\begin{aligned} \|g_1 \oplus \ldots \oplus g_L\| &\leq 1 \\ \sum_{i=1}^{L} \|g_i\| &\leq 1 \\ L \cdot |\det(AB^{-1})| &\leq 1 \end{aligned}$$

since each $\{(g_i)_{\mathbf{k},\mathbf{m}}\}$ is a Parseval frame for $\ell^2(\mathbb{Z}^d)$. Therefore, $|\det(AB^{-1})| \leq \frac{1}{L}$.

For $(ii) \implies (i)$, suppose $|\det(AB^{-1})| \leq \frac{1}{L}$. Then $L|\det(A)| \leq |\det(B)|$. Thus $L|\mathbb{Z}^d/A\mathbb{Z}^d| \leq |\mathbb{Z}^d/B\mathbb{Z}^d|$. By Theorem 5.4, there exists L sets of representatives $\{\Lambda_1, \ldots, \Lambda_L\}$ with $\Lambda_j = \{\mathbf{i}_{j1}, \ldots, \mathbf{i}_{j,|\det A|}\}$, each of which tiles $\mathbb{Z}^d$ by $A\mathbb{Z}^d$ and packs by $B\mathbb{Z}^d$. Therefore, by Corollary 5.6, there exists L Parseval frames for $\ell^2(\mathbb{Z}^d)$, with Gabor atoms $g_i = \frac{1}{\sqrt{|\det B|}}\chi_{\Lambda_i}$. Since Λ_i and Λ_j are $B\mathbb{Z}^d$-translation disjoint for any i, j, Corollary 5.7 implies $\{(g_i)_{\mathbf{k},\mathbf{m}}\}$ and $\{(g_j)_{\mathbf{k},\mathbf{m}}\}$ are orthogonal. Therefore, by Lemma 5.1, $g_1 \oplus \ldots \oplus g_L$ is a Gabor super-frame of length L.

Moreover, $g_1 \oplus \ldots \oplus g_L$ is an orthonormal Gabor super-frame only if equality holds.

■

Finally, we outline a proof which generalizes the so-called tight dual theorem to $\ell^2(\mathbb{Z}^d)$ (see [17]).

Theorem 5.7 *The following are equivalent*

(i) For every Gabor frame $\{g_{\mathbf{k},\mathbf{m}}\}$ with lower frame bound greater than 1, there exists a Parseval Gabor frame $\{h_{\mathbf{k},\mathbf{m}}\}$ such that (g, h) is a dual pair

(ii) $|\det(AB^{-1})| \leq \frac{1}{2}$

Proof: For $(i) \implies (ii)$, let $\{g_{\mathbf{k},\mathbf{m}}\}$ be a Gabor frame with frame operator S and lower frame bound $\frac{1}{\|S^{-1}\|} > 1$. By assumption, there is a Parseval frame $\{h_{\mathbf{k},\mathbf{m}}\}$ with (g, h) a dual pair. Let $\phi = h - S^{-1}g$. Then (g, ϕ) form an orthogonal pair. The frame operator for $\{\phi_{\mathbf{k},\mathbf{m}}\}$ is $\Theta_\phi^*\Theta_\phi = I - S^{-1}$, which is invertible, so that $\{\phi_{\mathbf{k},\mathbf{m}}\}$ is also a frame. Therefore, there are two orthogonal, Parseval frames, and so $|\det(AB^{-1})| \leq \frac{1}{2}$.

For $(ii) \implies (i)$, let $\{g_{\mathbf{k},\mathbf{m}}\}$ be a Gabor frame with frame operator S and lower frame bound $\frac{1}{\|S^{-1}\|} > 1$. From Lemma 3.7 in [17], there exists a Parseval frame $\{h_{\mathbf{k},\mathbf{m}}\}$ such that (g, h) is an orthogonal pair. Since $\|S^{-1}\| < 1$, $I - S^{-1}$ is a positive operator, and so consider $\phi = S^{-1}g + \sqrt{I - S^{-1}}h$. First, note that $\sqrt{I - S^{-1}}$ commutes with the modulation and translation operators. Also, $(g, \sqrt{I - S^{-1}}h)$ form an orthogonal pair, since

$$\sum_{\mathbf{k}\in\Omega}\sum_{\mathbf{m}\in\mathbb{Z}^d} \langle f, (\sqrt{I - S^{-1}}h)_{\mathbf{k},\mathbf{m}}\rangle g_{\mathbf{k},\mathbf{m}} = \sum_{\mathbf{k}\in\Omega}\sum_{\mathbf{m}\in\mathbb{Z}^d} \langle \sqrt{I - S^{-1}}f, h_{\mathbf{k},\mathbf{m}}\rangle g_{\mathbf{k},\mathbf{m}} = 0$$

Thus (g, ϕ) form a dual pair. It remains to show that $\{\phi_{\mathbf{k},\mathbf{m}}\}$ is a Parseval frame.

$$\begin{aligned}
\sum_{\mathbf{k}\in\Omega}\sum_{\mathbf{m}\in\mathbb{Z}^d} \langle f, \phi_{\mathbf{k},\mathbf{m}}\rangle \phi_{\mathbf{k},\mathbf{m}} &= \sum_{\mathbf{k}\in\Omega}\sum_{\mathbf{m}\in\mathbb{Z}^d} \langle f, (S^{-1}g + \sqrt{I - S^{-1}}h)_{\mathbf{k},\mathbf{m}}\rangle (S^{-1}g + \sqrt{I - S^{-1}}h)_{\mathbf{k},\mathbf{m}} \\
&= S^{-1}\left(\sum_{\mathbf{k}\in\Omega}\sum_{\mathbf{m}\in\mathbb{Z}^d} \langle f, (S^{-1}g)_{\mathbf{k},\mathbf{m}}\rangle g_{\mathbf{k},\mathbf{m}}\right) \\
&\quad + 0 + 0 + \sum_{\mathbf{k}\in\Omega}\sum_{\mathbf{m}\in\mathbb{Z}^d} \langle f, (\sqrt{I - S^{-1}}h)_{\mathbf{k},\mathbf{m}}\rangle (\sqrt{I - S^{-1}}h)_{\mathbf{k},\mathbf{m}} \\
&= S^{-1}f + (I - S^{-1})f \\
&= f
\end{aligned}$$

Therefore, $\{g_{\mathbf{k},\mathbf{m}}\}$ has a Parseval dual frame.

■

CHAPTER 6

FUTURE WORK

6.1 Further Research

The results of this work lead naturally to more questions.

First, optimal dual frames for 2 or more erasures need further study in those cases when the optinal dual frame is not unique. However, they are difficult to calculate using the operator norm. One proposed approach would be to calculate optimal duals with respect to a different metric of the error operator, for example, the trace norm, $\mathrm{tr}(TT^*)^{1/2}$.

Optimal dual frames also need further study in the infinite-dimensional case, for example, the discrete Gabor case. Also, a more in-depth problem would be the study of infinitely-many erasures.

The discrete Gabor case is one example of a projective unitary representation. Further study can be made of projective unitary representation frames in general.

Recently several researchers have been working on the Gabor frame theory for subspaces, and this theory can be studied in the $\ell^2(\mathbb{Z}^d)$ case.

In addition to these questions, the following two sections discuss some other problems in the area of frames.

6.2 Using the Löwdin Orthogonalization to Generate Parseval Frames

In [7], the authors give a generalization of the Gram-Schmidt orthogonalization which can be applied to a sequence of vectors to compute a Parseval frame for the subspace generated by the sequence, while preserving redundancy in the case of linearly dependent vectors. This procedure reduces to Gram-Schmidt orthogonalization if applied to a sequence of linearly independent vectors.

Another orthogonalization procedure, the Löwdin orthogonalization also yields Parseval frames in those instances when the vectors are linearly dependent.

Let $\{v_i\}_{i=1}^k$ be a sequence on the Hilbert space $\mathbb{C}^n$, with $k \geq n$. Then the synthesis operator of $\{v_i\}$ is the $n \times k$ matrix

$$\Theta^* = \begin{bmatrix} v_1 & v_2 & \dots & v_k \end{bmatrix}$$

and $\text{rank}(\Theta^*) = r \leq n$. By the singular value decomposition, $\exists U, V$ unitary and Σ diagonal so that

$$\Theta^* = U\Sigma V^*$$

In particular, there is a "reduced SVD" so that Σ contains only nonzero elements on the diagonal (since $n \leq k$, V may not be unitary, though it will have orthogonal columns), and then

$$\Theta^* = \underset{n\times r}{U} \underset{r\times r}{\Sigma} \underset{r\times k}{V^*}$$

The Löwdin orthogonalization is given by

$$L^* := UV^*$$

Note that the adjoint notation is used for L to keep consistent with the notation for synthesis operators. The first result shows that if $\{v_i\}_{i=1}^k$ is a frame this matrix is the synthesis operator of a Parseval frame.

Theorem 6.1 *If $\{v_i\}_{i=1}^k$ is a frame for $\mathbb{C}^n$, the columns of the matrix L^* form a Parseval frame.*

Proof: From the sizes of U and V, L^* is an $n \times k$ matrix, and

$$\begin{aligned} L^*L &= (UV^*)(UV^*)^* \\ &= UV^*VU^* \\ &= UU^* \\ &= I \end{aligned}$$

so that the associated frame operator is the identity. Therefore, the columns form a Parseval frame. Note that U is unitary if $\text{rank}(L^*) = r = n$, which is the case if $\{v_i\}$ is a frame.

■

Moreover, this frame is the same as $\{S^{-1/2}v_i\}$.

Theorem 6.2 *The Parseval frame given by the columns of L^* is the same as the Parseval frame given by $\{S^{-1/2}v_i\}$.*

Proof:

$$
\begin{aligned}
L^* - S^{-1/2}\Theta^* &= UV^* - (\Theta^*\Theta)^{-1/2}\Theta^* \\
&= UV^* - (U\Sigma V^* V \Sigma^* U^*)^{-1/2}(U\Sigma V^*) \\
&= UV^* - (U\Sigma^2 U^*)^{-1/2}(U\Sigma V^*) \\
&= UV^* - (U\Sigma^{-1} U^*)(U\Sigma V^*) \\
&= UV^* - (UV^*) \\
&= 0
\end{aligned}
$$

Therefore, $L^* = S^{-1/2}\Theta^*$.

■

There is still work to be done in the case when $\{v_i\}_{i=1}^k$ is not a frame, and rank(L^*)$= r < n$.

6.3 Mutually Unbiased Parseval Frames

Let H be a Hilbert space of dimension d. Then two sets of vectors $\{u_i\}_{i=1}^d$ and $\{v_i\}_{i=1}^d$ are called *mutually unbiased bases* (MUBs), if they satisfy

(i) $\{u_i\}$ and $\{v_i\}$ are both orthonormal bases for H.

(ii) $|\langle u_i, v_j\rangle|^2 = \frac{1}{d}$ for every i, j.

This naturally extends to the case for more than two sets of vectors, and finding the number of MUBs which exist for a given dimension is an active area of research.

Parseval frames share many of the nice properties of orthonormal bases, and so this naturally leads to the generalization of MUBs to mutually unbiased Parseval frames

Definition 6.1 *Two sequences of vectors* $\{u_i\}_{i=1}^n$ *and* $\{v_i\}_{i=1}^m$ *with* $n, m \geq d$ *are called* mutually unbiased Parseval frames *(MUPFs), if they satisfy*

(i) $\{u_i\}$ *and* $\{v_i\}$ *are both Parseval frames for* H*.*

(ii) $|\langle u_i, v_j\rangle|^2 = c$ *(a constant), for every* i, j*.*

The existence of such objects follows immediately from MUBs, since every MUB is also a MUPF.

It is known that in some dimensions of $\mathbb{R}^d$ no MUBs exist, see, for example, [4]. This leads to the following question

Question 1 *Do there exist MUPFs which are not MUBs, and, if so, can we find MUPFs in dimensions where no MUBs exist?*

We can find some necessary conditions for MUPFs.

Theorem 6.3 *If* $\{u_i\}_{i=1}^n$ *and* $\{v_i\}_{i=1}^m$ *are MUPFs with* $n, m \geq d$*, then each one is a uniform Parseval frame. Moreover, the constant* c *must be* $c = |\langle u_i, v_j\rangle|^2 = \frac{d}{nm}$

Proof: For any $1 \leq i \leq n$

$$\begin{aligned}\|u_i\|^2 &= \sum_{j=1}^m |\langle u_i, v_j\rangle|^2 \\ &= \sum_{j=1}^m c \\ &= mc\end{aligned}$$

Thus $\{u_i\}$ is a uniform Parseval frame, and a similar argument with u and v interchanged gives that $\{v_i\}$ is also a uniform Parseval frame, only with $\|v_i\|^2 = nc$.

For the moreover part, it is well known that for a uniform Parseval frame of length k, every vector in the frame has norm $\sqrt{\frac{d}{k}}$. Therefore, since $\{u_i\}_{i=1}^n$ is uniform

$$\frac{d}{n} = \|u_i\|^2 = mc$$

and so $c = \frac{d}{nm}$.

■

Note that for the orthonormal basis case, $n = m = d$, and then this simplifies to the usual $c = \frac{1}{d}$.

The first example, while somewhat trivial, shows that it is possible to have MUPFs which are not MUBs.

Example 6.1 *Let $\{v_i\}_{i=1}^4$ be the columns of*

$$\Theta_v^* = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

and $\{w_i\}_{i=1}^4$ be the columns of

$$\Theta_w^* = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2}i & -\frac{1}{2}i \end{bmatrix}$$

These are both Parseval frames for $\mathbb{C}^2$, with $\Theta_v^*\Theta_v = I$ and $\Theta_w^*\Theta_w = I$. Moreover, $|\langle v_i, w_j\rangle|^2 = \frac{1}{8}$ for all i, j.

The next example shows that it is possible for the frames to be of different lengths.

Example 6.2 *Let* $\{v_i\}_{i=1}^3$ *be the columns of*

$$\Theta_v^* = \begin{bmatrix} \sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

and $\{w_i\}_{i=1}^2$ *be the columns of*

$$\Theta_w^* = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}}i & -\frac{1}{\sqrt{2}}i \end{bmatrix}$$

These are both Parseval frames for $\mathbb{C}^2$*, with* $\Theta_v^*\Theta_v = I$ *and* $\Theta_w^*\Theta_w = I$*. Moreover,* $|\langle v_i, w_j\rangle|^2 = \frac{1}{3}$ *for all* i, j*.*

List of References

[1] R. V. Balan. *A Study of Weyl-Heisenberg and Wavelet Frames.* PhD thesis, Princeton University, 1998.

[2] G. Bhatt, L. Kraus, L. Walters, and E. Weber. On hiding messages in the oversampled Fourier coefficients. *J. Math. Anal. Appl.*, 320(1):492–498, 2006.

[3] B. G. Bodmann. Optimal linear transmission by loss-insensitive packet encoding. *Appl. Comput. Harmon. Anal.*, 22(3):274–285, 2007.

[4] P. O. Boykin, M. Sitharam, M. Tarifi, and P. Wocjan. Real mutually unbiased bases. Preprint, http://arxiv.org/abs/quant-ph/0502024v2.

[5] P. Casazza. Modern tools for Weyl-Heisenberg (Gabor) frame theory. *Adv. Imag. Elect. Phys.*, 115:1–127, 2001.

[6] P. Casazza and J. Kovaĉević. Equal-norm tight frames with erasures. *Adv. Comp. Math.*, 18:387–430, 2003.

[7] P. G. Casazza and G. Kutyniok. A generalization of Gram-Schmidt orthogonalization generating all Parseval frames. *Adv. Comp. Math.*, 27:65–78, 2007.

[8] O. Christensen. *An Introduction to Frames and Riesz Bases.* Birkhäuser, 2003.

[9] Z. Cvetković and M. Vetterli. Tight Weyl-Heisenberg frames in $\ell^2(\mathbb{Z})$. *IEEE Trans. Signal Proc.*, 46:1256–1259, 1998.

[10] J.-P. Gabardo and D. Han. Frame representations for group-like unitary operator systems. *J. Operator Theory*, 49:223–244, 2003.

[11] J.-P. Gabardo and D. Han. The uniqueness of the dual of Weyl-Heisenberg subspace frames. *Appl. Comput. Harmon. Anal.*, 17:226–240, 2004.

[12] D. Gabor. Theory of communication. *Journal of the Institution of Electrical Engineers*, 93(26):429–457, 1946.

[13] J. A. Goldstein and M. Levy. Linear algebra and quantum chemistry. *American Mathematical Monthly*, 98(8):710–718, Oct. 1991.

[14] P. Hall. On representatives of subsets. *J. London Math. Soc.*, s1-s10(37):26–30, 1935.

[15] D. Han. Approximations for Gabor and wavelet frames. *Trans. Amer. Math. Soc.*, 355:3329–3342, 2003.

[16] D. Han. Classification of finite group-frames and super-frames. *Canad. Math. Bull.*, 50(1):85–96, 2007.

[17] D. Han. Frame representations and parseval duals with applications to Gabor frames. *Trans. Amer. Math. Soc.*, 360(6):3307–3326, 2008.

[18] D. Han, K. Kornelson, D. Larson, and E. Weber. *Frames for Undergraduates*, volume 40 of *Student Mathematical Library*. American Mathematical Society, 2007.

[19] D. Han and D. R. Larson. Frames, bases, and group representations. *Mem. Amer. Math. Soc.*, 147(697):1–94, 2000.

[20] D. Han and Y. Wang. Lattice tiling and the Weyl-Heisenberg frames. *Geom. Funct. Anal.*, 11(4):742–758, 2001.

[21] R. Harkins, E. Weber, and A. Westmeyer. Encryption schemes using finite frames and Hadamard arrays. *Experimental Mathematics*, 14:423–433, 2005.

[22] E. Hewitt and K. A. Ross. *Abstract Harmonic Analysis*, volume I. Springer-Verlag, New York, second edition, 1979.

[23] R. B. Holmes and V. I. Paulsen. Optimal frames for erasures. *Linear Algebra and its Applications*, 377:31–51, 2004.

[24] W. Jing. *Frames in Hilbert C^*-Modules*. PhD thesis, University of Central Florida, 2006.

[25] D. Kalra. Complex equiangular cyclic frames and erasures. *Linear Algebra and its Applications*, 419:373–399, 2006.

[26] J. Kovačević and A. Chebira. Life beyond bases: The advent of frames. *IEEE Signal Proc. Mag.*, 2007.

[27] J. M. Morris and Y. Lu. Discrete Gabor expansion of discrete-time signals in $\ell^2(\mathbb{Z})$ via frame theory. *Signal Processing*, 40:155–181, 1994.

[28] O. Ore. On coset representatives in groups. *Proc. Amer. Math. Soc.*, 9(4):665–670, 1958.

[29] I. Shamam. Frames in hilbert spaces. Master's thesis, University of Central Florida, 2002.

[30] T. Strohmer and R. W. Heath, Jr. Grassmannian frames with applications to coding and communication. *Appl. Comput. Harmon. Anal.*, 14:257–275, 2003.

[31] S. Waldron and N. Hay. On computing all harmonic frames of n vectors in $\mathbb{C}^d$. *Appl. Comput. Harmon. Anal.*, 21(2):168–181, 2006.

[32] J. Wexler and S. Raz. Discrete Gabor expansions. *Signal Processing*, 21:207–220, 1990.

[33] P. Wocjan and T. Beth. New construction of mutually unbiased bases in square dimensions. *Quantum Info. Comp.*, 5(2):93–101, 2005.

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