2008

Integrability Of A Singularly Perturbed Model Describing Gravity Water Waves On A Surface Of Finite Depth

Steven Little
University of Central Florida

INTEGRABILITY OF A SINGULARLY PERTURBED MODEL DESCRIBING GRAVITY WATER WAVES ON A SURFACE OF FINITE DEPTH

by

STEVEN LITTLE
B.S. Electrical Engineering, Clemson University, 1985
M.S. Electrical Engineering, Clemson University, 1986

A thesis submitted in partial fulfillment of the requirements
for the degree of Master of Science
in the Department of Mathematics
in the College of Sciences
at the University of Central Florida
Orlando, Florida

Spring Term
2008

ABSTRACT

Our work is closely connected with the problem of splitting of separatrices (breaking of homoclinic orbits) in a singularly perturbed model describing gravity water waves on a surface of finite depth. The singularly perturbed model is a family of singularly perturbed fourth-order nonlinear ordinary differential equations, parametrized by an external parameter (in addition to the small parameter of the perturbations). It is known that in general separatrices will not survive a singular perturbation. However, it was proven by Tovbis and Pelinovsky that there is a discrete set of exceptional values of the external parameter for which separatrices do survive the perturbation. Since our family of equations can be written in the Hamiltonian form, the question is whether or not survival of separatrices implies integrability of the corresponding equation. The complete integrability of the system is examined from two viewpoints: 1) the existence of a second first integral in involution (Liouville integrability), and 2) the existence of single-valued, meromorphic solutions (complex analytic integrability). In the latter case, a singular point analysis is done using the technique given by Ablowitz, Ramani,

and Segur (the ARS algorithm) to determine whether the system is of Painlevé-type (P-type), lacking movable critical points. The system is shown by the algorithm to fail to be of P-type, a strong indication of nonintegrability.

ACKNOWLEDGMENTS

The author wishes to express his appreciation to the following:

Dr. Alexander Tovbis, for serving as chairman of the advisory committee and for the guidance and support he has provided throughout the development of this thesis.

Dr. Xin Li and Dr. John Cannon for their participation on the advisory committee.

My parents, Mr. and Mrs. L. M. Little, for their encouragement and support through the years.

TABLE OF CONTENTS

LIST OF FIGURES

1 INTRODUCTION

The problem of splitting of separatrices (breaking of homoclinic orbits) in a singularly perturbed model describing gravity water waves on a surface of finite depth has recently been studied by Tovbis and Pelinovsky [10]. They study the conditions for the existence of homoclinic orbits in the fourth-order equation

$$v^{(iv)}\left(z\right)+(1-\varepsilon^2)v''\left(z\right)-\varepsilon^2 v\left(z\right)=v^2\left(z\right)+\gamma\left(2v\left(z\right)v''\left(z\right)+\left(v'\left(z\right)\right)^2\right) \tag{1}$$

in the limit $\varepsilon \to 0$ where $\gamma, \varepsilon \in \mathbb{R}$. Their work was motivated by the relation of this equation to a travelling wave reduction of the fifth-order partial differential equation

$$r_t + \frac{2}{15} r_{xxxxx} - b r_{xxx} + 3 r r_x + 2 r_x r_{xx} + r r_{xxx} = 0, \tag{2}$$

which arises as a weakly nonlinear long-wave approximation to the gravity-capillary water-wave problem [2].

Equation 1 is a family of singularly perturbed nonlinear ordinary differential equations (ODEs), parametrized by the external parameter γ (in addition to the small parameter of the perturbations ε). It is known that, generally speaking, separatrices will not survive a singular perturbation. In [10], it was proved in regard to Equation 1 that the separatrices survive the perturbation only on a discrete set of exceptional curves $\gamma(\varepsilon)$ in the parameter space. The first five of these curves are shown in Figure 1.

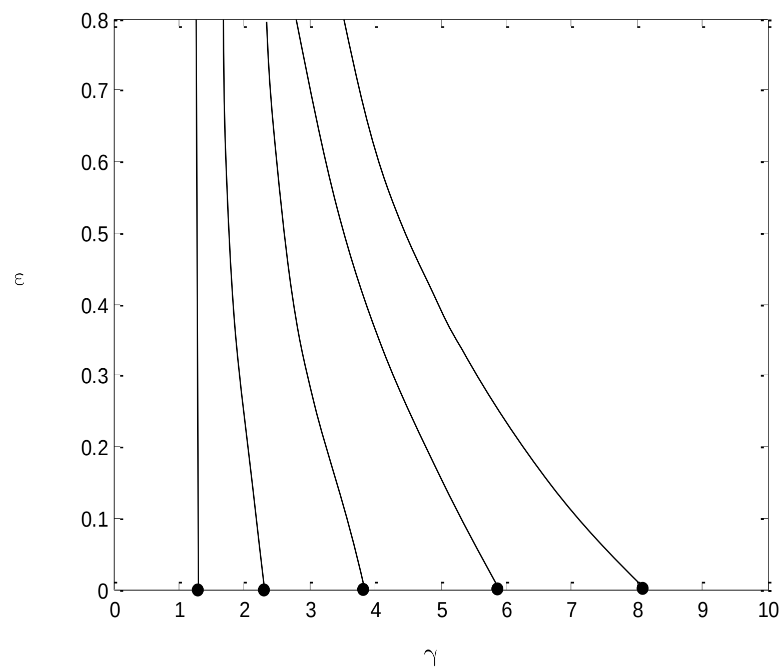

Figure 1 Five Curves $\gamma(\varepsilon)$ Where Separatrices Survive Perturbation

Equation 1 arises from the second-order nonlinear equation

$$y'' - y = y^2 \tag{3}$$

that undergoes a singular perturbation

$$\varepsilon^2 y^{(iv)} + \left(1-\varepsilon^2\right) y'' - y = y^2 + \varepsilon^2 \gamma \left(2yy'' + \left(y'\right)^2\right) \tag{4}$$

and the change of variables

$$v\left(z\right) = \varepsilon^2 y(x)\,, \quad x = \varepsilon z - c\,, \tag{5}$$

where $c \in \mathbb{C}$ is arbitrary.

A phase plot of solutions of the unperturbed equation (Equation 3) in the vicinity of the origin for different initial conditions is shown in Figure 2. The homoclinic solution, given by

$$y_1\left(x\right) = -\frac{3}{2}\cosh^{-2}\left(\frac{x}{2}\right) \tag{6}$$

is plotted in bold on the figure.

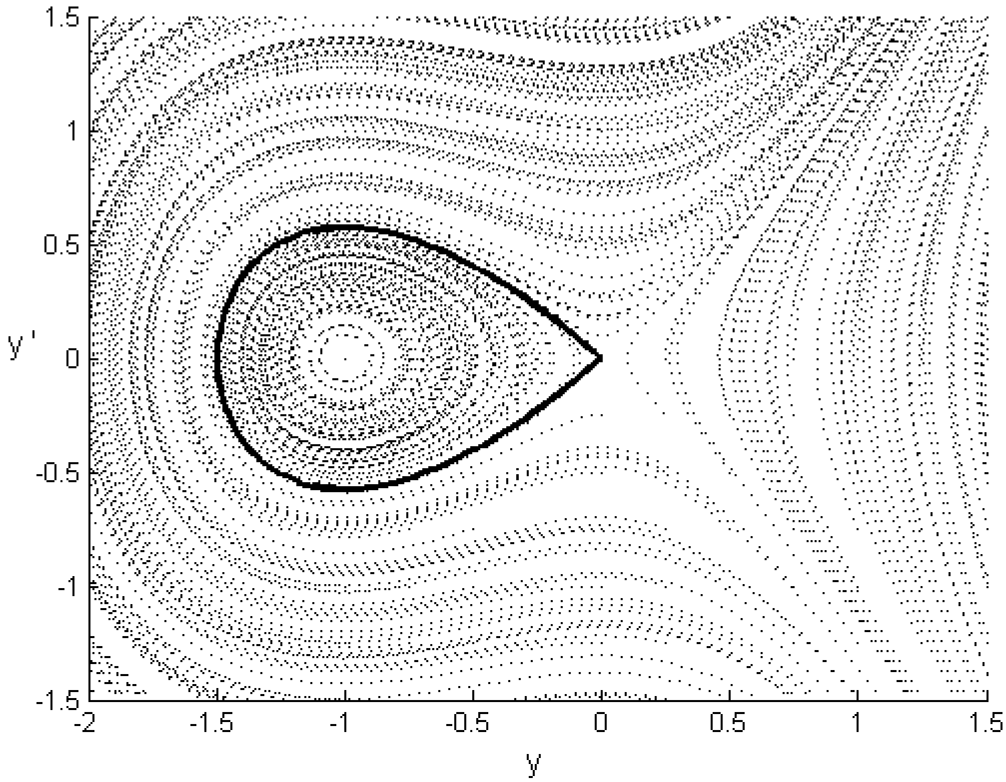

Figure 2 Phase Plot of Unperturbed Equation

Equation 3 can be transformed to the Hamiltonian form. If we let

$$q = y, \quad p = y', \tag{7}$$

and

$$H = \frac{p^2}{2} - \frac{q^2}{2} - \frac{q^3}{3}, \tag{8}$$

then

$$\frac{\partial H}{\partial p} = p = q', \quad \frac{\partial H}{\partial q} = -q - q^2 = -q'' = -p'. \tag{9}$$

The Hamiltonian is a first integral of the system, and the system is completely integrable.

A couple of questions can now be asked regarding the singularly perturbed model (Equation 1):

1) Is there a Hamiltonian for the perturbed system? It will be shown that the perturbed system does have a Hamiltonian representation.
2) Does survival of the separatrices imply integrability? In other words, are there any parameter curves $\gamma(\varepsilon)$ for which the complete integrability of the system survives the perturbation, and if so, do these values relate to the results proven in the aforementioned research? It will be shown that the perturbed system does not possess the Painlevé property, which is a strong indication that the perturbed system is not integrable.

In this thesis, the results of research regarding these questions will be presented. The Hamiltonian for the perturbed system is derived in Section 2. In Section 3, the concept of Liouville integrability and the unsuccessful search for a second integral for the perturbed system by various methods is discussed. The notion of complex analytic integrability is introduced in Section 4, and the singular point analysis technique introduced by Ablowitz, Ramani, and Segur (the ARS algorithm) [1] is presented and applied to the perturbed equation showing that the system is not of Painlevé-type.

2 HAMILTONIAN OF PERTURBED EQUATION

In this section the Hamiltonian for Equation 1 will be developed along with the associated Lagrangian and Hamilton-Jacobi equations. The primary reference used for the following material is Gelfand and Fomin [3] (with some changes of notation for convenience).

We desire to express the perturbed system in canonical form, with canonical variables $q_1,\ldots,q_n,p_1,\ldots,p_n$ and Hamiltonian $H = H\left(z,q_1,\ldots,q_n,p_1,\ldots,p_n\right)$ such that Hamilton's equations

$$\frac{dq_i}{dz} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dz} = -\frac{\partial H}{\partial q_i}, \quad \left(i = 1,\ldots,n\right) \tag{10}$$

are satisfied.

A first integral for Equation 1 can be derived by multiplying the equation by v' and integrating, as follows:

$$v'\left(v^2 + \varepsilon^2 v + \gamma\left(2vv'' + \left(v'\right)^2\right) - (1-\varepsilon^2)v'' - v^{(iv)}\right) = 0 \tag{11}$$

$$\int \left(v^2 v' + \varepsilon^2 v + \gamma \left(2vv'' + \left(v'\right)^2 \right) v' - \left(1-\varepsilon^2\right) v'' v' - v^{(iv)} v' \right) \, dz = C \tag{12}$$

$$\frac{v^3}{3} + \frac{\varepsilon^2 v^2}{2} + \gamma v \left(v'\right)^2 - \frac{1}{2}\left(1-\varepsilon^2\right) v'^2 - v''' v' + \frac{1}{2}\left(v''\right)^2 = C. \tag{13}$$

Assuming the Hamiltonian to be the negative of this conserved quantity C, a set of generalized coordinates $q_1, \ldots, q_n$ and associated momenta $p_1, \ldots, p_n$ must be defined which satisfy Hamilton's equations. Letting

$$\begin{gathered} q_1 = v, \qquad p_1 = v''' + \left(1-\varepsilon^2\right) v' - 2\gamma v v', \\ q_2 = v'', \quad p_2 = v', \end{gathered} \tag{14}$$

the Hamiltonian is then

$$\begin{aligned} H = -C &= -\frac{q_1^3}{3} - \frac{\varepsilon^2 q_1^2}{2} - \frac{q_2^2}{2} + \frac{p_1 q_1'}{2} + \frac{p_2 q_2'}{2} \\ &= -\frac{q_1^3}{3} - \frac{\varepsilon^2 q_1^2}{2} - \frac{q_2^2}{2} + \frac{p_1 p_2}{2} + \frac{1}{2} p_2 \left(p_1 - \left(1-\varepsilon^2\right) p_2 + 2\gamma q_1 p_2 \right) \\ &= -\frac{q_1^3}{3} - \frac{\varepsilon^2 q_1^2}{2} - \frac{q_2^2}{2} + p_1 p_2 - \frac{1}{2}\left(1-\varepsilon^2\right) p_2^2 + \gamma q_1 p_2^2 \end{aligned} \tag{15}$$

Verifying the canonical equations, we have

$$\frac{\partial H}{\partial q_1} = -q_1^2 - \varepsilon^2 q_1 + \gamma p_2^2 = -v^2 - \varepsilon^2 v + \gamma \left(v'\right)^2. \tag{16}$$

Using Equation 1, this can be written

$$\begin{aligned} \frac{\partial H}{\partial q_1} &= -v^{(iv)} - \left(1-\varepsilon^2\right) v'' + 2\gamma v v'' + 2\gamma \left(v'\right)^2 \\ &= -\frac{d}{dz}\left(v''' + \left(1-\varepsilon^2\right) v' - 2\gamma v v'\right) = -p_1'. \end{aligned} \tag{17}$$

Continuing, we have

$$\frac{\partial H}{\partial p_1} = p_2 = q_1', \tag{18}$$

$$\frac{\partial H}{\partial q_2} = -q_2 = -v'' = -p_2', \tag{19}$$

and

$$\begin{aligned} \frac{\partial H}{\partial p_2} &= p_1 - \left(1-\varepsilon^2\right) p_2 + 2\gamma q_1 p_2 \\ &= v''' + \left(1-\varepsilon^2\right) v' - 2\gamma v v' - \left(1-\varepsilon^2\right) v' + 2\gamma v v' \\ &= v''' = q_2'. \end{aligned} \tag{20}$$

Thus, Equation 1 has the Hamiltonian given by Equations 14 and 15, and this Hamiltonian is a first integral of the perturbed system.

The associated Lagrangian can now be computed as follows:

$$\begin{aligned} L &= -H + p_1 q_1' + p_2 q_2' \\ &= \frac{v^3}{3} + \frac{\varepsilon^2 v^2}{2} + \gamma v \left(v'\right)^2 - \frac{1}{2}\left(1-\varepsilon^2\right)\left(v'\right)^2 - v'''v' + \frac{1}{2}\left(v''\right)^2 + \\ &\quad v'''v' + \left(1-\varepsilon^2\right)\left(v'\right)^2 - 2\gamma v\left(v'\right)^2 + v'''v' \\ &= \frac{v^3}{3} + \frac{\varepsilon^2 v^2}{2} - \gamma v\left(v'\right)^2 + \frac{1}{2}\left(1-\varepsilon^2\right)\left(v'\right)^2 + v'''v' + \frac{1}{2}\left(v''\right)^2 . \end{aligned} \tag{21}$$

This Lagrangian is expressed in terms of the generalized coordinates and their derivatives as:

$$L = \frac{q_1^3}{3} + \frac{\varepsilon^2 q_1^2}{2} - \gamma q_1 \left(q_1'\right)^2 + \frac{1}{2}\left(1-\varepsilon^2\right)\left(q_1'\right)^2 + q_1' q_2' + \frac{1}{2} q_2^2 . \tag{22}$$

This Lagrangian does not explicitly depend on the independent variable z, consistent with the Hamiltonian being a first integral of the system.

The Euler-Lagrange equation for the Lagrangian given above is

$$L_v - \frac{d}{dz} L_{v'} + \frac{d^2}{dz^2} L_{v''} - \frac{d^3}{dz^3} L_{v'''} = 0 . \tag{23}$$

In this case,

$$L_v = v^2 + \varepsilon v - \gamma \left(v'\right)^2, \tag{24}$$

$$L_{v'} = -2\gamma v v' + \left(1 - \varepsilon^2\right) v' + v''' , \tag{25}$$

$$L_{v''} = v'' , \tag{26}$$

$$L_{v'''} = v' , \tag{27}$$

and

$$\begin{aligned} 0 &= L_v - \frac{d}{dz} L_{v'} + \frac{d^2}{dz^2} L_{v''} - \frac{d^3}{dz^3} L_{v'''} \\ &= v^2 + \varepsilon^2 v - \gamma \left(v'\right)^2 + 2\gamma \left(v'\right)^2 + 2\gamma v v'' - \left(1 - \varepsilon^2\right) v'' - v^{(iv)} + v^{(iv)} - v^{(iv)} \\ &= v^2 + \gamma \left(2 v v'' + \left(v'\right)^2\right) + \varepsilon^2 v - v^{(iv)} - (1 - \varepsilon^2) v'', \end{aligned} \tag{28}$$

which is Equation 1.

The Hamilton-Jacobi equation associated with the Hamiltonian (Equation 15) is

$$\begin{aligned} 0 &= \frac{\partial S}{\partial z} + H\left(q_1, q_2, \frac{\partial S}{\partial q_1}, \frac{\partial S}{\partial q_2}\right) \\ &= \frac{\partial S}{\partial z} - \frac{q_1^3}{3} - \frac{\varepsilon^2 q_1^2}{2} - \frac{q_2^2}{2} + \frac{\partial S}{\partial q_1} \frac{\partial S}{\partial q_2} - \frac{1}{2}\left(1 - \varepsilon^2\right)\left(\frac{\partial S}{\partial q_2}\right)^2 + \gamma q_1 \left(\frac{\partial S}{\partial q_2}\right)^2 . \end{aligned} \tag{29}$$

Since the Lagrangian does not depend on the independent variable, this can be written as

$$\alpha = -\frac{q_1^3}{3} - \frac{\varepsilon^2 q_1^2}{2} - \frac{q_2^2}{2} + \frac{\partial S}{\partial q_1}\frac{\partial S}{\partial q_2} - \frac{1}{2}\left(1-\varepsilon^2\right)\left(\frac{\partial S}{\partial q_2}\right)^2 + \gamma q_1 \left(\frac{\partial S}{\partial q_2}\right)^2, \tag{30}$$

where α is a constant.

3 LIOUVILLE INTEGRABILITY OF PERTURBED EQUATION

The integrability of a dynamical system does not have a unique, conclusive mathematical definition. The evolution of the concept of integrability and various proposed definitions are discussed in [4]-[9], among others. In this section, the concept of Liouville integrability is defined, and the search for a second constant of motion for the perturbed system is discussed.

3.1 Liouville Integrability

A Hamiltonian system with sufficiently many integrals of motion can be integrated by "quadratures"; that is, its solutions can be obtained by a finite number of algebraic operations and evaluation of integrals of given functions. This form of integrability is expressed in the theorem proved by Bour and Liouville, stated here for a time-independent Hamiltonian following the modern formulation given by Perelomov [8].

Theorem: Let $\mathbb{R}^{2n} = \left\{\left(p,q\right)\right\}$ be the phase space of a Hamiltonian system with the standard Poisson bracket and with Hamiltonian $H\left(p,q\right)$. Suppose that the system has n integrals of motion $F_1 = H,\ldots,F_n$ in involution, i.e.

$$\left[F_j, F_k\right] = \sum_{i=1}^{n} \frac{\partial F_j}{\partial q_i}\frac{\partial F_k}{\partial p_i} - \frac{\partial F_j}{\partial p_i}\frac{\partial F_k}{\partial q_i} = 0. \tag{31}$$

If the functions $F_1,\ldots,F_n$ are independent on the set

$$M_a = \left\{\left(p,q\right) \in \mathbb{R}^{2n} : F_j\left(p,q\right) = a_j, j = 1,\ldots,n\right\}, \tag{32}$$

then the solutions of Hamilton's equations

$$\dot{p}_j = -\frac{\partial H}{\partial q_j}, \quad \dot{q}_j = \frac{\partial H}{\partial p_j}, \tag{33}$$

lying in M_{a} can be obtained by quadratures.

3.2 Search for Second Integral of Motion for Perturbed Equation

In Section 2 it was shown that the perturbed model (Equation 1) has a Hamiltonian which is an integral of motion. Thus, for the system to be Liouville integrable as defined above, one additional integral of motion is needed.

The search for a second integral of motion associated with Equation 1 did not yield a positive result. Multiple approaches were investigated, including:

3) Using a canonical transformation to "simplify" the Hamiltonian.

4) Using Noether's theorem.

5) Using Whittaker's method.

3.2.1 Canonical Transformations

A canonical transformation is a transformation

$$Q_i = Q_i\left(z, q_1, \ldots, q_n, p_1, \ldots, p_n\right), \quad P_i = P_i\left(z, q_1, \ldots, q_n, p_1, \ldots, p_n\right), \tag{34}$$

which preserves the canonical form of Hamilton's equations, i.e.

$$\frac{dQ_i}{dx} = \frac{\partial H^*}{\partial P_i}, \quad \frac{dP_i}{dx} = -\frac{\partial H^*}{\partial Q_i}, \quad \left(i = 1,\ldots,n\right), \tag{35}$$

where $H^*\left(Q_1,\ldots,Q_n,P_1,\ldots,P_n\right)$ is some new function.

A generating function can be used to create a canonical transformation. For example, let $\Phi\left(q_1,\ldots,q_n,P_1,\ldots,P_n\right)$ be a generating function, then setting

$$p_i = \frac{\partial \Phi}{\partial q_i}, \quad \mathrm{Q}_i = \frac{\partial \Phi}{\partial P_i}, \quad H^* = H \tag{36}$$

will result in a canonical transformation. Similar equations apply for generating functions of the form $\Phi\left(q_1,\ldots,q_n,Q_1,\ldots,Q_n\right)$, $\Phi\left(Q_1,\ldots,Q_n,p_1,\ldots,p_n\right)$, and $\Phi\left(p_1,\ldots,p_n,P_1,\ldots,P_n\right)$.

In searching for an integral of motion, a canonical transformation can be used to cause the new Hamiltonian to depend only on the new generalized momenta:

$$H\left(q_1,\ldots,q_n,p_1,\ldots,p_n\right) \to H'\left(P_1,\ldots,P_n\right). \tag{37}$$

The new generalized momenta are the integrals of motion, as Hamilton's equations reduce to:

$$\dot{P}_i = -\frac{\partial H'}{\partial Q_i} = 0, \quad i = 1, \ldots, n, \tag{38}$$

$$\dot{Q}_i = \frac{\partial H'}{\partial P_i} = f_i\left(P_1, \ldots, P_n\right). \tag{39}$$

Alternatively, a canonical transformation can be used to make the Hamilton-Jacobi equation separable [9], i.e.

$$S\left(q_1, \ldots, q_n, P_1 \ldots, P_n\right) = \sum_{k=1}^{n} S_k\left(q_k, P_1 \ldots, P_n\right), \tag{40}$$

where S is a generating function and $P_1, \ldots, P_n$ are the new generalized momenta. Then from the canonical transformation relation

$$p_k = \frac{\partial}{\partial q_k} S_k\left(q_k, P_1, \ldots, P_n\right) \tag{41}$$

each p_k is a function of only one q_k. If the motion is periodic in each of the q_k, then action variables can be formed, with their associated angle-variable conjugates as follows:

$$I_k = \frac{1}{2\pi} \oint_{C_k} p_k(q_k, P_1, \ldots, P_n) dq_k, \tag{42}$$

$$\theta_k = \frac{\partial S}{\partial I_k} = \sum_{m=1}^{n} S_m \left(q_m, I_1, \ldots, I_n \right), \tag{43}$$

and the Hamiltonian $H'(I_1, \ldots, I_n)$ yields the canonical equations

$$\dot{I}_k = -\frac{\partial}{\partial \theta_k} H'\left(I_1, \ldots, I_n\right) = 0, \tag{44}$$

$$\dot{\theta}_k = \frac{\partial}{\partial I_k} H'\left(I_1, \ldots, I_n\right) = \omega_k \left(I_1, \ldots, I_n\right), \tag{45}$$

where ω_k is the "frequency" associated with each degree of freedom.

As an example transformation to be applied to the Hamiltonian (Equation 15) of the perturbed system, let

$$\Phi\left(q_1,\ldots,q_n,P_1,\ldots,P_n\right) = k_1 q_1 P_1 + k_2 q_1 P_2 + k_3 q_2 P_2, \tag{46}$$

then

$$p_1 = \frac{\partial\Phi}{\partial q_1} = k_1 P_1 + k_2 P_2, \quad p_2 = \frac{\partial\Phi}{\partial q_2} = k_3 P_2, \tag{47}$$

$$Q_1 = \frac{\partial\Phi}{\partial P_1} = k_1 q_1, \quad Q_2 = \frac{\partial\Phi}{\partial P_2} = k_2 q_1 + k_3 q_2. \tag{48}$$

Solving the Q_1, Q_2, equations for q_1, q_2, we have

$$q_1 = \frac{Q_1}{k_1}, \quad q_2 = \frac{1}{k_3}\left(Q_2 - \frac{k_2}{k_1} Q_1\right). \tag{49}$$

Plugging these into the Hamiltonian, we obtain

$$\begin{aligned} H^* = &-\frac{Q_1^3}{3k_1^3} - \frac{\varepsilon^2 Q_1^2}{2k_1^2} - \frac{\left(Q_2 - \frac{k_2}{k_1} Q_1\right)^2}{2k_3^2} + k_1 k_3 P_1 P_2 + k_2 k_3 P_2^2 \\ &-\frac{1}{2}\left(1-\varepsilon^2\right) k_3^2 P_2^2 + \gamma \frac{k_3^2}{k_1} Q_1 P_2^2. \end{aligned} \tag{50}$$

Setting

$$k_2 = -\frac{1}{2}\left(1 - \varepsilon^2\right) k_3 \tag{51}$$

eliminates two terms from the new Hamiltonian, but no real progress has been made as other terms have been created (compared to the original Hamiltonian).

3.2.2 Noether's Theorem

Consider the variational problem of finding necessary conditions for an extremal of the functional

$$J\left[q_1, \ldots, q_n\right] = \int L\left(z, q_1, \ldots, q_n, q_1', \ldots, q_n'\right)\, dz, \tag{52}$$

where the Lagrangian L depends on n continuously differentiable functions $y_1\left(z\right), \ldots, y_n\left(z\right)$. Noether's theorem [3] states that if the functional is invariant under the family of transformations

$$z^* = \Phi\left(z, q_1, \ldots, q_n, q_1', \ldots, q_n'; \varepsilon\right), \qquad q_i^* = \Psi_i\left(z, q_1, \ldots, q_n, q_1', \ldots, q_n'; \varepsilon\right), \tag{53}$$

(where the functions are differentiable with respect to ε and the value $\varepsilon = 0$ leads to the identity transformation), then

$$\sum_{i=1}^{n} L_{q_i'} \psi_i + \left(L - \sum_{i=1}^{n} q_i' L_{q_i'} \right) \varphi = const \tag{54}$$

along each extremal of the functional, where

$$\varphi\left(z, q_1, \dots, q_n, q_1', \dots, q_n'\right) = \frac{\partial \Phi\left(z, q_1, \dots, q_n, q_1', \dots, q_n'; \varepsilon\right)}{\partial \varepsilon} \Bigg|_{\varepsilon=0}, \tag{55}$$

$$\psi_i\left(z, q_1, \dots, q_n, q_1', \dots, q_n'\right) = \frac{\partial \Psi_i\left(z, q_1, \dots, q_n, q_1', \dots, q_n'; \varepsilon\right)}{\partial \varepsilon} \Bigg|_{\varepsilon=0} . \tag{56}$$

With regard to the Lagrangian of the perturbed system (Equation 22), note that it has no dependence on the independent variable z. Thus the transformation

$$z^* = z + \varepsilon, \qquad q_i^* = q_i, \tag{57}$$

is invariant, which leads to

$$L - \sum_{i=1}^{n} q_i' L_{q_i'} = const, \tag{58}$$

which verifies that the Hamiltonian is a constant of motion. However, the Lagrangian involves all of the generalized momenta and their derivatives, and no other invariant transformation was found.

3.2.3 Whittaker's Method

The technique known as Whittaker's method and described by Goriely [4] involves the direct use of the Poisson bracket condition in the Liouville integrability theorem above. In this approach the second integral of motion I is assumed to have a certain form, for instance, polynomial in p and q up to some order. In the case of a Hamiltonian H which does not depend on the independent variable, the condition $\left[I, H\right] = 0$ leads to a system of equations generated by collecting the coefficients of terms with the same power. If the system of equations have a non-trivial solution in which the resulting expression for I is independent of H, then I represents a second integral of motion.

This method was applied to the perturbed system for the case $\varepsilon = 0$, with the second integral of motion assumed to have the form

$$I = \sum_{i,j,k,l=0}^{(i+j+k+l)\leq 3} K_{ikjl} q_1^i q_2^j p_1^k p_2^l \ , \tag{59}$$

where K_{ijkl} are coefficients. The Poisson bracket expression is then

$$\begin{aligned}
0 &= \left[I,H\right] = \sum_{i=1}^{2} \frac{\partial I}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial I}{\partial p_i}\frac{\partial H}{\partial q_i} \\
&= K_{0001}q_2 + 2K_{0002}q_2p_2 + 3K_{0003}q_2p_2^2 + K_{0010}\left(q_1^2 - \gamma p_2^2\right) + K_{0011}\left(q_1^2p_2 - \gamma p_2^3 + q_2p_1\right) \\
&+K_{0012}\left(q_1^2p_2^2 - \gamma p_2^4 + 2q_2p_1p_2\right) + 2K_{0020}\left(q_1^2p_1 - \gamma p_1p_2^2\right) + 2K_{0021}\left(q_1^2p_1p_2 - \gamma p_1p_2^3 + q_2p_1^2\right) \\
&+3K_{0030}\left(q_1^2p_1^2 - \gamma p_1^2p_2^2\right) + K_{0100}\left(p_1 - p_2 + 2\gamma q_1p_2\right) + K_{0101}\left(p_1p_2 - p_2^2 + 2\gamma q_1p_2^2 + q_2^2\right) \\
&+K_{0102}\left(p_1p_2^2 - p_2^3 + 2\gamma q_1p_2^3 + q_2^2p_2\right) + K_{0110}\left(q_1^2q_2 - \gamma q_2p_2^2 + p_1^2 - p_1p_2 + 2\gamma q_1p_1p_2\right) \\
&+K_{0111}\left(q_1^2q_2p_2 - \gamma q_2p_2^3 + p_1^2p_2 - p_1p_2^2 + 2\gamma q_1p_1p_2^2 + q_2^2p_1\right) \\
&+K_{0120}\left(2q_1^2q_2p_1 - 2\gamma q_2p_1p_2^2 + p_1^3 - p_1^2p_2 + 2\gamma q_1p_1^2p_2\right) + 2K_{0200}\left(q_2p_1 - q_2p_2 + 2\gamma q_1q_2p_2\right) \\
&+K_{0201}\left(2q_2p_1p_2 - 2q_2p_2^2 + 4\gamma q_1q_2p_2^2 + q_2^3\right) \\
&+K_{0210}\left(q_1^2q_2^2 - \gamma q_2^2p_2^2 + 2q_2p_1^2 - 2q_2p_1p_2 + 4q_1q_2p_1p_2\right) + 3K_{0300}\left(q_2^2p_1 - q_2^2p_2 + 2\gamma q_1q_2^2p_2\right) \\
&+K_{1000}p_2 + K_{1001}\left(p_2^2 + q_1q_2\right) + K_{1002}\left(p_2^3 + 2q_1q_2p_2\right) + K_{1010}\left(p_1p_2 + q_1^3 - \gamma q_1p_2^2\right) \\
&+K_{1011}\left(p_1p_2^2 + q_1^3 - \gamma q_1p_2^3 + q_1q_2p_1\right) + K_{1020}\left(p_1^2p_2 + 2q_1^3p_1 - 2\gamma q_1p_1p_2^2\right) \\
&+K_{1100}\left(q_2p_2 + q_1p_1 - q_1p_2 + 2\gamma q_1^2p_2\right) + K_{1101}\left(q_2p_2^2 + q_1p_1p_2 - q_1p_2^2 + 2\gamma q_1^2p_2^2 + q_1q_2^2\right) \\
&+K_{1110}\left(q_2p_1p_2 + q_1^3q_2 - \gamma q_1q_2p_2^2 + q_1p_1^2 - q_1p_1p_2 + 2\gamma q_1^2p_1p_2\right) \\
&+K_{1200}\left(q_2^2p_2 + 2q_1q_2p_1 - 2q_1q_2p_2 + 4q_1^2q_2p_2\right) + 2K_{2000}q_1p_2 + K_{2001}\left(2q_1p_2^2 + q_1^2q_2\right) \\
&+K_{2010}\left(2q_1p_1p_2 + q_1^4 - \gamma q_1^2p_2^2\right) + K_{2100}\left(2q_1q_2p_2 + q_1^2p_1 - q_1^2p_2 + 2\gamma q_1^3p_2\right) + 3K_{3000}q_1^2p_2.
\end{aligned} \tag{60}$$

Collecting like terms results in the following set of equations:

$$\begin{aligned}
&K_{0011} = -3K_{3000} \qquad K_{0011} = \frac{1}{\gamma}K_{1002} \qquad \text{All other } K_{ijkl} = 0 \\
&K_{0011} = -2K_{0200} \qquad K_{0011} = -2K_{0200}.
\end{aligned} \tag{61}$$

Setting K_{0011} to unity and plugging the resulting coefficients back into the Poisson bracket expression yields the Hamiltonian. Thus, there is no integral of motion (other than the Hamiltonian) of the form given by Equation 59.

In general, the success in using canonical transformations or Noether's theorem to find a first integral is limited by the insight or luck of the searcher in finding symmetry to exploit; on the other hand, Whittaker's approach is a direct method but only applies to first integrals of specific forms and is computationally cumbersome. In light of the failure to find a second integral of motion to satisfy the necessary condition for Liouville integrability by these methods, the system was then approached from the viewpoint of complex analytic integrability using Painlevé analysis as described in the next section.

4 COMPLEX ANALYTIC INTEGRABILITY OF PERTURBED EQUATION

The notion of complex analytic integrability involves the analysis of the behavior of a dynamical system in the complex plane of the independent variable. In this context, the general solution of a system may exhibit singularities where it ceases to be analytic. These singularities may be poles, essential singularities, or branch points (algebraic or logarithmic). In addition, in the case of a nonlinear ODE, these singularities may be movable, i.e. dependent on the constants of integration, and therefore on the initial conditions of the system. The presence of movable critical points (essential singularities or branch points) is, in general, not compatible with the existence of single-valued, meromorphic solutions to the system. The existence of these solutions, in turn, is a strong indicator of the integrability of the system in a literal sense. As a result, testing for integrability in the complex analytic sense has focused on methods to determine whether the solutions of a dynamical system possess movable critical points. In this section the Painlevé method of singular point analysis for ordinary differential equations as embodied by the ARS algorithm [1] will be described and applied to the perturbed equation.

4.1 The Painlevé Property

The contributions of Sofya Kovalevskaya, Paul Painlevé, and others to the historical development of the Painlevé method, as well as the description of the ARS algorithm are given by [5], [7], and [9]. In the following the notation will generally conform to [7].

A family of solutions of an ODE is said to have the strong Painlevé property (strong P-property) if it has no movable critical points (i.e. branch points or essential singularities whose location in the complex plane depend on the constants of integration of the ODE). In this case the only movable singularities are ordinary poles, and the solution in the neighborhood of a singularity z_0 can be expressed as a Laurent expansion with the leading term proportional to

$$\lambda = \left(z - z_0\right)^{-p}, z \to z_0, \tag{62}$$

where p is a positive integer.

A family of solutions of an ODE is said to have the weak Painlevé property (weak P-property) if its only movable critical points are movable algebraic branch points and the general solution in the neighborhood of a movable singularity z_0 can be expressed as a Laurent expansion with the leading term proportional to

$$\lambda = \left(z - z_0\right)^{-1/n}, z \to z_0, \tag{63}$$

where n is a natural number.

An ODE is of strong (weak) Painlevé type (P-type) if all of its solutions have the strong (weak) P-property.

4.2 The ARS Algorithm

The ODE to undergo singular point analysis is assumed to be of the form

$$\frac{d^n w}{dz^n} = F\left(z; w, \frac{dw}{dz}, \ldots, \frac{d^{n-1} w}{dz^{n-1}}\right), \tag{64}$$

or equivalently,

$$\frac{dw_i}{dz} = F_i\left(z; w_1, w_2, \ldots, w_n\right), \quad i = 1,2,\ldots,n, \tag{65}$$

where F (or each F_i) is analytic in z and rational in its other arguments. The solution of Equation 64 or 65 is expanded as a Laurent series in a sufficiently small neighborhood of an arbitrary movable singular point z_0. The algorithm then consists of three steps:

1) Find the dominant behavior.

2) Find the resonances.

3) Find the constants of integration.

4.2.1 Finding the Dominant Behavior - Algorithm

It is assumed that the dominant behavior of $w(z)$ in a sufficiently small neighborhood of an arbitrary movable singularity z_0 is algebraic, i.e.

$$w\left(z\right) \approx a\left(z - z_0\right)^q \text{ as } z \rightarrow z_0, \tag{66}$$

where $\Re(q) < 0$. Substituting Equation 66 into Equation 64, all values of q are found such that two or more ODE terms' exponents are equal and more negative than all others (thus these terms are dominant and the others can be ignored as $z \to z_0$). For each qualifying value of q found above the corresponding value of a is computed causing the dominant terms to balance. Conclusions are then drawn from the leading order behavior as follows:

1) If all of the values of q are negative integers, then further steps of the algorithm are used to determine if the ODE exhibits the strong P-property.
2) If any of the values of q are irrational or complex numbers, then the ODE is not of P-type.
3) If any of the values of q is not a negative integer, but is instead a rational number, then from the dominant behavior (Equation 66) of $w(z)$ near z_0 the solution will have a movable algebraic branch point, possibly associated with the weak P-property.

For cases 1 and 3, for each value of q, Equation 66 may represent the first term of a Laurent series valid in a deleted neighborhood of z_0, and the Laurent series solution takes the form

$$w(z) = (z - z_0)^q \sum_{m=0}^{\infty} a_m (z - z_0)^m \tag{67}$$

in a sufficiently small deleted neighborhood of z_0.

4.2.2 Finding the Dominant Behavior – Application to Perturbed Equation

Under the assumption that the dominant behavior in a sufficiently small neighborhood of an arbitrary movable singularity z_0 is algebraic (i.e. $v(z) \approx a\tau^q, \tau = (z - z_0) \to 0$), then for Equation 1 we have

$$\begin{aligned} v^2 + \gamma\left(2vv'' + (v')^2\right) - \left(1 - \varepsilon^2\right)v'' - v^{(iv)} + \varepsilon^2 v = & \quad a^2\tau^{2q} \\ & + \gamma\left(2a^2 q(q-1) + a^2 q^2\right)\tau^{2q-2} \\ & - \left(1 - \varepsilon^2\right) aq(q-1)\tau^{q-2} \\ & - aq(q-1)(q-2)(q-3)\tau^{q-4} \\ & + \varepsilon^2 a\tau^q. \end{aligned} \tag{68}$$

Referring to the five terms (in powers of τ) on the right side of Equation 68 as term 1, 2, 3, 4, and 5, respectively, we desire to find (q,a) pairs for which the exponent of τ for two of the terms are equal and are more negative than the other terms. The exponent of τ for each term is plotted for negative values of q in Figure 3.

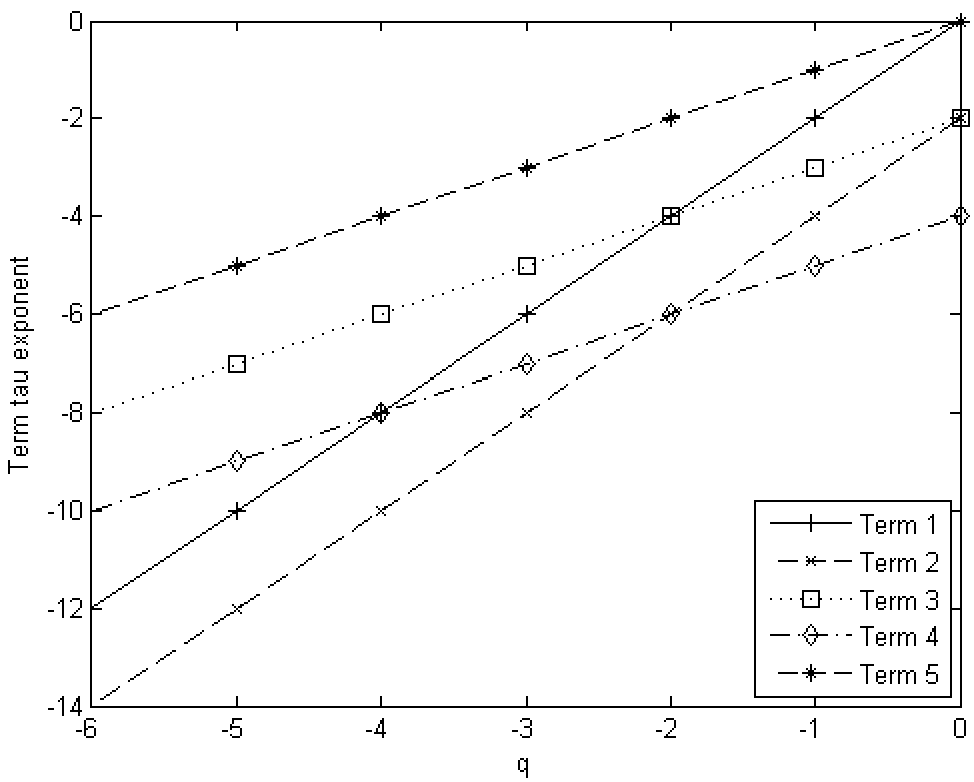

Figure 3 Equation Term Exponents vs. Value of q

Two cases will now be considered, 1) $\gamma \neq 0,$ and 2) $\gamma = 0.$

For the case $\gamma \neq 0,$ the only value of q where two lines cross below the others is -2, and in this case the balancing leading order terms are terms 2 and 4, for an appropriate value of a. To determine the value of a, we have

$$\gamma\left(2a^2q\left(q-1\right)+a^2q^2\right)\tau^{2q-2}-aq\left(q-1\right)\left(q-2\right)\left(q-3\right)\tau^{q-4}=0, \tag{69}$$

which for q=-2 yields

$$a=\frac{15}{2\gamma}. \tag{70}$$

As outlined in the previous section, since q is a negative integer for the sole allowed (q,a) pair, then $\frac{15}{2\gamma}\left(z-z_0\right)^{-2}$ may be the first term of a Laurent series in a deleted neighborhood of z_0.

For the case $\gamma = 0,$ the only value of q where two lines cross below the others is -4, and in this case the balancing leading order terms are terms 1 and 4, for an appropriate value of a. To determine the value of a, we have

$$a^2\tau^{2q} - aq\left(q-1\right)\left(q-2\right)\left(q-3\right)\tau^{q-4} = 0, \tag{71}$$

which for q=-4 yields

$$a = 840. \tag{72}$$

For this case, since q is again a negative integer for the sole allowed (q,a) pair, then

$840\left(z-z_0\right)^{-4}$ may be the first term of a Laurent series in a deleted neighborhood of z_0.

In both cases, to determine if this is an indication of the strong P-property, the resonances must next be examined.

4.2.3 Finding the Resonances - Algorithm

In Equation 67, z_0 is the position of the singularity and is an arbitrary constant. If $n-1$ of the coefficients a_m are also arbitrary, then these are the n constants of integration of the nth-order ODE (Equation 64), and Equation 67 is the general solution in the deleted neighborhood. The powers of $z-z_0$ at which the arbitrary constants appear are called resonances or Kovalevskaya exponents.

To find the resonances, for each (q,a) pair found above, a simplified equation is constructed retaining only the leading terms of the original ODE. The equation

$$w(z) = a\tau^{q} + \beta\tau^{q+r}, \quad \tau = (z - z_0) \to 0, \tag{73}$$

is substituted into the simplified equation, which to leading orders in β reduces to

$$Q(r)\beta(z - z_0)^{\hat{q}} = 0, \quad \hat{q} \geq q + r - n \tag{74}$$

If the highest derivative of the original equation is a leading term, $\hat{q} = q + r - n$ and $Q(r)$ is a polynomial of order n. If not, $\hat{q} > q + r - n$, and the order of the polynomial equals the order of the highest derivative among the leading terms. The roots of $Q(r)$ determine the resonances, and conclusions are drawn from the nature of the roots in accordance with the following:

1) One root is always -1, representing the arbitrariness of z_0.
2) If the value of a associated with q was found to be arbitrary in the leading order analysis, then another root is 0.

3) Any root r with $\Re\left(r\right) < 0$ can be ignored because it violates the hypothesis that τ^{q} is the dominant term in the expansion near z_0.

4) Any irrational or complex root r with $\Re\left(r\right) > 0$ indicates a movable branch point at z=z_0, and the solutions are not of P-type.

5) Any rational root $r = \dfrac{p}{q}$ with $\Re\left(r\right) > 0$ and with q as in the denominator of dominant behavior indicates in general a movable branch point which may be associated with the weak P-property.

6) If for every (q,a) found in the leading order analysis, all the roots of $Q\left(r\right)$ (except -1 and possibly 0) are positive integers, then there are no algebraic branch points, and the final step of the algorithm is needed to check for logarithmic branch points.

For the Laurent series expansion (Equation 67) to be the general solution of the ODE (Equation 64), $Q\left(r\right)$ must have n-1 non-negative distinct roots of real rational numbers including integers. If for every allowed (q,a) pair found in the leading order analysis,

$Q\left(r\right)$ has fewer than n-1 such roots, then none of the local solutions is general, suggesting that Equation 66 is missing an essential part of the solution.

4.2.4 Finding the Resonances – Application to Perturbed Equation

Recall from the previous section that to find the resonance values, we substitute

$$v\left(z\right) = a\tau^{q} + \beta\tau^{q+r} \tag{75}$$

(where $\tau = \left(z - z_0\right) \to 0$) into the original ODE omitting all but the leading order terms, to obtain an equation to leading order in B of the form

$$Q\left(r\right)\beta\tau^{\hat{q}} = 0 \tag{76}$$

where the roots of $Q\left(r\right)$ determine the resonance values. Thus, for the case $\gamma \neq 0$ we have

$$\begin{aligned}
&\gamma\left[2vv'' + \left(v'\right)^{2}\right] - v^{(iv)} = \\
&2\gamma\left(a\tau^{q} + \beta\tau^{q+r}\right)\left(aq\left(q-1\right)\tau^{q-2} + \beta\left(q+r\right)\left(q+r-1\right)\tau^{q+r-2}\right) \\
&+\gamma\left(aq\tau^{q-1} + \beta\left(q+r\right)\tau^{q+r-1}\right)^{2} - aq\left(q-1\right)\left(q-2\right)\left(q-3\right)\tau^{q-4} \\
&-\beta\left(q+r\right)\left(q+r-1\right)\left(q+r-2\right)\left(q+r-3\right)\tau^{q+r-4} = 0.
\end{aligned} \tag{77}$$

For $\left(q=-2, a=\dfrac{15}{2\gamma}\right)$, this expression simplifies to

$$\left(-r^4+14r^3-56r^2+49r+120\right)\beta\tau^{r-6}+O\left(\beta^2\right)=0. \tag{78}$$

Thus the resonances are the roots of the polynomial

$$Q\left(r\right)=-r^4+14r^3-56r^2+49r+120, \tag{79}$$

which are $r=-1,\ 8,\ \dfrac{7}{2}\pm i\dfrac{\sqrt{11}}{2}$.

For the case $\gamma=0$ we have

$$\begin{aligned} v^2-v^{(iv)} &= \left(a\tau^q+\beta\tau^{q+r}\right)^2-aq\left(q-1\right)\left(q-2\right)\left(q-3\right)\tau^{q-4} \\ &-\beta\left(q+r\right)\left(q+r-1\right)\left(q+r-2\right)\left(q+r-3\right)\tau^{q+r-4}=0. \end{aligned} \tag{80}$$

For $\left(q=-4, a=840\right)$, this expression simplifies to

$$\left(-r^4+22r^3-179r^2+638r+840\right)\beta\tau^{r-8}+O\left(\beta^2\right)=0. \tag{81}$$

Therefore the resonances are the roots of the polynomial

$$Q\left(r\right)=-r^4+22r^3-179r^2+638r+840, \tag{82}$$

which are $r = -1,\ 12,\ \dfrac{11}{2} \pm i\dfrac{\sqrt{159}}{2}$.

In both cases ($\gamma \neq 0$ or $\gamma = 0$) the polynomial has complex roots. Thus it can be concluded at this point, without proceeding to the final step of the ARS algorithm, that Equation 1 does not possess the Painlevé property, indicating the presence of movable branch points.

As previously discussed, the presence of movable branch points is a strong indication that the system is not integrable in a complex analytic sense, lacking single-valued, meromorphic solutions.

LIST OF REFERENCES

[1] M. J. Ablowitz, A. Ramani, and H. Segur, "A connection between nonlinear evolution equations and ordinary differential equations of P-type. I," *J. Math. Phys.*, vol. 21, no. 4, pp. 715-721, Apr. 1980.

[2] A. R. Champneys and M. D. Groves, "A global investigation of solitary-wave solutions to a two-parameter model for water waves," *J. Fluid Mech.*, vol. 342, pp. 199-229, July 1997.

[3] I. M. Gelfand and S. V. Fomin, *Calculus of Variations*, Mineola, New York: Dover, 1963.

[4] A. Goriely, *Integrability and Nonintegrability of Dynamical Systems*, River Edge, New Jersey: World Scientific, 2001.

[5] B. Grammaticos and A. Ramani, "Integrability – and how to detect it," in *Integrability of Nonlinear Systems* (Lect. Notes Phys. 638), Y. Kosmann-Schwarzbach, B. Grammaticos, K. M. Tamizhmani, Eds., Berlin Heidelberg, Germany: Springer, 2004.

[6] M. D. Kruskal and P. A. Clarkson, "The Painlevé-Kowalevski and Poly-Painlevé tests for integrability," *Stud. Appl. Math.*, vol. 86, pp. 87-165, 1992.

[7] M. Lakshmanan and S. Rajasekar, *Nonlinear Dynamics: Integrability, Chaos, and Patterns*, Berlin Heidelberg, Germany: Springer, 2003.

[8] A. M. Perelomov, *Integrable Systems of Classical Mechanics and Lie Algebras: Volume 1*, Berlin, Germany: Birkhäuser, 1990.

[9] M. Tabor, *Chaos and Integrability in Nonlinear Dynamics: An Introduction*, New York: Wiley, 1989.

[10] A. Tovbis and D. Pelinovsky, “Exact conditions for existence of homoclinic orbits in the fifth-order KdV model,” *Nonlinearity*, vol. 19, no. 10, pp. 2277-2312, Oct. 2006.