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Supporting A Standards-based Teaching And Learning Environment A Case Study Of An Expert Middle School Mathematics Teacher

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SUPPORTING A STANDARDS-BASED TEACHING AND LEARNING ENVIRONMENT: A CASE STUDY OF AN EXPERT MIDDLE SCHOOL MATHEMATICS TEACHER

by

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ABSTRACT

Although it has been more than 20 years since the publication of *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989) and 10 years since the second version of standards, *Principles and Standards for School Mathematics* (NCTM, 2000), the research underlines the lack of essential practices for standards-based teaching (Franke, Kazemi, & Battey, 2007). The literature also emphasizes the importance of planning in standards-based teaching, although few studies focus on the direct planning of the teacher (Kilpatrick, Swafford, & Findell, 2001). The aim of the current study was to conduct a case study to extract the planning and classroom practices of an expert seventh grade mathematics teacher. The extracted practices were interpreted using the teaching-in-context theory which is based on the beliefs, goals, and knowledge of the teacher. The case study was conducted in a design experiment environment where the instructional sequence was revised based on the classroom instruction.

The data were collected through different resources including videotapes of classroom sessions, teacher notes, students' artifacts, audiotapes of daily teacher interviews, weekly teacher meetings and classroom small groups in five weeks. Transcripts were used to observe the action patterns of the teacher during both planning and classroom practices. By triangulating the data, planning practices were separated into five categories: preparation, reflection, anticipation, assessment, and revision. These practices were interrelated in an environment of collaboration. Classroom practices also were categorized into five groups, namely creating and sustaining social norms, facilitating genuine mathematical discourse, supporting the development of sociomathematical norms, capitalizing on students' imagery to create inscriptions and notation, and developing small groups as communities of learners. Similar to the planning practices, these were also highly interrelated with social norms playing a key role in application of all other practices.

The results showed that the expert teacher used a diverse set of practices with each practice comprised of multiple actions to create and sustain a standards-based environment. The results also indicated that standards-based teaching requires a rich and connected body of knowledge about students, curriculum, content, and literature. It was found that the depth of the teacher's knowledge allowed her to develop practices that were consistent with her beliefs and goals. Finally, the planning and classroom practices were found to be highly interrelated. While effective planning practices facilitated the application of standards-based teaching, the classroom teaching practices equipped the teacher with the data necessary to perform effective planning practices.

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TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

LIST OF ACRONYMS/ABBREVIATIONS

IRB	Instructional Review Board
HLT	Hypothetical Learning Trajectory
RME	Realistic Mathematics Education

CHAPTER ONE: INTRODUCTION

A standards-based teaching and learning environment gives students the opportunity to develop robust understanding of mathematics by encouraging them to participate in mathematical discussions, ask mathematical questions, make conjectures about solving problems, and listen to mathematical arguments (National Council of Teachers of Mathematics [NCTM], 1989, 2000). Although the *Principles and Standards for School Mathematics* (NCTM) empowers teachers to create classrooms that facilitate students' mathematical inquiry, the challenges that need to be overcome for its successful implementation are numerous (Hufferd-Ackles, Fuson, & Sherin, 2004; Smith, 1996; Spillane & Zeuli, 1999). This study was a case study investigation of how an expert middle school mathematics teacher sustained effective instruction that was consistent with the tenets of NCTM standards documents (1989, 2000). The focus of the study was to extract and analyze the planning and classroom practices employed by the teacher, which played a crucial role in establishing a productive standards-based environment. The integers topic was chosen for analysis as this task and setting might pose various challenges both for the teacher and the students such as the concept of negative numbers and the arithmetic operations that can be applied on them (Ball, 1993).

Significance of Study and Research Questions

The successful implementation of mathematics education reform requires that teachers make significant changes in traditional teaching practices and develop a standards-based teaching and learning environment (NCTM, 2000). According to the reform movements, the student is not seen as a sole recipient that acquires knowledge directly from the teacher without questioning but as a learner who makes sense of mathematics by conjecturing, arguing, proving and discussing the ideas with his or her peers (Smith, 1996). However, creating and sustaining an effective standards-based environment is not trivial. The teacher needs to behave as a facilitator to create an environment that supports students to construct their own mathematical understanding rather than act as a knowledge source who only shows the procedures to get to the right answer (NCTM, 1991). Providing information by collectively making sense with the students rather than through direct instruction is a less clearly known method of instruction for many teachers. One of the important reasons for this is their past experience in traditional classrooms, which gives them little insight for teaching in a standards-based classroom (Ball, 1994; Goldsmith & Schifter, 1997; Schoenfeld, 1992).

Sometimes this kind of environment is mistakenly interpreted as a type of instruction where teachers need to withhold information and ideas from students and simply let them explore. However, students cannot be expected to discover everything. A teacher may need to show the commonly used words and symbols in mathematics, present an alternative method that has not been presented by a student, or restate what students explain in an organized and clear language (Hiebert, 2003). Teachers may also have misconceptions regarding the proper use of manipulatives and collaborative groups. Teachers may incorporate these tools without supporting students' understanding (NCTM, 2000). For example, many times manipulatives are given to students and the teacher tells them how to use the tools. However, students may have difficulty connecting or seeing the relationships between the manipulatives and the mathematics they are meant to embody since students do not have the same knowledge as the teacher (Cobb, Yackel, & Wood, 1992). The teacher needs to develop the mathematical understanding of students as they use manipulatives rather than just tell the students a relationship between the tool and the mathematics. Similarly, in small groups, students may not know what kind of questions to ask their friends or how to work collaboratively. Teachers also need to support students in small groups to create effective conversations and questions rather than just physically create the groups and leave students to work (Ball, 1990; Cohen, 1990; Cohen, 1994).

Even when teachers have an accurate understanding of the goals of the standards-based instruction, they often struggle with determining which practices they need to employ for successfully implementing them in their classrooms (Hufferd-Ackles, et al., 2004). This study aims to shed light on these problems by extracting and analyzing the planning and classroom practices of an expert middle school mathematics teacher who teaches consistent with the tenets of the NCTM *Standards* and facilitates students' mathematical inquiry.

Overview of the Study

Classroom practices can be defined as the pattern of routines and social interactions that take place between the teacher and the students and among the students themselves (Saxe, 1999). Planning practices, on the other hand, are comprised of the offline activities of the teacher as she analyzes the content that will be presented and determines its most important points (Ball, 1993). These practices may include the analysis of classroom lessons including planning for how to introduce the key ideas, how to choose the appropriate tasks that will achieve the desired objectives, and how to create assessments that adequately measure them. The practices that lie at the heart of a standards-based mathematical environment can be summarized as tasks, discourse, environment and analysis (NCTM, 1991).

Even when a teacher is knowledgeable about these practices, creating and sustaining a standards-based classroom is a difficult task which brings additional responsibilities (Ball, 1993; Franke, Kazemi, & Battey, 2007; Lampert, 2001). The teacher has to know how to create a good plan, how to revisit the plan and make adjustments based on the realities of the classroom, and how to carry out the plan despite the various challenges that may be faced in the classroom.

Furthermore, being aware of different types of practices may not be enough for applying them effectively. For instance, a teacher might recognize the importance of small group activities to create a productive standards-based environment, but she might not know what her role should be, or how to meaningfully engage her students in small groups.

Reform efforts emphasize the teacher's role as a facilitator who proactively supports students' learning (NCTM, 1989, 2000). The teacher is an active orchestrator that supports the development of mathematical practices as well as students' individual activities (Ball, 1993; Cobb, Wood, & Yackel, 1993; Lampert, 1990). The teacher needs to pay attention to the mathematical discussion in the classroom and think about how these conversations can be used in further mathematical constructions (Ball, 1993). Cobb et al. (1991) describes the teacher's role in creating and sustaining an environment that facilitates students' mathematical inquiry as:

> highlighting conflicts between alternative interpretations or solutions, helping students develop productive small group collaborative relationships, facilitating mathematical dialogue between students, implicitly legitimizing selected aspects of contributions to a discussion in light of their potential fruitfulness for further mathematical constructions, re-describing students' explanations in more sophisticated terms that are none the less comprehensible to students, and guiding the development of taken-as-shared[1] interpretations when particular representational systems are established (p. 7).

Several studies have aimed at understanding the teacher's role in standards-based mathematics education. These studies can be grouped into three main categories. In the first category, the teacher-researcher analyzes her own teaching and shares her experiences that she

[1] Members of the community have no direct access to other's understanding but may achieve the sense that some aspects of the understanding are shared (Cobb et al., 1992).

gained in the course of the study. A prominent example of this type of research was conducted by Lampert (2001), where she studied a year-long instruction of a 5th grade mathematics classroom and collected her experiences, insights, and ideas in her book “Teaching Problems and the Problems of Teaching”. Another example is Simon (1995), where he analyzed specific sessions (in which he was the teacher) of a prospective teacher education program aimed to increase preservice teachers’ mathematical knowledge and foster their views of mathematics teaching and learning consistent with reform standards. This type of research where one focuses on his or her own instruction is also called the first-person perspective (Ball, 2000).

In the second category, a research team supports and analyzes a teacher as the teacher tries to create a standards-based classroom atmosphere. The research team, in collaboration with the teacher, tries to identify the challenges faced, and develop strategies that can be useful in overcoming those challenges. Constant reflection and discussion of the teaching practice creates a supportive environment for the teacher where she can grow and learn (Cobb et al., 1991; Dixon, Andreasen, & Stephan, 2009; McClain, 1995).

In the third category, a third-person observer-researcher examines the teaching practices, decision making process, and interactions of the teacher with the students without exerting any influence on the teacher. Here, the teacher being examined can either be an expert or a non-expert teacher depending on the goals of the research (Schoenfeld, 1998; Schoenfeld, Minstrell, & van Zee, 2000). This type of research is usually conducted as a case study as the researcher merely observes and studies a teacher without affecting her natural teaching style.

The current study can be considered as a combination of the second and the third categories since it attempted to unravel the planning and classroom practices of an expert middle school mathematics teacher with the researcher being an observer in the classroom and a participant during the teacher meetings. The key differences of the current study from the earlier studies were that: (1) The current study focused on the implementation of a teaching and learning environment that supports students' mathematical inquiry for middle school whereas the earlier work mainly focused on elementary school. (2) The current study attempted to unravel both planning and classroom practices, and their relationship to each other, whereas the previous work predominantly investigated the classroom practices. (3) The current study investigated how the teacher's practices were affected by her beliefs, goals, and knowledge during the instruction of an entire unit, while the previous studies focused on short teaching intervals such as a part of a single classroom session.

Research Focus

The framework that guided this study was developed by the Teacher Model Group at Berkeley. The aim of this theory is to explain how and why teachers do what they do in the classroom. The model describes teachers' moment-to-moment decisions and actions based on their beliefs, goals, and knowledge (Schoenfeld, 1998).

This case study was conducted as part of a design research to investigate the expert teacher's natural teaching style during the planning and classroom practices she employed (Merriam, 2009). The analyses of the data were based on the grounded theory approach which was designed by Glaser and Strauss (1967). Grounded theory is a method for exploring hypotheses, concepts and propositions directly from the data rather than using prior assumptions. The current study used the constant comparative method, which involved simultaneously coding and analyzing data in order to develop concepts that represent the phenomena (Taylor & Bogdan, 1984).

The literature emphasizes the need for studies that elaborate on the core practices that are essential for successful implementation of a mathematical environment consistent with the tenets of the NCTM *Standards* (Franke, Kazemi, & Battey, 2007). This study aimed to address these issues through the following research questions:

1. What are the planning and classroom practices of an expert middle school mathematics teacher to sustain a standards-based environment?
2. What are the teacher's beliefs, goals, and knowledge that underlie the development of these practices?
3. What is the nature of the relationship between classroom and planning practices?

By answering these questions the researcher aimed to provide a detailed perspective into what it takes to teach consistent with the NCTM *Standards*. In the following chapter, studies that addressed standards-based teaching and the challenges teachers face when teaching in such an environment are explained in detail. Additionally, the importance of teachers' decision-making processes in order to cope with these challenges and the differences between novice and expert teachers' practices are summarized.

CHAPTER TWO: LITERATURE REVIEW

A major effort for setting comprehensive goals for students to learn mathematics with understanding was undertaken by NCTM in 1989 with the release of *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989). The core of the document was the development of mathematical power for all students including learning of mathematical value, becoming confident in the ability of doing mathematics, becoming problem-solvers, and learning to communicate and reason mathematically. Mathematics instruction would no longer focus on heavy memorization and drill procedures, but rather on conceptual understanding built on students' current knowledge (Goldsmith & Shifter, 1997). Since this movement required major changes in teachers' and students' practices, it is also known as the "reform movements" in teaching and learning mathematics. The standards documents emphasized the teacher's main role as a facilitator who develops students' mathematical reasoning, rather than a presenter that only shows the procedures for solving mathematical problems. NCTM later published *Professional Standards for Teaching Mathematics* (here after referred to as *Teaching Standards*) in 1991 that elaborated the purpose of standards-based teaching as:

> selecting mathematical tasks to engage students' interest and intellect; providing opportunities to deepen [students'] understanding of the mathematics being studied and its' applications, orchestrating classroom discourse in ways that promote the investigation and the growth of ideas; seeking and helping students seek, connection to previous and developing knowledge; and guiding individual, small group and whole class work (NCTM, 1991, p. 1).

Here it is important to note that the reforms do not prescribe a specific way of teaching but outline and illustrate the goals of the standards that teachers can use as a guide. Following the standards for teaching mathematics, NCTM published *Assessment Standards for School Mathematics* in 1995. These three documents (NCTM, 1989, 1991, 1995) were revised and resulted in the *Principles and Standards for School Mathematics* (here after referred to as "*Principles and Standards*") which was released in April 2000. This document refined the curricular aspects of previous documents, emphasizing five content standards (number, algebra, geometry, measurement, and data analysis and probability) highly related with five process standards: problem solving, reasoning and proof, connections, communications, and representations. The document also described the particular features of high quality mathematics education under six main categories (equity, curriculum, teaching, learning, assessment, and technology). Additionally, the NCTM *Curriculum Focal Points* were published as a guideline that supports three different big ideas for each grade level in grades Pre-K through 8 with several supporting ideas (NCTM, 2006). The intent of the Focal points was to emphasize the importance of teaching in a manner where students can explore the mathematical ideas deeply so that the same content would not have to be repeated every year. NCTM *Standards* documents (NCTM, 1989, 1991, 2000, here after referred as "NCTM *Standards*") underlined the importance of teaching and learning mathematics with deep and flexible understanding .

Principles and Standards state that, "Students must learn mathematics with understanding, actively building new knowledge from experience and prior knowledge" (NCTM, 2000, p.20). School experiences of students should support mathematics by engaging them in exploration of mathematical situations, communication of the ideas during this exploration, as well as modification and validation of those ideas in the classroom. Mathematics should not be understood as a body of knowledge that can be mastered only by some students. All students should be given the opportunity to learn mathematics meaningfully. Students need to experience mathematics as an authentic activity (Lampert, 1990). If students see mathematics as an activity rather than a series of procedural computations, they will take an active role in making conjectures and responding to the others' ideas (NCTM, 2000).

To accomplish these goals, teachers have a central role in changing the ways of teaching and learning mathematics (NCTM, 1989, 2000). The standards-based (or reform-based) teaching can be described as the kind of teaching that is supported by the NCTM *Standards* or reform movements (NCTM, 1989, 1991, 2000). The core of the standards-based teaching is to believe that "learning occurs as students actively assimilate new information and experiences and construct their own meanings . . . students do not learn simply a subset of what they have been shown" (NCTM, 1991, p. 2). The NCTM *Standards* endorse that classrooms should be organized as mathematical communities where students can work together in order to make sense of mathematics; rely on logic and mathematical evidence for verification; connect mathematical ideas; and make conjectures, invent, and solve problems. Thus, as Goldsmith and Shifter (1997) put it, in order to support the NCTM *Standards* "teachers, together with their students, [need to] create a culture of mathematical inquiry aimed at developing deep and flexible understanding of the domain" (p. 20).

However, many times teaching aligned with the tenets of the NCTM *Standards* are applied superficially only based on supporting the physical environment, giving problem-solving tasks or discourse with less attention to students' understanding of the mathematical content (NCTM, 2000). Standards-based teaching is more than only acquiring some instructional techniques. Teachers need to reflect on their beliefs about teaching and learning mathematics such as by focusing on the ways that students' mathematical understandings develop or by analyzing their own practice (NCTM, 1991).

In accordance with the NCTM *Standards*, many mathematics educators also argued that students should be given the opportunity to actively explore mathematical concepts by focusing on reasoning (e.g. why it works) and building on their personal knowledge (Ball, 1993; Cobb et al., 1993; Goldsmith & Schifter, 1997; Fennema et al., 1993; Lampert, 1991). In order to make sense of those concepts individually, students should be given opportunities to represent their ideas, make conjectures, collaborate with other students, and give explanations and arguments (NCTM, 1989; 2000). Studies show that when students give explanations or make conjectures, they elaborate, clarify, and reorganize their thinking. During the disagreements with others, they re-think and re-evaluate their ideas to support conceptual understanding (Ball, 1993; Cobb, 1988; Lampert, 1990).

Teachers' perspectives about mathematical proficiency also have a profound effect on the ways in which they apply the reforms supported by the NCTM *Standards*. The definition of *mathematical proficiency* presented in the consensus document *Adding It Up* addresses five interrelated strands: conceptual understanding such as comprehension of mathematical concepts; procedural fluency such as carrying out procedures flexibly; strategic competence such as the ability to formulate mathematics; adaptive reasoning such as explanation and justification; and productive disposition such as viewing mathematics as a sensible, useful, and worthwhile subject (Kilpartick, Swafford, & Findell, 2001). These strands do not develop in a linear order but with an interaction that goes back and forth among them.

There have been many studies that have provided essential contributions to understanding the teacher's practices that are aligned with the NCTM *Standards*. However, while some of the descriptions of practices were extracted from a single lesson in a teacher's classroom (Arcavi, Kessel, Meira, & Smith, 1998; Schoenfeld, Minstrell, & Van Zee, 2000; Silver & Smith, 1996);

some of them were derived from longitudinal studies that focused only on the elementary grades and were written from the classroom teacher's perspective rather than an outside observer (Ball, 1991; Lampert, 2001). It is also important to note that many of these studies only focused on classroom practices. The aim of this study was to identify a middle school expert teacher's routines of practices from the data that include detailed descriptions of classroom interactions, teacher meetings and notes, and one-on-one interviews with the teacher. Thus, the aim was to extract the details about the teacher's practices such as how she facilitated classroom conversations or how she prepared for the classroom. The study not only aimed to extract the teacher's practices but also her rationale for implementing those practices.

In the following sections, first practices that are consistent with the NCTM *Standards* are discussed. Second, the challenges that teachers face during the application of these practices are described. This is followed by a discussion of teachers' decision making processes and a model that aims to explain how and why teachers make those decisions. Next, the differences between expert and novice teachers' practices are investigated. Finally, co-teaching is described since the expert teacher collaborated with a special education teacher during the instruction.

Practices

In this section, the practices of a teacher who teaches consistent with the tenets of the NCTM *Standards* are organized under four areas: tasks, discourse, environment, and analysis. Tasks can be defined as vehicles used in developing students' understanding; discourse is the way teachers and students participate in the classroom; environment is social and physical characteristics within which learning occurs, and analysis means systematic reflection. These are the core practices that shape the mathematical classroom addressed in the *Teaching Standards* (NCTM, 1991) and in the *Principles and Standards* (NCTM, 2000).

Classroom practices can be defined as the pattern of routines and social interactions that take place between the teacher and the students and among the students themselves (Saxe, 1999). Planning practices can be defined as the pattern of routines between the teacher and her colleagues or within the teacher herself alone (Clark & Yinger, 1987). Pattern of routines stands for the “core activities (within mathematical domain and appropriate grade levels) that could and should occur regularly in the teaching of mathematics…such routines could become the central hub for teachers’ practice and should become the study of classroom practice” (Franke et al., 2007, p. 249). The roles of teachers during these practices will be defined in the next section.

Tasks

Tasks can be defined as “projects, questions, problems, constructions, applications and exercises in which students engage… provide the intellectual contexts for students’ mathematical development” (NCTM, 1991, p. 20). In standards-based teaching, teachers should engage students with the tasks whose solutions are not known in advance and the tasks that give rise to genuine curiosity for the students. These tasks can be connected to the real-world experiences of students as well as arise from the contexts of pure mathematics problems. The way a teacher presents the task is as important as the task itself. The tasks should not be seen only as practice problems that students solve in order to be proficient in applying a set of discrete procedures in order; rather they should be seen as activities designed to develop students’ mathematical understanding. In other words, the tasks should be supported with standards-based instruction instead of traditional instruction (Goldsmith & Shifter, 1997; Hiebert &Wearne, 1993). Here it is important to note that tasks should be aimed at improving both conceptual knowledge and procedural skills. However, several previous studies point out that when tasks are primarily aimed at improving conceptual understanding, they also help development of procedural skills while the opposite is not always true (Rittle-Johnson & Alibali, 1999; Rittle-Johnson, Siegler & Alibali, 2001).

As emphasized above, the way the tasks are presented in the classroom is very important. Even a traditional problem can be set up in a way that supports students' abilities to explain their answers using different strategies such as pictures or words. In other words, the teacher needs to problematize the mathematics with which students engage (Hiebert et al., 1996; Spillane & Zeuli, 1999). This means posing problems for which students do not know the answers or strategies in advance but are within students' reach, and allowing them to struggle with the problems to find the solutions and finally to examine the different solution methods in the classroom. The research shows that understanding can be best supported by engaging students with challenging tasks and allowing them to struggle (Hiebert, 2003). Providing students opportunities to struggle does not mean making mathematics extra difficult for them. Instead, it means giving them opportunities to see mathematics problems as real life problems that can be solved by different methods. Smith (2000) found that although one third of the U.S. teachers presented problems that may give students opportunity to develop deep mathematical understanding, they generally step in and solve the problems too quickly without giving enough time for students to work on the problems.

Examining different types of solutions is also very important to promote students' understanding. During the solution of the problem students should be allowed to use their own methods but be aware that they need to search for better ones. The discussion of the different methods also gives the students opportunity to evaluate the solutions that are different from their own. The teacher needs to support a comfortable environment where the students discuss different solutions but the solutions that have the greatest mathematical advantage win popularity among the classroom community (Hiebert, 2003). Providing information at the right time is another essential point to support students' mathematical understanding. While presenting too much information can prevent students from engaging in the problems, presenting too little information may make them feel frustrated (Dewey, 1933; Hiebert et al., 1996, Hiebert, 2003). Using open-ended tasks might help students explore their solutions that are described above and also give the teacher opportunity to support mathematical understanding without taking over the process of thinking for them.

The structures of the tasks are very important to support students' exploration and reinvention of mathematical ideas. Cognitively demanding tasks give the teacher and the students an opportunity to share the ideas, compare different strategies, make conjectures and generalize (Silver & Smith, 1996). The tasks should be open-ended in design to encourage students to find a solution by using strategies and different representations and they should be revisited by the teacher to create new activities based on the students' performance in the previous class (Martino & Maher, 1999). The types of questions as well as timing during the applications of the tasks are also crucial for the cognitive development of the students.

The instruction in this study involved supporting students' understanding of integer concepts and operations. Although the aim was to focus on teaching rather than the topic, integers is one of the topics that plays an important role in algebra, which is seen as a gateway to higher mathematics (NAEP, 2008). The concept of integers traditionally has not been an easy concept for children or adults. History shows that integers had not been fully comprehended by all mathematicians until only a few hundred years ago about the middle of the 18th century (Hefendendehl-Hebeker, 1991).

Many studies have investigated integer concepts to support students' understanding. These studies used different models including a micro-world, games, abacuses, charges, and debt and assets. The model that used a micro-world included a number line and a turtle aimed at conceptualizing of integers as operations, adding of integers as the composition of transformations and negation as unary operations upon integers and integers expressions (Thompson & Dreyfus, 1988). Some other studies used games as a model in teaching integers concept. One of these models was designed to use black (positive) and red (negative) cards in order to record students' progress during the game. For example, adding one red being equivalent to subtracting one black (Liebeck, 1990). Another model used a disco game concept with an abacus that had two wires and beads and was designed based on realistic mathematics education theory. The aim of the game was to control each disco gate in order to keep track of the number of people going in and out by using two wires with yellow beads going out and blue beads for going in (Linchevski, 1999). Similar to the abacus model, Battista (1983) used a model based on collections of electromagnetic charges. In this model two colors (positive-negative) of poker chips and two transparent jars were used. For example, while the white chips were used for positive charges, red chips were used as negative charges and the collection of charges were to be found by the students. Finally, some researchers used debt (negative) and asset (positive) concept to support calculations with integers (Semadeni, 1984). Many of these models are discussed by Streefland (1996) in order to show different aspects of teaching and learning negative numbers. These models were used to create different tasks to teach integers concepts and operations.

In this study the teacher created an instructional sequence for integers by using the Realistic Mathematics Education (RME) theory developed at the Freudenthal Institute

(Gravemeijer, 1994; Streefland, 1991). RME considers teaching and learning of mathematics as both a social and individual activity, where students not only learn as they work on problems individually but also as they engage in fruitful mathematical conversations (Cobb & McClain, 2001). An essential component of RME is that the instruction should be experientially real where students engage in personally meaningful activities (Gravemeijer, 1994). RME is based on the idea that mathematics is a human activity. Thus, students should be given opportunity to see the world through mathematics, which is also called mathematizing (Freudenthal, 1973). This way, students can naturally wonder about problems and find solutions starting from their informal knowledge. By supporting the instructional design with appropriate tasks, the teacher can give students the opportunity to build new knowledge from their experiences and previous knowledge, and help them translate their knowledge from informal to formal. The instructional sequence in this study was based on students' experiences about assets (positive), debts (negative) and net worth concepts where they can make sense of negative numbers and operations with them (Stephan, 2009).

In summary, the instruction can be supported with tasks that are experientially real to students where they can use their informal knowledge, use different strategies, and connect those strategies with the core mathematical ideas. On the other hand, tasks can be supported by the discourse that will direct students' attention to different ways of viewing the tasks and their explanations during the discussion of problems (Franke et. al., 2007; Hiebert & Wearne, 1993).

Discourse

Supporting a productive discourse is another important element of standards-based teaching. Discourse can be defined as "…the ways of representing, thinking, talking, agreeing and disagreeing that teachers and students use to engage in tasks" (NCTM, 1991, p. 20).

Traditionally, the discourse in U.S classrooms mainly includes emphasis on algorithms and the right answers; talking about mathematics is not common in many classrooms (Kilpatrick et al., 2001). The discourse in traditional mathematics classes is shaped by an Initiation-Response-Evaluation (IRE) sequence. The teacher initiates the conversation by asking questions that reveal the answer rather than guiding the students' reasoning to find a solution. Then the teacher focuses only on the answer of the problem instead of discussing the different strategies the students used and evaluating whether and why they are correct or not (Franke et al., 2007; Spillane & Zeuli, 1999). In contrast, standards-based teaching emphasizes the teacher's role in discourse as a facilitator that develops students' mathematical understanding rather than just an instructor that delivers information. The NCTM *Standards* emphasize the teachers' essential role in the discourse by stating that "teachers must [also] decide what aspects of a task to highlight, how to organize and orchestrate the work of students, what questions to ask to challenge those with varied levels of expertise, and how to support students without taking over the process of thinking for them and thus eliminating the challenge" (NCTM, 2000, p.19).

Silver and Smith (1996) also define the role of the teacher in discourse that supports standards-based teaching as posing questions and tasks that elicit students' reasoning, making students challenge their thinking; listening to the students' ideas carefully; asking students to explain and justify their answers; deciding how to connect students' ideas with mathematical notations; deciding when to clarify an issue, when to make students struggle with that issue, or when to provide information; monitoring students' participation and deciding when to make each student participate in the classroom discourse.

Discourse can be categorized in two groups: univocal and dialogic discourse (Knuth & Pressini, 2001). Univocal discourse can be characterized as a way the speaker intends to convey the ideas without considering the listener's ideas; once the intended meaning is conveyed, the goal is achieved. In this situation, the teacher focuses on whether the students understood her ideas rather than trying to understand the students' ideas. However, in the dialogic discourse the conversation occurs as a give and take between the speaker and the listener. Here, the teacher intends to incorporate the students' ideas into the discourse in order to use them as a tool to support a meaningful conversation. The dialogic discourse is the type of discourse that is supported by standards-based teaching since the core of the reforms is to teach mathematical tasks with intellectual integrity and give value to students' ideas (NCTM, 1989, 2000).

In standards-based teaching, students also need to know how to talk about mathematical ideas as well as think and act mathematically in different contexts (Rittenhouse, 1998). This way they can develop their mathematical literacy and fluency. The teacher's main role is to support a productive discourse environment. Since the students are not familiar with this environment, the teacher can be seen as the bridge between the expert and the novice. The teacher as an expert can "slow down the action" so that the novices can have a chance to better understand what is happening. Rittenhouse (1998) defines the expert teacher's role as to "step into and out" during the discourse. For example, a teacher may step into a conversation with students by listening to them and asking questions related to their answers and step out to talk about the classroom norms that the teacher wishes the classroom community to establish. Lampert (1990) defines the arguments' path during the discourse as a zig-zag that starts from making conjectures and results in proving or disproving them with different examinations. Since the mathematics develops with going back and forth between conscious guessing and proving, the path is zig-zag rather than linear.

Another important role of the teacher during discourse is “revoicing someone’s talk” (Franke et. al., 2007). The teacher has an important role in making students’ thinking clear to other students. The teacher needs to repeat, expand, rephrase or report what students say in order to clarify and advance the ideas by using mathematical vocabulary (O’Connor, 1998). Revoicing is also important to encourage students to develop mathematical terms and appreciate their activities that contribute to mathematical discourse.

Revoicing also can be used to shift the level of the classroom discourse by the teacher. For example, the teacher can restate students’ words clearly in order to help all students to understand mathematical ideas and reflect on them. As the students reflect on the mathematical ideas the level of the conversation may change. Here it is important to note that individual students have an important role in the development of the discourse that supports the shift of activity from individual to collective. As individual students participate in classroom social norms[2], students adapt to each other’s and the teacher’s mathematical activity and vice versa. However, mathematical understanding of others can only be taken-as-shared as the members of the community have no direct access to other’s understanding but may achieve the sense that some aspects of the understanding are shared (Cobb, Yackel, & Wood, 1992). This shift from individual activity to collective activity that occurs during discourse can be called reflective discourse or mathematizing discourse. During reflective discourse, actions and processes are transformed into conceptual mathematical objects. For example, when students are asked to find the different ways of writing five by adding two numbers, they can first individually come up with different solutions. They can then reflect on their solutions to discover why only some pairs

[2] Social norms can be briefly defined as the patterns and routines that are established by the classroom community. It will be explained in greater detail in the following section.

of numbers work (Cobb, Boufi, McClain, & Whitenack, 1997). Thus, reflective discourse can be supported by giving students the opportunity to reflect on and objectify their prior activity.

Reflective discourse is very important in developing students' mathematical understanding consistent with standards-based teaching, since it supports the relationship between classroom discourse and mathematical development. There are different ways of creating an environment that supports reflective discourse (Cobb et al., 1997). For example, the teacher's questions have an essential role in supporting the shift of the conversation level based on the students' answers and reorganizing what has been done up to that point. Another way to support reflective discourse is to symbolize students' thinking in order to make students' ideas clear to the other students. By supporting communication of students' mathematical ideas with symbolization, the teacher may help students understand each other and be able to continue the discourse in an effective way.

A good instructional design may also help teachers to be successful in promoting reflective discourse. For example, RME instructional theory supports symbolizations starting off from students' informal knowledge and moving to more formal as the instructional sequences progresses (Gravemeijer, 1994). Here it is important to note that the symbolizations used by the teacher do not have to be brought up by the students; they can be presented by the teacher to capture students' ideas. For example, in one of the studies that aimed to help students understand double-based thinking strategies (e.g., 4+4=8 to find 4+6), the teacher introduced a story where each twin has a set of markers with 10 markers in each set. When she asked students to find the total when each twin has nine markers, by internalizing the imagery, a student could find the answer by stating that he knew ten plus ten was twenty thus nine plus nine could not be nineteen since there were two missing. Next, the teacher folded back to context in order to make his explanation clear to other students. Thus, it showed that using a real context where students can imagine helped them mathematize the context and allowed the teacher to fold back to the context as necessary (McClain & Cobb, 1998).

In order to create a discourse where mathematical ideas are explored, the teacher also needs to create a productive environment where ideas can be discussed freely. The following section describes environment that the teacher needs to support in order to teach aligned with the NCTM *Standards*.

Environment

NCTM (1991) defines environment as a setting for learning and states that "it is the unique interplay of intellectual, social and physical characteristics that shape the ways of knowing and working that are encouraged and expected in the classroom" (p. 20). Standards-based teaching does not only support the change in the physical environment of the classroom such as creating small groups or using of manipulatives but it also supports a socio-intellectual environment. Teachers have an essential role in creating a risk-free environment that gives students opportunities to share their ideas, make conjectures, and challenge and evaluate other students' ideas in the classroom. Social norms create a risk-free environment where the students and the teacher focus on reasoning rather than just answer (Stephan & Whitenack, 2003).

Social norms which are a crucial component of standards-based classrooms can be defined as patterns and routines that are established by the classroom community where the patterns become hidden-regularities that guide the classroom members' actions and are taken-for-granted during the interaction in the classroom culture (Wood, 1998). For example, social norms for whole classroom discussion may consist of explaining and justifying solutions, understanding other students' explanations, stating agreement and disagreement, and questioning different solutions in a situation where conflicts can occur (Cobb et al., 1991). The teacher's role is to establish the norms by leading the negotiation and renegotiation of them with her students. Many times students do not explain their ideas or try to make justifications on their own which may cause difficulty in supporting the ongoing discourse in the classroom. Especially the students who are in middle and high schools and have a background with traditional school experiences may pose difficulties in creating such a classroom culture (Silver & Smith, 1996). Thus, the teacher's role in creating and sustaining this environment is even more important in upper grades.

The teacher also needs to guide the students to help them understand what counts as mathematically different in their explanation, what counts as an efficient mathematical solution, what counts as a sophisticated solution or what counts as an acceptable mathematical solution, which collectively can be named as socio-mathematical norms (Yackel & Cobb, 1996). Socio-mathematical norms can also support students' development of intellectual autonomy, the belief that students are responsible for creating their own solution and meaning for mathematics, not relying on an authority to tell them what to do (Kamii, 1982). Mathematical activity in the classroom occurs within the mathematical practices that are established by the community members and they are taken-as-shared by its members. Here classroom mathematical practices refer to content specific ways of reasoning and arguing that become normative through interactions (Stephan & Cobb, 2003). Tools have an important role in establishing classroom mathematical practices as the students' activity with tools support conceptual and symbolic meanings (Stephan, 2003). Tools can be separated into two categories namely signs or semiotic systems such as inscriptions on paper, and technical or instrumental tools such as computers and calculators (Meira, 1998, 2002). These tools can be used to scaffold students' internalization process or reorganize their understanding. The tools become meaningful when students reason using these tools. Here it is important to note that the inscription (symbolization) becomes a signifier when it is used for reasoning. Gravemeijer (2002) explains the difference between inscription and signifier by giving an example that the inscription refers to the actual mark on the paper while the signifier refers to what the person perceives from that mark. Thus, what the mark signifies for the students can be thought as students "imagery" (Gravemeijer, 2002).

Pirie and Kieren (1989) emphasize the importance of supporting imagery with a recursive theory where the effective actions function with the initiation of imagery. The first level of the theory includes images that are based on situation-specific activity which constitutes the initial step for abstraction. The researchers state that "it is the learner who makes this abstraction by recursively building on images based in action" (p. 8). The initial development of students' thinking starts with taking the images that they created before (without creating a new one each time) and directing their thinking in order to make sense of the subsequent task. For example, if students have the image of multiplication as making copy of the given number as many times as the first number (e.g., 4 x 5 as four copies of five), then they can make sense of fractions in the same way; they can imagine 3 ½ x 2/3 as a sum of three copies of 2/3 and a half copy of it. Mathematical development continues as students reason by using or adapting the images' properties and as they engage in more instructional tasks that start to formalize these properties (McClain & Cobb, 1998). By folding back to the initial image students can create new meaning for the objects they are working with and move to more sophisticated levels of thinking. Here it is important to note that students use or transform their initial images to make sense of more complicated ideas. Thus, when those images are imposed from outside or given ready-made to students without regard for students' own images, this may not contribute to the development of their mathematical reasoning (Pirie & Kieran, 1989). Development of imagery helps students create meaning for mathematical activities and adds richness to their mathematical reasoning and understanding (Thompson, 1996). Thus, the teacher has an important responsibility in creating an environment that supports students' imagery.

Studies showed that teachers may consider that they teach consistent with the standards since the students use manipulatives and work in small groups in the classroom. However, the researchers that observed these teachers stated that those teachers' teaching methods were far from the standards (Ball, 1990; Cohen, 1990). One of the reasons for this discrepancy is that many times the manipulatives are given directly to the students and they are asked to make sense of them according to the context. Cobb, Yackel and Wood (1992) describe this situation as a learning paradox where "the assumption that students will inevitably construct the correct internal representation from the materials presented implies that their learning is triggered by the mathematical relationship they are to construct before they have constructed them." (p. 5). For example, Dienes blocks reflect the base-ten number system according to teachers' knowledge but for the students they are just wooden blocks. When students have difficulties seeing what the teachers want them to see, the teachers generally tell them the relationship between the manipulative and the algorithm. But mapping the steps during instruction does not result in understanding or proficiency since the relationship between the algorithm and the blocks is not clear to the students. One way to solve this problem is to create an instructional design that develops a bottom-up approach (e.g., RME) where students' make sense of the tools as they build on their informal knowledge to gain formal knowledge (Gravemeijer & Stephan, 2002; Gravemeijer, 2007).

Small groups can also help create an environment where students make sense of mathematics collaboratively. However, the teacher has an important role in supporting the small groups. Although there have been many studies that investigate students' learning in small groups, there are only a few studies that identify the teacher's role in small groups. Blunk (1998) investigated Lampert's role, who is an expert teacher, in supporting a small group environment. She describes the teacher's role as the following: first, the teacher needs to explain why working in small groups can be useful for the students; second, the teacher should emphasize to the students the importance of practicing their own skills to work with other people. Finally, the teacher should convince the students that she is not the sole resource of knowledge in the classroom.

Yackel, Cobb, and Wood (1991) state that the teacher needs to create cognitive activities that will stimulate small group discussions. This type of activity can result in discussion in the small groups that may create conflicts if the answers of the students do not match. This disequilibrium requires the students to reorganize their activities in order to reach an agreement, which helps them to gain a better conceptual understanding of the task. Here it is important to note that students should have enough time to work on the questions before they interact with other students or the teacher. This time is important for students to conjecture; later they can check their answers with the group members and follow up with the other members of the community during the classroom discussion.

The teacher's role during the classroom practice, which includes supporting the appropriate use of tasks and discourse and establishing and maintaining a positive environment plays an essential role in standards-based mathematics teaching. However, successful implementation of classroom practices also requires the teacher to plan, reflect on and refine her instructional practice. These are explained in the following section.

Analysis

NCTM (1991) defines analysis as "[a] systematic reflection in which teachers engage [in] ongoing monitoring of classroom life [such as] how well the tasks, discourse, and environment foster the development of every student's mathematical literacy and power" (p. 20). Teachers need to reflect on and refine what they and their students are doing during classroom instruction and how their actions impact students' learning (NCTM, 2000). Standards-based teaching also empowers teachers to do analysis not only individually but also by coming together with their colleagues and creating a community of teachers. The importance of collaboration with the other teachers is emphasized by the study of Stigler and Hiebert (1999): "Collaborating with colleagues regularly to observe, analyze, and discuss teaching and students' thinking or to do 'lesson study' is a powerful, yet neglected, form of professional development in American Schools" (as cited in NCTM, 2000, p. 19). In order to observe the process of collaboration, one of the aims of the current study was to extract the practices of the expert teacher's analysis not only when she was working individually but also while she was collaborating with other teachers.

Teacher planning has a significant influence on creating opportunities for students to learn with understanding, covering the content, managing transition from one activity to another, and the general focus of classroom processes (Clark & Yinger, 1987). Studies show that many teachers see planning mainly as the selection of students' learning activities and preparation of the content that will be taught, and rarely the specification of learning objectives (Fuchs, Fuchs, Hamlett, & Stecker, 1991; Peterson, Marx, & Clark, 1978). Reform movements also encourage an essential shift in teacher planning that includes ensuring that students learn mathematics and have a positive disposition toward mathematics; creating tasks that challenge and extend students' ideas; adapting and modifying the activities as they are applied in the classroom; and including both short and long term plans and commenting on students' learning (NCTM, 1991). The importance of designing a lesson plan that elaborates ones' content goals and focuses on the students' anticipated learning is highlighted in *Adding It Up* which reflects the ideas consistent with the tenets of standards-based teaching. The authors also stated that "Planning can profitably be seen as a detailed form of instructional design aimed at reducing the uncertainties of one's practice, centered on the continual adjustment, improvement of instruction, and informed by a close scrutiny of what happens as the lesson unfolds" (Kilpatrick et al., 2001, p. 337).

Simon's (1995) model provides a foundation for the practices that support standards-based teaching. The social-constructivist learning theory guides the model he created. The social constructivist theory, which is also known as the emergent perspective, combines a constructivist psychological perspective with an interactionist social perspective (Cobb & Yackel, 1996). This perspective abandons the idea that teachers can give ready-made knowledge to students and supports that students learn only when they construct the ideas. However, it is not enough to say that students will construct the ideas themselves. The teacher needs to plan in order to support the students' construction of the mathematical ideas that took thousands of years to develop in history and the teacher needs to modify the plan constantly after interaction with students. Simon (1997) examined the teacher's role aligned with this perspective using a cognitive and social theoretical framework. The teacher's role from a cognitive view can be summarized in two parts as identifying student tasks that pose appropriate challenges for students and should result in re-organization of students' ideas and constructing more sophisticated mathematics. The second role of the teacher is to generate reflective tasks that support generalization, abstraction and organization of classroom activities. From the social view, the teacher's role is to see the classroom as a mathematics learning community and support the community to create and sustain the norms and mathematical practices that are taken-as-shared by all community members as well as to orchestrate discussions. Simon's (1995) schematic model explains the critical role of a teacher in developing the students' understanding of mathematics. This model, referred to as the mathematics teaching cycle (MTC) addresses the relationship between the teacher's knowledge, the hypothetical learning trajectory (HLT), and the interactions with students.

An HLT can be defined as the teacher's anticipated path by which students' learning might occur. The generation of an HLT prior to classroom instruction is the main part of planning for the mathematics lessons. As the teacher interacts with the students, both the teacher and students have different experiences. These experiences may lead to the modification of the teacher's goals and knowledge. One characteristic of the MTC is that the teacher predicts how students' thinking and understanding will evolve during the classroom activities. Teachers need to make decisions about the possible mathematical knowledge of their students by using their own knowledge to interpret students' actions and language (Steffe & D'Ambrosio, 1995).

Three aspects of teachers' knowledge can support the creation of an HLT. These are the teacher's sense of students' learning, the teacher's understanding of learning a particular mathematical topic, and the teacher's perspective of the role of a mathematics teacher (Simon, 1997). Although an HLT consists of goals for student learning and the mathematical tasks that will be used to support them, it does not describe a framework for thinking about the learning process, the selection of mathematical tasks, and the role of the tasks in students' learning. To this end, the instructional sequence designed according to the tenets of RME theory can support teachers since it does not provide a ready-made script but offers additional resources for teachers so that they can make their own adaptations according to their students (Gravemeijer, 2007).

Revising the tasks based on the classroom and making adaptations is a highly emphasized but yet neglected practice of the teachers in the United States (Stigler & Hiebert, 1999). When the planning of teachers in the United States was compared with the planning of teachers in Japan (one of the countries where students have the top scores on international assessments), the researchers found significant differences (Stigler, Fernandez, & Yoshida, 1996). For example, while student thinking played an important role in the planning activities of Japanese teachers, American teachers mainly listed the objectives that they planned to address during instruction. The plans of Japanese teachers included four main categories: learning activities on which the teacher expected students to work, expected student reactions during these activities including different solution ways, the time that is planned to apply for each activity, and the guidance that the teacher provided to support students to understand the activities (Stigler et al., 1996). In other words, the teacher's planning can be defined as the teacher's lesson image for the class. A lesson image is the teachers' envisioning of the possibilities related to the lesson that includes the students' knowledge, the anticipated reaction of the students to the planned lesson, students' difficulties, and more (Schoenfeld, 1998). Another important planning difference was that Japanese teachers came together more often to make a trial of the lesson images and to work collaboratively to improve them. This practice is highly recommended by the NCTM *Standards*.

Students' thinking also became the core of many important projects in United States (Carpenter et al., 1989; Franke et al., 1996; Schifter & Fosnot, 1993; Schifter, 1998). The Cognitively Guided Instruction Project (CGI) was one of those projects that bought teachers together and gave them the opportunity to explore about the students' thinking in a specific context (Carpenter et al., 1989; Franke et al., 1996). Another study known as Developing Mathematical Ideas (DMI) also provided different experiences for teachers to develop skills that were consistent with standards-based teaching such as focusing on students' thinking, assessing the mathematical validity of students' ideas, listening to students' mathematical ideas and trying to make sense of them (Schifter & Fosnot, 1993; Schifter, 1998). Thus, using student thinking was one of the most important elements of analysis in the teacher's planning emphasized by researchers as well as the teachers whose students performed top in international evaluations. In order to teach aligned with the NCTM *Standards*, the practice of analyzing the class based on students' understanding and revising it as necessary needs to be an important part of the teacher's practices.

Up to now, the roles of teachers during the practices are summarized in order to achieve the successful implementation of standards-based teaching. These practices are grouped under four categories: tasks, discourse, environment, and analysis. For each group, the teacher's role is described to teach aligned with the NCTM *Standards*. However, it is important to note that applying these practices is not trivial; they pose several challenges for novice as well as expert teachers (Chazan & Ball, 1999). The next section describes some of the most important challenges.

Challenges in Practice

Studies such as Trends in International Mathematics and Science Study (TIMSS) shows that most teachers' instructions in the U.S. reflect the core of traditional practices although teachers believe that they are changing the way they teach. (Hiebert & Stigler, 2000; Hiebert et al., 2005; Stigler & Hiebert, 1997). Many classrooms include Initiation-Response-Evaluation

(IRE) patterns and the teachers focus on the correct answers rather than discussing different strategies. The type of questions teachers generally ask include a single correct answer where the teacher knows the answer and the students are responsible for finding this answer. Students have little opportunity to discuss their ideas and students' incorrect answers are generally left alone without focusing on them while, in fact, discussing those misconceptions can have an important effect on students' conceptual development (Franke et al., 2007). However, changing existing practices has been found to be challenging, it not only requires understanding the new practices supported by the reform movement but also requires actively engaging students with those practices. For example, many times teachers who want to promote student thinking and enhance understanding put students in cooperative groups and use manipulatives (Stigler et al., 1996). However, the studies showed that although the teachers created small groups, they did not change the very essence of their practice (Ball, 1990; Cohen, 1990; Webb, Nember, & Ing, 2006).

In order to create an environment consistent with the NCTM *Standards*, the classroom community needs to establish the norms for expected student behavior, the teacher needs to focus on students' thinking and use different strategies to reflect on students' ideas. Cobb, Wood, Yackel, and McNeal (1992) give an example of a teacher who learned to overcome these challenges as she changed her practice. First, the teacher learned that children may have their own methods; she encouraged children to explain their mathematical ideas and respected students' answers even when they were not correct. However, accepting wrong answers also created another fundamental confliction that she had to overcome. She resolved this conflict by realizing that students actually learn when they understand why their solutions were incorrect rather than when simply presented with the correct solution.

The challenge of applying the practices consistent with standards is not only difficult for novice teachers but also for expert teachers. Chazan and Ball (1999) discussed the challenges that they had while trying not to “tell” when they taught in a high school and elementary school respectively. During the discussion of an algebra task, Chazan had encouraged students to explain their ideas and feel confident during the solution of the problem. But he had wondered if students could reach a common ground, and hesitated about how he should intervene if they could not. During a lesson taught by Ball, students seemed to agree on an incorrect solution although she attempted to induce disequilibrium. She was concerned about students not being able to resolve an unproductive agreement. They address three factors with which the teacher needs to be careful during the discussion: the mathematics under discussion (e.g., if students reach the task meaningfully with some help), the nature and direction of class discussion (e.g., engaging pace), and the social and emotional climate of the class (e.g., discussion can become personally unpleasant).

Chazan and Ball (1999) also discuss what it means for a teacher to “tell”. Many teachers believe that they should not give any information to students in order to be consistent with the reform movements. However, the researchers emphasize that there are other types of telling such as restating a comment that is given by a student, stating in clear language the conclusion that students agreed upon; asking students to make themselves clearer; controlling the focus of discussion; and asking students to sit down, come to the board, etc. In order to resolve some of the concerns with teaching by telling, the researchers also reformulated telling by emphasizing the importance of teacher’s actions where initiating and eliciting goes hand in hand, in terms of relationship of these actions within each other and in terms of exploiting telling to promote conceptual growth (Lobato, Clarke, & Ellis, 2005). For example, the teacher can open classroom discussion by initiating (e. g. she can summarize students' work in an advance level by inserting new information) and then follow this by eliciting (e.g. arrange situations so that students can reflect upon their understanding), and these two actions might form a cyclic pattern.

Smith (1996) states that beliefs such as "telling" or providing the steps clearly with the procedures step-by-step or considering that students only learn from the teacher's demonstration support teachers' sense of efficacy. According to this, students cannot be expected to find the solutions themselves and thus their achievement of mathematics can be attributed to the teacher. They face a dilemma as the teachers feel a sense of reduction in their efficacy as they try to teach without "telling".

Studies showed that many teachers find it challenging to open up their classrooms to mathematical ideas since it is difficult to manage the direction of instruction with students' wrong answers and anticipate where a lesson will go as well as to create instructions for this type of lesson (Sherin, 2002; Silver & Smith, 1996). Finally, the researchers also found that only around 15% of teachers in a study could manage to teach consistent with the reform movement. The rest of them either had difficulty supporting discourse norms although they created tasks to promote problem solving (around 40%), or they could not create effective tasks nor support discourse (around 45%) (Spillane & Zeuli, 1999).

The challenges that are described above, changing existing practices, teaching without telling, anticipating students' answers, adapting the lesson based on students' interaction, can be overcome with a productive decision making process. In the following section the decision making process which is mainly based on teachers' beliefs, goals, and knowledge is elaborated.

Decision Making

Decision making is the basic skill of teaching (Shavelson, 1973). Teachers make a large number of important decisions that can be categorized in two groups: planning (preactive) and teaching (interactive). During planning or the preactive decision making process, teachers decide "what to teach, how they are going to teach, how to organize the classroom, what routines to use, and how to adapt instruction for individuals" (Fennema & Franke, 1992, p. 156). During preactive decision making, teachers also select the materials that they will use, how to assess students' written work, and other activities to reflect on and use as part of their decision making for subsequent classes (Fennema & Franke, 1992; Grouws, 1991).

The process of decision making includes interactive decisions that are made during instruction as the teacher interacts with the whole class, small groups, or individuals. Among many of the choices teachers make decisions "…to modify their plans, to respond to a child in a particular way, to call on a given child for an answer, to reward or reject certain answers, to discipline an unruly child, to encourage a shy child, and to speed up or slow down a lesson" (Fennema & Franke, 1992, p.156). Interactive decisions must generally be made faster and with less information and resources than the decisions that are made during planning. Teachers also need to make decisions for classroom management, classroom organization, time allocation, and previous instruction both during preactive and interactive decision making processes.

One of the teachers' challenges is to make productive decisions in a short time. They may have difficulties making decisions about how to start the discussion and how to support students' thinking once the task is introduced (Schifter, 1996). Another difficulty teachers may have is how to decide on the task selection. Simon (1995) examined his decision making during task selection and revision of the task during his teaching of preservice teachers. He used the mathematics teaching cycle model that includes teacher's knowledge, student assessment and an HLT to explain his decision making process in a classroom that is consistent with the tenets of reform movements. The model emphasizes the importance of the relationship between planning and classroom instruction in order to be effective during the selection and enactment of the task.

There have been many studies that focus on teachers' decision-making processes in traditional classroom settings. In their study, Peterson and Clark (1978), considered a decision as a choice between two or more alternatives. In these decisions the only input was student's behavior. If the student behavior was within tolerance, then the teacher continued. Otherwise, if the teacher had an alternative strategy, she applied it; if not, the teacher continued to do her planned activity. The results of the study showed that many teachers do not have alternative strategies, thus they followed the path as if the student behavior was in tolerance. Following this study, Shavelson and Stern (1981) created an alternative model by including the research based on teacher planning and routines. Similar to the former study, if the students' behavior was in tolerance the teacher continued along her path. If not, the teacher evaluated if there was a routine that could be applied. She applied the routine if she knew of a routine for that particular situation, otherwise she continued with her original route and stored the information for future use. Many of these studies were content-independent where the researchers only focused on the teacher's decisions based on students' behavior rather than students' learning of specific context.

Another study that provides insight into teachers' decision-making processes was conducted by Leinhardt and Greeno (1986). Their study mainly focused on describing the mental structures of skilled teachers and comparing the behaviors of expert and novice teachers. The model specified the skill according to the subject matter knowledge and lesson structure knowledge such as agendas, scripts, and routines. An agenda was described as a dynamic plan that includes overall goals and the lesson activities that can be modified. A script was defined as the teacher's goals and actions for a particular topic, and routines were described as activities that are performed by the teacher and the students frequently. The model emphasized that the teacher starts the class with the goals and related routines that will help her to move smoothly. Additionally, the study showed the importance of teacher's subject matter knowledge in supporting agendas and scripts. However, this model mainly focused on teacher's subject matter and the lesson structure knowledge and gave less attention to teacher's pedagogical knowledge and beliefs in analyzing the decision making process.

Another model was developed by the Teacher Model Group at Berkeley and became known as the teaching-in-context theory after having been applied many times on different teachers that had different teaching styles. This model describes teachers' moment to moment decisions and actions based on their beliefs, goals and, knowledge (Schoenfeld, 1998). In order to understand the teacher's beliefs, goals, and knowledge, one needs to have the teacher's lesson images which include a series of action plans. Action plans can be described as "… a set of actions intended to be taken in order to work toward the achievement of a constellation of currently high priority goals" (Schoenfeld, 1998, p.28). Action plans can typically be found in forms such as scripts, routines, and mini-lectures. The action plans are the backbone of the lesson image and the lesson image is an important component used in the teaching-in-context model. As described earlier, lesson images can be defined as the teacher's envisioned path that is planned to be taken during the related lesson. The teacher's lesson image may include aspects related to her knowledge of the students, the anticipation of students' reactions (e.g. answers, misconceptions, difficulties), and how she will deal with these reactions. As the current study also used this model in explaining the teacher's decisions and practices, the individual components of this model; knowledge, beliefs, and goals, are described in greater detail in the following sections.

Knowledge

A teacher's knowledge is comprised of multiple components. Shulman (1986) distinguishes between subject matter knowledge, pedagogical knowledge and curriculum knowledge. Subject matter knowledge refers to both the content and the structure of the subject. The teacher needs to know what is true as well as why it is true. Pedagogical content knowledge goes beyond the subject matter knowledge by addressing how to teach the subject matter. This includes knowledge of students' mathematical thinking and learning, being able to provide multiple representations of mathematical ideas in ways that students can grasp, and being aware of common student errors and misconceptions. Curriculum knowledge involves awareness of curriculum materials and the knowledge of how topics are arranged within the years. Studies show that in order to teach effectively, teachers need to combine these different types of knowledge together (Ball & Cohen, 1999; Hill, Rowen & Ball, 2005).

Recently, researchers also conceptualized and developed measures of teachers' combined knowledge of content and students (Hill, Ball, & Schiling, 2008). They re-defined mathematical knowledge for teaching and illustrated how it is related to knowledge of content and students. According to this model, mathematical knowledge for teaching is categorized in two groups: subject matter knowledge and pedagogical knowledge. While subject matter knowledge includes common content knowledge, knowledge at the mathematical horizon (i.e. a broad view of mathematical ideas and practices), and specialized content knowledge; pedagogical knowledge includes knowledge of content and students, knowledge of content and teaching, and knowledge of curriculum.

Ernest (1989) also elaborated the categories of the teacher's knowledge in detail such as knowledge of mathematics, knowledge of other subject matter, knowledge of teaching mathematics, knowledge of organization for teaching mathematics (e.g. management and classroom routines), knowledge of the context of teaching (e.g. knowledge of students in the classroom) and knowledge of education (e.g. knowledge acquired from the research).

Despite the variety of knowledge definitions, these knowledge types cannot be separated from each other. A teacher needs to have a "knowledge package" that combines knowing key ideas, sequences for developing these ideas, and connecting them efficiently. In order to use the knowledge package, teachers also need to decompose or "unpack" the knowledge to apply them in the classroom to support students' conceptual understanding (Ma, 1999). Many studies have showed the relationship between teachers' knowledge and their practices. Many of them illustrated the teachers' difficulties during instruction because of the teachers' lack of rich and connected mathematical knowledge (Ball, 1990; Borko et al., 1992; Thompson & Thompson, 1996).

Besides knowledge, teachers' beliefs have an important effect on teachers' practice (Ernest, 1989; Fennema & Franke, 1992). For example, it is possible for teachers to have the same knowledge but teach differently because of their different beliefs about teaching and learning. Next, the effects of beliefs on teachers' practice are discussed.

Beliefs

Beliefs can be defined as the understandings, premises or propositions that are held and thought to be true. Beliefs also can be thought of as one's perspective related with some aspect of the world or as dispositions toward a behavior (Philipp, 2007). Teacher's beliefs can be categorized as mathematics, teaching, and learning beliefs (Ernest, 1989; Thompson, 1992).

Studies show that one of the challenges to applying reform practices is teachers' beliefs about how to do mathematics that are acquired from their school experience through the years of watching, listening to and practicing teaching (Lampert, 1991).

In order to understand a teacher's practice, it is important to know her beliefs and goals as well as her relevant knowledge (Schoenfeld, 1998). Studies show that teacher's actions are highly related with these concepts. For example, although a preservice teacher believed in teaching mathematics with understanding, she could not apply the practice aligned with her beliefs because her lack of a conceptual understanding (Borko et al., 1992). The researchers also showed that practice affects knowledge and beliefs. That is, as a teacher continues to apply new practices her knowledge and beliefs about teaching and learning may change. As such, the relationship is dialectical (Cobb, Yackel & Wood, 1991).

Studies show that there is a strong relationship between beliefs and practice (Thompson, 1992). However, the relationship is not always consistent in that teachers sometimes carry out practices that are not aligned with their beliefs (Cooney, 1985; Thompson, 1984). For example, in a study even though the teacher believed in making mathematics meaningful to students, and thus wanted to teach division of fractions in a way that students can make sense of it, she failed to achieve this because of her lack of a conceptual knowledge base (Borko et al., 1992). These inconsistencies are explained by different studies. One of the important reasons is not having the required knowledge as given in the example above. Raymond (1997) also addresses two points that account for such inconsistencies. First, teachers' beliefs are mostly aligned with their mathematical beliefs rather than teaching and learning beliefs. Second, time constraints, resources, standardized tests, and student behavior can cause teachers' to conduct practices that are discrepant from their beliefs. The inconsistency of the beliefs can also be explained with the levels of beliefs which can be categorized in two groups: surface beliefs and deep beliefs (Kaplan, 1991). While surface beliefs can be more consistent with superficial practices (e.g. tasks and organization planned by the teacher such as creating small groups, using manipulatives), deep beliefs are consistent with pervasive behaviors (e.g. the way the teacher actually communicates with students).

It is important to understand how the beliefs are structured and held in order to explain the teacher's action with the beliefs. Green (1971) structured the beliefs based on the role of evidence in the analysis. According to this, if the beliefs are held without evidence, then it is difficult to modify them with new evidences since the teacher has already made up her mind. However, if the beliefs are held by the evidences, they can be modified and criticized with different reasoning. The study also defined the three different aspects of the belief structure: quasi-logical relationship (e.g. primary and derivative belief), psychological strength (e.g. teacher's commitment to use of tools), and the beliefs isolated from the others (e.g. belief only about mathematics). Build on this study and the other schemes that are used by researchers, Cooney, Shealy and Arvold (1998) categorized the teachers for supporting professional development. In the study, the preservice teachers reflected on their teaching including how they thought about the content as well as how they pedagogically supported students during the course of a four-quarter sequence. At the end of the study, the researchers categorized the teachers into four groups: isolationist, naïve idealist, naïve connectionist, and reflective connectionist. The isolationist refers to the group of teachers whose beliefs are structured in a way that remain separated from the others and are held without evidence. The naïve idealists are the people who absorb what others believe but fail to analyze their own beliefs. Naïve connectionist and reflective connectionist pay attention not only to other peoples' reflection and beliefs but also compare them to their own beliefs. Although, the former struggles to resolve the conflict or differences in beliefs the latter is adept in integrating others' beliefs into his or her own belief system. Cooney (1999) also points out that categorizing teachers based on their belief structures might help to understand the foundation for conceptualizing teachers' ways of knowing which is essential for professional development. For example, by using the conceptualized differences, a facilitator can decide how to encourage reflection and adaptability. Based on the different categories she can find a way to influence and stimulate teachers' reflective thinking about their own beliefs on mathematics, teaching and learning.

The beliefs used in the teaching-in-context model stand for attributed beliefs rather than professed beliefs. Attributed beliefs are extracted from the teacher's actions. These attributions may or may not reveal teacher's professed beliefs. Thus, it would not be correct to claim that these beliefs are exactly the same as the teacher's actual beliefs. However, by collecting as much data as possible and triangulating within it, the attributed beliefs can be very close to the teacher's real beliefs (Schoenfeld, 1998, 2000).

The attributed beliefs or belief bundles are important factors in explaining the teacher's actions. A belief bundle can be described as a collection of beliefs and has two main characteristics: it connects a particular belief with the entire body of beliefs, and the levels of certain beliefs are used to explain the new goals of instruction as well as emergent goals (Aguirre & Steer, 2000). The bundle also connects the different beliefs of the teacher in order to analyze the teacher's actions. For example, a bundle may be composed of beliefs such as mathematics as making-sense, students learn working in groups, and the teacher is not the only source of knowledge in the class and each belief may also create a new bundle with other beliefs such as mathematics as making-sense and students learn by constructing their own understanding.

It is also important to consider teachers' beliefs with the circumstances and constraints in the setting. Many times beliefs are affected by educational priorities such as classroom management or building students' confidence (Hoyle, 1992; Skott, 2001). They are also affected by teachers' goals that change for particular activities or students (Skott, 2001). Next, the effects of goals on practice are described together with their relationship to beliefs.

Goals

Goals can be defined as cognitive constructs that explain what the teacher wants to achieve for the class (Aguirre & Speer, 2000). Often teachers are expected to accomplish complex and even conflicting goals (Clark & Lampert, 1986). For example, although a teacher may want to teach the task with conceptual understanding, the necessity to finish it on time may result in developing a conflicting goal.

As with beliefs and knowledge, goals have an important effect on the prediction of teachers' actions. Different from beliefs, goals are more specialized and targeted to accomplish certain outcomes for different situations. In order to determine one's goals from her actions, one has to have the set of beliefs of that person (Schank & Abelson, 1977). A change in the goals can help researchers to observe the beliefs and investigate the effect of these beliefs on new goals. The formulation of goals can be explained by belief bundles (Aguirre & Speer, 2000). For example; a belief such as "mathematics as making-sense" can be associated with another belief such as "students learn by working in groups" and these two beliefs can be used to explain the formation of a new goal "give enough time for students to work on activity sheet".

The goals of a teacher can be very useful in predicting her actions. However, it is important to note that the goals cannot be considered as fixed ideals that stand without being influenced by the teacher's day- to-day experiences. In fact, the goals are continually modified during interactions with the students (Simon, 1994; Steffe, 1991).

Emergent goals can arise due to unforeseen incidents within or outside of the classroom. Saxe (1991) defined four parameters that have an effect on the emergence of individuals' goals: cultural activities (e.g., mathematical goals that emerge in exchange may be guided by an economic motive); social interactions (e.g., through negotiation and assistance); prior understanding (e.g., the understanding that one brings to bear on practice); and particular sign forms (e.g., the body-parts counting system developed by some tribes) and cultural artifacts (e.g., historically elaborated sign forms).

Similar to the beliefs, the goals used in the teaching-in-context model stand for the attributed goals based on the teacher's actions. Thus, it cannot be claimed that the goals that were stated in the study were exactly the teacher's goals. However, by member checking and triangulating within the data, the attributed goals can be very close to the teacher's real goals.

This study used the teaching-in-context model in order to explain an expert teacher's planning and classroom practices. However, it is also important to know how expert and novice teachers' practices may differ from each other. The next section elaborates this difference in greater detail.

Teaching with Experience

There have been many studies that investigate the differences between expert teachers' and novice teachers' practices (Leinhardt & Greeno, 1986; Livingston & Borko, 1989, 1990) and decision making (Westerman, 1991). In these studies, the expert teacher is defined based on cognitive aspects such as teachers' knowledge and skills. Experts' knowledge is defined as better integrated, more accessible, and organized in specific ways such as ideas are connected and the relationship between the ideas can be clearly specified. Studies also show that expert teachers' knowledge has a more refined hierarchical structure that includes not only the procedural rules but also the interrelationship of the procedures (Leinhardt & Smith, 1985). Experts also build upon the instructional topics that are introduced in previous lessons and display considerable sophistication of the subject matter during instruction.

The essential elements that create expert mathematics lessons are defined in three parts: "rich agendas (lesson images), consistent but flexible lesson structures, and explanations that meet the goals of clarifying concepts and procedures and having students learn and understand them" (Leinhardt, 1989, p.52). Expert teachers have complex cognitive skills that allow creating plans and making quick online decisions. In other words they have organized actions which are called schemata. They also have a large list of actions that they apply fluently. These activities are referred to as routines and they are developed by the teacher as a result of the organization of schemata for the action that is performed (Leinhardt & Greeno, 1986). Another difference between novice and expert teachers are the latter's use of well-practiced routines. The routines help the teacher to collect information and use that information during the instruction (e.g., focusing on problems that students had difficulties with in their homework assignments). Expert teachers' routines are also very flexible in that they can be adapted to different situations.

The act of reflection has a central place in the practices of an expert teacher. Schon (1983, 1987) defines reflection in two categories: reflection-in-action and reflection-on-action. While the former refers to thinking about the action while it is performed, the latter refers to reflecting on the action after it is performed. These concepts emphasize the idea of reflective thinking that might lead to an adaptive teacher who pays attention to the context. This way, the teacher can transform her knowledge and apply it in dynamic situations. Schon (1983) argues that over time as the person reflects *on* the actions that she performed, she acquires knowledge and starts to be able to reflect *in* the action by asking herself what she is doing at that point and what her purpose in doing that is. In other words, the experience of reflecting on action allows the teacher to become adept at reflecting in action. Although experience is very important to reflect-in-action, knowing about more about students' thinking as well as the mathematical content are also crucial components of reflecting in action (Cooney et al., 1998).

There are also some studies that define an expert teacher only based on the number of years the teacher has spent in the classroom (Burden, 1981). These studies show that teacher's level of teaching characteristics tends to improve over the years.

The teacher analyzed in this study is considered to be an expert teacher primarily based on her cognitive aspects of teaching. The teacher's students have shown consistent growth scores over time, and she has been teaching for more than five years using standards-based practices. Although the time in service was not a main requirement, the assumption that the expert teacher had developed effective routines in the teaching and planning practices during this time was an important factor in choosing the teacher. Another important consideration was the teacher's active involvement in research related with supporting students' thinking during the mathematical instruction.

Co-teaching

Although the aim of the current study was not to investigate the effect of co-teaching in a standards-based classroom, it is important to note that the expert teacher collaborated with a special education resource teacher since three students in the classroom had mild learning disabilities. The two teachers used the method of co-teaching in their instruction. The origin of the term co-teaching is based on cooperative teaching where the general and special educator provide direct educational programming to all students within a general setting (Bauwens, Hourcade, & Friend, 1989). Cook and Friend (1995) defined co-teaching as “two or more professionals deliver substantive instruction to a diverse or blended group of students in a single physical space” (p. 2).

Three essential elements of co-teaching consist of cooperating in the planning, the instruction of students and assessment (Murawski & Dieker, 2004). For example, during planning the teachers need to come together and talk about the instructional methods that might be efficient and effective in helping all of the students meet the academic standards. During these meetings while the special education teacher can provide goals and objectives for any individualized education program (IEP), the general teacher can describe the content goals. The teachers, then together can plan for how to introduce the content in order to help all students in learning. Second, during the instruction both of the teachers need to actively engage with students. The studies showed that peer tutoring and cooperative learning (e.g., small groups) can be very effective when used with co-teaching, since the co-teacher can perform similar practices as the general teacher such as effective observing and data collection (Dieker & Murawski, 2003). Finally, the teachers need to come together to assess the students’ progress and development to revise the instruction and decide whether some students need more individual accommodations.

The practices outlined above were followed by the expert teacher and the co-teacher in the current study. Prior to the beginning of the instruction they held meetings together with other teachers who were to teach the same unit to define the instructional sequence, the goals of instruction and the projected learning route. During the instruction, the expert teacher took a leading role while the co-teacher provided the necessary support through collecting data and moving around the classroom to ensure that the lesson was meeting the needs of all students. For the current study, the actions of the co-teacher were consistent with the actions of the expert teacher in managing a standards-based classroom. In other words, both teachers adhered to the core practices that are required to teach in a standards-based environment. It is important to note that the current study focused only on the expert teacher's practices; finding the co-teacher's practices were outside the scope of this study.

Summary

In summary, it is important to understand the key practices of standards-based teaching in order to use them effectively. Although, there have been many studies that describe the goals of standards and the teacher's role in the class, the studies that investigate both classroom and planning practice for a specific context are rare. Moreover, many of the existing studies only observed the classroom interactions without interviewing the teacher for long periods to understand the purpose of the teacher's actions. The current study not only investigated the expert teacher practices inside the classroom but also the teacher's role during the planning stage while she interacted with other teachers.

CHAPTER THREE: METHODOLOGY

In this chapter, first the characteristics of design research and case study are discussed and the rationale for applying the case study methodology to investigate an expert teacher's practices is provided. Then the sampling methods for selecting the teacher and the instructional sequence are outlined. This is followed by a discussion of the data sources used and the methods for collecting them. In the subsequent sections, the interpretive framework and the procedures that were used for data analysis are explained. Finally, trustworthiness of the study are discussed.

Design Research

The main characteristic of design research can be defined as testing and revising the conjectures inherent in initial designs in order to develop theories within the context the phenomena occurs (Cobb, 2003). Design research is highly interventionist in that it allows the researcher to develop improvements in a natural setting. It also has both prospective and reflective characteristics that create a cyclic process. It is prospective, as it is implemented with a hypothetical learning trajectory considering potential pathways of learning and development, and it is reflective since during the experiment the conjectures are tested and if these are refuted new ones are generated and tested again (Cobb et al., 2003).

In fact, this cyclic process is compatible with Simon's (1995) mathematics teaching cycle (Cobb, 2000). As defined earlier, a mathematics teaching cycle includes learning goals for students, planned instructional activities, and an envisioned learning process in which the teacher anticipates students' reasoning during the enactment of the activities. The teacher revises her trajectory based on students' current understanding in the class. Thus, the cycle emphasizes the importance of anticipation as well as revision in teaching practice aligned with design research.

Design experiments can be conducted in different settings such as classroom experiments, preservice teacher development experiences, inservice developmental studies, one-to-one experiments, and school district restructuring experiments (Cobb et al., 2003). The design experiment in the current study was a classroom experiment in which an instructional design was tested and revised on a daily basis. Typically, this type of experiment is conducted by a research team that collaborates with a teacher whose primary responsibility is implementing the instruction (Cobb, 2000).

Different from many of the other design-based research studies, the aim of the research team in this study was to improve teaching practice rather than to test a theory. The design research team included the expert teacher (Dr. Stephan), the co-teacher (Mrs. Taylor), two other seventh grade teachers (Mr. Jones and Mrs. Wilson) that were teaching the same unit and the researcher of this study. The research team met three times before the instruction began and every week after the beginning of the instruction. Additionally, the expert teacher and the researcher met after every classroom session. The expert teacher who had previous experience in conducting design research served as the team leader during these meetings. The teacher created a hypothetical learning trajectory (HLT) and based on that she also designed an instructional sequence on integer concepts and operations (Stephan, 2009). In the meetings, the expert teacher led discussions on the HLT as well as the activities and also encouraged the teachers to make decisions based on students' understanding. The HLT played an important role during the conversation about students' understanding and the big ideas for the integers topic.

The main focus of the teachers and the researcher during these meetings was student learning. The team anticipated the different strategies students might invent, talk about the imagery and inscriptions that might support students' understanding, and conjectured about the possible topics that might evolve during the enactment of the activities in the classroom. The team also worked through the tasks of the sequence to anticipate how students might reason and to clarify the intent of the activities. These teacher meetings where the team anticipated and revised the conjectures became an important part of the current study. Thus, the design research supported an environment where the researcher could conduct a case study to investigate the cyclic process that consists of classroom and planning practices.

Case Study

Case studies have been commonly used in education research where the aim is to examine a specific phenomenon in a bounded system (Merriam, 2009). Case studies are generally preferred when the research focus is to extract the holistic and meaningful characteristics of real-life events (Yin, 1984). If the researcher selects the case because it is intrinsically interesting and the researcher wants to have a full understanding of the phenomena, then these studies can also be characterized as particularistic or intrinsic case studies (Merriam, 2009; Stake, 1995). In this situation, the investigator studies the case to learn more about the case itself rather than learning about a general problem. This study can be characterized as a particularistic case study since the aim is to investigate an expert teacher’s planning and classroom practices rather than investigating it for general understanding.

The overall intent of this study can be defined as an interpretive case study. An interpretive case study includes thick and rich description of a phenomenon similar to a descriptive case study. However, in an interpretive case study, the data can be used to develop conceptual categories or to test the theoretical assumptions that were held before the data collection (Merriam, 2009). In this study, the data were analyzed by taking apart observations, interviews, and field notes in order to describe the teacher’s practices and interpret them based on her beliefs, goals, and knowledge. The coding process was guided by two main analytic processes namely making comparisons and asking questions. This analytic procedure is called the constant comparative method which constitutes an integral part of the grounded theory. In grounded theory, hypotheses, concepts, and propositions are directly derived from the data rather than using prior assumptions (Glaser & Strauss, 1967). Although grounded theory research aims to discover categories among the variables and find relationships by solely using the data itself, ignoring theories related to the study can result in undesirable results. However, it is important to note that, depending too much on the existing theories may constrain the researcher's creativity and her ability to find new categories from the data. One way to use the literature is to extend an existing theory and investigate how it can be applied to new or different situations (Strauss & Corbin, 1990).

In this study, the analysis primarily used the teaching-in-context theory (Schoenfeld, 1998). This theory, developed by the Teacher Model Group at Berkley, attempts to understand a teacher's decisions based on beliefs, goals, and knowledge. The current study applied the general framework of teaching-in-context but extended it over a period of an entire instructional unit rather than applying it to a single lesson.

Case studies may include different disciplinary perspectives. A study that focuses on the culture of a classroom or a group of students can be described as an ethnographic case study. Merriam (2009) states that, "While the culture has been variously defined, [ethnographic study] essentially refers to beliefs, values, and attitudes that structure the behavior patterns of a specific group of people" (p. 27). An ethnographic study can help readers to understand how to behave in a cultural setting appropriately (Bogdan & Biklen, 1982). This study can be considered as an ethnographic case study, since it aimed to extract an expert teacher's practices based on her beliefs, goals, and knowledge in a standards-based classroom culture. Thus, ethnographic tools were used to collect data such as classroom observations, interviews, and field notes. Although ethnographic research can be analyzed in many ways, the analytic approach of grounded theory is particularly suitable for ethnographic studies (Charmaz & Mitchell, 2001).

In summary, this study was part of a larger design research and in order to understand the nature of the expert teacher's practices, a particularistic and interpretative ethnographic case study was conducted. The data were analyzed by using grounded theory to test and extend the teaching-in-context theory.

Sampling

Teacher selection

The expert teacher, Stephan, holds a doctorate degree in Mathematics Education from Vanderbilt University, and she actively continues to carry out research on student learning. She has been successfully designing instruction that is consistent with the tenets of the NCTM *Standards* and that facilitates students' mathematical inquiry. The effectiveness of her instruction is reflected by the high achievements of her students on standardized tests and on the teaching awards she has received.

The researcher of the current study observed Stephan for one full academic year as she taught a seventh grade mathematics class in a public middle school in Central Florida. During this period, many visitors including teachers of different levels, principals, and district coordinators visited her class to observe how she created and maintained a standards-based, effective teaching and learning environment.

Stephan follows an instructional theory that is based on Realistic Mathematics Education (RME) developed at the Freudenthal Institute (Gravemeijer, 1994; Streefland, 1991). RME considers teaching and learning of mathematics as both a social and individual activity, where students not only learn as they individually work on problems but also as they engage in fruitful mathematical conversations (Cobb & McClain, 2001).

Stephan believes in a socio-constructivist learning theory also known as the emergent perspective. The emergent perspective combines the social and psychological aspects of students' learning (Cobb & Yackel, 1996). While the social perspective is an interactionist view of communal and collective processes (Bauersfeld, Krummheuer, & Voigt, 1988), the psychological perspective is a constructivist view of the individual student's (or the teacher's) activity as they participate in and contribute to the development of these communal processes (von Glasersfeld, 1992). Stephan's belief in the socio-constructivist learning theory has a profound effect on her teaching. She is not the only authority in the classroom; she considers the classroom as a community where everyone has an opportunity to learn and share. Thus, one of the important reasons for observing Stephan was to understand what it means to teach every day in a way that is described in the socio-constructivist learning theory.

Instructional sampling

This study focused on the practices of an expert teacher as she prepared and carried out her instruction on integers. The concept of Integers were chosen as a suitable setting that poses various challenges both for the teacher and the students such as the concept of negative numbers and the arithmetic operations that can be applied on them (Ball, 1993).

Linchevski and Williams (1999) state that "traditionally negative numbers introduce a new aspect into the study of mathematics: for the first time reasoning in an algebraic frame of reference seems to be required. While counting numbers are constructed as abstracted from real objects and quantities…. negative numbers and properties of these numbers are traditionally given meaning through formal mathematical reasoning" (p. 134). The obstacles of understanding integers had continued approximately 1500 years in mathematics history (Hefendendehl-Hebeker, 1991). Similarly students generally have difficulties in understanding integers and the operations on them.

The expert teacher also faced challenges to find an appropriate context within which to teach integers in her seventh grade classroom (Stephan, 2009). Prior to this study there were many models including number lines, temperature, and two colored chips that were introduced in the textbooks. Although these models seemed to help students remember how to perform the mathematical operations, the reasoning behind them might not always have been clear. In order to support her instruction, and help students to reinvent the rules of integer operations, the expert teacher designed an instructional sequence by using realistic mathematics education theory (Gravemeijer, 1994; Stephan et al., 2003).

Thus the reasons to focus on the integers topic can be summarized as: 1) Integers has been a challenging task during the history of mathematics that present unique difficulties. 2) It may be difficult to make sense of negative numbers and the operations on them using real life examples, as opposed to other topics such as natural numbers. 3) The teacher being examined designed the instructional sequence using RME, and it has been used to teach integers for the past two years. This gave the researcher a unique opportunity to observe how an instructional sequence consistent with the NCTM *Standards* is implemented, how it can be revised as needed, and how it can be effectively applied in a real classroom environment.

Setting

Classroom setting

This study was conducted in a public middle school (grades 6-8) in Central Florida. Among the approximately 1500 total students, the percent of students eligible for free lunch was around 10% while the percent of students eligible for reduced lunch was 7%. When the experiment was conducted, the teacher was teaching two seventh grade classrooms during the second and third periods. Among these, the third period was chosen because the teacher was available right after the class period to reflect on the classroom session.

The classroom environment was designed by Stephan to support students' learning during instruction. The class had a white board, projector, and one computer. Thus the teacher was able to project the activity sheets on the board and the students could directly work on the activity projected on the white-board as they explained the solutions of problems. This was a useful time saver for the teacher.

There were twenty one students in the classroom including fourteen boys and seven girls. Three of the boys had identified learning disabilities. Only one student did not want to participate in the audio and video recordings. There was also an experienced co-teacher, Taylor, who specialized in special education to support Stephan during the instruction. Taylor was also a teacher with 10 years of teaching special education experience. Stephan and Taylor had co-taught in an inclusion setting for three years.

The desks were arranged in groups of three to allow the students to work collaboratively. The students had different responsibilities that changed every day in the small groups: policeman, author, and leader. The leader's job was to tell other students in the group how they need to share the task that was given by the teacher. The policeman's job was to make sure that everyone participated in the solution of the task. The author's job was to write down the results that were found by the group members before submitting the paper to the teacher. The responsibilities of the students in each small group were rotated every day by the teacher.

At the back of the class there were two big labels on the wall: conjectures and theories. Under these labels the students' conjectures that were made during the class were posted. Once the students proved the conjectures and all the community members agreed, these conjectures were moved to the theories section of the wall.

Next to the white board, the teacher kept a word wall where she posted the words that were discussed in the classroom related to the tasks such as "positive", "negative", "asset", and "debt". Also the teacher had a list that included the norms that she expected the classroom community to establish such as "listening to each other" and "explain your solutions". She also had many books related to the reforms that support students' learning and manipulatives such as unifix cubes and geometric shapes.

Realistic mathematics education

In this study, the teacher used an instructional theory based on Realistic Mathematics Education (RME) developed at the Freudenthal Institute (Gravemeijer, 1994; Streefland, 1991) to design the integer unit. The roots of RME are based on Freudenthal's idea of mathematics as a human activity (Freudenthal, 1973). He stated that people need to see mathematics "not as a closed system, but rather as an activity, the process of mathematizing reality and if possible even that of mathematizing mathematics" (Freudenthal, 1968, p. 7). In order to mathematize, "one sees, or organizes and interprets the world through and with mathematical models. Like language, these models often begin simply as representations of situations, or problems, by learners" (Fosnot & Dolk, 2005, p. 189).

RME suggests three design heuristics that must be considered in designing a conjectured learning path with a set of activities that will support this learning path (Gravemeijer, 2004). These heuristics are: guided reinvention, didactical phenomenology, and emergent model which are aligned with the heuristics used by Stephan in designing the instructional sequence. These heuristics are defined in the following sections.

Reinforcing with a real context

One of the important heuristics of RME is that the instructional sequence should start with an experientially real context (Gravemeijer, 1994). Students do not necessarily have to experience the activity themselves but they should be able to imagine the activity. This experience allows students to use their informal knowledge of mathematics as a starting point in developing progressively more formal mathematical reasoning and supports reinventing formal mathematical activities. Thus, the learning route should be designed in a way that allows students to invent the intended mathematics. For example, the instructional sequence in this study started with a context that involves determining a person’s financial net worth. The focus of the beginning activities was to understand how debts and assets were quantitatively composed in a net worth statement and how integer operations worked in this context (Stephan, 2009). In other words, the students were given an opportunity to mathematize the net worth statements in order to understand integers and the operations on them. These operations were primarily comprised of addition and subtraction, but the sequence involved multiplication toward the end as well.

Accounting in the abstract

The instructional design should support the students’ reasoning from concrete to abstract. In order to develop this type of instructional design, the designer needs to envision a path that the class may follow (Gravemeijer, 2004). The instructional sequence can be supported with the knowledge of history of mathematics and the prior research that illustrate how students' may develop their thinking for that topic. Based on these, the teacher can create a hypothetical learning path. This path should be based on didactical phenomenology which emphasizes the importance of progressive mathematization. Rather than concretizing abstract mathematical knowledge, didactical phenomenology gives students opportunity to create mental objects in order to mathematize the phenomena that is introduced as a starting point. For example, students can symbolize taking away a debt of $300 as –(-$300) and explain their reasoning as the transaction which is the minus sign stands for take away actions and the negative sign in the parenthesis refer to debt (Stephan, 2009).

Using symbols

The third heuristic focuses on transitioning students' informal activity to a more formal one. To achieve this transition instructional activities should be designed to encourage students to model their informal mathematical activity and more progressively to modeling their formal mathematical activity which is also called emergent modeling (Gravemeijer, 2004). Gravemeijer and Stephan (2002) state that, "The students are expected to develop formal mathematics by way of mathematizing their informal mathematical activities" (p.148). During the transition from informal to formal, the designer should support students' reasoning with certain tools such as physical devices, inscriptions, and symbols that can be shared by students to explain their mathematical reasoning. For example, as students reasoned with the vertical number line as a model of their operations with assets and debts, the model evolved to become a model for formal integer operations. Students were encouraged to write the word problem, *Linda has a net worth of $1500 and she adds a debt of $600*, in terms of symbols such as $1500 + (-$600). Later as the instructional sequence progressively moved to promote more abstract reasoning, students were only given numbers and symbols such as -3 + (-7) - (-5) (Stephan, 2009).

These three heuristics, reinforcing with a real context, accounting in the abstract, and using symbols, of RME were used by the expert teacher to develop an HLT and instructional sequence as explained in the following sections.

Hypothetical learning trajectory

Simon (1995) used the term Hypothetical Learning Trajectory (HLT) as a theoretical construct that refers to “the teacher’s prediction as the path by which learning might proceed” (p. 135). An HLT has three main properties: a learning goal, learning activities, and a hypothetical learning process. A teaching cycle is developed from the combination of an HLT, students’ interactions, and the teacher’s knowledge. According to Simon’s model, the teacher anticipates what kind of mental activities students may engage in during participation in the envisioned instructional activities and considers how the activities align with the teacher’s end goals. Since the teacher envisions these activities, she can assess the actual learning of the students and compare it with the HLT only in the classroom environment. In this process, the co-teacher can play an important role by sharing experiences from the previous years and observing to what extent the envisioned learning route is materialized by the current instruction. The new experience and the insight that the teacher gains in the classroom result in the modification of the HLT for subsequent activities. The modification of the HLT, its application, and the ensuing revision as a result of the change in the teacher’s knowledge creates a teaching cycle (Simon, 1995). The HLT is consistent with the emergent perspective in that it offers a basis for elaboration of the teaching cycle from a socio-constructivist perspective (Clement & Sarama, 2004; Gravemeijer, 2007).

The teacher's role is to help students transform their current mathematical activity to a more sophisticated one and in order to achieve this she can use an HLT as part of planning. RME theory can be used to create HLTs for longer term tasks (Gravemeijer & Stephan, 2002). However, designing long-term learning trajectories can be more difficult than designing short-term trajectories.

The expert teacher created the HLT by using a task that was designed according to the RME theory. The HLT was separated into five categories: tools, imagery, activity/taken-as-shared interest, possible topics of mathematical discourse and possible gestures (Rasmussen, Stephan, & Allen, 2004). Tools were created to support students' cognitive development. They can be categorized under two groups: signs or semiotic systems such as inscriptions on paper and technical tools or instrumental tools such as computers or calculators (Meira, 1998, 2002). In this study, tools refer to inscriptions. Although the term inscription primarily means marks on the paper, it is generally used to indicate any sign that is on the computer screen, paper, or other materials. When the tools are used in order to make sense of a context, they often stand for a signifier rather than an inscription. The difference between a signifier and an inscription is that the former refers to a person's perception of a mark, which can also be thought as the students' "imagery" (Gravemeijer, 2002). The important point is to make sure that the symbolizations (inscriptions) emerge from students' reasoning and have significance for them. In other words, mathematical symbols and models should be developed in a bottom-up manner. RME theory is one way of supporting the new tools in order to help students reinvent mathematics. Thus, the tools in the HLT help students solve the problems by using their informal knowledge and the knowledge from the preceding activities (Gravemeijer & Stephan, 2002).

The word "taken-as-shared" in the HLT indicates that the members of the classroom community achieve a sense of some aspect of the knowledge is shared but one never knows that this knowledge is exactly the same for everybody since they do not have direct access to each other's understanding. For example, as the students found the net worth of two people they began to understand that debts reduce the net worth while assets increase it. Thus, the meaning of assets and debts as having positive effect and negative effect were taken-as-shared by the students. That is, members of the community had no direct access to other's understanding but achieved the sense that some aspects of the knowledge were shared (Cobb et al., 1992). The category named "taken-as-shared interests" stands for the hypothetical interests (e.g. students might be interested in which famous person has more net worth) that can emerge as students' reason with tools and imagery.

The teacher also included possible topics of mathematical discourse that might emerge during the use of the tools. For example, during the comparison of the two people's net worth, Stephan focused on different solutions of the comparison methods. Finally, possible gesturing was also an important part of the HLT since gestures, which are often externalization of mental imagery may support students' development of imagery. For example, illustrating going up and down with arms might help students visualize actions on the vertical number line in their mind. The HLT became a guide for the teacher in designing the instructional sequence. As the instructional sequence was applied in the classroom, the teacher first revised the HLT based on students' learning and then the activities that composed the instructional sequence.

Data Collection

The data collected during the instruction of the integers topic included interviews with the teacher, audio and video tapes of the classroom sessions, field notes, teacher notes, research meetings, and a collection of students' work. The university's Institutional Review Board (IRB) approved all aspects of this study (see Appendix A). The teachers in the study as well as the students are referred to by pseudonyms. Since the expert teacher was also a researcher in this study, she is referred to by her real name.

Schedule

Data collection started approximately two weeks before teaching of the integers topic with meetings that included Stephan, three other seventh grade mathematics teachers, and the researcher of the current study. One of the teachers was a co-teacher (specialized in special education) who assisted Stephan during the instruction and the other two teachers had been preparing to teach integers using the same instructional sequence with their own classes. In the first meeting, Stephan introduced the HLT and explained how it could be used to help them during the instruction. She also explained the difficulties that emerged during the development of integers in the history of mathematics based on the related research she studied. The main aims of the meeting were to understand the big ideas of the unit, share experiences from the previous year, and talk about the following week's classroom practices based on students' envisioned thinking and potential misconceptions. The teachers and the researcher met once a week formally and then informally throughout the week. All teacher meetings were audio taped in order to document any changes in the plan of instruction together with the rationale for such changes. These meetings continued to take place on a weekly basis until the end of the study.

Once the integer instruction started in March, 2009, the data collection continued for five weeks with the class observations through video and audio tapes, field notes taken by the researcher, daily interviews with the teacher, and the weekly meetings discussed above. The advisor of the researcher also observed several of the classroom sessions to take field notes and extracted questions regarding the teacher's decisions. Additionally, the researcher watched each day's video on the same day to extract any remaining questions pertaining to Stephan's decisions. The researcher then asked Stephan the questions during the interviews. The interviews were not time constrained, which allowed the teacher to explain her thoughts and comments thoroughly regarding the observed class and the next day's plan. For example, if the teacher introduced a novel way of symbolizing students' thinking on the board, the teacher was asked how she decided to introduce the particular inscription.

Classroom sessions

Data from the classroom sessions were collected in four ways: video recordings, field notes, audio recordings of the students in small groups, and students' work. All sessions were videotaped by using two cameras: one camera was placed at the back of the classroom and focused mainly on the teacher, and the other was placed in the front to record both the teacher's and students' activities.

All small groups were audio taped during their activities in order to observe how the teacher interacted with the students in small groups. Additionally, each student's work was collected every day at the end of the class. These data were later used to analyze the effects of students' verbal and written activities on the teacher's decision making processes. Field notes were taken by the researcher in each session (and once a week by the researcher's advisor) to document the teacher's practices and extract questions to be asked in the subsequent interviews.

Formal and informal meetings

Stephan, the other three teachers, and the researcher met every week to discuss that week's experiences and make plans for the following weeks. This formed a small learning community where the teachers shared the difficulties faced by their students and reflected on what they could do to address them. The teachers also discussed the progress they made in the instructional sequence and any challenges they faced in applying the sequence. These meetings generally resulted in plans to address these problems. In addition to these formal meetings, informal discussions took place between the teachers as questions arose related to the instructional sequence. These meetings were used to describe Stephan's planning practices, and document their relationship with the classroom practices.

Formal and informal interviews

After each classroom session, Stephan was interviewed and these interviews were audio taped. During the interviews, she was asked to interpret the lesson that she had just taught. She also explained her plan and the big ideas for the following day. Additionally, the researcher asked questions about how some of her specific decisions contributed to an environment that supported students' mathematical understanding. The intent here was to understand the teacher's beliefs, goals, and knowledge and connect them with the specific events from the planning and classroom. Since the teacher also explained her plan for the following day in these interviews, the data were used to extract the planning practices. Finally, the teacher and the researcher had informal meetings before the class if the teacher decided to make some changes in her plans. However, these meetings occurred rarely, as the teacher generally applied the plan from the previous day.

The data collected through the classroom interactions, interviews, field notes and teacher meetings were used to describe the nature of the expert teacher's planning and classroom practices, explore the relationship between them based on the teacher's beliefs, goals, and knowledge. Table 1 below summarizes the relationship between the data and the research questions.

Table 1: The research questions and the related data set

Research Questions	Related Data Set
What are the planning and classroom practices of an expert middle school mathematics teacher to sustain a standards-based environment?	Classroom Sessions/ Field Notes Teachers' Meetings Teacher Interviews
What are the teacher's beliefs, goals, and knowledge that underlie the development of these practices?	Teacher Interviews/ Notes, HLT, Instructional Sequence Classroom Sessions/ Teachers' Meetings
What is the nature of the relationship between classroom and planning practices?	Classroom Sessions/ Teachers' Meetings, Teacher, Interviews/ Notes, HLT, Instructional Sequence

Interpretative Framework

Theory of teaching-in-context

The main framework that guided this study was developed by the Teacher Model Group at Berkeley (Schoenfeld, 1998). The aim of this theory is to explain how and why teachers do what they do in the classroom. The model describes teachers' moment-to-moment decisions and actions based on their knowledge, beliefs, and goals. When a teacher enters a classroom, she carries a substantial body of knowledge which may include knowledge about the students, the school environment, and the content. She also carries a set of beliefs about how teaching should be done. Based on these, the teacher develops goals that she wants to achieve through executing a series of actions. It is these actions and decisions that the theory of teaching-in-contexts attempts to explain using the teacher's knowledge, beliefs, and goals (Schoenfeld, 1998, Aguirre & Speer, 2000).

Knowledge can be defined as the set of intellectual resources that the teacher uses in any given situation. This includes knowledge of the students, of the context, and of the content. Shulman (1986) categorizes the knowledge under three categories, namely subject matter knowledge, pedagogical content knowledge, and curricular knowledge. Subject matter content knowledge refers to both the content of the subject and the structure of the subject. Teachers need to know what is true as well as why it is true. Pedagogical content knowledge includes having different representations available for specific topics, understanding the difficulties that students can have, and being aware of misconceptions, preconceptions, and conceptions of students. Curricular knowledge includes the knowledge of how topics are arranged within the years and the knowledge of curriculum materials for specific topics. Teachers need to have multiple types of knowledge in order to deal with the complexity of the classroom.

A teacher's beliefs play an important role in determining the levels of priorities of her goals. There have been different definitions of beliefs. In this study, beliefs refer to personal philosophies that include conceptions and values that reflect one's world view (often implicitly held) such as students learn by listening or students learn by constructing their own understanding (Ernest, 1989; Thompson, 1992; Aguirre & Speer, 2000). Beliefs can encompass several areas such as beliefs about students' understanding and learning, beliefs about the individual or group work for the class to engage in, and the beliefs about the importance of formulas and explanations. However, they can be described mainly under three categories: beliefs about teaching, beliefs about learning, and beliefs about mathematics. Teachers' beliefs can be shaped by the goals that they want to achieve as well as the knowledge that they bring to support those beliefs. The effect of knowledge on beliefs, however, depends on the structure of the beliefs as well as whether they are held with evidence (Cooney, 1999). The structure of the beliefs illustrate how the beliefs are held: whether they are shaped the results of people's telling or rooted in rationality. For example, one can have a belief because an authority stated it to be so or because of rational evidence based on one's own experiences or logical deductions.

The beliefs used in the teaching-in-context model stand for the *attributed* beliefs rather than professed beliefs (Aguirre & Speer, 2000). Attributed beliefs are inferred from the teacher's actions. These attributions may or may not reveal a teacher's professed beliefs. Thus, it would not be correct to claim that these beliefs are exactly the same as the teacher's actual beliefs. However, by triangulating as much as possible within the data using the interviews, teacher's notes, teacher meetings and class sessions the attributed beliefs can be very close to the teacher's real beliefs (Schoenfeld, 1998).

Goals are "cognitive constructs that describe what the teacher wants to accomplish" (Aguirre & Steer, 2000, p. 332). Decisions are also shaped by the goals of the teacher that are held at a high level at any given moment. Similar to beliefs, the goals are attributed to the teacher's actions as it is often not possible to know the goals and beliefs precisely. Goals can be classified as overarching goals as well as content specific goals or student related goals. Goals may be pre-determined or emergent (Schoenfeld, 1998). Pre-determined goals are set by the teacher as a result of her current knowledge and beliefs, whereas emergent goals can arise due to unforeseen incidents within or outside of the classroom. For example, a teacher might decide to introduce a new mathematical vocabulary based on the students' answers that are discussed in the classroom even though she did not plan it before the classroom.

As with beliefs and knowledge, goals are an important factor in explaining the teacher's practices. Many times the goals can be explained with belief bundles. For example, a belief can be associated with another belief in the formation of a new goal. Although it is impossible to observe all the beliefs that the teacher has during her decision making, a shift in goals may be helpful in understanding the teacher's beliefs (Aguirre & Speer, 2000). For example, in one of the studies the teacher's goal at the beginning of the class was to allow students to make presentations on their own since she believed that students learn by presenting their ideas. During the instruction since the students could not present correctly, the teacher decided to directly guide the students since she thought that some students might get confused during this process. To this end, the shift in the teacher's goal from asking students to present their ideas to directly presenting the solution herself revealed that her belief that students learn by presenting their ideas was superficial (Aguirre & Speer, 2000).

Thus, the theory brings together several pieces from the literature including teacher knowledge (Fennema & Franke, 1992; Shulman, 1986), beliefs (Calderhead, 1996; Thompson, 1992), goals (Clark & Peterson, 1986; Saxe, 1991; Schank & Abelson) and a descriptive model of the teaching process (Borko & Putnam, 1996). It is important to note that the theory mainly explains the reasoning behind the actions from the teacher's perspective since it is based on the teacher's knowledge, beliefs, and goals.

Data Analysis

In this study, data analyses were performed using the constant comparative method of grounded theory (Glaser & Strauss, 1967). Grounded theory can be described as detecting relationships, discovering theories and concepts, and drawing conclusions directly from the data. Grounded theory can be used to verify, amend, and extend the existing theories in light of new situations and data (Strauss & Corbin, 1990).

A grounded theory includes categories, hypotheses, and the conceptual links among the categories. The main strategy of grounded theory is the constant comparative analysis which starts with a particular incidence from the interviews, field notes, and documents that can be compared with other incidents from the data. These comparisons help to create tentative categories and later on these categories are compared within themselves. Finally, a theory can be generated by the constant comparison of data within and between other categories, and reformulating hypotheses by constantly confronting the theory with negative cases (Glaser, 1965).

The data analysis of this study was categorized under two parts: planning and classroom practices. Planning practices are comprised of the offline activities of the teacher as she carefully analyzes the content that will be presented and determines its most important points (Ball, 1993). Classroom practices, on the other hand, can be defined as the patterns of routines and social interactions that take place between the teacher and the students and among the students themselves (Saxe, 1999).

Prior to analyzing the teacher's planning and classroom practices, an initial list of the teacher's relevant knowledge, beliefs, and goals were extracted by using interviews with the teacher, the teacher notes, the HLT, and the teacher meetings. This list was expanded during the analysis in light of new data supporting different knowledge, beliefs, and goals.

Planning practices were extracted from the teacher's actions during the teacher meetings and the daily planning sessions. Classroom practices, on the other hand, were extracted from the teacher's actions in the classroom using the transcripts of the videotapes as well as audiotapes from small groups. Both planning and classroom practices were interpreted using the teacher's knowledge, beliefs, and goals.

In summary, during the analysis of the data first the teaching-in-context theory was used in order to interpret the teacher's actions based on her beliefs, goals, and knowledge. These elements were extracted from the teacher meetings, interviews, classroom sessions, and teacher notes using the constant comparative method. During the grouping of data into sub-categories (i.e. beliefs into learning, teaching, and mathematics; goals into overarching, content, and local;

knowledge into content, pedagogical, and curriculum) the teaching-in-context theory as well as the literature was used. To identify the teacher's actions that are part of her planning and classroom practices, grounded theory was used to compare and contrast the teacher's behaviors during planning and teaching. Once the actions were extracted, they were grouped under different practices which themselves are created from the logical relationship between the actions, the teacher notes and the literature. Finally, all of the identified actions are interpreted using the teaching-in-context theory. The table below illustrates the summary of analyses that were applied in this study:

Table 2: Phases of the analysis and the methods and data sources for each phase

Phase	**Source/Method**
Determination of using beliefs, goals, and knowledge as the fundamental element of the analysis	Teaching-in-context theory
Finding belief, goals, and knowledge of the teacher	Constant-comparative method of ground theory using data from teacher meetings, interviews, classroom sessions, and teacher notes
Creation of subcategories for beliefs, goals, and knowledge	Teaching-in-context theory and the literature
Identifying actions for planning and classroom practices	Constant-comparative method of ground theory using data from planning and classroom sessions
Grouping actions under different practices	Logical relationship between the actions, teacher notes, and the literature
Interpretation of actions	Teaching-in-context theory

Extraction of beliefs, goals, and knowledge

To explore the planning and classroom practices of the expert teacher, first an initial list of beliefs, goals, and knowledge was created using the transcripts of daily interviews with the teacher and the teacher meetings. This list was augmented by the teacher notes that gave an overview of the teacher's overall philosophy about teaching and learning mathematics. Additionally, the HLT was used to derive the teacher's goals for the integers topic. Beliefs were derived from the daily interviews and the teacher meetings only if the teacher revealed her general views about teaching, learning, and mathematics. If the teacher specifically talked about the events that occurred (or the events she expected to occur) in the class and the actions that she was planning to take, these were included in the teacher's goals.

The beliefs were categorized under three groups: beliefs pertaining to teaching, beliefs pertaining to learning, and beliefs pertaining to mathematics (Ernest, 1989; Thompson, 1992). For example, a belief such as "students learn mathematics with understanding" was placed under the learning beliefs while "the teacher is not sole source of knowledge" was categorized under teaching beliefs, and "mathematics is a sense-making activity" was put under the mathematics beliefs.

The goals were also categorized under three groups as overarching goals, content goals, and local goals (Schoenfeld, 2000). For example a general goal such as "focusing on the big idea of the lesson" was included in overarching goals, whereas a goal that depends on the current content such as "thinking of net worth as intangible quantity" was considered as a content goal. The goals that occur based on the needs of the current situation (which usually occur in response to students' actions) such as "bringing a student's idea from another period to classroom" were considered as local goals.

The teacher's relevant knowledge was also categorized under three groups namely the subject matter, the pedagogical, and the curriculum knowledge (Shulman, 1986). Although determining the entire body of a teacher's knowledge is not a straightforward task, the current study attempted to describe it from the teacher's belief, goals, and actions. For example, the teacher's belief that an environment that nurtures respect and a sense of community plays an essential role in students' learning was considered as an indicator of the teacher's knowledge about the environment aspect of the NCTM *Standards*.

Analysis of planning practices

To analyze the expert teacher's planning practices, first the teacher meetings prior to the beginning of the study, the weekly teacher meetings that were held during the study, as well as the daily interviews with the teacher were transcribed. These transcripts were studied to find the patterns of actions that the teacher regularly employed during planning sessions. Using the constant comparative method, the observed actions were put under investigation to determine if they repeated sufficiently throughout the data set to warrant them a practice status.

Once the initial list of the actions was created, the actions were studied to find the logical relationships between them. Based on this, the actions were categorized into five practices namely preparation, reflection, anticipation, assessment and revision. This grouping was performed for the purpose of facilitating the analysis by collecting similar actions together.

After the grouping, each action was interpreted using the selected dialogues from the data set that epitomized the main features of that action. The interpretation was based on the teaching-in-context theory which uses the teacher's beliefs, goals, and knowledge to explain the teacher's practices. During the interpretation, the emphasis was put on the highest priority beliefs and goals and the most relevant knowledge of the teacher regarding the action in question.

Analysis of classroom teaching practices

Similar to the analysis of planning practices, first the videotapes of classroom sessions as well as the audio recordings of small groups were transcribed. Using the constant comparative method, these transcripts were used to find the regularly occurring patterns of actions that the teacher exhibited during the instruction.

The extracted actions were categorized into five different practices based on the logical relationships among them as well as the teacher notes, interviews with the teacher, and the literature. These practices were creating and sustaining social norms, facilitating genuine mathematical discourse, supporting the development of sociomathematical norms, capitalizing on students' imagery to create inscriptions and notation, and developing small groups as communities of learners. After grouping the actions, the classroom dialogues that illustrated the main features of each action were interpreted based on the teacher's beliefs, goals, and knowledge.

Trustworthiness

An essential component of a qualitative study is the supporting evidence for its trustworthiness (Lincoln & Guba, 1986). In order to ensure the trustworthiness of the current study multiple methods were used. These included observations, interviews, meetings, field notes, and documents. Triangulation among these multiple data sources was an important element in reducing the limitations of the study (Mathison, 1988). Interviews enabled the researcher to learn the expert teacher's perspective, how she made decisions, what she believed about teaching and learning in a standards based environment, her comments related to the observed classroom and how the instruction proceeded compared to her expectations and plans. The teacher meetings were used to understand Stephan's perspective more clearly as she expressed her ideas related to the instructional sequence, students' work, and the classroom environment. All the data that were extracted from the interviews, teacher meetings, teacher notes, and classroom sessions were checked for consistency by comparing the same lesson within itself as well as between different lessons whenever possible.

Member checking was also used to enhance the trustworthiness of the study. Member checking can be described as taking the data and interpretations to the people who participated in the study during the analysis and asking if the results are plausible (Guba & Lincoln, 1981). The practices as well as the beliefs, goals, and knowledge extracted by the researcher were shown to the expert teacher to get her opinion.

The study was conducted over a five-week-long period. This allowed the researcher to see the patterns in the data more clearly and accurately. Comparing the data within itself during the same day and across different days permitted the use of the constant comparative method and allowed for the creation of more reliable categories.

Being a participant in the study can also enhance the trustworthiness as it gives an insiders' perspective to the researcher (Merriam, 2009). In this study, the researcher attended all of the teachers' meetings as a participant rather than an observer. This gave the researcher an opportunity to put herself in the role of a teacher and acquire a better understanding of what is perceived from that perspective. Additionally, high level of detail and data provided by the study such as the list of the teacher's practices, beliefs, goals, and knowledge as well as the HLT and the instructional sequence provide further support for the reproducibility of the study. To this end, by using the same instructional sequence and investigating the same expert teacher's practices, it is expected that one can find results that are consistent with those of the current study. Additionally, one can use the same methodology to research another teacher's practices and compare it with the findings of the current study.

CHAPTER FOUR: INTERPRETATIVE FRAMEWORK

The aim of this chapter is to give a detailed description of the interpretative framework by using specific examples from the data set as well as to describe the process by which the teacher's beliefs, goals and knowledge were extracted. During the analysis, teaching-in-context theory was used to explain the expert teacher's actions and practices. For the purposes of the current analysis, the term action is used to indicate a repeating way of behavior adopted by the teacher, while practice stands for a collection of related actions.

Teaching-in-Context

The intent of the analysis was to extract the teacher's practices and explain how and why she applied them. The study used the theory of teaching-in-context as a framework to explain the teacher's decision making process and actions based on the teacher's beliefs, goals, and knowledge (Schoenfeld, 1998). The framework supports that one can provide a detailed explanation of what the teacher does and why she does it with a good understanding of the teacher's beliefs, goals, and knowledge. The beliefs serve as a background for the goals that the teacher wants to accomplish and the teacher accesses different types of knowledge to achieve those goals. To determine the teacher's beliefs, goals, and knowledge one needs to have the teacher's lesson image which can be obtained by interviewing the teacher. Lesson images include the teacher's knowledge about the students, the ways that they are likely to think in certain scenarios, and how she might react to those strategies. Lesson images also contain the action plans where the teacher defines the set of actions that she needs to go through in order to achieve the goals. It is also important to note that, in any given moment, beliefs, goals, and knowledge may have different activation levels (e.g. a particular goal may have a higher priority than other goals in that moment). An unexpected situation in the classroom can cause a change in the activation levels. Although the analysis of the planning and teaching practices included all of the beliefs, goals, and knowledge during the interpretation, in the following, each component of the model is shown individually using examples from the data for illustration purposes.

Beliefs

Beliefs are “mental constructs that represent the codifications of people’s experiences and understandings” (Schoenfeld, 1998, p. 23). The teacher may have different kinds of beliefs such as beliefs about learning, teaching, and mathematics. Beliefs shape what the teacher considers about the appropriate behaviors in different circumstances based on her goals that she wants to accomplish and the knowledge that she brings to those circumstances. The framework considers attributed beliefs rather than professed beliefs. The attributed beliefs are those that are extracted from the interviews with the teacher, teacher meetings, and teacher notes. Although one cannot assume that attributed beliefs are exactly the same as the teacher’s real beliefs, they can be made as close as possible by triangulating among multiple data sources as well as member checking. The teacher’s beliefs have an important role not only in understanding the teacher’s goals, but also in assessing her relevant knowledge.

The following excerpt is taken from the first day of the experiment where the teacher re-emphasized the way the students need to behave in the classroom:

> T: Let’s remind ourselves how we act in here. How do we participate in here?
>
> Seth: Respectful.
>
> T: You are respectful to each other. That means I do not want to hear the “s” word. What is the “s” word?
>
> Sts: Shut up, stupid.
>
> Seth: You can say “please be quiet”.
>
> T: Yes you can say “please be quiet”. Do it respectfully. I am starting to hear you guys getting disrespectful to each other, I want it to stop. We need to argue in productive ways. Right, Mark?

This discussion took place at the beginning of the instructional sequence as the teacher wanted to make sure early on that students did not behave in disrespectful ways toward each other. Although, arguing respectfully was an already negotiated norm at this point in the semester, the teacher started to hear some undesirable words, and therefore reminded students that it is not appropriate in this classroom. The teacher's practice here is to renegotiate the norms about how to behave in the classroom. Her practice can be explained by her belief that students learn better in an environment where they respect each other. This environment allows them to explain their ideas without worrying that they will be criticized by the other students disrespectfully.

Goals

A goal can be described succinctly as "something you want to accomplish" (Schoenfeld, 1998, p. 25). The teacher may have different goals such as overarching, content, and local goals (Schoenfeld, 2008). Overarching goals are the general goals of the teacher that she wants to accomplish in her planning and instruction. For example, teaching in a way that supports a discourse that stays alive and interesting can be considered as an overarching goal. Content goals, on the other hand, are those that capture the teacher's aims for a given instructional context. For instance, students' understanding of net worth as an abstract (intangible) quantity can be counted as a content goal for the integers topic. Local goals are more limited in their scope; they may remain valid only for a short duration such as one classroom session. As an example, the teacher may want to bring a student's idea into the discussion since it might be useful to other students. Additionally, emergent goals might arise in response to unexpected situations. For instance, a student may ask an interesting question or offer an alternative explanation to a question, which may result in the creation of a new goal to respond to those situations.

Goals may play different roles in the teacher's decision making process. Similar to beliefs, they are also attributed to the teacher by triangulating within the data as much as possible. Often goals can be extracted from the teacher's lesson images. However, in reaction to unexpected situations, new goals may emerge and these may have higher priority than the planned goals. Teacher's decisions are primarily shaped by the set of currently active high priority goals. The excerpt below is taken from a classroom discussion in the first episode where the focus of the discussion was on understanding the meaning of net worth:

> T: What does it mean to be worth 2.7 billion dollars? What does it mean when we say someone is worth that much money? Do you have any idea?
>
> Dusty: Like [pause]. No.
>
> T: All right. Who has not gone today?
>
> Gage: That means like if someone like Oprah sold all she had that is how much money she would get.
>
> T: That is a pretty neat idea. Isn't it? If she were to sell everything that is how much she would be worth. Brad?
>
> Brad: How much she made over her life time.
>
> T: Anything else?
>
> Tisha: I think if she is sponsored or something like that, that it is how much they would pay for her.
>
> T: All right. You know how somebody described it from the last period. Do you guys know Danny?
>
> Sts: Yes.
>
> T: He described it as if she was kidnapped, that is how much money they would ask for her.

During the discussion of the net worth of a famous person, the teacher first asked students meaning of the net worth. Gage explained that it is how much money a person would get back during her life time if she were to sell everything. Then the teacher asked different students to explain the meaning of net worth. Brad explained it similar to Gage by stating that it would be the money that she made over her life. Tisha also claimed that it would be the money that a sponsor would pay to her. Finally, the teacher brought up one of the student's idea from another class period to the discussion by stating that it would be the amount of money that would be asked if she were to be kidnapped.

The content goal of the teacher during this discussion was to emphasize net worth as an abstract quantity. In order to achieve this goal, she first asked students to explain the meaning of net worth and capitalized on some students ideas in order to emphasize the big idea of the activity. An active overarching goal of the teacher was to support a discourse that stays alive and interesting. Thus she connected the context with a famous person's financial transactions. Once the students produced different explanations in the discussion, she concluded it by giving a further example from a previous period which emphasized the idea of net worth as an abstract quantity.

Knowledge

Schoenfeld (1998) considers teacher's knowledge as not only what she knows about subject matter, pedagogy, and curriculum, but also how this knowledge is organized and accessed. Teacher's knowledge includes abstractions of the real life experiences that are encountered on a day-to-day basis. The teacher uses those abstractions in order to perceive and interact with new situations. Despite that the teacher may have a large body of knowledge, she may only use a small and fundamental subset of it when making decisions (Schoenfeld, 1998).

Knowledge may be considered as the point of reference upon which beliefs and goals are based. For instance, in order for a teacher to believe that mathematical inquiry is important for students' learning, she needs to know what mathematical inquiry is as well as why it is important for learning. Knowledge may be acquired through multiple sources such as books, journals, communicating with others, and by simply observing. Similar to the beliefs and goals, the teacher's knowledge was also extracted from the data corpus.

The following excerpt taken from the second day serves to illustrate how the teacher's knowledge affected her practice during the instruction. In this discussion, the teacher asked students to find Brad's net worth when his assets were $600,000 and his debts were $790,000. After a classroom discussion most of the students agreed that the net worth was negative $190,000. At this point the students encountered negative net worth for the first time:

> T: If he sells all the stuff he has, he would get cash and pay off his debt but he would still be in debt. Everybody okay with that … I have one last question for you. Somebody told me Brad is worth nothing. Is that true? [She gives students time to think] I heard "nothing" and "less than nothing".
>
> T: All right. Adam which one do you think it is?
>
> Adam: Less than nothing.
>
> T: What do you think Gage?
>
> Gage: Like less than nothing.
>
> T: Charlie?
>
> Charlie: Less than nothing.

In this discussion, the teacher first summarized students' findings that Brad will still be in debt after selling all his belongings and using the money towards paying off his debts. She then wanted to determine if students considered this situation as Brad's net worth being nothing or less than nothing. At this point many students correctly identified that the answer should be less than nothing. The teacher's knowledge based on her previous experiences showed that many times students have misconceptions in understanding the negative numbers and their relationship to zero due to their abstract nature. In questions that require subtracting a bigger number from a smaller one, students may mistakenly find the answer as zero because, like the history of integers, many students have difficulty conceptualizing that there exist quantities below zero. The teacher's knowledge about this misconception might have been the driving factor for the teacher's inquiry in this discussion.

In this section, the teacher's beliefs, goals, and knowledge are considered in isolation to interpret the teacher's actions for illustration purposes. In reality, all of these sources are likely to come into play when shaping the teacher's decisions. However, in different situations, certain beliefs, goals, or knowledge can have higher priority than others. Thus when interpreting the teacher's actions the focus was placed on those high priority ones. In the following section, the researcher's methods of extracting the teacher's beliefs, goals, and knowledge from the collection of data sources is described.

Extraction of the Teacher's Beliefs, Goals and Knowledge

Extraction of the teacher's beliefs, goals, and knowledge began by transcribing the multiple sources which were comprised of teacher notes, audio tapes of teacher interviews, and teacher meetings as well as audio and video tapes of classroom sessions. Once the transcripts were completed, the researcher examined each source to find repeating patterns of behaviors and statements made by the teacher. When a certain type of behavior which may indicate a belief, goal, or knowledge was encountered, the researcher made a note of it and kept it under investigation to determine if the same behavior would be observed multiple times and across multiple data sources. In other words, the researcher triangulated across multiple data sources to make sure that a belief, goal, or knowledge that was attributed to the teacher was actually in the teacher's repertoire. When it was necessary, the researcher directly asked the teacher in the interviews to explain why she behaved in a certain way, which revealed more information about the teacher's beliefs, goals, and knowledge. As the majority of beliefs, goals, and knowledge were extracted in this way, they can be called *attributed* as they were attributed to the teacher by the researcher. However, sometimes the teacher directly revealed her beliefs, goals, and knowledge (although this happened predominantly for goals rather than beliefs and knowledge).

These types of *professed* statements were also included among the teacher's beliefs, goals, and knowledge (Aguirre & Speer, 2000).

Once the initial list of beliefs, goals, and knowledge was extracted as outlined above each of them was organized into sub-categories using the categories in the literature (Schoenfeld, 2008). Beliefs were categorized into beliefs about learning, beliefs about teaching, and beliefs about mathematics. Goals were organized as overarching goals, contents goals, and local goals. Knowledge was partitioned as subject matter knowledge, pedagogical knowledge, and curricular knowledge. To facilitate the analysis, the teacher's beliefs, goals, and knowledge were coded in the following fashion. Beliefs are represented with the capital letter B followed by the subscript L for learning beliefs, T for teaching beliefs, and M for beliefs about mathematics. For example, B_L1 represents the teacher's first belief related with learning in the order extraction (thus the number does not indicate priority). For goals, G is used to represent the general category while, the subscripts O, C, and L are used to represent overarching, content, and local goals. Similarly, knowledge is indicated by K with subject matter, pedagogical, and curriculum knowledge distinguished by subscripts S, P, and C respectively (see Table 3). All of these abbreviations are given in Appendix B.

Table 3: Coding of beliefs, goals, and knowledge

Beliefs	**Goals**	**Knowledge**
Learning (B_L)	Overarching (G_O)	Subject matter (K_S)
Teaching (B_T)	Content (G_C)	Pedagogical (K_P)
Mathematics (B_M)	Local (G_L)	Curriculum (K_C)

After the teacher's beliefs, goals, and knowledge were coded, the actions that she commonly applied during the planning and classroom sessions were also extracted from the data set. To determine whether an action was part of the teacher's practice, rather than being an isolated occurrence, the researcher investigated how consistently this action was being repeated by the teacher. This action was judged by whether the teacher exhibited this action consistently when faced with similar situations. After the extraction of the teacher's actions, the actions that are related to each other were collected under a practice that defines the overall aims of those actions. In naming these practices the literature as well as the teacher notes were used. The following examples are given to illustrate how a belief, a goal, and knowledge went through this process.

In one of the interviews the teacher stated that, "I have in my mind all the time. Are they explaining? Are they asking questions? ". The researcher took a note of these behaviors and kept them under observation to understand whether the statement indicates one of the beliefs of the teacher. Later in the interview, the teacher stated that "Those norms [explaining one's reasoning, asking questions to understand others, etc.] are what make an inquiry classroom drive." Based on these and other occurrences of similar statements in teacher meetings and the teacher's behaviors in classroom sessions, the researcher attributed a belief to the teacher that "the teacher is responsible for creating and supporting an environment where ideas can be discussed freely and respectfully." This belief was then categorized into beliefs about teaching as it is the teacher's responsibility to support creation and sustainment of social norms.

In the first teacher meeting, the teacher defined the intent of the net worth statement template activity as "The imagery we are trying to develop initially is assets and debts are the quantities that have opposite effect on the net worth statement. One of the big ideas of the template is conceptualizing asset as something owned, debt as something owed and net worth as an abstract quantity ". Based on this statement in the teacher meeting, other sources such as the HLT (the table given in the next session), teacher interviews, and classroom sessions were examined. Since similar statements and behaviors that support this view were seen in other data sources, the researcher attributed several content goals such as “developing the imagery where assets and debts have the opposite effect on net worth”, “defining asset as something owned and debt as something owed” and “conceptualizing net worth as an abstract quantity”.

In another meeting, the teacher anticipated the students’ answers and how she might react to them: “if they [students] do not say that the net worth is the amount of money left over, they will say, especially our procedural kids, that it is what you get when you subtract those two [assets and debts]” and then she emphasized that “it is not clear to me what they mean by money, they mean physical money and coins?” and then she stated that she would focus on it during the discussion of net worth problems. The teacher made similar statements during an interview by stating that “I am not exactly sure how he was conceptualizing the net worth at that point … perhaps they were just trying to perform some procedure to be successful, not necessarily thinking through what those quantities stand for”. The classroom sessions also illustrated actions consistent with these statements where the teacher focused on students’ understanding of net worth. As evidenced by these actions, the teacher had knowledge about the students’ possible misinterpretations of net worth. This information was recorded under the pedagogical knowledge of the teacher as it relates to anticipating students’ misconceptions.

The following example is given to illustrate how the teacher’s actions were interpreted based on her beliefs, goals, and knowledge. Here, the action that is interpreted is the teacher’s encouragement of the students to express agreement or disagreement, and explain why they agree or disagree during a mathematical discussion. This action was observed to be consistently repeated by the teacher throughout the instructional sequence. It was categorized under the practice of establishing and sustaining the social norms as it contributed to creation of a productive discussion environment. The following dialogue is taken from the discussion of a net worth problem:

> Dusty: The negative $190,000 is how much Brad is in debt.
>
> T: Dusty says it is how much Brad is in debt [she writes this on the board]. You all agree with that?
>
> Sts: Yes.
>
> T: Who doesn't agree with that? Nathan?
>
> Nathan: I think it is how much net worth Brad has.
>
> T: It is the net worth [the teacher writes on the board]. Why don't you think it is his debt? Hold on… [Another student wants to answer].
>
> Nathan: Because [he pauses]. I do not know. I just don't get it.
>
> T: If you disagree with Dusty than you need to explain why you do not think it is right. Sally what do you think?

In the discussion above, the teacher's decisions were primarily shaped by the goal that she should renegotiate the social norms as necessary during the instruction (G_O1). This decision was emphasized by the teacher's last statement that if students disagree, they need to explain the reasons for their disagreement. This overarching goal was actually general goal of the teacher that she wanted to accomplish throughout her instruction. When Nathan could not explain why he disagreed with Dusty's argument, consistent with her goal, she reminded students that they should try to explain their reasoning. This goal might be linked to the teacher's belief about learning that students reorganize their ideas when they explain (B_L9). This belief enabled her to try to elicit an explanation from the students instead of merely accepting their statements. Additionally, an important reason for the teacher to engage the students in a discussion about the meaning of negative net worth seemed to be based on her pedagogical knowledge of students' difficulties in conceptualizing negative net worth (K_P2).

In the following two chapters, the teacher's planning and classroom practices are analyzed in detail using the interpretative framework discussed above.

CHAPTER FIVE: ANALYSIS OF PLANNING PRACTICES

The aim of this chapter is to document and explain the expert teacher's planning practices, both prior to the beginning of the instruction as well as during the instruction, with respect to the teacher's beliefs, goals, and knowledge. The planning practices of the teacher were extracted by looking for the common patterns in the teacher meetings as well as in the daily planning sessions. The extracted practices were grouped under five main categories: preparation, reflection, anticipation, assessment, and revision. It is important to note that the teacher's actions in these practices were not mutually exclusive; an action that belonged to assessment necessarily required reflection on the classroom interactions as well. Furthermore, most of the actions that comprised these practices were conducted in an environment of collaboration with two other teachers who were teaching the same topic, the co-teacher, and the researcher. Although the expert teacher acted as the team leader, any one of the teachers could, and did, call for formal or informal meetings where everyone's participation was equally valued. Together, they created a community of learners who shared experiences, reflections, and anticipations.

Practice One: Preparation

Preparation prior to the beginning of the instruction had a central place in the teacher's planning. It included practices such as creating a hypothetical learning trajectory, designing an instructional sequence, studying the relevant literature, and contemplating on the big ideas of the unit.

Action 1: Creating a hypothetical learning trajectory

A Hypothetical Learning Trajectory (HLT) is a route that the teacher imagines that her classroom will go through during the instruction. It is called hypothetical because the teacher is anticipating what she expects to happen; whether deviations from it will occur or the route will be entirely different from the one planned will not be known until the instruction occurs. The HLT gives the intent of the instructional sequence by including various elements of the instruction such as the big ideas of the lessons, the tools that will be used, the imagery that will be developed, and the potential gesturing that may aid students in developing that imagery.

The HLT created by the teacher in the current study, which is shown in Table 3, was based on a financial context where the main interest was determining a person's net worth. The progression of the rows from top to bottom was in the order of their appearance in the instructional sequence. As can be seen from the table, it included separate columns for tools, imagery, taken-as-shared interests, topic for mathematical discourse, and possible gesturing which is often an externalization of mental imagery that might support students' development of imagery. The tools column represented the inscriptions and notations[3] that would be used to develop students' imagery. In general, this column would include notations, inscriptions (e.g. graphs), and physical devices, such as calculators. In order to give students opportunities to explore integer concepts and operations, calculator use was relegated to the end of the unit and outside the confines of the sequence.

To illustrate what an entry in the tool column would mean, the net worth template can be used as an example (see Table 4). The net worth statement template was an inscriptional device that was used to develop the imagery of assets and debts as quantities that have an opposite effect on net worth. The imagery column included the list of the images that the teacher anticipated students to develop during the instructional sequence. The purpose of the images was to serve as a reference point the students could fold back to as they were trying to make sense of

[3] While inscriptions are considered as any form of mathematical writing, notations refer to specialized forms of written mathematics that can be used to generate new mathematical objects and define mathematical objects uniquely (Lehrer, Storm & Confrey, 2002).

more complicated tasks. The purpose of the taken-as-shared interests was to anticipate the motivation that students would have to generate conversations as they worked on the tasks. For example, as shown in the second row of the HLT, students might be motivated to calculate the net worth if their taken-as-shared goal was to compare net worths of two celebrities or their friends from class. This was motivational for students if they were interested in determining which of their friends had a higher net worth. Possible topics for mathematical discourse included a set of topics, the discussion of which as a whole class could be instructive and illustrative for the students to understand the big ideas of the lessons. An example of a discourse topic is in the second row of the table where the topic for mathematical discourse is discussing the different strategies for finding the net worth. During this discussion, students could achieve a better sense of what the net worth is and in which ways it can be computed with some ways being more efficient or simpler than others. Finally, possible gesturing contained certain body movements which can support tactile imagery and could allow students to connect movement to the mathematics of integer operations. Here, the tactile imagery refers to mental images that students might develop by the sense of seeing the body movements related with the context. For example, the teacher's movement of her arms up and down during the discussion of net worth problems (e.g. adding assets make net worth to go up with the arm movements) might help students to develop imagery where + indicates increases in net worth and - indicates decreases.

Table 4: Hypothetical Learning Trajectory

Tool	Imagery	Activity/Taken-as-shared Interests	Possible Topics of Mathematical Discourse	Possible gesturing
Net Worth Statement Template	Assets and debts are quantities that have opposite effect on net worth		Conceptualizing an asset as something owned, a debt as something owed Conceptualizing a net worth as an abstract quantity (not tangible)	
Net Worth Notations	Differences in Collections of assets & Collections of debts	Determining a person's net worth Comparing net worth	Different strategies for finding net worth	
Symbols (+ and -)	+ indicates an asset and – indicates a debt	Determining net worth Transactions	Different strategies for finding net worth Creating additive inverses as objects How do various transactions affect net worth	Arms moving up and down to indicate positive or negative affects
Net Worth Tracker (vertical number line)	Empty number line to express (+ and -) movements	Reasoning with number line to find a net worth after a transaction has occurred Determining difference between net worth	Going through zero The affect of different transactions Different strategies for finding net worth	Up and down movement with arms

It is important to note that the HLT is not static; it can be revised after the beginning of the instruction based on the interaction with the students. For instance, the teacher may find that more activities are needed to support the desired imagery which may result in the addition of a new row to the HLT table. The initial HLT shown above had actually undergone modifications by the teacher in the current study in order to better support students' understanding of integers. One of these modifications was to add a red-black number line as a tool before the introduction of + and – symbols. This tool served to support the image of a structured space of positive and negative integers. A second change was the addition of an activity where students matched the result of a transaction with operations (e.g., -2(-25)) into a single transaction (e.g., +50).

The motivation for Stephan to create the HLT can be explained by different knowledge, beliefs, and goals. Her knowledge of the mathematics education literature, particularly design research, and HLT (K_P8) as well as her knowledge of integers (K_S1) appeared to be two of the important reasons for creating the HLT. Also her belief that effective teaching requires careful planning (B_T5) seemed to be another affecting factor.

Action 2: Creating instructional sequence (Choosing tasks that will achieve objectives)

The instructional sequence was designed by the teacher prior to the beginning of the study (the entire instructional sequence is given in Appendix C). The primary goal of the sequence was to help students reinvent the rules of integer operations while developing imagery that they could use to make sense of these operations. To achieve this goal, the teacher created an instructional sequence within the context of determining a person's net worth. The instructional sequence exhibited three important characteristics of the RME theory which can be named as reinforcing with a real context, accounting in the abstract, and using symbols (Stephan, 2009).

Reinforcing with a real context

The experientially real context that served as the anchoring context for the instructional sequence was based on determining a person's financial net worth. To make it interesting for the students, the instructional sequence started with a whole class discussion about what might be the net worth of a famous person such as Oprah. At this point, the students did not know what was meant by net worth, but the teacher warmed them up to the idea by saying: "I Googled Oprah this morning before you came to class and found that her net worth is $1.5 billion. Can you imagine that? How many zeroes are in that number? What does it mean to say that she is worth $1.5 billion?" (Stephan, 2009, p. 19). The students responded to this by enumerating the things she owns such as cars, boats, magazines, etc. The teacher then asked them what kind of things she might owe money on. Students gave examples such as mortgages and loans. After this, the teacher split the board into two parts and wrote students' guesses about what she owns to one side and what she owes to the other. Then, she asked them if they knew the financial terms for these two ideas, and then introduced the terms *assets* and *debts*. Stephan's strong belief that students learn better with an instruction that relates to their informal knowledge (B_L5) had an essential role in creating an instructional sequence based on the financial context. She thought that this would help students connect the subsequent tasks with their informal knowledge. Her belief that problem solving is important to explore mathematical ideas also seemed to motivate her to create an instructional sequence that included different problem solving tasks within the financial context (B_M3).

In order to engage students in meaningful activities, the teacher asked students to describe their own assets and debts. Next, the teacher asked students to define *net worth* and challenged them to describe it in terms of assets and debts. The instructional sequence continued with another example where the teacher told students that she was trying to save money for her son's college payments and therefore her financial advisor gave her a net worth statement to fill out. The class went over this statement and discussed the terms that were not known by the students. This activity was the underpinning of the instructional sequence where the students could develop imagery to support that assets and debts are quantities that affect net worth in opposite directions, one positively and one negatively. The following activities continued with determining a person's financial net worth using the net worth statement and comparing net worths of two or more people. Many of the students tried to solve them by adding up the assets and debts and finding their difference. The intention of the instructional design at this point was for the students to understand the ideas that (1) assets increase net worth while debts decrease it; that is they have opposite effect on a person's net worth and (2) net worth is an abstract quantity that may be described by both positive and negative values. Because these were crucial ideas to grasp to be able to work with integer operations within this context, the teacher continued with more activities that use net worth statements to assess students' understanding of these ideas.

Accounting in the abstract

Once the students were comfortable with the ideas of assets, debts, and net worth, the instructional sequence progressively moved to a more integer-like form where assets and debts were represented with + and – signs respectively. This introduction happened during an activity called the dating game where students were asked to determine and compare the net worths of three bachelors. Different from the previous activities, this activity did not explicitly use the words asset and debt but involved numbers such as a bank balance of +$1000 and a car loan of -$15,000[4]. Students solved this activity in different ways; some totaled the assets and debts and found their difference while others sequentially worked through the list of numbers. Some students also noticed that some numbers cancel out each other so they did not have to operate on them redundantly.

The instructional sequence continued with transaction activities where students take an already computed net worth and alter it by using a transaction. To make sense of this activity, the students needed to understand that net worth itself is a quantity which can directly be worked on. To emphasize this, the teacher gave a net worth statement (see Appendix C, "Don't Cry Over

[4] Although some activities included the debt as well as the minus sign, this did not mean double negation. The meaning of it as a debt was taken-as-shared.

Spilled Milk”) which is largely covered by what was to represent milk stains that prevented students from seeing the individual items (they could still see the total net worth). The activity asked students what would be the new net worth if they took away some assets. Although some students argued that they needed to see the individual quantities that make up the net worth, by discussing it in small groups they realized that it would reduce the total net worth as taking away an asset is a bad transaction.

This introductory activity served as a basis for another activity called good and bad decisions where students were asked to judge the nature of various transactions. The students went through the list of the transactions and marked each one as a good or bad decision. During the classroom discussion, the teacher introduced a symbolization to represent different types of transactions – for instance adding a debt of $650 was written as +(-$650). The sign outside the parenthesis indicated the type of the transaction (here “add”), while the sign inside represented what type of quantity that is “debt”. Once students were comfortable with converting a verbal description to a numerical form using this symbolization, the ensuing activity asked them to do the opposite. The students were then given number sentences such as –(-$8250) and asked to verbally describe them by using the words *add, take away, asset,* and *debt.*

To help students model their reasoning and support their imagery, a vertical number line[5] was introduced in this part of the instructional sequence. Using this tool, students could more easily record the effects of various operations especially those that involve crossing zero. The choice of a vertical number line (as opposed to the horizontal number line) was motivated by the following: (1) it better fit to the supported imagery that net worth goes *up* by adding assets and

[5] All the number lines used in the instructional sequence were open number lines where proportionality was not required.

down by adding debts; and (2) it avoided students' potential misconception of placing negatives to the right of zero on a horizontal number line. Stephan did not introduce the number line as ready-made but brought it up to support students' reasoning. This appeared to be based on her belief that introducing tools as ready-made might cause difficulty for students as the relationship between the tool and the task would not be clear (B_T15).

Using symbols

The next step in the instructional sequence was to move progressively from word problems to number sentences. To this end, students were first given a word problem such as "Monica has a net worth of -$7400. An asset of $3000 is taken away. Is this good or bad? What is her net worth?" (Stephan, 2009, p. 22). This procedure was followed by problems that progressively used symbols such as, (1) Net worth: $1500: transaction adds a debt of $600; (2) $45: add an asset of +$5; (3) 360: add -160; (4) -$90 – (-$100); (5) -50 – (-50); (6) -3 + 4 – (-23) – 10 (Stephan, 2009, p. 22). As illustrated by this example at this stage of the instructional sequence students started working on textbook-like integer problems. Thus the instructional sequence gradually transitioned the students from concrete to abstract reasoning. The teacher's belief that students learn better with an instruction that moves from concrete to abstract gradually had an important effect to create this type of instructional design (B_L6).

The following tasks in the instructional sequence focused on students' meaningful understanding of transactions such as why double negatives result in a positive number while opposite signs result in a negative one. At this point, students could relate to the imagery of good and bad decisions and argued that a double negative is like taking away debt which is a good decision so it should result in the net worth going up. Similarly, opposite signs indicate either adding debt or taking away asset both of which are bad decisions that result in the net worth going down. To emphasize the equivalence of different types of transactions, an activity was created where the original (e.g., $10,000) and the new net worth (e.g., $12,000) were given and students were asked to list the possible transactions. Here students came up with answers such as + (+$2000), - (-$2000), or multiple transactions such as +$3000 - $1000 (Stephan, 2009). Here it is important to note that the sequence mainly focused on addition and subtraction of integers. However, the teacher also introduced multiplication toward the end of the sequence. For example, the teacher introduced that -2(25) can be defined as taking away two sets of $25 assets and -2(-25) can be thought as taking away two sets of $25 debts. The instruction did not include the division operation.

The instructional sequence was designed to move the students from concrete reasoning with assets and debts to a progressively more abstract one that solely uses symbols. In doing so, it supported various images and tools that help students reinvent the rules of integers and make sense of integer concepts and operations. The teacher's knowledge of RME (K_p9) as well as her knowledge related with content (K_s1) and the state standards (K_c2) that elaborate the objectives of the integers unit was essential in creating the instructional sequence.

Action 3: Reading and adapting relevant research into classroom

Another important practice that emerged during the planning was to talk about the difficulties that students might have for a given activity. Based on her previous knowledge, Stephan anticipated that students might initially struggle in activities where debts overwhelm assets resulting in negative net worths (K_P4). She conjectured that it may be difficult for the students to grasp the idea of negative numbers unless they think of net worth as an abstract quantity. She pointed out that going under zero has historically been a difficult concept for people (K_P6) as demonstrated by the following excerpt taken from a teacher meeting:

Stephan: In history people said you cannot take something from nothing so if you think about these things as physical, concrete things you are right. You cannot take 790,000 things from 600,000 things; you do not have enough things to take away from. So what do these numbers mean? You can only go down to zero, if you think of these as physical concrete things. You cannot go under zero. So that is why it is so important to think about these are not things, this is what you're worth, this is what you're worth negatively and if you go under zero you are in debt. Debt is an abstract quantity. So how does it look for kids?

Why it is so important for me is because historically that is where they got stuck, too. You might hear kids saying something like this "how can he be negative 190,000 instead of zero?"…I heard kids saying "he is worth nothing" instead of worth negative, because they were thinking such as if you have $100 and you add debt of $200, you will be at zero.

As she knew that the mathematics community struggled with the idea of going under zero for concrete objects, the instructional sequence involved an abstract quantity. However, the teacher recognized that the students may not readily think of net worth as an abstract quantity and therefore it is crucial for them to help students develop that understanding. The teacher's belief that students learn mathematics when they understand (B_L1) seemed to have played an important role in her emphasis for understanding net worth as an abstract quantity. Additionally, her subject matter knowledge about integers (K_S1) seemed to have an essential role in her actions, which was revealed by her questioning of what it means to be in the negative.

Action 4: Unpacking big ideas of the tasks

One of the advantages of knowing the possible discourse was to decide whether the teacher should focus on the ideas that students bring into the discussion. In the first meeting, one of the teachers was not sure about focusing on transactions in the first days. Since the expert teacher created the instructional sequence she knew the aims of the activities as well as the subsequent tasks in the sequence (K_C1). She explained the difference between finding the net worth from the given assets and debts and the following transaction problems as:

Stephan: Negative numbers and positive numbers are not actions. We won't get to actions until transactions. That will take a little while…

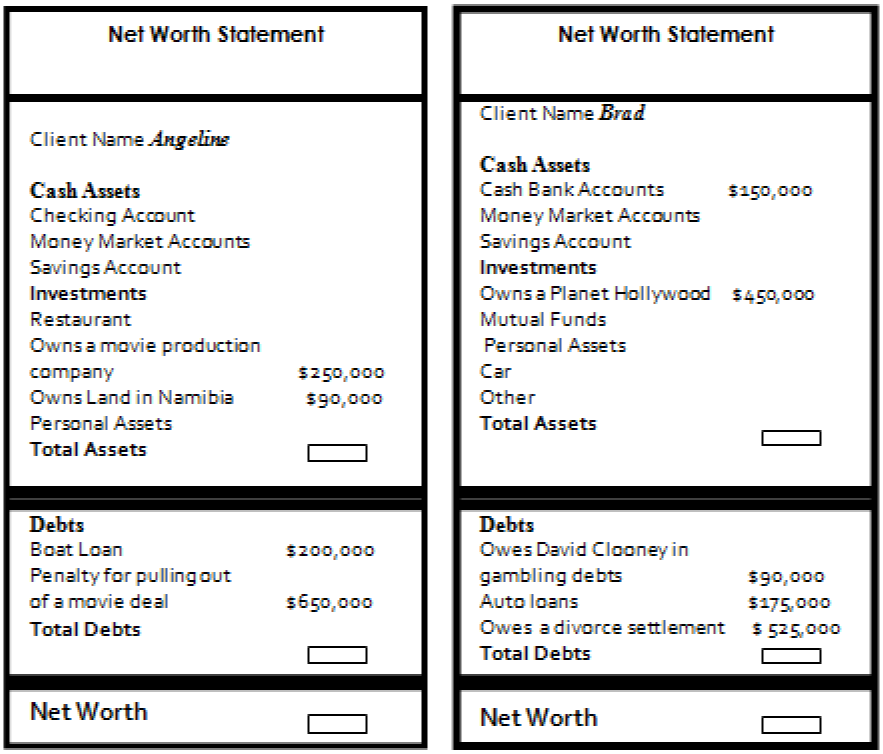
Jones: We are formally doing transactions.

> Stephan: But, they are not doing integer operations. Generally, what I see on these pages is to add assets and add debts and find the difference…We are not concentrating on meaning of those actions per se. The main goal on these pages are understanding the meaning of net worth, the integer operations are not main intent but they are doing some operations…

The conversation above shows that the teacher was planning to introduce the transactions in another activity after the students understood the concepts of asset, debt, and net worth. In the beginning activities where students found the total assets and debts, the teacher's aim was to focus on the meaning of the net worth which most students find by adding up all the assets and adding up all the debts then finding the difference between them. The teacher's goal to tackle one big idea at a time (G_O4) seemed to be the driving factor in her separation of transactions from assets and debts. Her careful planning (B_T5) and knowledge of the instructional sequence (K_C1) also played an important role in these decisions as the teacher had a clear idea of when the transactions would be introduced.

The teachers continued to talk about the big ideas of the activities and the questions that teachers might ask during the discourse as well as the possible gestures that might support students' images:

> Jones: The big question here is to ask the students "who is worth more?" One of them gives the negative net worth…
>
> Stephan: You said in the Angelina and Brad problem, one gives you the negative net worth that might be worth talking more about (See Figure 1). What does the negative net worth mean? I think it is important to put possible topics of discourse here under the second net worth statements right here "What does a negative net worth mean?" [she writes it down on the HLT]. I am trying to guess possible gestures that you can use here. I think those are important to develop images. It would not hurt to start talking about Angelina's way up here [gesturing with her arm up] and Brad is way down here [gesturing with her arm down]. Some of our kids are very gestural in our class…
>
> Taylor: I think we can do gestures up and down.



Net Worth Statement

Client Name *Angeline*

Cash Assets
Checking Account
Money Market Accounts
Savings Account
Investments
Restaurant
Owns a movie production company $250,000
Owns Land in Namibia $90,000
Personal Assets
Total Assets

Debts
Boat Loan $200,000
Penalty for pulling out of a movie deal $650,000
Total Debts

Net Worth

Net Worth Statement

Client Name *Brad*

Cash Assets
Cash Bank Accounts $150,000
Money Market Accounts
Savings Account
Investments
Owns a Planet Hollywood $450,000
Mutual Funds
Personal Assets
Car
Other
Total Assets

Debts
Owes David Clooney in gambling debts $90,000
Auto loans $175,000
Owes a divorce settlement $ 525,000
Total Debts

Net Worth

Figure 1: Net worth statement

In this excerpt, Jones stated that they need to ask questions about who is worth more to understand whether students can see the difference between negative and positive net worths. Stephan acknowledged that it would be valuable to do a formative assessment at this point to determine what students understand by a negative net worth. Her beliefs that formative assessments are an essential element of effective teaching (B_T8) might have helped her to decide to ask students what they understand about negative net worth. Therefore, the teachers added the meaning of negative net worth under the possible discourse in the HLT. Next, the gestures, which are very important for supporting the imagery, were also discussed and the teachers decided that they could move their arms up and down to emphasize the difference between positive and negative net worths. The teacher's goal of supporting tactile imagery (G_O5) appeared to be the driving factor for using gestures containing certain body movements.

In summary, the practice of *preparation* included important actions such as creating an HLT, creating an instructional sequence, reading and adapting relevant literature, and unpacking the big ideas in order to be prepared for the classroom instruction. Although each action was based on the teacher's different beliefs, goals, and knowledge, it was observed that the teacher's knowledge of content as well as her knowledge of curriculum and related literature had an essential role in her preparation. Additionally, her belief about teaching that it requires careful planning and students learn mathematics with understanding were important factors to prepare an instructional sequence where students can explore mathematical ideas. Finally, her overarching goals of tackling one big idea at a time and supporting students' ideas with the tools when they have the reasoning seemed to have shaped her preparation to achieve these goals during the instruction.

Practice Two: Reflection

Reflection involved contemplating on the classroom interactions of the current as well as the previous years to evaluate if the instruction was happening as planned. It also included evaluating whether the teachers were faced with unexpected situations, what misconceptions students developed, and, in general, what could be done to make the instruction more effective.

Action 1: Reflecting on previous year's instruction

After Stephan introduced the HLT, the teachers started to talk about how they introduced the beginning activity in the previous year and the difficulties that they had during the instruction. Since the task was to start with a discussion of a famous person's net worth, the teachers tried to remember how they facilitated the introduction in order to help students to understand the words asset and debt. The following excerpt shows how the expert teacher and the co-teacher together tried to remember and talked about the previous year's instruction to help the others imagine what the beginning discussions might look like:

> Stephan: So their conception of somebody's net worth is initially to name up all the things that they owned and the teacher has to suggest "do you think that they might have any debts or any loans?" Then kids say "oh yes, she probably may have loans for her million dollar house...". We did not introduce them as assets and debts. We did put categories on the board and when kids name the things that we know as assets, we put them on one side of the board intentionally so that when we ask them about "Does she owe anything?" then we could start to list those things on the other side, it was almost like having two columns on the board.
>
> Taylor: I think we put owe for the things she kinds of pay for it. And then somebody said that she has to pay for the car loan, pay for the studio…
>
> Stephan: And then I think we put the title of each column, say "what you owe" and "what you own" and then we introduced the vocabulary words [debts and assets]. This helped them to make sense of this net worth statement... The intent of this discussion is to get them familiar, bring out their knowledge of these owes and owns thing.

The aim of this conversation was to decide on a possible discourse that could be used during the introduction. The teacher's knowledge from the previous year (K_P4) helped her to anticipate students' possible answers during the discussion. Another aim of the discussion was to talk about the big ideas of the activity and to determine the supportive conversations that the teachers wanted to engage in with their students. Her goal of focusing on the big idea of the lesson (G_O3) seemed to have shaped her suggestions for the opening discourse where students could explore the asset and debt concepts. The teacher's belief that students learn better with an instruction that relates to their informal knowledge (B_L5) also served as a fundamental reason to start the instruction with a discussion where students could guess what a famous person might own and owe.

Action 2: Making sense of students' solutions

During the teacher meetings, many times the teachers discussed what kind of solutions the students came up with in the classroom and tried to make sense of those solutions and identify the students' misconceptions. One of the misconceptions that the teachers encountered was the use of faulty algorithms that students used such as reverse subtraction. The following excerpt illustrates the teachers' discussion of how the students might have found the answer as -\$1250 to the question where assets were \$3750 and debts were \$4500. Here it is important to note that the faulty algorithm refers to writing 3750 on the top and 4500 on the bottom, and subtracting the greater digits from the lesser ones:

> Stephan: Can we try to make sense of what they are doing, the algorithm that is wrong?
>
> Jones: They follow the rule for subtraction until the final one.
>
> Stephan: But why isn't that working? That is my question, mathematically why is it not that?
>
> Jones: Because there is…I do not know the answer.
>
> Stephan: I do not know the answer either. I am trying to figure out myself. Pretend that is 3500 [instead of 3750] for a second than it would work.
>
> Jones: Yes, there are lots of numbers that work. One of my students told me yesterday "I have been doing this every time and it works".
>
> Stephan: It is until…
>
> Jones: The numbers cannot be the same. Last three digits cannot be the same.
>
> Stephan: What about 2150 (instead of 3750)?
>
> Jones: No, it is 100 more.
>
> Stephan: Are you also getting this reverse algorithm?
>
> Wilson: Yes.

As it was seen in the excerpt above, the teachers first tried to make sense of a faulty subtraction algorithm that sometimes gave correct answers. They tried it with different numbers that would result in both correct and incorrect answers to see the pattern in the algorithm. After experimenting with a few numbers, they decided that the algorithm might work if the last three digits of the numbers are the same, otherwise it usually does not. The teacher's practice of making sense of students' misconceptions was an essential part of her planning together with the other teachers. Her belief that students learn from their misconceptions (B_L8) is also likely to have had an important role in her spending time to understand the misconceptions and creating tasks in order to address them in the class discussion.

The teacher's belief about valuing students' ideas in the classroom as well as during the planning (B_T4) and her belief in mathematics as a sense making activity (B_M1) seemed to be the reason for bringing students' solutions to the planning meetings. This helped the teachers to think about and understand the flaws of these algorithms and prepare questions that include the types of numbers for which the algorithms would fail. Creating questions where students can explore their misconceptions was an important practice since by doing this the students could learn from their mistakes and reorganize their ideas. In the rest of the discussion, the teachers talked about how they could help students to make sense of the computations and prevent them from using the reverse subtraction algorithms:

> Stephan: [Based on Jones's statement that he has a student that believes his faulty algorithms work all the time, Stephan makes some suggestions] You can have two different solutions up there by contrasting... Seeing the other ways up there can make him think about it [his solution].
>
> Jones: What makes him to see the other way makes more sense?
>
> Stephan: I do not know. All I did is to take Sienna's idea of number line. So I stole from the third period I said do you guys know Sienna? When they talked about the pay back and I introduced the number line that helped some kids. That is what I have done. Introduce the number line model during this pay back discussion.
>
> Jones: Finding the net worth?
>
> Stephan: Yes, or to make sense of why-750 is right and the other one is wrong.
>
> Jones: Which is essentially finding the correct net worth? A reasonable way to explain it.

The expert teacher suggested that writing different solutions on the board might help the students who had correct solutions by using the reverse algorithms analyze the other solutions. She also added that modeling students' solutions on the vertical number line might help students make sense of their computations. She stated that this way students could see why -750 was correct and -1250 was not. These statements of the teacher seemed to be connected to her belief that students need to analyze different solutions in order to make sense of mathematics (B_M1).This way students could see their mistakes and also think about others' solutions.

Although introducing the number line in the classroom was not the goal of the teacher in that particular class, she introduced it as a student reasoned with the pay off idea during the discussion of a net worth problem. The number line actually seemed to help many students visualize the pay off idea as well as communicate with each other during their explanations since many of them explained their reasoning using the number line. Stephan's extensive knowledge related to students' imagery and use of tools was essential in enabling her to connect the student's reasoning with the number line to remediate students' misconceptions (K_p7).

Action 3: Thinking about the big ideas of the following day's classroom

After the first day's lesson the teacher noticed some students saying in their small groups that a person is worth nothing when the net worth is negative. Therefore, she decided to discuss the meaning of the negative net worth to evaluate how students conceptualized the difference when the debts were bigger than the assets during the daily planning session:

> Stephan: On the Brad/Angelina page, that was our first time getting a negative net worth and not everyone had a way to deal with that. Some students added Brad's total assets (TA) and total debts (TD), others found the difference and said he was worth a positive amount; others found the difference and said he was in debt $190,000 and others said he was worth nothing. I think I will lead off tomorrow's class with the Brad problem and ask questions about his net worth (NW). Some big ideas involved in tomorrow are how to deal with someone's net worth when his debts outweigh his assets. How do students conceive of this "gap" between TA and TD, especially when the TD is bigger? What does it mean to have a negative net worth? What does it mean to have no worth? Then, I will have them create their own NW statement for others to solve.

As it was seen in the excerpt, the teacher first reflected on students' solutions that they came up with during class, which led her to decide what she would do the following day. In order to elucidate the difference between negative net worth and no worth she also planned to ask specific questions that focus on the difference. Her knowledge of students' common difficulties was an important factor in deciding which misconceptions she needed to focus on more and discuss with the whole class (K_P2).

Stephan's goal of focusing on the big idea of the lesson (G_O3) might have driven her to reflect on the big ideas of the following day in her daily planning. Since Stephan valued students' ideas in the classroom as well as planning she reflected on and used both to make decisions for the following day's class (B_T4). For example she stated that she wanted to clarify the meaning of negative net worth and zero since some students seemed to have difficulty discerning the difference between them.

Action 4: Analyzing students' works to initiate discourse with different solutions

To initiate the following day's discussion, the teacher often analyzed students' solutions from the current day and selected the important and different ones that were related to the big idea of the lesson. Her knowledge of *Teaching Standards* (NCTM, 1991) where the analysis of teaching and learning was emphasized had an important role in motivating her reflection after each class and on how she analyzed the instructions as well as the homework that she collected from the students (K_C4). The excerpt below demonstrates the teacher's practice as she was trying to choose students' solutions in the daily planning to highlight during the following day's instruction from the students' work. The question was to find out the difference between two net worths when one person has -$2000 and the other $3000:

> Stephan: This one, he is thinking about going down, I don't know if I would do this one, it is similar. But we might need to encourage Danny a little bit, because he has been so out of the class before this unit. Alice: "1,000 more because the positive number is greater than the negative number". I might put that one up there. Anyway I will probably look across these and pick some out in order to contrast solutions and strategies as well.

As it is seen in the excerpt, during her selection she not only considered the right answers but also some misconceptions such as Alice's answer. She wanted students to compare and contrast the different solutions in order to make sense of them (B_M4). The reason for selecting the wrong answer besides the right ones seemed to be based on her belief that students' learn from their mistakes (B_L8). By discussing them in the classroom students would have an opportunity to understand why they were not correct. The teacher also chose multiple correct solutions to present to the students. By doing that she valued students' ideas and gave credit to them for coming up with different ways (B_T4). Emphasizing the different solutions was an essential practice of the teacher since it encouraged students to think in different ways.

Additionally, she also evaluated students' participation in the classroom and decided to encourage those that tend to be passive in the classroom. Her belief that the teacher should be fair and provide equal opportunities for all students might have motivated her to think about the student participation in the classroom (B_T6). Reflecting on the students' participation was an essential practice of the teacher, since the standards-based environment could only be created in a productive way when all students had equal opportunities to explain their ideas.

Action 5: Reflecting on the classroom environment

As it was stated above, during planning the teacher was reflecting on the current day's lesson and using her reflections to plan for the following day. Different goals took priority in this process. Although the goals were mainly related to the content, sometimes they also included goals such as renegotiating social norms (G_O1) to support a classroom environment where students can discuss their ideas freely and respectfully. The teacher's knowledge about the standards-based environment defined in *Principles and Standards* as well as in *Teaching Standards* (K_C3, K_C4) had an important role in including the classroom environment in her reflection:

> Stephan: Alice and Tisha were stuck in the symbol world without understanding their meaning. I love being in the symbol world only if it has meaning and it does not. I am trying to force them through putting their work into public like I did for Tisha at the end. Making meaning of some quantities other than just rules and memorizing. So it is why I picked Tisha at the end but she didn't want to be wrong in front of anybody. I do not know that is the case for everybody in here, it might be the case for the girls accept for Marsha; she is strong enough to take on the boys. I also told Stuart at the end if he comes about all cocky he is going to embarrass himself. But if he comes up with humility then we are all okay with that. I might have this discussion tomorrow more out and open for the difference between being arrogant and being humble. So that we can get some more girls feeling free, I have to call on the girls. I do not have to call boys.

In the excerpt above the teacher first reflected on the students' solutions in the class. She stated that Tisha and Alice were using the symbols + and - without making sense. Thus, she added that in order to encourage Tisha to defend her ideas and learn from her mistakes she invited her to explain her solution. The behaviors of Tisha seemed to indicate to the teacher that she did not want to be wrong in front of the classroom. This also made the teacher conjecture that whether most girls in the classroom might be feeling in a similar way because of the active participation of boys. Stephan's belief that the teacher is responsible for creating and sustaining an environment where ideas can be discussed freely and respectfully (B_T2) motivated her to plan on talking about the norms in the following day's classroom. Thinking about participation in the classroom and deciding to encourage especially the quiet students was an important part of her planning. Based on the events of the last class, she decided to focus on talking about how students need to behave in class on the following day.

In summary, the practice of *reflection* included actions of contemplating on the previous year's instruction, working on the problems before the instruction, making sense of students' solutions, thinking about the big ideas of the following class, analyzing students' works to initiate discourse, and reflecting on the classroom environment. During the reflection it was observed that the teacher's knowledge of curriculum, in particular the instructional sequence, had an important role on the teacher's actions. This allowed her to recognize whether an issue that students struggled with would be the big idea of later activities, or it would not come back later and therefore needs to be remediated in the next classroom session. Additionally, her pedagogical knowledge about students' possible solution strategies and misconceptions were helpful to understand students' ideas. Her knowledge about the tools and how students can reason with them also helped her to reflect on the different uses of the tools, recognize their incorrect uses, and create more activities to remediate those misconceptions. Her belief about teaching that valuing students' ideas and explanations empowers them to participate in the classroom and her belief of mathematics as a sense-making activity were also essential in her practice. These beliefs enabled her to contemplate on different solutions proposed by the students and bring the ones that she thought would help other students to the class discussion.

Practice Three: Anticipation

Anticipation was closely related to reflection, but it went one step further, by trying to conjecture what potential actions could take place in the classroom, so that when they happened the teacher was not caught by surprise and was prepared to respond effectively. In other words, while reflection involved looking back on the past events, anticipation involved looking forward to what might happen in the subsequent classes.

Action 1: Anticipating possible discourse during the introduction of the tasks

In the planning meetings, the teachers also talked about how to introduce shifts in symbolization. At this point they were talking about how to introduce a net worth statement activity that left off the words such as asset, debt, bank balance, car loan, etc. Until this point, all activities made explicit use of these terms such as the example shown on the left side of Figure 2.

Bank Balance: +$1000 Car Loan: -$15,000 Boat Loan: -$45,000 Retirement Fund: +60,000 Net Worth: $	+$5900 - $1700 -$2000 +$800 Net Worth: $

Figure 2: Shift in symbolization

However, from this point on the intent of the teachers was to drop the words and just use the symbols and the numbers as shown on the right side of the same figure. The excerpt below is taken from this discussion:

> Researcher: So here not only do we add the zero and negative net worths but also we remove the words [such as assets, debts, bank balance, car loan, etc.]
>
> Stephan: Yes, that is a shift in symbolization and it is very intentional here from the designers' point of view. I don't think I've ever had a classroom discussion about no names being there such as bank balance, car, etc.
>
> Jones: I think somebody in the other class said "Oh, make up your own [example]". Because once you did this blank page [referring to net worth statement problem], they kind of get the idea.
>
> Stephan: I don't know if it is so much of a discussion topic in class, but we'll see. I usually introduce this as "Is that going to throw you off guys if I just don't put these?"
>
> Jones: Or you can say people forgot to tell what that asset is for or what that debt is for.
>
> Stephan: I think I will introduce it by saying that "Guys, I am so tired of writing these up and I just left them all out. Is it ok if I just put + for assets and – for debts?" That is what I do. I always make up a story like this and say "Is that going to give you trouble?"

As illustrated by this excerpt, the teachers talked about how to introduce shifts in symbolization, how students might react to these shifts, and what the teacher can do to make these shifts as natural as possible for the students. The teacher's knowledge of the previous year's instruction had an important role in anticipating the possible discourse that might occur during the instruction (K_P4). Here the expert teacher revealed that she usually introduces this shift in symbolization by asking students if it is okay for them to not keep using the words for assets and debts as she is getting tired of writing them all the time. However, it is important to note that even though the numbers were standing without context in the activity, students could easily make up a story that might correspond to the numbers given by the teacher.

Stephan's belief that students learn better with an instruction that moves progressively from concrete to abstract (B_L6) activated her goal of supporting the development of meaningful imagery (G_O5). By thinking about the shift in the symbolization, the teacher showed that smooth transitions from context to symbols are important for her to give students opportunities to develop meaningful imagery. This way, the students could connect the symbols with the previous images that they developed and she could fold back to those images when necessary.

Action 2: Working on the problems before the instruction

One of the teachers' practices during the planning was to solve the problems themselves before starting the discussion to see the different strategies they developed. This action allowed them to anticipate the solutions that students might come up with and the images they might develop during the solution of the problems. This practice was likely rooted in the teacher's knowledge of the importance of anticipating students' answers (K_P1). The following excerpts were taken from the teachers' discussion after they solved the task individually that asked for two different people's net worth where one of them was in the positive and the other one was in the negative (this was the first time the students from the previous years encountered negative net worth):

> Taylor: [During the explanation of the problem last year] someone said for Angelina [who has positive net worth] you cash all her assets, you pay it off and this money is left [see Appendix C, "Angelina and Brad Net Worth Problem"].
>
> Stephan: That is right, somebody showed all those things[e.g. bank accounts, mutual funds…] and said she can pay off her debts but Brad cannot.
>
> Jones: That makes perfect sense "paying it off". I am trying to figure out what is the mathematical sense of paying it off? Is there any value to bringing it in that we can use later on?

Stephan: Two things I can come up with. One is when we get to integer operations and let's say we got $600,000 and pretend this is a transaction she goes in the debt of $790,000 and pay it off, you got more debt left.

Jones: So when you put it on the number line you only have enough to get going to zero but there is still more left over. That is the whole number line idea that we want to develop.

Taylor: You only have this much to pay off [$600,000] but you still owe this [$190,000].

Stephan: That is the idea of going under zero that took the math community so long to get. [Secondly] students also get some kind of sense of interpreting a negative net worth of $190,000.

Jones: The only thing I do not know is if they need a tool to model it.

Stephan: Not on the number line yet. This will be the long term benefit of having someone in class to contribute by saying that you can pay that much and you still have debt because it is the imagery for later on.

During the discussion both of the general education teachers anticipated the solutions as well as the images that students might bring into the classroom discourse such as the pay off idea and number line. The expert teacher suggested that it might be too early to introduce the number line unless the students bring it up, since the aim of these activities is not the transactions themselves but the meaning of net worth. This can be explained by the teacher's goal of focusing on the big idea of the lesson (G_O3) as well as introducing a tool when most students have developed the related reasoning (G_O6). The knowledge of the instructional sequence (K_C1) also helped the teacher to assess an appropriate time when introducing a tool.

Action 3: Anticipating students' thinking

When the teachers were talking about the activities that they would use in the classroom, they also anticipated students' possible solutions to the questions in the activity. The teacher's knowledge about students' possible answers (K_P1) based on the previous years of instruction was effective in anticipating students' solutions. The following excerpt shows the teacher's discussion for the question that asks for a new net worth when original net worth was -$5000 and $3000 debt is taken away:

Stephan: I have got both -8000 and -2000. They could also put 2000.

Jones: I tried to do what my students are going to do, I think. Everyone is going to put -5000 that is clear but some of them are going to put -3000 and then come up with -8000. Some of them are going to put -5000 and take away a debt of 3000 and I believe they are going to put 2000, but they may not know what to put as a sign. Somebody is going to do -5000 plus 3000 and get -2000 by writing the bigger number on top subtracting that and putting the sign of the bigger number.

Researcher: For the second one if they subtract the small number from the bigger and put the sign it won't work [the second one was: Meagan has a net worth of -$4300. A debt of $3000 is added. Is this good or bad? What is her net worth now?]

Stephan: That is right; they have to add in this case. Here is my thought for why we would go here [this page] first. It is because you want the number line up there now (see Appendix C, "Net Worth Problems").

Jones: It is going to support these things. Because some kids at this point still struggle and this will become a useful tool.

Stephan: And the teacher may reintroduce the concept if they need.

Jones: If students need it, you have to highlight it.

Stephan: But they probably won't. As they struggle through this, I will say "okay since you are struggling with this I am going to put it on the number line. So where do we start? -5000. I have been labeling the number line [i.e. writing the original net worth, transaction and the new net worth], I don't know if you have been doing this and I ask "what does it stands for? That is the original net worth" so I write original net worth, and then they will say "okay that is debt of $3000 taken away" and then I say what do I do here? How do I symbolize that on the number line?

After the teachers had worked on the problem for a few minutes, they came up with the possible strategies that students may create to solve transaction tasks. Stephan suggested that students might come up with answers such as -8000 and -2000 as well as 2000. Jones also anticipated similar types of answers and also stated that some students might find the answers by just making a computation such as subtracting the smaller number from the bigger one and writing the bigger number's sign. Next, the researcher stated that even though the students might find the correct answer by just using the computation without making sense, this would not work for the second problem. In the rest of the discussion Stephan suggested to use the number line to make sense of those problems. The reason that she suggested the use of the number line can be explained with her belief that mathematics is a sense-making activity where students analyze their solutions (B_M4). Since she noticed that students had difficulty deciding whether they need to add or subtract the given numbers in the context, she decided that putting a number line on the board and visualizing their solutions on the number line might help students to make sense of the problems. Stephan's knowledge about the possible students' solutions and the ways of supporting those solutions by using the tools were critical in her decision of how she could use the number line to make sense of their solutions for all students in the class and also to encourage them to use a number line in their explanations.

In another teacher meeting, in order to help make sense of numbers with one sign (such as -50) instead of two signs such as –(+50) or +(-50), Stephan added an activity to the sequence and brought it to the teacher meeting (see Figure 3). The activity involved matching different transactions with +50 and -50 (e.g. -2(-25) or -25+75). In the excerpt below, the teachers were talking about the activity and anticipated students' answers as well as the difficulties students might have:

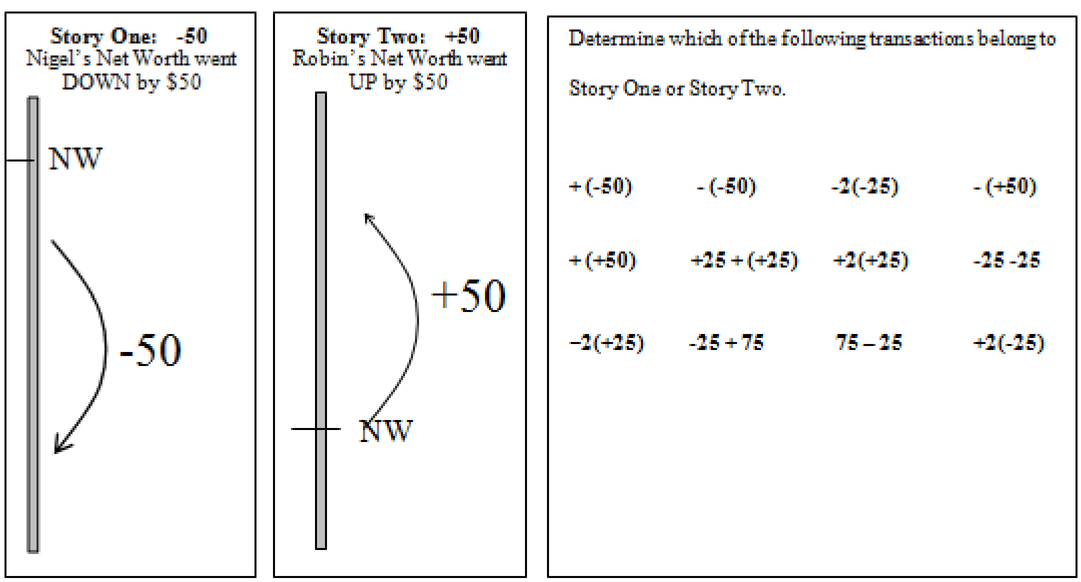


Figure 3: Net worth story problem

Wilson: Some of them may want to put the transaction on the number line. I am afraid for this one [the activity] because it is going to be hard for them to make that transaction as this is not a number sentence [e.g. 75-25] original net worth and transaction equal new net worth.

Researcher: If you say like go up 75 and down 25 with the gestures and totally it goes up 50 then they might understand better.

Stephan: You know what, why we won't do this? Because of this conversation [the teachers were talking on the same question for a while] when you introduce this task, we will introduce it like this. Story one, it has -50, what that means is Nigel had some net worth and this is in the introduction of the task and all we know is what happened to the net worth is it went down by 50. Which one of these transactions could have made his net worth go down by 50? Over here he started with some net worth and this time he goes up by 50. Which one of these transactions would you put over here to match that? What do you think about that? (See Figure 3)

Researcher: I think it is a good idea because they always refer to up and down and then they may say 25 down and 75 up and 50 up.

Wilson: I think so.

Jones: It might also help too if we do that on the overhead projector where they can write[their thinking on the picture of the number line] and when they go to the board say he is taking away two 25 dollar debts and they can see how it fits versus what somebody says.

Stephan: They can say "we go down this much and come back 75".

During the discussion of the task one of the teachers stated that students might think of the given transactions as number sentences that include the original net worth and transactions. Because they wanted to avoid this interpretation, Stephan suggested using two different stories where in the first one the net worth went down by some amount and up in the second. Her belief that the teacher is responsible for clarifying the tasks that students work on might have been influential in her thinking about the introduction of the task in detail (B_T12). Additionally, putting those increments and decrements on the number line and projecting it during the class, would help students visualize the amount of change on the net worth. The reason for including the number line might be to continue supporting students' understanding with the imagery that they already developed (G_O5). By giving students opportunity to connect the problem with the number line imagery where they used going up and down concepts, the teachers also wanted students to make sense of how a transaction can be interpreted with one sign. It was important for students to understand the equivalence of those expressions since many textbooks include examples that use only one sign such as -25 - 40 instead of -25 + (-40).

To summarize, the practice of *anticipation* included anticipating possible discourse during the introduction of the tasks as well as anticipating students' thinking. This practice was different from reflection in that it involved looking forward to what might happen next, while reflection involved looked back to what happened. The teacher's pedagogical knowledge particularly knowledge of students' possible answers and knowledge from the previous years of instruction played an important role in anticipation. Her belief that not all mathematical solutions are equal seemed to be activated in her anticipation of different solutions. Based on this belief, she not only thought about the most likely solution that students might come up with, but also other more or less efficient solutions. Her goal of supporting students' ideas with tools was also essential in anticipating students' reasoning with the tools.

Practice Four: Assessment

Using effective assessment practices was crucial for the teacher to understand the students' progress and identify their misconceptions and difficulties. These practices included creating effective formative and summative assessment tasks as described below.

Action 1: Creating formative assessment tasks

One of the advantages of holding teacher meetings after the class was that the teacher had a chance to think more about the difficulties that students had or might have based on their interactions with the students and create new tasks that would help students to overcome those difficulties. The following excerpt shows the teacher's assessment of a class interaction and her decision-making process to create new tasks based on her reflection:

Stephan: For that first dating game hardly anything came up differently than I thought. Most of the students put the assets together and put the debts together and found the difference. There was no negative net worth, it was not that complicated. The second one that we spent a lot more class time on and I saw some mistakes in calculations. Flora and some others put the lower number on the top and said it is supposed to be always assets on the top and debts on the bottom. Nobody said that and nobody made that rule. So I think I need to come up with a special page for that using net worth on the number line like the Sam, Sue problem [i.e. contrast different solutions].

While she was reflecting on the class, she stated that it went in the way that she expected. However, she also emphasized that in the second part of the activity while students were finding the total net worth, some students tried to subtract the smaller numbers from the bigger one regardless of which one was asset and which one was debt. Thus, she planned to create a new activity to get feedback from the students and help them remediate their mistakes by analyzing different solutions. In the lesson that the teacher reflected on above, since focusing on the computation was not the big idea, she did not want to spend too much time on the computations (G_O3). However, she knew from the HLT that she would encounter students' computational errors in the following lessons where the focus would be on integer operations (K_C1). Thus, she planned to create an activity where students could see their mistakes and make sense of why their solutions did not work.

In the weekly meetings the teachers continued to talk about the formative assessment tasks that might inform them about students' understanding. Since the students started to use the number line often in finding solutions to the problems and proving their solutions on the board, the teachers talked about the ways students used the number line in the classroom and also some students' misconceptions in using it. They decided to create a new task to assess informally students' understanding of the number line and help those who were struggling to make sense of it. The task that was created by the teachers included a different representation of the problem where the net worth was $4000 and the transaction was +(-$8000):

Jones: Is there any task focusing on these things [number line]?

Stephan: Maybe like the Sam, Sue problem. Three different students' solutions, who do you agree with and why? And focus on those gaps there.

Jones: Especially after coming from spring break.

Stephan: I am right here on page 21 (see Appendix C, "Alice's Net Worth").

Jones: I have not started that one yet.

Stephan: Are you all in net worth problems?

Wilson: Yes.

Stephan: So we are all around the same place. Maybe on Monday we can give a bell work problem, you all help. [Bell work activities are the activities that are given to the students as soon as class begins. These activities particularly aim to a review the material from the previous day].

Taylor: Sam and Sue problem for bell work?

Stephan: Something like that [Everyone tries to prepare a bell work question for a few minutes].

Jones: What if, we had a couple of number lines with the stuff on it incorrectly and had a story and say "which number line tells the story?" It is what you did?

Stephan: Pretty much. Here is the story, which number line tells the story? So mine is adding debt and I made one of them go up 12,000 on purpose, one goes down to -12,000 and I purposely did not do two jumps not to give it away. And then you can defend which one you agree with, somebody will hopefully go through zero to defend this one (see Figure 4).

Jones and Taylor: I like that.

Wilson: So you kind of picked one right answer and two incorrect ones.

Stephan: I do not always pick one answer. Sometimes I have two right answers. I do not want to put the other right answer that goes through zero because I want that to come back from them, it is too leading for me.

Once the teachers talked about where they were in the instructional sequences, they first worked on the bell work problems individually and then collectively. They decided to create an activity that represented a given transaction with multiple number lines with only one representation being correct (Figure 4)[6]. The teacher's belief that formative assessment plays a crucial role in effective teaching (B_T8) might have motivated her to create a task together with the other teachers to evaluate students' understanding of integer operations and whether they could use the

[6] The number lines were red and black in the original activity. Here they are shown in monochrome for printing purposes.

number line in a correct manner. Another reason to create formative assessment tasks seemed to stem from her goal that the teacher should ensure that students understand the mathematical solutions that are discussed in the classroom (G_O7).

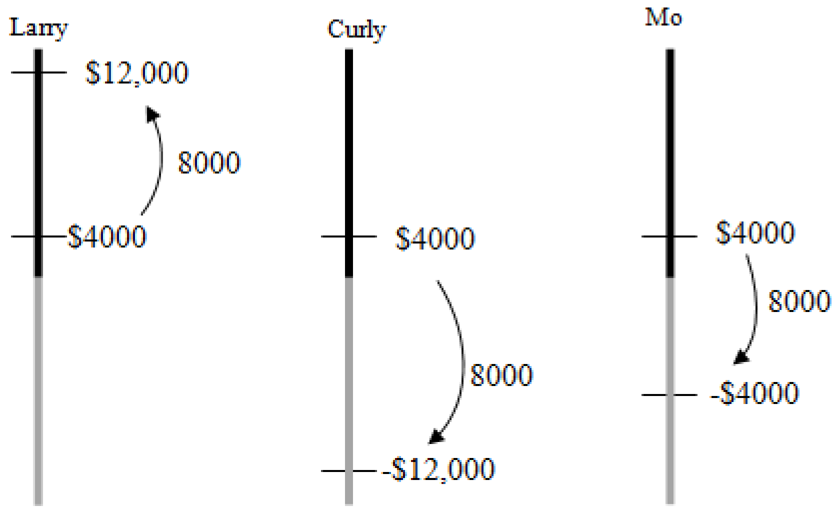


Figure 4: The formative assessment task

During the discussion, Stephan also suggested that the number lines given in the solutions should not include two arrows (going down to zero and the left over) since it might be too leading for the students. The reason that the teacher did not want to include too much detail on the number line might have stemmed from her goal that students should use the tools the way they reason with them (G_O6). For example, in this problem although some students solved the problem by going first to zero and then the left over, some of them symbolized their solutions by only one arrow without stopping on the zero. Creating problem solving tasks was an important action of the teacher since she believed that those tasks might help students to understand mathematical ideas (B_M3) as well as give the teacher more accurate feedback about students' thinking.

Action 2: Summative assessment

Stephan created a quiz two weeks after the instructional sequence started and shared it with the other teachers that were present in the teacher meetings. Once they agreed on the questions, all the teachers applied the quiz to assess their students' development. The questions in the quiz were similar to the tasks that were solved in class. For example she included the tasks where students were asked to find the total net worth such as with the Angelina-Brad problem, comparing net worths and some transaction problems. Additionally, the word problems included a supportive vertical number line for the students who might need it in the solution of the problems (see Figure 5).

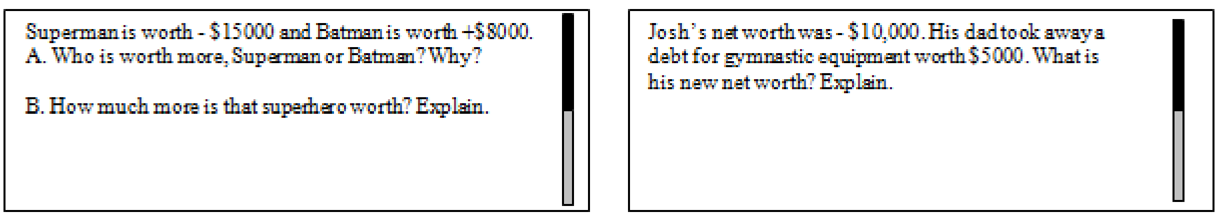


Figure 5: Example from quiz

The quiz was important for the teacher to understand how students were solving the tasks and if they understood the problems that were discussed in class. It also gave insight about the students' difficulties as well as the strategies that they used during the solutions to the problems. Based on the assessment, the teacher revised her planning and discussed more problems similar to the ones with which the students had the most difficulty. During the instructional sequence, the teacher continued to create formative assessment tasks almost every day such as bell work that students solved at the beginning of the lesson.

One of the teacher's practices related to assessment was to create a notebook quiz every Thursday and apply it on Friday. The aim of the notebook quiz was to ask questions from the activity sheets in order to encourage students to keep their activities organized and also write the answers on it in the classroom. The time for the quiz was only five minutes which was more than enough for a student who had their papers with the answers as they did not have to explain the answers. During these quizzes the students were not allowed to talk or use each other's paper. The preparation of these notebook quizzes seemed to be motivated by the teacher's belief that taking responsibility is crucial for one's learning (B_L3). As the teacher continued to apply the notebook quizzes, the students started to become more responsible to learn the tasks that they missed when they were out of school and to take notes during the discussion of the problems in the classroom without the teacher reminding them.

To prepare a comprehensive assessment task at the end of the instructional sequence, each teacher took one or two big ideas of the integers units and created (or adapted from the textbook) problems to evaluate those ideas. Once the teachers selected the problems, they came together in a meeting and discussed the problems that they chose. During the selection of the problems, they considered students' anticipated answers and misconceptions related with those tasks from different perspectives. Stephan's belief that collaborating with other teachers is essential in improving one's instruction and assessment seemed to have an important role in coming together with other teachers to create a comprehensive test that included different types of problems (B_T14). The teachers together created problems that evaluated conceptual understanding of integers as well as computational skills. In the unit test, the teachers included problems with a financial context as well as different contexts such as filling the spaces in a given temperature table and bare number problems. The teacher's knowledge of content and students' possible misconceptions allowed her to create assessment tasks that would evaluate students' understanding of mathematics effectively (K_S1, K_P2). By planning together all teachers applied the same unit test in their classrooms and had a chance to evaluate and compare the students' achievement after the test.

In summary, the practice of *assessment* included actions of creating summative as well as formative assessment tasks. The teacher's knowledge of content and curriculum, in particular the instructional sequence, was crucial in preparing both procedural and conceptual assessment tasks that would help her understand and evaluate the students' mathematical thinking. Additionally her pedagogical knowledge particularly the knowledge of students' misconceptions was important to create formative assessment tasks that would remediate students' difficulties. Since she believed that problem solving is important to understand mathematical ideas, she created assessment tasks that would help her understand students' thinking in the solutions of the problems. Her goal of ensuring that students understand the mathematical ideas that were discussed in the classroom also seemed to be highly activated during the planning of her formative assessment to get feedback from the students.

Practice Five: Revision

Revision was a key part of planning where teachers improved the instructional sequence and their methods of teaching based on the feedback acquired through reflection, anticipation, and assessment. It involved adding new tasks to the instructional sequence, changing the order of the existing tasks, and making changes to the HLT to include new images, gestures, tools and possible discourse.

Action 1: Revising the instructional sequence

One of the main practices that emerged during the teacher meetings was to revise the instructional sequence by adding new tasks or changing the order of the activities. In one of the meetings the teachers stated that they needed a task to evaluate students' understanding related to ordering the numbers. Thus, they talked about how they could assess students' understanding of correctly ordering integers, negative numbers in particular. The expert teacher suggested after the activity which includes positive, negative, and zero net worths, that they might ask a problem that contained two negative net worths. The teacher's practice of talking about the objectives of tasks and how they are related to the following tasks played an essential role in helping the teachers see the route of the instruction and make decisions accordingly:

> Jones: Why do you want to use more than one negative net worth in here?
>
> Stephan: Because to me it is the same idea and they work on integers not integer operations so we want to confront integer values before we are going to start with integer operations.
>
> Wilson: Is it coming later in the sequence?
>
> Stephan: It will come up later in operations with addition and subtraction rather than before. But then there are multiple ideas we are going to have to deal with. They are going to have to talk about "how do I take away an asset?" That is a big conceptual step to do, just to be able to do that operation. All of a sudden they will also talk about this negative 500 and negative 600. Which one is bigger or smaller is another huge idea and I do not want those two to happen at the same time. Integer concepts and operations, there are two parts of that. There is a concept of integer, the values of integers how one integer is related with another one, for example in our mind 100 is bigger than zero and negative 100 is in our mind clearly bigger than negative 200. But it is just related to understanding integers then you start operating with them and doing some adding and subtracting. So we will work on the concept part first, so I think we need to introduce it here.

As it is seen in the excerpt above, Stephan stated that ordering of negative numbers is an important conceptual step where the teacher needs to make an assessment about whether the students are comfortable with it. She emphasized that it might not be appropriate to assess students' thinking about this during the transaction tasks since those tasks themselves are also big ideas. She added that it might be difficult for students to tackle two important ideas at the same time. The teacher's goal of tackling one big idea at a time (G_O4) seemed to have shaped the teacher's decision about the timing of evaluating students' understanding of ordering negative numbers. Her knowledge about the tasks that would follow as well as their purposes (K_C1) also motivated her to deal with the order of integers before operating on them. To her, the integers were composed of two different parts: concepts and operations. While ordering of the numbers belonged to the concepts, transactions were part of the operations. In the following discussion the teachers decided to create a task that satisfied the objective of ordering negative numbers as well as comparing them:

> Jones: I want to be sure that when I put bell work up there… all the kids realize that zero is not same as negative numbers. Then they understand the abstract value; you can go below zero and it makes you go in more debt.
>
> Researcher: Are you going to make them to compare negative numbers also?
>
> Jones: Yes… I think you cannot compare negative numbers until they conceptualize it as abstract value.
>
> Stephan: Because you cannot conceptualize anything below zero you have difficulty on that…I am really rethinking the order of sequence now. I think the page after this (see Appendix C, "In the red"). I think we need a number line here. I really think so because even with concept of integer forget operation right now, it is a big deal to order those numbers on the number line.
>
> Wilson: I want to see order on the number line. I want to see that negative 4000 is below negative 1000 and I want them to see that, too.
>
> Stephan: I think we need a zero on that, then we have a page asking to order a bunch of positive and negative numbers. And we may ask how much Juli's worth more than Didem? And then operations. This way, they can start making objects out of distances from zero and they might also start thinking about those pay off ideas.

During the discussion Jones stated that he wanted to know if his students were able to understand the difference between negative numbers and zero as well as other negative numbers. He stated that he wanted the students to be able to conceptualize the negatives as an abstract idea. The expert teacher emphasized that the number line can be helpful in ordering the numbers. Another teacher, Wilson, also agreed and stated that she wanted to know if her students were correctly able to order the negative numbers on the number line. The discussion motivated the expert teacher to create a new task where students could show whether they were comfortable with ordering the numbers on the number line and also whether they could see the gap between two numbers. She used a vertical number line since it is consistent with the idea of "the more debt you have, the deeper you go financially". This way the activity would be connected with the students' previous images (e.g. assets, debts) and it would also support development of meaningful imagery where students can relate the ordering of integers with asset and debt concepts (G_O5). The teacher's knowledge of the instructional sequence as well as the imagery had a crucial role in creating this activity (K_C1, K_P7). Since she knew the overall sequence she could connect the previous tasks with the subsequent tasks and recognize what was missing. Her knowledge of imagery also allowed her to create activities that included the number line, which students could use as a supportive reasoning device and develop images that would help them understand the subsequent tasks. Additionally, her belief that mathematics is a sense-making activity (B_M1) allowed her to create new tasks to give students opportunities to explore mathematical ideas and realize their potential misconceptions.

In the meetings, the teachers also changed the order of the instructional sequence as necessary based on the interactions with the students. After the "good and bad decision" where the students decided whether a given transaction was good or bad, the instructional sequence was followed with the net worth trackers that included vertical number lines and the transactions below them without linking them to the context (see Figure 6).

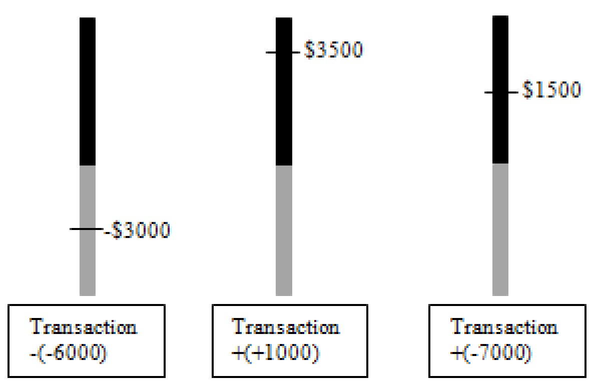


Figure 6: Net worth trackers

The teachers decided to give this activity to the students in the following class since they wanted students to think of using the number lines themselves when working on the problems related to the situation-specific imagery. Instead of the activity above they decided to use the activity that included word problems such as "Donald has a net worth of -$5000. A debt of $3000 is taken away. Is this good or bad? What is his net worth now?" The following excerpt shows the practice of teachers' revising the sequence:

> Stephan: I am going to make a suggestion. Because of the fact that students are not naturally using the net worth tracker or the number line to reason with, this page is premature [net worth tracker activity] I think. I want to go to the page 20 (see Appendix C, "Net Worth Problems") first, have them do it themselves just with context like it used to be in the sequence. This was a logical next page, and then my strategy is going to be, let's pretend everybody gets the same answer on that number one for Donald… I am going to work it out real quick [teachers work on the problem for a few minutes].

As it was seen in the excerpt the teacher suggested changing the order of the instruction since the students were not using the number line naturally. Thus, instead of giving the activity illustrated in the figure, she argued that it would make more sense to continue with a context where students might themselves prefer to use the number line to find the solutions. The teacher's goal of supporting development of meaningful imagery (G_O5) as well as supporting students' ideas with tools when they have the reasoning (G_O6) might have motivated her to change the order of the activities. This way, the students would be given more opportunities to use the number line in their solutions since they needed it rather than giving it to them ready-made. Once they felt more comfortable using the tool, they were given the net worth trackers. This allowed students to draw back to their previous images that they developed during the solution of the problems similar to the one above.

The practice of *revision* mainly included revising the instructional sequence. The teacher's knowledge of the sequence allowed her to notice what was missing and whether the activities in the sequence were in the right order to support students' mathematical development.

Her knowledge related with the use of tools also had an important role, as this knowledge allowed her to create new activities to remediate incorrect reasoning with the tools. The teacher's belief that mathematics is a sense-making activity was also central in her revising the sequence by adding new tasks or changing the orders of the problems in order to help students explore mathematical ideas. Her goals of supporting the development of imagery and supporting students' ideas with the tools were also highly activated in the application of this practice.

Summary

Planning is an essential part of standards-based teaching (NCTM, 2000). Teachers need to prepare for what they teach before they actually teach it. During this study, the expert teacher exhibited important planning practices. Before starting the integers sequence, she studied the relevant literature to learn more about the different methods of teaching integers as well as the history of mathematics related to how the mathematicians developed the integer concepts and operations. She adapted the knowledge she derived from the literature into the instructional sequence to determine the ideas that she needed to emphasize and anticipated the difficulties her class might encounter during instruction.

The teacher arranged meetings to come together with the other teachers to co-plan the instructional sequence. This created an environment of sharing and learning where the teachers shared their ideas about the activities and organized their thinking before instruction. During the meetings, they talked about the previous year's instruction such as what kind of ideas students brought to the class during whole class discussions. It was helpful for the teachers to remember the previous year and reflect on and also change some parts of their instructions based on their sharing of experiences. They also talked about each activity to unpack their big ideas, objectives, and their relationships with each other. During the discussion of the big ideas, they also talked about what kind of questions they could ask to the students to assess their understanding of these big ideas. These questions served as formative assessments where the teachers could get feedback from the students and revise the instructional sequence based on their interaction.

In order to facilitate an effective discourse, the teachers also talked about the different ways of introducing various activities. They first individually worked on the problems of the activities to come up with different solution strategies and then compared them with each other to anticipate students' solutions. This helped them to not be caught by surprise when the ideas were later brought up by the students. The planning meetings gave the teachers opportunities to review the objectives of the instruction. During this review, if they noticed that the current set of activities did not cover all the objectives they created new activities to fill in those gaps

In the analysis, the planning actions of the teacher were divided into five practices named preparation, reflection, anticipation, assessment, and revision. These categories were not isolated from each other; in fact they belonged to a planning cycle as depicted in Figure 7.

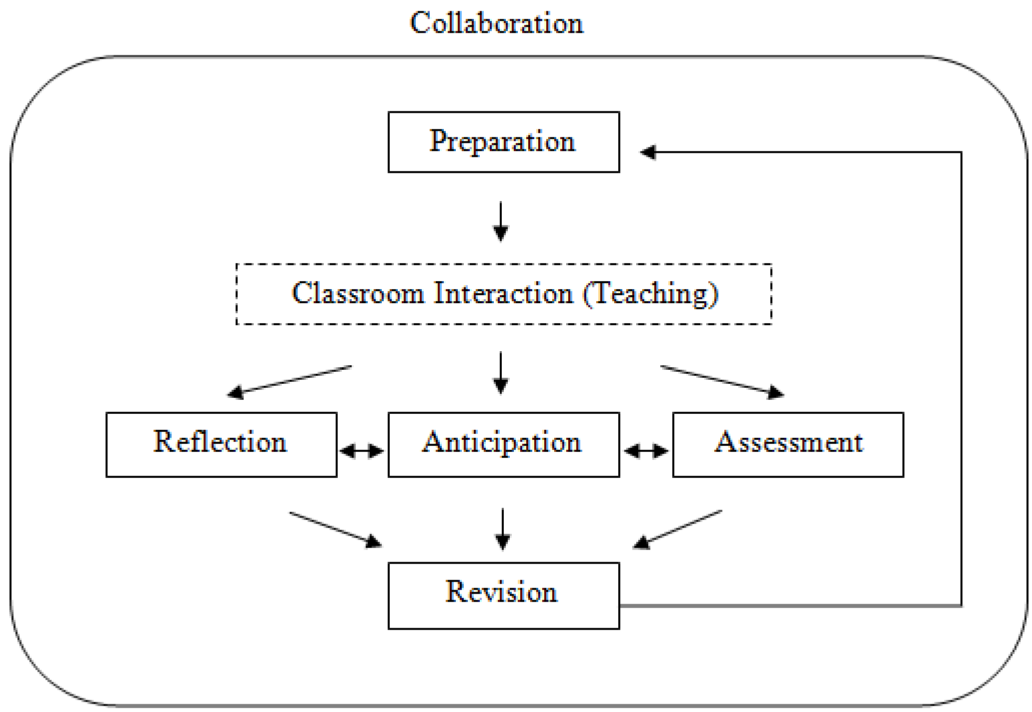


Figure 7: Planning cycle

In this figure, all practices are shown within the circle of collaboration, which supported and enriched all of the practices. The cycle started with the teacher's initial preparation, which also included reflection, anticipation, and assessment (arrows not shown). This stage was performed before the beginning of the instruction. After the classroom interaction, the teacher performed planning practices that involved reflection, anticipation, and assessment. These practices resulted in the revision of the instructional sequence which was done during the preparation for the next day thus completing the cycle. All of these practices were conducted within the learning community created by the participation of several teachers and the researcher.

CHAPTER SIX: ANALYSIS OF CLASSROOM TEACHING PRACTICES

To find the general categories of teaching practices employed by the expert teacher, the classroom data were analyzed to identify the actions that she performed during the classroom instruction. Using the relationships between these actions, interviews with the teacher, teacher notes, and the literature, these actions were then grouped into five practices: (1) creating and sustaining social norms, (2) facilitating genuine mathematical discourse, (3) supporting the development of sociomathematical norms, (4) capitalizing on students' imagery to create inscriptions and notation, and (5) developing small groups as communities of learners. The classroom dialogues that illustrate the teacher's practices were selected from the entire instructional sequence by triangulating among the data to make sure that the selected examples actually represent the teacher's practices rather than being outliers. The extracted practices were then analyzed using the teacher's beliefs, goals, and knowledge.

Practice One: Creating and Sustaining Social Norms

Social norms are the structures of classroom participation that are established by the teacher and students (Cobb & Yackel, 1996). The teacher has an important role in creating and sustaining the norms that become taken-as-shared by the classroom community. Since the study was conducted in the second semester, the data did not include the teacher's role in establishing the norms but included how she re-established and sustained them during the integer sequence.

Action 1: Encouraging students to express agreement and disagreement

As it was explained in Chapter 5, the instructional sequence started with introducing the context of assets and debts and continued with net worth problems. The following excerpt is taken from the second day during the discussion of a problem asking students to compare Angelina's net worth ($90,000) to Brad's (-$190,000):

Dusty: The negative $190,000 is how much Brad is in debt.

T: Dusty says it is how much Brad is in debt [she writes this on the board]. You all agree with that?

Sts: Yes.

T: Who doesn't agree with that? Nathan?

Nathan: I think it is how much net worth Brad has.

T: It is the net worth [the teacher writes on the board]. Why don't you think it is his debt? Hold on… [Another student wants to answer].

Nathan: Because [he pauses]. I do not know. I just don't get it.

T: If you disagree with Dusty than you need to explain why you do not think it is right. Sally what do you think?

During the discussion, Dusty stated that the total net worth was negative because it showed how much debt Brad had. The teacher restated his words and asked if students agreed with Dusty. At this point, although many students seemed to agree, Nathan stated that the number shows the net worth not the debt. The teacher asked him to explain why he did not agree that it was debt as Dusty had suggested. Nathan answered that he did not know the reason and he had difficulty understanding. The teacher used this opportunity to remind students that it is important to explain their reasoning when they disagree with an argument. Her decision was primarily shaped by a goal, namely that the teacher should remind students of the expectations as necessary during the instruction (G_O1). When Nathan could not explain why he disagreed with Dusty's argument, consistent with her goal, she reminded students that they should try to explain their reasoning.

Another goal in this episode was to make sure that all students could understand negative net worth as an abstract (intangible) quantity (G_C3). Her knowledge of students' difficulties in explaining negative net worth (K_P2) prompted her to discuss its meaning and ask students to state their agreement or disagreement during the discussion. Also her knowledge of the instructional sequence (K_C1), where understanding the meaning of net worth was an essential element, seemed to allow her to focus on the idea of negative net worth.

Action 2: Encouraging students to understand others' solutions

During the third day students continued to solve problems where they worked on finding different net worths and comparing them with each other. In these activities one of the goals of the teacher was to support students to discover the additive inverse method of integer addition and subtraction (G_C4). Although some students could see the additive inverse and cancel out the numbers that are inverses, many students continued to find the net worth by finding the difference between assets and debts. In the whole class discussion some students insisted that their methods were easier, and did not want to pay attention to other students' methods. At this point the teacher reminded them of the importance of understanding others' solutions:

> T: How many people came up with the strategy where they added all the assets and all the debts and find the difference? Raise your hand [Many students raise their hands].
>
> Tisha: That is easier for me.
>
> T: That is easier for you, isn't it? Gage you did not do that? All right listen to Gage he says he has a quicker way than you. Let's see.
>
> Gage: Does everyone see? [He goes to the board and asks whether everyone sees what he writes on the board]
>
> T: It is a great question. Does everyone see? I expect all of you to listen to Gage and make sense of what he is saying and if you have a question Brad what are you supposed to do?
>
> Brad: Ask him.
>
> T: You better ask him. Go ahead Gage.

In this discussion, Tisha stated that the way she computed the net worth, which is finding the differences between assets and debts, was easier for her. At this point, the teacher asked Gage who solved the problem by using the "cancel out" method to explain his solution on the board. She also reminded students to listen to Gage and try to make sense of his reasoning, and ask questions if they do not understand. This seemed to be motivated by her belief that the teacher is responsible for giving students opportunities to see efficient and different ways of solving problems during the class discussion (B_T9). Asking Gage to explain his solution and encouraging students to carefully listen and ask questions was also motivated by the teacher's knowledge about students' tendency (K_P3) to stick with their own ways even when they are not efficient.

Action 3: Encouraging students to explain their solutions

During the explanation of the "cancel out" method in day three, Mark stated that he had difficulty understanding when students explain by just using words:

> T: Mark, any questions for him?
>
> Mark: I just don't like when people explain. I just know it is working out.
>
> Gage: You can also … [he is trying to explain the problem]
>
> T: Wait, what are you talking about? [she is asking Mark]
>
> Mark: I get more confused when people explain. If they just show what they do, it is enough for me.
>
> T: You can get it just by symbols.
>
> Mark: I do not need them to explain.
>
> T: It is a good point Mark now you can see why writing down really helps. Because Mark is not really good at processing with what you say but he is really good at processing what he sees so it would help him to write something down. But for some of us who do not get it only with symbols, it really helps to hear something doesn't it Nathan? Since you are saying "say it again because I did not catch you the first time". So both of them are pretty important: writing them down and speaking it out loud. Brad do you want to say something else?

When this discussion took place, Brad was explaining his method but Mark interrupted by saying that he gets confused when someone explains by just using the words and he would rather see the symbols. He insisted that he would rather not hear the solution as it makes him more confused. Although this was a negative reaction that the teacher did not anticipate, she managed to turn this into a constructive argument by first acknowledging that Mark has a point. The teacher converted a negative reaction by a student to an opportunity to remind the students of the norms again (G_O1). If she had not encouraged Mark by acknowledging his point, he could have been discouraged to openly express his disagreement in further discussions.

She then explained to the students that some people are better at understanding what they see and some are better at understanding what they hear, therefore they should do both when explaining their solutions. She also reminded students that during a verbal explanation, they have the opportunity to ask if they miss some parts of the explanation. The belief that each student may learn differently (B_L2) had an important effect on the teacher's decision-making process. This belief seemed to be activated by her knowledge that some people are better at processing auditory information while others may feel more comfortable with visual information (K_P10).

Action 4: Asking students to repeat other students' solutions

In the same day (day 3), students were given an activity where they were asked to find out the total net worth of a person. Since the net worth was negative, the teacher asked students explain what a negative net worth might mean:

T: What does it mean? Marsha what does his net worth mean?

Marsha: If he uses all his assets and pays off his debts that is how much he would need more.

T: Stuart what did she just say?

Stuart: If he paid off all his assets to … [inaudible]

T: He pays off. Say it again Stuart, real loud.

Stuart: He pays off debts by using assets.

T: So he uses his assets to pay off his debts.

Stuart: Yes.

T: Can you tell them, what happens? Brad, finish that.

Brad: He actually has assets but his debts overwhelmed his assets

T: Norman, say what Brad just said?

Norman: When he pays off debts from his assets that is how much he had left.

The teacher's goal in this discussion was for the students to understand what it means to have a negative net worth (G_C3). During the discussion, the teacher first asked Marsha to explain the meaning of negative net worth. She stated that it would be the money he needs to pay after he paid off his debts by using all his assets. The teacher asked Stuart to repeat what Marsha said. However, since his voice was not loud enough for everyone to hear the teacher asked Stuart to tell it "real loud". He stated that he would use his assets to pay off his debts.

Next, the teacher asked Brad to finish what Stuart just said. He said that his debts overwhelmed his assets and then the teacher asked Norman to say what Brad said. He stated that it would be the money left over once he paid off his debts from his assets. From the point of view of sustaining social norms, asking students to repeat what they have heard was done to help them understand their obligation to listen to and understand each other. Thus, the practice of asking students to repeat each other's solutions seemed to be underpinned by the teacher's goal of reminding students the expectations (Go1) and ensuring that students understand each other (G_O7).

Action 5: Encouraging students to behave respectfully

During day five, the students were given an activity where they were asked to compare two people's net worths on a vertical number line. The question was to find how much more Gilligan's net worth of $3000 was than Marry Ann's net worth of -$2000:

> T: Let's hear from Stuart. This might sort this out. All right Stuart you are up.
>
> Stuart: I will restate the question for the people that do not understand what the correct answer is
>
> T: All right please state what 1000 is [which as Stuart's answer]
>
> Stuart: How many more, more is even in bold letters, is Gilligan worth than Mary Ann and show on net worth line. That is net worth line, correct? More which means basically you have $2000 in debt and you got 3000 here, we are trying to find how much….wait a second, now I understand, never mind.

> T: Never mind, what? You were sure before. All right time out. Thank you. I think what we just saw, I know you are ambitious to prove the other one but listen for a second. Stuart should not walk away embarrassed, too, because he changed his mind when he is up there. Is that okay to do?
>
> Sts: Yes.
>
> T: Maybe why he is a little embarrassed is he was so cocky before, right? He was talking about like you all did not know what you are talking about. So may be when you come up here you should not be so cocky but it is okay to change your mind. Exactly, what he did is okay to do , all right?

In the classroom discussion of the problem although many people argued that the answer was 5000, Stuart and his group stated that the answer should be 1000. The teacher asked Stuart to explain why it was 1000. The fact that she did not immediately dismiss an incorrect answer but asked the student to explain his reasoning stemmed from her belief that students learn better when they discover their own mistakes. This behavior was also consistent with her belief that mistakes can be used as learning opportunities (B_L8).

In his explanation Stuart started his argument by stating that he would repeat the question to the students who did not understand. He did this in an over-confident manner. After that, during his explanation, he noticed that his answer was incorrect and went back to his place without saying anything. The teacher commented that Stuart should not be embarrassed because he did it incorrectly. However, she also emphasized that had he not been so cocky when he came to the board, he would not have been embarrassed upon realizing his mistake. She reminded students to be more humble but at the same time not to be shy to admit when they make a mistake. Stephan's belief that an environment that nurtures respect and a sense of community plays an essential role in students' learning (B_L7) had an important role in the conversation emphasizing the importance of being humble during their explanations. Her knowledge of standards-based environments which supports a sense of community where students can express their ideas honestly and openly without fear of ridicule also seemed to have an essential role on her actions (K_C3).

Action 6: Encouraging students to use mistakes as learning opportunities

During the same day (day five) the teacher asked students similar types of problems where they could work on finding the difference between two net worths. The following excerpt is taken from the discussion of a problem where students were asked to find who is worth more if Paris has -$20,000 and Nicole has -$22,000:

> T: We have a lot of things to talk about. Nathan, have you figured it out? [he shakes his head, meaning no] Come on up. Nathan says I cannot come to grips with this problem. He is looking at us to present because he does not have any answer to this one. He needs some help. He is going to tell you what he is thinking so far and write up there what he is thinking so far and when he gets stuck he will say "here's where I am stuck guys, help me out" Let's hear from him first. Nathan tell everybody what you did and why you did it and then we will start helping [Nathan puts the numbers on the number line in a wrong order where -22,000 is closer to zero than -20,000].
>
> Nathan: I do not know, I just put the numbers.
>
> Gage: I just want to show quickly.
>
> T: Wait do not help him, so he knows what he has done.
>
> T: Tell us why you put Nicole there and why you put Paris there? Is that logical question to ask?
>
> Sts: Yes.
>
> Nathan: I do not know.
>
> T: He says that he does not know. Sit Gage, I appreciate it. We want to hear from new people. All right, Flora help him out.

In this episode, Nathan was explaining his solution on the board but he seemed to get stuck and asked for help. The teacher's belief that students learn from their mistakes (B_L8) motivated her to invite Nathan to the board to explain his reasoning even though his solution was incorrect. At this point, Gage, who was very active in the classroom, wanted to come to the board and show his mistakes. The teacher did not want Gage to interfere immediately because she first wanted Nathan to try to explain where he had difficulty. She asked Nathan why he wrote the numbers in that order. Since he said he did not know why he wrote it like that the teacher asked the other students to help out. This decision was consistent with the teacher's belief that a sense of community plays an essential role in students' learning (B_L7).

At this point, Gage immediately went to the board to explain without asking for permission, which caused the teacher to say that while she appreciated his enthusiasm, she wanted to give other students an opportunity to explain as well (since Gage had already contributed to the discussion several times). This seemed to be motivated by her belief that the teacher should be fair and provide equal opportunities for all students (B_T6). Here it is important to note that the teacher was careful about doing this without making Gage feel upset, which might have caused him to shut himself down in the following discussions.

Action 7: Renegotiating the social norms

Since some students started to behave disrespectfully to each other during the classroom discussion, the teacher needed to have a conversation with the class on day six to remind students how they should behave in the classroom:

> T: That is exactly what I was thinking. When I announced that some people have a hard time with this, what was the natural response I got from you? About 4 or 5 people said oh that is so easy. What does it imply?
>
> Sts: Cocky.
>
> T: How does it make people feel?
>
> Sts: Sad.
>
> T: Do you think they want to contribute to our class now?
>
> Sts: No.
>
> T: I would not either, if I were them. Norman, what would you like to say about humility?
>
> Norman: I do not know.
>
> T: Humility sort of relates to humiliating somebody but it is positive. When you come here I want you to act with humility and be humble. You guys ever heard the word humble? Be humble, do not act like you all figured it out, because as soon as you do Stuart what happens?
>
> Stuart: You get proven wrong.

> T: How does that make you feel? Some people with good self-confidence are okay after that. Do not come to class thinking you figured it all out. Also think about how it makes other people feel. I do not want to hear anyone say "this was easy, what is the problem?" It is not nice.

This discussion was triggered when some students claimed that it was "so easy" for a question for which other students had difficulty understanding. The teacher asked students what this kind of behavior implies and how it makes other people feel. She connected the situations with her knowledge of what happened in the classroom in the previous days (K_P11) to students who were cocky and used it to remind them of the importance of being humble.

She reminded students that when some students feel that their ideas are not valued by others, they might not want to contribute to the classroom discussions anymore. Stephan's beliefs that the teacher is responsible for creating and sustaining an environment where ideas can be discussed freely and respectfully (B_T2) allowed her to talk about the ways students need to behave. She explained the difference between humiliate and humility, and asked them to behave with humility and humbleness in the classroom. This way they would not feel shy and embarrassed when they realize their mistakes, and other people would not be offended by their behavior.

Action 8: Encouraging students to take responsibility/ownership for their learning

On the following day, just before the spring break the teacher made an activity named "chalk-talks" where students wrote their ideas about the classroom on the board:

> T: I have new activity for you today. We have been hearing some positive comments and some negative comments around the room lately. It might be fun to get that frustration out and get some enjoyment out. Here is what we are going to do. This is a silent activity. You cannot talk. If you talk during the game, you have to sit. Each person gets a marker and on this side of the board we will right some joys like I heard in the second period "we like arguing and arguing is fun in this class" if it is what you think then you will put "arguing is fun in this classroom". It is called chalk-talk by the way or marker-talk now. A concern in one of my classroom was they said "the teacher does not teach us". Write that, I am not going to be offended. Or you can write "Mrs. Taylor does not help me"; write that only if you believe it. Everybody will have a chance to write something. Do not write anything about other people. We are not going to be negative and attack each other. You can respond to someone who writes such as "the teacher does not help me", you can response "but that is help". You can say "I agree" or "I don't agree."

After the introduction of the activity she separated the board into two parts for the things students liked to do in the class and those that they did not. Once the students finished writing their ideas, she started a conversation about them (see Figure 8). The following excerpt is taken from the discussion of the idea "figuring out the things on their own first in the classroom" for which some students indicated dislike:

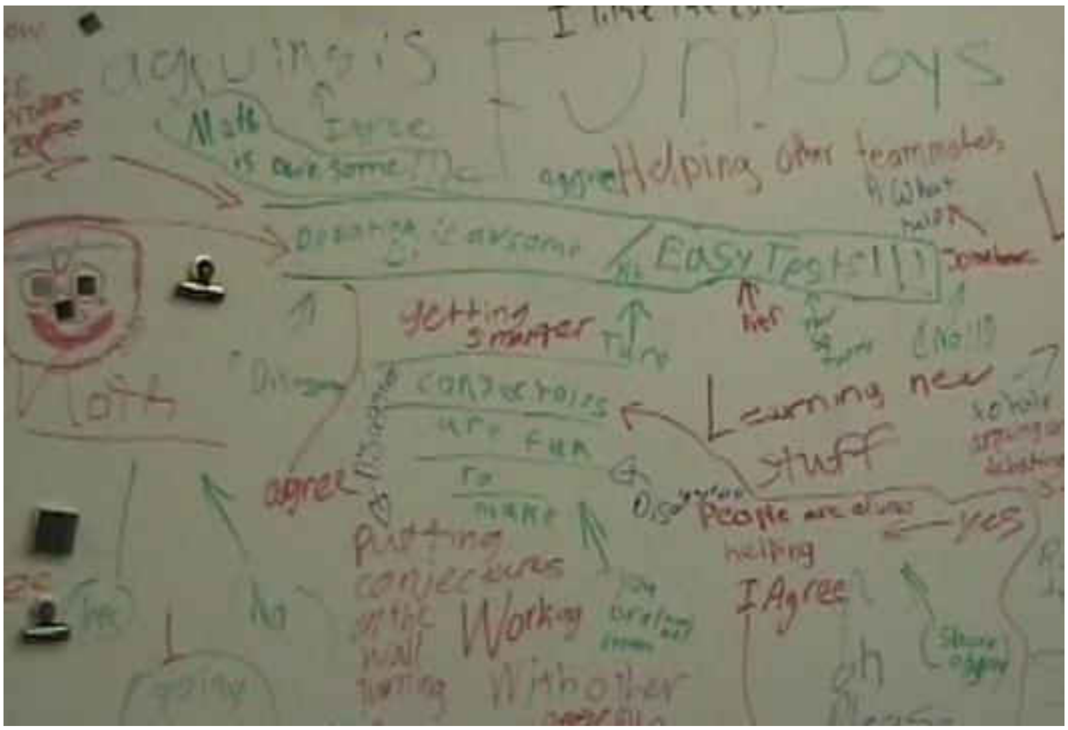


Figure 8: Example from "chalk-talk" activity

T: "Having the teacher not help you". What do you think about that?

Marsha: You say that you need to figure it out on your own.

T: Why do you think we do that?

Charlie: So that we can learn ourselves.

Nathan: You can ask people at your table.

T: What if they do not know? What do you do? Ask somebody else. Has that happened to you, nobody in this room knew the answer? That might happen once or twice but not very often. Usually around the room you can find someone that may help you. And if it occurs once or twice, don't we always talk about it in the class? I am interested in the idea why you think we do not help you. It is not a traditional help, is it?

Usually the teacher comes up and answers for you, when you get stuck. When you get stuck what do we do?

Charlie: Tell us to talk to our group.

Seth: Or to a friend.

T: I know some of you do not like talking with your groups.

Anthony: If I cannot figure out something alone I should come and knock another door to find the answer.

T: It is one of the ideas. When you come to 8th grade, high school or adult, you get a job and you have to work with people, even those we do not like. I know you like your group mates but you also have to work with people that you do not like. How do you do that? That is the key. Also we learn a lot from each other. And we do not give you the answer immediately on purpose because like Charlie said we want you to figure it out. We know you got the brain for it. So if you figure it out yourself it is much more meaningful than if I show you how to do it. You do not need to memorize it, right? So we feel like we are helping you but it is a different kind of help. We are helping you become good thinkers instead of relying on me for how to do that or rely on Mrs. Taylor. If I were you and if the teacher didn't tell me how to do things, I would get frustrated, too. We are trying to help you in a different way. So just try to get comfortable with that, if you have not.

Here, the teacher selected a negative idea, which stated that the teacher does not help them. The reason this idea appeared was most likely due to students being used to the traditional style of teaching in which students are given ready-made knowledge by the teacher. Stephan asked them why they think they do not do it in this classroom. Some students suggested that this gives them an opportunity to learn on their own and to learn how to find answers by communicating with other people. Based on these, the teacher explained that taking responsibility for their own learning is a crucial part of life that they will need even more in the future. She reminded them that they will have to work with people even if they may not like them, and doing it properly is a key for being successful. She told them that it is more meaningful if they try to figure things out on their own rather than the teacher just telling them the answers. This way they wouldn't have to memorize but rather learn with understanding. She also explained that both teachers in the classroom are helping them but it is a different kind of help than they are used to. She advised them try to get comfortable with this kind of environment if they had not already.

This communication was primarily based on the teacher's belief that students should take responsibility and ownership for their learning (B_L3). This belief seemed to originate from her belief that when people figure things out on their own or through asking questions, they learn it with understanding (B_L1). She also knew that when students get to higher grades or graduate, there won't always be someone to give them the answers they need, so they would be more successful if they learned to figure things out by themselves from the early ages. Through this approach she emphasized the importance of learning how to learn rather than learning just the correct answers. This type of thinking is essential in standards-based teaching as the *Principles and Standards* emphasize the need for instruction that enables students to fulfill personal ambitions and career goals in an ever-changing world (K_c3).

In summary, the practice of *creating and sustaining social norms* included encouraging students to express their agreement and disagreement, understand as well as repeat other's solutions, behave respectfully, use their mistakes as learning opportunities, take responsibility for their learning, and renegotiate social norms as necessary. It was observed that in order to execute all of these actions effectively the teacher had a diverse combination of beliefs, goals and knowledge. Particularly for this practice, it was noticed that the teacher's knowledge related with the NCTM *Standards* as well as her pedagogical knowledge about the students' behavior and her current students seemed to help her sustain social norms. The other motivating factor in supporting social norms seemed to be her beliefs related to learning: she believed that an environment that nurtures respect and a sense of community plays an essential role in students' learning. Also, her belief that students learn from their mistakes and taking responsibility is crucial for one's learning seemed to be highly activated in this practice. Finally, her overarching goal of renegotiating the social norms as necessary seemed to have a central part in applying this practice.

Practice Two: Facilitating Genuine Mathematical Discourse

The NCTM *Standards* emphasize that teachers need to facilitate discourse to support students' mathematical understanding rather than just deliver information. The teacher has an essential role in discourse such as asking questions that promote higher-level thinking, restating students' explanations in clearer language, introducing vocabulary and using students' solutions effectively to promote discourse.

Action 1: Introducing mathematical vocabulary when students have invented an idea

The following excerpt was taken from the first day where the students were guessing the things that a famous person might own and owe. The reason that the teacher initiated the discussion based on a famous person's net worth was to keep the discourse interesting and alive (G_O2). The teacher wrote the guesses related to own to the left of the board, while she wrote the guesses related to owe to the right. After students came up with various items some tangible such as cars, money, and boats and some intangible such as stocks, mortgages, and loans. The teacher asked students how they might name each part:

> Mark: That list on the left is how much money she gets or how much she has (T writes at the top of the list "how much she owns") and the right side is how much money she loses (T writes "how much she owes").
>
> T: Someone used the word "debts"
>
> Sts: Danny used it.
>
> T: You do not have to write this list down. The title of this list is Danny's title "Debts". Those are debts right? Anybody here in debt? (Marsha raises her hand.) Marsha how much money are you in debt?
>
> …
>
> T: You probably do not know the title for this column [T shows the column named "how much owns"] but you might have heard it, it is actually called assets. Actually, these two words are used in finance. You are going to learn a little bit about the finance. How much people are worth? Are you curious how much are you worth?

During the discussion the teacher first asked students to guess the possible name of each part on the board. One of the students suggested that one part was for how much money she has and the other for how much money she loses. The teacher reworded this and wrote on the board how much money she owns and how much money she owes. Both own and owe were words that were easily recognized by the students. Her goal was to support students' understanding of the financial terms assets and debts based on their informal knowledge of owning and owing (G_C1). This goal seemed to be activated by her belief that students learn better when they can connect the newly introduced concepts with their informal knowledge (B_L5).

Next, the teacher capitalized on Danny's answer where he used the term "debts". She said they would use the term debt to represent the items on the right side of the board. She asked if any of the students were in debt. Then she introduced the word "asset" as the counterpart of debt and explained to students that these two words were commonly used in finance. The teacher's belief that she should introduce new vocabulary when students invent an idea allowed her to introduce the term asset (B_T10). Introducing the new mathematical terms once students understood the mathematical ideas seemed to help discourse to move smoothly since students did not have difficulty understanding the meaning of those new words. In order to promote the discourse, the teacher also considered the students' taken-as-shared interests that would motivate them to generate conversations as they worked on the tasks related with net worth. Thus, at the end of the conversation she asked students if they would be curious about how much they were worth as well as the other people. Next, she distributed the new activity that asked students to find famous people's net worth (see Appendix C, "Angelina and Brad").

Similar to the situation described above, in day three, the teacher gave students an activity where they could use additive inverses. Once the students explored the idea and talked about their method where they named it as the "cancel out" method, the teacher introduced the mathematical vocabulary for those numbers in the activity:

> T: Any other ideas, you want to name these two things? The name you might think of is opposites but the technical term is called inverses [she writes opposites and inverses]. You should write these terms down, it means they are opposite of each other. Do you know what does it mean?
>
> Dusty: One is positive and the other one is negative such as +$1000 and -$1000.
>
> T: What else here are inverses? [she shows the numbers on the activity]
>
> Dusty: They should be the same.
>
> T: Yes, opposites and inverses are the same numbers. Help me with the definition. How do you know if the numbers are inverses?
>
> Marsha: The same number with different symbols
>
> T: The same numbers with different symbols (+,-). Another way you might say it is canceling each other out [writes on the board]. That was Seth's contribution.

During this activity that included positive and negative numbers, the teacher introduced the vocabulary "inverses". Before she defined the meaning of the word inverse, she asked students if they could guess what it means. Dusty gave an example from the activity by showing that numbers such as 1000 and -1000 are inverses. Then the teacher asked them to define what makes two numbers inverses. Marsha responded by saying that the numbers should be the same but have different signs. The teacher capitalized on her ideas and wrote the definition of the inverse on the board. Her belief that the teacher is responsible for introducing mathematical vocabulary seemed to motivate her to introduce the mathematical word 'inverse" at this point. (B_T10). She also stated that inverses cancel each other out as Seth stated before. She first connected the more technical term inverse to opposite, which was part of the students' informal knowledge. She then helped students develop the idea of additive inverse through examples. Once the students invented the idea, she formally defined the mathematical term. Introducing the term "inverse" after students explored the relationship between the numbers had an important role in facilitating the discourse. As the students understood the meaning of the word, they started to use it during their explanation of solutions and this carried the discourse to a more advanced level. The teacher's knowledge of content had an essential role in deciding which mathematical word she needs to introduce and when she introduces them (K_S1).

Action 2: Asking questions that promote higher level thinking

During the discussion of a problem in day six about finding the gap between -10,000 and +8000, Seth stated that he noticed a pattern when two numbers have different signs; they just need to add those two numbers together. The following is taken from that discussion, where the teacher helped students to generalize their ideas:

> Seth: This works when one number is positive [8000] and the other one is negative [-10,000].
>
> T: What is the "this" part of this works?
>
> Seth: 10,000 plus 8000.
>
> Anthony: It is even easier when they are both the same.
>
> T: Anthony, can you give an example of what you mean? Go ahead make it up.
>
> Anthony: For example if she did not owe money, she had 5000 [instead of -10,000] then all you need to do is add 3000.
>
> Sts: That is the difference.
>
> T: So did you hear what he said? If she has 5000 instead of way down here, she would only have 3000 more. How did you get 3000?
>
> Anthony: Use your common sense.
>
> T: Remember what we said [she reminds the students about how they need to behave]
>
> Anthony: You have 5000 and you want to get 8000, it is how much more you need to get 8000.
>
> Seth: You have to subtract 5000 from 8000.

T: I think what you mean by common sense was most people can see it is 3000 without doing any calculation. Seth made a good point if you did not just see this is 3000. Here is how you can teach somebody to get it. You just find the difference [she writes 8000-5000=3000].

Seth: That only works when both numbers have the same signs.

T: So if they are both positive you're just going to find the difference. Seth says if they are both negative let's try it out. Let's do a different one. How about -8000? How much more is this person's net worth? Brad? [T asks students to compare -8000 and -10,000].

Brad: I did it a little different than this.

T: Back to this, remember Seth's idea what is happening when they are both negative?

Sts: $2000.

Tisha: Positive or negative?

T: We are not going to worry about the signs right now.

Sts: If the numbers are in different sections [positive and negative sections of the number line] you add them; if the numbers are in the same section you subtract them [the teacher writes it on the board].

After Seth's conjecture about defining the structured gap between two different net worths, the teacher wanted to continue the discussion by restating Anthony's words where he said that it would be easier to find the gap when both numbers have the same sign. The teacher's goal in this activity was to help students understand how they could find out the difference between any two net worths (G_C5). She asked Anthony if he could give an example that supports his conjecture. Anthony replaced -10,000 with 5000 in the given question and stated that the difference in net worth between 8000 and 5000 would be 3000. The teacher asked if he could explain why it was $3000, but he had difficulty explaining his reasoning and said that he used common sense. At this point, Seth stated that it could be found by subtraction, which was supported by the teacher by showing it on the board. Next, the teacher wanted to make students think about a case where both numbers were negative with the intention of helping students generalize the pattern. Stephan's belief that the teacher should pursue the students' ideas when appropriate might have motivated her to ask students to think about a situation when both numbers are negative (B_T16).

The question this time was how much more would a person's net worth be who has -$8000 than a person's net worth who has -$10,000. Most students agreed that it should be $2000. As can be seen from the excerpt above the teacher avoided talking about the sign of the difference since it was a big idea of later activities. Her goal was for students to understand and internalize the distance between these numbers on the vertical number line.

At the end of this discussion, many students arrived at the conclusion that to find the difference between two integers, they have to add them up if they are in different sections on the number line, and subtract them if they are in the same section. Thus, capitalizing on Anthony's conjecture about a different case where both numbers were positive and asking students to consider the case where both numbers were negative helped them see all of the possible cases and engineer a general rule to find the difference between any two integers. The teacher's extensive knowledge of content allowed her to use students' solutions to initiate higher-level questions where students could explore mathematical ideas (K_S1). By asking higher-level questions, the teacher supported a discourse where students came up with different ideas and reflected on the ideas that were stated by their friends. This way the discourse stayed alive as the teacher asked them to think about different mathematical ideas rather than the ones that they already knew.

Action 3: Restating students' explanation in a clearer/advanced language

In the next class students were given a net worth problem where the question included multiple positive and negative numbers, and students were asked to find the net worth of the person. Although some students solved the problem by finding total assets and total debts and then finding the difference, the teacher wanted to focus on the "cancel out" method. The teacher asked Brad to explain his solution since he found the answer by canceling out the inverse numbers and added the remaining ones. Although many students seemed to understand his method, Tisha stated that she was confused. As Brad had already repeated his explanation twice and many students understood him, the teacher summarized Brad's solution:

> Brad: I canceled those and added the ones that are left.
>
> Tisha: I am confused. Do you add all debts together?
>
> T: Do we need to start over? I will quickly go over it again for the people who missed it for the first time. Because I don't want him to go all over again. He started by looking at the asset of $9000 and debt of $9000 and he says why even worry about those. They cancel each other out (she shows it for the other pairs that cancel out each other). Whose cancel out method was it? Seth's, Gage's from our class. He gets rid of those; I do not have to worry about them now. You okay with that Tisha?
>
> Tisha: Yes.

During the classroom discussion, Brad explained his cancel out method twice since some students had difficulty understanding. This caused some loss of time for the next part of the problem, which the teacher wanted to focus on as well. However, Tisha stated that she got confused by Brad's explanation and wanted him to explain again. At this point, the teacher intervened and said that she would summarize Brad's ideas quickly for the people who missed it for the first time. After her explanation, she asked Tisha if it now made sense to her.

Here, almost all the students seemed to understand Brad's reasoning except Tisha. This seemed to activate the teacher's belief that she is responsible for efficiently managing the time (B_T13). Therefore she rephrased Brad's main points more quickly. This illustrated the teacher's practice of restating students' explanations in a clear language to make it understandable to everybody as necessary in order to focus on the big idea of the current lesson (G_O3).

Action 4: Using solutions effectively to engineer the teacher's summary

The teacher continued to present different types of problems in the following days. During the discussion of a problem on day sixteen, which was ▢ - (-7000) = 2000, the students agreed that the answer was -5000. At this point Anthony asked if switching the place of the first and the second numbers would affect the result:

> Anthony: They are both debt. Does it matter to change the order of numbers?
>
> T: Let's see does it matter? Can you have this -7000 - (-5000) = 2000, is this going to get you the same result? [Students talk in their small groups for a few minutes].
>
> Sts: No, it is negative 2000.
>
> T: I think we all agreed that -5000 is the beginning net worth. Anthony asked a good question. Can we just change the order of these? [She also asked if they can switch the numbers in the first question where the operation was addition and students said they can do it in that one].
>
> T: -5000 - (-7000) does not seem to work when you switch the numbers. What was this one 2000 and this one was -2000. This does not work. Why is it not working with subtraction but working with addition?
>
> Tisha: It works for addition because it does not matter how you switch the numbers. Like you can do 7+5 and 5+7, and you get the same answer. But when you are doing subtracting and you have a lower number or bigger number you cannot switch them because if you did 7-5 you get 2 but if you do 5-7 you totally get different answer
>
> T: What would you get?
>
> Tisha: -2.
>
> T: But in addition it does not make a difference. Why?
>
> Brad: For the subtraction you take away the small number in the first one and the bigger number in the second one.
>
> T: Yes but does it make a difference in addition?
>
> Sts: No…
>
> T: This is called the commutative law. Do you know that? You guys just discovered the commutative law of addition. In this example, when adding two negatives, the commutative law holds, doesn't it? You can switch these two with an adding symbol and get the same answer. Seth is asking does the commutative law hold when you use negative and positive such as -7000 + (+5000). Can you switch them and get the same answer? I am not going to explore that in the class but if you want to get extra credit you can investigate the commutative law with two different signs with addition. Does it hold if you switch these two numbers and it is an addition? Okay? If you want to write that down I will leave it for you.

For Anthony, it seemed like switching the numbers would not affect the answer of the problem. The teacher asked Anthony's question to all students and gave them some time to think about it. The students found that changing the order resulted in a different answer. The teacher then asked what would have happened if they changed the order in the earlier question where the operation was addition instead of subtraction. All students agreed that changing the order did not matter for that question. Based on this, the teacher asked if anybody could explain why changing the order matters for subtraction but not for addition. Tisha gave an example using smaller numbers and stated that changing the order for subtraction might result in subtracting the bigger number from the smaller one, when the original question was the other way around. Brad restated this by using take away instead of subtraction. At this point, the teacher told the students that they have just discovered a law called the Commutative Law of Addition, which states that the order of the terms does not matter for addition. Seth asked if this law would hold when one of terms was negative and the other was positive (in the previous question they solved, both terms were negative). The teacher left the exploration of this question as an exercise to the students.

The teacher had an essential role in engineering the discourse effectively by giving students opportunities to explore the mathematical ideas by using the students' questions (i.e. whether it differs to change the order of numbers in the subtraction) that came up in the solution of the problem. Finding the answers of the questions that were raised in the classroom made students more enthusiastic about the exploration and explaining their ideas during the discourse. Secondly, the teacher also engineered the discourse by introducing the mathematical concept after students explored the mathematical ideas. Finally, learning these new ideas encouraged students to raise more questions in the discourse (e.g. whether commutative law of addition holds when both numbers have different signs).

This episode illustrates several of the teacher's beliefs, goals, and knowledge related to discourse. When Anthony asked whether the order of the numbers was important, the teacher gave students a few minutes to think about the question since she believed that having sufficient time was important before discussing the ideas (B_T7). Thus, rather than asking for a quick response from the students or answering the question herself, she waited for other students to analyze the question. This allowed students to absorb what was being asked and be able to contribute to the classroom discussion. This can also be related to the teacher's goal of involving as many students as possible in the classroom discourse to support the discourse staying alive and interesting (G_O2). If the teacher expected an immediate response, it is likely that only a few students would have been able to contribute to the discussion.

When students found that changing the order matters for subtraction, the teacher asked them to try another example that involved addition. This was based on her practice of asking questions that promote higher level thinking by creating opportunities for analyzing, comparing, and synthesizing. This practice seemed to stem from her belief that mathematics is a sense-making activity (B_M1). The teacher then inquired if the students could explain why the order mattered for subtraction but not for addition. After students developed some explanations, the teacher introduced the mathematical terminology for the commutative law and restated students' explanations in a more advanced way. These actions illustrate her practice of introducing new vocabulary when students have invented an idea, and that she should restate students' explanations in a more sophisticated way. The main reason that shaped her practice might be explained with her belief that students learn better with instruction that relates to their informal knowledge (B_L5). Discussing about the commutative law was not a planned goal of the teacher at that point in the discussion; it became an emergent goal with the questions of the students. Although they made an exploration into the commutative law, the teacher did not want to spend too much time on it in order to continue with the big idea of the current lesson. Therefore, she assigned the further questions of the students about the commutative law as an exercise. This could be explained by the teacher's goal of focusing on the big idea of the lesson while allowing room for useful explorations (G_O3).

In summary, the practice of *facilitating genuine mathematical discourse* included introducing mathematical vocabulary when students invented an idea, asking questions that promote higher level thinking, restating students' explanation in a clearer/advanced language, and using students' solutions effectively to engineer a summary. During this practice it was observed that the teacher's subject-matter knowledge had an essential role in facilitating genuine mathematical discourse as this allowed her to pursue students' ideas comfortably. Her belief that students learn better with instruction that relates to their informal knowledge and students learn mathematics with understanding seemed to be highly activated during this practice. Her overarching goal of supporting a discourse that stays alive and interesting appeared to be a driving factor in the application of this practice. Finally, focusing on the big idea of the lesson was highly influential as this allowed her to prevent the discussion from straying away from the big ideas of the current class.

Practice Three: Supporting the Development of Sociomathematical Norms

One of the important roles of the teacher in standards-based instruction is to support sociomathematical norms which refer to understanding what counts as a different, efficient, sophisticated, and acceptable mathematical solution (Cobb & Yackel, 1996). The teacher also needs to encourage students to make and investigate mathematical conjectures and recognize reasoning and proof as fundamental aspects of mathematics (NCTM, 2000). Since in this study the expert teacher co-taught with a special education teacher, the co-teacher also supported her to sustain the social norms.

Action 1: Encouraging students to give conceptual explanations

As explained in the earlier sections, the instructional sequence started with different net worth problems. On the second day of the sequence students were working on one of the problems that asked for finding two different net worths, students were faced with a predicament since one of the person's total debts was overwhelming the assets (i.e. the total asset was $600,000 and total debt was $790,000). The teacher's main goals in this episode was for the students to be able to correctly determine the net worth (G_C2) when debts are bigger than the assets (i.e. when the net worth is negative) as well as to understand what it means to have a negative net worth (G_C3). Until this point, all the net worths had been positive and the students were used to finding the net worth by subtracting the debts from the assets. In other words, they were performing a vertical subtraction operation by writing the bigger number above the smaller number then subtracting. However, writing the numbers in this order did not make sense to some of the students in this problem as shown by the following excerpt:

> Anthony: Angelina is worth more because his [Brad's] asset is $600,000 so his debt is more than his asset so it is a problem like this [he writes on the board 600,000-790,000 = -$190,000 in vertical order as shown on the left of Figure 9)
>
> T: Say something about this Anthony. I have not seen a subtraction problem something like that before.
>
> Sts: Me neither [some of them].
>
> T: Have you guys seen subtraction like that before?
>
> Sts: Yes, I have [some of them].
>
> T: Some of you did like that [subtracting lesser number from the greater number] but we want to hear what you are doing [T wants students a mathematical explanation for the calculation on the left of Figure 9].
>
> Anthony: See if you put this [$790,000] on top I tried to do this at home, since the bottom is bigger you put it on top of this one [$600,000] but then it does not make sense.
>
> T: Show us, how you did.
>
> Anthony: If you did this 790,000- 600,000 [writing this in vertical form, see the middle of Figure 9] than that [$790,000]one would be positive and this [$600,000] one is negative.

Here, Anthony seemed to be confused by the fact that he needed to do a subtraction operation but for the first time the subtrahend was greater than the minuend. He sensed that something was wrong by doing a digit-by-digit vertical subtraction in this case, but at the same time he realized that reversing the order of the numbers would not make sense as it would result in a positive net worth (he recognized that the result should be negative but did not know how to get there). His attempts are shown on the left and middle of Figure 9. The teacher wanted him to clarify how he performed the operation shown on the left, but he explained why the middle one, where the greater number was on the top, did not make sense due to the net worth being positive. During the discussion, Mark came to the board and revised Anthony's attempt (right image in Figure 9) by subtracting from bottom to top as shown on the right of Figure 9.

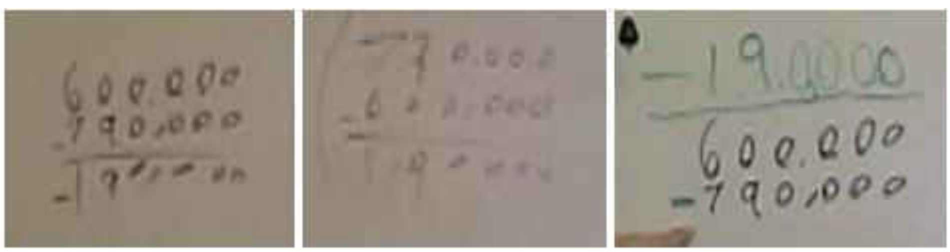


Figure 9: Different solutions

Next the teacher asked if anybody found the net worth as in the middle example. Norman said that he did it that way and initially left the result as positive but then realized that the debts were larger so he changed his mind:

> Norman: I put positive but then I changed my mind.
>
> T: You changed your mind. Norman says I put as positive but then I changed my mind. What did you change your mind to?
>
> Norman: Negative.
>
> T: You changed your mind to negative. Do you want to explain that? You get the same number but the sign is different tell us about that.
>
> Norman: That is like how much money you own like the total asset like and the debts are more than what you owe.

T: Norman says that debts are more than what you have so what?

Norman: It will be like owing.

T: So you are gonna be owing $190,000. Can Norman do it in this way?

Sts: Yes.

T: What do you guys need to remember?

Sts: Negative.

As can be seen in the excerpt above, the teacher asked Norman if he could explain why he changed his mind. Norman, stated that because the debts were larger than the assets the result should be negative. The teacher then asked the class if this would be an acceptable solution, a question to which the students agreed. She emphasized that they could do it this way but they would have to remember to use the correct sign at the end of the operation. The teacher concluded this discussion by making the following comment:

> T: Whatever order you put it if you put this [pointing to lesser number] on top and this [greater number] on bottom or you put this [lesser number] on bottom and this [greater number] on top you did seven minus six and you got one. So was he right? This way you have to remember like Norman said this number is debt. I like how Anthony said. He said your debt overwhelmed your assets so you are going to end up with debt. All right, no more different ways, let's just go on. We're gonna come back to this later.

This episode primarily highlights the teacher's practices of emphasizing what counts as an acceptable mathematical solution. When Anthony presented a solution where he vertically subtracted the bigger number from the smaller one, the teacher said that she had not seen subtraction done in that way and asked Anthony to explain his method. By capitalizing on the comments of the other students, they established a sociomathematical norm as the classroom community agreed that in these kinds of situations they can subtract the lesser number from the greater one as long as they remember that the greater number represents debts, and therefore the result should be negative. Her beliefs that not all mathematical solutions are equal (B_M6) and the teacher is responsible for giving opportunities to students to see efficient and different ways during the classroom (B_T9) seemed to have an essential role in this discussion.

After students felt comfortable with finding a person's net worth in different ways, the teacher gave an activity where the students needed to compare two people's net worth. The question was comparing Paris's net worth, which was -$20,000 to Nicole's net worth of -$22,000 and finding the gap between them. During the discussion, some students had difficulty in ordering the numbers on the vertical number line. Nathan was one of the students who had difficulty. The teacher encouraged him to come to the board and explain what he was thinking and asked students to help him once he presented his ideas. Her knowledge about students' possible misconceptions (K_P2) enabled her to capitalize on Nathan's mistakes. Nathan placed the two net worths on the number line in opposite order, -22,000 being above -20,000, and could not explain why he did that. At this point one of the students stated that the order of the numbers should be reversed:

> Adam: -20,000 is supposed to be before -22,000. Paris is a little more closer to zero.
>
> T: Adam says Paris is a little bit closer to zero. What do you want to say Charlie?
>
> Charlie: Because Nicole owes more so she has to be in red more
>
> T: She has to be in red more he says, because Nicole owes more than Paris. How do you know she owes more?
>
> Charlie: Because the question already says one of them is negative 22,000 and the other one is negative 20,000.

The teacher's goal in the discussion above was to support students' to be able to order the numbers on the vertical number line in a meaningful way (G_C5). During the discussion, Adam stated that Paris is closer to zero so she should be before (above) Nicole. Charlie also replied by arguing that Nicole is more in red since she owes more[7]. When the teacher asked him if he could explain why she owes more, he stated that it was given in the example. Therefore, the teacher continued to ask questions to bring out a logical explanation. She asked students how they knew that they would put the numbers in that order. Students continued to say that negative 20,000 was closer to zero. However, the teacher persisted by asking them how they knew it was closer. The teacher's belief that students learn mathematics with understanding (B_L1) motivated her to ask students why -20,000 was closer to zero than -22,000. Marsha stated that the negative part of the vertical number line could be thought of as a reflection of the positive part. She added that since 20,000 is closer to zero the reflection of it, -20,000, also would also be closer to zero. At this point the teacher noticed that several students could not visualize and make sense of Marsha's reflection analogy even when Marsha tried to draw a reflected line on the board. Therefore, the teacher decided to focus more on Charlie's idea:

> T: Charlie you had an idea that you were relating it to debt how much people are worth. Why this number is below this number. Can you say it again? Perhaps that might help Nathan, too.
>
> Charlie: Because negative 20,000 is being closer to out of debt than negative 22,000.
>
> T: Did you hear that?
>
> T: 20,000 is closer to being out of debt than 22,000.
>
> T: Anybody state it differently?
>
> Marsha: It is closer to zero.
>
> T: Without close to zero I think we got that idea out there. Talk about debts and negatives.
>
> Brad: The reason 22,000 should be farther down because she is further down the hole.
>
> T: Do you know what he is talking about? What do you mean by further down in the hole?
>
> Brad: Like you owe more money than the other person.

---

[7] The imagery of black-red number line will be explained in more detail in the next practice. For now, it is sufficient to assume that black represents the positive parts of the number line whereas red represents the negative.

T: So are you saying Brad the further down you are going in the hole [T makes a gesture by pointing down] the worse your debts get. What do you think about that idea?

Nathan: I agree with that idea.

During the discussion the teacher decided to go back to Charlie's idea to connect the numbers with assets and debts as Marsha's reflection idea seemed too sophisticated for some students as expressed by the teacher during the interview. Another important reason for why she focused on Charlie's idea seemed to be her belief that students learn better with an instruction that relates to their informal knowledge (B_L5), rather than an abstract idea. In other words, by using Charlie's explanation she folded back to the context.

In the following discussion, she capitalized on Charlie's ideas by asking him if he could define why -20,000 is closer to zero than -22,000. He explained his reasoning by stating that -20,000 would be closer to the zero since it is closer to being out of debt. The teacher then asked how it could have been explained differently by using the assets and debts concepts. Brad stated that the person who has -22,000 would be much further in the hole, which means he would owe more than the person who has a net worth of -20,000. The teacher concluded the discussion by restating his reasoning in a clear language and continued with a similar question.

Action 2: Encouraging efficient solutions

During day sixth, the students were given activities that included finding a person's net worth and then applying the given transaction to the original net worth that they found. The net worth statement that the student worked on is shown in Figure 10. In solving this problem, most of the students continued to use their original strategy of adding up assets and debts and finding their difference. Stephan's primary goal in this first part of the activity was for the students to explore the idea of additive inverse (G_C4). Thus, she encouraged students to think of more efficient solutions.

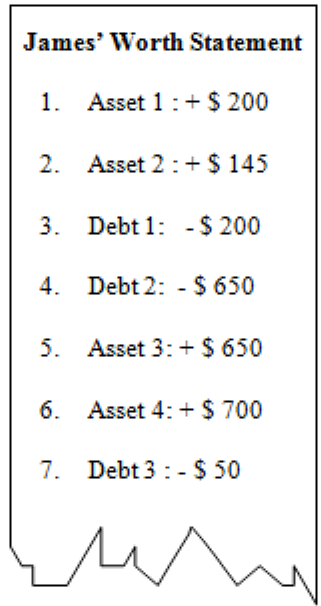


Figure 10: James's net worth statement

T: Danny tell us your way.

Danny: I just added all the positives and got $1695 that is total asset [T writes on the board]. I got 900 for total debts. And then I got $795 [the teacher writes on the board 1695-900=795].

T: Anybody have quicker way? If you have to add, how many things do you have to add together? [T shows the positive numbers in Figure 10].

Tisha: It is not harder.

T: Okay, if you think that is quick, it is fine. Does anyone have a way that is quicker than that?

Brad: Me.

T: You have a quicker way?

Brad: Here is +$200 and debt of $200 cross that off. And then $650 and -$650, cross that off, too [T crosses through those numbers on the board]. And then I added 700 plus 145 and got $845 [T combines those two numbers with arrows and write $845].

T: Where did you go next?

Brad: Subtract 50 and got 795 [T writes 845-50=795]

T: Which one is quicker to you?

Sts: I like Brad's way.

In this discussion, Danny explained his solution by stating that first he added the positives to find the total assets and then he found the total debts and computed the difference to find the answer. When the teacher asked them if they could think of a more efficient way, Brad stated that his method was easier since he did not need to add all those numbers but crossed out ones that were the same but had opposite signs. To make the idea clear to the students, the teacher crossed the numbers that are inverses of each other as Brad described his solution. She then asked which way was quicker and most students agreed that Brad's way was a quicker and a better solution.

Here, the teacher's practice of encouraging students to use efficient solutions helped students to see the advantage of the cancel out method which was a big idea of the lesson. Stephan's belief that the teacher is responsible for giving opportunities to students to discover efficient and different ways during the classroom discussion (B_T9) seemed to be the driving factor in her practice. Her belief that students should be able to analyze the different methods in order to make sense of mathematics (B_M4) might also have encouraged her to ask students which method was quicker.

Action 3: Encouraging students to make conjectures

On the ninth day, the teacher gave students an activity that included verbal descriptions of transactions and asked them to write if each transaction is stupid or smart. For example, one of the statements was "Christian: He took away an asset of (+$50) from his net worth statement". Students could easily decide that this decision was bad since he took away an asset. Once the students decided whether the given transactions were good or bad decisions, the teacher introduced the symbolization that would be used in the rest of the unit such as taking away would be shown with a minus sign and debt would be shown with a negative sign. Then, the teacher asked students to write all the statements by using the symbols. For instance, the students wrote the example given above as –(+50). The excerpt below is taken from the classroom where the students were discussing the problem of whether the situation "Donnie takes away a debt of $500" was good or bad. At this point Stuart came up with a conjecture on which the teacher capitalized:

> Stuart: I think we should make a class conjecture[8] about this. If we have the signs both inside and outside the parenthesis that are the same then you are making money; if they are like negative sign and negative sign.
>
> T: I am not going to make this a class conjecture this is you talking, here. Stuart's conjecture [she writes on the board].
>
> Sts: I came up with the same [students start to talk at the same time].
>
> T: Wait, wait one person at a time.
>
> Stuart: If the negative sign is outside the parenthesis and inside the parenthesis you are making money. You are making money because you are taking away debt and it is the same if you have both positive signs inside and outside [The teacher writes on the board –(-), +(+), you are making money, smart].
>
> T: Do you want to finish, Stuart? Is it the only part of your conjecture?
>
> Stuart: If you have positive outside and negative inside it is not smart, it is stupid and you have a negative outside and positive inside, it is stupid too [The teacher writes on the board +(-), -(+) stupid].
>
> T: Do you think you can defend this conjecture? I am going to leave it up there right now and I know some of you may already have figured it out. And Stuart is just the first person to get it out. Do not let it bother you that you knew it already; my name is not going up there. So we are going to call it as Stuart's conjecture right now. At lunch time we are going to write it on the poster board put it under conjecture perhaps during lunch you can talk about it [this was a block period split by the lunch break].

Stuart argued that he noticed a pattern and he also included his friends in the conjecture since he saw that many of them also noticed the same pattern. The teacher capitalized on Stuart's ideas and gave his name to the conjecture since he was the first one to say it out loud. Her belief that valuing students' ideas and explanations empowers them to participate in the classroom discussion (B_T4) might have played an important role in naming the conjecture after Stuart and putting it up on the poster board. Stuart continued his conjecture by stating that if the signs are

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[8] The teacher separated the wall at the back of the classroom as conjectures and theories. When students made a conjecture it was written under the conjectures section with the name of the student making the conjecture. It stayed there until it was proved, then it was moved to the theories section.

the same such as –(-) and +(+) the person is making money. He added that if they are different such as –(+) and +(-), it is a bad decision because you are taking away asset or adding debt. Thus the first two make the net worth go up and the last two make it go down. Next, the teacher encouraged students to prove it. Here it is important to note that since many students at that point noticed the pattern, the teacher reminded students not to get upset because their names were not written on the board. She encouraged them to be the one to tell the conjecture next time. The teacher's belief that an environment that nurtures a sense of community (B_L7) plays an essential role in students' learning might have affected her to say to the students to not get upset because their names were not up there. She emphasized that she wrote Stuart's name since he was the first one to say it out loud in the community even though many of them also discovered the same ideas. Following this, she told students that she would write the conjecture on the poster board and asked the students to prove it when they came back after the lunch. Two of the reasons for the teacher to leave the proof of the conjecture to the second part of the lesson might have been to give sufficient time to think (B_T7) and to support her goal of not disrupting the flow of the activity (G_C6).

After the break, the students could easily prove it using the vertical number line. The teacher's practice of encouraging students to conjecture and prove their solutions supported students to think at a more sophisticated level and analyze the problems that were solved in the classroom. The teacher's belief that mathematics is a sense-making activity that includes making conjectures and proving arguments (B_M2, B_M5) might have motivated her to focus on Stuart's conjecture and cement his ideas on the board to help other students notice the same pattern.

Action 4: Encouraging students to express their reasoning and proof

In the following class sessions, the teacher continued to provide more questions that included finding the new net worth after applying a transaction. In order to save time, the teacher listed six questions' answers (as found by students) on the board without individually discussing each. She then asked if there were any answers with which that they did not agree. One of the students, Nathan, stated that he had a question related to the fifth one where it asked for the new net worth with the original net worth being -$7400 and the transaction taking away an asset of $3000 (see Figure 11):

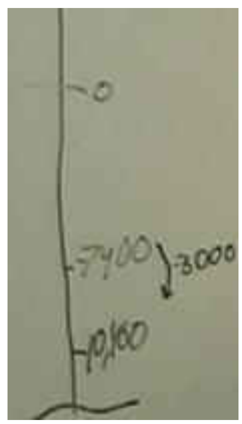


Figure 11: Dusty's number line

Nathan: Number five.

T: Thank you for saying that. I also saw different answers on number five, too. Can you do it? [Asks Dusty and he nods his head]. Dusty did many of them on the number line. Do you want me to put a number line for you?

Dusty: [He draws the number line] The original net worth is negative $7400 and the asset $3000 is taken away, so you have to go down since it is taking away and you get $10,400 [he draws Figure 11].

T: Nathan is it enough for you to change your solution or do you have any question?

Nathan: Aren't you subtracting 3000 from 7400?

Sts: No you are adding.

T: Why are you adding? Tisha can you explain?

Tisha: Because you are adding more debt to your net worth.

T : You are adding more debt to your net worth. It says taking away assets, we are not adding any debts.

Tisha: But since you said the net worth is already in debt and when you take away asset that makes your net worth go even more down [Tisha seems to have understood that taking away asset is the same as adding debt].

T: What do you think about that? [T asks Nathan]

Nathan: I agree, I thought it in a different way.

T: If we put the number line on there you say ooh I understand. What I recommend to you is to use the number line. And that also goes for other people who struggle with it. Use number line to help you and it really helps defend the answer. So if you are up here and you say the answer is -2000, when people say "what are you talking about?" you might use the number line to defend it to just help people to see.

The teacher's belief that she is responsible for sustaining an environment where ideas can be discussed freely (B_T2) might have motivated her to show an appreciation to Nathan when he stated that he did not understand one of the questions. This way she might also have wanted to demonstrate to the other students that they can share their difficulties without being shy.

The teacher's practice of encouraging students to give conceptual explanations when appropriate shaped the discussion above. At the end of the conversation, the teacher reminded the class that some students who had difficulty solving those problems could have used the number line to make it easier to see the effect of a transaction. Additionally, she suggested that students could use the number line to prove their solutions. Stephan's belief that students might learn differently (B_L2) encouraged her to bring up the number line solution to the classroom discussion to help those who could not solve it without visual representation. One of the important reasons for the teacher's suggestion of using the tool might have stemmed from her knowledge that students perform better when they reason with the number line (K_P11). Here it is important to note that in proving similar problems in the following sessions most students adopted this strategy - when a student went to the board to explain the solution, he or she used the number line to convince the other people and when that did not happen the community members asked him to show the solution on the number line. Thus, using the number line to test conjectures and prove solutions of net worth problems became a sociomathematical norm as it became the accepted method of explanation by the classroom community.

In the following days, students continued to solve the transaction problems and many times they used the number line in their proofs. The following excerpt is taken from the classroom discussion (day 10) during Norman's explanation of the question where the original net worth was -$5000 and the transaction was –(-$1750). His original solution is shown on the left and middle parts of Figure 12.

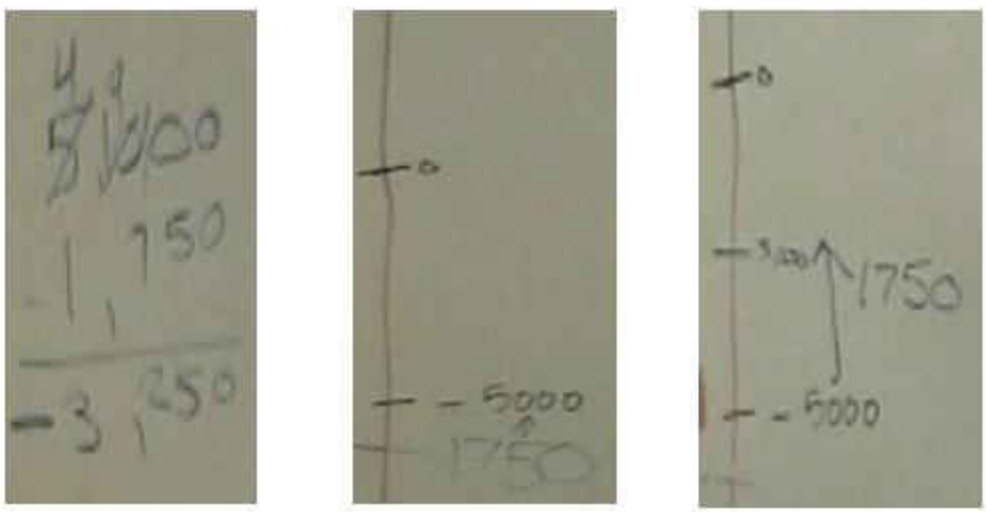


Figure 12: Norman's solution

T: What do you think about his number line symbolization?

Sts: I agree with the answer but not the number line.

T: Give him some advice. Dusty? You agree with the answer but not the symbolization?

Dusty: You take away a debt of $1750, it should be the opposite way.

Norman: [Erases his symbols. This time he just switches the places of -5000 and -1750]

T: Is that right?

Dusty: 1750 should not be on the line, put an up arrow from -5000 and then you put 1750 on the side way

T: Then, where do you land?

Norman: [He writes -3250 on the number line as shown on the right of Figure 12]

T: Let's make sure how we all know how to use the number line so that we all use the number line in the same way, right? We always start with the original net worth and then Dusty said what do you put on the arrow?

Dusty: Transaction.

T: You always put the transaction on the arrow. Dusty said do not put1750 on the number line. That is the transaction that is what happens to the person. You go up 1750 and where does it land? It is the new net worth. We are always going to do that in this way in the class so that we do not confuse each other.

During the explanation of the problem, Norman first showed his computation and then inserted the numbers on the number line as shown in the middle of Figure 12. The teacher's belief that mathematics is a sense-making activity (B_M1) and that students learn from their own mistakes (B_L8) motivated her to ask students if they agreed with Norman's symbolization. At this point many students stated that they agreed with his answer but not with the symbolization on the number line.

Many students wanted to explain but the teacher asked Dusty to state why he did not agree since he was the first one to say it out loud. Since she also believed that students learn in an environment that nurtures community (B_L7), she asked other students to give advice to him with using the number line. Dusty asked Norman to erase 1750 and put it on the arrow and write the final answer above of 5000. Actually, this was how the teacher and many students used the number line up to this point in their proofs. The teacher restated Dusty's explanation in order to emphasize the acceptable way of using the number line and to support the conceptual explanation of proofs in transaction problems. The teacher's practice of encouraging students to use the number line in a correct and acceptable way led to the establishment of a sociomathematical norm. Her goal of establishing sociomathematical norms as necessary during the instruction (G_O1) allowed her to discuss the acceptable ways of using the number line by the classroom community. The teacher's behaviors showed that she views reasoning as an integral part of mathematics as it was emphasized in the process standards in *Principles and Standards* (NCTM, 2000). Her extensive knowledge of content as well as tools also allowed her to pursue the views that are highlighted in the standards (K_S1, K_P7).

In the following days the teacher gave students bare number problems that included missing terms. The goal of the episode given below was to help students to understand missing addend tasks (G_C7). The excerpt is taken from the whole class discussion for the problem -7000 - $\square$ = -5000. While some students argued that the answer was -2000, some others stated that it should be 2000:

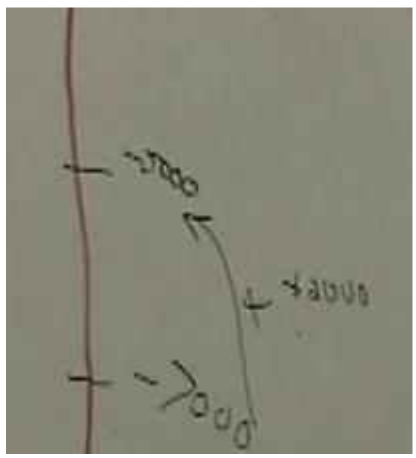


Figure 13: Gage's number line

Gage: The reason I thought it is +$2000 is because this person right here has -$7000 and his friends have $5000 debt. So this person [shows -7000] owes more than that person has. So in order for him to get as much as this person's net worth he had to pay 2000 already. So if he has - $7000 and he pays $2000 to others than he will get worse than his friend [shows -5000]

Brad: Do it on the number line [Gage shows on the number line as in Figure 13].

T: Yes, it is helpful to see things, isn't it?

T: What I want all to do is analyze what he has written, analyze his thinking and see if you agree with him. Make sure I have got this right. You said you are starting with a debt of $7000. You have to get up to -$5000 that is how much the friend owes he says, so to do that you add an asset of $2000[T shows this on Gage's number line]. Is that your argument? You are adding an asset of $2000 [writes + (+2000) on the number line].

At this point Stuart stated that he wants to point out something on the board and the teacher encouraged him to do so:

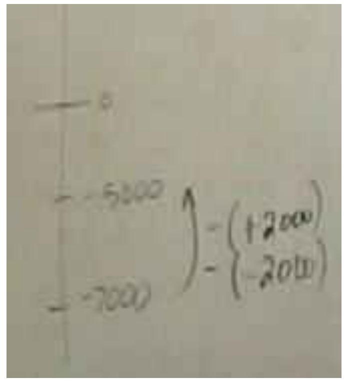


Figure 14: Teacher's symbolization on the number line

Stuart: I just thought this [the box] is like parenthesis

T: The box you mean?

Stuart: Yes, and we are trying to do – (+2000) and it is almost I would say you are taking away an asset thus it will get you down to -9000 [he shows this on the number line, see Figure 14].

Seth: Stuart is true.

Gage: Wouldn't it be 11,000 if the answer is 5000?

T: No, he is saying you started here Gage [shows -7000] but you are actually taking away an asset of $2000

Stuart: You guys do not look at the part here [he shows the – sign in the middle]

T: Do you understand the difference? Let me make sure I can say it right, Stuart. You said this is a take away sign and that is fixed so you cannot change that. The only option you have is to change is what is in the box. Has Gage got rid of that sign? [Students say "yes"] He turned it into positive, didn't he? You turned it to add. You took away my minus sign and turned it into an add sign. I want to see what you have to take away from that [-7000] to get to -5000. Do you think you should take away asset of $2000? Will it get to there? Stuart said no. What do you think? Do you change it? Let me make sure you got the idea [she draws Figure 14] I think we are evenly split, some of you thought it will be taking away asset of $2000, and some of you thought it will be taking away debt of $2000. Let's figure out which one is it? Which one is the smart decision? Taking away asset of $2000 and think parenthesis as the box [she writes – (+2000)] or taking away debt of $2000[she writes –(-2000)]. We cannot change this take away sign because it is not in the box that is what Stuart said right? All we can change it what is inside the box. And there are two choices either asset or debt? Gage which one are you with right now?

Gage: Debt.

The discussion started with Gage's argument that the answer should be +2000 since the person who had -$7000 was already more in debt and to match the other person who had -$5000 he had to add \$2000 more. As he explained verbally at the beginning, some students had difficulty understanding his ideas. As it was a previously established sociomathematical norm, they asked him to prove it on the number line. The teacher also supported students' ideas and emphasized that it might be helpful for most of the people to see his solution visually. Once he drew his number line, the teacher asked students to analyze it and explain whether it is an acceptable solution. At this point, Stuart stated that Gage did not notice the operation sign in the middle and therefore +\$2000 would not be the correct answer. Although some students noticed why the positive answer was not correct, the teacher wanted to make sure that everyone understood Stuart's argument. Thus, she restated Stuart's language clearly and at the same time provided a visual representation for both solutions on the board to help students see why +\$2000 did not make sense while -\$2000 did. The teacher's practice of clarifying the acceptable solutions and encouraging students to give conceptual explanations was essential in establishing the sociomathematical norms of paying attention to the operation sign before finding the unknown number inside the box. After the teacher's clarification, many students agreed that they should be careful about the given operations and what they change and what is fixed when they work on these types of problems.

Since the students were given missing addend number sentences for the first time in this episode, some of them had difficulty finding the correct answer. One of the reasons was that students ignored the operation sign in the middle. When Stuart pointed this out, the teacher capitalized on it in order to help students to see why Gage's solution was not acceptable. The teacher emphasized Stuart's argument of "changing the operation sign in the middle is not allowed, you can only change what is in the box". The teacher's knowledge of students' misconceptions (K_P2) about ignoring the operation sign might have had an important effect on her decision to focus on the acceptable solution.

Action 5: Encouraging different solutions

As the sequence moved forward, students were given activities (day 13) that asked for writing the same transaction in different ways. During these activities, the teacher employed the practice of emphasizing efficient and alternative solutions. The excerpt below was taken from the classroom discussion where the teacher asked the students to find the possible answers for Fantasia's missing transaction that represents a coffee spill (see Figure 15). First, the teacher asked the students the easier ones (an answer that includes only one transaction) and one of the students stated that the transaction was +(+$2000):

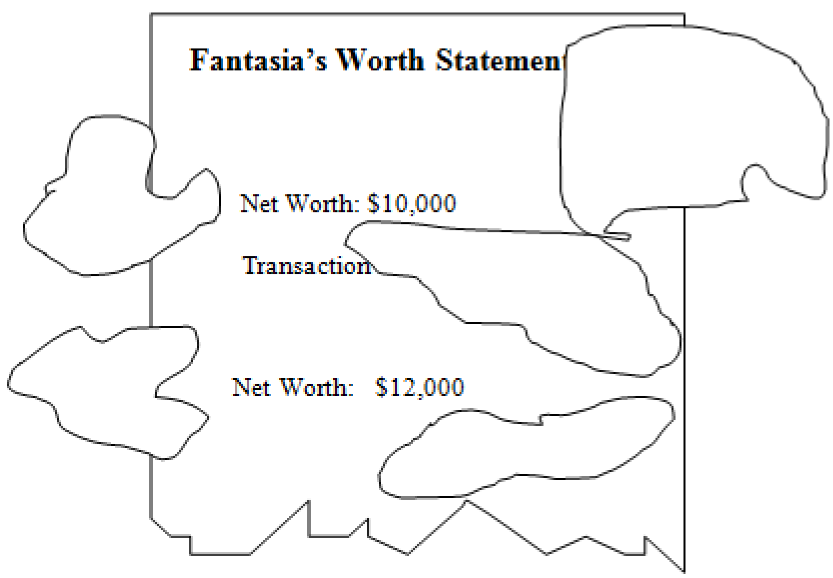


Figure 15: Coffee Spill Problem

T: What is the other easy one?

Charlie: That is not easy but…

T: Hold on then.

Dusty: Minusing debt of 2000 [T writes –(-2000)]

T: Anybody else got that one on the paper? Do you agree with this one Brad or did you just put it because Dusty said?

Brad: I agree.

Charlie: I do not agree.

T: You do not agree? Okay, talk about it Charlie.

Charlie: Because you are minusing…never mind I agree.

T: You do. You just changed your mind. Why do you agree now?

Charlie: Minusing debt is like she owed $2000 and then she did not have to pay it so she went up.

T: So it will be a good thing. Like we did in our stupid and smart decision thing it would be a good decision so you go up $2000.

As it can be seen from the excerpt, the teacher first asked students to find out the easier (quicker, or more efficient) ways that were discussed in the previous sessions. By asking the students to start with the more efficient solutions, the teacher reminded them of the sociomathematical norm that transactions that include one component (i.e. +(+2000) or –(-2000)) are quicker. After Dusty stated that –(-2000) would also be a quick solution, the teacher asked students whether it was on their paper and also whether they agreed or disagreed. At this point Charlie stated he did not agree without analyzing the answer. When the teacher asked him to explain his reasoning, he changed his mind. The teacher this time asked him the reason for changing his mind. Charlie explained it by using the context where he stated that the net worth would go up since her debt was taken away. It is important to note that explaining one's reasoning using take away, add, assets and debt was already established as an acceptable way of reasoning by the classroom community. By capitalizing on his words the teacher folded back to the good and bad decisions and rephrased Charlie's explanation.

As the discussion continued, students also came up with the answers that included two transactions such as –(+3000) + (+5000). For each different answer, the teacher asked the students to analyze the transaction on the board and state their reasoning for whether they agreed or not. During the explanations of their ideas students used the context, the number line, and previously proven conjectures such as –(-) and +(+) are good decisions. After the teacher brought up different transactions in the classroom discussion and received students' ideas about them, she introduced an alternative way that was suggested in her previous period:

> T: Let me share with you what one student in our last class period did. It is very similar to Cody's. Start off with 10,000 and added plus 500 plus another 500… [she writes on the board 10,000 + (+500) + (+500) + (+500) + (+500)]. Is that possible for Fantasia?
>
> Sts: Yes.
>
> T: It equals to 12,000. It is doable isn't it? He just thought about breaking 2000 as 500, 500, … Let me show you the cool way of writing this, instead of writing 500, 500... I do not want to do it every time [she writes on the board 10,000 + 4(+500)]. What do you think that means?
>
> Sts: Four five hundreds.
>
> T: Four $500 asset are being added. All right? Do you see that? This is going to be a short cut. $10,000 net worth and you add four sets of $500 asset, and then you will get $12,000 at the end. If you ever see something like that with the multiplication, you'll know that is how many sets of asset or debts that you are doing in the transaction.

Once the students came up with the different answers the teacher introduced a solution from the previous period to share a sophisticated way of writing the same expression. As it can be seen in the excerpt above, she introduced a different symbolization and supported how students might make sense of it. The teacher's encouragement of using sophisticated solutions motivated students to use multiplication in writing the transaction in different ways in the following classrooms. Since some students were not very comfortable with the expressions that include multiplication, each time a student brought up a solution that involving multiplication the teacher asked students to explain what it means.

In the episode above the teacher's goal was to focus on the activities that aim to investigate equivalent transactions (G_C6). During the first part of the discussion, the teacher's belief that students should know what counts as efficient and different solutions (B_M6) was highly activated. Thus, the teacher encouraged the students first to look for the efficient solutions and then the different ones. Additionally, her belief that the teacher should remind students of the socimathematical norms (G_O1) might have played a role in her folding back to the reasoning that students used in the good and bad decision activity for acceptable solutions. The reason for the teacher to bring up a sophisticated solution that includes multiplication might have resulted from her belief that the teacher is responsible for introducing the different and efficient methods if students cannot come up with them (B_T9). Another reason might be her local goal of bringing a student's idea from the previous period (G_L1), since she might have thought that the idea could be useful for other students as well.

Action 6: Encouraging sophisticated solutions

Next, students were given the problem, ▢ + (-5000) = +5000. Although many students argued that the answer was 10,000, Brad argued that the answer should be zero. At that point the teacher asked Marsha if she could show her solution on the board using the number line (see Figure 16)

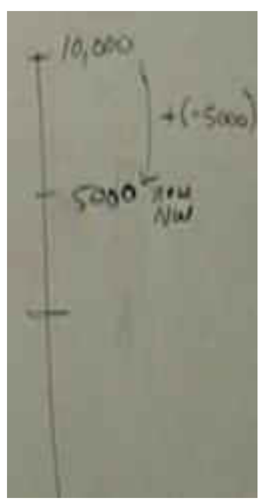


Figure 16: Teacher's symbolization during restatement

Marsha: I will show it on the number line [She put 10,000 on the number line. Made a downward arrow and wrote +(-5000) to land at 5000]. So I added negative 5000 and then got 5000.

T: Thank you. Is that enough? Can you see why he [referring to Brad] might say zero?

Sts: Yes.

T: I can see why he might say zero. Here is the question I have for you guys. Marsha did not know where she starts, because that is the blank. How did you guys get 10,000? She proved 10,000 is the answer but that is only if you knew that. How did you get it? Mark?

Mark: I did up like backwards, I started from the answer 5000

T: You started plus 5000 because that was the new net worth [she shows on the number line].

Mark: I knew that I had to work backwards so I did 5000 as my original net worth and I changed the sign in minus 5000 and made it plus, I just like added 5000 to the other 5000 and I got10,000.

T: He knows that this transaction does not make him to go up, does he? He is saying I have to work backwards. He knew he had to be higher so he says what number started with get me going down 5000 to that number. So he says he worked backwards on that. 5000 plus 5000 has to start with 10,000. And then you can double check yourself right? 10,000 add a debt of 5000, yes it is 5000 the new net worth. Do you understand that working backwards thing? He said I started here, and I know I had to start top because it is a bad transaction it is going to bring me down so to find the original he said he worked backwards, he added to get up there.

During the discussion Marsha proved that the answer is 10,000 rather than zero on the number line. This helped Brad to see that his solution was incorrect. The teacher and other students also stated that they understand why Brad might have said zero. At this point the teacher stated that although Marsha proved 10,000 is correct, she did not show how she found 10,000 in the first place. Thus, she asked how would a person, who did not know the answer was 10,000 find the answer. Stephan's beliefs related to mathematics that not all mathematical solutions are equal (B_M6) seemed to have shaped her to ask students if they could find the answer in another way. This way, she encouraged students to think about a more sophisticated solution instead of the guess-and-check method. Mark then explained his method of working backwards, which was restated by the teacher in clear language.

Her belief related to teaching that the teacher is responsible to provide opportunities to students to see sophisticated ways of solving problems during the classroom discussion (B_T9) seemed to have affected her decisions. This belief also activated the goal of clarifying the ideas by restating Mark's solution since sometimes students had difficulty expressing their ideas and thus the other students might not understand the mathematical ideas stated by their classmates (G_O7).

Close to the end of the instructional sequence (day 17), the teacher provided students problems that included only one sign. The following excerpt was taken from a discussion of the problem -7 – 9 = ?:

> Anthony: There is a positive sign in front of 9, isn't there? [Before Anthony's question some students state there is no parenthesis]
>
> T: Some of you say there is no parenthesis here and there is no sign whether it is asset or debt.
>
> Brad: I did not know what it was.
>
> T: You did not know what it was. A bunch of people in the second period did not know what it was either. Remember yesterday we worked so hard on the problems like this, you take away asset of 9 [she writes –(+9)] what is another way of rewriting this?
>
> Sts: Minus 9 [she writes -9]
>
> T: Can you write it like that (-9)?
>
> Tisha: Yes, but it would be negative 9.
>
> T: Whatever word you want to use, can you rewrite these two? Do they mean the same thing?
>
> Sts: Yes
>
> T: That is what we worked on yesterday. Anthony says you can put parenthesis in and he said it is kind of like -7 - (+9), as in taking away asset of 9. Did I get you right Anthony?
>
> Anthony: Yes.
>
> T: All right if you do not see the parenthesis, feel free to put them in. Or if you recognize it as taking away 9 is the same thing as taking away an asset of 9, it is okay too. Many of you did not have trouble with these questions as I thought.

During the discussion the teacher capitalized on one of the student's sophisticated solutions related with the symbolization, which might also help other students with solutions to integer problems that include only one sign. Thus, she asked students what would be another way to rewrite –(+9) and students recognized it as -9 without much difficulty. The teacher reminded students that they could put the parenthesis and add another sign if they felt more comfortable with that representation. The teacher's goal in this activity was to help students to be comfortable with the problems that include only one sign (G_C7). Since she knew that students would run into those types of problems in the following classes as well as in the textbooks, she wanted to emphasize the equivalence of those expressions.

In summary, the practice of *supporting the development of sociomathematical norms* included encouraging students to give conceptual explanations; seek efficient, different and sophisticated solutions; make conjectures; and express their reasoning. During this practice the teacher's content knowledge as well as her pedagogical knowledge, in particular the knowledge of students' misconceptions and current students were essential. This body of knowledge allowed her to establish what counts as mathematically acceptable, different, and sophisticated solutions. Her beliefs related with mathematics particularly that mathematics is a sense making activity and making conjectures is important to understand mathematical ideas seemed to have played an important role in this practice. Additionally, her content goals such as ensuring that students understand net worth as an abstract quantity and overarching goals such as renegotiating sociomathematical norms as necessary during the instruction were highly activated in this practice.

Practice Four: Capitalizing on Students' Imagery to Create Inscriptions and Notation

The reform movement supports that all students should be given opportunity to learn mathematics meaningfully (NCTM, 2000). Students need to see mathematics as an authentic activity where they take active roles in making conjectures and responding to others' ideas. Imagery helps students give meaning to mathematical activities, which adds richness to their mathematical reasoning and understanding. Imagery can encompass various forms such as personal images, background images, notational images, and situation specific images (Thompson, 1996). In this study the teacher's role in supporting students' development of both situation-specific and notational imagery were investigated. While the former refers to mental representations conjured up in a person's mind within a given situation or context, the latter refers to what a person perceives from the given tools.

The context constitutes a setting where students can ground their mathematical activity to develop situation-specific imagery (McClain & Cobb, 1998). Tools, on the other hand, serve as a resource that students can use to construct and communicate their mathematical reasoning and develop notational imagery (Confrey, 1990). Imagery helps students ground their mathematical activity with meaningful "anchors" that serve as the basis for more sophisticated, abstract mathematical reasoning. Focusing on students' imagery can create opportunities for class discussion in which students can develop meaningful solution procedures, which are likely to be remembered and productively used by them in future activities.

Action 1: Building the sequence on a context that students can imagine

In the beginning of the sequence, the teacher asked students to determine a famous person's properties that she may own and owe. During the classroom discussion, students guessed different things that she may own such as houses, boats, and loans as well the things that she may owe such as car loans, credit cards, and mortgages. The teacher intentionally listed the things related with the assets on the left side of the board and the things related with the debts on the right side of the board. Next, she asked students if anyone knew the mathematical terms for each list. Since Danny used the word debt during the discussion, the teacher capitalized on it and stated the title of the list as Danny's title "debts" and then asked students if anyone in the classroom had any debts. After the students talked about debts and related it to their own experiences, this time the teacher introduced the word asset by stating that "You probably do not know the title for this column but you might have heard it, it is actually called assets" and added that they would learn a little bit about finance in this unit. In order to engage students in personally meaningful activities, the teacher asked them to describe their own assets and debts.

Next, the teacher asked students the meaning of net worth and whether they could describe it in terms of assets and debts. The aim of starting the unit with a situation where students could imagine assets, debts, and the effect of them on net worth was to develop a situation-specific imagery. Her belief that students learn better with instruction that relates to their informal knowledge (B_L5) seemed to encourage the teacher to create an instructional sequence based on a familiar financial context. Rather than introducing the ideas as ready-made she tried to support students to develop the ideas gradually from concrete to abstract. This also gave her the opportunity to fold back to those ideas in order to support more complex ones. Thus, starting a task with an experientially-real context and moving gradually to more abstract ideas was an important practice of the teacher. Here it is important to note that experientially real does not mean that students have to experience the event but they should be able to imagine the given scenario told by the teacher (Gravemeijer, Bowers, & Stephan, 2003).

Once the students became familiar with the words assets and debts, the teacher provided an activity that included a net worth statement. During the introduction of this activity, she told the students that her accountant asked her to fill out the net worth statement (an example is shown in Figure 17) in order to help her to show how to save enough money for her son's college payment.

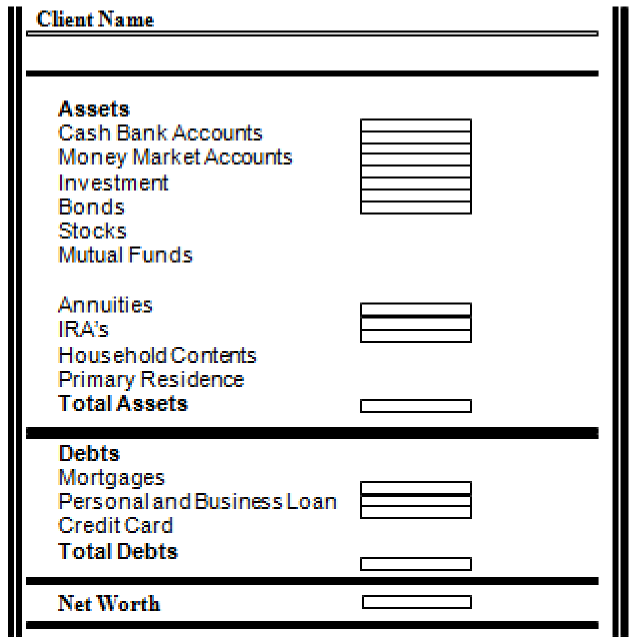


Figure 17: Net worth statement

Students reviewed the statement and found the words that were familiar and unfamiliar to them. The goal of this activity was to make students familiar with words such as loans, stocks, and mortgage and to develop the idea that net worth is an intangible quantity (G_C3) - not the money that a person has in his or her pocket. She introduced the task as follows:

> T: Maybe you can figure it out in your own time. Let me tell you the story, four years ago I found that I was pregnant with my son and I freaked out. You know why I freaked out? Because I thought, all right I am going to have this baby. 18 years from now, what I am thinking with 18 years?
>
> Sts: College.
>
> T: You guys have any family members in college; you know how expensive it is. By the time you get 18 it is gonna be at least twice what it is now probably or more than that. So I freaked out about how I am going to pay for his college money in 18 years.
>
> Adam: Scholarship.
>
> T: So what I did Adam, I hope he can get scholarship but if it is not I don't want to go in debt myself. So Seth what I did was I went to a financial advisor. Do you know what that is Norman?
>
> Norman: Someone that gives you advice for money.
>
> T: Something like that. He advises on guess what?
>
> Sts: Your finances.

> T: I asked him to help me. How in the world will I pay for this kid in 18 years for his college. It is what he said. He sent me on this[see Figure 17]by email called net worth statement, you see the client name so I put my name and he said to fill this out so that you can find out what your net worth is. He needed a starting place to be able to advise me and he said he needs to know what I have and what I owe. So he sent me this and asked me to fill it out. I am going to give you a copy of it and I want you to go through this net worth statement. Do not fill it out because you probably do not have a lot of these things but I want you to look through this and pick out the words that you do not know what it means. Let's take one minute to do it quietly. Put a little star to the words that you do not know. Put your name on it, you can be a client. Do not fill out the money part just look at the words that you do not know [students take few minutes to mark the words that they do not know].
>
> T: Okay. Were there any words that you guys did not know in the statement?

As it can be seen from the excerpt, the teacher introduced the net worth statement by connecting it with real life where she stated it was the document that was given to her by the financial advisor. In order to familiarize the students with the words on the list she discussed the ones that they do not know the meanings to in the classroom. This activity was the underpinning of the instructional sequence where the students could develop situation-specific imagery where assets and debts are quantities that affect net worth in opposite directions (G_C1). By providing this activity she helped students to develop meaning for the words assets and debts so she could draw back to those images to support subsequent activities. This way, she started to support the development of meaningful imagery (G_O5). In the following class sessions students folded back to this activity to make sense of the given numbers. The teacher's knowledge of the mathematics education literature, in particular the RME instructional design, and her content knowledge had an essential role in creating a sequence based on the idea of a familiar financial context (K_P9, K_S1).

Action 2: Modeling students' informal thinking

In the following days (beginning with day 3) students continued to solve problems related to net worth. At this point when students brought up the idea of pay off, the teacher capitalized on it and modeled students' thinking with the vertical number line:

T: Brad, can you tell them, what happens?

Brad: He actually has assets but his debts overwhelmed his assets.

T: Nathan, say it again what Brad just said?

Nathan: When he pays off his debts from his assets that is how much he has left.

T: All right. What is his pay off amount? How much is the pay off?

Sts: 8400.

T: 8400? If he pays that off, he is…

Sts: In the negative 400.

Nathan: He still has to pay.

T: So he still has to pay, all right. Do you know Sienna in my second class? She said this in really cool ways. If he had $8000 [the teacher writes it on the board], she kind of modeled it on the up and down vertical number line. If he has $8000 and pays it off as much as he can, what would it take him to (See Figure 18)?

Dusty: Zero.

T: What do you think about it Betty? [She agrees] Tisha, you do not agree with that?

Tisha: No, I do.

T: What are you agreeing with?

Tisha: That is when he pays it off, he will go down to zero [the teacher writes zero on the number line].

T: If he paid it off, he would go down to zero. Can he pay off all his debt? [Students say "no"] She said no. What is left? [T shows arrows 8000 to 0 on the vertical number line]

Seth: Negative 400.

T: So she said negative 400. She called this a pay off [the teacher writes "pay off" next to 8000]. That will take him $8000 to pay off and then he is still in debt, right?

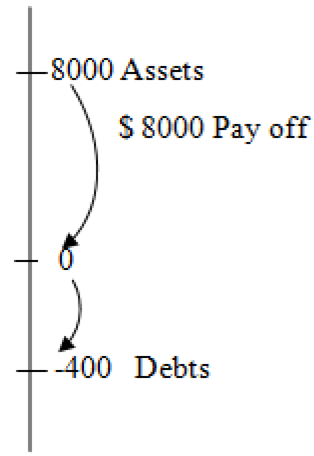


Figure 18: Demonstration of pay off idea on the vertical number line

During the argumentation, the students claimed that the person could use his assets to pay off his debts. But they also recognized that he had more debts than assets so he could not pay off all his debts. To emphasize the pay off idea, the teacher asked how much he could actually pay off. Nathan responded that he could pay off $8000 if he used all his assets, which would leave him $400 in debt. Next, the teacher stated that one of her students from the last period modeled this pay off idea with the vertical number line. She asked students that if he pays off 8000 where would this take him. Students said that it would take him to zero and the teacher provided a visual representation for it on the number line with the down arrow. Then, she asked if he could pay off all his debt when he paid $8000. Seth stated that it would leave negative $400. The teacher showed the money that was left over by going down to -400 on the number line.

As it was seen above, in this episode the teacher introduced the *notational imagery,* which was the vertical number line (and arrows) in this case to make students' thinking visible. However, it is important to note that the teacher did not introduce the number line as ready-made but introduced it in a way that captured students' ideas and reasoning. She knew that the number line would connect with students' reasoning because Sienna, a student in another 7th grade class, created it. Introducing tools based on and to facilitate students' reasoning was an important part of the teacher's practice. This allowed students to better understand why that tool was needed and how it could be used to develop and communicate their ideas. Stephan's overarching goal that the teacher supports students' ideas with tools when they have the reasoning that can be modeled easily with that tool (G_O6) shaped the discussion above. Once the students started to explain the pay off concept the teacher decided to bring another student's solution into the classroom (G_L1). This way, she could model students' thinking on the board, which helped many students to understand the ideas that were presented by their friends. The practice of capturing students' ideas with a model was effective during the instruction since it facilitated the communication of students with each other as well as the teacher.

Using the vertical number line helped students make sense of computations. Especially at the beginning of the instructional sequence many students tried to find the answers by always subtracting the lesser number from the greater one, and using the sign of the greater number. By capitalizing on students' imagery of paying off and left over, the vertical number line became an inscriptional device that made students' imagery visible. In the example above, the teacher capitalized on the idea that the students subtracted one number from the other because they wanted to know what was left over after paying off their debts. For example in this problem he had to pay $8400 but he only had $8000. Thus after paying $8000 he went down to zero and he was left with $400 in debt. Thus, the pay off image made visible by the number line also supported students' understanding of why and when to subtract. Helping students to make sense of their computations by using imagery of paying off was an important part of the teacher's practice. Her knowledge about the literature (K_P7) that the tools are useful when they are used to support students' reasoning had an essential role in deciding when to introduce the vertical number line. In these activities the number line served as a tool to help students visualize the positive and negative net worths. Using the number line as a computational device was later introduced in the transaction activities.

Action 3: Introducing a tool as a supportive reasoning device

In the following lesson, the teacher gave an activity that supported students' notational imagery but with different reasoning. The aim of the teacher in these activities was to help students compare two different net worths and find the difference between them. For this reason, the teacher introduced a black and red number line where the positive parts of the number line were shown in black and the negative parts were shown in red. Defining a new number line in this way built on the financial context as well as on students' reasoning; the teacher also connected this idea with the financial concept of "being in the black" and "being in the red" as shown in the following excerpt (see Figure 19):

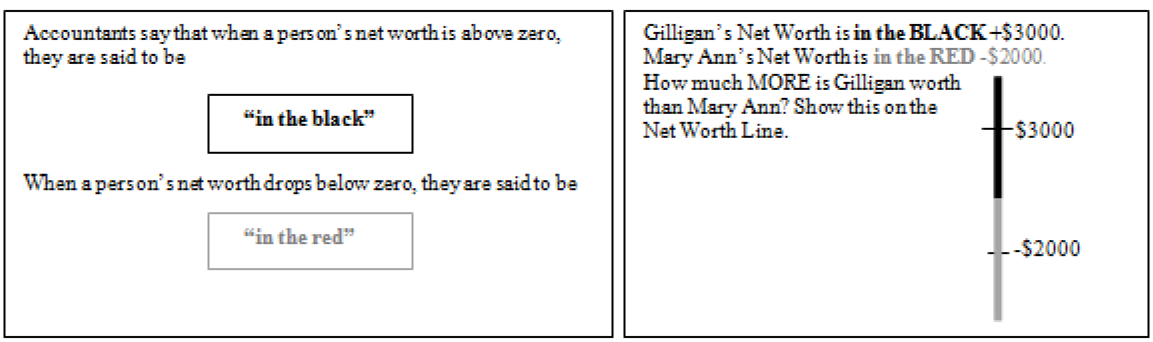


Figure 19: Red and black number line

T: Has anyone heard the names in the black or red? Have you ever heard it?

Charlie: I heard in the red before.

T: Do you know what it means?

Flora: In the black something to do with assets and in the red something to do with debts.

T: Dusty what do you want to say?

Dusty: In the black you get stuff and in the red you lose stuff.

> T: Something like that. I think we all know that black has something to do with assets and red has something to do with debts. So honestly I am not making this up when accountants say a person's net worth is above zero that means they are in the black and when a person's net worth drops below zero, they are in the red. Where do you want to be, if you are that person?

Here, the teacher first asked if the students knew what it means to be in the black and to be in the red. Some students conjectured that being in the black is associated with having assets and being in the red is associated with having debts. The teacher capitalized on these thoughts and supported students in developing both the situation specific imagery of being in the red and black and a notational image where black and red colors are associated with positive and negative parts of the vertical number line.

The teacher's goal in introducing the black and red number line (G_C5) was to support the students' notational imagery where they could make sense of ordering the net worths as well as finding the difference between them. Here it is important to note that the same vertical number line where the teacher captured students' ideas related to the pay off idea served to support reasoning related with ordering the numbers and finding the difference between them. Thus, although the tool remained the same, the reasoning with the tool shifted. The teacher introduced a black and red number line where the positive parts of the number line were shown in black and the negative parts were shown in red. To build on the context, she also connected this idea with the financial concept of "being in the black" and "being in the red". Her goal that the teacher should support tools that will help students visualize their reasoning (G_O6) motivated her to create this activity. Additionally, the teacher's knowledge about students' difficulties in understanding zero and negative numbers might have encouraged her to introduce the black and red number line (K_P2). Another important point is that this activity was not a part of the instructional design at the beginning of the instruction. The teacher felt it necessary to support students' reasoning in ordering numbers on the number line as it was discussed in Chapter 5 under the "revision" practice.

Action 4: Discussing correct and incorrect uses of a tool

During the solution to the problem above (see Figure 19) the students came up with different strategies. The teacher collected students' solutions at the end of the class and planned to discuss some of them on the following day. During the selection of the solutions, she focused on the ones that include common misconceptions and different correct approaches. Figure 20, shows some of the students' solutions:

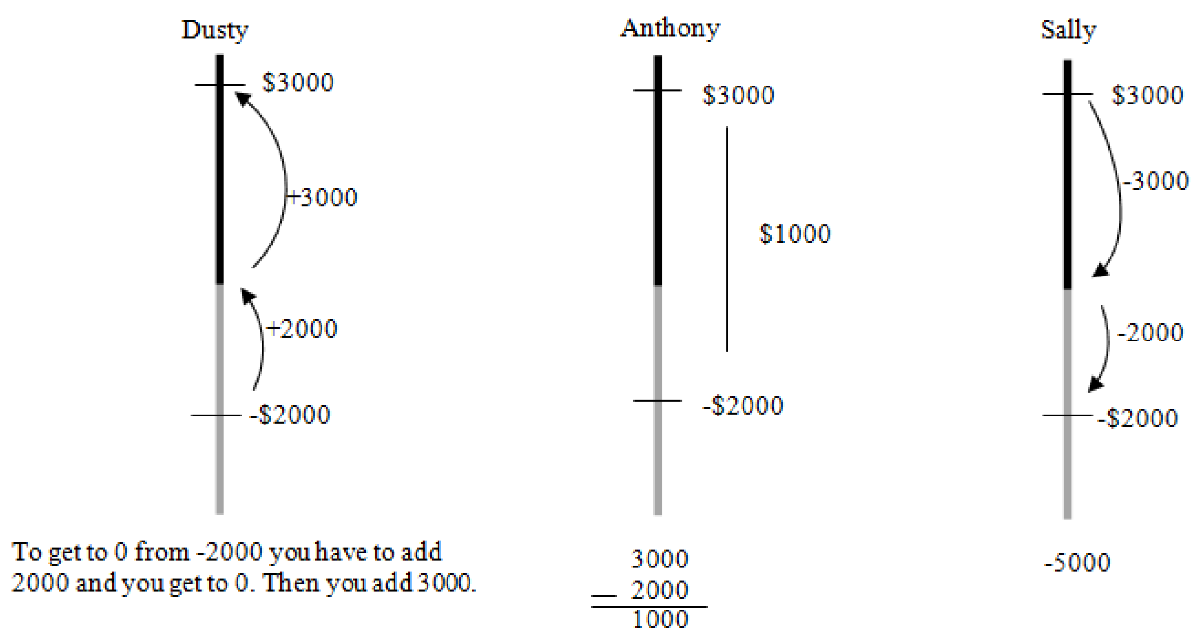


Figure 20: Students' solutions

After projecting different solutions on the board, the teacher asked students to analyze each solution and explain whether they thought it made sense. The teacher's belief that mathematics is a sense-making activity that involves analyzing solutions (B_M4) might have motivated her to project the different answers so students could compare and contrast those solutions. During the discussion, some students claimed that the answer should be 1000 by subtracting 2000 from 3000. However, some other students challenged this solution and argued that the correct answer was 5000 since they first had to go up by 2000 to reach zero and then go up another 3000. Another student found the answer as -5000 by first going down from 3000 to zero and from zero to -2000. The teacher accepted both of the latter two solutions as legitimate since her intent was for the students to structure the gap between the two net worths quantitatively, not based on the actual sign of the result. Here, the number line served as a tool for the students to see the gap between two given numbers; going up or down was not the focal point at this point in the sequence. After projecting the solutions on the board the teacher asked students to state if they agreed or disagreed with each one. The reason for including not only the correct solutions but also the erroneous solutions seemed to have stemmed from her belief that students can learn from their mistakes (B_L8). The action of the teacher that is discussing both right and wrong solutions helped students to notice their mistakes and why they did not make sense.

Additionally, the teacher's action of illustrating different correct answers could be explained by her belief of valuing students' ideas (B_T4) and her goal of supporting the previously established practice of exploring multiple solutions (G_O1). She not only showed Dusty's solution, but also Sally's solution as well since they both could be accepted as correct solutions depending on who they took as reference. At this point students' reasoning with notational imagery was based on defining the gap between two net worths. The teacher's role in supporting discussions of uses of a tool was essential in students' development of useful imagery. The class discussion seemed to help many students to make sense of the answer of the problem as well as the ways that would help them to solve the problems by using the number line.

Action 5: Encouraging students to draw on previous images

In the following class sessions (beginning with day 5), the teacher continued to support the use of notational imagery with the problems where she gave the black and red vertical number line and asked students to order the numbers on it. The excerpt below was taken from the discussion of ordering two negative net worths, one of them was -$20,000 while the other one was -$22,000. Since some students had difficulty ordering those numbers correctly, the teacher encouraged students to remember their previous images:

> T: Have you guys ever heard about an expression climbing your way out of debt?
>
> Sts: Yes [some of them say “no”].
>
> T: If you have heard that expression, why do they say you are climbing your way out of debt? Alice why do they say that?
>
> Alice: Mmmm. I don’t know.
>
> T: What about Sally, do you have an idea?
>
> Sally: Like you are climbing higher and higher.
>
> T: Yes, if you are climbing usually you go higher and higher and you are getting out of the hole. I like the idea of hole down there. Charlie?
>
> Charlie: It is like you are digging yourself into a hole.
>
> T: You heard that before, digging yourself into the hole? That is the idea. The more you charged on the credit card, the deeper you are going in debt - digging yourself into a deeper hole [she makes a gesture of pointing down]. You get worse in debt.

The teacher initiated the discussion by asking students if they had heard of the expression “climbing your way out of debt”. The goal of the teacher during this discussion was to support the development of situation-specific imagery (G_O5) in order to make sense of the order of negative numbers. Sally stated that it is like climbing higher and higher and the teacher emphasized her words and said it is like climbing higher as in trying to get out of a hole. At this point Charlie stated that if one’s debt increases, it can be considered as digging oneself into a hole. Finally, the teacher focused on his ideas with an example such as if they get charged more on the credit card, they go more into debt and it would be like digging themselves a deeper hole.

Stephan included the number line in the activity and connected it with the context (assets and debts) to fold back to students' previous imagery. Supporting development of students' imagery helped them make sense of why negative numbers that have greater absolute value are under the others that have lesser absolute values. The teacher related this to the more debt they have, the deeper the hole they go into. The teacher's belief that students should make sense of mathematics (B_M1) might have been influential in creating this activity to connect the notational imagery with the situation-specific imagery. Here it is important to note that the teacher's knowledge related to students' misconception in ordering the numbers (K_P2), especially negative numbers, motivated her to talk about these kinds of activities in the teacher meetings and make them part of the instructional sequence.

As the instructional sequence moved forward, the teacher continued to draw back to students' previous images in the given tasks. One of the tasks that the teacher gave students was related to the effect of actions on the net worth (e.g., taking away debt is a good thing). Students used this later when making sense of integer operations. The teacher's goal of the episode below was to support students to find the effect of transactions on the net worth (G_C6). Thus the activity included the action words such as taking away and adding besides asset and debt with which the students were already familiar. The following excerpt shows how the teacher introduced the activity to support the imagery:

> T: Ann here, she says [in the activity] that she took away an asset of $200 from her net worth statement [see Appendix C "Smart or Stupid Decision"]. Do not say the answer. But do you think Ann's decision is a smart decision or a stupid decision? That is what I want you to write down. What is her decision, stupid or smart?
>
> Charlie: This is easy.

T: Is Bradley's decision stupid or smart? [She shows the rest of the decisions in the activity] I want you to write it down for all and check with your partner. Take just two minutes. Charlie said it is too easy I can do it in one minute?

Charlie: One second [T goes around the class].

T: How many smart decisions did you get? [Whole classroom discussion]

Sts: Two.

T: Anybody get more or less than two smart? Who were the smartest?

Sts: Ernie and Bradley.

T: I have a lot of students in the past that said stupid decision for Ernie. Do you know why? [Ernie: He took away a debt of (-$5400) from his net worth statement].

Dusty: Because it says debt.

T: Yes it is debt and automatically people think debt is a stupid decision.

Sts: I put smart.

T: That is good. If you just look at the fact that it is a debt you might say that it is stupid, what makes it a smart decision?

Danny: Taking away debt.

After the teacher introduced the task, she asked students if they could find whether transactions in the given activity were good or bad. In order to make the problem clear Stephan elaborated the first one and asked students to write the effect of each transaction next to the given verbal statement. She encouraged students to fold back to the concept of assets and debts and decide whether it was a good or bad decision to add or take them away. Since the students already had the image of assets increasing the net worth, while the debts decrease it, they were able to find the effects of the actions easily. She continued to support situation-specific imagery where this time students could imagine the effects of transactions, since her goal was to develop meaningful imagery (G_O5).

Then, she gave the students a few minutes to work on the statements. At this point many students stated that it was an easy activity. Once the students finished all of them the teacher started from Ernie's decision since in the previous years some students thought that his decision was bad when they saw the word debt. However, many students in the class stated that it was a good decision because it represented taking away a debt. During the introduction of the activity the teacher used words "stupid and smart", since those words were likely to appeal to students' attention and therefore keep the discourse alive (G_O2).

Action 6: Introducing mathematical symbolizations

Once the students decided whether the transactions were good or bad, and each transaction's result was discussed in the classroom, the teacher continued by introducing how students could rewrite each sentence by using mathematical symbolization. The following excerpt shows the teacher's introduction of symbols:

> T: Here is a very important part. Everybody take page 15 (see Appendix C, "Smart or Stupid Decisions") out and I want you to start taking notes. Because I will introduce the way we are going to symbolize these things. These decisions people are making are affecting people's net worth aren't they? When you make a decision that affects your net worth either good or bad we will call that a transaction. Something happens, some transaction happens and it changes your net worth. You guys said this was a smart or stupid transaction?
>
> Sts: Stupid.
>
> T: It is a stupid transaction because what happens to his net worth, Brad?
>
> Brad: It goes down.
>
> T: Whatever it is, it is going to go down, isn't it? All right, so we need a way to start symbolizing these so that we do not have to write all these words. Here's how we are going to symbolize that. So I did not let you go out [one student wanted permission from teacher to temporarily leave the class before she started], because I knew if you miss it, you miss a lot. As much as possible we are going to always use these two symbols. The first symbol is going to stand for the action that takes place like Tisha said it is take away. So if it is take away the action is we are going to take away something from the net worth statement. That is going to be what symbol?
>
> Sts: Down arrow.
>
> T: You can have a down symbol, couldn't you? What mathematical symbol you think they use?
>
> Sts: Minus.
>
> T: Minus right? I like that idea down symbol, but we will use the mathematical symbol that is the minus sign. So you can think of this as the action, the take away action, subtract or minusing [she writes these on the board].

Charlie: Adding...

T: Well, that is another transaction, that could happen, isn't it? You could add somebody's net worth. We will get to that one in a minute. But just for this one we take away, we're always going to put parenthesis, what we're taken away Marsha? You guys taking note? So what is it taken from Ann here?

Sts: Money.

T: What kind of money?

Sts: An asset.

T: So guess this sign what is it going to be? Plus [she writes on the board -(+200)]. That is how I am going to start symbolizing this. We do not have to write any word at all. This sign [she points to the sign inside the parenthesis] is either going to be asset which is the case here, isn't it? What else could it be?

Sts: Debt.

Tisha: How? Ohh, adding a debt?

T: Yes you can add a debt, could you not? So I am saying in general Tisha this second sign either going to be an asset as in this example, it was an asset, wasn't it? Or it could be a debt sign. What sign do you think we use for debt?

Sts: Minus [some of them say negative]

T: Here since it is taking away an asset, what is another action we could have done? Charlie you said that

Charlie: Add.

T: So you might see plus or minus here [outside the parenthesis] or plus and minus here [inside the parenthesis]. The first sign always stands for the action that takes place. Taking away or adding somebody's net worth. The second sign and always put in parenthesis stands for whether that number is asset or debt. That is a lot of me talking so I'd rather stop talking, what I would like you to do is symbolize Bradley and the others [one student symbolizes Bradley and she writes it down on the board and students start to work on it for the rest of them].

In this episode, the teacher introduced symbolizations for the transactions. She stated that in order to symbolize the transaction they would use two signs, where the first one outside the parenthesis shows the action. The second one inside the parenthesis indicates whether the quantity is an asset or a debt. During the introduction, she asked students to guess the sign that they would use for take away. Many students stated that it should be the minus sign. The teacher then pointed out that although it is the minus sign for this problem, it could be the plus sign if the transaction was addition. Similar to the actions she asked students what sign they would use for debts and assets. Finally, she summarized how they could write the given transactions by using mathematical symbols and asked the students to symbolize the other problems given in the activity.

The teacher's goal in the discussion above was to introduce the mathematical symbolization (B_T10). Since the teacher believes that students should make sense of the symbols (B_M7), instead of introducing them ready-made, she introduced them gradually by connecting them with students' situation-specific imagery. This allowed the students to fold back to their images if they had difficulty deciding what those symbols meant in the given problems. Moreover, they were able to see the patterns in the given transactions and could discover rules for integer operations. As an example, Stuart's conjecture given in the previous section (i.e. –(-) and +(+) are good decisions whereas –(+) and +(-) are bad decisions) emerged on the following day during the discussion of symbolizing transactions. By connecting the mathematical symbols that were introduced by the teacher and the "good and bad decision" imagery many students could come up with the conjecture of –(-) and +(+) have the same effect on the net worth which is good and +(-) and -(+) have the same effect on the net worth which is bad. Students were able to come up with these ideas that lie at the heart of the integers topic as a result of the instructional design that gradually moved them from concrete to abstract relating concepts with their informal knowledge (B_L6). Here it is important to note that the teacher's knowledge pertaining to students learning from the relevant literature might have also played a role in forming her beliefs about the characteristics of instruction that would be most effective (K_P5).

Action 7: Encouraging students to record their thinking

During the following sessions the teacher continued to support using the number line as an acceptable explanation. She asked students which number lines would match the question where the original net worth was $4000 and the transaction was +(-$8000). The reason for the teacher to bring up this activity was to assess whether students were using the tool with understanding. In order to achieve this she included possible number lines that students could use in their solutions. Next, she asked them to compare and contrast the solutions on the board and find which one (or ones) might belong to the question given in Figure 21. During students' explanations the teacher encouraged them to record their thinking on the number line.

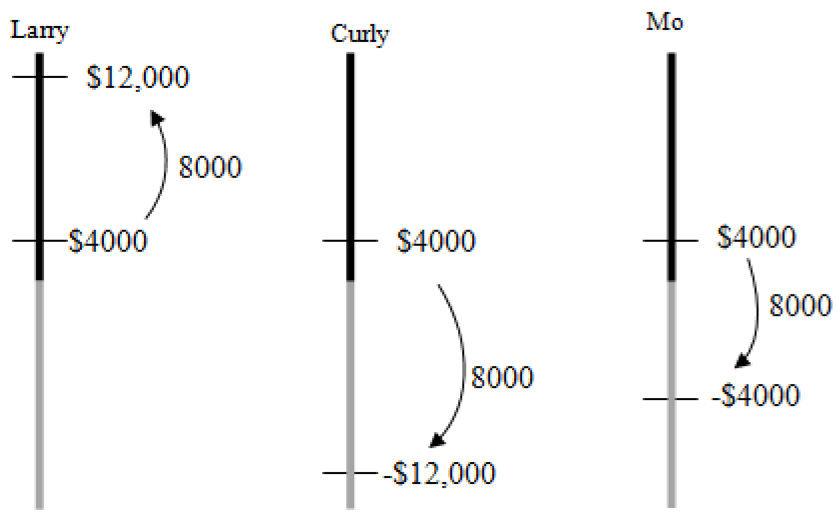


Figure 21: Bell work problem

Although many students chose the representation of Mo, some students stated that Curly's number line was correct. At that point, the teacher asked the students in this latter group to defend their reasoning. However, none of them were able to provide any explanation. As Stephan believes that students learn from their mistakes (B_L8), she encouraged them to come to the board and explain their reasoning but none of those students were comfortable enough to explain their solutions. Since Stephan also believes that the teacher needs to support a sense of community where ideas can be discussed freely with a mutual trust (B_T2), she did not want to force those students but continued to encourage them to explain their ideas. The teacher's encouragement allowed those students who were shy to participate to become more open in the subsequent classroom sessions.

Since no one wanted to prove Curly’s solution, the teacher asked the students to prove Mo’s number line. Norman was one of the students who argued that Mo’s number line was correct. The teacher encouraged him to show his solution on the number line:

> Norman: Since it is adding a debt he goes down 4000 and then 4000 more to get 8000 in total (See Figure 22)
>
> T: Dusty what did he say?
>
> Dusty: He goes down from 4000 to zero and then from zero to another 4000. So if you add 4000 plus 4000 you get 8000. Is that what you said?
>
> Norman: Yes.
>
> T: Mariana, Alice do you want to talk about this? [Mariana and Alice agreed with Curly’s number line at the beginning] What do you want to reject about what Norman said? What did you add?
>
> T: What did you add?
>
> Mariana: I did not use two jumps.
>
> T: Is it necessary to use two jumps? If you can see it a little easier, use two jumps. Does anyone want to add that? You liked what Norman said?
>
> Sts: Yes.
>
> T: His symbols were really helpful aren’t they? [T shows the image in Figure 22]

Figure 22: Norman's solution

Norman showed his solution on the number line by first going down from 4000 to zero and from zero to -4000. He then added 4000 to 4000 and got 8000 (see Figure 22). His symbols were helpful to all students, thus they did not ask any questions regarding the solution. The reason why the teacher asked Norman to symbolize his thinking seemed to be her goal of clarifying the mathematical ideas that are presented by the students and supporting the use of tools with reasoning (G_O6, G_O7).

Next, the teacher encouraged Mariana and the others to use the number line to solve the problem, since she observed that when they use the number line they could make sense of the problem and find the correct answer. In the following of the same session, the teacher asked the students if they could prove why Curly's number line was not correct. At this point Brad started explaining verbally rather than using symbols. The teacher encouraged him to record his thinking on the number line and to draw on the previous images:

> T: My next question is you know how Charlie rejected Larry's and he said why, who can point out why Curly is not working? Because I think Mariana is now convinced with this one, so Mariana my tip to you is start using the number line to figure out this kind of stuff, okay? Alice, are you convinced? Brad can you tell the basis for rejecting this one?
>
> Brad: What I did is same thing here [points out Norman's solution]. I went 4000 to 0.
>
> T: Can you write it down so everybody can see what you are saying?
>
> Brad: Then I added 4000 to 12,000.
>
> T: What does that 16,000 stand for Sally?
>
> Sts: It is the total of two jumps.
>
> T: What does it stand for in the money world? What does this 16,000 stand for? Why is 16,000 important?
>
> Brad: Because it shows what debt he [Curly] added on
>
> T: How much was added Mariana?
>
> Mariana: 16,000.
>
> T: 4000 Brad said to get to zero and 12,000 to get here. You added on $16,000 that is not very nice is it? How much were we supposed to add on?

Sts: 8000.

T: You doubled the debt that you were supposed to add on.

Brad disproved Curly's solution on the number line similar to Norman's proof. He first went down from 4000 to zero and then from zero to 12,000. Next, he added 4000 and 12,000 which resulted with a total 16,000 debt. He stated that the solution was not correct since the total debt given in the problem was 12,000. The teacher's belief that mathematics is a sense-making activity where students analyze the mathematical solutions might have encouraged her to ask students to disprove the wrong answers (B_M4).

Recall in a previous example (imagery-action 4), the teacher orchestrated a similar type of discourse by asking students to compare and contrast the number lines that she chose from their artifacts. The difference between these two examples from the point of imagery is that, in the first one the teacher provided the activity to support students' understanding of comparing two different numbers and seeing the gap between them, in the second one the aim was first to find the direction to go on the number line and then to arrive at the correct number. Thus, although in the first one the direction, operation, and the landing point was not an issue, in the second one they were the main points of the discussion. In other words, the teacher supported different reasoning by using the same tool (shift in reasoning). Her goal of remediating the possible incorrect images (G_O5) that the students might have developed was also important in asking this problem in the classroom. The teacher's knowledge of her current students' misconceptions (K_P11) as well as her knowledge about the importance of using the tools with understanding (K_P7) had a crucial place in her practices.

Action 8: Recording students' ideas on the board

One of the teacher's important actions was to record students' thinking during their verbal explanations as well as during her restatement of those explanations. The excerpt below was taken from the discussion of a question which asked the new net worth when the original net worth was 298 and the transaction was +(-427). After Mark explained his solution, the teacher restated his answer to make it clear to the other students. During her restatement of Mark's explanation, she also recorded his ideas clearly on the number line since his symbolization was not clear (see Figure 23):

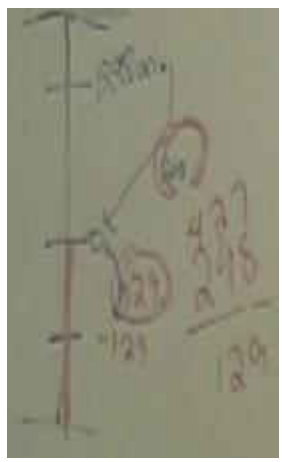


Figure 23: Teacher's symbolization of Mark's solution on the number line

T: Where did he get 129 from? I need you to try to make sense of this. He starts off, here is the original net worth, is it good or bad?

Sts: Good.

T: He is in the black, I will actually make this red so you can see [she colors red the negative part of the number line]. And then what will his first jump be, Norman?

Norman: 298.

T: Has he added all his debts so far?

Sts: No

T: How much debt has he added so far? [students have difficulty answering this]

T: Does Mark start here, all the way down to 427? In one big jump?

Sts: No.

T: How many jumps Anthony does it take? [The teacher asks students to analyze Mark's graph]

Anthony: Two.

T: What is his first jump?

Anthony: Going down to zero.

T: How much?

Anthony: Going down 298

T: Norman is that the whole jump he needed to go? No, he is a little bit more. The question is how much is that little bit more, pretend we did not know what it was. How much is that little bit more? He has already gone down 298. How much he had to go altogether? [students say 427] He had to go 427, he went down a little bit, how much he has left? He cannot count his fingers, can he? 298, 299. What is the quick way Sally?

Sally: He subtracts 298.

T: 298 from what?

Sts: 427

T: That is how much he had left to jump. He has to go 129 more. Where is that going to land him? (see Figure 23)

Sts: -129.

After Mark explained his reasoning, the teacher felt it necessary to restate the students' words in a clear language to help all the classroom members understand his solution (G_O7). During her restatement she used the number line that was drawn by Mark. While she was restating, she first folded back to the imagery of good and bad decisions from the previous class sessions. Once students could find the effect of the transaction thus identifying the direction that they needed to go, the teacher asked them what might be the first jump they might do. At this point the students stated that they would need to do two jumps, the first one down to zero and the second one could be found by subtracting the original net worth from the given number.

The teacher's goal in this discussion was also to support developing the notational imagery of the vertical number line to help students solve integer operation problems with understanding (G_O5). However, it is important to note that although the tool, the vertical number line, was the same as the tool they used in ordering and comparing the numbers, the reasoning with the tool was different. Here, the students integrated the tool with the imagery of good and bad decisions which helped them to determine whether to go up or down on the vertical number line. Additionally, the teacher tried to develop the image of using zero as a break point, which could be used to split up a large transaction that goes through zero into two smaller transactions.

To support the imagery of good and bad decisions, as well as the previous images, the teacher continued to engage students in various activities where they could use these images collectively. The following example demonstrates how the teacher recorded students' thinking in the discussion of an activity which involved finding the original net worth given the new net worth and the transaction. The task was to find what the box should contain to satisfy ▢ – (-30) = 10. Mark explained his answer of -20 but it was mostly verbal and his notation did not make sense to students. Therefore, the teacher attempted to summarize Mark's solution while notating his reasoning on a number line.

> T: I will summarize it to make sure Mark I got you right. I think Mark said you start on the number line by putting your new net worth, the answer you called it [she writes "10 new net worth" on the number line]. That is where I want to end my jump at +10 that is my new net worth. He said, I have got to figure out where it started, what was the beginning net worth so that when I jump I will get at 10. He said I look at the transaction and the transaction says take away a debt of 30. Is that stupid or smart?
>
> Sts: Smart.
>
> T: Which way would we go? Up or down?
>
> Sts: Up.
>
> T: So which way do we have to go to get 10? We had to go up to get 10. So which arrow is it? It is really this arrow, isn't it [showing the arrow pointing up]? He said I want to land at 10 and I want to go up to get there. Because take away debt of 30 means he went up 30 to land at10. So where do I need to start? So he says well to get there since I do not know what this number is [the original net worth] I have to go

backwards to find that, I wanted to start here add 30 to get 10. He said I have to work backward. What is the first jump that you make?

Sts: 10 [she writes on the number line, see Figure 24]

T: And then what?

Sts: 20.

T: So he says that it's got to be -20. How did I do in summarizing?

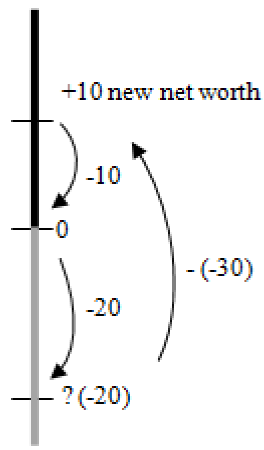


Figure 24: Finding the original net worth from the new net worth and transaction

The teacher initiated this discussion by elaborating the explanation of the students to make sure that the explanation could be clearly understood by everyone (G_O7). During her statement she also re-symbolized the student's solution on the number line since it was not clear before (see Figure 24). She first wanted students to determine whether taking away a debt of 30 was a good or bad decision. Students had no difficulty in identifying it as a good decision as this was already taken-as-shared. The teacher then asked students whether this transaction would move the net worth up or down. It was also established that a good decision moves the net worth up. However, because the original net worth was unknown, they had to work backwards starting from 10 and going down to first 0 and then to -20 to complete the total transaction. By going from a positive number to a negative one in two steps, the teacher also emphasized the image of structured space.

The teacher's belief that students can make sense of the problems by developing meaningful imagery (G_O5) might have prompted her to explain the problems on the number line and also connect them with the previous images such as good and bad decisions. Using images of good and bad decisions helped students to develop another image that was going up or down on the number line and finding the point where they needed to land. As it was stated before during the solution of those problems students also used the zero as a break point and they stopped at zero first if they needed to go through it. The situation specific imagery (e.g. good and bad decision) helped students to make sense of the notational imagery (e.g. going up and down arrows and where to land on the number line). The teacher's extensive knowledge of content and different solution ways allowed her to understand the students' ideas easily and record them on the board during her statement (K_S1, K_P1).

Action 9: Continuing to support more abstract ideas with imagery

After students became comfortable with the transaction problems, the teacher gave students a new activity (day 14) in order to help them to be comfortable with the problems that included only one sign. The following excerpt is taken from a discussion of the problem where the teacher asked students to match different transactions to either +50 or -50. During the introduction of the problem the teacher stated that -50 and+50 are the changes in two different people's net worths. After students worked on the problems, they stated the transactions that they matched. If there was anyone who disagreed with a given matching then those solutions were either proved or disproved. One of the expressions that the students did not agree on was -2(-25). Brad and some other students stated that it is same thing as going down 50:

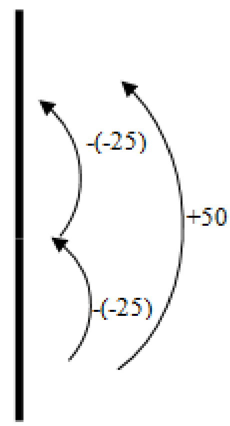


Figure 25: Teacher's visualization of symbolization

T: Brad says it goes down.

Stuart: Aren't you taking away a debt?

Brad: Ohh, yes.

T: He left out a sign [she shows the sign outside of –2(-25)]. Taking away two debts of 25; is that stupid or smart?

Sts: Smart.

T: Taking away 2 debts of 25 not just one debt but two of them we have to take it away. That is going to make us go up [she shows on the number line, see Figure 25].

Sts: A lot.

T: See the two 25 jumps there [T shows the image in Figure 25].

After Brad stated that -2(-25) would be the same as going down 50, Stuart challenged him by stating that the number sentence meant taking away two debts of 25. At that point, Brad understood his mistake. In order to emphasize the mathematical idea that was presented by Nathan, the teacher restated his words and folded back to the image of good and bad decisions as it might have helped students to understand why the answer was positive rather than negative. The goal of the teacher in this question was to support students to make sense of equivalent transactions especially when one term had only one sign (G_C6).

Since the students would investigate the effect of different number sentences in the following tasks (e.g. -25-50), the teacher wanted to give this activity to help students see why -50 might be equal to -(+50) or +(-50) or other transactions such as those given in the activity. The teacher's goal was to support meaningful imagery (G_O5) by folding back to good and bad decision where students could visualize that -50 means going down independent of how many smaller transactions it might involve. This way, if students did not feel comfortable dealing with one sign they could translate the question to have two signs such as rewriting -25 - 50 as -25 – (+50). The teacher's belief that students should make sense of the symbols (B_M7) also might have motivated her to create a task where students could develop meaning for the numbers that include only one sign.

The teacher's practice of recording the equality of different transactions on the number line also supported students' notational imagery, which included the vertical number line with up and down arrows (see Figure 25). Here it is important to note that this time an arrow did not show a single transaction as it was used previously, but it corresponded to a set of transactions with their net effects being the same (i.e. taking away a debt of -50 was shown by the same arrow as adding an asset of 50).

After the activity above the students were given different bare number sentences. During the solution of the problem, 5 –(-3 – 2) = ?, the teacher first reminded the students to pay attention to the parenthesis to be careful about the order of operations and then encouraged them to draw on their previous images to make sense of the mathematical symbols that were given in the question (see Figure 26). The excerpt below was taken from the discussion during the restatement of Mark's explanation by the teacher since some students had difficulty understanding his explanation:

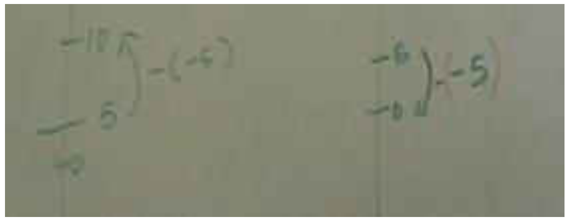


Figure 26: Teacher's record of students' solutions on the number line

T: I think a lot of people's minds changed. Mark I am going to repeat what you said and see if you agree. I know Charlie and some of you said you do not agree and I appreciate that. I am talking to everybody but I will check with you guys to see if you agree with what Mark says. Mark agreed with you guys that you do parenthesis first, and he said if you rewrite that it will end up like this 5-(-5). Is everybody okay with that? I am checking with three of you guys [the students who said they did not agree]. Because you also wrote this. Danny wrote the same thing horizontally. Tisha are you okay with that?

Tisha: I changed my mind.

T: Mark, you said if we talk about money situation, what does this stand for [shows 5]?

Mark: 5 bucks, net worth.

T: What does this stand for?

Sts: Take away debt of 5.

T: [T writes on the board under the numbers] Mark said when you take away debt of 5 it is a good thing. Would you like someone to take away your debt of 5?

Sts: Yes.

T: Which direction are we gonna go?

Sts: Up.

T: That is what Mark says. He said it is like starting here at 5.[she shows on the number line] and take away debt he says makes you go up. Let's put yours on the number line Danny. You started at 5 and what is the transaction for you?

Danny: Take away.

T: Take away 5 like that? [She shows on the number line, writes -5 on the arrow] Which one is correct?

Brad: Mark's.

T: How come you changed your mind?

Brad: Because when you take away a debt you go up.

T: What is Danny doing here?

Sts: He is adding a debt.

T: You forgot the other negative sign Danny [she writes –(-5) on his number line, see Figure 26].

During the discussion of the problem many students agreed that the answer should be zero without thinking. Next, the teacher asked Mark to explain his reasoning and she restated what he said in a clear language. She first asked students if it is okay to write the question as 5 – (-5). Many students agreed that it is okay to write it like that. At this point some students already changed their minds. The teacher asked what that number sentence means in terms of money and Mark stated that it is taking away a debt of $5. The teacher then asked students if they would like their debt to be taken away. All students agreed and stated that the net worth would go up in that situation. She asked Danny to put his answer on the number line and compare it with Mark's to find out which one made more sense to him. Danny also stated that he changed his mind and now agreed with Mark's solution.

The goal of the teacher in this episode was to support students to solve different types of bare number problems (G_C7), especially the ones that included only one sign as the symbolization for them was different than the problems that students solved in the previous class sessions. The teacher's belief that students learn better when they can make sense of the symbols seemed to have motivated her to encourage students to connect those new symbols with their previous images and make sense of the given number sentences (B_M7). This way even when students forgot the rules for integer operations, they could make sense of the number sentences using their imagery. Actually, in the classroom many times she observed that students who could not find the answer could solve the problems if they drew the number line and folded back to their previous images. This way the teacher could support students' learning of mathematics with understanding rather than just memorizing, which is a fundamental idea emphasized in the NCTM *Standards*.

The practice of *capitalizing on students' imagery to create inscriptions and notation* contained building the context on a sequence that students can imagine modeling students' thinking, introducing the vertical number line as a supportive reasoning device, discussing correct and incorrect uses of the number line, encouraging students to draw on previous images and record their thinking, introducing mathematical symbolizations, recording students' ideas on the board, and continuing to support more abstract ideas with imagery. During this practice the teacher's content knowledge, knowledge of possible misconceptions, tools, and imagery seemed to have an essential role in the teacher's practice. Her belief about mathematics that analyzing solutions is important to understand the mathematical ideas and students should use symbols with understanding were highly activated. This was evidenced by the teacher's gradual transitioning to symbolic notation by capitalizing on the students' imagery. Her goals during this practice were to support the development of meaningful imagery by folding back to previous images and using students' informal knowledge, supporting students' ideas with tools when they have the reasoning, and ensuring that students understand the mathematical ideas that are discussed in the classroom.

Practice Five: Developing Small Groups as Communities of Learners

Teachers need to support students in small groups to create effective conversations where they can develop their mathematical understanding (NCTM, 2000). Although, the teacher may know that a collaborative working environment can be useful for students, she might not know what to do to facilitate discussions in these environments. The teacher has essential roles in supporting the small groups such as encouraging students to explain their results to each other, collecting data for the whole class discussion, and asking students to explain their solutions.

The expert teacher in this study formed the small groups heterogeneously. During the formation of the small groups, she tried to bring together the students who have different academic success. She also considered the students' personalities and placed the ones that would be more likely to collaborate with each other in the same group. Although the groups occasionally changed within the academic year, they stayed constant during the instruction of the integers topic. Additionally, sometimes the teacher used different strategies such as asking students to form a pair in the classroom and discuss their solutions with their partner to keep the discussions interesting.

Action 1: Encouraging students to explain to each other

In the current study, one of the teacher's small group actions was to encourage students to compare their answers and explain their solutions to each other as illustrated by the following excerpt. Here, the students were asked to find the net worth of a person who had of the following debts: a -$1000 bank loan, a -$15,000 car loan, a boat loan of -$45,000, and an asset of $60,000 retirement fund. The teacher first encouraged the students to find the net worth on their own and then to check their answers in their small groups:

> T: When you get done you can start talking to your partners and see if you got the same answers or different answers. If you got different answers, check your calculations.
>
> T: Did you get $121,000 for this one?
>
> Tisha: Yes.
>
> T: You all need to talk. Charlie, you got way less than her. You got -1000, right?
>
> Charlie: Yes.
>
> T: Tisha talk to him. How did he get that?
>
> Charlie: You add the positives and you subtract negatives that is how much she's worth.
>
> Tisha: That is what I did. Ohh, yes, I did not see the signs, I got it.

After the introduction of the task, the teacher encouraged students to check their answers with their partners and if they got different answers she wanted them to go over their calculations. Comparing the answers with their partners was useful in different ways. First of all, it gave the students a chance to see their mistakes before the classroom discussion and to try to think about if it was a conceptual error or a computational error. This way, the different answers that were openly discussed in the class did not include many computational mistakes but the mistakes that stemmed from students' conceptual difficulties about that task.

However, sometimes students were apprehensive about explaining their ideas within their groups. As it was seen in the excerpt below, although Tisha and Charlie were in the same group and had totally different answers, they did not talk to each other. Actually, they even did not know what the other's answer was. In these situations, the teacher facilitated the discussion by asking what Charlie's answer was and telling him Tisha's answer. After she encouraged them to explain to each other she left the group and the students started to talk about their answers and solution methods. In this particular example, Tisha saw her mistake where she forgot the signs in the question. The teacher's belief that learning is both an individual and a social process (B_L4) and explaining answers to each other helps students to reorganize their ideas (B_L9) seemed to motivate her to first ask students to work on the activity individually and then share their answers with their friends. These beliefs also activated the goal of reminding students of the social norms as necessary (G_O1). Here, consistent with the NCTM *Standards*, the teacher's knowledge about student learning that students organize their mathematical thinking through communication appeared to have a crucial role in her practice (K_C3).

Action 2: Asking students about their solution methods

On the following days students were given problems where they were asked to compare net worths and find the differences between them. Once the students started working in small groups, the teacher walked around the classroom and stated that she saw $1000 and $5000 as answers (the question asked for the difference in net worth between asset of $3000 and debt of -$2000) and reminded students to prove which one they agree with and be ready to defend it for the classroom discussions:

> T: I keep seeing around the room. Some of you said he is worth $1000 more and some of you said he is worth $5000 more. Show on your paper which one you agree with and defend it by writing an explanation, why you believe [she goes around the classroom].
>
> Gage: I already got it.
>
> T: I do not see any words.
>
> Gage: It is the same thing with this [he shows some calculations].
>
> T: Say out loud what you did?
>
> Gage: I got -2000 plus 2000 to get to zero.
>
> T: Okay.
>
> Gage: And then I added 3000 to get up there [pointing to 3000].
>
> T: Why did you do all that? You did not say why?

While she was walking around the small groups, Gage stated that he already got the answer. At this point the teacher encouraged him to write down his explanation on the paper. However, instead of writing his explanation he showed his calculations and stated his answer. The teacher then asked him to explain why he did those calculations. One of the teacher's practices was to ask students to explain their reasoning of the answers while she was walking around the room and then to bring those answers to the whole class discussions. In doing so she was collecting data for the discourse. However, since there was not enough time for this activity to make a classroom discussion, the teacher encouraged students to write down those explanations on their papers. This way at the end of the class when she collected the students' papers she could analyze their solutions and decide on the following day's classroom conversations based on them.

The teacher's belief that being able to explain shows students' understanding (B_T11) seemed to motivate her to ask students to write their explanations to defend their answers. As she walked around the room, she noticed that some students just wrote the answer and thought that they were finished. In order to encourage those students to think more on the question, first the teacher asked them to explain what they did on their paper then if they just wrote the answer she asked them why they did it.

As was seen in the excerpt above, after the student showed his answer the teacher noticed that he did not explain his solution. By inquiring more about his reasoning she motivated him to find a conceptual explanation that he could defend. After, the teacher left Gage's group, he went back to study the question and tried to explain his reasoning. The teacher's knowledge of content as well as her knowledge of students' possible solutions for the given task allowed her to encourage students to think deeper and make sense of their solutions rather than just doing calculations (K_S1, K_P7).

Action 3: Encouraging students to draw on their previous images

Another action of the teacher was to encourage students to draw on their previous images. The following excerpt was taken from the classroom discussion where students solved transaction problems. The problem asked for the new net worth when a person had a starting net worth of -$10,000 and lost $8000 worth of coins. The teacher asked students to work on the problem in their small groups and started to circulate around different groups:

T: How did you find that?

Charlie: She lost an asset and I put negative so now she owes $18,000 because she lost $8000 which was an asset.

T: Can you show it on the number line? Work on the number line. I need someone to help me in that [she walks away from the group].

Charlie: How did you do it? [He asks Tisha since she used the number line in her solution and they talk about it].

Tisha: They are both negative and you have to add on the number line.

T: [She comes back after few minutes] Tisha you have the same number line as him. Is this your writing? Tell me what you have written?

Tisha: Isn't this positive [she shows 8000]?

T: It is $8000 how much the coin is [she walks away from the group, this time Charlie explains to Tisha]

Charlie: And then she lost it. That means her asset is lost, it is like taking away asset.

While the students worked on the problems first individually and then with their friends, the teacher walked around the classroom to examine their solutions. In the example above, she first asked Charlie how he solved the problem. After he explained his answer, the teacher encouraged him to prove his solution by using the number line and then left the groups to visit the other groups. The reason that the teacher was stopping by the small groups and asking questions might have stemmed from her knowledge about the students' behavior (e.g. some students were shy) and her belief that the teacher needs to help students to become comfortable with this environment (K_P3, B_T2). Additionally, the teacher's practice of not spending too much time in one group and visiting all groups appeared to be a result of her belief that the teacher should be fair in the classroom and provide equal opportunities for all students (B_T6).

After the teacher left from the group, her statement caused Charlie to think more about the problem and ask the other members if they used the number line in their solutions. Since Tisha used it in her solution, they started to talk about how the answer could be proved by using the number line. When the teacher returned to the group she saw that Tisha had a number line on her paper. Thus, she asked her to explain how she used the number line. During her explanation Tisha hesitated about the meaning of losing 8000 coins and asked the teacher. This started another conversation between Tisha and Charlie after the teacher left. Since Stephan believed that being able to explain helps students realize their mistakes (B_T11), she asked Tisha to explain her number line but at that point she noticed that she did something wrong with "losing 8000 coins" so she started to work on it again and started to talk to her partner. Here, the teacher's questions helped the students to start talking to each other.

The teacher's goal of supporting the development of imagery (G_O5) might have encouraged her to ask Charlie to explain his solution by using the previous images. One of the reasons the teacher asked Charlie to use the number line might have been to bring the number line idea to the classroom in order to help students who struggle with the computations. Another reason is that she might have noticed different uses of the number line where students used it without making sense. This gave the teacher an opportunity to remind students of acceptable ways of using the number line.

Action 4: Collecting data

As the teacher walked around the classroom during the small group discussions, she learned more about the different kinds of solutions that students came up with. Often, she used this action to help her engineer whole class discussions both for the current class and subsequent classes by incorporating this data into her planning. The solutions that she focused on changed with her goals about the big ideas of the classroom. For example, while sometimes she focused on the misconceptions at other times she focused on the solutions that included conceptual explanations. The following excerpt shows the teacher's practice of collecting data during the small group discussions and bringing it up in the whole class discussion. In this episode, the students were asked to determine the outcome when the original net worth was $298 and the transaction was +(-$427). During the small group discussion, the teacher first asked Marsha to explain her work to the small group and then asked Marsha if she would share what she said with the entire class later:

> T: Why do you subtract? Talk about these jumps?
>
> Marsha: Because when you add a debt you have to take away part of your asset so that is why you subtract
>
> T: But how do you know 129, the second jump? [Marsha has number line on her paper that goes from 298 to zero and zero to -129. The teacher shows the second arrow on her number line that goes from 0 to 129].That is what I am asking about.
>
> Marsha: Because to get to zero he has to minus 298 and he subtracts 298 from the four hundred number[she means 427].
>
> T: Why?
>
> Marsha: Just to know how much he has to jump up.
>
> T: Can you say it in whole class?
>
> Marsha: Yes.
>
> T: [T starts whole classroom discussion] Marsha is going to explain that. All right Marsha we'll count on you. Why don't you listen to Marsha and ask her questions if you do not get what she says?

During the small group discussion, the teacher asked Marsha why she did two jumps on her number line. Marsha explained that she subtracted 427 since it was adding a debt. However, the teacher did not seem satisfied with this answer and asked her how she knew that she had to go from 0 to -129 on her number line in the second jump. She stated that she found that by subtracting 298 from 427. This time the teacher asked her if she could explain the reasoning behind her computation. As Marsha could explain it the teacher wanted to bring it to the class discussion and encouraged other students to listen to her and ask questions if they do not understand. Thus, circulating around the classroom and learning as much information as possible (i.e. collecting data) about students' solutions by asking them questions and highlighting ones that she thought might further her agenda was an essential action of the teacher in the small groups.

The teacher's practice of asking students how they solved the problems as well as asking them to explain their reasoning was prominent in the dialogue above. This was primarily rooted in her belief that students learn mathematics by making sense (B_M1). The teacher's goal in this problem was to help students understand how to solve the transaction problems especially by using the number line (G_C6). Since the teacher saw that Marsha used the number line in her solution and also she could explain her reasoning, she wanted to focus on her solution in order to help other students to make sense of the subtraction by using the number line. The reason the teacher might ask students to listen to her and ask questions might be based on her goal of reminding students of the norms (G_O1). One of the reasons she asked Marsha if she would like to share her solution in the whole classroom discussion might be due to her belief in supporting an environment where students represent their ideas willingly (B_T3).

Action 5: Clarifying the task before students start to work in their small groups

When students worked in small groups, they had different roles in the group namely the leader, the policeman, and the author. Since the small groups included three students, the teacher categorized each student's place as windows, wall, and door based on their proximities to those parts of the physical arrangement of the classroom and then posted the role of each place on the board (e.g. windows: leader, wall: policeman, door: author). The role assigned to each place changed everyday allowing students to take on different roles. The leader's role was to decide the group members' jobs such as how they should share the workload of the activity. The authors were supposed to write the final answers that the group members agreed on and give them to the teacher at the end of the class if the teacher asked for them. The policeman's job, on the other hand, was to check if everyone shared the work equally. The reason that the teacher gave different jobs to the students seemed to have originated from her belief that taking responsibility is important for learning (B_L3) and the teacher should be fair in the classroom and provide equal opportunities (B_T6). The teacher's knowledge of students' behavior such as the tendency of not sharing the jobs equally seemed to have an important role in giving different assignments to the students in collaborative groups (K_P3). Additionally, in order to keep the discourse alive and interesting in small groups (G_O2), sometimes the teacher used different strategies such as asking students to form a pair in the classroom and discuss their solutions with their partner.

The excerpt below shows how the teacher introduced the task and addressed students' roles in their small groups. Here the students were asked to translate several statements to number sentences and find the net worths. An example statement from the activity is "net worth: $1500 and transaction: adds a debt of $600" (see Appendix C, "Alice's Net Worth"):

> T: Here is what I would like you to do. You will get an activity page. Leaders, I want you to decide if you want to work on them individually and talk later or start talking on all the questions. Leaders are in charge of that. But each person needs to put answers on their paper. Here are the directions: It says write number sentences for the following changes that occur to Alice's net worth and use the net worth tracker if you want to. I am telling you some of you need to. I tell you this, if you get an answer let's say Mark and Norman gets different answers, one way to prove it will be the net worth number line. That is a good proof because Dusty just showed a proof on the number line. If you get stuck, go to the number line. What does it mean to write a number sentence? What it means is start with the original net worth that will be 1500 the transaction Tisha said and then
>
> Tisha: You add debt of 600.
>
> T: How about write it with symbols?
>
> Tisha: 1500 + (-600).
>
> T: Equals whatever your answer is, you find what the answer is. Do not tell it now. Consider it as a practice test; we will have the unit exam soon. Practice writing these in number sentences with the answer. If you get stuck and don't know how to find it, the net worth number line [she draws one on the board] is a good idea to use. Who has questions about this?

During the introduction, the teacher first reminded the leaders of their role and clarified what the "number sentence" meant in the problem since some students had difficulty understanding. Thus, for the first problem she asked Tisha was what the statement was and then she asked her how she could write it with symbols. After Tisha described it, the teacher wrote it on the board. She also clarified that the students needed to complete the number sentences that they wrote and encouraged students to use the number line if they got stuck or to prove their solutions. Once the students understood the aim of the activity the teacher distributed the papers and started to walk around. Stephan's belief that the teacher is responsible for clarifying students' roles and the tasks that they will work on in their small groups (B_T12) seemed to have shaped her practice of introducing the task before the students started to work on it. As the teacher's goal in this activity was to continue supporting the development of images related with situation-specific imagery as well as notational imagery (G_O5), she gave the statements verbally and asked students to write them with the symbols and find the answers of them based on that. She also supported the use of the number line as she observed that many students who had difficulty could solve the problems by using the number line.

Action 6: Encouraging students to ask each other for help

Once the students were given the net worth activity that asked for applying different transactions to the original net worth (Appendix C, "Alice's Net Worth"), they started to solve the problems and the teacher circulated around the small groups. There were around 10 problems in the activity, and the students first started to work on them individually within their small groups. While the teacher was walking around the room, she noticed that Gage seemed not to help his group friends even though he solved many of the problems confidently and his group mates had some difficulties in some of the problems. The following excerpt is taken during the students' working of the number sentence problems:

T: Are you done with the whole page?

Gage: This is so easy.

Mark: I do not understand this, number seven [net worth $800, transaction: take away debt of $200].

T: Talk to your partner, he thinks it is so easy.

Mark: He does not help.

T: That is not acceptable Gage, even though you might not have finished, you can still help.

Gage: Okay [He starts to work with Mark].

While the teacher circulated around the classroom observing how students were dealing with the activity, she asked Gage if he had finished all of the problems. He stated that the problems in the activity were very easy for him. At this point, the teacher saw that Mark was struggling on one of the problems on the page and he stated that he did not understand number seven. The teacher encouraged him to ask his partner to get help and Mark stated that he did not help him. The teacher emphasized that it was not an acceptable behavior and encouraged Gage to help his partner. Stephan's belief that the teacher is not the only source of knowledge in the classroom (B_T1) as well as her belief that an environment that nurtures a sense of community plays an essential role in learning (B_L7) seemed to have been the reason for her encouragement of the students to ask each other for help. After the teacher left, they talked about the problem and agreed the answer was 1000 by talking to each other and then they started to work on the rest of the problems.

Action 7: Encouraging students to come to an agreement

As the instructional sequence moved forward, students continued to solve the problems related with transactions. The excerpt below is taken from the discussion of a problem where the teacher asked students to find the different ways of writing +50 and -50 with multiple transactions. She asked them to share their solutions with their group members and decide whether they agree or not for the each different way. She also added that during the classroom discussion she would ask each group to state one transaction that they all agreed on thus agreement within a group was important:

> T: If you got the first one (+50) start the second one (-50). What are the different ways to write the second one?
>
> Brad: We got the first one altogether. But I think mine is wrong.
>
> T: Which one is yours? [Since there are three different ones, the teacher asks which one is his].
>
> Brad: This one.
>
> T: You do not like that one? You guys like the first one? When I mean "like", I do not mean whether you like it or not. Do you think it works mathematically?
>
> Danny: Yes [he thinks Brad's solution is right].
>
> T: If you all figure out the first one, start the second one.

While students were working in small groups, the teacher walked around the room to see the different solutions. She encouraged students to start working on the second one if they solved the first one. At this point, Brad stated that although each member of the group including himself found one transaction equivalent to +50, he stated that he was not sure about his solution. The teacher encouraged the group members to work on it together and decide whether each expression was mathematically correct and then move on to the next one. After the teacher left the group, the students continued to work on the expressions to come to an agreement on each representation.

One of the reasons that the teacher asked them to come to an agreement regarding the solutions seemed to stem from her belief that analyzing solutions is important to understand mathematical ideas (B_M4). By analyzing the different ways, students would have opportunities to make sense of mathematical expressions and also would have the idea that there could be different correct answers for the same mathematical problem. The teacher's belief that taking responsibility is important in learning (B_L3) might also have encouraged her to ask all group members to decide whether each solution was correct or not. Because when the teacher asked each group to give one answer, it was important that all students agreed with and shared the responsibility for that answer. Since the teacher believed that mathematics is a sense-making activity (B_M1), during her instruction she included the problems that have multiple answers. For example, in this question all members found a different solution and they had to analyze each of them. These types of problems allowed the students to first work on multiple solutions within their groups and then discuss the solutions of the other groups efficiently.

Here it is important to note that the routine of the teacher in small groups was to first introduce the task and then ask students to work on it individually for a few minutes, and then share their answers with the group members. However, in order to make the small group discussions alive, she used different strategies such as asking the leader to decide how the groups need to work or asking students to find a partner to work together instead of their group members. In one of the activities toward the end of instruction (see Figure 27), the teacher asked the students to work on each box for two minutes individually. She set the clock and asked students to switch to the next box every two minutes.

Sam said, "*I think that **adding a debt** is the same thing as **taking away an asset**.*"
Sue said, "***Adding a debt** is totally different than **taking away an asset**.*"

Who do you agree with and why?

Dave said, "*I think that **taking away a debt** is the same as **adding an asset**.*"
Donna said, "*I think that **taking away a debt** is like **subtracting a debt**.*"
Drake said, "*I think that **two minuses make a plus**.*"

Who do you agree with and why?

Chuck said, "*This problem doesn't make sense. You don't know if that – sign in front of the 7 is a **subtraction sign** or a **negative sign**.*"
Cherry said, "*I don't think there's a difference between a **subtraction sign** and a **negative sign**. They're the same thing.*"

Who do you agree with and why?

Figure 27: The activity in the small group

Once the students, recorded their answers for each box, the teacher gave them new directions for how they should work in their small groups during the next six minutes:

> T: Now you got 2 minutes to talk about box 1. For example, in your group, person number one will say I think I agree with Sam, person two says I agree with Sue, and then person three will agree with Sam. So quickly state your conclusion and then spend the rest of the two minutes to see if you can come to an agreement about the answer. Because when it is all over, we are going to have a discussion and I am going to select with the random selector the team that will present, so be ready to explain. Then I will give you another two minutes for box 2 and then another 2 minutes to do box 3. Two minutes for each box, start with your conclusion around the table and then debate. Any question? Do box 1, right now.

As it was seen in the excerpt, once the students finished working individually on each box, the teacher asked them to state own position first and then explain their reasoning to each other and come to an agreement at the end of two minutes for each box. The reason that the teacher wanted students to agree on one answer was based on her goal of supporting the social norms (i.e. state disagreement and its reasoning) as necessary during the instruction (G_O1). Thus she encouraged students to defend their answers to promote their mathematical reasoning in a social setting.
In the classroom discussion, she used random selector to choose one of the groups and then one person from the group would reflect the group's decision.

The teacher's belief that students should have sufficient time to think individually and work with the group members before starting to discuss their ideas with the whole class (B_T7) played an important role in her introduction of the task above. Also her belief that the teacher is responsible for efficiently managing the time (B_T13) seemed to be the reason for restricting the time. The teacher wanted the students to spend no more than two minutes for each question, since she knew that students had a tendency to get stuck on one question and not even have a chance to work on the others (K_P3). However, it is important to note that, restricting the time in this manner was not a typical practice of the teacher – she seemed to use this strategy when she wanted to leave more time for the whole class discussion.

Action 8: Not correcting students' mistakes immediately

Towards the end of the instruction, the students were given problems that included only one sign such as 5 – (-3 – 2) =? After the teacher asked the question, she started to walk around the classroom:

> T: You got some proof? Show me your proof.
>
> Gage: Here [He shows the answer on his paper].
>
> T: It is your answer, where is your proof?
>
> Gage: I can prove it. It is like negative 3 minus positive 2 equals to negative 1.
>
> T: Prove that to me.
>
> Gage: Because if you have a debt of $3 and you're paying $2 of it you can have still one dollar to owe.
>
> T: Why are you paying $2?
>
> Gage: I said if you have a debt of like $3 and you pay off $2.
>
> T: Why are you paying? I do not get that [she encourages Gage to use asset and debt concepts to help him make sense of "pay off" idea is not correct here as -2 indicates going more in debt instead of paying off debt].
>
> Gage: Pay some of your debt.
>
> T: All right, we will see. Do you have some proof? [to Mark who is in the same group with Gage]

Mark: Yes, I do.

T: Explain to me how that part is on the number line.

Mark: It is first negative 3.

T: So you put it first.

Mark: Yes, and then I saw the next symbol is negative and there will be also a positive sign in front of 2[Marks seems to think -2 as -(+2)].

T: How do you say that?

Mark: Taking away asset and I knew that it is bad.

T: So you went up? [She shows Mark's number line]

Mark: So negative 3 minus asset of $2 so you will go up, you are confusing me!

T: I am confused by what you are saying. You start with -3 and then taking away asset and you go up, is it what you said?

Mark: I think, I need to go down.

While the teacher was circulating the small groups, one of her practices was to ask students to prove their solutions. When she asked Gage to prove his solution, he just showed his answer but had difficulty proving it. Here it is important to note that although Gage's answer was incorrect, the teacher did not try to correct him at this point. She encouraged him to use the previous images of assets and debts, but when he still had difficulty she continued with Mark who was in the same group. Mark's solution was different than Gage's, rather than a computation there was a number line on his paper. While Mark was explaining his solution, he stated that he first went down 3 and then stated that -2 is the same thing as taking away asset which is bad. However, in his paper the arrow was going up 2 instead of going down. Thus, the teacher asked him whether he must be going up or down. At that point Mark stated that he was confused since probably his words contradicted with his number line. After the teacher left the group, Mark continued to work on the problem and decided that he needed to go down.

The teacher's belief that students learn from their mistakes (B_L8) might have motivated her to not to immediately correct the mistakes but let students think more on their own and discuss with their group members. This was an important practice of the teacher. She knew that the mistakes would be discussed in the whole class discussion and would play an important role in contrasting different solutions. She also thought that if all mistakes were corrected on the spot, then the students might not absorb what was wrong and there wouldn't be much to discuss in the whole class discussion. The teacher's belief that students reorganize their thoughts when they explain (B_L9) seemed to be the reason for why she asked Mark to explain his solution. During his explanation he noticed that he was supposed to go down two instead of up. The teacher's practice of encouraging students to draw on previous images when they had difficulty had been helpful to the students to discover and remediate their own mistakes. Her knowledge of students' misconceptions and possible answers also seemed to help her grasp students' solutions during their statement and ask questions that would support them to solve the contradictions that they had instead of correcting them directly (K_P1, K_P2).

The practice of *developing small groups as communities of learners* consisted of encouraging students to explain to each other and ask each other for help when needed, draw on their previous images, reach an agreement for the solution of problems, asking students how they solved the problems, collecting data, clarifying tasks before students start to work in small groups, and not immediately fixing students' mistakes (i.e. use these as opportunities for class discussion). During this practice the teacher's knowledge of the NCTM *Standards*, her content knowledge, her pedagogical knowledge about students' possible answers and behaviors had an important role on teacher's practice. Her beliefs that mathematics is a sense-making activity, the teacher is not the sole source of knowledge, and students reorganize their ideas when they explain seemed to be highly active in the teacher's small group practice. Her primary goals during this practice was to remind students the expectations, their roles in the small groups, and collect data to support class discussion that stay alive and interesting.

Summary

In this section, the teacher's roles in the classroom instruction were identified and analyzed under five practices namely creating and sustaining social norms, facilitating genuine mathematical discourse, supporting the development of sociomathematical norms, capitalizing on students' imagery to create inscriptions and notation, and developing small groups as communities of learners. The teacher performed several actions in each practice to sustain a standards-based classroom instruction (see Chapter 7 for the full list). However, the teacher's actions in each category should not be thought of in isolation from the actions in other practices. Often, effective action in one category allowed the teacher to support the actions in other practices as well. For example, it would not have been possible to maintain a productive discourse without establishing and sustaining social norms. Similarly, without developing students' imagery and establishing sociomathematical norms, the discourse as well as small group discussions would be bland and ineffective as the norms support students to express their mathematical arguments and rationales honestly and openly (NCTM, 2000) and imagery creates meaning for mathematical activities and adds richness to their understanding (Thompson, 1996). To this end, the actions in these five practices should be considered as highly interrelated. In particular, social norms appeared to be the glue that held together all the actions as standards-based teaching can only be applied effectively in an environment where ideas can be discussed freely and respectfully (NCTM, 2000).

The teacher had an essential role in supporting the social norms to create an environment where students could discuss their ideas freely and respectfully. Since many students in the classroom had different experiences in their past academic lives, during the beginning of the semester the teacher worked with the students to establish the norms to create an environment where students could learn mathematics meaningfully. Since the current study was conducted in the second semester, the teacher's establishment of the social norms was not covered by the data set. However, the current study documented how the teacher sustained that environment.

In order to help students to learn mathematics with understanding she asked students to explain their answers. She also encouraged them to express their agreement or disagreement with the given explanations. This way, the students had a chance to discuss the mathematical ideas that were brought up by different students. It helped students not only learn the answers but also understand why some solutions worked and others didn't. In the whole class discussions, the teacher encouraged the students to explain their answers even when they solved the problems incorrectly and asked the other students to help out their friends. This strengthened the sense of community where students help each other and share ideas without being shy.

As the students were seventh graders, they sometimes did not behave appropriately. At these points, the teacher reminded them of how they needed to behave in the classroom and encouraged them to discuss in productive ways. This caused those students who did not behave appropriately to change their behaviors, which resulted in the creation of a comfortable discussion environment. As Stephan knew that students not only needed to learn mathematics but also how to take responsibility, she often encouraged them to be responsible for their learning through asking questions when they did not understand or finding out the necessary works that were done in the classroom when they were absent.

Another important role of the teacher was to facilitate a genuine mathematical discourse where the students could explore the mathematical ideas. In order to improve the communication of ideas, the teacher often restated students' solutions in a clear (or sophisticated) way to help the other students understand those ideas. Additionally, in order to help them stay focused on each other's solutions, she asked them to restate their friends' explanations. This helped them to grasp the ideas that they had not thought of themselves. During the discourse the teacher also introduced the mathematical vocabulary that the students needed to know when they invented new ideas. This was helpful for students to increase the sophistication of their explanations and to learn about the mathematics terminology. Stephan also asked questions that promoted higher-level thinking, which the students could answer after comparing and contrasting multiple solutions. These types of questions not only helped the students to develop a deeper understanding but also allowed for lively discussions.

One of the teacher's practices was to support the development of sociomathematical norms in order to facilitate the mathematical communication among the community members. To this end, the teacher encouraged the students to give conceptual explanations with their answers. In the classroom during the discussion of the solutions of the problems, Stephan encouraged the students to come up with mathematically efficient and/or sophisticated ways as well as different ways. The aim of the teacher was to help students see the different possible answers of the problems and give them opportunity to select the ones that made more sense to them during the discussion. She also organized the tasks such that the students could notice a pattern, make conjectures, and then prove them. This way, the students discovered the important rules themselves rather than being given them ready-made. It also helped the students to remember the rules easily by folding back to the images that they developed. In order to help students make conjectures and prove their solutions, the teacher also introduced tools as supportive reasoning devices.

Capitalizing on students' imagery to create inscription and notation was also an essential classroom practice of the teacher. To support imagery, the teacher first created an instructional sequence grounded in a financial context where students made sense of positive and negative numbers as well as the integer operations. During the classroom instruction the teacher supported the students to develop situation-specific imagery with a discourse where the students could imagine the situation given in the problems. These situation-specific images were used as a resource to fold back to in the subsequence tasks. As the students explained their solutions using the financial context, the teacher introduced tools in order to support students' reasoning. The tools helped students to explain their reasoning to others and to organize students' ideas. The most important of these tools was the vertical number line and it supported notational imagery. By using both situation specific and notational imagery, the students could make sense of integer operations and even reinvent some of the rules of integers.

Finally, the teacher also had an important role in developing small groups as communities of learners. In order to keep the small group discourse alive and give equal responsibilities to the students the teacher applied several strategies. In general the teacher's practice was to ask students to think about the problems first individually and then share their answers with their friends in the small group. To keep the discussions interesting and allow students to interact with their friends other than those in their small groups, the teacher also sometimes asked the students to find themselves a partner to discuss their solutions. If she asked them to submit one work from each group, she gave each student a different role and switched them every day in order to give equal opportunity for all students. Before asking students to work in their small groups, she always explained the task as well as the students' role very clearly. During the discussion in small groups, she walked around the classroom to collect data rather than fixing students' answers. In order to understand students' solutions, she also asked them to explain their answers. This helped her to understand their reasoning and decide whether she needs to bring those ideas to the whole classroom discussions. When she was walking around the classroom, as she noticed that students had difficulties solving the problems, she encouraged them to talk to their group members and compare their answers.

In summary, all the practices discussed in this chapter were interrelated to each other. The teacher created and sustained a productive standards-based teaching environment by applying all those practices without abandoning one for other. For example, since students explained their solutions (social norms), the teacher could understand their ideas and posed higher-level questions to support discourse. This gave the students opportunities to clarify their ideas and generalize them by making and proving conjectures (sociomathematical norms). Since the teacher supported the development of imagery, the students could prove those conjectures by using the images that they developed. Finally, the small groups allowed mini discussion environments where students cooperated and organized their ideas before the whole class discussions.

CHAPTER SEVEN: CONCLUSION

In this study, the planning and classroom practices of an expert middle school mathematics teacher were analyzed in order to shed light on what the teacher does and why she does them for a successful implementation of a mathematical environment consistent with the tenets of the NCTM *Standards* (NCTM, 1989, 1991, 2000). While the answers to these questions are given in the analysis, this chapter reflects on the findings and draws overall conclusions. The main conclusions that emerged from the current study are:

1) The teacher used a diverse set of practices with each practice comprised of multiple actions to create and sustain a standards-based environment
2) Teaching in line with the NCTM *Standards* require a rich and connected body of knowledge about students, curriculum, content, and literature
3) The depth of the teacher's knowledge allowed her to develop practices that are consistent with her beliefs and goals; and
4) Planning and classroom practices are highly interrelated

Each of these conclusions is elaborated in the following sections by using examples and reflections from the literature and the analysis.

Practices

The planning practices of the teacher included preparation, reflection, anticipation, assessment, and revision. Preparation prior to the beginning of instruction had a central place in planning. Preparation included actions such as creating an HLT, designing an instructional sequence that would achieve objectives and contemplating on the big ideas of the unit. Reflection involved thinking about the classroom interactions of the current year as well as the previous years to evaluate if the instruction was taking place as planned. Anticipation was highly related to reflection but it went one step further by trying to conjecture what potential actions may transpire in the classroom. While reflection involved looking back to on the past events, anticipation involved looking forward to what might happen next. Assessment was another important practice of the expert teacher and was comprised of both summative and formative assessment. It helped the teacher understand the students' progress and identify their misconceptions and difficulties. During the revision the teacher improved her instructional sequence and her methods of teaching based on the feedback acquired through reflection, anticipation, and assessment. All these practices were conducted in learning communities. In the teacher meetings the teachers applied all those practices. Thus, although collaboration was not considered as a separate practice it empowered the other practices. Each of these practices was comprised of multiple actions which were discussed in detail in Chapter 5. Table 5 given below summarizes these actions and indicates which beliefs, goals, and knowledge drove the formation of these actions.

Table 5: Planning practices (refer to Appendix B for the codes)

PRACTICE 1: PREPARATION			
Actions	**Knowledge**	**Beliefs**	**Goals**
Creating HLT	K_S1, K_P8	B_T5	
Creating Instructional Sequence	K_P9, K_S1, K_C2	B_L5, B_T15, B_M3	
Reading and adapting relevant literature	K_P4, K_P6, K_S1	B_L1	
Unpacking big ideas	K_C1	B_T5, B_T8	G_O4, G_O5
PRACTICE 2: REFLECTION			
Reflecting on previous year's instruction	K_P4	B_L5	G_O3
Making sense of students' solutions	K_P7	B_T4, B_M1	
Thinking about the big ideas of the following day's classroom	K_P2	B_T4	G_O3
Analyzing students' works to initiate discourse for the following day	K_C4	B_M4, B_L8, B_T4, B_T6	
Reflecting on the classroom environment	K_C3, K_C4	B_T2	G_O1
PRACTICE 3: ANTICIPATION			
Anticipating possible discourse during the introduction of the task	K_P4	B_L6	G_O5

Working on the problems before the instruction	K_P1, K_C1		G_O3, G_O6
Anticipating students' thinking	K_P1	B_M4, B_T12	G_O5
PRACTICE 4: ASSESSMENT			
Creating formative assessment tasks	K_C1	B_T8, B_M3	G_O3,G_O6,G_O7
Summative assessment	K_S1, K_P2	B_L3, B_T14	
PRACTICE 5: REVISION			
Revising the instructional sequence	K_C1, K_P7	B_M1	G_O4, G_O5, G_O6

Classroom practices included creating and sustaining social norms, facilitating genuine mathematical discourse, supporting the development of sociomathematical norms, capitalizing on students' imagery to create inscriptions and notation, and developing small groups as communities of learners. The expert teacher had an essential role in sustaining the norms such as encouraging students to explain their reasoning, stating their agreement and disagreement and encouraging them to take responsibility. During the discourse the teacher supported students' mathematical understanding by asking higher-level questions, restating students' words in a clear language and introducing the vocabulary. The expert teacher supported the sociomathematical norms by several actions such as encouraging students to give conceptual explanations, and seek efficient and different solutions. Imagery helps students give meaning to mathematical activities which adds richness to their mathematical reasoning and understanding. The teacher supported students' imagery by creating inscriptions and notation based on students' reasoning. For example, the teacher introduced the vertical number line after the students brought the "pay off" idea. Finally, the teacher also supported development of small groups as communities of learners by encouraging students to help each other, discuss their solution and come to an agreement. Table 6 illustrates the actions that the expert teacher performed during the execution of the classroom practices together with the highly activated beliefs, goals, and knowledge that underlie these actions.

Table 6: Classroom practices (refer to Appendix B for the codes)

PRACTICE 1: CREATING AND SUSTAINING SOCIAL NORMS			
Actions	**Knowledge**	**Beliefs**	**Goals**
Encouraging students to express agreement or disagreement	K_C1, K_P2		G_O1, G_C3
Encouraging students to understand others' solutions	K_P3	B_T9	G_C4
Encouraging students to explain their solutions	K_P10	B_L2	G_O1
Asking students to repeat other students' solutions			G_C3, G_O1, G_O7
Encouraging students to behave respectfully	K_C3	B_L7, B_L8	
Encouraging students to use mistakes as learning opportunities		B_L7, B_L8, B_T6	
Renegotiating social norms	K_P11	B_T2	
Encouraging students to take responsibility/ownership for their learning	K_C3	B_L1, B_L3	
PRACTICE 2: FACILITATING GENUINE MATHEMATICAL DISCOURSE			
Introducing mathematical vocabulary when students have invented an idea	K_S1	B_L5, B_T10	G_O2, G_C1,
Asking questions that promote higher level thinking	K_S1	B_T16	G_C5
Restating students' explanation in a clearer/advanced language		B_T13	G_O3
Using solutions effectively to engineer the teacher's summary		B_T7,B_M1, B_L5	G_O2, G_O3
PRACTICE 3: SUPPORTING THE DEVELOPMENT OF SOCIOMATHEMATICAL NORMS			
Encouraging students to give conceptual explanations	K_P2	B_M6, B_T9, B_L1, B_L5	G_C2, G_C3, G_C5
Encouraging efficient solutions		B_T9, B_M4	G_C4
Encouraging students to make conjectures		B_M2, B_M5, B_L7, B_T7	G_C6
Encouraging students to express their reasoning and proof	K_p2, K_P11,K_S1, K_P7	B_M1, B_L2, B_T2, B_L7, B_L8	G_O1 G_C7
Encouraging different solutions		B_M2, B_T9	G_C6,G_O1,G_L1
Encouraging sophisticated solutions		B_M2,B_T9	G_O7, G_C7
PRACTICE 4: CAPITALIZING ON STUDENTS' IMAGERY TO CREATE INSCRIPTIONS AND NOTATION			
Building the sequence on a context that students can imagine	K_P7, K_S1	B_L5	Gc1, Gc3, G_O5
Modeling students' informal thinking	K_P9		G_O6, G_L1

Introducing a tool as a supportive reasoning device	K_p2		G_C5, G_O6
Discussing correct and incorrect uses of a tool		B_M4, B_L8, B_T4	G_O1
Encouraging students to draw on previous images	K_P2	B_M1	G_O2, G_O5, G_C6
Introducing mathematical symbolizations	K_P5	B_M7, B_L6, B_T10	
Encouraging students to record their thinking	K_P7, K_P11	B_L8, B_T2, B_M4	G_O6, G_O7, G_O5
Recording students' ideas on the board	K_S1, K_P1		G_O7, G_O5
Continuing to support more abstract ideas with imagery		B_M7	G_O5, G_C6, G_C7
PRACTICE 5: DEVELOPING SMALL GROUPS AS COMMUNITIES OF LEARNERS			
Encouraging students to explain to each other	K_C3	B_L4, B_L9	G_O1
Asking students about their solution methods	K_S1, K_P1	B_T11	
Encouraging students to draw on their previous images	K_P3	B_T11, B_T2,B_T6	G_O5
Collecting data		B_M1, B_T3	G_O1, G_C6
Clarifying the task before students start to work in small groups	K_P7	B_T12, B_L3, B_T6, B_M4	G_O1, G_O2, G_O5
Encouraging students to ask each other for help		B_L7, B_T1	
Encouraging students to come to an agreement	K_P3	B_L3, B_M1, B_T7, B_T13	
Not correcting students' answers immediately	K_P1, K_P2	B_L8, B_L9	

Knowledge and Standards-Based Teaching

The analysis illustrated that the teacher's various types of knowledge played key roles in her successful implementation of a standards-based classroom. In particular, the teacher's knowledge about content, students, curriculum, and literature shaped her practice of establishing a productive teaching and learning environment. In the following, the effects of each type of knowledge on the teacher's practice are discussed and supported with examples from the analysis.

Content knowledge

The content knowledge of the teacher provided the necessary foundation upon which the instructional sequence was built. It included knowledge about positive and negative numbers, comparing and ordering them; operations such as addition, subtraction, multiplication, and their relationships; the properties of operations such as distribution and association properties; additive inverses; and the mathematical symbols and notations.

The teacher's rich knowledge of content helped her think flexibly during the discussion and ask higher level questions when she found the opportunity. This helped students explore important mathematical ideas and come up with conjectures. For instance, in one of the class sessions after the teacher gave a task where the aim was comparing the given net worths and finding the difference using a supportive number line, one student stated that he found a pattern when two numbers were in different sections (when one number was positive and the other was negative). Another student added that he also noticed a pattern when both numbers were positive. In order to encourage students to make sense of what those patterns suggested and to reach a general consensus the teacher asked students to analyze the two negative numbers. The teacher's content knowledge of the positive and negative numbers, the relationship with each other, the effect of operations applied on them, and the use of the number line helped her understand the students' conjectures and ask question that led students to discover a general pattern. When this happened, the teacher was not expecting students to put forward these conjectures but her content knowledge allowed her to easily understand what students said and facilitate the discourse with higher-level questions where students could analyze different situations and arrive at a conclusion.

Knowledge about students

The teacher's knowledge about students was accumulated through various sources such as her experiences from previous years, studying of literature about student learning, interaction with other teachers and direct communication with her current students. This body of knowledge allowed her to anticipate students' behaviors, solution strategies, difficulties and misconceptions as well as the potential images that they might develop.

The teacher's knowledge about students' difficulties and misconceptions helped her decide what to focus on during the instruction to help students overcome those misconceptions. For instance in one of the planning sessions, she anticipated that students might have difficulty in ordering the numbers especially the negative ones on the number line. Thus, she suggested creating a task that would focus on comparing two different net worths and finding the difference between them. In order to help students make sense of the numbers she supported it with a vertical number line that students could use for reasoning. The class sessions showed that some students had misconceptions related with ordering the numbers and using the vertical number line together with the context helped them to make sense of why the numbers that have greater absolute value are below the numbers that have lesser absolute value.

Having knowledge of students from previous years also helped the teacher to decide how to orchestrate the classroom discourse to bring out students' ideas. For instance, in the first teacher meetings Stephan together with the other teachers reflected on how their students interacted with the sequence in the previous year. This enabled them to come up with effective ways to introduce the instructional sequence. Once the instruction started they used both the knowledge of previous years as well as the current students to decide on different ways of introducing tasks. For instance, in one of the daily planning sessions the teacher stated that during the small groups she observed that some students equated negative net worth to no net worth. Based on this knowledge, she included questions in her plan that involved both negative and zero net worth and encouraged students to compare them in classroom discussions. The teacher often gathered her knowledge about the students from the classroom discussions, small groups and homework assignments and used this knowledge to plan for the following day's instruction.

The teacher also acquired knowledge about students by reading the relevant literature pertaining to how they learn mathematics in general as well as how they learn particular topics and their difficulties with these topics. She utilized these difficulties in planning sessions. For example, in the teacher meetings she expressed the difficulties of understanding negative numbers in the history of mathematics and used this knowledge to create activities to help students conceptualize the net worth as an abstract quantity. Her general knowledge about learning theories also had an important effect on her practice. Since from the literature, her past teaching experiences, and her previous research she knew that learning is both a social and individual activity, she tried to create and sustain that kind of environment in her instruction. For example, after introducing tasks to the students, she gave them time to think on them individually and then talk with their partners and share their ideas. Additionally, her knowledge about students' possible behavior such as being reluctant to explain their ideas and ask for help played a key role in her small group practices.

Knowledge of curriculum and standards

The teacher's curriculum knowledge included the knowledge of state standards, knowledge of standards for *Teaching Standards* (NCTM, 1991) and the NCTM *Standards* (NCTM, 1989, 2000). She incorporated this knowledge into the creation of the instructional sequence as well as her planning and classroom practices. For example, during the creation of the instructional sequence she checked whether she covered the objectives that were supported by the state standards (e.g. whether it supports students to use and justify the rules of integers).

The knowledge of *Teaching Standards*, which emphasizes the importance of creating worthwhile mathematical tasks, teacher and students' roles in discourse, tools for enhancing the discourse, and analysis of teaching and learning had an essential impact on the teacher's practice. During the selection and creation of the tasks, she considered whether the task would give students opportunity to make sense of mathematics. For example, the first activities that were given to the students included a financial context including assets, debts, and net worth where students could connect it with their own lives and fold back to it in the activities that followed if they had difficulties. In the planning the teachers also talked about how to initiate a discourse after giving an activity or how they could support the shift in symbolization with effective discourse. For instance, in one of the early planning sessions, Stephan stated that she would initiate the discussion by asking students about a famous person's net worth and then continue by asking them to list the things the person might own and owe and stated that she would write them on the board. To keep the discourse alive she supported students to understand each other by restating their explanations herself in a clear language. Knowing about her role and the students' role in the discourse was helpful for Stephan to facilitate the discourse.

The teacher's knowledge about the tools enhancing the students' understanding and discourse was important in her planning as well as classroom practices. She planned to introduce the vertical number line which became an inscriptional device that made students' imagery visible. The reason she selected the vertical number line over the horizontal number line was that it fit better to the supported situation specific imagery where the net worth went *up* by adding assets and *down* by adding debts. Once she introduced the tools in the classroom based on students' reasoning, during her classroom instruction she also supported their use as the students needed.

After each class session, the teacher reflected on the lesson and analyzed students' solutions and behaviors as well as her instruction. For example, in one of the reflections she stated she planned to talk about students' behavior since she noticed that during the whole class discussion some students started to behave without respect to each other. The analysis of teaching and learning during the planning was essential in her actions in the classroom.

During the planning as well as in the classroom the teacher's knowledge of process standards were also effective in her decisions (NCTM, 2000). The process standards include problem solving, reasoning and proof, communication, connections, and representations. In the planning the teacher created tasks that would give students opportunities to inquire, develop, and deepen their understanding of mathematical ideas. For example she created tasks that asked students to find whether a given situation (e.g. taking away debt) was good or bad and to find the new net worth after they applied the given transactions. During the solutions to these activities students made conjectures related to the integer rules and proved them in the classroom. Additionally, in the classroom when students came up with conjectures, the teacher encouraged them by writing these conjectures on the poster board and asking to prove them.

In the planning Stephan also considered the questions she might ask during the discourse that mainly focused on explanations of the reasoning such as asking the meaning of negative net worth or the difference between having no net worth and negative net worth. In the classroom, when the result of a problem was found to be negative she asked students to explain what it meant to be negative. Another example is that in one of the classroom sessions a student came up with an additive inverse method during the solution of a net worth problem. The teacher capitalized on it and gave all students the opportunity to see this efficient method. The knowledge of communication where the community members feel confident to express their ideas and analyze the different strategies as well as the different answers had an important role in her actions.

Additionally her knowledge about connections and representations was essential in her planning and classroom practices. In the instructional sequence each task was connected to the next one and the students built on one another to develop a coherent understanding. For example, in the beginning of the instruction the students developed the concepts of assets, debts and net worths. Then the teacher provided activities to help students understand comparing two numbers and finding the difference between them using these concepts. Also posing questions on the vertical number line as a supportive reasoning device where the positive and negative parts were shown in black and red color was a very helpful visual tool for students to make sense of their answers. In their reasoning, the students could connect this tool with the activities that they solved before by associating *being in the black* with having assets and *being in the red* with having debts. The students seemed to develop an imagery that as they go down on the number line they go more into debt (get into a deeper hole) and if they go up on the number line they climb out of debt (out of the hole). Thus, at the end of the discussion they stated that the numbers that are "more" negative should be below the numbers that are "less" negative. Thus the teacher's knowledge about the importance of connection as well as representation had played an essential role in her creation of the tasks that were connected to each other and supported with visual representation.

Knowledge of literature

The knowledge of literature that Stephan obtained by reading articles and books related to mathematics educations as well as the research she conducted during and after her doctoral studies had important effects on her practices. Before the study started she created an instructional sequence designed using the Realistic Mathematics Education theory. By using RME, she created an instructional sequence grounded in an experientially real context, which in this case was determining a person's financial net worth. She intentionally designed the sequence to gradually move from concrete to abstract. In order to achieve this, she supported tools to reinforce students' imagery.

The teacher's knowledge about the importance of a hypothetical learning trajectory played a vital role in her planning and instruction. During the planning, she used the HLT that she created to define the taken-as-shared interests, possible topics of discourse, tools, imagery, and gestures for the next class sessions. For example, in the HLT for the first activity the tool was the net worth statement template and the possible topic of discourse was conceptualizing the net worth as an abstract quantity. Thus, during the instruction most of the decisions the teacher gave were aligned with supporting this idea. For instance, she intentionally did not focus on students' difficulties about the signs since she knew from the HLT that they would be the focus of later activities.

Practices Consistent with Beliefs and Goals

Previous studies point out that teachers' practices are not always consistent with their beliefs and goals (Thompson, 1992). Although many teachers believe that they apply the practices that are essential for a standards-based classroom, they do not change the very essence of their practice (Stigler & Hiebert, 1997; Hiebert & Stigler, 2000; Hiebert et al., 2005). For instance, even though teachers physically support the environment that standards suggest such as creating small groups or using manipulatives, they do not use this environment effectively to promote students' understanding. This is ascribed to several reasons such as not having the required knowledge (Borko et al., 1992), having superficial beliefs (Kaplan, 1991), relying too much on mathematical beliefs while undermining teaching and learning beliefs, time constraints, and the pressures of standardized tests (Raymond, 1997).

In the current study, however, the teacher's practices were found to be consistent with her beliefs and goals. One reason for that could be the rich and connected body of knowledge she brought to both classroom and planning pertaining to knowledge about students, mathematics, curriculum and standards, and literature. This knowledge helped her overcome the various difficulties that are outlined in the literature. For example, due to her knowledge of the big idea and goals of each class session from the HLT and the instructional sequence, she did not let the discussion stray away from them, which allowed her to better cope with time constraints. She did not remediate all the misconceptions or address all the questions that the students came up with in the class but rather focused on the ones related to the big ideas and the current goals. She knew from the HLT that dealing with those misconceptions and questions would be the focal points of future class sessions.

Secondly, the teacher's knowledge allowed her to develop deep beliefs rather than surface beliefs which also supported her practice of being consistent with her beliefs and goals. In the literature, surface beliefs are found to give rise to practices that support the physical environment suggested by the standards such as creating small groups or giving tasks that might support conceptual understanding. However, these environments are not supported by the necessary communication and teacher behaviors that actually render those environments useful

(Kaplan, 1991). In the current study, it was observed that the teacher's knowledge helped her not only create these environments but also orchestrate effective communication within these environments. For example, during small group activities she circulated the room and encouraged students to draw on their previous images and explain their reasoning rather than simply correcting their mistakes. This empowered students to come up with their own methods and identify the problems in their own reasoning. In the whole class discussions, she knew through her knowledge of the literature and experience that social norms are crucial for a productive discussion environment. Thus she first established these norms and renegotiated them as necessary to support an effective and respectful classroom environment. These practices illustrated that the teacher in the current study had deep beliefs rather than surface beliefs, which allowed her practice to be aligned with her goals and beliefs.

Another reason shown in the literature for inconsistency of beliefs and practices is the demands of standardized tests, which primarily assess the procedural skills rather than conceptual understanding (Raymond, 1997). This creates a dilemma for teachers as they think that spending too much time to develop conceptual understanding might not leave enough time to practice procedural skills. In this study, the teacher created an instructional sequence that included activities to develop not only imagery to support making sense of integers and integer operations but also the activities that included various textbook-like bare number problems where students could improve their procedural skills. Because the teacher knew from the beginning that these procedural activities were part of the instructional sequence, and students would be more successful in them once they develop the supportive imagery, she did not have to change her practice due to the pressures of standardized tests.

Previous studies also cite the inconsistency between teachers' beliefs about the nature of mathematics and their teaching and learning beliefs as a reason for the conflict between practice and beliefs. Raymond (1997) found that when teachers' beliefs about the nature of mathematics are primarily traditional despite having non-traditional beliefs about teaching and learning of mathematics, their practice also tends to be traditional. This creates a conflict with their non-traditional beliefs about how mathematic should be taught and how students learn mathematics. In the current study, the teacher's mathematics beliefs were consistent with her teaching and learning beliefs. In fact, her most fundamental belief about mathematics, which is that mathematics is a sense making activity that involves problem solving, making conjectures, analyzing solutions, and proving arguments was supported by her learning and teaching beliefs. For example, her belief that students learn mathematics when they understand was in direct support of her belief that mathematics is a sense making activity. This fundamental mathematical belief was also supported by her teaching beliefs such as valuing students ideas and explanations empowers them to participate in classroom discussions through making conjectures and analyzing solutions. This coherency between the teacher's beliefs about mathematics, teaching, and learning appeared to be an important factor in her practice to be consistent with her beliefs and goals.

Relationship between Planning and Classroom Practices

Although many teachers consider planning at the heart of effective teaching, studies involving American teachers show that during planning most of the emphasis is put on the activities that students will go through rather than the goals of these activities as well as how students will interact with them (Shavelson & Stern, 1981; Clark & Yinger, 1987; Kilpatrick et al., 2001). The research also suggests that inflexible lesson plans may prevent teachers from adapting to the inevitable changes that come up during instruction. Despite these challenges, planning has an ever more central place to be successful in a standards-based teaching and learning environment. However, planning to teach in line with the NCTM *Standards* as well as its relationship with classroom practices have rarely been explored in literature (Kilpatrick et al., 2001). Shedding more light into this area has been one of the goals of the current study. In this section, examples from the analysis are used to illustrate how the teacher's planning practices affected her classroom practices as well as how the classroom practices helped the teacher to revise her planning.

In the study, it was observed that the teacher performed effective planning practices that facilitated her instruction in the classroom. First, she created an HLT that outlined the possible route of students' learning by including the tools, supported imagery and gestures, taken-as-shared interests, and possible discourse topics (i.e. big ideas). Based on the HLT, she created an instructional sequence, which was reinforced with a financial context. The instructional sequence was designed to connect the students' informal knowledge with progressively more abstract mathematical ideas. The creation of the HLT and the instructional sequence were crucial for the teacher in different ways. Based on them, she knew what the students needed to learn during the entire course of the unit from the beginning, which allowed her to organize the big ideas that she wanted to emphasize during the sequence. As she knew the big ideas of each class and what comes next, she could make better decisions during her instruction about what to focus on in the classroom and what could be deferred to later sessions. For example, during the initial part of the instruction some students had difficulty in their calculations when finding the net worth. Since at this point the teacher's aim was to develop the big idea that the assets and debts are intangible quantities that affect the net worth in opposite directions, she did not focus on those calculational mistakes. She knew that performing calculations correctly and efficiently were the big ideas of later activities related to transactions.

The teacher meetings that were held before and during the instruction gave the teachers opportunities to talk about each activity in detail. In these meetings, the teachers worked on the activities individually and collaboratively to anticipate students' possible solutions. For example, the teachers anticipated that students might have difficulty in understanding the negative net worth. To help students overcome this difficulty, they talked about how they could introduce this concept, what kind of tools and imagery might support students' understanding of negative net worth and what types of questions they could ask to assess students' understanding. In these meetings, the teachers decided to introduce the vertical number line as a supportive reasoning device and thought that it would match well with the "pay-off" idea that students were likely develop. These anticipations actually materialized in Stephan's class and the work they did during the planning allowed her to effectively deal with students' misconceptions.

The teacher's reflection after each day's class was also an essential part of her planning practice. During this reflection, she analyzed the events of the class session to evaluate if a renegotiation of social norms was necessary, what kinds of misconceptions and difficulties the students experienced, whether she was able to achieve her goals, and whether she was on-track with regards to the instructional sequence. She used this self-analysis to revise her plan for the following day's class session. In one of the class sessions, she noticed that some students started to behave in arrogant ways, which was unacceptable according to the social norms that they had established. The teacher was worried that this type of behavior could affect the quality of the whole class discussions as it might cause the other students to be timid about sharing their ideas. Although she talked to the students about the inappropriateness of this behavior during the class session in which it occurred, in her daily reflection she decided to talk more about the importance of humility and being respectful the following day.

The teacher's planning practices facilitated and gave direction to the class instruction. The teacher's classroom practices, on the other hand, equipped her to do more effective planning; thus the relationship between them was found to be mutual as shown in Figure 28. During the class sessions the teacher collected data and used this data to revise her plan. For example, at the beginning of the instructional sequence after the teacher gave students some problems that included different types of net worths such as positive, negative, and zero, she started to walk around the small groups. One of her practices was to ask students to explain their solutions. She noticed that some students had difficulty distinguishing between negative and zero net worth. Based on this observation she decided during the planning to include more questions in the next day's activity that emphasize the difference of these two types of net worths. She also sometimes collected students' works from the class activities and made plans according to them such as creating bell work questions that would help students to focus on some of their misconceptions.

Another example of the teacher's classroom practices affecting her planning was from the classroom discourse where the teacher encouraged students to use the vertical number line in solving the problems with integer operations. Although the vertical number line helped many students make sense of the operations and correctly solve the problems, the teacher noticed that in the whole class discussion that some students used it in ineffective ways. She brought this into the planning meetings held with the other teachers and they decided to create a task to resolve this problem. The task showed different solutions of an integer operations problem on the number line and asked students which solution was correct and why. During the discussion of this task, the students who had problems with using the number line effectively realized the appropriate ways of using this tool.

In the class sessions, the teacher also encouraged students to conjecture related to the mathematical ideas that they noticed in the solution of the problems. When the teacher noticed in the class discussion that some students were close to detecting a pattern or a general rule but needed more support, she prepared tasks in the planning to help students discover these patterns or rules. For example, in order to support the students' imagery in one of the class sessions, the teacher gave a task that asked students to find out whether a person's decision was good or bad based on the given statement and then she introduced the mathematical symbols + and -. During the solution of the problems that were given in the classroom some students noticed a pattern, but they could not generalize it (i.e. they observed that taking away debt was a good decision but they did not think of the other combinations). The teacher used this in her plan for the next day and created a bell work activity that includes the other combinations such as adding debt and taking away asset during which students discovered that +(-) and –(+) are bad decisions while –(-) and +(+) are good decisions with the same effect.

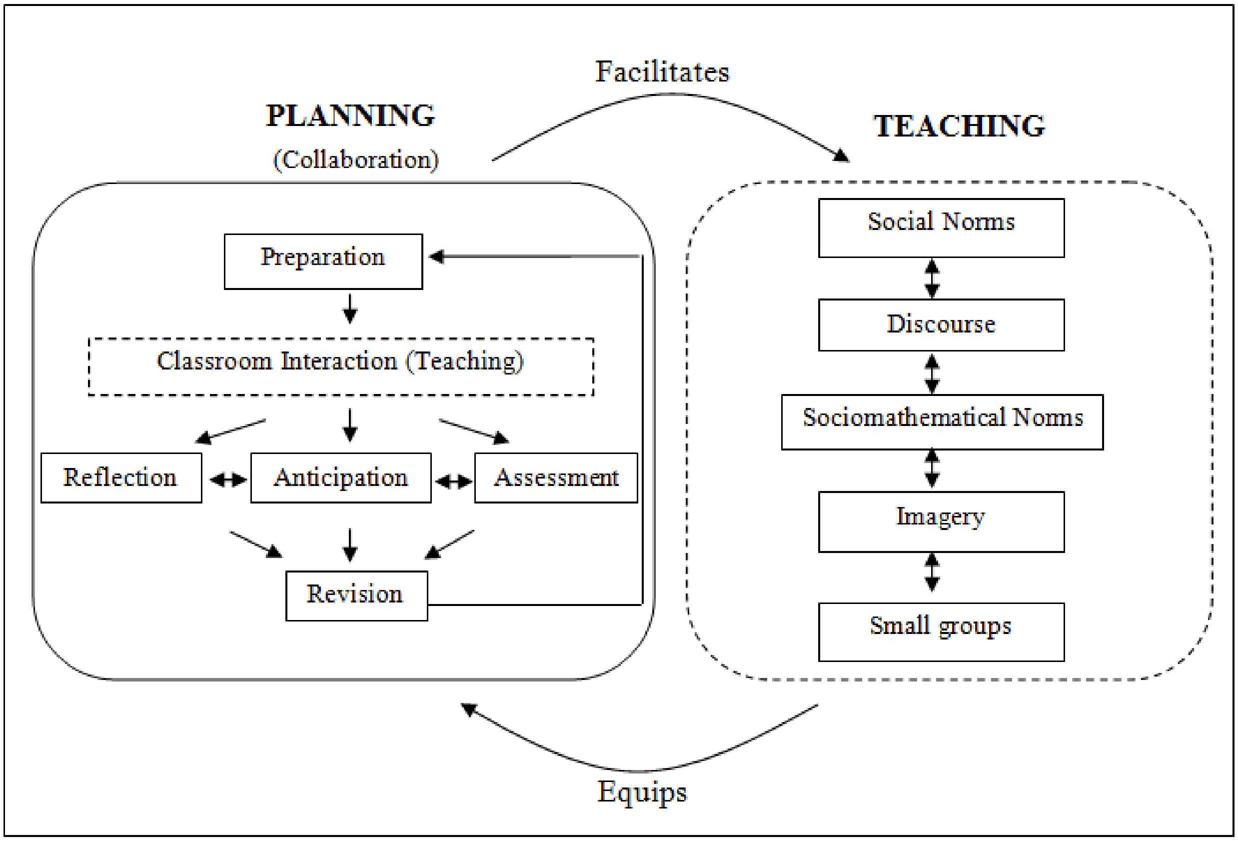


Figure 28: Planning and teaching cycle

As illustrated by the examples above, the teacher collected data during her classroom practices and used them to support her planning. Thus while her planning practices facilitated the classroom instruction, her classroom practices equipped her with the knowledge to refine her plans.

Implications

This study analyzed an expert teacher's planning and classroom practices and extracted the beliefs, goals, and knowledge that underlay the development of those practices. It was found that the teacher's rich and connected body of knowledge as well as her wide set of beliefs and goals were essential to teach consistent with the NCTM *Standards*. During planning, it was observed that effective collaboration within a learning community of teachers resulted in more effective planning. This highlights the importance of giving teachers sufficient collaborative planning time as part of their regular schedule.

In this study, the teaching-in-context theory was used to explain the expert teacher's practices during a five week long period. Therefore, different from earlier studies which used the theory in explaining the teacher's decision making process in short periods (e.g. one classroom session), this study showed that the theory can be used to interpret the teacher's behaviors in a longer period.

The findings of this study can be used by teacher educators to create an environment in professional development as well as pre-service teacher courses to help teachers understand the dynamics of a standards-based environment and what it takes for them to teach consistently with NCTM *Standards*. Educators might create similar types of environments to those described in this study and observe which practices and actions the teachers have difficulty to apply, and give them opportunities to improve on those actions. Additionally, the lists of practices and actions that are found by this study might help educators to evaluate whether pre-service teachers teach aligned with standards during their internship.

In future studies, the sources of difficulties that hinder teachers from performing effective practices might be investigated based on their beliefs, goals, and knowledge. An important question might be whether difficulties arise primarily due to teachers' lack of knowledge or beliefs and goals that are inconsistent with standards-based teaching. Once the reasons are determined, future studies may investigate how to help teachers change their beliefs, goals, and knowledge to make them aligned with standards-based teaching.

Another future investigation may involve evaluating the student achievement between the teachers who took professional development courses that include the application of the practices that were found in this study and those teachers who teach traditionally. Finally, the effect of the co-teacher in supporting a standards-based teaching environment can also be investigated.

Conclusion

This study investigated an expert middle school mathematics teacher who teaches consistent with the tenets of the NCTM *Standards* during the instruction of a seventh grade mathematics class on integers with the purpose of documenting her planning and classroom practices; understanding their relationship; and explaining them based on the teacher's beliefs, goals, and knowledge. The analysis demonstrated that the teacher draws from a rich and connected body of knowledge of content, students, curriculum, and standards as well as literature in her planning and teaching. This knowledge allowed her to develop practices that are consistent with her beliefs and goals. It was found that planning and classroom practices were highly interrelated with one another in that, they often provided the necessary means of subsistence to each other. For example, the collection of data by the teacher through various classroom practices provided her the material needed for planning, and the plan in turn, facilitated the application of classroom practices. This relationship created a planning and teaching cycle that lies in the heart of a productive inquiry and standards-based classroom.

APPENDIX A: IRB APPROVAL

University of Central Florida

University of Central Florida Institutional Review Board
Office of Research & Commercialization
12201 Research Parkway, Suite 501
Orlando, Florida 32826-3246
Telephone: 407-823-2901, 407-882-2012 or 407-882-2276
www.research.ucf.edu/compliance/irb.html

Notice of Expedited Review and Approval of Requested Addendum/Modification Changes

From: **UCF Institutional Review Board**
FWA00000351, Exp. 10/8/11, IRB00001138

To: **Didem Akyuz and Juli K. Dixon, Ph.D.**

Date: **January 29, 2009**

IRB Number: **SBE-08-05977**

Study Title: **Developing the concept of integers for seventh grade students by using classroom norms that support an inquiry atmosphere**

Dear Researcher:

Your requested addendum/modification changes to your study noted above which were submitted to the IRB on 01/27/2009 were approved by **expedited** review on 1/28/2009.

Per federal regulations, 45 CFR 46.110, the expeditable modifications were determined to be minor changes in previously approved research during the period for which approval was authorized.

Use of the approved, stamped consent document(s) is required. The new forms supersede all previous versions, which are now invalid for further use. Only approved investigators (or other approved key study personnel) may solicit consent for research participation. Subjects or their representatives must receive a copy of the consent form(s).

This addendum approval does NOT extend the IRB approval period or replace the Continuing Review form for renewal of the study.

On behalf of Tracy Dietz, Ph.D., IRB Chair, this letter is signed by:

Signature applied by Janice Turchin on 01/29/2009 09:20:20 AM EST

Janice M Turchin

IRB Coordinator

Internal IRB Submission Reference Number: 005000

APPENDIX B: BELIEFS, GOALS, AND KNOWLEDGE

Beliefs

Beliefs about learning

- B_L1: To learn mathematics with understanding
- B_L2: Each student may learn differently
- B_L3: Taking responsibility is important in learning
- B_L4: Learning is both an individual and social process
- B_L5: Students learn better with an instruction that relates to their informal knowledge
- B_L6: Students learn better with an instruction that moves from concrete to abstract gradually
- B_L7: An environment that nurtures respect and a sense of community plays an essential role in students' learning
- B_L8: Students learn from their mistakes
- B_L9: Students reorganize their ideas when they explain

Beliefs about teaching

- B_T1: The teacher is not sole source of knowledge
- B_T2: The teacher is responsible for creating and sustaining an environment where ideas can be discussed freely and respectfully
- B_T3: Teacher needs to support an environment where mutual trust is important and students represent their ideas willingly
- B_T4: Valuing students ideas and explanations empowers them to participate in class discussions
- B_T5: Effective teaching requires careful planning

- B_T6: The teacher should be fair in the class and provide equal opportunities for all students to participate
- B_T7: Having sufficient time to think (and talk to group members) is important before discussing solutions (or proving conjectures)
- B_T8: Formative and summative assessments play a crucial role in effective teaching.
- B_T9: Teacher is responsible to provide opportunity to students to see efficient, different and sophisticated ways during the class discussion
- B_T10: The teacher is responsible to introduce mathematical symbolization(words)
- B_T11: Being able to explain shows students' understanding and notice their mistakes
- B_T12: The teacher is responsible to clarify students' role and the task that they will work on in their groups
- B_T13: The teacher is responsible for efficiently managing the time
- B_T14: Collaborating with other teachers essential to improve instruction and assessment
- B_T15: Introducing tools as ready-made might cause difficulty as the relationship between the tool and the task would not be clear
- B_T16: The teacher should pursue the ideas in the students comments when appropriate

Beliefs about mathematics

- B_M1: Mathematics is a sense-making activity
- B_M2: Making conjectures is important to understand mathematical ideas
- B_M3: Problem solving is important to understand mathematical ideas
- B_M4 Analyzing solutions (compare-contrast) is important to understand mathematical ideas

- B_M5: Proving arguments is important to understand mathematical ideas
- B_M6: Not all mathematical solutions are equal. Students should know what counts as an acceptable, sophisticated, different and efficient solution
- B_M7: Students should make sense of the symbols

Goals

Overarching goals

- G_O1: Renegotiate expectations (social and sociomathematical norms) as necessary during the instruction
- G_O2: Support a discourse that stays alive and interesting
- G_O3: Focus on(support) the big idea of the lesson/ following lesson
- G_O4: Tackle one big idea at a time
- G_O5: Support development of meaningful imagery by folding back to previous images
- G_O6: Support students' ideas with tools when they have the reasoning
- G_O7: Ensuring that students understand the mathematical ideas that are discussed in the class

Content goals

- G_C1: Define the terms *assets* as what you own, and *debts* as what you owe
- G_C2: Define the net worth as difference between assets and debts
- G_C3: Think of net worth as intangible quantity
- G_C4: Explore additive inverse
- G_C5: Model structured gap using black-red number line
- G_C6: Explore transactions

- G_C7: Solve bare number sentences with integer operations

Local goals

- G_L1: Bring a student's idea from another period to the class

Knowledge

Pedagogical knowledge

- K_P1: Knowledge of possible students' answers (including possible images)
- K_P2: Knowledge of possible misconceptions (difficulties)
- K_P3: Knowledge of possible behavior (tendency)
- K_P4: Knowledge from the previous years of instruction
- K_P5: Knowledge of relevant literature related to integers sequence
- K_P6: Knowledge of history of mathematics
- K_P7: Knowledge of tools and imagery
- K_P8: Knowledge of design research and HLT
- K_P9: Knowledge of RME
- K_P10: Knowledge of different representations
- K_P11: Knowledge of current students(or events) in the class

Subject-matter knowledge

- K_S1: Knowledge of integers (knowledge of net worth, knowledge of additive inverses, knowledge of integer rules etc.)

Curriculum knowledge

- K_C1: Knowledge of the instructional sequence (e.g. goal of each activity)
- K_C2: Knowledge of state standards
- K_C3: Knowledge of national Principles and standards
    - Content standards (number and operations, algebra, geometry, measurement…)
    - Process standards (problem solving, reasoning and proof, communication, connection, and representation)
    - Principles (equity, curriculum, teaching. learning, assessment, technology)
- K_C4: Knowledge of Professional Standards for Teaching Mathematics (referred as Teaching Standards)
    - Tasks
    - Discourse
    - Environment
    - Analysis of teaching and learning

APPENDIX C: INSTRUCTIONAL SEQUENCE

Net Worth Statement

Client Name

Cash Assets

Cash Bank Accounts
Money Market Accounts
Other Cash

Investments

Bonds
Stocks
Mutual Funds
Annuities
IRA's
Retirement Plans
Real Estate
Other Investments

Personal Assets

Household Contents
Primary Residence
Automobiles
Other

Total Assets

Debts

Mortgages
Personal or Business Loans
Automobile Loans
Credit Cards/Charge Accounts
Other Debts

Total Debts

Net Worth

Net Worth Statement

Client Name *Cindy*

Cash Assets

Checking Account	$110
Money Market Accounts	
Savings Account	$55
Investments	
Bonds	
Mutual Funds	
Real Estate	
Other	
Personal Assets	
Income for tutoring Bobby in math	$80
Received check for	
Babysitting Marsha's kids	$60
Total Assets	

Debts	
Credit card charge for clothes	$75
Charge on Target card	$115
Total Debts	

NET WORTH

Net Worth Statement

Client Name *Bobby*

Cash Assets

Cash Bank Accounts	$100
Money Market Accounts	
Savings Account	$105
Other	
Investments	
Bonds	
Motorcycle	$600
Mutual Funds	
Real Estate	
Other	
Personal Assets	
Received check from Greg for fixing his car	$900
Other	
Total Assets	

Debts	
Mortgages	
UCF Loan for Books	$1500
Auto Loans	
Purchased surfboard	$200
Total Debts	

NET WORTH

Who is worth more, Cindy or Bobby? Explain your conclusion in complete sentences.

Net Worth Statement		Net Worth Statement	
Client Name *Angelina*		Client Name *Brad*	
Cash Assets		**Cash Assets**	
Checking Account		Cash Bank Accounts	$150,000
Money Market Accounts		Money Market Accounts	
Savings Account	$100,000	Savings Account	
		Other	
Investments		**Investments**	
Restaurant	$500,000	Owns a Planet Hollywood	$450,000
Owns a movie production company	$250,000	Mutual Funds	
Owns Land in Namibia	$90,000	Real Estate	
Other		Other	
Personal Assets		**Personal Assets**	
		Car	
		Other	
Total Assets		**Total Assets**	
Debts		**Debts**	
Boat Loan	$200,000	Owes David Clooney in gambling debts	$90,000
Penalty for pulling out of a movie deal	$650,000	Auto Loans	$175,000
		Owes Emily Anniston a divorce settlement	$525,000
Total Debts		**Total Debts**	
NET WORTH		**NET WORTH**	

Who is worth more money when Brad and Angeline get married? Explain in complete sentence.

Net Worth Statement

Client Name

Cash Assets

Investments

Personal Assets

Total Assets

Debts

Total Debts

Net Worth Statement

Client Name

Cash Assets

Investments

Personal Assets

Total Assets

Debts

Total Debts

Make up a story and at least five statements for each person you choose above. Make sure you write a question below that has to be solved, then trade your paper with another group so they can solve your story problem.

The Dating Game

If you were making a decision based upon financial worth **only**, who would you choose to date, bachelor number one, two or three? A positive sign indicates an asset, a negative sign indicates a debt.

Bachelor Number One's Worth Statement:

Bank Balance: +$1000
Car Loan: -$15,000
Boat Loan: -$45,000
Retirement Fund: +$60,000
Net Worth:$

Bachelor Number Two's Worth Statement:

Bank Balance: +$10,000
Investment in Offshore Oil: +$25,000
Loss in Stock Market: -$50,000
Retirement Fund: +$20,000
Net Worth:$

Bachelor Number Three's Worth Statement:

Bank Balance: -$100
Investment in Energy Efficient Fuel: +$20,000
Organic Sweet Potato Farm: +$5000
Stock Market Loss on Mushrooms that power cars: -$20,000

Net Worth:$

Explanation

The Dating Game

If you were making a decision based upon financial worth **only**, who would you choose to date, bachelorette number one, two or three? A positive sign indicates an asset, a negative sign indicates a debt.

Bachelorette Number One's Worth Statement:

+$5900
- $1700
-$2000
+$800
Net Worth: $

Bachelorette Number Two's Worth Statement:

-$2900
-$3700
-$1800
+$8000
Net Worth: $

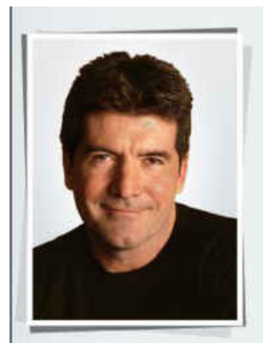
Bachelorette Number Three's Worth Statement:

-$13,000
+$2000
+$18,000
-$7000

Net Worth:

Explanation

Who's Worth More?

Paula	Randy	Simon
		
+$1000	+$20000	+$20000
-$1000	-20000	-20000
+2000	+7000	+7000
+500	-$2000	-$2000
-2000	-5000	-5000
-500		
+$60		

Show your work and/or state your strategy:

In the Red!!!

Accountants say that when a person's net worth is above zero, they are said to be

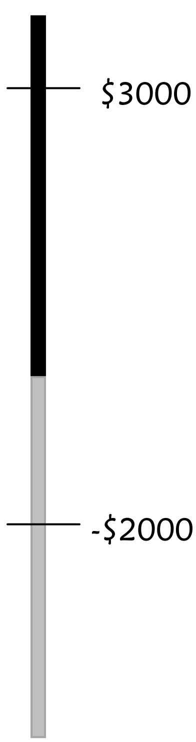
"in the black"

When a person's net worth drops below zero, they are said to be

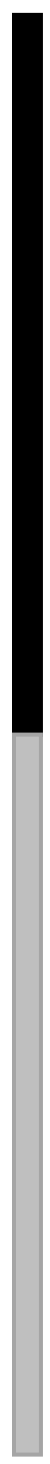
"in the red"

Gilligan's Net Worth is **in the BLACK** +$3000.

Mary Ann's Net Worth is in the RED -$2000.



How much MORE is Gilligan worth than Mary Ann? Show this on the Net Worth Line.



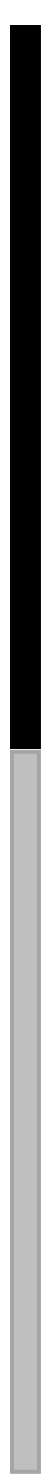
Paris' Net Worth is in the red -$20,000.

Nicole's Net Worth is in the red -$22,000.

Who is worth more?

How much more?

Show this on the Net Worth Line.



M. C. Hammer's Net Worth is in the red -$100,000.

Michael Jackson's Net Worth is in the black +$220,000.

Who is worth more?

How much more?

Show this on the Net Worth Line.

Net Worth Comparisons

Use a Net Worth Number Line, if necessary, to solve each of the following problems.

1. Sally is worth +$200 and Melissa is worth -$200. Who is worth more and how much more?

2. Betty is worth -$1000 and Maria is worth -$1500. Who is worth more and how much more?

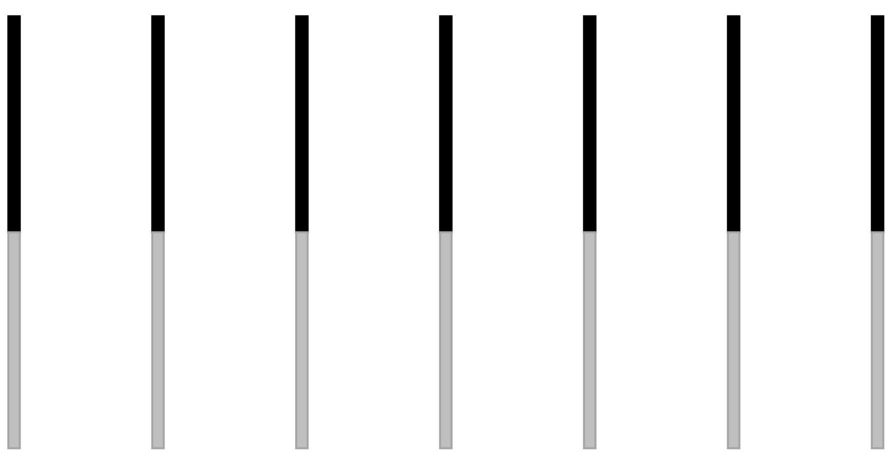
3. Charlie is worth $500 and Brandon is worth -$700. Who is worth more and how much more?

4. Gage is worth $900 and Nicholas is worth -$200. Who is worth more and how much more?

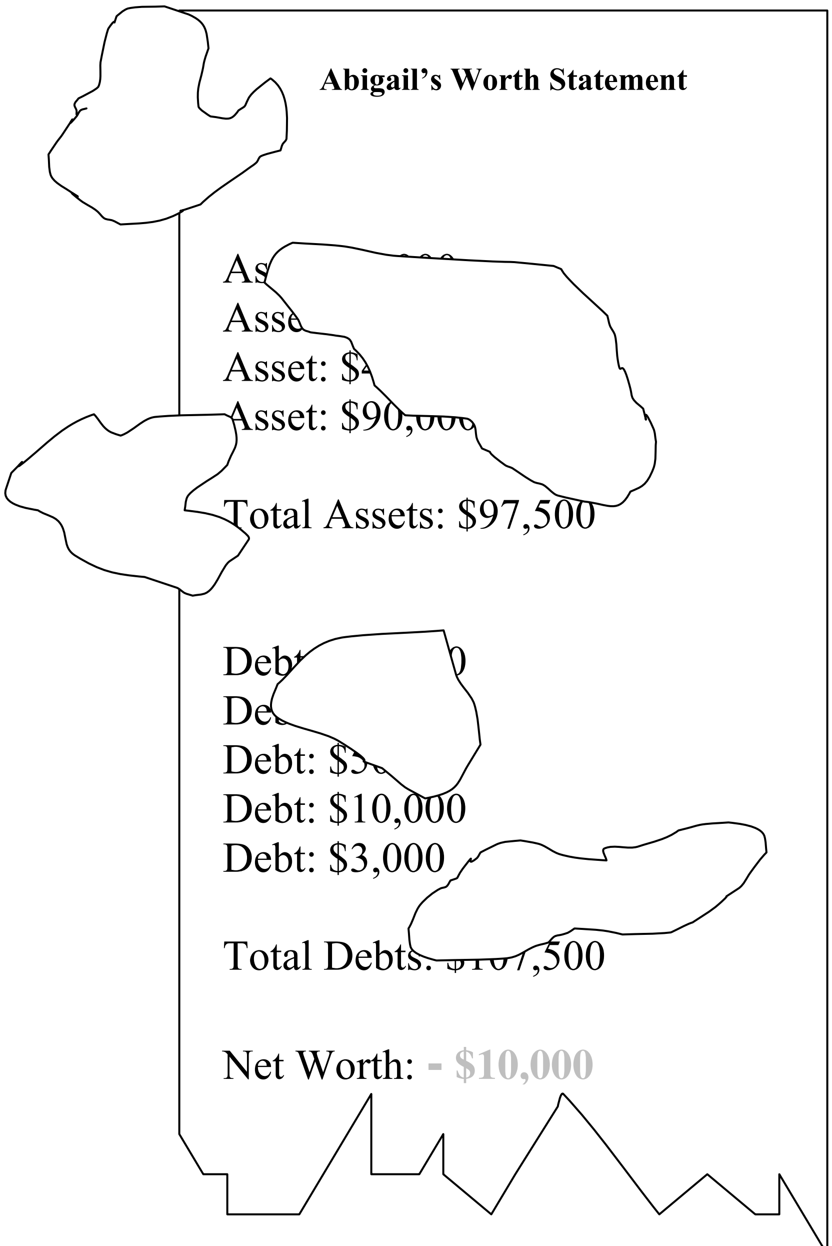
5. Danny is worth -$50 and Andrew is worth -$250. Who is worth more and how much more?

6. Stuart is worth $450 and Sergio is worth -$40. Who is worth more and how much more?

7. Tisha is worth -$5000 and Anastasia is worth $1000. Who is worth more and how much more?



!!Don’t Cry Over Spilled Milk!!



Abigail’s Worth Statement

As
Asse
Asset: $4
Asset: $90,

Total Assets: $97,500

Deb 0
De
Debt: $5
Debt: $10,000
Debt: $3,000

Total Debts: ,500

Net Worth: **- $10,000**

Abigail lost an asset (a valuable coin) worth $8000. She wanted to figure out what she was worth now that the asset was taken away from her net worth statement. BUT the only copy of her net worth statement she could find has milk stains on it. Can you help her figure out her net worth now?

James has the following Worth Statement:

James' Worth Statement

1. Asset 1: +$200
2. Asset 2: +$145
3. Debt 1: -$200
4. Debt 2: -$650
5. Asset 3: +$650
6. Asset 4: +700
7. Debt 3: -$50

1. Find his net worth:

2. If he added another asset ($500) to his statement, how would that affect his net worth?

3. What would be his net worth now?

Frank has the following Worth Statement:

Frank's Worth Statement

1. Asset 1: +$1000
2. Debt 1: -$1000
3. Asset 2: +$2000
4. Debt 2: -$650
5. Asset 3: +$650
6. Debt 3: -500
7. Debt 4: -$500

1. Find his net worth:

2. Seminole County Bank sent him a letter saying that their computer made an error. They will be taking away that $650 asset (asset number 3 above). If the bank removes the $650 asset, how will that affect his net worth?

3. What would be his net worth now?

Kim has the following Worth Statement:

Kim's Worth Statement

1. Debt 1: -$1000
2. Asset 1: +$2500
3. Debt 2: -$6000
4. Asset 2: +$9000
5. Asset 3: +$7000
6. Debt 3: -9000
7. Debt 4: -$3000

1. Find her net worth:

2. Debt 4 was a loan that Kim took out from her dad. However, Kim's dad decided to be nice (WAY nice!) and told her that she didn't ever have to pay him back. Therefore, we can take away the $3000 debt from Kim's worth statement. If we remove the 3000 debt, how would that affect her net worth?

3. What would be her net worth now?

!!??Smart or Stupid Decision??!!

Which of the following students made bad decisions about their finances?

Ann: She took away an asset of (+$200) from her net worth statement

Bradley: He added an asset of (+$3000) to his net worth statement

Christian: He took away an asset of (+$50) from his net worth statement

Devon: He added a debt of (-$650) to his net worth statement

Ernie: He took away a debt of (-$5400) from his net worth statement

Fran: She took away an asset of (+$201) from her net worth statement

Gracie: She added a debt of (-$67) to her net worth statement

Herbert: He took away an asset of (+$450) from his net worth statement

Writing Money Transactions with Symbols

TO SHOW A TRANSACTION TAKING PLACE WE WILL USE TWO SIGNS:

The **first** sign will stand for the transaction (adding or taking away)

and

The **second** sign will stand for whether the amount is an asset or a debt.

EXAMPLE: John ADDS a DEBT of \$300 would be written as follows + (-300)

EXAMPLE: Sal TAKES AWAY a DEBT of \$400 would be written as follows - (- 400)

Jake takes away an asset of \$4500. This can be written as: – (+4500)

Try the following, then check your answers.

A. Frank adds a debt of \$530.
B. David adds an asset of \$783
C. Diane takes away an asset of \$3420
D. Michelle adds a debt of \$624
E. Deborah takes away a debt of \$352

ANSWERS:
A. + (-530)
B. + (+783)
C. – (+3420)
D. + (- 624)
E. – (- 352)

Using the words ADD, TAKE AWAY, DEBT and ASSET, describe each transaction below:

1. – (+300)

2. + (-340)

3. + (+534)

4. + (342)

5. – (-7344)

6. – (+1200)

7. – (890)

8. + (– 6832)

9. – (– 566)

10. – (-1)

NET WORTH PROBLEMS

Nancy has a net worth of $5000! A debt of $3000 is TAKEN AWAY. Is this good or bad? What is her net worth now! Draw your own Net Worth Trackers to help you figure these out, if you need them.

1. Donald has a net worth of -$5000! A debt of $3000 is TAKEN AWAY. Is this good or bad? What is his net worth now!

2. Meagan has a net worth of -$4300! A debt of $3000 is ADDED. Is this good or bad? What is her net worth now?

3. Melanie has a net worth of +$600! A debt of $1000 is ADDED. Is this good or bad? What is her net worth now!

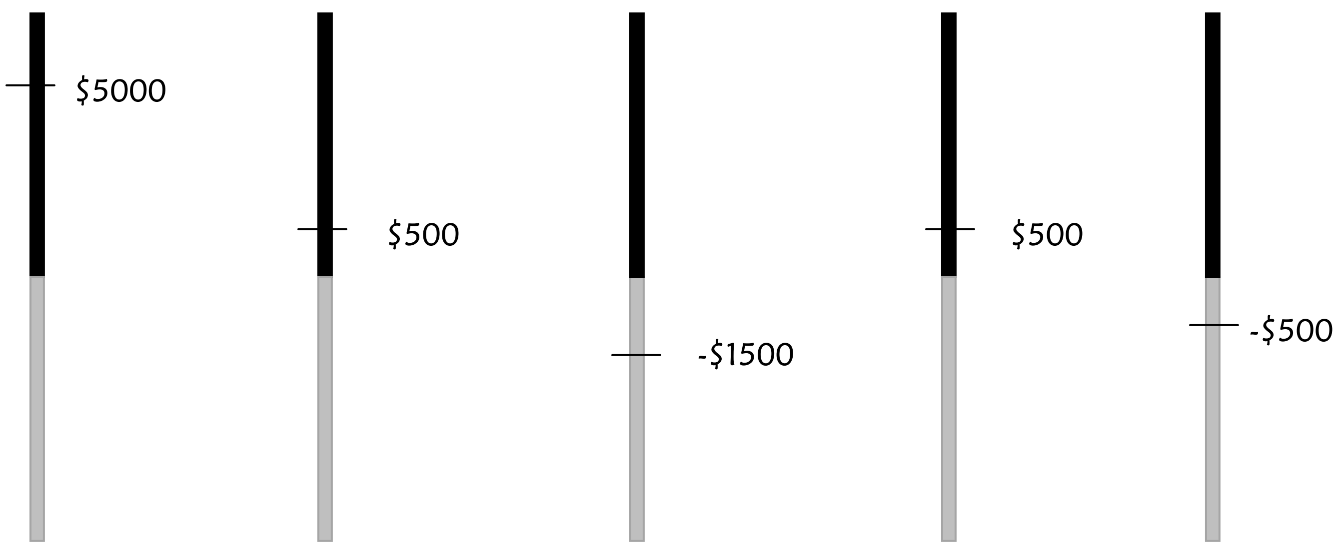
4. Todd has a net worth of +$10,000! An asset of $3000 is ADDED. Is this good or bad? What is his net worth now!

5. Monica has a net worth of -$7400! An asset of $3000 is TAKEN AWAY. Is this good or bad? What is her net worth now!

6. Andrea has a net worth of +$2200! A debt of $3000 is ADDED. Is this good or bad? What is her net worth now!

Net Worth Trackers

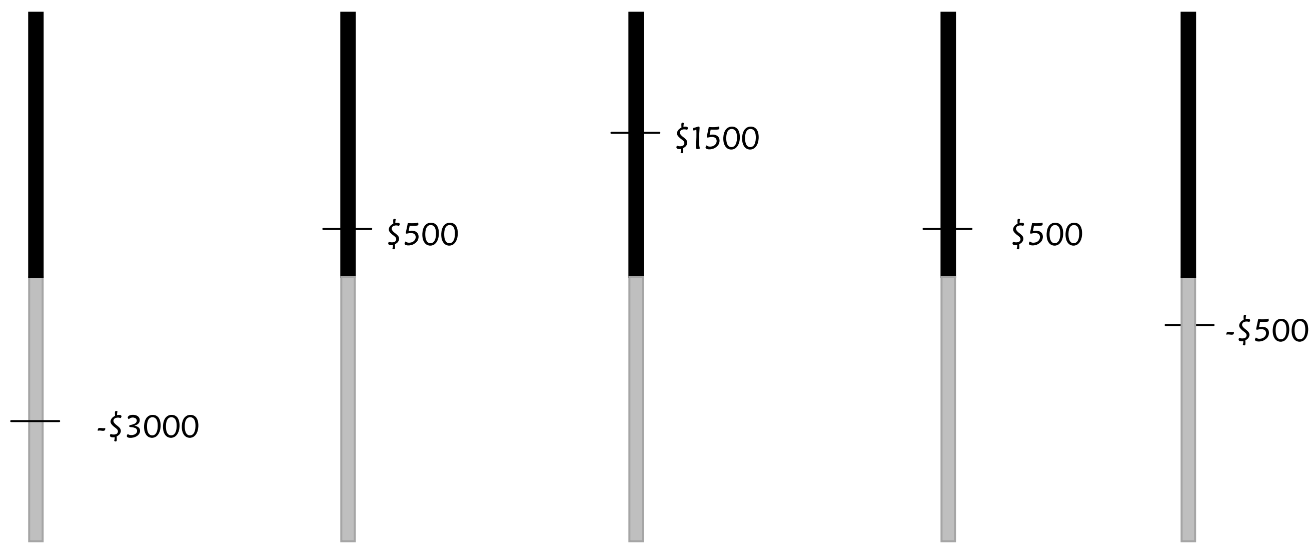
For each of the problems below, use the Tracker to show whether each person is in the black, in the red, or breaking even.



Transaction: + (- 4000)	Transaction: - (+ 1000)	Transaction: + (- 1000)	Transaction: - (- 2000)	Transaction: - (+ 4000)

Net Worth Trackers

For each of the problems below, use the Tracker to show whether each person is in the black, in the red, or breaking even.



Transaction: - (- 6000)	Transaction: + (+ 1000)	Transaction: + (- 7000)	Transaction: - (+ 2000)	Transaction: - (- 500)

Alice's Net Worth

Write number sentences for the following changes that occur to Alice's net worth (Use a net worth tracker if you need to):

1. Net Worth: $1500
   Transaction: Adds a debt of $600

2. Net Worth: $600
   Transaction: Adds a debt of $1100

3. Net Worth: - $400
   Transaction: Adds a debt of $450

4. Net Worth: - $550
   Transaction: Adds an asset of $1900

5. Net Worth: $1250
   Transaction: Adds an asset of $350

6. Net Worth: $1600
   Transaction: Adds a debt of $400

7. Net Worth: $800
   Transaction: Takes away a debt of $200

Write number sentences for the following changes that occur to Alice's net worth (Use a net worth tracker if you need to):

8. Net Worth: - $2000
   Transaction: Adds an asset of $600

9. Net Worth: - $1800
   Transaction: Adds a debt of $1200

10. Net Worth: - $2600
    Transaction: Takes away a debt of $300
11. Net Worth: - $2000
    Transaction: Takes away a debt of $500

After each transaction below, record the new NET WORTH (Use a net worth tracker if you need to):

1. \$45 adds an asset of (+\$5)
2. \$50 adds an asset of (+\$70)
3. \$100 adds a debt of (-\$75)
4. \$200 adds a debt of (-\$225)
5. -\$200 adds a debt of (-\$105)

A. \$255 adds a debt of (-\$200)

B. \$155 adds an asset of (+\$50)

C. \$110 adds an asset of (+\$15)

D. \$125 adds a debt of (-\$325)

E. -\$100 adds a debt of (-\$150)

a) \$255 adds (-80)

b) \$300 takes away (+100)

c) \$500 adds (-500)

d) -\$100 adds (-300)

e) -\$400 takes away (+200)

f) -\$800 takes away (-100)

g) -\$200 takes away (-200)

After each transaction below, record the new NET WORTH:

1. 95 add (+15)
2. 150 add (+250)
3. 360 add (-160)
4. 225 add (-125)
5. 75 add (-175)

A. \$600 + (- \$250)

B. \$1200 + (+ \$150)

C. \$500 - (+\$105)

D. \$605 + (- \$305)

E. \$700 + (- \$100)

I. - \$250 + (+\$60)

II. - \$1900 - (- \$100)

II. - \$90 + (+\$100)

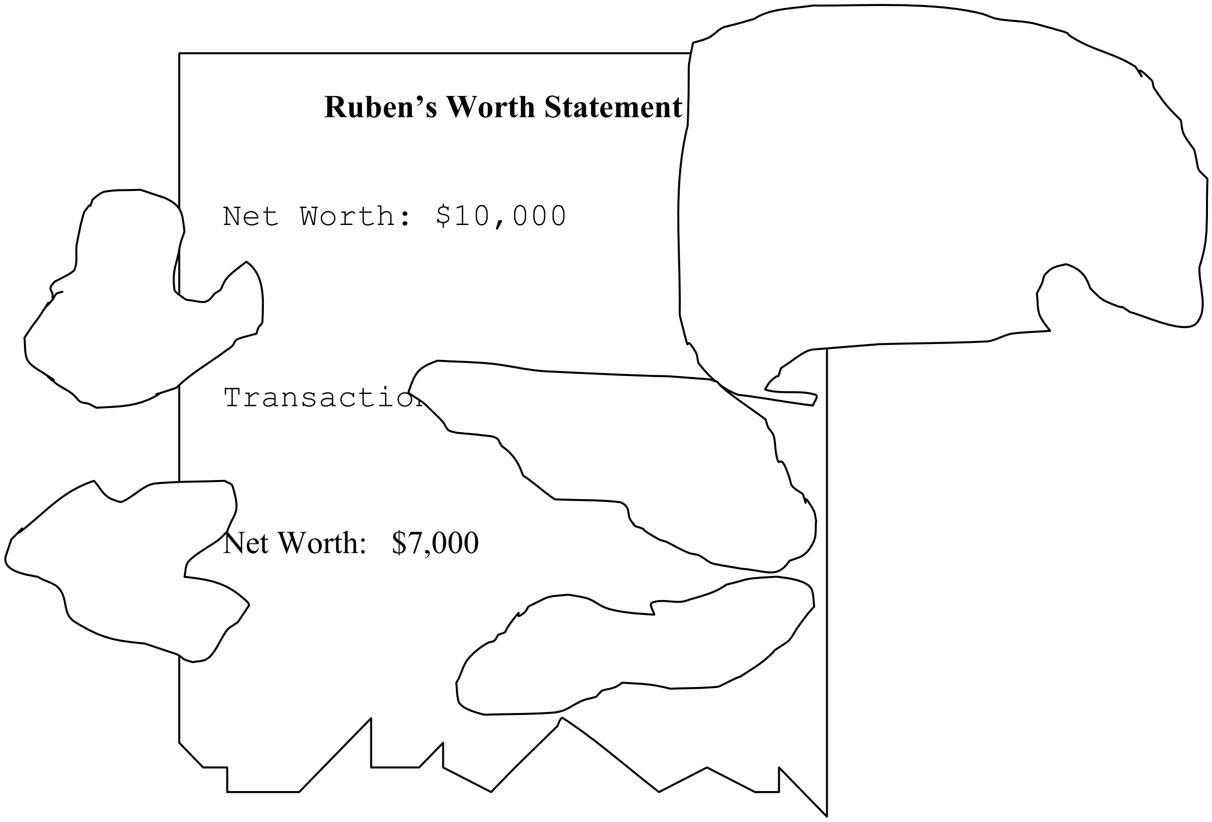
IV. \$100 + (+\$235)

V. \$245 + (-\$145)

VI. \$100 + (-\$180)

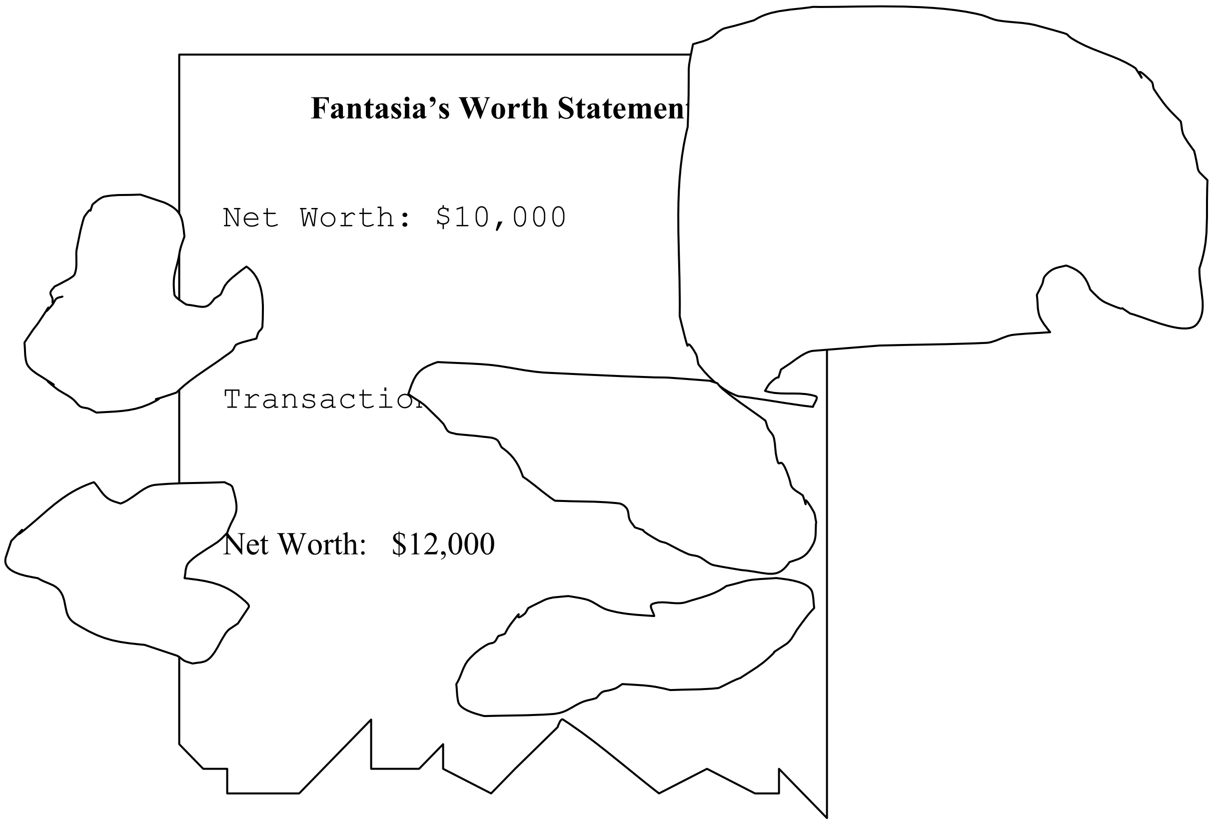
VII. \$ 80 + (-\$180)

Oops! Coffee Spill!



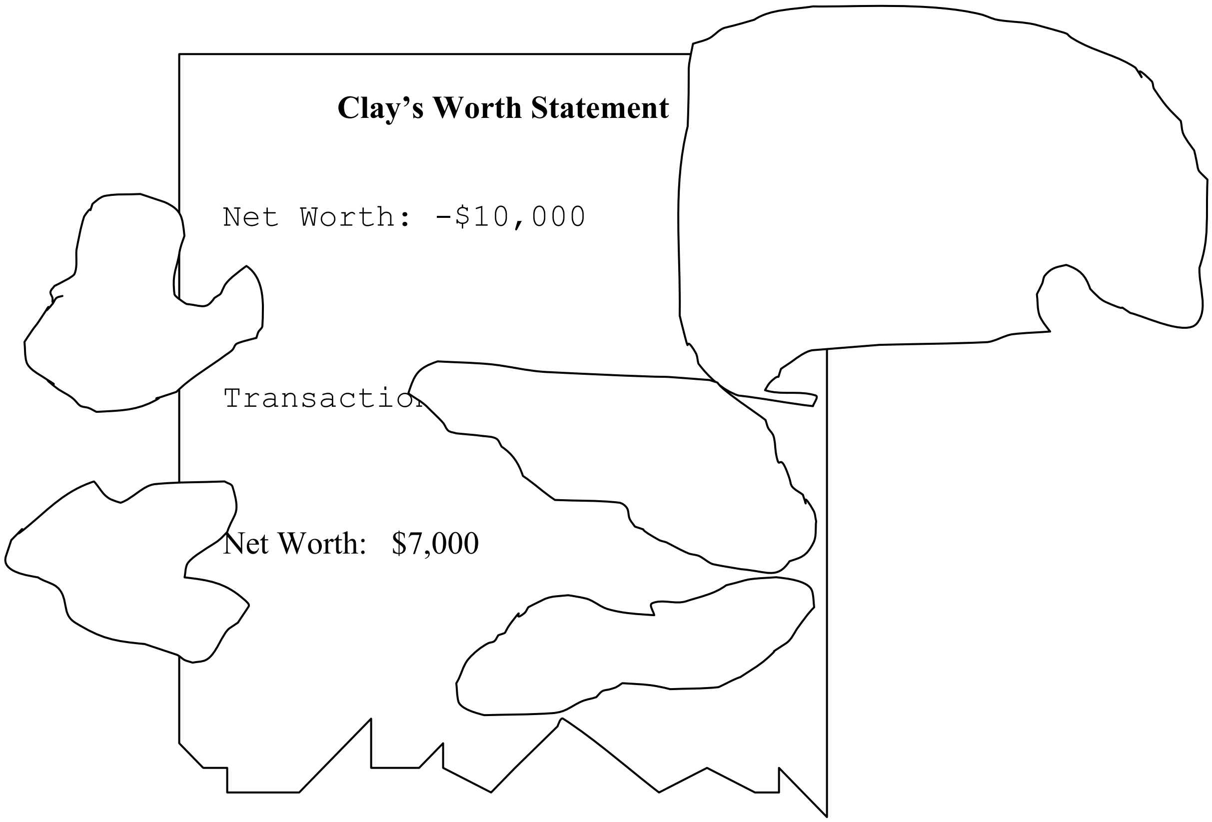
Somebody spilled coffee on Ruben's Net Worth Statement. He is trying to figure out what transaction took place to give him a new net worth of $7000. What could it have been? List as many as you can think of.

Oops! Coffee Spill!

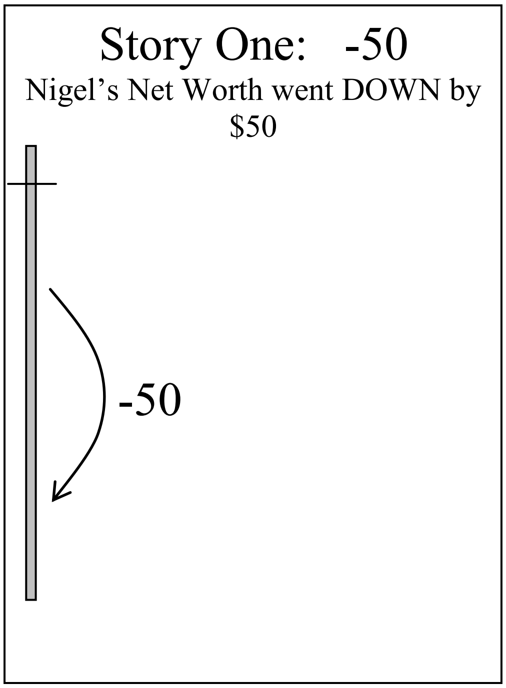


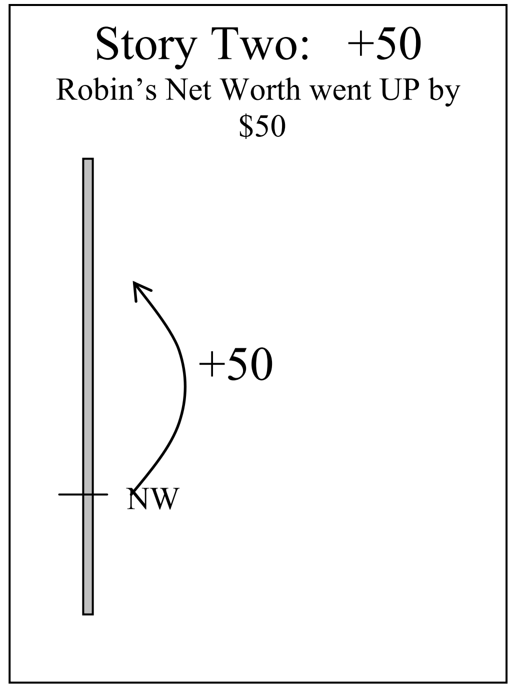
Somebody spilled coffee on Fantasia's Net Worth Statement. She is trying to figure out what transaction took place to give her a new net worth of $12,000. What could it have been? List as many as you can.

Oops! Coffee Spill!



Somebody spilled coffee on Clay's Net Worth Statement. He is trying to figure out what transaction took place to give him a new net worth of $7000. What could it have been? List as many as you can think of.





Determine which of the following transactions belong to story One or story Two.

+(-50)	-(-50)	-2(-25)	-(+50)
+(+50)	+25 + (+25)	+2(+25)	-25 -25
-2(+25)	-25 + 75	75 – 25	+2(-25)

For each problem below, state the person's beginning NET WORTH, whether the change is a good or bad change, and their new NET WORTH.

a. 17 + (-5)

b. -23 + (+11)

c. 250 + (-250)

d. 325 – (-100)

e. -117 + (-23)

f. -50 – (-50)

g. -154 + (-26)

h. 153 + (524)

i. 619 – (235)

Solve the following problems:

1. -45 + (-16)
2. -4 + 2 + (-5)
3. -5 + (-5)
4. 10 + 24 + (-12)
5. 22 – (-10)
6. -3 + 4 – (-23) –10

Solve the following problems:

1. 20 – 5
2. 18 + 7
3. -17 + 7
4. -20 + (-3)
5. 25 – (-10)
6. -25 – (-20)
7. -100 – 50
8. -45 + 70
9. 20 + (-35)
10. -45 + (-20)

Sam said, "*I think that **adding a debt** is the same thing as **taking away an asset**.*"
Sue said, "***Adding a debt** is totally different than **taking away an asset**.*"

Who do you agree with and why?

Dave said, "*I think that **taking away a debt** is the same as **adding an asset**.*"
Donna said, "*I think that **taking away a debt** is like **subtracting a debt**.*"
Drake said, "*I think that **two minuses** make a **plus**.*"

Who do you agree with and why?

Mrs. Robinson's class was given the following problem to solve: -2 - 7.

Chuck said, "*This problem doesn't make sense. You don't know if that – sign in front of the 7 is a **subtraction sign** or a **negative sign**.*"
Cherry said, "*I don't think there's a difference between a **subtraction sign** and a **negative sign**. They're the same thing.*"

Who do you agree with and why?

Ryan's Rules

+(-) means that your net worth goes down so +(-) is a -.
-(+) means that your net worth goes down so -(+) is a -.
-(-) means that your net worth goes up so -(-) is a +.
+(+) means that your net worth goes up so +(+) is a +.

What do Ryan's Rules mean? Do you agree? Explain.

REFERENCES

Aguirre, J., & Speer, N. (2000) Examining the relationships between beliefs and goals in teacher practice, *Journal of Mathematical Behavior*, *18*(3), 327–356.

Arcavi, A., Kessel, C., Meira, L., & Smith, J.P. (1998). Teaching mathematical problem solving: An analysis of an emergent classroom community. In A. Schoenfeld, J. Kaput, & E. Dubinsky (Eds.), *Research in collegiate mathematics education III* (pp. 1-70). Washington, DC: Conference Board of the Mathematical Sciences.

Ball, D. (1990). Reflections and deflections of policy: The case of Carol Turner. *Educational Evaluation and Policy Analysis, 12,* 241-245.

Ball, D. (1991). Research on teaching mathematics: Making subject matter knowledge part of the equation. In J. Brophy (Ed.), *Advances in research on teaching: Vol. 2, Teachers' subject matter knowledge and classroom instruction* (pp. 1-48). Greenwich, CT: JAI Press.

Ball, D. (1993). With an eye on the mathematical horizon: Dilemmas of teaching elementary school mathematics. *The Elementary School Journal*, *93*(4), 373-397.

Ball, D.L. (1994, November). *Developing mathematics reform: What don't we know about teacher learning--but would make good working hypotheses?* Paper presented at Conference on Teacher Enhancement in Mathematics K-6, Arlington, VA.

Ball, D. L., & Cohen, D. K. (1999). Developing practice, developing practitioners: Toward a practice-based theory of professional education. In L. Darling-Hammond and G. Sykes (Eds.), *Teaching as the learning profession* (pp. 3–31). San Francisco, CA: Jossey-Bass.

Ball, D. (2000). Working on the inside: Using one’s own practice as a site for studying teaching and learning. In R. Lesh&A. E. Kelly (Eds.), *Research on design in mathematics and science education* (pp. 365-402). Hillsdale, NJ: Lawrence Erlbaum Associates, Inc.

Bauersfeld, H., Krummheuer, G., & Voigt, J. (1988). Interactional theory of learning and teaching mathematics and related microethnographical studies. In H. G. Steiner & A. Vermandel (Eds.), *Foundations and Methodology of the Discipline Mathematics Education* (pp. 174-188). University of Antwerpen.

Bauwens, J., Hourcade, J. J., & Friend, M. (1989). Cooperative teaching: A model for general and spe- cial education integration. Remedial and Special Education, 10, 17-22.

Battista, M. T. (1983). A complete model for operations on integers. *Arithmetic Teacher*, 30(9), 26-31.

Blunk, M.L. (1998). Teacher Talk About How to Talk in Small Groups. In M. Lampert & M. L. Blunk (Eds.), *Talking mathematics in school: Studies of teaching and learning* (pp. 190–212). Cambridge, England: Cambridge University Press.

Bogdan, R. C. & Biklen, S. K. *Qualitative Research for Education: An Introduction to Theories and Methods.* (5th ed.) Needham Heights, Mass: Allyn & Bacon, 2007.

Borko, H., Eisenhart, M., Brown, C. A., Underhill, R. G., Jones, D., & Agard, P. C. (1992). Learning to teach hard mathematics: Do novice teachers and their instructors give up too easily? *Journal for Research in Mathematics Education, 23*, 194-222.

Borko, H., & Putnam, R. (1996). Learning to teach. In D. Berliner & R. Calfee (Eds.)*, Handbook of educational psychology* (pp. 673–708). New York: Macmillan.

Burden, P. (1981, November). Teacher's perceptions of their personal and professional development. Paper presented at the annual meeting of the Midwestern Educational Research Association, Des Moines, IA. (ERIC Document Reproduction Service No. ED 210258).

Calderhead, J. (1996). Teachers: Beliefs and knowledge. In D. Berliner & R. Calfee (Eds.), *Handbook of educational psychology* (pp. 709–725). New York: Macmillan.

Carpenter, T. P., Fennema, E., Peterson, P. L., Chiang, C., & Loef, M. (1989). Using knowledge of children's mathematical thinking in classroom teaching: An experimental study. *American Educational Research Journal*, *26*, 499–532.

Charmaz, K., & Mitchell, R. G. (2001). Grounded theory in ethnography. In: P. Atkinson, A. Coffey, & S. Delamont (Eds.), *Handbook of ethnography* (pp. 160–174). London: Sage.

Chazan, D., & Ball, D. L. (1999). Beyond being told not to tell. *For the Learning of Mathematics, 19*(2), 2–10

Clark, C.M., & Yinger, R.J. (1987). Teacher planning. In J. Calderhead (Ed.), *Exploring teachers' thinking* (pp. 84–103). London: Cassell Educational.

Clark, C., & Lampert, M., (1986). The study of teacher thinking: Implications for teacher education. *Journal of Teacher Education, 37*(5), 27–31.

Clements, D., & Sarama, J. (2004). Learning trajectories in mathematics education. *Mathematical Thinking and Learning, 6*(2), 81-90.

Cobb, P. (1988). The tensions between theories of learning and instruction in mathematics education. *Educational Psychologist, 23*, 78–103.

Cobb P., Yackel E., & Wood T. (1991).Curriculum and Teacher Development: Psychological and Anthropological Perspectives . In E. Fennema, T.P. Carpenter, & S.J. Lamon (Eds.), *Integrating research on teaching and learning mathematics* (pp. 83-119). Albany, NY: SUNY University Press.

Cobb, P., Wood, T., Yackel, E., Nicholls, J., Wheatley, G., Trigatti, B., & Perlwitz, M. (1991). Assessment of a problem-centered second grade mathematics project. *Journal for Research in Mathematics Education, 22*(1), 3-29.

Cobb, P., Yackel, E., & Wood, T. (1992). A constructivist alternative to the representational view of mind in mathematics education. *Journal for Research in Mathematics Education, 23,* 2-33.

Cobb, P., Wood, T., Yackel, E., & McNeal, B. (1992). Characteristics of classroom mathematics traditions: An interactional analysis. *American Educational Research Journal, 29*(3), 573-604.

Cobb, P., Wood, T., & Yackel, E. (1993). Discourse, mathematical thinking, and classroom practice. In N. Minick, E. Forman, & A. Stone (Eds.), *Education and mind: Institutional, social, and developmental processes* (pp. 91-119). New York: Oxford University Press.

Cobb, P., & Yackel, E. (1996). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. *Educational Psychologist*, *31*, 175-190.

Cobb, P., Boufi, A., McClain, K. and Whitenack, J. (1997) Reflective Discourse and Collective Reflection. *Journal for Research in Mathematics Education, 28*(3), 258–277.

Cobb, P. (2000). Conducting classroom teaching experiments in collaboration with teachers. In A. Kelly & R. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 307–334). Mahwah, NJ: Lawrence Erlbaum Associates, Inc.

Cobb, P., & McClain K. (2001). An Approach for Supporting Teachers' Learning in Social Context. In F. Lin & T. J. Cooney (Eds.), *Making Sense of Mathematics Teacher Education* (pp. 27-231). Netherlands: Kluwer Academic Publishers.

Cobb, P. (2002). Modeling, symbolizing, and tool use in statistical data analysis. In K. P. E. Gravemeijer, R. Lehrer, B. v. Oers & L. Verschaffel (Eds.), *Symbolizing, modeling and tool use in mathematics education* (pp. 171-195). Dordrecht: Kluwer Academic Publishers.

Cobb, P., diSessa, A., Lehrer, R., Schauble, L. (2003). Design experiments in educational research. *Educational Researcher, 32*(1), 9–13.

Cobb, P. (2003). Investigating students' reasoning about linear measurement as a paradigm case of design research. In M. Stephan, J. Bowers, P. Cobb & K. Gravemeijer (Eds.), *Supporting Students' Development of Measuring Conceptions: Analyzing Students' Learning in Social Context* (pp. 1-16). Reston, VA: National Council of Teachers of Mathematics.

Cohen, D. K. (1990). A revolution in one classroom: The case of Mrs. Oublier. *Educational Evaluation and Policy Analysis, 12*(3), 311–329.

Cook, L., & Friend, M. (1995). Co-teaching: Guidelines for creating effective practices. Teaching Exceptional Children, *28*, 1-16.

Cooney, T. J. (1985). A beginning teachers' view of problem solving. *Journal for Research in Mathematics Education*, *16*(5), 324-336.

Cooney, T.J., Shealy, B.E.& Arvold, B. (1998). Conceptualizing belief structures of preservice secondary mathematics teachers. *Journal for Research in Mathematics Education*, *29*, 306–333.

Cooney, T. (1999). Conceptualizing teachers' ways of knowing. Educational Studies in Mathematics, *38*, 163–187.

Dewey, J. (1933). *How we think: A restatement of the relation of reflective thinking to the educative process.* Lexington, MA: Heath.

Dieker, L. A., & Murawski, W. W. (2003). Coteaching at the secondary level: Unique issues, current trends, and suggestions for success. *The High School Journal, 86*, 1- 13.

Dixon, J. K., Andreasen, J. B. & Stephan, M. (fall 2009). Establishing social and sociomathematical norms in an undergraduate mathematics content course for prospective teachers: The role of the instructor. *AMTE Monograph VI: Scholarly practices and inquiry into the mathematics preparation of teachers.*

Ernest, P. (1989). The knowledge, beliefs and attitudes of the mathematics teacher: A model. *Journal of Education for Teaching, 15*, 13–34

Fennema, E., & Franke, M. L. (1992). Teachers' knowledge and its impact. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 147–164). New York: Macmillan.

Fennema, E., Franke, M. L., Carpenter, T. P., & Carey, D. A. (1993). Using children's mathematical knowledge in instruction. *American Educational Research Journal, 30,* 555–583.

Fennema, E., Carpenter, T. P., Franke, M. L., Levi, L., Jacobs, V. R., & Empson, S. B. (1996). A longitudinal study of learning to use children's thinking in mathematics instruction. *Journal for Research in Mathematics Education*, *27*, 403–434.

Fosnot, C.T., & Dolk, M. (2005). “Mathematics” or “mathematizing?”. In. C.T. Fosnot (Ed.). *Constructivism: Theory, perspectives and practice* (2nd ed.) (pp. 175-191). NY: Teacher’s College Press.

Franke, M., Kazemi, E., & Battey, D. (2007). Understanding teaching and classroom practice in mathematics. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 225–256). Charlotte, NC: Information Age Publishing and the National Council of Teachers of Mathematics.

Freudenthal, H.(1968). Why to teach mathematics so as to be useful?. *Educational Studies in Mathematics, 1*, 3–8.

Freudenthal, H. (1973). *Mathematics as an educational task.* Dordrecht, Holland: Reidel.

Fuchs, L.S., Fuchs, D., Hamlett, C.L., & Stecker, P.M. (1991). Effects of curriculum-based measurement and consultation on teacher planning and student achievement in mathematics operations. *American Educational Research Journal*, *28*, 617–641.

Fuson, K. C., Wearne, D., Hiebert, J. C., Murray, H. G., Human, P. G., Olivier, A. I., Carpenter, T. P., & Fennema, E. (1997). Children's conceptual structures for multidigit numbers and methods of multi- digit addition and subtraction. *Journal for Research in Mathematics Education*, *28*, 738-760.

Glaser, B., & Strauss, A. (1967). *The discovery of grounded theory*. Chicago: Aldine

Glaser, B. G. (1965). The constant comparative method of qualitative analysis. *Social Problems*, *12*, 436–445.

Goldsmith, L., & Schifter, D. (1997). Understanding teachers in transition: Characteristics of a model for developing teachers. In E. Fennema, & B. Scott Nelson (Eds.), *Mathematics teachers in transition* (pp. 19-54). New Jersey: Lawrence Erlbaum.

Gravemeijer, K. (1994). *Developing realistic mathematics education*. Utrecht, Netherlands: CD-Press.

Gravemeijer, K. (2002). Preamble: from models to modeling. In K. P. E. Gravemeijer, R. Lehrer, B. v. Oers & L. Verschaffel (Eds.), *Symbolizing, modeling and tool use in mathematics education* (pp. 7-22). Dordrecht: Kluwer Academic Publishers.

Gravemeijer, K & Stephan M. (2002). Emergent Models as an instructional design heuristic. In K. P. E. Gravemeijer, R. Lehrer, B. v. Oers & L. Verschaffel (Eds.), *Symbolizing, modeling and tool use in mathematics education* (pp. 145-169). Dordrecht: Kluwer Academic Publishers.

Gravemeijer, K. (2004). Local instruction theories as means of support for teachers in reform mathematics education. *Mathematical Thinking and Learning, 6*(2), 105-128.

Gravemeijer, K. (2007). *Emergent modeling and iterative processes of design and improvement in mathematics education*. Paper presented at Tsukuba International Conference (III), Tokyo, Kanazawa and Kyoto, Japan.

Green, T. (1971). The activities of teaching. New York: McGraw-Hill.

Grouws D.A. (1991). Improving Research in Mathematics Classroom Instruction. In E. Fennema, T.P. Carpenter, & S.J. Lamon (Eds.), *Integrating research on teaching and learning mathematics* (pp. 199-215). Albany, NY: SUNY University Press.

Guba, E.G., & Lincoln, Y.S. (1981). *Effective evaluation.* San Francisco: Jossey-Bass.

Hefendehl-Hebeker, L. (1991). Negative numbers: Obstacles in their evolution from intuitive to intellectual constructs. *For the Learning of Mathematics, 11*(1), 26–32.

Hiebert, J., & Wearne, D. (1993). Instructional tasks, classroom discourse, and students' learning in second-grade arithmetic. *American Educational Research Journal*, *30*, 393- 425.

Hiebert, J., Carpenter, T. P., Fennema, E., Fuson, K., Human, P., Murray, H., Olivier, A., & Wearne, D. (1996). Problem solving as a basis for reform in curriculum and instruction: The case of mathematics. *Educational Researcher, 25*(4), 12–21.

Hiebert, J. (2003). Signposts for teaching mathematics through problem solving. In F. K. Lester & R. Charles (Eds.), *Teaching mathematics through problem solving: Prekindergarten-grade 6* (pp. 53-61). Reston, VA: National Council of Teachers of Mathematics.

Hiebert, J., Stigler, J. W., Jacobs, J. K., Givvin, K. B., Garnier, H., Smith, M., et al. (2005). Mathematics teaching in the United States today (and tomorrow): Results from the TIMSS 1999 Video Study. *Educational Evaluation and Policy Analysis, 27,* 111–132.

Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal, 42*(2), 371-406.

Hill, H. C., Ball, D. L., & Schilling, S. G. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education, 39*(4), 372–400

Hufferd-Ackles, K., Fuson, K., & Sherin, M. (2004). Describing levels and components of a math-talk learning community. *Journal for Research in Mathematics Education, 25*(2), 81-116.

Hoyles, C. (1992). Mathematics teaching and mathematics teachers: A meta-case study. *For the Learning of Mathematics*, *12*(3), 32–44.

Kamii, C. (1982). *Number in preschool and kindergarten*. Washington, DC: National Association for the Education of Young Children.

Kaplan, R. G. (1991, October). *Teacher beliefs and practices: A square peg in a square hole*. Proceedings of the Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Blacksburg, VA.

Kilpatrick, J., Swafford, J., & Findell, B. (Eds.) (2001). *Adding it up: Helping children learn mathematics*. Washington, D.C.: National Academy Press.

Knuth, E., & Peressini, D. (2001). Unpacking the nature of discourse in mathematics. *Mathematic Teaching in the Middle School, 6*(5), 320-325.

Lampert, M. (1990). When the problem is not the question and the solution is not the answer: Mathematical knowing and teaching. *American Educational Research Journal*, *27*(1), 29-63.

Lampert M. (1991). Connecting Mathematical Teaching and Learning. In E. Fennema, T.P. Carpenter, & S.J. Lamon (Eds.), *Integrating research on teaching and learning mathematics* (pp. 121-152). Albany, NY: SUNY University Press.

Lampert, M. (2001). *Teaching problems and the problems of teaching*. New Haven: Yale University Press.

Lehrer, R., Strom, D., & Confrey, J. (2002). Grounding metaphors and inscriptional resonance: Children's emerging understanding of mathematical similarity. *Cognition & Instruction, 20* ,359–398.

Leinhardt, G., & Smith, D. (1985). Expertise in mathematics instruction: Subject matter knowledge. *Journal of Educational Psychology, 77*(3), 247-271.

Leinhardt, G., & Greeno, J. G. (1986). The cognitive skill of teaching. *Journal of Educational Psychology, 78,* 75-95.

Liebeck, P. (1990). Scores and forfeits: An intuitive model for integer arithmetic. *Educational Studies in Mathematics, 21*(3), 221–239.

Linchevski, L., & Williams, J. D. (1999). Using intuition from everyday life in "filling" the gap in children's extension of their number concept to include the negative numbers. *Educational Studies in Mathematics*, *39*, 131–147.

Lincoln, Y.S., & Guba, E.G. (1986). But is it rigorous? Trustworthiness and authenticity in naturalistic evaluation. New Directions for Program Evaluation, *30*, 73 -84.

Livingston, C., & Borko, H. (1989). Expert-novice differences in teaching: A cognitive analysis and implications for teacher education. *Journal for Research in Mathematics Education*, *21*, 372–387.

Livingston, C., & Borko, H. (1990). High school mathematics review lessons: Expert–novice distinction. *Journal for Research in Mathematics Education*, *21*, 372–387.

Lobato, J., Clark, D. J., & Ellis, A. B. (2005). Initiating and eliciting in teaching: A reformulation of telling. *Journal of Research in Mathematics Education, 36,* 101–136.

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States.* Mahwah, NJ: Lawrence Erlbaum.

Martino, A. M., & Maher, C. A. (1999). Teacher questioning to promote justification and generalization in mathematics: what research practice has taught us. *Journal of Mathematical Behavior*, *18*(1), 53–78.

McClain, K. (1995). *The teacher's proactive role in supporting students' mathematical growth.* Unpublished Dissertation, Vanderbilt University, Nashville, TN.

Meira, L. (1998). Making sense of instructional devices: The emergence of transparency in mathematical activity. *Journal for Research in Mathematics Education, 29,* 87-103.

Meira, L. (2002). Mathematical representations as systems of notations-in-use. In K. P. E. Gravemeijer, R. Lehrer, B. v. Oers & L. Verschaffel (Eds.), *Symbolizing, modeling and tool use in mathematics education* (pp. 171-195). Dordrecht: Kluwer Academic Publishers.

Merriam, S. B. (1988). *Case study research in education: A qualitative approach*. San Francisco: Jossey-Bass.

Merriam, S.B. (2009). *Qualitative Research: A Guide to Design and Implementation.* San Francisco: Jossey-Bass.

Murawski, W. W., & Dieker, L. A. (2004). Tips and strategies for co-teaching at the secondary level. *Teaching Exceptional Children*, *36*(5), 52–58.

National Mathematics Advisory Panel. (2008). *Foundations for success: The final report of the National Mathematics Advisory Panel.* Washington, DC: U.S. Department of Education.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics. (1991). *Professional standards for teaching mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics. (2006). *Curriculum focal points for prekindergarten through grade 8 mathematics: A quest for coherence*. Reston, VA: Author.

Peterson P.L, Marx R.W, and Clark C.M. (1978). Teacher planning, teacher behavior, and student achievement, *American Educational Research Journal*, *15*, 417-432.

Perry, M. (1991). Learning and transfer: Instructional conditions and conceptual change. *Cognitive Development*, *6*, 449–468.

Philipp R.A (2007). Mathematics Teachers' Beliefs and Affect. In F. Lester (Eds.), *Second handbook of research on mathematics teaching and learning* (pp. 257–315). Charlotte, NC: Information Age Publishing and the National Council of Teachers of Mathematics.

Pirie, S., & Kieren, T. (1989). A recursive theory of mathematical understanding. *For the Learning of Mathematics*, *9*(3), 7–11.

Rasmussen, C., Stephan, M., & Allen, K. (2004). Classroom mathematical practices and gesturing. *Journal of Mathematical Behavior*, *23*, 301–323.

Raymond, A. M. (1997). Inconsistency between a beginning elementary school teacher's mathematics beliefs and teaching practice. *Journal for Research in Mathematics Education, 28*, 550-576.

Rittenhouse, P. S. (1998). The teacher's role in mathematical conversation: Stepping in and stepping out. In M. Lampert & M. L. Blunk (Eds.), *Talking mathematics in school: Studies of teaching and learning* (pp. 163–189). Cambridge, England: Cambridge University Press.

Rittle-Johnson, B., & Alibali, M. W. (1999). Conceptual and procedural knowledge of mathematics: Does one lead to the other? *Journal of Educational Psychology*, *91*, 175–189.

Rittle-Johnson, B., Siegler, R. S., & Alibali, M. W. (2001). Developing conceptual understanding and procedural skill in mathematics: An iterative process. *Journal of Educational Psychology*, *93*, 346–362.

Saxe, G. B.., Gearhart, M., & Seltzer, M. (1999). Relations between classroom practices and student learning in the domain of fractions. *Cognition and Instruction, 17,* 1–24.

Schank, Roger C., & Abelson, Robert P. (1977). *Scripts, plans, goals, and understanding: An inquiry into the human knowledge structures.* Hillsdale: Lawrence Erlbaum Associates.

Schifter, D. (1996). *What's happening in math class?* New York: Teachers College Press.

Schoenfeld, A. (1992). Learning to think mathematically: Problem solving, metacognition and sense making in mathematics. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 334-370). New York: Macmillan.

Schoenfeld, A. H. (1998). Toward a theory of teaching-in-context. *Issues in Education*, *4*, 1-94.

Schoenfeld, A. H. (2000). Models of the Teaching Process. *Journal of Mathematical Behavior, 18(3)*, 243-261.

Schoenfeld A. H., Minstrell J., & van Zee E. (2000). The detailed analysis of an established teacher's non-traditional lesson. *Journal of Mathematical Behavior*, *18*(3), 281-325.

Schon, D. A. (1983*). The reflective practitioner: How professionals think in action*. New York: Basic Books.

Schon, D. A. (1987). *Educating the reflective practitioner: Toward a new design for teaching and learning in the professions*. San Francisco: Jossey-Bass.

Semadeni, Z. (1984). A principle of concretization permanence for the formation of arithmetical concepts. *Educational Studies in Mathematics, 15*(4), 379–395.

Shavelson. R.J. (1973).What is The Basic Teaching Skill? *The Journal of Teacher Education,* 144-151.

Shavelson R.J & Stern P. (1981). Research on teachers' pedagogical thoughts, decisions, and behavior, *Review of Educational Research*, *51*, 455-498.

Sherin,M.G. (2002)A balancing act: Developing a discourse community in a mathematics classroom', *Journal of Mathematics Teacher Education, 5*(3), 205–233.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, *15*(2), 4-14.

Simon, M. (1994). Learning mathematics and learning to teach: Learning cycles in mathematics teacher education. *Educational Studies in Mathematics*, *26*(1), 71-94.

Simon, M. A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education*, *26*(2), 114-145.

Simon, M.A. (1997). Developing new models of mathematics teaching: An imperative for research on mathematics teacher development. In E. Fennema & B. S. Nelson (Eds.), *Mathematics Teachers in Transition.* Hillsdale, NJ: Lawrence Erlbaum Associates.

Silver, E.A. & Smith, T (1996). Building discourse communities in mathematics classrooms: A worthwhile but challenging journey. In P. C. Elliott (Ed.), *1996 Yearbook: Communication in mathematics, K–12 and beyond* (pp. 20–28). Reston, VA: NCTM.

Skott, J. (2001). The emerging practices of a novice teacher: The roles of his school mathematics images. *Journal of Mathematics Teacher Education, 4*(1), 3–28.

Smith, J.P. (1995). *The effects of a computer micro-world on middle school students' use and understanding of integers.* Unpublished Dissertation, Ohio State University, Columbus, Ohio.

Smith, J.P. III (1996). Efficacy and teaching mathematics by telling: A challenge for reform. *Journal for Research in Mathematics Education*, *27*, 387–402.

Smith, M. S. (2000). Balancing old and new: An experienced middle school teacher's learning in the context of mathematics instructional reform. Elementary School Journal, *100*(4), 351–375.

Spillane, J. P., & Zeuli, J. S. (1999). Reform and teaching: Exploring patterns of practice in the context of national and state mathematics reforms. *Educational Evaluation and Policy Analysis, 21*(1), 1–27.

Stake, R.E. (1995*). Art of case study research: Perspectives on practice*. Thousand Oaks, CA: Sage Publications.

Steffe, L. P. (1991). The learning paradox: A plausible counterexample. In L. P. Steffe (Ed.), *Epistemological foundations of mathematical experience* (pp. 26-44). New York: Springer-Verlag

Steffe, L.P. & D'Ambrosio, B.S. (1995). Toward a working model of constructivist teaching: A reaction to Simon. *Journal for Research in Mathematics Education*, *26*, 114–145.

Stein, M. K., Grover, B. W., & Henningsen, M. (1996). Building student capacity for mathematical thinking and reasoning: An analysis of mathematical tasks used in reform classrooms. *American Educational Research Journal*, *33*(2), 455–488.

Stein, M. K., & Lane, S. (1996). Instructional tasks and the development of student capacity to think and reason: An analysis of the relationship between teaching and learning in a reform mathematics project .*Educational Research and Evaluation*, 2, 50-80.

Stephan, M. (2009). What are you worth*? Mathematics Teaching in Middle School, 15*(1), 16-23.

Stephan, M., Janet B., Paul C., & Gravemeijer K.(2003). *Supporting Students' Development of Measuring Concepts: Analyzing Students' Learning in Social Context.* Journal for Research in Mathematics Education Monograph, No. 12. Reston, VA: National Council of Teachers of Mathematics.

Stephan, M. L. (2003). Reconceptualizing Linear Measurement Studies: The Development of Three Monograph Themes. In M. L. Stephan, J. S. Bowers, P. Cobb & K. P. E. Gravemeijer (Eds.), *Supporting students' development of measuring conceptions: Analyzing students' learning in social context* (*Journal for Research in Mathematics Education*, Monograph number 12, pp. 17-35). Reston, VA: National Council of Teachers of Mathematics.

Stephan, M. L., & Cobb, P. (2003). The methodological approach to classroom-based research. In M. L. Stephan, J. S. Bowers, P. Cobb & K. P. E. Gravemeijer (Eds.), *Supporting students' development of measuring conceptions: Analyzing students' learning in social context* (*Journal for Research in Mathematics Education*, Monograph number 12, pp. 36-50). Reston, VA: National Council of Teachers of Mathematics.

Stephan, M.L., & Whitenack J.(2003). Establishing Classroom Social and Sociomathematical Norms for Problem Solving.In F. K. Lester & R. Charles (Eds.), *Teaching mathematics through problem solving: Prekindergarten-grade 6* (pp. 149-168). Reston, VA: National Council of Teachers of Mathematics.

Stigler, J. W., Fernandez, C., & Yoshida, M. (1996). Traditions of school mathematics in Japanese and American elementary classrooms. In L. P. Steffe & P. Nesher (Eds.), *Theories of mathematical learning* (pp. 149–175). Mahwah, NJ: Lawrence Erlbaum.

Stigler, J. W., & Hiebert, J. (1997). Understanding and improving classroom mathematics instruction: An overview of the TIMSS video study. *Phi Delta Kappan*, *79*(1), 14–21.

Strauss, A. & Corbin J. (1990). *Basics of Qualitative Research.* Newbury Park, CA: Sage.

Stigler, J. W., & Hiebert, J. (1999). *Teaching gap.* New York: Free Press.

Streefland, L. (1991). *Fractions in Realistic Mathematics Education: A paradigm of developmental research*. Dordrecht, Netherlands: Kluwer Academic Publishers.

Streefland, L. (1996). Negative numbers: Reflections of a learning researcher. *Journal of Mathematical Behavior, 15*(1), 57-79.

Taylor, S. J., & Bogdan, R. (1984). *Qualitative research methods: The search for meaning*. New York: John Wiley.

Thompson, A. (1984). The relationship of teachers' conceptions of mathematics and mathematics teaching to instructional practice. *Educational Studies in Mathematics*, *15*, 105-127.

Thompson, A.G. (1992). Teachers' beliefs and conceptions: A synthesis of the research. In D. Grouws (Ed.) *Handbook of research on mathematics teaching and learning* (pp. 127–146). NewYork: Macmillan.

Thompson, P. W., & Dreyfus, T. (1988). Integers as transformations. *Journal for Research in Mathematics Education*, *19*(2), 115-133.

Thompson, P. W. (1996). Imagery and the development of mathematical reasoning. In L. P. Steffe, P. Nesher, P. Cobb, G. A. Goldin & B. Greer (Eds.), *Theories of mathematical learning* (pp. 267–285). Mahwah, NJ: Erlbaum.

Thompson, A. G., & Thompson, P. W. (1996). Talking about rates conceptually, Part II: Mathematical knowledge for teaching. *Journal for Research in Mathematics Education, 27,* 2–24.

O'Connor, M.C. (1998). Language socialization in the mathematics classroom: Discourse practices and mathematical thinking. In M. Lampert and M. Blunk (Eds.), *Talking Mathematics, Studies of Teaching and Learning in School* (pp.17–55). Cambridge University Press, Cambridge, NY.

Peterson P.L., & Clark C.M (1978) Teachers' reports of their cognitive processes during teaching. *American Educational Research Journal*, *15*, 555-565.

von Glasersfeld, E. (1992). Constructivism reconstructed: A reply to Suchting. *Science and Education*, *1*, 379-384.

Webb, N. M., Nemer, K. M., & Ing, M. (2006). Small-group reflections: parallels between teacher discourse and student behavior in peer-directed groups. *The Journal of Learning Sciences*, *15*(1), 63–119.

Westerman, D. A. (1991). Expert and novice teacher decision making. *Journal of Teacher Education*, *42*(4), 292–305.

Wood, T. (1998). Alternative patterns of communication in mathematics classes: Funneling or focusing? In H. Steinbring, M. G. B. Bussi, & A. Sierpinska (Eds.), *Language and communication in the mathematics classroom* (pp. 167–178). Reston: National Council of Teachers of Mathematics.

Yackel, E. & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education*, 27(4), 458-477.

Yackel, E., Cobb, P., & Wood, T. (1991). Small-group interactions as a source of learning opportunities in second-grade mathematics. *Journal for Research in Mathematics Education, 22*, 390-408.

Yin, R.K. (1984). *Case Study Research, Design and Methods,* Sage Publications, Beverly Hills, California.