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ON HALL MAGNETOHYDRODYNAMICS: X-TYPE NEUTRAL POINT AND PARKER PROBLEMS

by

KYLE REGER
B.S. University of Central Florida, 2011

A thesis submitted in partial fulfillment of the requirements
for the degree of Master of Science
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Major Professor:
Bhimsen K. Shivamoggi

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ABSTRACT

The framework for the Hall magnetohydrodynamic (MHD) model for plasma physics is built up from kinetic theory and used to analytically solve problems of interest in the field. The Hall MHD model describes fast magnetic reconnection processes in space and laboratory plasmas. Specifically, the magnetic reconnection process at an X-type neutral point, where current sheets form and store enormous amounts of magnetic energy which is later released as magnetic storms when the sheets break up, is investigated. The phenomena of magnetic flux pile-up driving the merging of antiparallel magnetic fields at an ion stagnation-point flow in a thin current sheet, called the Parker problem, also receives rigorous mathematical analysis.

ACKNOWLEDGMENTS

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CHAPTER 1

INTRODUCTION

The Hall Magnetohydrodynamic (MHD) model is a two-fluid model often used to describe the equilibrium state of the plasma. It becomes especially useful when considering problems containing resistivity due to the equations' relative simplicity when compared to the equations that the Hall MHD equations are deduced from. In particular, Hall MHD is used extensively when studying the process of magnetic reconnection due to its ability to accurately describe plasmas with large magnetic field gradients [2]. Magnetic reconnection occurs as a result of non-ideal effects in Ohm's law. Physically, the close encounter of magnetic field lines causes the magnetic field gradients to become locally strong, thus enhancing the typically weak non-ideal process in Ohm's law. Hence, reconnection is a localized process [3].

Rapid reconnection of magnetic fields in the neighborhood of neutral points plays a central role in the explosive energy release of the solar flare phenomenon. The dynamics of the fluid motion and field reconnection remains a subject of interest ever since Dungey [11] first pointed out the exotic nature of the X-type magnetic neutral point. The solution of the problem is essential to understanding the absence of small-scale fields in interstellar space and the solar corona [18].

1.1 X-type Neutral Point

X-type neutral points are essentially where hyperbolic magnetic fields meet and create magnetic neutral lines within the plasma flow in the form of an X-point [26]. A thin neutral current sheet is then formed when plasma collapses near the neutral line of the applied magnetic field. In resistive magnetohydrodynamics (MHD) the ion inflow is the only means to transport magnetic flux into the reconnection layer. As the magnetic flux continually accumulates in the region of the neutral sheet, the total current and the sheet width increase until large magnetic pressure gradients develop, which inhibit the ion inflow - the "bottleneck" problem [27]. The Hall effect [29] can overcome this [9, 16], thanks to the decoupling of electrons from ions on length scales below the ion skin depth d_i. If the reconnection layer width is less than d_i, the electron inflow can keep transporting the magnetic flux into the reconnection layer and hence reduce the flux pile-up.

Previous numerical work [22, 21, 12, 30] indicated that the dissipation in Hall MHD, as d_i increases, changes from an elongated sheet geometry (Sweet-Parker type [17, 19]) to a more open X-point geometry (Petschek type [8]). However, fully kinetic simulations [8, 15] and EMHD-based treatments [6] have shown that elongated current sheets are also possible. To further the controversy, more recent particle-in-cell simulations [23] show spatial localization of the out-of-plane current to within a few d_i's of the X-line.

More recently, Shivamoggi [25] considered a non-resistive Hall plasma near an X-type neutral point. Therein, asymptotic solutions are provided for the resulting nonlinear ordinary differential equations. We continue the development of this problem by working with those nonlinear differential equations and finding suitable transformations to allow a more detailed analysis of their behavior.

1.2 Parker Problem

When two opposite magnetic fields are pressed together via a stagnation-point plasma flow, the fields annihilate - the Parker problem [18]. Dorelli [10] gave analytic generalizations while considering Hall effects to the previous flux pile-up merging solution [18]. The solutions therein exhibited the quadrupolar structure of the toroidal magnetic field that is characteristic of the Hall effect and has been confirmed in laboratory experiments [20, 13].

Shivamoggi [24] generalized some of Dorelli's results by adding a poloidal shear to the toroidal ion flow. The differential equation characterizing the magnetic field profile was found to have a triple-deck structure as in fluid boundary layer theory. We seek to provide an exact solution to the magnetic field profile to better understand its characteristics.

CHAPTER 2

DERIVATION OF HALL MAGNETOHYDRODYNAMICS

We seek to apply our knowledge of fluids to approximate the motions of collections of charged particles moving in an electromagnetic field (a plasma). Specifically, we will be deriving the Hall Magnetohydrodynamic (MHD) model for plasmas with the help of kinetic theory, classical electromagnitism, and a variety of simplifying assumptions.

2.1 Mass and Momentum Conservation in Plasma

To derive the Hall MHD equations we follow the steps taken by Braginskii [5] and Goedbloed and Poedts [14]. First we start with time-dependent particle probability distribution functions, $f_a(t, \mathbf{r}, \mathbf{v})$, where the subscript a denotes which species of particles. The probable number of particles of species a in the six-dimensional volume element $d\mathbf{r}d\mathbf{v}$ centered at $(\mathbf{r}, \mathbf{v})$ will then be $f_a(t, \mathbf{r}, \mathbf{v})d\mathbf{r}d\mathbf{v}$. The time evolution of these distribution functions is determined by the Boltzmann equation.

$$\frac{\partial f_a}{\partial t} + \frac{\partial}{\partial x_\beta}(v_\beta f_a) + \frac{\partial}{\partial v_\beta}\left(\frac{F_{a\beta}}{m_a} f_a\right) = C_a. \tag{2.1}$$

Since we're modeling plasma, the particles in question are affected by electric and magnetic fields so the force applied is the Lorentz force

$$\mathbf{F}_a = e_a \left(\mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{B} \right), \tag{2.2}$$

where e_a is the charge of particle species a, $\mathbf{E}$ is the applied electric field, and $\mathbf{B}$ is the applied magnetic field. Note that $f_a(t, \mathbf{r}, \mathbf{v})$ is a smoothed density averaged over a volume containing a large number of particles (fluid element approximation). The same smoothing applies to $\mathbf{F}_a$ as it does not account for the microfluctuations caused by individual particles.

The collision term, C_a, in (2.1) accounts for collisions between particles of the same species and between other species. For simplicity, we assume collisions are elastic and do not convert particles from one species to another. This coincides with a plasma that is characterized by being "not too hot" and "not too dense". This Goldilocks plasma carries several physical limitations on the phenomena that can be modeled by the equations, namely

(a) The long-range Coulomb interaction between charged particles should dominate over the short-range binary collisions with neutrals. Meaning, the time scale on which collective oscillatory motion occurs is much smaller than the mean time between collisions of charged plasma particles with neutrals. Explicitly, if τ is the time scale of the collective oscillatory motion, then we require

$$\tau \ll \frac{1}{n_n \sigma v_{th}},$$

where n_n is the density of neutral particles, σ is the cross sectional area of an atom, and v_{th} is the thermal speed of the particles.

(b) The length scale of the plasma dynamics should be much larger than the minimum length at which a quasi-neutrality condition holds. A quasi-neutral plasma satisfies the relation

$$\frac{|Zn_i - n_e|}{n_e} \ll 1,$$

where Z is the ion charge number and n_i, n_e are the ion and electron densities respectively, but the quasi-neutrality condition requires that locally within a certain volume scale the plasma has a almost neutral charge. If the charge is imbalanced, then huge electric fields produce accelerations in the particles so that the imbalance is neutralized almost instantaneously. However, local charge imbalances may be produced by thermal fluctuations. To estimate their size, one should compare the thermal energy kT of the particles with their electostatic energy $e\Phi$. The latter can be estimated through Poisson's law, so that if λ is our typical length scale then

$$\lambda \gg \lambda_D \equiv \sqrt{\frac{\epsilon_0 kT}{e^2 n}},$$

where λ_D is called the Debye length, ϵ_0 is the permittivity of free space, k is the Boltzmann constant, T is the temperature of the plasma measured in Kelvin, e is the unit charge, and n is the density of the plasma.

(c) The plasma should have sufficiently many particles present in a Debye sphere, i.e.

$$N_D \equiv \frac{4}{3}\pi\lambda_D^3 n \gg 1.$$

This consideration is made so that the statistical approximations are valid.

While these may seem like a large number of conditions to satisfy, many important plasmas carry these properties. For example, both the sun's corona and the plasma in a Tokamak fusion reactor fit these limitations, but very hot and dense plasmas like the core of the sun do not fit within this regime. Figure 2.1 illustrates the regime in which these conditions hold.

Coming back to the issue of the collision term, C_a, we decompose it into the contributions C_{ab} made by the collisions of particles of species a with particles of species b:

$$C_a = \sum_b C_{ab},$$

where $C_{ab} = C_{ab}(f_a, f_b)$. For our plasma model, we only consider two species of particles - that of ions (i) and electons (e), so in the ion Boltzmann equation we have C_{ie}, C_{ii} and the electron Boltzmann equation contains C_{ei}, C_{ee}. From conservation principles, we may now observe some conditions that the collision terms must satisfy. Since the total number of particle species a at a certain position is not changed by collisions with particles of species b (only their velocities change), we have

$$\int C_{ab} d\mathbf{v} = 0. \tag{2.3}$$

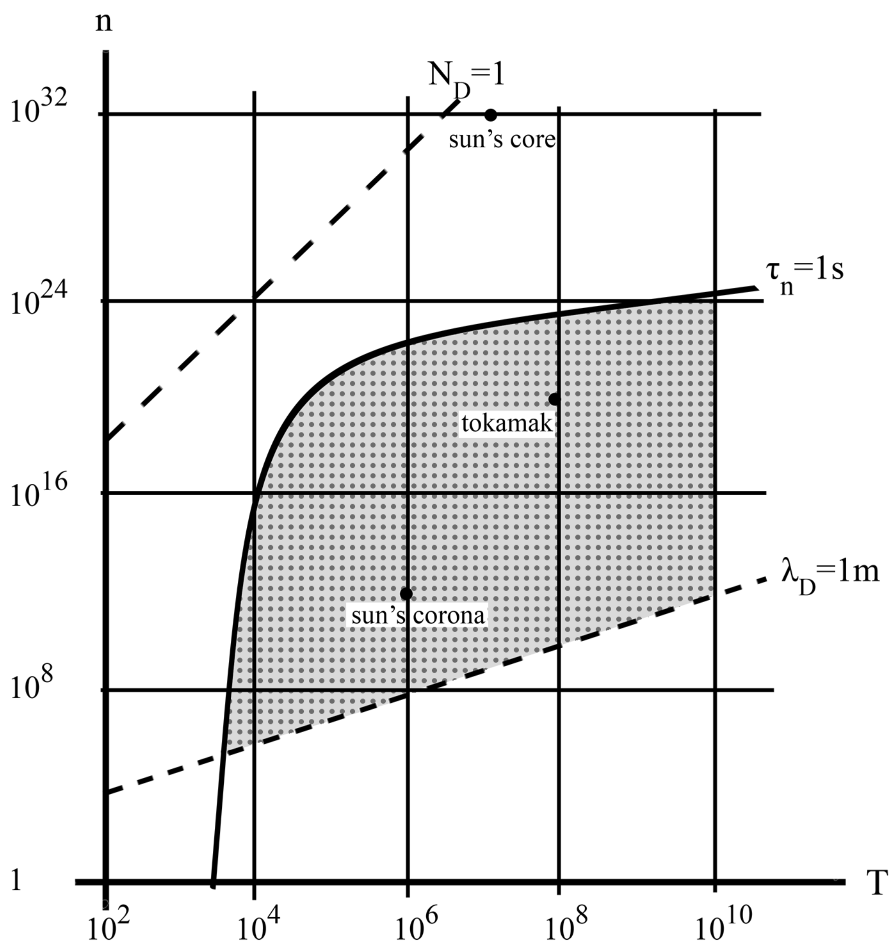


Figure 2.1: Goldilock's Plasma

This figure contains the conditions for the plasma fluid approximation, which Hall MHD is based upon, to hold in terms of density of the plasma, n, and temperature, T. This Goldilock's plasma, which is not too hot or too dense, lives in the shaded area between the curve of the mean time between collisions of charged plasma particles with neutrals, τ_n, equal to one; the line of Debye length, λ_D, equal to one; and sufficient particles living in a Debye sphere, $N_D \gg 1$. These conditions are required so that Coulomb collisions drive the dynamics of the system, the plasma's microfluctuations may be neglected, and enough particles exist for the statistical approximations to be valid.

Momentum and energy are conserved for collisions between particles of the same species, or in equation form,

$$\begin{aligned} \int m_a \mathbf{v} C_{kk} d\mathbf{v} &= 0, \\ \int \tfrac{1}{2} m_a v^2 C_{kk} d\mathbf{v} &= 0, \end{aligned}$$

where $k = e, i$. Between the ions and electrons, the total momentum and energy must be conserved, so we have

$$\int m_e \mathbf{v} C_{ei} d\mathbf{v} + \int m_i \mathbf{v} C_{ie} d\mathbf{v} = 0,$$

$$\int \tfrac{1}{2} m_e v^2 C_{ei} d\mathbf{v} + \int \tfrac{1}{2} m_i v^2 C_{ie} d\mathbf{v} = 0.$$

Now, to promote clarity of writing, we make the following definitions. The number of particle species per unit volume

$$n_a(t, \mathbf{r}) = \int f_a(t, \mathbf{r}, \mathbf{v}) d\mathbf{v}. \tag{2.4}$$

The mean velocity of the particles

$$\mathbf{u}_a(t, \mathbf{r}) = \frac{1}{n_a} \int \mathbf{v} f_a(t, \mathbf{r}, \mathbf{v}) d\mathbf{v} = \langle \mathbf{v} \rangle_a. \tag{2.5}$$

Now we multiply (2.1) by $d\mathbf{v}$ and integrate to find the zeroth moment of the Boltzmann equation.

$$\int \frac{\partial}{\partial t} f_a d\mathbf{v} + \int \frac{\partial}{\partial x_\beta} (v_\beta f_a) d\mathbf{v} + \int \frac{\partial}{\partial v_\beta} \left(\frac{F_{a\beta}}{m_a} f_a \right) d\mathbf{v} = \int C_{ab} d\mathbf{v} \tag{2.6}$$

On the first two pieces on the LHS, we assume proper conditions on the probability distribution function so that the derivatives may be taken out of the integral. In the third term on the LHS, we integrate it by parts and assume that the probability distribution decays quickly enough at infinity, so that it goes to zero. The RHS is simply equation (2.3), so it too goes to zero. What we're left with, after using the definitions (2.4) and (2.5), is the conservation of mass equation

$$\frac{\partial n_a}{\partial t} + \nabla \cdot (n_a \mathbf{u}_a) = 0. \tag{2.7}$$

In literature, it is also referred to as the equation of continuity. When analysing an incompressible plasma n_a is time-independent, and (2.7) reduces to just a solenoidal condition on (2.5).

$$\nabla \cdot \mathbf{u}_a = 0. \tag{2.8}$$

Now we calculate the first moment of the Boltzmann equation by multiplying (2.1) by $\mathbf{v}$ and integrating over $d\mathbf{v}$. This results in the following terms

$$\begin{aligned}
\int \frac{\partial f_a}{\partial t} \mathbf{v} d\mathbf{v} &= \frac{\partial}{\partial t}(n_a \mathbf{u}_a), \\
\int \mathbf{v} \cdot \frac{\partial f_a}{\partial \mathbf{r}} \mathbf{v} d\mathbf{v} &= \nabla \cdot \int \mathbf{v}\mathbf{v} f_a d\mathbf{v} = \nabla \cdot (n_a \langle \mathbf{v}\mathbf{v} \rangle_a), \\
\int \frac{e_a}{m_a}(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f_a}{\partial \mathbf{v}} \mathbf{v} d\mathbf{v} &= -\frac{e_a n_a}{m_a}(\mathbf{E} + \frac{1}{c}\mathbf{u}_a \times \mathbf{B}), \\
\int C_a \mathbf{v} d\mathbf{v} &= \int C_{ab} \mathbf{v} d\mathbf{v} \quad (b \neq a).
\end{aligned}$$

Adding these equations together we obtain the momentum equation for particles of species a

$$\frac{\partial}{\partial t}(n_a m_a \mathbf{u}_a) + \nabla \cdot (n_a m_a \langle \mathbf{v}\mathbf{v} \rangle_a) - n_a e_a (\mathbf{E} + \frac{1}{c}\mathbf{u}_a \times \mathbf{B}) = \int C_{ab} m_a \mathbf{v} d\mathbf{v}. \tag{2.9}$$

In principle we may continue and calculate the nth moment of the Boltzmann equation and get an infinite hierarchy. However, for our purposes we will truncate at the first moment. Now we perform some more manipulations and definitions to get this equation to look more familiar to us. First, let us separate thermal fluctuations from the macroscopic movement of the plasma as a whole by defining the random velocity $\mathbf{w}$ of the particles with respect to the average velocity $\mathbf{u}_a$:

$$\mathbf{w} \equiv \mathbf{v} - \mathbf{u}_a, \quad \text{where} \quad \langle \mathbf{w} \rangle = 0.$$

The random velocity part of the term $\langle \mathbf{vv} \rangle$ in the momentum equation (2.9) gives rise to the stress tensor $\mathbf{P}_a$ defined as

$$\mathbf{P}_a \equiv n_a m_a \langle \mathbf{ww} \rangle = p_a \mathbf{I} + \pi_a,$$

where the isotropic part $p_a(\mathbf{r}, t)$ is pressure and the traceless stress tensor $\pi_a(\mathbf{r}, t)$ contains the anistropic effects of the distribution function and acts like a viscosity. The collision term may also be simplified with the use of the random velocity and equation (2.3).

$$\int C_{ab} m_a \mathbf{v} d\mathbf{v} = \int C_{ab} m_a (\mathbf{w} - \mathbf{u}_a) d\mathbf{v} = \int C_{ab} m_a \mathbf{w} d\mathbf{v} = \mathbf{R}_{ab}$$

This term acts as a frictional force in the model and is the mean momentum transfer from particles of species b to species a. Under our assumptions, the vast majority of collisions in the plasma we are modeling are Coulomb collisions. Therefore, on physical grounds, we expect $\mathbf{R}_{ab}$ to be proportional to the Coulomb force, which is proportional to e^2 for singly charged ions. Further, $\mathbf{R}_{ab}$ must be also proportional to the density of electrons n_e, to the density of scattering centers n_i, and to the relative velocity of the two fluids. Thus we may write $\mathbf{R}_{ei}$ as

$$\mathbf{R}_{ei} = \eta e^2 n^2 (\mathbf{u}_i - \mathbf{u}_e), \tag{2.10}$$

where η is the proportionality constant called the specific resistivity of the plasma. Use of the continuity equation (2.7) allows us to now rewrite our equation of momentum conversation (2.9) as

$$n_a m_a \left[\frac{\partial}{\partial t} + (\mathbf{u}_a \cdot \nabla)\right] \mathbf{u}_a = -\nabla p_a - \nabla \cdot \pi_a + n_a e_a (\mathbf{E} + \frac{1}{c}\mathbf{u}_a \times \mathbf{B}) + \mathbf{R}_{ab}. \qquad (2.11)$$

In literature, the equation of momentum conservation is also referred to as the equation of motion.

2.2 Magnetohydrodynamic Approximation

Now that we have established the all-important equations of continuity and momentum conservation for plasmas using a model of two interpenetrating fluids, let us combine them together with a variety of assumptions so that they are easier to manipulate [7]. We now assume explicitly that the plasma is quasi-neutral with singly charged ions $Z = 1$. Thus we have $e_i = -e_e = e$ and $n_i \approx n_e = n$. Using the notation $m_i = M$ and $m_e = m$ for the mass of the ions and electrons respectively, we now define the mass density ρ and current density $\mathbf{j}$ as follows:

$$\begin{aligned} \rho &\equiv n_i M + n_e m \approx n(M + m) \\ \mathbf{j} &\equiv e(n_i \mathbf{u}_i - n_e \mathbf{u}_e) \approx ne(\mathbf{u}_i - \mathbf{u}_e) \end{aligned} \qquad (2.12)$$

Ignoring viscious effects (π_a) the ion and electron equations of momentum conservation (2.11) can be written as

$$Mn\left[\frac{\partial}{\partial t}+(\mathbf{u}_i\cdot\nabla)\right]\mathbf{u}_i = en(\mathbf{E}+\frac{1}{c}\mathbf{u}_i\times\mathbf{B})-\nabla p_i+\mathbf{R}_{ie} \tag{2.13}$$

$$mn\left[\frac{\partial}{\partial t}+(\mathbf{u}_e\cdot\nabla)\right]\mathbf{u}_e = -en(\mathbf{E}+\frac{1}{c}\mathbf{u}_e\times\mathbf{B})-\nabla p_e+\mathbf{R}_{ei} \tag{2.14}$$

We now neglect electron inertia by taking the limit $m\to 0$, which causes the LHS of (2.14) to go to zero. We then add equations (2.13), (2.14) to obtain

$$Mn\left[\frac{\partial}{\partial t}+(\mathbf{u}_i\cdot\nabla)\right]\mathbf{u}_i=\frac{en}{c}(\mathbf{u}_i-\mathbf{u}_e)\times\mathbf{B}-\nabla p,$$

where $p\equiv p_i+p_e$. Note that the electric field has cancelled out as well as the collision terms since $\mathbf{R}_{ie}=-\mathbf{R}_{ei}$. Rewriting using the defined quantities in (2.12), we have

$$\rho\left[\frac{\partial}{\partial t}+(\mathbf{u}_i\cdot\nabla)\right]\mathbf{u}_i=\frac{1}{c}\mathbf{j}\times\mathbf{B}-\nabla p. \tag{2.15}$$

This is the ion equation of motion specific to the MHD approximation. The electric field does not appear explicitly because the fluid is neutral.

Let us now take a different linear combination of our equations of momentum conservation. Multiplying (2.13) by m and (2.14) by M and subtracting the two yields

$$Mmn\left[\tfrac{\partial}{\partial t}(\mathbf{u}_i-\mathbf{u}_e)+(\mathbf{u}_i\cdot\nabla)\mathbf{u}_i-(\mathbf{u}_e\cdot\nabla)\mathbf{u}_e\right] = -m\nabla p_i+M\nabla p_e+ne(m+M)\mathbf{E} + \tfrac{ne}{c}(m\mathbf{u}_i+M\mathbf{u}_e)\times\mathbf{B}-m\mathbf{R}_{ie}-M\mathbf{R}_{ei}. \tag{2.16}$$

The term being cross multiplied with the magnetic field may be reexpressed through clever manipulations and use of (2.12)

$$\begin{aligned} m\mathbf{u}_i + M\mathbf{u}_e &= M\mathbf{u}_i + m\mathbf{u}_e + M(\mathbf{u}_e - \mathbf{u}_i) + m(\mathbf{u}_i - \mathbf{u}_e) \\ &= M\mathbf{u}_i + m\mathbf{u}_e + (m - M)\frac{\mathbf{j}}{ne}. \end{aligned}$$

Additionally using our phenomenological approximation to the collision term (2.10) and neglecting electron inertia by taking the limit $m \to 0$, (2.16) becomes

$$\begin{aligned} 0 &= M\nabla p_e + neM\mathbf{E} + \frac{ne}{c}\left(M\mathbf{u}_i - M\frac{\mathbf{j}}{ne}\right) \times \mathbf{B} - neM\eta\mathbf{j} \\ 0 &= \frac{1}{ne}\nabla p_e + \mathbf{E} + \frac{1}{c}\mathbf{u}_i \times \mathbf{B} - \frac{1}{nec}\mathbf{j} \times \mathbf{B} - \eta\mathbf{j}. \end{aligned}$$

Our final assumption is that the electron pressure term here can be neglected to yield a generalization to Ohm's law, which describes the electrical properties of the plasma.

$$\mathbf{E} + \frac{1}{c}\mathbf{u}_i \times \mathbf{B} = \eta\mathbf{j} + \frac{1}{nec}\mathbf{j} \times \mathbf{B}. \tag{2.17}$$

The $\mathbf{j} \times \mathbf{B}$ term is called the Hall current term. In the ideal MHD model, this term is small enough to be neglected, but it becomes especially important in certain magnetic field geometries that the full Hall MHD model is most suited towards, like current sheet formation where there are large magnetic field gradients. Combined with Maxwell's equations, the MHD equations form a system describing the dynamics of a single ion fluid.

CHAPTER 3

ANALYSIS

3.1 X-type Neutral Point

We now extend the analysis of Shivamoggi [25] on the unsteady-state properties of an incompressible plasma near a 2D X-type magnetic neutral point. For a more detailed discussion on the physics and previous work on the X-type neutral point problem, see the section 1.1. The governing equations for the dynamics of an incompressible quasineutral plasma under appropriate assumptions (see chapter 2 for development) take the form

$$nM\left[\frac{\partial}{\partial t} + (\mathbf{u}_i \cdot \nabla)\right]\mathbf{u}_i = -\nabla p + \frac{1}{c}\mathbf{j} \times \mathbf{B}, \tag{3.1}$$

$$\mathbf{E} + \frac{1}{c}\mathbf{u}_i \times \mathbf{B} = \eta\mathbf{j} + \frac{1}{nec}\mathbf{j} \times B, \tag{3.2}$$

$$\nabla \cdot \mathbf{u}_e = 0, \tag{3.3}$$

$$\nabla \cdot \mathbf{u}_i = 0, \tag{3.4}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{3.5}$$

$$\nabla \times \mathbf{B} = \frac{1}{c}\mathbf{j}, \tag{3.6}$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \tag{3.7}$$

where $\mathbf{B}$ is the magnetic field, $\mathbf{E}$ is the electric field, $\mathbf{u}_i$ is the fluid velocity of the ions, $\mathbf{u}_e$ is the fluid velocity of the electrons, $p \equiv p_i + p_e$ is the sum of the ion and electron pressures, η is the plasma resistivity constant, n is the density of the plasma, e is the unit charge, c is the speed of light, M is the ion mass, and $\mathbf{j} \equiv ne(\mathbf{u}_i - \mathbf{u}_e)$ is the current density.

We now nondimensionalize the system with a characteristic length scale a, magnetic field strength B_0, and Alfvén time scale $\tau_A \equiv a/V_{A_0}$, where $V_{A_0} \equiv B_0/\sqrt{nM}$. We assume that all quantities do not vary in the z-direction and take the magnetic field and ion-fluid velocity to have the form

$$\mathbf{B} = \nabla\psi \times \hat{\mathbf{k}} + b\hat{\mathbf{k}}, \qquad \mathbf{u}_i = (\hat{\mathbf{k}} \times \mathbf{u}_i) \times \hat{\mathbf{k}} + w\hat{\mathbf{k}} \equiv \mathbf{u} + w\hat{\mathbf{k}}, \tag{3.8}$$

where $\hat{\mathbf{k}}$ is the unit vector in the z-direction. Utilizing (3.6) to eliminate the current density, (3.1) yields

$$\left[\frac{\partial}{\partial t} + (\mathbf{u}\cdot\nabla)\right]\mathbf{u} = -\nabla(p + b^2) - (\nabla^2\psi)\nabla\psi, \tag{3.9}$$

$$\left[\frac{\partial}{\partial t} + (\mathbf{u}\cdot\nabla)\right] w - [b, \psi] = 0. \tag{3.10}$$

The curl of (3.2) along with (3.6) and (3.7) gives

$$\left[\frac{\partial}{\partial t} + (\mathbf{u}\cdot\nabla)\right]\psi + \sigma[b, \psi] = \hat{\eta}\nabla^2\psi, \tag{3.11}$$

$$\left[\frac{\partial}{\partial t} + (\mathbf{u}\cdot\nabla)\right] b + \sigma[\psi, \nabla^2\psi] + [\psi, w] = \hat{\eta}^2 b, \tag{3.12}$$

where $[F, G] \equiv (\nabla F \times \nabla G)\cdot\hat{\mathbf{k}} = F_x G_y - F_y G_x$, $\sigma \equiv \frac{d_i}{a} = \frac{c}{a\omega_{pi}}$, $\hat{\eta} \equiv \frac{\eta c^2 \tau_A}{a^2}$. Consider the initial-value problem for (3.9)-(3.12) near an X-type magnetic neutral point with initial conditions,

$$t = 0 : \mathbf{u} \cdot \hat{\mathbf{i}} = -\dot{\gamma}_0 x,\ \mathbf{u} \cdot \hat{\mathbf{j}} = \dot{\gamma}_0 y,\ w = \tfrac{-(kx^2 - y^2)}{\sigma},$$
$$\psi = kx^2 - y^2 + \mu_0 x^2 y^2,\ b = \dot{C}_0 xy, \tag{3.13}$$

where

$$\dot{\gamma}_0 > 0,\ k > 1,\ \dot{C}_0 > 0,\ \mu_0 < 0 \tag{3.14}$$

are externally determined parameters and $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ are the unit vectors for the x and y directions respectively. This initial condition describes a stagnation-point plasma flow impinging transversely onto the $x = 0$ plane and incorporates solenoidal constraints on $\mathbf{u}$. The parameters have such constraints because we wish to direct the initial Lorentz force of the system,

$$\begin{aligned} t = 0 : \mathbf{j} \times \mathbf{B} \ = \ & 4x \left\{ (1-k)k + \left[\mu_0 (1-2k) - \left(\tfrac{\dot{C}_0}{2} \right)^2 \right] y^2 - k\mu_0 x^2 - \mu_0 y^2 (x^2 + y^2) \right\} \hat{\mathbf{i}} \\ & 4y \left\{ (k-1) + \left[\mu_0 (2-k) - \left(\tfrac{\dot{C}_0}{2} \right)^2 \right] x^2 + \mu_0 y^2 - \mu_0^2 x^2 (x^2 + y^2) \right\} \hat{\mathbf{j}} \\ & -2\dot{C}_0^2 (kx^2 + y^2) \hat{\mathbf{k}}, \end{aligned} \tag{3.15}$$

in a matter that maintains the prescribed initial stagnation-point flow. As is characteristic of Hall effects, the out-of-plane magnetic field b prescribed above exhibits a quadrupolar spatial structure.

Let us assume that the solution of equations (3.9)-(3.12) with the initial conditions (3.13) has a self-similar form

$$
\begin{aligned}
&\mathbf{u}(x,y,t)\cdot\hat{\mathbf{i}} = -\dot{\gamma}(t)x,\ \ \mathbf{u}(x,y,t)\cdot\hat{\mathbf{j}} = \dot{\gamma}(t)y,\\
&w(x,y,t) = \tfrac{1}{\sigma}\left[\beta(t)y^2 - k\alpha(t)x^2\right],\\
&\psi(x,y,t) = k\alpha(t)x^2 - \beta(t)y^2 + \mu(t)(x^2+y^2),\\
&b(x,y,t) = \dot{C}(t)xy,\\
&P(x,y,t) = -\tfrac{1}{2}\nu(t)(x^2-y^2) + P_0,\ \ \nu(t) > 0,
\end{aligned}
\tag{3.16}
$$

where we have the initial conditions

$$
t = 0 : \alpha = \beta = 1,\ \ \dot{\gamma} = \dot{\gamma}_0,\ \ \dot{C} = \dot{C}_0,\ \ \mu = \mu_0. \tag{3.17}
$$

For this ansatze to be valid, we require the resistivity $\hat{\eta}$ to be zero. Substituting our assumed solution (3.16) into (3.9) yields

$$
\ddot{\gamma} = 2(k^2\alpha^2 - \beta^2), \tag{3.18}
$$

(3.11) gives

$$
\dot{\alpha} - 2(\dot{\gamma} + \sigma\dot{C})\alpha = 0, \tag{3.19}
$$

$$
\dot{\beta} + 2(\dot{\gamma} + \sigma\dot{C})\beta = 0, \tag{3.20}
$$

$$
\dot{\mu} = 0 \text{ or } \mu = \mu_0, \tag{3.21}
$$

(3.12) turns into

$$
\ddot{C} + 8\sigma\mu(k\alpha + \beta) = 0, \tag{3.22}
$$

and (3.10) is identically satisfied. Integrating (3.19) and (3.20) produces the solutions

$$
\alpha = \exp(2(\gamma + \sigma C)), \tag{3.23}
$$

$$\beta = \exp(-2(\gamma + \sigma C)). \tag{3.24}$$

Multiplying (3.18) and (3.22) by the integration factor $(\dot{\gamma} + \sigma\dot{C})$ and substituting (3.23) and (3.24) (See Appendix A for details), we find the first integral to be

$$(\dot{\gamma} + \sigma\dot{C})^2 - (k^2\alpha^2 + \beta^2) + 8\sigma^2\mu_0(k\alpha - \beta) = q \tag{3.25}$$

with

$$t = 0 : \gamma = C = 0, \tag{3.26}$$

where q is our constant of integration given by

$$q = (\dot{\gamma}_0 + \sigma\dot{C}_0)^2 - (k^2 + 1) + 8\sigma^2\mu_0(k - 1). \tag{3.27}$$

To analyze the behavior of this function, we introduce the change of variables $\ln(f) = -2(\gamma + \sigma C)$, which transforms (3.25) into

$$\dot{f}^2 = 4N(f) \equiv 4(f^4 + 8\sigma^2\mu_0 f^3 + qf^2 - 8\sigma^2\mu_0 kf + k^2), \tag{3.28}$$

with initial condition $f(0) = 1$. The equation admits an implicit solution

$$t \pm \frac{1}{2}\int_1^{f(t)} (N(z))^{-1/2} dz = 0, \tag{3.29}$$

where the lower bound of integration is from the initial condition. The integral is in general an elliptic form and, in certain cases, may be inverted through the use of elliptic functions once the parameters have been given a fixed value. Note that when $f(t) \to 0$, $|\gamma + \sigma C| \to \infty$ so we may analytically find the time t^* when the first singularity occurs

$$t^* = \frac{1}{2}\int_0^1 (N(z))^{-1/2} dz. \tag{3.30}$$

We see that if the polynomial $N(z)$ has a root of multiplicity two in the interval [0,1], then the integral diverges and the finite-time singular collapse of the current sheet is prevented. By utilizing the constraints on the parameters of the system, namely $\dot{\gamma}_0 > 0$, $k > 1$, $\dot{C}_0 > 0$, $\mu_0 < 0$, and $\sigma > 0$, we note that the coefficient of z^3 is negative, the coefficient of z is positive, and $k^2 > 1$. Thus, by Descartes's rule of signs, there can be either two or zero positive roots to the polynomial and either two or zero negative roots to the polynomial. We break the analysis down into cases.

Case 1: Two roots at $z = 0$. This case is trivially false because the constant term is strictly positive.

Case 2: Two roots at $z = 1$. Synthetic division yields the conditions for this to occur.

$$\begin{aligned} \dot{\gamma}_0 + \sigma \dot{C}_0 &= 0 \\ k^2 - 4\sigma^2 \mu_0 (k+1) &= 1. \end{aligned}$$

Neither of these equations can be satisfied due to the constraints on the parameters of the system, so this case leads to falsehood. Although it may be noted that if one does satisfy the above equations, then the system is in equilibirum, i.e. $\gamma(t) + \sigma C(t) = 0$ for all time.

Case 3: A repeated root at $z = a \in (0, 1)$ with two purely imaginary complex roots.

$$\begin{aligned} N(z) &= (z-a)^2(z^2+b^2) \\ &= z^4 - 2az^3 + (a^2+b^2)z^2 - 2ab^2 z + a^2 b^2. \end{aligned}$$

Because $a > 0$ and $b^2 > 0$ the coefficient of z cannot be positive, so this case also has no solutions.

Case 4: A repeated root at $z = a \in (0, 1)$ with two negative roots b and c.

$$\begin{aligned} N(z) &= (z-a)^2(z+b)(z+c) \\ &= z^4 + (b+c-2a)z^3 + (a^2 - 2ac - 2ab - bc)x^2 + a(a(b+c) - 2bc)x + a^2bc. \end{aligned}$$

From the z^3 coefficient we have $2a > b + c$, the z coefficient yields $a(b+c) - 2bc > 0$, and the constant term gives $a^2bc > 1$. Manipulation of these relations yields a false statement.

Case 5: A repeated root at $z = a \in (0, 1)$ with two complex conjugate roots b and $\bar{b}$.

$$\begin{aligned} N(z) &= (z-a)^2(z+b)(z+\bar{b}) \\ &= z^4 + 2(\Re(b) - a)z^3 + (a^2 - 4a\Re(b) - |b|^2)z^2 + 2a(a\Re(b) - |b|^2)z + a^2|b|^2, \end{aligned}$$

where $\Re(b)$ is the non-zero real part of b and $|b|^2 = b\bar{b}$. From our coefficients we have the relations $a > \Re(b)$, $a\Re(b) > |b|^2$, and $a^2|b|^2 > 1$. The first and second relations imply that $1 > a > \Re(b) > 0$. A more detailed analysis is required to determine whether the final condition that the coefficient of z^3 and z are related by a factor of $-k$ when the constant term of the equation is k^2. Upon defining $2p \equiv -8\sigma^2\mu_0 > 0$, $N(z)$ may be expressed as

$$N(z) = z^4 - 2pz^3 + qz^2 + 2pkz + k^2.$$

Defining the change of variables $\zeta = z - \frac{k}{z}$, we set $N(z)$ equal to zero then divide by z^2 to get

$$\begin{aligned} 0 &= \tfrac{N(z)}{z^2} \\ &= z^2 - 2pz + q + 2p\tfrac{k}{z} + \tfrac{k^2}{z^2} \\ &= (z^2 + \tfrac{k^2}{z^2}) - 2p(z - \tfrac{k}{z}) + q \\ &= (\zeta^2 + 2k) - 2p\zeta + q. \end{aligned}$$

Solving the quadratic in ζ we get the roots

$$\zeta_{+,-} = p \pm \sqrt{p^2 - q - 2k}.$$

Solving the change of variables relation, we get the four roots of $N(z)$ to be

$$\begin{aligned} z_{1,2} &= \tfrac{1}{2}(\zeta_- \pm \sqrt{\zeta_-^2 + 4k}), \\ z_{3,4} &= \tfrac{1}{2}(\zeta_+ \pm \sqrt{\zeta_+^2 + 4k}). \end{aligned}$$

Note that, by symmetry, if $z_1 \notin \mathbb{R}$, then $z_1 = \bar{z}_3$. Also if $z_2 \notin \mathbb{R}$, then $z_2 = \bar{z}_4$. Assume $z_1 = z_3 \in \mathbb{R}$ and $z_2 = \bar{z}_4 \notin \mathbb{R}$. Then we have, upon defining $r \equiv \sqrt{\zeta_-^2 + 4k}$ and $s \equiv \sqrt{\zeta_+^2 + 4k}$,

$$\begin{aligned} \zeta_- + r &= \zeta_+ + s \\ \zeta_- - r &= \bar{\zeta_+} - \bar{s}, \end{aligned}$$

where subtracting these equations yields $r = \Im(\zeta_+) + \Re(s) \in \mathbb{R}$. Taking the complex conjugate of the first equation and then subtracting them gives $\bar{s} = \Im(\zeta_-) + \Re(r) \in \mathbb{R}$. By assumption, $\Im(\zeta_+ + s) = 0$, which implies $\Im(\zeta_+)$ because s is real. This is a contradiction. A similar outcome occurs when assuming $z_1 = \bar{z}_3 \notin \mathbb{R}$ and $z_2 = z_4 \in \mathbb{R}$. Thus, the finite-time singularity cannot be prevented while still matching the initial conditions to preserve the initial stagnation-point flow.

Since the finite-time singularity cannot be prevented, let us try to characterize its behavior. We have from (3.30)

$$\frac{\partial t^*}{\partial k} = -\frac{1}{4}\int_0^1 \frac{8\sigma^2\mu_0(z^2 - z) + 2k(1 - z^2)}{(N(z))^{3/2}} dz \tag{3.31}$$

Assuming $N(z)$ has no roots in $[0,1]$ the integral will converge and $T^{-} < N(z) < T^{+}$ for all $z \in [0,1]$, where $T^{-,+}$ are some positive numbers. Thus we have

$$\begin{aligned}
\frac{\partial t^*}{\partial k} &= -\frac{1}{4}\int_0^1 \frac{8\sigma^2\mu_0(z^2-z)+2k(1-z^2)}{(N(z))^{3/2}}dz \\
&\leq -\frac{1}{4T^-}\int_0^1 [8\sigma^2\mu_0(z^2-z)+2k(1-z^2)]dz \\
&= -\frac{1}{3T^-}\left(k-\sigma^2\mu_0\right) \\
&< 0
\end{aligned} \tag{3.32}$$

Similarly under the same assumption on $N(z)$,

$$\frac{\partial t^*}{\partial \mu_0} \geq \frac{1}{6T^+}\sigma^2(1+2k) > 0, \tag{3.33}$$

$$\frac{\partial t^*}{\partial(\dot{\gamma}_0+\sigma\dot{C}_0)} \leq -\frac{\dot{\gamma}_0+\sigma\dot{C}_0}{2T^-} < 0, \tag{3.34}$$

$$\frac{\partial t^*}{\partial \sigma} \leq \frac{1}{6T^-}\left(2\sigma\mu_0(1+2k)-\dot{C}_0(\dot{\gamma}_0+\sigma\dot{C}_0)\right) < 0, \tag{3.35}$$

Thus an increase in k, $\dot{\gamma}_0$, $\dot{C}_0$, or σ decreases the time until singularity while an increase in μ_0 delays the singularity when $N(z)$ has no roots in $[0,1]$ as is confirmed in figures 3.1-3.4.

Indeed, following the methodology of [1], we may express (3.30) in terms of elliptic functions by using the change of variables

$$t = \frac{z-\tau_-}{z-\tau_+},$$

where

$$\begin{aligned}
\tau_{+,-} &\equiv \sqrt{\tfrac{z_1 z_3+\lambda_{+,-} z_2 z_4}{1+\lambda_{+,-}}}, \\
\lambda_{+,-} &\equiv \tfrac{-b\pm\sqrt{b^2-ac}}{a}, \\
a &\equiv (z_2-z_4)^2, \\
b &\equiv (z_1+z_3)(z_2+z_4)-2(z_1 z_3+z_2 z_4) \\
c &\equiv (z_1-z_3)^2,
\end{aligned} \tag{3.36}$$

which yields

$$t^* = \frac{|\lambda_+ - \lambda_-|}{2}\int_{\frac{\tau_-}{\tau_+}}^{\frac{1-\tau_-}{1-\tau_+}} \frac{dt}{\sqrt{(Ft^2-G)(Ht^2-J)}}, \tag{3.37}$$

where

$$\begin{aligned}
H &\equiv 1+\lambda_+, \\
J &\equiv 1+\lambda_-, \\
F &\equiv \lambda_+ H, \\
G &\equiv \lambda_- J.
\end{aligned} \tag{3.38}$$

Now (3.37) can be integrated to give an elliptic form

$$t^* = -\frac{|\lambda_+ - \lambda_-|}{2\sqrt{FJ}}\sqrt{\frac{(Ft^2+G)(Ht^2+J)}{(Ft^2-G)(Ht^2-J)}}\,\text{EllipticF}\left(t\sqrt{\frac{F}{G}},\sqrt{\frac{\lambda_-}{\lambda_+}}\right)\Bigg|_{t=\frac{\tau_-}{\tau_+}}^{t=\frac{1-\tau_-}{1-\tau_+}} \tag{3.39}$$

Here we have assumed that the roots of $N(z)$ all have multiplicity one, $\lambda_+ \neq \lambda_-$, $\lambda_+ \neq -1$, and $\lambda_- \neq -1$. EllipticF is the incomplete elliptic integral of the first kind, meaning

$$\text{EllipticF}(z,k) \equiv \int_0^z \frac{1}{\sqrt{1-m^2}\sqrt{1-k^2m^2}}dm.$$

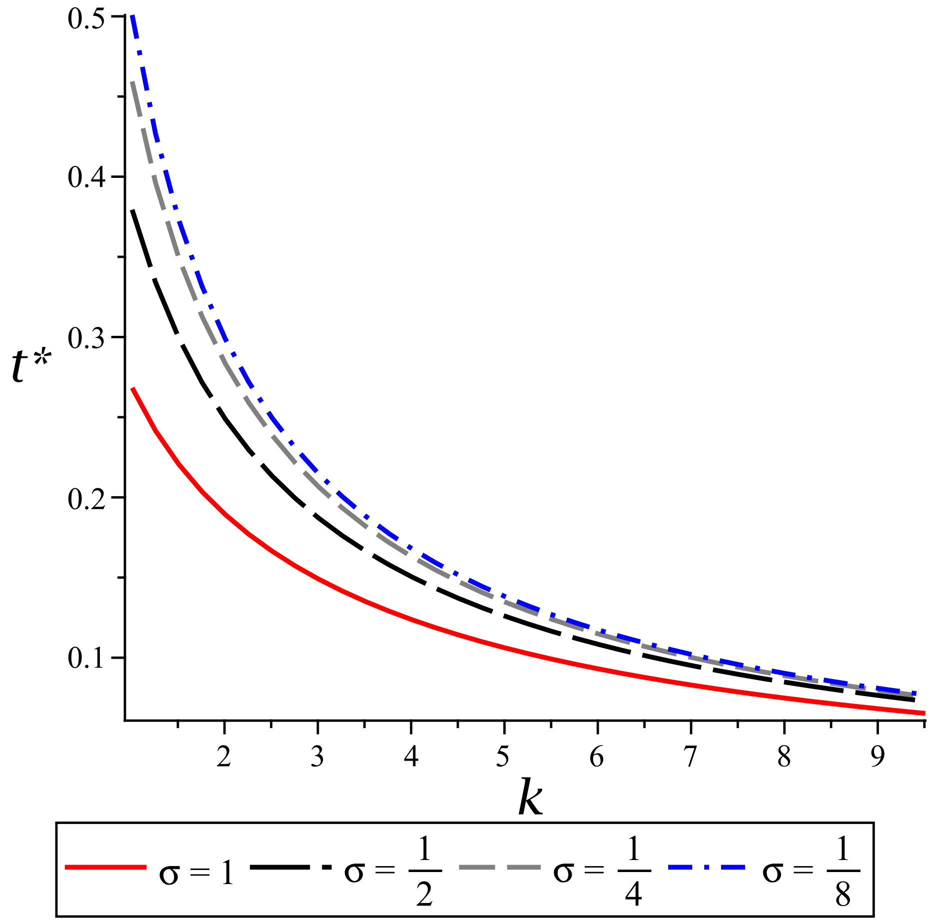


Figure 3.1: Initial Current Density's Effect on Sheet Collapse

We fix the initial in-plane ion velocity and out-of-plane magnetic field strength to unity and higher order in-plane magnetic field perturbation to negative unity, i.e. $\dot{\gamma}_0 = \dot{C}_0 = -\mu_0 = 1$, and vary the initial current density k and strength of the Hall parameter σ and note their effects on the time for the current sheet to collapse t^*, which is calculated from (3.30). As expected from our analytical results (3.35) and (3.32), increasing k and σ decreases t^*.

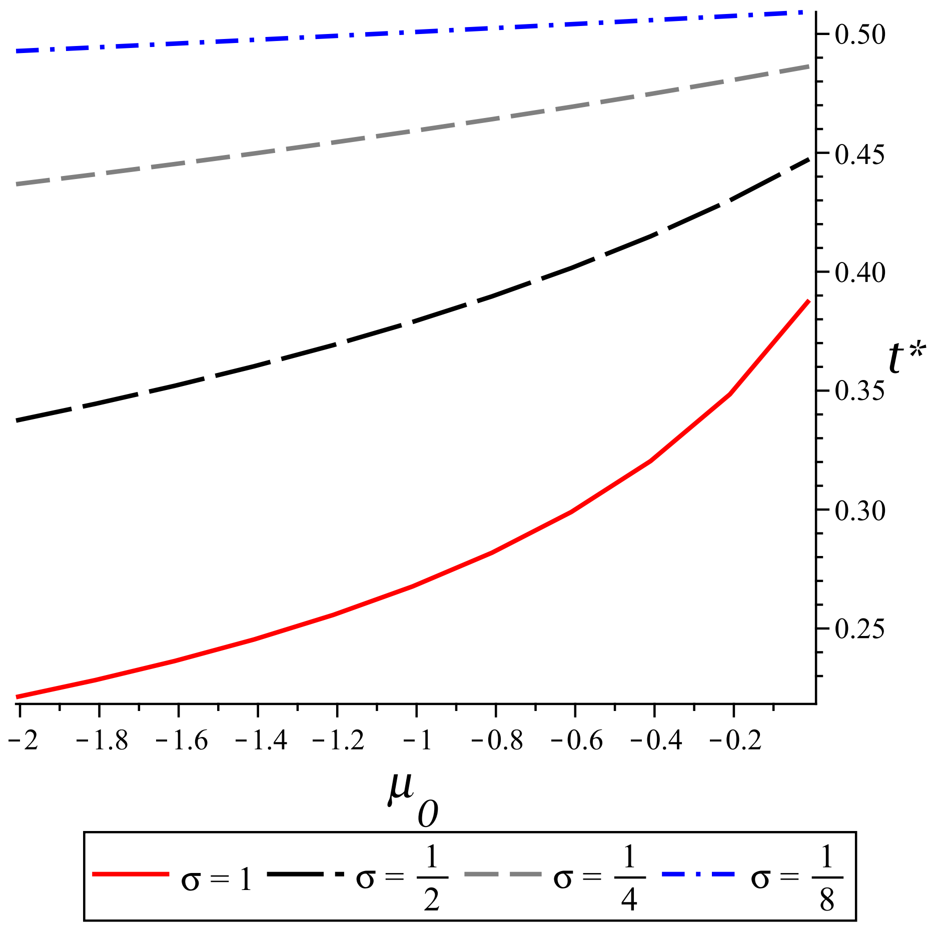


Figure 3.2: High Order Perturbation's Effect on Sheet Collapse

We fix the initial in-plane ion velocity and out-of-plane magnetic field strength to unity and initial current density to two, i.e. $\dot{\gamma}_0 = \dot{C}_0 = 1$ and $k = 2$, and vary the strength of the higher order perturbation of the in-plane magnetic field μ_0 and strength of the Hall parameter σ and note their effects on the time for the current sheet to collapse t^*, which is calculated from (3.30). As expected from our analytical results (3.35) and (3.33), decreasing μ_0 and increasing σ decreases t^*. When σ is small, changes in μ_0 have little effect on t^* because of the heavy amount of coupling of μ_0 and σ in (3.30). Indeed, μ_0 only appears in t^* in $\sigma^2\mu_0$ pairs.

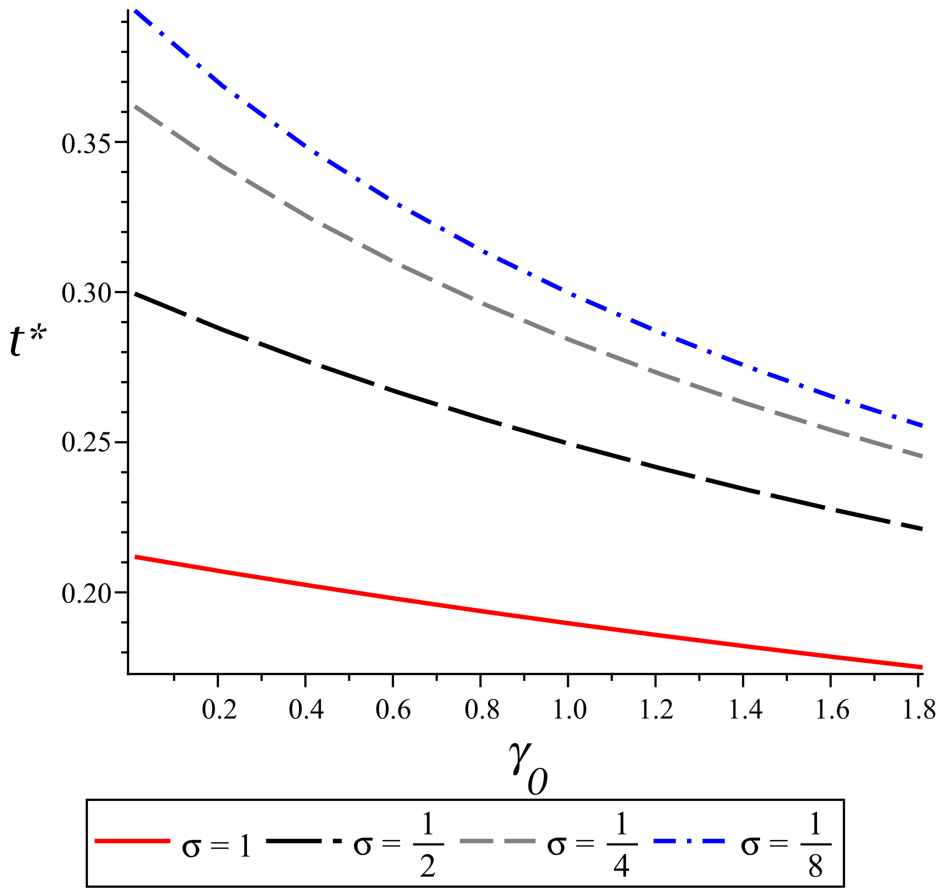


Figure 3.3: In-Plane Ion Velocity's Effect on Sheet Collapse

We fix the initial out-of-plane magnetic field strength to unity and higher order in-plane magnetic field perturbation to negative unity and initial current density to two , i.e. $\dot{C}_0 = -\mu_0 = 1$ and $k = 2$, and vary the initial in-plane ion velocity $\dot{\gamma}_0$ and strength of the Hall parameter σ and note their effects on the time for the current sheet to collapse t^*, which is calculated from (3.30). As expected from our analytical results (3.35) and (3.34), increasing $\dot{\gamma}_0$ and σ decreases t^*.

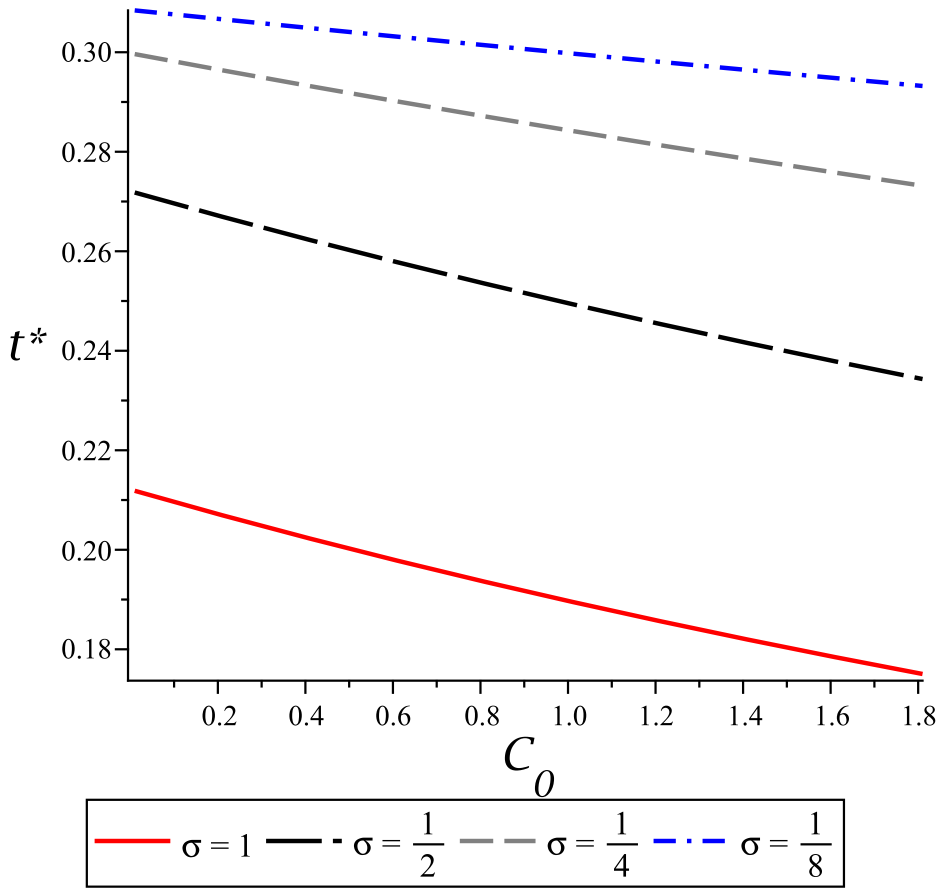


Figure 3.4: Out-of-plane Magnetic Field Strength's Effect on Sheet Collapse

We fix the initial in-plane ion velocity to unity and higher order in-plane magnetic field perturbation to negative unity and initial current density to two , i.e. $\dot{C}_0 = -\mu_0 = 1$ and $k = 2$, and vary the initial out-of-plane magnetic field strength $\dot{C}_0$ and strength of the Hall parameter σ and note their effects on the time for the current sheet to collapse t^*, which is calculated from (3.30). As expected from our analytical results (3.35) and (3.34), increasing $\dot{C}_0$ and σ decreases t^*. When σ is small, $\dot{C}_0$ has very little effect on t^* because it is directly tied to σ and only appears in $\sigma\dot{C}_0$ pairs.

3.2 Parker Problem

We now extend the analysis of Shivamoggi [24] on the Parker problem where magnetic flux pile-up drives the merging of antiparallel magnetic fields at a ion stagnation-point flow in a thin current sheet. For a more detailed discussion on the physics and previous work on the Parker problem, see section 1.2.

The governing equations for the dynamics of an incompressible quasineutral plasma under appropriate assumptions (see chapter 2 for development) take the form

$$nM\left[\frac{\partial}{\partial t} + (\mathbf{u}_i \cdot \nabla)\right]\mathbf{u}_i = -\nabla p + \frac{1}{c}\mathbf{j} \times \mathbf{B}, \tag{3.40}$$

$$\mathbf{E} + \frac{1}{c}\mathbf{u}_i \times \mathbf{B} = \eta\mathbf{j} + \frac{1}{nec}\mathbf{j} \times B, \tag{3.41}$$

$$\nabla \cdot \mathbf{u}_e = 0, \tag{3.42}$$

$$\nabla \cdot \mathbf{u}_i = 0, \tag{3.43}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{3.44}$$

$$\nabla \times \mathbf{B} = \frac{1}{c}\mathbf{j}, \tag{3.45}$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \tag{3.46}$$

where $\mathbf{B}$ is the magnetic field, $\mathbf{E}$ is the electric field, $\mathbf{u}_i$ is the fluid velocity of the ions, $\mathbf{u}_e$ is the fluid velocity of the electrons, $p \equiv p_i + p_e$ is the sum of the ion and electron pressures, η is the plasma resistivity constant, n is the density of the plasma, e is the unit charge, c is the speed of light, M is the ion mass, and $\mathbf{j} \equiv ne(\mathbf{u}_i - \mathbf{u}_e)$ is the current density.

We now nondimensionalize the system with a characteristic length scale a, magnetic field strength B_0, Alfvén time scale $\tau_A \equiv a/V_{A_0}$, where $V_{A_0} \equiv B_0/\sqrt{nM}$. We assume that all quantities do not vary in the z-direction and take the magnetic field and ion-fluid velocity to have the form

$$\begin{aligned} \mathbf{B} &= \nabla\psi \times \hat{\mathbf{k}} + b\hat{\mathbf{k}} \\ \mathbf{u}_i &= \nabla\phi \times \hat{\mathbf{k}} + w\hat{\mathbf{k}}, \end{aligned} \tag{3.47}$$

where $\hat{\mathbf{k}}$ is the unit vector in the z-direction. Using (3.45) to eliminate current density, (3.40) yields

$$\frac{\partial w}{\partial t} + [w, \phi] = [b, \psi]. \tag{3.48}$$

Taking the curl of (3.41) and substituting (3.45) and (3.46) gives

$$\frac{\partial \psi}{\partial t} + [\psi, \phi] + \sigma[b, \psi] = \hat{\eta}\nabla^2\psi, \tag{3.49}$$

$$\frac{\partial b}{\partial t} + [b, \phi] + \sigma[\psi, \nabla^2\psi] + [\psi, w] = \hat{\eta}\nabla^2 b, \tag{3.50}$$

where $[F, G] \equiv (\nabla F \times \nabla G) \cdot \hat{\mathbf{k}} = F_x G_y - F_y G_x$, $\sigma \equiv \frac{d_i}{a} = \frac{c}{a\omega_{pi}}$, $\hat{\eta} \equiv \frac{\eta c^2 \tau_A}{a^2}$.

Consider a stagnation-point ion flow at a current sheet separating plasmas of opposite magnetizations (the Parker problem) in Hall MHD. Let us assume that the magnetic field lines are straight and parallel to the current sheet. Here, pure resistive annihilation without reconnection of antiparallel magnetic fields (in the x,y-plane) occurs. Specifically, consider a unidirectional applied magnetic field

$$\mathbf{B}_0 = B_0(x)\hat{\mathbf{j}} \tag{3.51}$$

with boundary condition $B_0(0) = 0$ which is carried towards a neutral sheet at $x = 0$ by a stagnation-point ion flow,

$$\mathbf{u}_i = -ax\hat{\mathbf{i}} + ay\hat{\mathbf{j}} + w\hat{\mathbf{k}}, \tag{3.52}$$

where $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ are the unit vectors for the x, y and z-directions respectively.

The process in question is steady and when the magnetic field is prescribed as in (3.51), equations (3.48)-(3.50) become

$$E + \frac{\partial \psi}{\partial x}\frac{\partial \phi}{\partial y} - \sigma \frac{\partial b}{\partial y}\frac{\partial \psi}{\partial x} = \hat{\eta}\frac{\partial^2 \psi}{\partial x^2}, \tag{3.53}$$

$$\frac{\partial b}{\partial x}\frac{\partial \phi}{\partial y} - \frac{\partial b}{\partial y}\frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial x}\frac{\partial w}{\partial y} = \hat{\eta}\nabla^2 b, \tag{3.54}$$

$$\frac{\partial w}{\partial x}\frac{\partial \phi}{\partial y} - \frac{\partial w}{\partial y}\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial x}\frac{\partial b}{\partial y} = 0, \tag{3.55}$$

where $E \equiv \frac{\partial \psi}{\partial t}$. We look for a solution of the form

$$b = yf(x), \tag{3.56}$$

$$w = \alpha g(x) + \frac{\beta}{2}y^2 h(x), \tag{3.57}$$

where α and β are constants that characterize the toroidal flow and poloidal shear respectively. Using (3.56), (3.57), and (3.52) on (3.55) gives

$$(\alpha g' + \frac{\beta}{2}y^2 h')(-ax) - (\beta y h)(-ay) + \frac{\partial \psi}{\partial x}f' = 0, \tag{3.58}$$

where prime denotes differentiation with respect to x, which has possible solution

$$(-ax)(\alpha g') = -\frac{\partial \psi}{\partial x}f', \tag{3.59}$$

$$h(x) = x^2. \tag{3.60}$$

Substituting (3.56), (3.57), (3.59), (3.60) into (3.54) we obtain

$$f'' + \frac{a}{\hat{\eta}}(xf' - f) = \frac{\beta}{\hat{\eta}} x^2 \frac{\partial \psi}{\partial x}, \tag{3.61}$$

which has approximate solution

$$f(x) \approx Ax - \frac{\beta}{a} x^2 \frac{\partial \psi}{\partial x}, \tag{3.62}$$

where A is a constant. Using equations (3.56) and (3.62) gives

$$E = \hat{\eta} B_0' + (a + \sigma A) x B_0 + \sigma \frac{\beta}{a} x^2 B_0^2. \tag{3.63}$$

Defining $\epsilon \equiv \frac{\hat{\eta}}{E}$, $\alpha \equiv \frac{\beta}{a^2}$, and letting $\frac{a}{E} = 1$ and $\frac{A}{E} = 1$, this equation takes the form

$$\epsilon B_0' + (1 + \sigma) x B_0 + \sigma \alpha x^2 B(x)^2 = 1. \tag{3.64}$$

Noting that this equation is of special Riccati-type form, we may use the substitution $B_0(x) = -\frac{\epsilon C'(x)}{\sigma \alpha x^2 C(x)}$ to convert it into a linear second-order ODE

$$C''(x) + \left(\frac{1 + \sigma}{\epsilon} x - \frac{2}{x} \right) C'(x) - \frac{\sigma \alpha}{\epsilon^2} C(x) = 0. \tag{3.65}$$

Because this equation is linear, we may solve it and transform the problem backwards to get a solution to the Ricatti equation. Doing so gives a solution

$$B_0(x) = -\frac{c\epsilon(\sigma\alpha + \chi) K_{U,9} + (\gamma x^2 - \epsilon)\chi(cK_{U,5} + K_{M,5}) + (\chi - 6\gamma) K_{M,9}}{2\gamma^2 \sigma \alpha x^3 (cK_{U,5} + K_{M,5})}, \tag{3.66}$$

where

$$\begin{aligned} \gamma &\equiv \sqrt{(1+\sigma)^2 + 4\sigma\alpha}, \\ c &\equiv \tfrac{3}{4\sqrt{\pi}} \tfrac{(5\gamma - \sigma - 1)\Gamma\left(\frac{3\gamma - 1 - \sigma}{4\gamma}\right)}{\chi + \sigma\alpha}, \\ \chi &\equiv \gamma^2 + (1+\sigma)\gamma, \\ K_{G,a}(x) &\equiv \text{KummerG}\left(\tfrac{a\gamma - 1 - \sigma}{4\gamma}, \tfrac{5}{2}, \tfrac{\gamma x^2}{2\epsilon} \right), \end{aligned} \tag{3.67}$$

$\Gamma(x)$ is the Gamma function, and the KummerU(a, b, z) and KummerM(a, b, z) functions satisfy the differential equation

$$z\frac{d^2w}{dz^2} + (b - z)\frac{dw}{dz} - aw = 0.$$

Taking a Taylor series expansion of the solution (3.66) about $x = 0$ recovers Parker's solution

$$B_0(x) \approx \epsilon x = \frac{\hat{\eta}}{E}x.$$

When x is large, KummerU $\ll$ KummerM so we have the asymptotics

$$B_0(x) \approx \frac{(6\gamma - \chi)K_{M,9}}{2\gamma\sigma\alpha x^3 K_{M,5}},$$

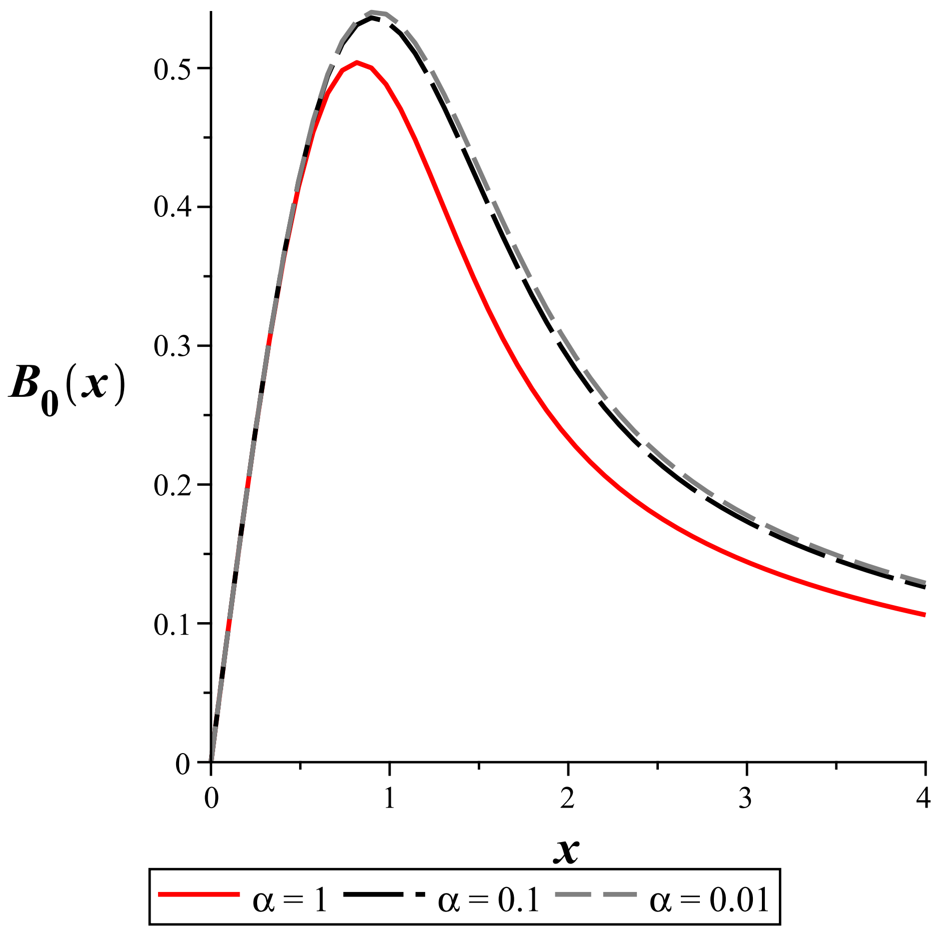
which is comparable to the asymptotic behavior of the solution given previously by Shivamoggi [24] as

$$B_0(x) \approx \left[\frac{\sigma\beta}{2a(a + \sigma A)}\right]\left[-1 + \sqrt{+\frac{4E(a + \sigma A)a^2}{\sigma^2\beta^2}}\right]\frac{1}{x}$$

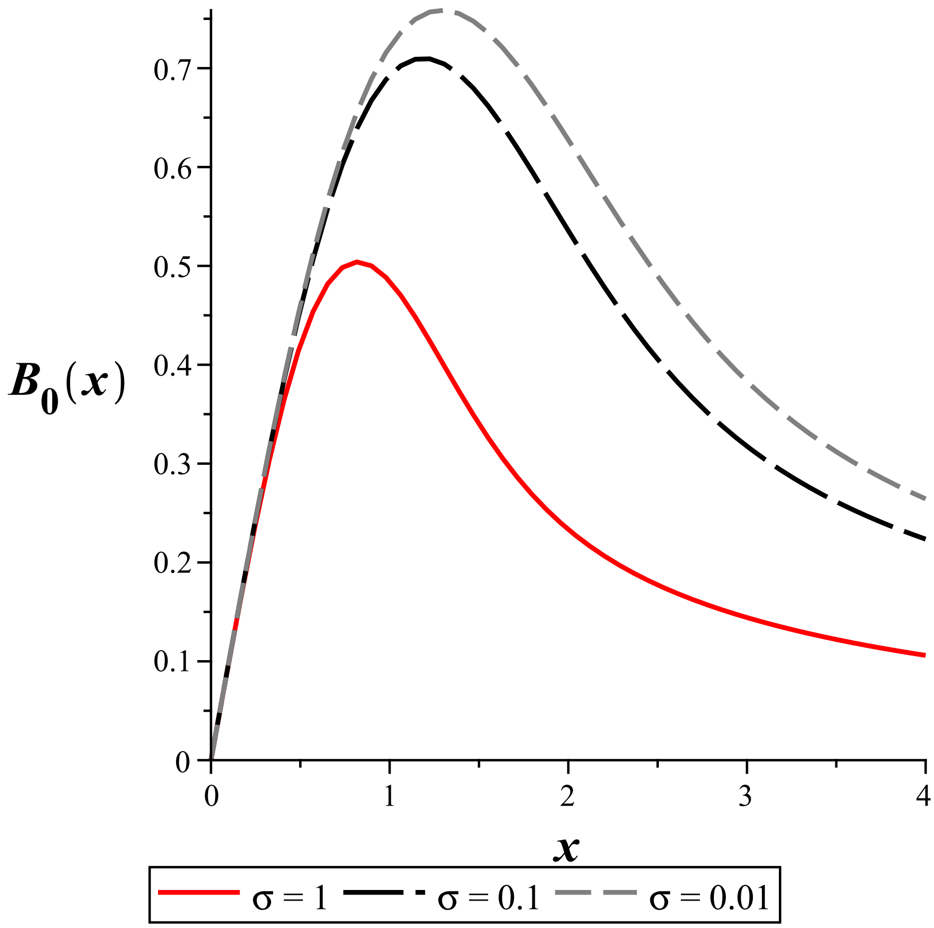
because $K_{M,9}/K_{M,5}$ grows on the order of x^2 for large x.

To get a picture on how the solution behaves with modification of the parameters ϵ, α, and σ we may simply plot the numerical solutions, as in figure 3.5. Overall, the behavior of the solution does not change with the parameters. It follows the same curve: a linear climb from the origin, which tapers off at a peak that transfers into behavior on the order $1/x^3$, and finally a slow decay on the order of $1/x$, which is the triple-deck structure of the Parker problem for Hall MHD as originally proposed by Terasawa [31].

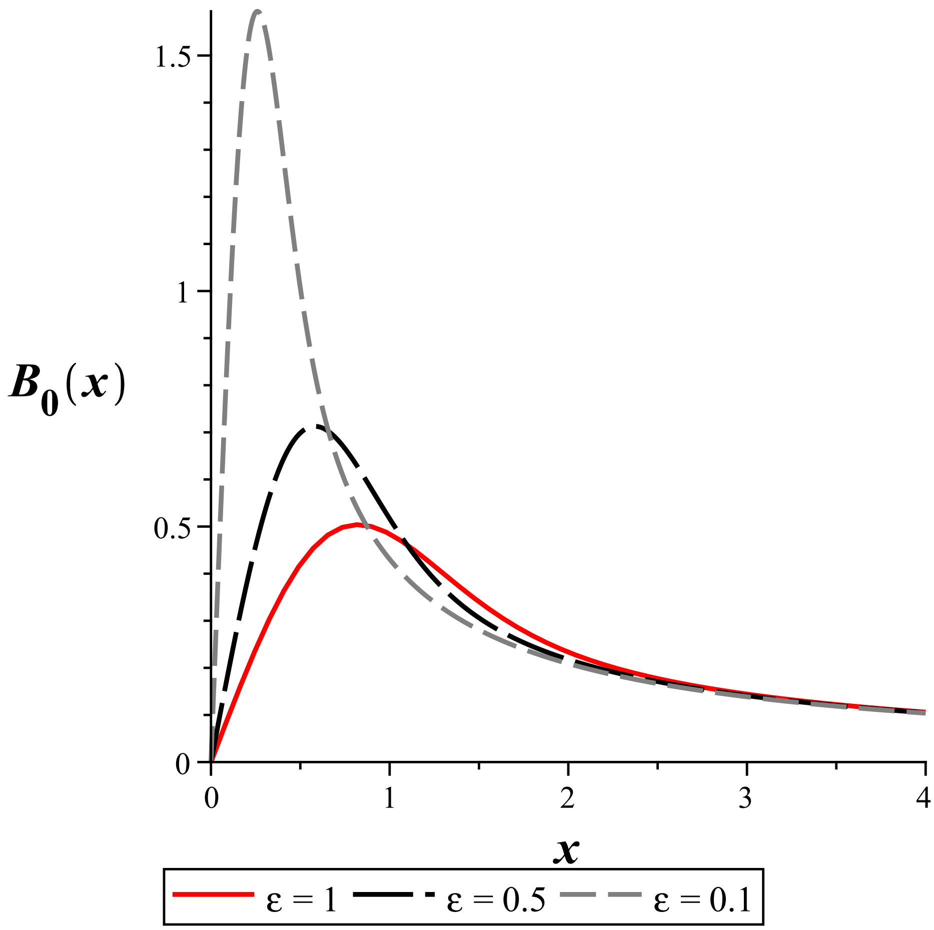
Neither α nor σ significantly affect the rate at which $B_0(x)$ increases near the origin. Instead, increasing α and/or σ decreases the x value at which $B(x)$ changes its behavior from a rapid linear climb from the origin to the $O(1/x^3)$ descent, and hence affect the maximal value attained by $B_0(x)$, although α seems to have much less of an effect than σ. The plasma resistivity parameter ϵ has the strongest effect on the solution $B_0(x)$. This results from the fact that the Kummer functions in the analytic solution spatially scale on the order $1/\epsilon$.



(a) Poloidal Shear



(b) Hall Effect



(c) Plasma Resistivity

Figure 3.5: Effect of Parameter Values on Parker Magnetic Field Profile

In these graphs, the unspecified parameters are set to unity and we observe how the magnetic field profile $B_0(x)$ changes as the third parameter is varied. The strength of the poloidal shear and Hall parameter, α and σ respectively, both increase the peakedness of the solution profile as they increase in value with the Hall parameter having a more significant effect. Plasma resistivity ϵ is the only parameter that changes how localized the peak is to the origin and affects the slope of the solution near the origin.

CHAPTER 4

CONCLUSIONS

In our analysis of the dynamics of an incompressible, nonresistive plasma at an X-type neutral point, we derived an implicit solution to the system, which indicated a finite-time collapse of the current sheet. An exact equation for the time of collapse (3.30) was found and characterized by the roots of the fourth-degree polynomial $N(f)$ (3.28). For all initial conditions satisfying the constraints to preserve the initial stagnation-point flow (3.14), the singular finite-time collapse of the current sheet cannot be prevented. Relations between initial conditions to the system and the time of singularity were also deduced (3.32)-(3.35). A strengthening of the Hall effect, which is parametrized by σ, appears to speed up the collapse of the current sheet.

We presented an analytic solution to the profile of magnetic field for the phenomena of magnetic flux pile-up driving the merging of antiparallel magnetic fields at an ion stagnation-point flow in a thin current sheet, called the Parker problem. It exhibits the triple-deck structure proposed by Terasawa [31] and shares asymptotic properties of the solution previously given by Shivamoggi [24].

APPENDIX

X-TYPE NEUTRAL POINT FIRST INTEGRAL

In this section we fill in the details to get from (3.18)-(3.22) to (3.25). We begin with the four coupled differential equations

$$\dot{\alpha} - 2(\dot{\gamma} + \sigma\dot{C})\alpha = 0 \tag{A.1}$$

$$\dot{\beta} + 2(\dot{\gamma} + \sigma\dot{C})\beta = 0 \tag{A.2}$$

$$\ddot{C} + 8\sigma\mu_0(k\alpha + \beta) = 0 \tag{A.3}$$

$$\ddot{\gamma} - 2(k^2\alpha^2 - \beta^2) = 0 \tag{A.4}$$

with the initial conditions for $t = 0$:

$$\alpha = \beta = 1;\ C = \gamma = 0;\ \dot{C} = \dot{C}_0;\ \dot{\gamma} = \dot{\gamma}_0 \tag{A.5}$$

With the initial conditions (A.5), (A.1) and (A.2) immediately lead to

$$\alpha(t) = \exp(2\gamma + 2\sigma C) \tag{A.6}$$

$$\beta(t) = \exp(-2\gamma - 2\sigma C) \tag{A.7}$$

Now we multiply both equations (A.3) and (A.4) by the factor $2(\dot{\gamma} + \sigma\dot{C})$ and integrate the equations. For $C(t)$ we get

$$\begin{aligned} \int 2(\dot{\gamma} + \sigma\dot{C})\ddot{C} + 16(\dot{\gamma} + \sigma\dot{C})\sigma\mu_0(ke^{2(\gamma+\sigma C)} + e^{-2(\gamma+\sigma C)}) &= 0 \\ \sigma\dot{C}^2 + 8\sigma\mu_0(ke^{2(\gamma+\sigma C)} - e^{-2(\gamma+\sigma C)}) + 2\int \dot{\gamma}\ddot{C} &= q_1 \end{aligned} \tag{A.8}$$

where q_1 is the constant of integration. Similarly for $\gamma(t)$ we have

$$\begin{aligned} \int 2(\dot{\gamma} + \sigma\dot{C})[\ddot{\gamma} - 2(k^2e^{4(\gamma+\sigma C)} - e^{-4(\gamma+\sigma C)}] &= 0 \\ \dot{\gamma}^2 - k^2e^{4(\gamma+\sigma C)} - e^{-4(\gamma+\sigma C)} + 2\sigma\int \dot{C}\ddot{\gamma} &= q_2 \end{aligned} \tag{A.9}$$

where q_2 is the constant of integration. We may integrate the $\int \dot{C}\ddot{\gamma}$ term in (A.9) by parts to get

$$\dot{\gamma}^2 - k^2 e^{4(\gamma+\sigma C)} - e^{-4(\gamma+\sigma C)} + 2\sigma \dot{C}\dot{\gamma} - 2\sigma \int \dot{\gamma}\ddot{C} = q_2 \tag{A.10}$$

Now we may solve for the $\int \dot{\gamma}\ddot{C}$ term in (A.8) and substitute it into (A.10) to form our exact invariant

$$\begin{aligned} (\dot{\gamma} + \sigma\dot{C})^2 + 8\sigma^2\mu_0(ke^{2(\gamma+\sigma C)} - e^{-2(\gamma+\sigma C)}) - k^2 e^{4(\gamma+\sigma C)} - e^{-4(\gamma+\sigma C)} &= q \\ (\dot{\gamma} + \sigma\dot{C})^2 + 8\sigma^2\mu_0(k\alpha - \beta) - k^2\alpha^2 - \beta^2 &= q, \end{aligned} \tag{A.11}$$

where $q = (\dot{\gamma}_0 + \sigma\dot{C}_0)^2 - (k^2+1) + 8\sigma^2\mu_0(k-1)$ is the constant of integration from our initial conditions (A.5).

LIST OF REFERENCES

[1] M. Abramowitz and I. A. Stegun, *Handbook of mathematical functions*, U.S. Government Printing Office, 1972.

[2] D. Biskamp, *Nonlinear magnetohydrodynamics*, Cambridge University Press (1993).

[3] D. Biskamp, *Magnetic reconnection in plasmas*, Cambridge University Press (2000).

[4] D. Biskamp, *Magnetic reconnection via current sheets*, Physics of Fluids **29** (1986) 1520.

[5] S. Braginskii, *Transport processes in a plasma*, Reviews of Plasma Physics **1** (1965) 205-311.

[6] L. Chacón, et. al., *Steady-state properties of driven magnetic reconnection in 2D electron magnetohydrodynamics*, Physical Review Letters **99** (2007) 235001.

[7] F. Chen, *Introduction to plasma physics and controlled fusion*, Plenum Press (1984).

[8] W. Daughton, et. al., *Fully kinetic simulations of undriven magnetic reconnection with open boundary conditions*, Physics of Plasmas **13** (2006) 072101.

[9] J. Dorelli and J. Birn, *Whistler-mediated magnetic reconnection in large system: Magnetic flux pileup and the formation of thin current sheets*, Journal of Geophysical Research **108** (2003) 1133.

[10] J. Dorelli, *Effects of Hall electric fields on the saturation of forced antiparallel magnetic field merging*, Physics of Plasmas **19** (2003) 3309-3314.

[11] J. Dungey, *Conditions for the occurrence of electrical discharges in astrophysical systems*, Philosophical Magazine **44** (1953) 725.

[12] R. Fitzpatrick, *Scaling of forced magnetic reconnection in the Hall-magnetohydrodynamic Taylor problem*, Physics of Plasmas **11** (2004) 937-946.

[13] A. Frank, et. al., *Hall currents in a current sheet: Structure and dynamics*, Physics of Plasmas **15** (2008) 092102.

[14] H. Goedbloed, S. Poedts, *Principles of magnetohydrodynamics*, Cambridge University Press (2004).

[15] H. Karimabadi, at. al., *Multi-scale structure of the electron diffusion region*, Geophysical Research Letters **34** (2007) 235001.

[16] D. Knoll and L. Chacón, *Coalescence of magnetic islands, sloshing, and the pressure problem*, Physics of Plasmas **13** (2006) 032307.

[17] E. Parker, *Sweet's mechanism for merging magnetic fields in conducting fluids*, Journal of Geophysical Research **62** (1957) 509-520.

[18] E. Parker, *Comments on the reconnection rate of magnetic fields*, Journal of Plasma Physics **9** (1973) 49-63.

[19] H. Petchek, *ASS/NASA Symposium on the Physics of Solar Flares*, NASA, Washington, DC (1964) 425.

[20] Y. Ren, et. al., *Experimental verification of the Hall effect during magnetic reconnection in laboratory plasma*, Physical Review Letters **95** (2005) 055003.

[21] B. Rogers, et. al., *Role of dispersive waves in collisionless magnetic reconnection*, Physical Review Letters **87** (2001) 195004.

[22] M. Shay, et. al., *The scaling of collisionless, magnetic reconnection for large systems*, Geophysical Research Letters **26** (1999) 2163-2166.

[23] M. Shay, et. al., *Two-scale structure of the electron dissipation region during collisionless magnetic reconnection*, Physical Review Letters **99** (2007) 155002.

[24] B. Shivamoggi, *Parker problem in Hall magnetohydrodynamics*, Physics of Plasmas **16** (2009) 0521.

[25] B. Shivamoggi, *Steady and unsteady Hall magnetohydrodynamics near an X-type magnetic neutral line*, Physics of Plasmas **18** (2011) 052304.

[26] B. Shivamoggi, *Magnetohydrodynamic theories of magnetic field reconnection*, Physics Reports **127** (1985) 99-184.

[27] B. Shivamoggi, *Hall magnetohydrodynamics near an X-type magnetic neutral line*, Europhysics Letters **85** (2009) 25001.

[28] B. Shivamoggi, *Current-sheet formation near a hyperbolic magnetic neutral line*, Springer Lecture Notes in Physics **571** (2001) 381-388.

[29] B. Sonnerup, *Solar System Plasma Physics*, North-Holland (1979).

[30] P. Sweet, *Electromagnetic Phenomena in Cosmic Physics*, Cambridge University Press (1985).

[31] T. Terasawa, *Hall current effect on tearing mode instability*, Geophysical Research Letters **10** (1983) 475-478.