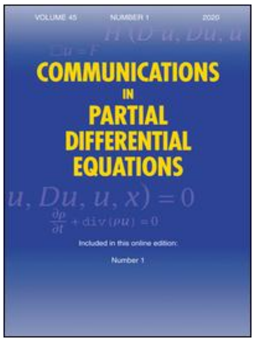

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Integro-Differential Harmonic Maps into Spheres

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For $s \in (0,1)$ we introduce (integro-differential) harmonic maps $v : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$, which are defined as critical points of the Gagliardo/Slobodeckij energy

$$\int_\Omega \int_\Omega \frac{|v(x)-v(y)|^p}{|x-y|^{n+sp}}\, dx\, dy,$$

with the condition that $v(\Omega) \subset \mathbb{S}^{N-1}$, for the $(N-1)$-sphere $\mathbb{S}^{N-1} \subset \mathbb{R}^N$. If $p=2$ these are the classical fractional harmonic maps first considered by Da Lio and Rivière. For $p \neq 2$ this is a new energy which has degenerate, non-local Euler-Lagrange equations. They are different from the n/p-harmonic maps introduced by Da Lio and the author, and have to be treated with new arguments, which might be of independent interest for further applications on geometric energies. The main result is Hölder continuity for these maps in the critical case $p = \frac{n}{s}$.

Keywords Harmonic maps; Nonlinear elliptic PDE; Regularity of solutions.

2010 Mathematics Subject Classfication 58E20; 35B65; 35J60; 35S05.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be an open domain. Initiated by the studies of Rivière and Da Lio [15] recent works [13, 14, 16, 36–38] explored regularity for critical points of the energy $\mathcal{F}_{s,p}$ acting on maps $v : \mathbb{R}^n \to \mathbb{R}^N$, $n, N \in \mathbb{N}$, for closed manifolds $\mathcal{N} \subset \mathbb{R}^N$,

$$\mathcal{F}_{s,p}(v) := \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} v|^p, \quad v(x) \in \mathcal{N} \text{ a.e. in } \Omega. \tag{1.1}$$

Here, the operator $(-\Delta)^{\frac{s}{2}} v$ is defined as a multiplier operator with symbol $|\xi|^s$, that is, denoting the Fourier transform and its inverse by $()^\wedge$ and $()^\vee$, respectively,

$$(-\Delta)^{\frac{s}{2}} v = (-|\xi|^s v^\wedge)^\vee .$$

For the definition of the fractional Laplacian $(-\Delta)^{\frac{s}{2}}$ as integral operator see (2.1).

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The energies $\mathcal{F}_{s,p}(\cdot)$ can be seen as non-local alternatives to the energy

$$\mathcal{D}_{1,p}(v) = \int_{\mathbb{R}^n} |\nabla v|^p, \quad v \in \mathcal{N} \subset \mathbb{R}^N \text{ a.e..}$$

The most classical situation is the Dirichlet energy $\mathcal{D}_{1,2}$. For $n = 2$, $p = 2$ the Dirichlet energy has a critical Euler-Lagrange equation: the scaling of the highest order terms and the scaling of the lower-order terms is the same, and it is not possible to use general regularity theory based just on the right-hand side growth. A finer analysis of the equation is necessary. In $\mathbb{R}^2$ Rivière's celebrated [34] showed for $\mathcal{D}_{1,2}$, and in fact for all conformally invariant variational functionals, that the Euler-Lagrange equations exhibit an antisymmetric structure. This again is then shown to be related to the applicability of integrability-by-compensation arguments which in turn induce regularity of the critical points. In $\mathbb{R}^n$, for the critical scaling $s = \frac{n}{2}$ and $p = 2$, this argument was extended to the energies $\mathcal{F}_{\frac{n}{2},2}$ in [13, 14].

If $p \neq 2$ much less is known. The critical scaling is $p = \frac{n}{s}$. In the subcritical setting, $p > \frac{n}{s}$ for $\mathcal{F}_{s,p}$ or $p > n$ for $\mathcal{D}_{1,p}$, Hölder regularity is immediate from the Sobolev embedding: indeed any map $v : \mathbb{R}^n \to \mathbb{R}^N$ with finite energy $\mathcal{F}_{s,p}(v) < \infty$, or $\mathcal{D}_{1,p}(v) < \infty$, is Hölder continuous if $p > \frac{n}{s}$, or $p > n$, respectively. In the supercritical case $p < \frac{n}{s}$ it is an interesting open question whether critical points exhibit any kind of (partial) regularity. However, the theory of classical harmonic maps, in particular [33] for $\mathcal{D}_{1,2}$ on $\mathbb{R}^n$, suggests that even partial regularity might fail for maps which are not minimizers or stationary.

For the critical scaling $p = \frac{n}{s}$, regularity theory is restricted to symmetric targets such as the sphere $\mathbb{S}^{N-1} \subset \mathbb{R}^N$ or compact Lie groups, see [20, 41, 46] for the energy $\mathcal{D}_{1,n}$, and [16] for $\mathcal{F}_{s,\frac{n}{s}}$. The main difficulty is that in the resulting Euler-Lagrange equations the leading order differential operator is degenerate, which heavily complicates the arguments related to possible effects of integrability by compensation. In this article we consider for $s \in (0, 1)$ critical points of the energy

$$\mathcal{E}_{s,p}(v) := \int_\Omega \int_\Omega \frac{|v(x) - v(y)|^p}{|x - y|^{n+sp}}\, dx\, dy, \quad v(x) \in \mathbb{S}^{N-1} \text{ a.e. in } \Omega. \tag{1.2}$$

Maps with finite energy $\mathcal{E}_{s,p}(v) < \infty$ belong to the homogeneous fractional Sobolev space $\dot{W}^{s,p}(\Omega)$, endowed with the seminorm

$$[f]_{\dot{W}^{s,p}(\Omega)} \equiv [f]_{s,p,\Omega} := \left(\int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x - y|^{n+sp}}\, dx\, dy \right)^{\frac{1}{p}}.$$

For $p = 2$ and $\Omega = \mathbb{R}^n$, the energies $\mathcal{E}_{s,p}$ and $\mathcal{F}_{s,p}$ coincide. But for $p \neq 2$ they are different: Indeed, while $\mathcal{F}_{s,p}(v)$ is finite if and only if

$$\|(-\Delta)^{\frac{s}{2}} v\|_p < \infty,$$

the energy $\mathcal{E}_{s,p}(v)$ is finite if and only if $(-\Delta)^{\frac{s}{2}} v \in \dot{F}^0_{p,p}$, where $\dot{B}^0_{p,p} = \dot{F}^0_{p,p}$ is the homogeneous Besov/Triebel-Lizorkin space, cf. [24, 47]. The latter space is difficult to handle: just to give a flavor of the problems one runs into, it is in general not true that $\dot{F}^0_{p,p} \subset L^1_{loc}$.

The following is our main result, which holds for *any* open, bounded or unbounded, domain $\Omega \subset \mathbb{R}^n$.

Theorem 1.1. *Assume that $p = \frac{n}{s}$ and $u : \Omega \to \mathbb{S}^{N-1}$ is a critical point of* (1.2), *i.e. for any $\psi \in C_0^\infty(\Omega, \mathbb{R}^N)$*

$$0 = \frac{d}{dt}\Big|_{t=0} \mathscr{E}_{s,\frac{n}{s}}\left(\frac{u + t\psi}{|u + t\psi|}\right). \tag{1.3}$$

Then u is locally Hölder continuous in Ω. That is, for any compact $K \subset \Omega$ there is a Hölder exponent $\alpha > 0$ so that $u \in C^{0,\alpha}(K)$.

As in the case of $\mathscr{F}_{s,p}$ we focus on the situation of $p = \frac{n}{s}$. Again if $p > \frac{n}{s}$ then $\mathscr{E}_{s,p}(u) < \infty$ implies Hölder continuity via Sobolev embedding, and if $p < \frac{n}{s}$ it is dubious whether one can expect any kind of regularity.

For $s \to 1$ and $\Omega = \mathbb{R}^n$ the rescaled energy $(1-s)^{\frac{1}{p}} \mathscr{E}_{s,p}(v)$ converges to $\mathscr{D}_{1,p}(v) = \int_{\mathbb{R}^n} |\nabla u|^p$, see [10]. Our arguments, however, seem to rely heavily on $s < 1$. Only in a very formal way one can see how our techniques reduce to arguments in [40] as $s \to 1$.

Once the initial regularity is obtained from Theorem 1.1, higher regularity is a further interesting open problem: From classical analogues in the regularity theory of elliptic systems, cf. [45], one might suspect that from a general growth argument *continuous* solutions to (1.3) are as smooth as solutions $v : \Omega \to \mathbb{R}$ to the scalar problem

$$0 = \frac{d}{dt}\Big|_{t=0} \mathscr{E}_{s,p}(v + t\psi) \quad \forall \psi \in C_0^\infty(\Omega). \tag{1.4}$$

This scalar equation has recently been studied in [17], and they found results on Harnack's inequality and Hölder regularity. We refer also to [2, 26]. But unless $p = 2$, it is unknown whether solutions to (1.4) are more regular. As mentioned above, up to rescaling the energy $\mathscr{E}_{s,p}$ approximates $\int |\nabla v|^p$ for $s \to 1$, and there are continuous, but non-smooth solutions to the p-Laplace equation

$$0 = \frac{d}{dt}\Big|_{t=0} \int |\nabla(v + t\psi)|^p = \operatorname{div}(|\nabla u|^{p-2} \nabla u)[\psi].$$

So for now it is unclear how much of higher regularity to expect for solutions to (1.4), and thus for our fractional harmonic maps from Theorem 1.1.

Before we remark on some ingredients of the proof, let us first mention a motivation for considering $\mathscr{E}_{s,p}$: this energy is closely related to curvature energies of knots and surfaces.

Let $\gamma : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^N$ be a knot, i.e. an embedded curve. The so-called *Möbius energy* introduced by O'hara in [30] is given by

$$\mathscr{M}_2(\gamma) := \int_{\mathbb{R}/\mathbb{Z}} \int_{\mathbb{R}/\mathbb{Z}} \left(\frac{1}{|\gamma(x) - \gamma(y)|^2} - \frac{1}{d_\gamma(x,y)^2}\right) |\gamma'(x)|\,|\gamma'(y)|\; dx\, dy.$$

Here $d_\gamma(x, y)$ denotes the intrinsic distance between $\gamma(x)$ and $\gamma(y)$ on the curve γ. The factors $|\gamma'(x)||\gamma'(y)|$ guarantee invariance under reparametrization. In particular we shall assume w.l.o.g. $|\gamma'| \equiv 1$.

In [19] Freedman and colleagues discovered that $\mathscr{M}_2$ is invariant under Möbius transformations, and using this they showed that *minimizers* of $\mathscr{M}_2$ are smooth. In [8]

Blatt, Reiter and the author gave a new regularity proof and extended this to critical points of $\mathcal{M}_2$. The key observation is that critical points of the Möbius energy are at least philosophically related to fractional harmonic maps into spheres. Indeed, $\mathcal{M}_2(\gamma) < \infty$ is equivalent to $\mathcal{E}_{\frac{1}{2},2}(\gamma') < \infty$ for $\Omega := \mathbb{R}/\mathbb{Z}$, and the condition $|\gamma'| \equiv 1$ is the same as saying $\gamma' \in \mathbb{S}^{N-1}$. Thus, if γ is a critical point of the Möbius energy $\mathcal{M}_2$, then at least formally its derivative γ' exhibits features of a critical point to $\mathcal{E}_{\frac{1}{2},2}$ – and indeed regularity then follows by extending the arguments developed in the theory of fractional harmonic maps into spheres [14, 15, 36].

Considering other curvature energies, such as general O'Hara knot energies [30], generalized versions of the tangent-point energy [6] and Menger curvature [23], one observes that these energies are also related to $\mathcal{E}_{s,p}$ (1.2) for some s and some p with possibly $p \neq 2$, [3–5, 25]. For several of these curvature energies, regularity even of minimizers is not understood, and arguments as in [19] seem not to work, since also the invariance class is not known. See [7, 9, 31, 32]. Thus, the present work is also intended to deliver a framework which hopefully will lead to substantial progress in this area.

Let us now focus on the ingredients of the proof of Theorem 1.1. The Euler-Lagrange equations of $\mathcal{E}_{s,p}$ take the form

Proposition 1.2 (Euler-Lagrange Equations). *Any critical point as in Theorem* 1.1 *satisfies*

$$\omega_{ij} \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^{p-2}(u^i(x)-u^i(y))(u^j(x)\varphi(x)-u^j(y)\varphi(y))}{|x-y|^{n+sp}}\, dx\, dy = 0 \quad (1.5)$$

for any $\varphi \in C_0^\infty(\Omega)$ *and any constant* $\omega_{ij} = -\omega_{ji} \in \{-1, 0, 1\}$. *Here and henceforth we use Einstein's summation convention.*

Consequently, Theorem 1.1 can be rewritten as

Theorem 1.3. *Assume that* $u : \Omega \to \mathbb{R}^N$, $|u| \equiv 1$ *on* Ω *and* u *is a solution to the integro-differential equation* (1.5). *Then* u *is locally Hölder continuous in* Ω.

One important tool in the study of $\mathcal{E}_{s,p}$ [13–15, 36, 37] are estimates on three-commutators

$$H_\alpha(f,g) := (-\Delta)^{\frac{\alpha}{2}}(fg) - f(-\Delta)^{\frac{\alpha}{2}}g - g(-\Delta)^{\frac{\alpha}{2}}f, \quad (1.6)$$

first introduced in [14, 15], see Theorem A.1. In some sense H_α measures how far away the differential operator $(-\Delta)^{\frac{\alpha}{2}}$ is from having a product rule. The intuition for H_α should come from classical operators $\alpha \in 2\mathbb{N}$, e.g.,

$$H_2(f,g) = 2\nabla f \cdot \nabla g :$$

The important observation for H_2 and the main ingredient for estimates on three-commutators H_α as in Theorem A.1 is that H_α behaves like a product of two differential operators of order less than α applied to f and g, respectively.

This intuition, which leads to pointwise estimates for H_α, [37], needs to be extended to our nonlinear situation:

Theorem 1.4 (Commutator Estimates). *Fix $s \in (0,1)$ and let $f, g, h \in C_0^\infty(\mathbb{R}^n)$. For all $t < s$ large enough, let for*

$$\Upsilon_1(z) := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^{\frac{n}{s}-1}\ |\Gamma(x, y, z)|}{|x - y|^{n+s\frac{n}{s}}} dx\, dy,$$

where

$$\Gamma(x, y, z) = |g(x) + g(y) - 2g(z)|\ \| x - z|^{t-n} - |y - z|^{t-n}|.$$

Then

$$\|\Upsilon_1\|_{\frac{n}{n-t}} \precsim [f]_{s,\frac{n}{s},\mathbb{R}^n}^{\frac{n}{s}-1}\ [g]_{s,\frac{n}{s},\mathbb{R}^n}$$

Moreover let

$$\Upsilon_2 := \int_{B_\rho} \int_{B_\rho} \frac{|f(x) - f(y)|^{\frac{n}{s}-1}\ |\Theta(x, y)|}{|x - y|^{n+s\frac{n}{s}}} dx\, dy,$$

where, I^t is the Riesz-Potential, see (2.2).

$$\Theta(x, y) = I^t(g(-\Delta)^{\frac{t}{2}} h)(x) - I^t(g(-\Delta)^{\frac{t}{2}} h)(y) - \frac{1}{2}(h(x) - h(y))(g(x) + g(y)).$$

Then

$$\Upsilon_2 \precsim \|(-\Delta)^{\frac{t}{2}} g\|_{\frac{n}{t}} [f]_{s,\frac{n}{s},\mathbb{R}^n}^{\frac{n}{s}-1} [h]_{s,\frac{n}{s},\mathbb{R}^n}.$$

We prove a localized version of Theorem 1.4 in Lemma 6.5 and Lemma 6.6.

Both, bilinear and non-linear commutator estimates share similar features to the so-called "integration by compensation" arguments leading to higher integrability of Jacobians [12, 29] and Wente's inequality [11, 42, 48]. We will shed more light on this relation in the outline of the proof of Theorem 1.3, Section 3, after Lemma 3.4.

Another important tool is the Sobolev inequality: We will use it in the following form, which comes from the boundedness of the Riesz potential I^t from $L^p(\mathbb{R}^n)$ into $L^{\frac{np}{n-tp}}(\mathbb{R}^n)$.

Theorem 1.5 (Classical Sobolev Inequality). *For any $s \geq t \geq 0$, $p \in (1, \frac{n}{s-t})$, setting $p_{s,t}{}^* = \frac{np}{n-(s-t)p}$ we have for any $f \in C_0^\infty(\mathbb{R}^n)$*

$$\|(-\Delta)^{\frac{t}{2}} f\|_{p_{s,t}{}^*,\mathbb{R}^n} \precsim \|(-\Delta)^{\frac{s}{2}} f\|_{p,\mathbb{R}^n},$$

or in other words

$$\|I^{s-t} g\|_{p_{s,t}{}^*,\mathbb{R}^n} \precsim \|g\|_{p,\mathbb{R}^n},$$

We will also use a refined and not so well known version of Theorem 1.5. It is adapted to our Gagliardo seminorms, and might be useful for other problems, too.

Theorem 1.6 (Sobolev Inequality). *For any* $s > t \geq 0$, $p \in (1, \frac{n}{s-t})$, *setting* ${p_{s,t}}^* = \frac{np}{n-(s-t)p}$ *we have for any* $f \in C_0^\infty(\mathbb{R}^n)$

$$\|(-\Delta)^{\frac{t}{2}} f\|_{p^*_{s,t},\mathbb{R}^n} \precsim \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}} \, dz \, dy \right)^{\frac{1}{p}}.$$

Theorem 1.6 follows from the Sobolev embedding for Triebel spaces $\dot{F}^s_{p,q}$, namely

$$\|f\|_{\dot{F}^t_{p^*_{s,t},2}} \precsim \|f\|_{\dot{F}^s_{p,p}}.$$

For the theory of Triebel- and Besov spaces we refer to [24]. To the best of our knowledge it was first proved in [27], see also the presentation in [47, Theorem 2.71]. The proof for our special situation simplifies, see [39]. It is important to remark that the last inequality and thus Theorem 1.6 is false for $s = t$ if $p > 2$. In particular, the constants of Theorem 1.6 will blow up as $t \to s$, in contrast to Theorem 1.5 which also holds for $s = t$.

The paper is organized as follows: First we introduce some notation and conventions in the next Section 2. In Section 3 we state the main decay estimate from which Theorem 1.3 follows, namely Lemma 3.1. Before proving Lemma 3.1 we first give an outline which explains the main steps and ideas of the proof. The remaining part of the paper are then the proofs of several steps of Lemma 3.1: Section 4 treats what would be left-hand side estimates, giving the precise relation between the localized Gagliardo-seminorm, and the operator $T_{s,B}$. In Section 5 we give the details on the tangential estimate of $T_{s,B}$. In Section 6 we prove Theorem 1.4, in Lemma 6.5 and Lemma 6.6.

2. Preliminaries and Notation

2.1. *Norms*

For any measurable set $B \subset \mathbb{R}^n$, $t \in (0, 1)$, $p \in (1, \infty)$ recall the Gagliardo or Slobodeckij semi-norm,

$$[f]_{t,p,B} := \int_B \int_B \frac{|f(x) - f(y)|^p}{|x-y|^{n+pt}} \, dx \, dy.$$

The set of mappings $f : B \to \mathbb{R}$ with finite Gagliardo-norm $[f]_{t,p,B}$ is the homogeneous Sobolev space $\dot{W}^{s,p}(B)$. This space is equivalent to the homogeneous Triebel-space $\dot{F}^s_{p,p}(\mathbb{R}^n)$, [24, 47]. Apart from the special Sobolev inequality, Theorem 1.6, we will make no use of this identification.

We denote the L^p norms with

$$\|f\|_{p,B} := \|f\|_{L^p(B)}.$$

When we drop the set B, we mean integration over $\mathbb{R}^n$, i.e.,

$$[f]_{t,p} := [f]_{t,p,\mathbb{R}^n}, \quad \|f\|_p := \|f\|_{p,\mathbb{R}^n}.$$

From now on,

$$p_t = \frac{n}{t}.$$

Operators. The fractional Laplacian $(-\Delta)^{\frac{t}{2}} f$ for $t \in (0,1)$ and f in the Schwartz class $\mathscr{S}(\mathbb{R}^n)$ is given by

$$(-\Delta)^{\frac{t}{2}} f(x) = c_t \int_{\mathbb{R}^n} \frac{f(y) - f(x)}{|x-y|^{n+t}}\, dy. \tag{2.1}$$

The inverse of $(-\Delta)^{\frac{t}{2}}$, the Riesz potential, is denoted by I^t, and is for $t \in (0,n)$ given by

$$I^t F(x) = \tilde{c}_t \int_{\mathbb{R}^n} |y-x|^{t-n} F(y) dy. \tag{2.2}$$

For more details and arguments on these operators and related norms, we refer to, e.g., [18, 28, 35, 43, 44]. For a ball B, a map $u : B \to \mathbb{R}^N$, and $t > 1 - (1-s)p_s$, we denote

$$(T_{t,B}u)^i(z) := \int_B \int_B \frac{|u(x)-u(y)|^{p_s-2}(u^i(x)-u^i(y))\ (|x-z|^{t-n} - |y-z|^{t-n})}{|x-y|^{n+sp}}\, dx\ dy. \tag{2.3}$$

The lower bound on t guarantees the convergence of these integrals at least for Schwartz functions $u \in \mathscr{S}(\mathbb{R}^n, \mathbb{R}^N)$. Note that $1-(1-s)p_s < s$. That is, it is always possible to find $t < s$ arbitrarily close to s for which $T_{t,B}u$ is well-defined.

The main motivation for $T_{t,B}u$ is the following identity, which holds for any $\varphi \in C_0^\infty(\mathbb{R}^n)$

$$\begin{aligned}&\int_{\mathbb{R}^n} (-\Delta)^{\frac{t}{2}} \varphi(z)\ T_{t,B}u^i(z)\ dz \\ &\quad = c \int_B \int_B \frac{|u(x)-u(y)|^{p_s-2}(u^i(x)-u^i(y))\ (\varphi(x)-\varphi(y))}{|x-y|^{n+sp}}\, dx\ dy.\end{aligned} \tag{2.4}$$

The above representation follows by replacing $\varphi(x)$ and $\varphi(y)$ by, cf. (2.2),

$$\varphi(x) = c \int_{\mathbb{R}^n} |x-z|^{t-n}\ (-\Delta)^{\frac{t}{2}} \varphi(z) dz,$$

and then using Fubini's theorem.

Also note that for any $\tilde{t} > 0$,

$$I^{\tilde{t}} T_{t,B}u^i(z) = T_{t+\tilde{t},B}u^i(z). \tag{2.5}$$

Cutoff Functions. We denote the open ball centered at x_0 with radius R as $B_R(x_0)$. We will denote with $A_R(x_0)$ the annulus $A_R(x_0) = B_R(x_0) \backslash B_{\frac{R}{2}}(x_0)$. Often we will drop the center of the balls and annuli, and all balls/annuli we use will be concentric.

On balls and annuli we use two types of cutoff functions: For any measurable set C we denote with χ_C the characteristic function on C.

We will also need smooth cutoff functions, which we denote with η_{B_R}:

$$\eta_{B_R} \in C_0^\infty(B_{2R}(x_0)), \quad \eta_{B_R} \equiv 1 \text{ on } B_R(x_0) \quad \forall i \in \mathbb{N}: \quad \|\nabla^i \eta_{B_R}\|_\infty \precsim R^{-i},$$

and

$$\eta_{A_R} := \eta_{B_R} - \eta_{B_{\frac{R}{2}}}.$$

We will denote the mean value

$$(f)_{B_R} := |B_R|^{-1} \int_{B_R} f.$$

3. Proof of Theorem 1.3

Here we present the proof of Theorem 1.3, which relies on a decay estimate of the Gagliardo-norm on small balls, Lemma 3.1. In Section 3.1 we state the decay estimate and show how Theorem 1.3 follows.

Since the proof of Lemma 3.1 is quite technical, we first present in Section 3.2 the outline and main steps behind the proof, before giving the precise estimates in Section 3.3.

3.1. *The Decay Estimate: Proof of the Theorem*

Theorem 1.3 is a consequence of the following decay estimate. Recall the definition of the $\dot{W}^{s,p}$-seminorm $[u]_{s,p,B}$ from Section 2.

Lemma 3.1. *Let u be as in Theorem* 1.3 *and $K \subset \Omega$ be compact. Then there exist $\tau \in (0, 1)$, $\sigma > 0$, $L_0 \in \mathbb{N}$, $C > 0$, $\rho_0 > 0$, such that for any $B_\rho(x_0) \subset K$, $\rho < \rho_0$ and any $L \geq L_0$ such that $B_{2^L \rho}(x_0) \subset \Omega$, we have for $p_s = \frac{n}{s}$*

$$[u]^{p_s}_{s,p_s,B_\rho(x_0)} \leq \tau [u]^{p_s}_{s,p_s,B_{2^L\rho}(x_0)} + C \sum_{l=1}^{\infty} 2^{-\sigma(L+l)} [\tilde{u}]^{p_s}_{s,p_s,B_{2^{L+l}\rho}(x_0)}.$$

Here, $\tilde{u}$ is an extension of u from K to $\mathbb{R}^n$.

From Lemma 3.1 one can obtain Theorem 1.3 as follows:

Proof of Theorem 1.3. The crucial point is that in Lemma 3.1 $\tau < 1$. We employ an iteration argument, see [21, Chapter III, Lemma 2.1] and also the presentation in [8, 14, 15], and from $\tau < 1$ we obtain $\theta > 0$ so that

$$[u]^{p_s}_{s,p_s,B_r} \leq C_u\, r^\theta \quad \text{for all } B_r \subset K. \tag{3.1}$$

Now we use Jensen's inequality and the fact that $|x-y|^{n+sp_s} \precsim r^{2n}$ for $x, y \in B_r$, the latter because $p_s = \frac{n}{s}$. Recall also that $(f)_{B_r}$ is the mean value of f in B_r, then for any $B_r \subset K$

$$\begin{aligned} r^{-\theta-n} \int_{B_r} |u(x) - (u)_{B_r}|^{p_s}\, dx &\precsim r^{-\theta-2n} \int_{B_r} \int_{B_r} |u(x) - u(y)|^{p_s} dx\, dy \\ &\precsim r^{-\theta} \int_{B_r} \int_{B_r} \frac{|u(x)-u(y)|^{p_s}}{|x-y|^{n+sp_s}} dx\, dy = r^{-\theta} [u]^{p_s}_{s,p_s,B_r} \overset{(3.1)}{\leq} C. \end{aligned}$$

This implies that u belongs to the Campanato space $\mathscr{L}^{p,n+\theta}(K)$ which is isomorphic to the space $C^{0,\frac{\theta}{p}}(K)$, see, e.g., [21, Chapter III, Theorem 2.1] or [22, Theorem 5.5]. Alternatively, regularity also follows from (3.1) via the general theory on Riesz potentials on Sobolev-Morrey spaces, cf. [1, Theorem 3.1 and Corollary after Proposition 3.4]. This concludes the proof of Theorem 1.3 assuming Lemma 3.1. □

3.2. *Outline of the Proof of Lemma 3.1*

In this section we highlight the main arguments in the proof of Lemma 3.1. The precise proof is given in Section 3.3. As usual with arguments involving non-local operators, the precise estimates contain tails, i.e. well-behaved terms which have to be carried through the computations. Since those tails sometimes cloud the main arguments, in this outline we hide them behind the label "...".

In this spirit, we shall establish the main ideas of Lemma 3.1 by explaining the following inequality

$$[u]_{s,p_s,B_\rho} \leq \tau[u]_{s,p_s,B_{5\rho}} + \cdots , \tag{3.2}$$

for some $\tau < 1$ and a small ball B_ρ.

Step 1: Left-Hand Side Estimates: Lemma 3.2. The first step in obtaining (3.2) is to estimate the Gagliardo norm $[u]_{s,p_s,B_\rho}$ in terms of the operator $T_{t,B}u$ defined in Section 2. This estimate is the first crucial idea, since it later allows us to split the estimate of Gagliardo norm into two parts – firstly the projection of $T_{t,B}u$ in the linear space spanned by u and secondly the projection of $T_{t,B}u$ into the linear space orthogonal to u (See the next step).

Lemma 3.2. *For any $\varepsilon > 0$, $K, L \in \mathbb{N}$, $0 < t < s$, and $p_t = \frac{n}{t}$, there are constants $C, \sigma > 0$ depending only on s, t and the dimension, and C_ε depending on ε, such that*

$$\begin{aligned}[u]^{p_s}_{s,p_s,B_\rho} \leq{} & (\varepsilon + C\, 2^{-K\sigma})\, [u]^{p_s}_{s,p_s,B_{2^L\rho}} \\ & + C_\varepsilon \left([u]^{p_s}_{s,p_s,B_{2^L\rho}} - [u]^{p_s}_{s,p_s,B_\rho}\right) \\ & + C[u]_{s,p_s,B_\rho}\, \|\chi_{B_{2^K\rho}}(z) T_{t,B_{2^L\rho}} u\|_{p_t'}.\end{aligned}$$

Recall that p' always denotes the Hölder dual $p' = \frac{p}{p-1}$.

For the proof of Lemma 3.2 we refer to Section 4, but let us give an idea for such an estimate: let us motivate why the following estimate holds.

$$[u]^{p_s-1}_{s,p_s,B_\rho} \precsim \|T_{t,B_{4\rho}} u\|_{p_t',B_{3\rho}} + \ldots . \tag{3.3}$$

To obtain (3.3), we observe that for some smooth φ, compactly supported in, say, $B_{2\rho}$, $[\varphi]_{s,p_s,\mathbb{R}^n} \leq 1$,

$$[u]^{p_s-1}_{s,p_s,B_\rho} \precsim \int_{B_{4\rho}} \int_{B_{4\rho}} \frac{|u(x)-u(y)|^{p_s-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+sp_s}} dxdy + \ldots . \tag{3.4}$$

This follows from an adaption of the usual arguments of taking the test function φ to be essentially $\eta_{B_\rho}(u-(u)_{B_{2\rho}})$, see Lemma 4.1.

In the case $p_s = 2$, the double-integral term on the right-hand side of (3.4) essentially corresponds to

$$\int (-\Delta)^{\frac{s}{2}} u \cdot (-\Delta)^{\frac{s}{2}} \varphi + \cdots .$$

For $p_s \neq 2$ we still would like to use this kind of distributional representation, i.e. a weak equation tested by a (pseudo-)derivative of φ. For this let $T_{t,B_{4\rho}} u$ be defined as in (2.3) and use (2.4). Then it is established that for some smooth φ compactly supported in $B_{2\rho}$ and $[\varphi]_{s,p_s,\mathbb{R}^n} \leq 1$, and all $t \leq s$ sufficiently close to s,

$$\begin{aligned} [u]^{p_s-1}_{s,p_s,B_\rho} &\precsim \int_{\mathbb{R}^n} (-\Delta)^{\frac{t}{2}} \varphi(z)\ T_{t,B_{4\rho}} u(z) + \cdots . \\ &\precsim \|(-\Delta)^{\frac{t}{2}} \varphi\|_{p_t,B_{3\rho}}\ \|T_{t,B_{4\rho}} u\|_{p_t',B_{3\rho}} + \cdots , \end{aligned} \tag{3.5}$$

by Hölder's inequality. Recall that $p_t = \frac{n}{t}$. In the last step we also used the following intuition: Since the support of φ is contained in $B_{2\rho}$, the support of $(-\Delta)^{\frac{t}{2}} \varphi$ is *essentially* contained in $B_{3\rho}$; that is to say, the estimates $(-\Delta)^{\frac{t}{2}} \varphi$ in $\mathbb{R}^n \backslash B_{3\rho}$ belong to the well-behaved terms "...".

The above estimate holds for any $t \leq s$ sufficiently close to s. To reach the conclusion (3.3), one now would like to estimate

$$\|(-\Delta)^{\frac{t}{2}} \varphi\|_{p_t,B_{3\rho}} \precsim [\varphi]_{s,p_s,\mathbb{R}^n} \leq 1.$$

If $p_s \leq 2$ the last estimate is true for the natural choice $t := s$. But in the case $p_s > 2$, this is false for $t = s$. We have to choose $t < s$, for which we can apply Theorem 1.6. This establishes (3.3). Note that since in general we are forced to pick $t < s$, the operator $T_{t,B}$ is more singular than $T_{s,B}$, and one crucial strength of our arguments below is that they can treat even this increased singular setting.

Step 2: Splitting Argument. In the last part of the proof, we went to great lengths to make the operator $T_{t,B} u$ appear in (3.3) and Lemma 3.2. In this part we collect the fruits of this work:

Lemma 3.3 (The Splitting). *If $|u| \equiv 1$ in B_{ρ_1}, then for any $p \in (1, \infty)$, $t > 0$, $\rho_2 > 0$,*

$$\|\chi_{B_{\rho_1}} T_{t,B_{\rho_2}} u\|_p \precsim \|\chi_{B_{\rho_1}} u^i (T_{t,B_{\rho_2}} u)^i\|_p + \max_{\omega_{ij}} \|\chi_{B_{\rho_1}} u^j \omega_{ij} (T_{t,B_{\rho_2}} u)^i\|_p,$$

where the maximum is taken over all matrices $\omega \in \{-1, 0, 1\}^{N \times N}$ with $\omega_{ij} = -\omega_{ji}$. Recall that we use Einstein's summation convention.

Indeed, this is true for any unit vector $u \in \mathbb{R}^N$, $|u| = 1$:

$$|v|_{\mathbb{R}^N} \precsim |u^i\ v^i| + \max_{\omega_{ij}} |u^j \omega_{ij} v^i| \quad \text{for any } v \in \mathbb{R}^N. \tag{3.6}$$

Note that these are finitely many ω_{ij}. (3.6) can be seen as a consequence of what is sometimes called the Lagrange-identity. A direct proof for (3.6) can be found in, e.g., [16]. This kind of decomposition has been used in this form in [36], motivated from a very similar decomposition in [15].

Consequently, with the help of Lemma 3.3, we have arrived in our simplified presentation at

$$[u]^{p_s-1}_{s,p_s,B_\rho} \precsim \|u^i T_{t,B_{4\rho}} u^i\|_{p'_t,B_{3\rho}} + \max_{\omega} \|u^j \omega_{ij} T_{t,B_{4\rho}} u^j\|_{p'_t,B_{3\rho}} + \cdots . \tag{3.7}$$

Geometrically, $u^i T_{t,B} u^i$ measures the part of $T_{t,B}u$ which is orthogonal to the tangential space of the sphere $T_u\mathbb{S}^{N-1}$, and the collection of $u^j \omega_{ij} T_{t,B} u^j$ measures the part of $T_{t,B}u$ which belongs to the tangential space $T_u\mathbb{S}^{N-1}$.

We treat each part independently: Both will rely on an "integrability by compensation"-effect in the form of non-linear commutators of Theorem 1.4. To make this effect appear, we will need to add information from geometry ($u \in \mathbb{S}^{N-1}$), or the Euler-Lagrange equation in the right way.

Step 3: The Orthogonal Part. For the orthogonal part we have the estimate

Lemma 3.4. *Assume that* $\chi_{B_{2^L\rho}}|u| = \chi_{B_{2^L\rho}}$, $L \in \mathbb{Z}$, *then for some* $\sigma > 0$

$$\|u^i T_{t,B_{2^L\rho}} u^i\|_{p'_t} \precsim [u]^{p_s}_{s,p_s,B_{2^{2L}\rho}} + \sum_{l=1}^{\infty} 2^{-\sigma(L+l)} [u]^{p_s}_{s,p_s,B_{2^{2L+l}\rho}}.$$

The proof is given below, but let us first compare this estimate to effects of integration by compensation.

Note that we have the exponent p_s on the right-hand side. This is a higher exponent than $p_s - 1$, the one we would expect from formal considerations: A formal application of Hölder inequality gives

$$\|u^i T_{t,B_{4\rho}} u^i\|_{p'_t} \precsim \|u\|_\infty \, \|T_{t,B_{4\rho}} u\|_{p'_t}.$$

Now using the definition of $T_{t,B_{4\rho}}$ the power of u should be $p_s - 1$, i.e. the best estimate one might think one could expect is an estimate of the form

$$\|u^i T_{t,B_{4\rho}} u^i\|_{p'_t} \precsim \|u\|_\infty \, [u]^{p_s-1}_{s,p_s,B_{5\rho}} + \cdots .$$

(Actually even this holds only in the case $t > s$ which is not our setting). Compare this to the estimate suggested by Lemma 3.4:

$$\|u^i T_{t,B_{4\rho}} u^i\|_{p'_t,B_{3\rho}} \precsim [u]^{p_s}_{s,p_s,B_{5\rho}} + \ldots$$

For any given $\delta > 0$, if the radius ρ is small enough, the absolute continuity of integrals implies

$$\|u^i T_{t,B_{4\rho}} u^i\|_{p'_t,B_{3\rho}} \precsim \delta [u]^{p_s-1}_{s,p_s,B_{5\rho}} + \ldots \tag{3.8}$$

The appearance of this δ is what makes the decay estimate (3.2) work (via (3.3)). And it is thanks to the exponent p_s instead of $(p_s - 1)$ that we can produce this δ.

This higher exponent is similar to the effect of integration by compensation, probably first used in Wente's inequality, [11, 42, 48]: Let $a, b, c \in W^{1,2}(\mathbb{R}^2)$, then

$$\int_{\mathbb{R}^2} (\partial_x a \partial_y b - \partial_y a \, \partial_x b) c \precsim \|\nabla a\|_2 \|\nabla b\|_2 \|\nabla c\|_2.$$

Again, this is a deeper estimate than the one from a simple application of Hölder inequality which would only imply

$$\int_{\mathbb{R}^2} (\partial_x a\ \partial_y b - \partial_y a\ \partial_x b)\ c \precsim \|\nabla a\|_2\ \|\nabla b\|_2\ \|c\|_\infty.$$

Here, the former estimate replaces the L^∞-norm of c with the L^2-norm of the gradient of c. And again, the advantage of having a L^2-norm is that it is small on small sets, which is certainly false for the L^∞ norm.

Proof of Lemma 3.4. From $u(\Omega) \subset \mathbb{S}^{N-1}$ we have the following identity

$$(u(x) - u(y)) \cdot (u(x) + u(y)) = |u|^2(x) - |u|^2(y) = 1 - 1 = 0, \tag{3.9}$$

for any $x, y \in B_{2^L\rho} \subset \Omega$. Thus,

$$u^i(z)(T_{t,B_{2L}}u)^i(z)$$
$$= \int_{B_{2^L\rho}} \int_{B_{2^L\rho}} \frac{|u(x) - u(y)|^{p_s-2}(u(x) - u(y)) \cdot \Gamma(x, y, z)(|x - z|^{t-n} - |y - z|^{t-n})}{|x - y|^{n+sp_s}} dx\, dy, \tag{3.10}$$

where

$$\Gamma(x, y, z) := -\frac{1}{2}(u(x) + u(y) - 2u(z)).$$

This means that in some sense $u(z) \cdot T_{t,B_{2^L\rho}}u(z)$ can be interpreted as a product of lower-order operators, in view of Theorem 1.4, and more precisely we conclude with Lemma 6.5. □

To give the reader a better intuition how the arguments of Lemma 6.5 work in our situation, let us motivate this effect by the following formal argument: For the Hardy-Littlewood maximal function $\mathcal{M}$ we essentially have

$$\frac{u(x) - u(y)}{|x - y|^t} \precsim \mathcal{M}(-\Delta)^{\frac{t}{2}}u(x) + \mathcal{M}(-\Delta)^{\frac{t}{2}}u(y),$$

and thus for $\Gamma(x, y, z)$

$$\frac{\Gamma(x, y, z)}{|x - y|^t} \precsim \mathcal{M}(-\Delta)^{\frac{t}{2}}u(x) + \mathcal{M}(-\Delta)^{\frac{t}{2}}u(z) + \mathcal{M}(-\Delta)^{\frac{t}{2}}u(y).$$

In the actual proof one has to be more careful about the interplay of the variables x, y, z, but what this means is essentially

$$\frac{|u(x) - u(y)|^{p_s-2}(u(x) - u(y)) \cdot \Gamma(x, y, z)}{|x - y|^{tp_s}} \precsim \text{“}(\mathcal{M}(-\Delta)^{\frac{t}{2}}u)^{p_s}(x, y, z)\text{”}.$$

Here, with “$(\mathcal{M}(-\Delta)^{\frac{t}{2}}u)^{p_s}(x, y, z)$” we mean a sum of certain products of $\mathcal{M}(-\Delta)^{\frac{t}{2}}u(x)$,$\mathcal{M}(-\Delta)^{\frac{t}{2}}u(y)$,$\mathcal{M}(-\Delta)^{\frac{t}{2}}u(z)$ to certain exponents which add up to p_s.

Integrating the remaining part of the kernel in (3.10), we arrive formally at

$$|u^i T_{t,B_{4\rho}} u^i| \precsim |\cdot|^{(t+tp_s-sp_s)-n} * (\mathcal{M}(-\Delta)^{\frac{t}{2}} u)^{p_s} + \dots.$$

If we now choose $t < s$ sufficiently close to s, then $t + tp_s - sp_s > 0$. Thus $|\cdot|^{(t+tp_s-sp_s)-n}$ is the kernel of the Riesz potential $I^{t+tp_s-sp_s}$ from Definition 2.2, and the last estimate becomes

$$|u^i T_{t,B_{4\rho}} u^i| \precsim I^{t+tp_s-sp_s}((\mathcal{M}(-\Delta)^{\frac{t}{2}} u)^{p_s}) + \dots.$$

Now, by the classic Sobolev embedding, Theorem 1.5, the Hardy-Littlewood maximal theorem, and then the special Sobolev embedding, Theorem 1.6,

$$\|I^{t+tp_s-sp_s}((\mathcal{M}(-\Delta)^{\frac{t}{2}} u)^{p_s})\|_{p_t'} \precsim \|(\mathcal{M}(-\Delta)^{\frac{t}{2}} u)^{p_s}\|_{\frac{s}{t}} \precsim \|(-\Delta)^{\frac{t}{2}} u\|_{p_t}^{p_s} \precsim [u]_{s,p_s,\mathbb{R}^n}^{p_s}.$$

Up to localization and a replacement for the maximal function $\mathcal{M}$ with Riesz potentials this essentially explains the proof of Lemma 6.5.

Step 4: The Tangential Part. It remains to obtain for some small $\delta > 0$ and all sufficiently small radii ρ the estimate

$$\|\omega_{ij} u^j T_{t,B_{4\rho}} u^i\|_{p_t', B_{3\rho}} \precsim \delta [u]_{s,p_s,B_{5\rho}}^{p_s-1} + \dots, \tag{3.11}$$

where $\omega_{ij} = -\omega_{ij} \in \{-1, 0, 1\}$ is arbitrary.

As above for the orthogonal part, this estimate follows by absolute continuity for small radii ρ from

Lemma 3.5. *Let $K \in \mathbb{Z}$, $B_{30K} \subset \Omega$ and assume that u satisfies* (1.5). *If $t < s$ is close enough to s, then for some uniform $\sigma > 0$,*

$$\begin{aligned}\|\chi_{B_{2^K\rho}} \omega_{ij} u^j T_{t,B_{2^{10K}\rho}} u^i\|_{p_t'} &\precsim [u]_{s,p_s,B_{2^{20K}\rho}}^{p_s} + 2^{-\sigma K} [u]_{s,p_s,B_{2^{20K}\rho}}^{p_s-1} \\ &\quad + [u]_{s,p_s,\mathbb{R}^n} \sum_{k=1}^{\infty} 2^{-\sigma(K+k)} [u]_{s,p_s,B_{2^{20K+k}\rho}}^{p_s-1}\end{aligned}$$

The proof can be found in Section 5. Again observe the exponent p_s instead of $p_s - 1$ for the first term of the right-hand side. We now present the main ideas leading to this kind of estimate, and motivate

$$\|\omega_{ij} u^j T_{t,B_{4\rho}} u^i\|_{p_t', B_{3\rho}} \precsim [u]_{s,p_s,B_{5\rho}}^{p_s} + \dots.$$

Firstly, by duality there is some smooth φ, compactly supported in $B_{5\rho}$ and $\|(-\Delta)^{\frac{t}{2}} \varphi\|_{p_t,\mathbb{R}^n} \le 1$, such that

$$\begin{aligned}\|\omega_{ij} u^j T_{t,B_{4\rho}} u^i\|_{p_t', B_{3\rho}} &\precsim \int ((-\Delta)^{\frac{t}{2}} \varphi)\ \omega_{ij}\ u^j T_{t,B_{4\rho}} u^i + \dots \\ &= -\int (-\Delta)^{\frac{t}{2}} (\varphi \omega_{ij} u^j) T_{t,B_{4\rho}} u^i \\ &\quad - \int \varphi \omega_{ij} ((-\Delta)^{\frac{t}{2}} u^j) T_{t,B_{4\rho}} u^i - \omega_{ij} \int H_t(\varphi, u^j) T_{t,B_{4\rho}} u^i + \dots\end{aligned}$$

where the bi-commutator $H_t(\cdot,\cdot)$ is the one from (1.6). The estimates on $H_t(\cdot,\cdot)$ have been already used in the fractional harmonic map case (i.e., the L^2-case), and also here it can be dealt with in a more subtle yet similar fashion, using Theorem A.1. For the remaining parts we firstly use again that $I^t(-\Delta)^{\frac{t}{2}} f = f$, basically inverting the argument leading to (3.5),

$$\omega_{ij} \int (-\Delta)^{\frac{t}{2}}(\varphi u^j)(T_{t,B_{4\rho}} u)^i$$
$$= \omega_{ij} \int_{B_{4\rho}} \int_{B_{4\rho}} \frac{|u(x)-u(y)|^{p_s-2}(u(x)-u(y))(\varphi u^j(x)-\varphi u^j(y))}{|x-y|^{n+sp_s}} dxdy + \dots$$

This is where the Euler-Lagrange equation (1.5) plays its role: The right-hand side integrated over all of Ω is zero for arbitrary antisymmetric matrix $\omega \in \{-1,0,1\}^{N\times N}$. Since moreover the support of φ is $B_{3\rho}$, the remaining integral is well-behaved,

$$\omega_{ij} \int_{\Omega\setminus B_{4\rho}} \int_{B_{3\rho}} \frac{|u(x)-u(y)|^{p_s-2}(u^i(x)-u^i(y))\ \varphi u^j(x)}{|x-y|^{n+sp_s}} dxdy = \dots,$$

since here the kernel $|x-y|^{-n-sp_s} \gtrsim \rho^{-n-sp_s}$ is non-singular.

Lastly, we need to treat

$$\int \varphi\omega_{ij}\ ((-\Delta)^{\frac{t}{2}} u^j)\ T_{t,B_{4\rho}} u^i.$$

In the classical or even n/p-fractional harmonic map setting this term is zero since ω is antisymmetric and in this setting essentially $T_{t,B_{4\rho}} u^i = |(-\Delta)^{\frac{s}{2}} u|^{p_s-2}(-\Delta)^{\frac{s}{2}} u^i$. This is not true anymore in the integro-differential case, and we write this term as

$$\int_{B_{4\rho}} \int_{B_{4\rho}} \frac{|u(x)-u(y)|^{p_s-2}(u^i(x)-u^i(y))\Theta^i(x,y)}{|x-y|^{n+sp_s}}\ dx\ dy, \tag{3.12}$$

with

$$\Theta^i(x,y) = I^t(\varphi\omega_{ij}((-\Delta)^{\frac{t}{2}} u^j))(x) - I^t(\varphi\omega_{ij}((-\Delta)^{\frac{t}{2}} u^j))(y).$$

This time we use that by the antisymmetry of ω.

$$(u^i(x)-u^i(y))\omega_{ij}(u^j(x)-u^j(y))(\varphi(x)+\varphi(y)) = 0.$$

Then we can replace $\Theta^i(x,y)$ by

$$\omega_{ij}(I^t(\varphi((-\Delta)^{\frac{t}{2}} u^j))(x) - I^t(\varphi((-\Delta)^{\frac{t}{2}} u^j))(y) - \frac{1}{2}(u^j(x)-u^j(y))(\varphi(x)+\varphi(y)). \tag{3.13}$$

This term again falls under the nonlinear commutator estimates in form of Theorem 1.4, more precisely we will use Lemma 6.6. The main observation is that

$$\Theta^i(x,y) = -\frac{1}{2}\int_{\mathbb{R}^n} (|x-z|^{t-n} - |y-z|^{t-n})\ (-\Delta)^{\frac{t}{2}} u(z)\ (\varphi(x)+\varphi(y)-2\varphi(z))\,dz.$$

Following the formal analysis as we did in the orthogonal part, the expression in (3.12) is roughly controlled by

$$\|I^{t+(t-s)p_s}(|\mathcal{M}(-\Delta)^{\frac{t}{2}}u|^{p_s}\ \mathcal{M}(-\Delta)^{\frac{t}{2}}\varphi)\|_{1,\mathbb{R}^n}.$$

This is almost the correct estimate, but a more refined analysis in the proof of Lemma 6.6 actually obtains an estimate of roughly the form (for some $r \in (0, t + (t-s)p_s)$),

$$\|I^{t+(t-s)p_s-r}(|\mathcal{M}(-\Delta)^{\frac{t}{2}}u|^{p_s})\ I^r(\mathcal{M}(-\Delta)^{\frac{t}{2}}\varphi)\|_{1,\mathbb{R}^n}.$$

Now again using Hölder inequality and Sobolev embedding, we obtain an estimate of the expression in (3.12) by $[u]^{p_s}_{s,p_s}\|(-\Delta)^{\frac{t}{2}}\varphi\|_{p_t}$, and a localization and absolute continuity of integrals gives (3.11).

Together, the estimates (3.8) and (3.11) plugged into (3.7) then imply (3.2).

Remark 3.6. Before considering to estimate $T_{t,B}$ for $t < s$, one might consider estimating

$$\|T_{s,B}u\|_{\dot{F}^0_{p_s',p_s'}},$$

for the homogeneous zero-order Triebel-Lizorkin space $\dot{F}^0_{p_s,p_s}$, see e.g. [24, 47]. Indeed, one could justify this idea, because of

$$\|(-\Delta)^{\frac{s}{2}}u\|_{\dot{F}^0_{p_s,p_s}} \approx [u]_{s,p_s,\mathbb{R}^n}.$$

However, estimates in the space $\dot{F}^0_{p_s,p_s}$ leads to problems if $p_s > 2$: it is not necessarily true that $\dot{F}^0_{p_s,p_s} \subset L^1_{loc}$ or $L^\infty \cdot \dot{F}^0_{p_s,p_s} \subset \dot{F}^0_{p_s,p_s}$. Moreover $f \precsim g$ does not necessarily imply that $\|f\|_{\dot{F}^0_{p_s,p_s}} \precsim \|g\|_{\dot{F}^0_{p_s,p_s}}$. This makes $\dot{F}^0_{p_s,p_s}$ unsuitable for our estimates, for example for the splitting argument in Lemma 3.3.

3.3. *Precise Proof of the Decay Estimate: Lemma 3.1*

Proof of Lemma 3.1. Let us first mention some assumptions which simplify the presentation. Since the claim of Theorem 1.3 is local in nature those pose no restriction to the generality. Firstly, we are going to assume that u is defined everywhere on $\mathbb{R}^n$, $u \in L^p \cap L^\infty(\mathbb{R}^n)$, and that

$$[u]_{s,p_s,\mathbb{R}^n} < \infty.$$

This can be justified by cutting off u in a strict subset Ω_2 of Ω, such that $K \subset \Omega_2 \subset \Omega$. The resulting error terms can be controlled since they are of lower order. This argument has been detailed in [8] for $p_s = 2$, we leave the adaption to the reader.

Next, for some $\delta > 0$ to be determined later, let $\rho_0 > 0$ be so that

$$\sup_{r<\rho_0, x_0\in\mathbb{R}^n} [u]_{s,p_s,B_r(x_0)} < \delta. \tag{3.14}$$

Such a ρ_0 exists by absolute continuity of the integrals. We will assume $\rho_0 = 1$ and show the claim for $B_\rho(x_0) = B_1(0)$. Also we may assume that $B_{2^{L_0}}(0) \subset \Omega$ for a huge

$L_0 \in \mathbb{N}$, where L_0 is determined from the applications of the following Lemmas. We define

$$\mathrm{Tail}(\sigma, L, C) := C \sum_{l=1}^{\infty} 2^{-\sigma(L+l)} [u]^{p_s}_{s,p_s,L+l}.$$

The claim of Lemma 3.1 takes the form

$$[u]^{p_s}_{s,p_s,B_1} \leq \tau [u]^{p_s}_{s,p_s,B_{2L}} + \mathrm{Tail}(\sigma, L, C). \tag{3.15}$$

Note that for any $\varepsilon > 0$, if L is large enough (depending on σ and C), we have

$$\begin{aligned} \mathrm{Tail}(\sigma, L, C) &\leq \varepsilon [u]^{p_s}_{\tilde{L}} + C \sum_{l=1}^{\infty} 2^{-\sigma(\tilde{L}+l)} [u]^{p_s}_{s,p_s,\tilde{L}+l} \\ &\leq \varepsilon [u]^{p_s}_{\tilde{L}} + \mathrm{Tail}(\sigma, \tilde{L}, C). \end{aligned}$$

This means that the tail can be shifted from L to $\tilde{L} > L$ without causing much harm in terms of obtaining (3.15). In the following we thus consider σ, C, and even L a constant that can increase (in the case of C, L) or decrease (in the case of σ) as the proof progresses.

The first step for (3.15) is Lemma 3.2. Let $K > 0$ and $L = 10K$,

$$\begin{aligned} [u]^{p_s}_{s,p_s,B_1} \leq{}& (\varepsilon + C\, 2^{-K\sigma})\, [u]^{p_s}_{s,p_s,B_{2L}} + C_\varepsilon \left([u]^{p_s}_{s,p_s,B_{2L}} - [u]^{p_s}_{s,p_s,B_1} \right) \\ &+ C\, [u]_{s,p_s,B_1} \| \chi_{B_{2K}}(z) T_{t,B_{2L}} u \|_{p_t'}. \end{aligned}$$

Next we employ the usual hole-filling trick, often credited to Widman [49]: Add $C_\varepsilon [u]^{p_s}_{s,p_s,B_1}$ to both sides and divide by $C_\varepsilon + 1$. Then

$$[u]^{p_s}_{s,p_s,B_1} \leq \frac{\varepsilon + C\, 2^{-K\sigma} + C_\varepsilon}{C_\varepsilon + 1} [u]^{p_s}_{s,p_s,B_{2L}} + \frac{C}{C_\varepsilon + 1}\, [u]_{s,p_s,B_1}\, \| \chi_{B_{2K}}(z) T_{t,B_{2L}} u \|_{p_t'}.$$

Taking K large enough, and ε small enough, so that $\varepsilon + C\, 2^{-K\sigma} < 1$. Then,

$$\tau := \frac{\varepsilon + C\, 2^{-K\sigma} + C_\varepsilon}{C_\varepsilon + 1} < 1,$$

and we have

$$[u]^{p_s}_{s,p_s,B_1} \leq \tau [u]^{p_s}_{s,p_s,B_{2L}} + C\, [u]_{s,p_s,B_1}\, \| \chi_{B_{2K}}(z) T_{t,B_{2L}} u \|_{p_t'}. \tag{3.16}$$

We know that $\chi_{B_{2K}} |u| = \chi_{B_{2K}}$, because we assume that $B_{2L} \subset \Omega$ and $u(\Omega) \subset \mathbb{S}^{N-1}$. Thus Lemma 3.3 is applicable.

We obtain from Lemma 3.4,

$$[u]_{s,p_s,B_1} \| \chi_{B_{2K}}(z) u^i T_{t,B_{2L}} u^i \|_{p_t'} \precsim [u]_{s,p_s,B_1}\, [u]^{p_s}_{s,p_s,B_{22L}} + [u]_{s,p_s,B_1} \mathrm{Tail}(\sigma, L, C).$$

and from Lemma 3.5

$$
\begin{aligned}
&[u]_{s,p_s,B_1}\|\chi_{B_{2K}}(z)u^j\omega_{ij}T_{t,B_{2L}}u^i\|_{p_t'}\\
&\precsim [u]_{s,p_s,B_1}[u]^{p}_{s,p_s,B_{2L}}+2^{-\sigma L}\,[u]_{s,p_s,B_1}\,[u]^{p_s-1}_{s,p_s,B_{2L}}+[u]_{s,p_s,B_1}\,[u]_{s,p_s,\mathbb{R}^n}\,\sum_{k=1}^{\infty}2^{-\sigma(L+k)}[u]^{p_s-1}_{L+k}\\
&\precsim [u]_{s,p_s,B_1}\,[u]^{p}_{s,p_s,B_{2L}}+2^{-\sigma K}[u]^{p_s}_{s,p_s,B_{2L}}+[u]_{s,p_s,\mathbb{R}^n}\sum_{k=1}^{\infty}2^{-\sigma(L+k)}[u]^{p_s}_{L+k}
\end{aligned}
$$

Thus, in view of the splitting lemma, Lemma 3.3, the estimates on orthogonal part, Lemma 3.4, and on the tangential part, Lemma 3.5, (3.16) becomes

$$
\begin{aligned}
[u]^{p_s}_{s,p_s,B_1} \le \tau[u]^{p_s}_{s,p_s,B_{2L}} &+ C([u]_{s,p_s,B_1}+2^{-\sigma K})[u]^{p_s}_{s,p_s,B_{22L}}\\
&+\mathrm{Tail}(\sigma, L, C+[u]_{s,p_s,\mathbb{R}^n}+[u]_{s,p_s,B_1}).
\end{aligned}
$$

Taking K large enough, and $\delta > 0$ from (3.14) small enough, there is $\tilde{\tau} \in (\tau, 1)$, so that

$$
[u]^{p_s}_{s,p_s,B_1} \le \tilde{\tau}[u]^{p_s}_{s,p_s,B_{22L}} + \mathrm{Tail}(\sigma, L, \tilde{C}).
$$

This proves Lemma 3.1. □

4. Left-Hand Side Estimates: Proof of Lemma 3.2

The proof of Lemma 3.2 consists of two parts. Firstly, by the classical argument of using a localized version of the function itself as a testfunction we have:

Lemma 4.1. *For any $p \in (1,\infty)$, $s \in (0,1)$, for any $\rho > 0$ and any $L \in \mathbb{N}$ the following holds: For any $\varepsilon > 0$ there is $C_\varepsilon > 0$ so that for any map $u : B_{2^L\rho} \to \mathbb{R}^N$,*

$$
\begin{aligned}
[u]^{p}_{s,p,B_\rho} \precsim [u]_{s,p,B_{2\rho}} \sup_{\varphi}\int_{B_{2^L\rho}}\int_{B_{2^L\rho}} &\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+sp}}dx\,dy\\
&+ C_\varepsilon([u]^{p}_{s,p,B_{2^L\rho}} - [u]^{p}_{s,p,B_\rho}) + \varepsilon[u]^{p}_{s,p,B_{2^L\rho}}.
\end{aligned}
$$

The supremum is taken over all $\varphi \in C_0^\infty(B_{2\rho}, \mathbb{R}^N)$, with $[\varphi]_{s,p,\mathbb{R}^n} \le 1$.

Proof. Recall that $\eta_{B_\rho} \in C_0^\infty(B_{2\rho})$ is a cutoff-function so that $\eta_{B_\rho} \equiv 1$ in B_ρ. Denoting

$$
\psi(x) := \eta_{B_\rho}(x)(u(x) - (u)_{B_\rho}),
$$

we have

$$
[u]^{p}_{s,p_s,B_1} \le \int_{B_{2^L\rho}}\int_{B_{2^L\rho}} \frac{|u(x)-u(y)|^{p-2}(\psi(x)-\psi(y))\,(\psi(x)-\psi(y))}{|x-y|^{n+sp}}dx\,dy
$$

Now we write

$$
\begin{aligned}
\psi(x)-\psi(y) = (u(x)-u(y)) &- (1-\eta_{B_\rho}(x))(u(x)-u(y))\\
&+ (\eta_{B_\rho}(x)-\eta_{B_\rho}(y))(u(y)-(u)_{B_\rho}).
\end{aligned}
$$

so using that $\eta_{B_\rho} \equiv 1$ on B_ρ,

$$[u]^p_{s,p_s,B_1} \precsim I + II + III,$$

where

$$I := \int_{B_{2^L\rho}} \int_{B_{2^L\rho}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\psi(x)-\psi(y))}{|x-y|^{n+sp}} dx\, dy,$$

$$II := \int_{B_{2^L\rho}} \int_{B_{2^L\rho}\backslash B_\rho} \frac{|u(x)-u(y)|^{p-1}|\psi(x)-\psi(y)|}{|x-y|^{n+sp}} dx\, dy,$$

and using that $\eta_{B_\rho}(x) - \eta_{B_\rho}(y) = 0$ if both $x, y \in B_\rho$,

$$\begin{aligned} III \precsim & \int_{B_{2^L\rho}\backslash B_\rho} \int_{B_{2^L\rho}} \frac{|u(x)-u(y)|^{p-2}|\eta_{B_\rho}(x)-\eta_{B_\rho}(y)||u(y)-(u)_{B_\rho}||\psi(x)-\psi(y)|}{|x-y|^{n+sp}} dx\, dy \\ & + \int_{B_{2^L\rho}} \int_{B_{2^L\rho}\backslash B_\rho} \frac{|u(x)-u(y)|^{p-2}|\eta_{B_\rho}(x)-\eta_{B_\rho}(y)||u(y)-(u)_{B_\rho}||\psi(x)-\psi(y)|}{|x-y|^{n+sp}} dx\, dy. \end{aligned}$$

Since

$$|\psi(x)-\psi(y)| \leq |\eta_{B_\rho}(x)-\eta_{B_\rho}(y)|\ |u(y)-(u)_{B_\rho}| + |u(x)-u(y)|.$$

we have for $X = (B_{2^L\rho}\backslash B_\rho \times B_{2^L\rho}) \cup (B_{2^L\rho} \times B_{2^L\rho}\backslash B_\rho)$

$$\begin{aligned} II + III \precsim & \int\int_X \frac{|u(x)-u(y)|^{p-2}|\eta_{B_\rho}(x)-\eta_{B_\rho}(y)|^2|u(y)-(u)_{B_\rho}|^2}{|x-y|^{n+sp}} dx\, dy \\ & + \int\int_X \frac{|u(x)-u(y)|^{p-1}|\eta_{B_\rho}(x)-\eta_{B_\rho}(y)||u(y)-(u)_{B_\rho}|}{|x-y|^{n+sp}} dx\, dy \\ & + \int\int_X \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dx\, dy. \end{aligned}$$

Using Hölder's inequality, Proposition D.1, and Proposition D.2, and then Young's inequality for any $\varepsilon > 0$,

$$II + III \precsim C_\varepsilon([u]^p_{s,p,B_{2^L\rho}} - [u]^p_{s,p,B_\rho}) + \varepsilon[u]^p_{s,p,B_{2^L\rho}}.$$

It remains to treat I, where by Proposition D.3

$$I \precsim [u]_{s,p,B_{2\rho}} \sup_{\varphi\in C_0^\infty(B_\rho),[\varphi]_{s,p,\mathbb{R}^n}\leq 1} \int_{B_{2^L\rho}} \int_{B_{2^L\rho}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+sp}} dx\, dy.$$

□

Having Lemma 4.1, Lemma 3.2 follows from

Lemma 4.2. *Fix $0 < t < s$ close enough to s, and $p_t = \frac{n}{t}$. Then for any $L, K \in \mathbb{N}$,*

$$\sup_{\varphi \in C_0^\infty(B_1), [\varphi]_{s,p_s,\mathbb{R}^n} \le 1} \int_{B_{2L}} \int_{B_{2L}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y)) \cdot (\varphi(x)-\varphi(y))}{|x-y|^{n+sp}} dx\, dy$$
$$\precsim \|\chi_{B_{2K+1}} T_{t,B_{2L}} u\|_{p_t'} + 2^{-K} [u]^{p_s-1}_{s,p_s,B_{2L}}.$$

Proof. We use (2.4), and need to estimate

$$I := \int_{\mathbb{R}^n} \eta_{A_{2K}}(z) (-\Delta)^{\frac{t}{2}} \varphi(z)\ T_{t,B_{2L}} u(z) dz$$

and

$$II := \sum_{k=1}^{\infty} \int_{\mathbb{R}^n} \eta_{A_{2K+k}}(z) (-\Delta)^{\frac{t}{2}} \varphi(z)\ T_{t,B_{2L}} u.$$

As for I,

$$|I| \precsim \|(-\Delta)^{\frac{t}{2}} \varphi\|_{p_t}\ \|\eta_{A_{2K}} T_{t,B_{2L}}\|_{(p_t)'}$$

and by Theorem 1.6,

$$\|(-\Delta)^{\frac{t}{2}} \varphi\|_{p_t} \precsim [\varphi] \precsim 1.$$

The remaining term II is treated as follows

$$\begin{aligned} II &= \sum_{k=1}^{\infty} \int_{\mathbb{R}^n} (-\Delta)^{\frac{2s-t}{2}} (\eta_{A_{2K+k}}(z) (-\Delta)^{\frac{t}{2}} \varphi)(z)\ I^{2(s-t)} T_{t,B_{2L}} u(z) dz \\ &\overset{(2.5)}{=} \sum_{k=1}^{\infty} \int_{\mathbb{R}^n} (-\Delta)^{\frac{2(t-s)}{2}} (\eta_{A_{2K+k}}(z) (-\Delta)^{\frac{t}{2}} \varphi)(z)\ T_{2s-t,B_{2L}} u(z) dz \\ &\precsim \sum_{k=1}^{\infty} \|(-\Delta)^{\frac{2(t-s)}{2}} (\eta_{A_{2K+k}} (-\Delta)^{\frac{t}{2}} \varphi)\|_{\frac{n}{2s-t}}\ \|T_{2s-t,B_{2L}} u\|_{\frac{n}{n-2s+t}} \end{aligned}$$

Proposition D.4, for $\delta = s - t > 0$ small enough,

$$\|T_{2s-t,B_{2L}} u\|_{\frac{n}{n-2s+t}} \precsim [u]^{p_s-1}_{s,p_s,B_{2L}}.$$

Proposition B.2 implies that

$$\|(-\Delta)^{\frac{2(t-s)}{2}} (\eta_{A_{2K+k}} (-\Delta)^{\frac{t}{2}} \varphi)\|_{\frac{n}{2s-t}} \precsim 2^{-\sigma(K+k)}\ \|(-\Delta)^{\frac{t}{2}} \varphi\|_{p_t} \precsim 2^{-\sigma(K+k)}. \qquad \square$$

5. Estimate of $T_{t,B}u$: Proof of Lemma 3.5

In this section we show how the expression $T_{t,B}u$ can be controlled. We saw in Section 3 how to estimate the orthogonal part $u^i T_{t,B} u^i$, Lemma 3.4. Below are the arguments for the tangential part $u^j \omega_{ij} T_{t,B} u^i$.

Proof of Lemma 3.5. For presentation of the proof let us assume $\rho = 1$. Let $L := 10K$. We have for some $g \in L^{p_t}$

$$\|\chi_{B_{2K}} \omega_{ij} u^j T_{t,B_{2L}} u^i\|_{p_t'} \precsim \int (\chi_{B_{2K}} g)\ \omega_{ij} u^j\ T_{t,B_{2L}} u^i = I + II,$$

where, using again $f = (-\Delta)^{\frac{t}{2}} I^t f$,

$$I := \int (-\Delta)^{\frac{t}{2}} (\eta_{B_{22K}} I^t(\chi_{B_{2K}} g))\ \omega_{ij} u^j\ T_{t,B_{2L}} u^i,$$

$$II := \sum_{k=1}^{\infty} \int (-\Delta)^{\frac{t}{2}} (\eta_{A_{22K+k}} I^t(\chi_{B_{2K}} g))\ \omega_{ij} u^j\ T_{t,B_{2L}} u^i.$$

As for II, (we make sure that $s < 2s - t < 1$)

$$\int (-\Delta)^{\frac{2s-t}{2}} (((-\Delta)^{\frac{t}{2}} (\eta_{A_{22K+k}} I^t(\chi_{B_{2K}} g))\ \omega_{ij} u^j))\ I^{2(s-t)} T_{t,B_{2L}} u^i$$

$$\overset{(2.5)}{=} \int (-\Delta)^{\frac{2(s-t)}{2}} (((-\Delta)^{\frac{t}{2}} (\eta_{A_{22K+k}} I^t(\chi_{B_{2K}} g))\ \omega_{ij} u^j))\ T_{2s-t,B_{2L}} u^i$$

$$\precsim \|(-\Delta)^{\frac{2(s-t)}{2}} (((-\Delta)^{\frac{t}{2}} (\eta_{A_{22K+k}} I^t(\chi_{B_{2K}} g))\ \omega_{ij} u^j))\|_{\frac{n}{2s-t}}\ \|T_{2s-t,B_{2L}} u^i\|_{\frac{n}{n-2s+t}}$$

$$\precsim \|(-\Delta)^{\frac{2(s-t)}{2}} (((-\Delta)^{\frac{t}{2}} (\eta_{A_{22K+k}} I^t(\chi_{B_{2K}} g))\ \omega_{ij} u^j))\|_{\frac{n}{2s-t}}\ [u]^{p_s-1}_{s,p_s,B_{2L}}$$

In the last step we used Proposition D.4. It remains to estimate

$$\|(-\Delta)^{\frac{2(s-t)}{2}} (((-\Delta)^{\frac{t}{2}} (\eta_{A_{22K+k}} I^t(\chi_{B_{2K}} g))\omega_{ij} u^j))\|_{\frac{n}{2s-t}}.$$

By Proposition B.2,

$$\|(-\Delta)^{\frac{t+\delta}{2}} (\eta_{A_{2K+L}} I^t(\chi_{B_{2K}} g)_{\frac{n}{t+\delta}} \precsim 2^{-(K+k)\frac{n}{p_t}} \|g\|_{p_t}.$$

Moreover, we assumed w.l.o.g $\|u\|_\infty \le 1$, so

$$\begin{aligned}
&\|(-\Delta)^{\frac{2(s-t)}{2}} (((-\Delta)^{\frac{t}{2}} (\eta_{A_{22K+k}} I^t(\chi_{B_{2K}} g))\omega_{ij} u^j))\|_{\frac{n}{2s-t}} \\
&\precsim \|u\|_\infty \|(-\Delta)^{\frac{t+2(s-t)}{2}} (\eta_{A_{22K+k}} I^t(\chi_{B_{2K}} g)))\|_{\frac{n}{2s-t}} \\
&\quad + \|(-\Delta)^{\frac{2(s-t)}{2}} u\|_{\frac{n}{2(s-t)}} \|(-\Delta)^{\frac{t}{2}} (\eta_{A_{22K+k}} I^t(\chi_{B_{2K}} g)))\|_{\frac{n}{t}} \\
&\quad + \|H_{2(s-t)}(u, (-\Delta)^{\frac{t}{2}} (\eta_{A_{22K+k}} \|I^t(\chi_{B_{2K}} g)))\|_{\frac{n}{2s-t}} \\
&\precsim (\|(-\Delta)^{\frac{t}{2}} u\|_{p_t} + \|u\|_\infty)\ 2^{-(K+k)\sigma}\ \|g\|_{p_t}.
\end{aligned}$$

In the last step we used estimates on the three-term-commutator H, Theorem A.1, and Sobolev inequality.

The I case remains, and setting $\varphi := \eta_{B_{22K}} I^t(\chi_{B_{2K}} g)$, we have $\|(-\Delta)^{\frac{t}{2}} \varphi\|_{p_t} \precsim 1$. Indeed, this again follows from Theorem A.1 and the following estimate which works for any $q \in (1, p_t)$ such that $\frac{nq}{n-tq} \in [p_t, \infty)$

$$\|(-\Delta)^{\frac{t}{2}} \eta_{B_{22K}} I^t(\chi_{B_{2K}} g)\|_{p_t} \precsim 2^{2K(\frac{n}{p_t} - \frac{n}{q})}\ \|I^t(\chi_{B_{2K}} g)\|_{\frac{nq}{n-tq}} \precsim 2^{(2K-K)(\frac{n}{p_t} - \frac{n}{q})} \precsim 1.$$

Then $|I| \leq |I_1| + |I_2| + |I_3|$, with

$$I_1 := \omega_{ij} \int (-\Delta)^{\frac{t}{2}} (\varphi u^j)\ T_{t,B_{2L}} u^i,$$
$$I_2 := \omega_{ij} \int \varphi (-\Delta)^{\frac{t}{2}} u^j\ T_{t,B_{2L}} u^i,$$
$$I_3 := \omega_{ij} \int (-\Delta)^{\frac{2(s-t)}{2}} H_t(\varphi, u)\ T_{2s-t,B_{2L}} u^i.$$

For term I_3, if $(s-t)$ is small enough, we can apply the localized version of Theorem A.1, as well as Proposition D.4, and then Theorem 1.6 (here we need to assume that L is a multiple of K, say $L = 10K$)

$$|I_3| \precsim \|(-\Delta)^{\frac{2(s-t)}{2}} \|H_t(\varphi, u)_{\frac{n}{2s-t}} \|T_{2s-t,B_{2L}} u\|_{\frac{n}{n-2s+t}} \precsim [u]^p_{s,p_s,B_{2L}} + \sum_{l=1}^{\infty} 2^{-\sigma(L+l)} [u]^{p_s-1}_{L+l}.$$

Now we take care of I_1, employing (2.4),

$$I_1 = \int_{B_{2L}} \int_{B_{2L}} \frac{|u(x)-u(y)|^{p-2}(u^i(x)-u^i(y))\omega_{ij}(\varphi(x)u^j(x)-\varphi(y)u^j(y))}{|x-y|^{n+\alpha p}} dxdy.$$

We use the Euler-Lagrange system (1.5), also using that if $L \geq 10K$, the support of $\operatorname{supp} \varphi \in B_{2K}$ is rather small,

$$\begin{aligned}
|I_1| &\leq \int_{\Omega \backslash B_{2L}} \int_{\Omega} \frac{|u(x)-u(y)|^{p_s-1}|\varphi(x)u^j(x)-\varphi(y)u^j(y)|}{|x-y|^{n+\alpha p}} dx\ dy \\
&\leq \int_{\Omega \backslash B_{2L}} \int_{B_{2K}} \frac{|u(x)-u(y)|^{p_s-1}|\varphi(x)u^j(x)|}{|x-y|^{n+\alpha p}}\ dx\ dy \\
&\precsim \|u\|_{\infty,\Omega} \int_{\mathbb{R}^n \backslash B_{2L}} \int_{B_{2K}} \frac{|u(x)-u(y)|^{p_s-1}|\varphi(x)|}{|x-y|^{n+\alpha p}}\ dx\ dy \\
&\precsim [\varphi]_{s,p_s,B_2} \sum_{l=1}^{\infty} 2^{-\sigma(L+l)} [u]^{p_s-1}_{s,p_s,B_{2L+l}}.
\end{aligned}$$

In the last step we used Proposition D.5 and that w.l.o.g $\|u\|_\infty \precsim 1$. It remains to treat

$$\begin{aligned}
I_2 &= \omega_{ij} \int \varphi (-\Delta)^{\frac{t}{2}} u^j\ T_{t,B_{2L}} u^i \\
&= \omega_{ij} \int_{B_{2L}} \int_{B_{2L}} \frac{|u(x)-u(y)|^{p-2}(u^i(x)-u^i(y))\omega_{ij}(I^t(\varphi(-\Delta)^{\frac{t}{2}} u^j)(x) - I^t(\varphi(-\Delta)^{\frac{t}{2}} u^j)(y))}{|x-y|^{n+sp}} dx\,dy.
\end{aligned}$$

We proceed as in (3.12), (3.13), and now I_2 falls under the realm of Lemma 6.6 and this concludes the proof. □

6. Compensation Effects for Commutator-Like Expressions

6.1. *Preliminary Estimates*

Many arguments in the following proofs are based on the following case study. We used this kind of argument in [37, Chapter 3] to obtain estimates for H_s as in Theorem A.1.

Proposition 6.1. *For almost every $x, y, z \in \mathbb{R}^n$, we have three cases*

Case 1: $|x-y| \leq \frac{1}{2}|x-z|$ *or* $|x-y| \leq \frac{1}{2}|y-z|$,
Case 2: $2|x-y| \geq \max\{|x-z|, |x-z|\}$ *and* $|x-z| \leq |y-z|$,
Case 3: $2|x-y| \geq \max\{|x-z|, |x-z|\}$ *and* $|x-z| > |y-z|$,

and for arbitrary $\beta \in (0, n)$, $\varepsilon \in (0, 1]$:
In Case 1, $|x-z| \approx |y-z|$, *and*

$$||x-z|^{\beta-n} - |y-z|^{\beta-n}| \precsim |x-y|^{\varepsilon} \min\{|x-z|^{\beta-\varepsilon-n},\ |y-z|^{\beta-\varepsilon-n}\}.$$

In Case 2,

$$||x-z|^{\beta-n} - |y-z|^{\beta-n}| \precsim |x-y|^{\varepsilon}|x-z|^{\beta-\varepsilon-n}.$$

In Case 3,

$$||x-z|^{\beta-n} - |y-z|^{\beta-n}| \precsim |x-y|^{\varepsilon}|y-z|^{\beta-\varepsilon-n}.$$

From Proposition 6.1 and the definition of Riesz potentials, (2.2), we have the following β-Hölder-continuity estimates for $\beta \in (0, \alpha)$

Proposition 6.2. *For any $\alpha \in (0, 1)$, $\beta \in (0, \alpha)$, for almost every $y, z \in \mathbb{R}^n$ and for any $f = I^{\alpha}F$,*

$$|f(x) - f(y)| \leq C_{\alpha-\beta}\ |x-y|^{\beta}\ (I^{\alpha-\beta}|F|(x) + I^{\alpha-\beta}|F|(y)).$$

From Proposition 6.2, we deduce

Proposition 6.3. *Let $\beta \in (0, 1)$, $\alpha \in (0, 1)$ and $\varepsilon \in (0, 1-\alpha)$ such that $\varepsilon < \min\{1-\alpha, \beta - \frac{\alpha}{2}\}$. Then,*

$$\begin{aligned} &|f(x) + f(y) - 2f(z)|\, \big||x-z|^{\beta-n} - |y-z|^{\beta-n}\big| \\ &\quad \precsim \left(I^{\beta-\frac{\alpha}{2}}\right)|I^{\beta}f|(y) + I^{\beta-\frac{\alpha}{2}}|I^{\beta}f|(x) + I^{\beta-\frac{\alpha}{2}}|I^{\beta}f|(z)\ |x-y|^{\alpha+\varepsilon}k_{\beta-\frac{\alpha}{2}-\varepsilon,\beta}(x, y, z), \end{aligned}$$

where $k_{s,\gamma}$ has the form,

$$k_{s,\gamma}(x, y, z) := \min\{|y-z|^{s-n}, |x-z|^{s-n}\} \tag{6.1}$$

$$+ \left(\frac{|y-z|}{|x-y|}\right)^{\gamma-s} |y-z|^{s-n}\chi_{\{|y-z|<2|x-y|\}} \tag{6.2}$$

$$+ \left(\frac{|x-z|}{|x-y|}\right)^{\gamma-s} |x-z|^{s-n}\chi_{\{|x-z|<2|x-y|\}}. \tag{6.3}$$

Proof. Let

$$F := I^{\beta} f.$$

We have the following simple estimate

$$|f(x)+f(y)-2f(z)| \le \begin{cases} |f(x)-f(z)|+|f(y)-f(z)|,\\ |f(x)-f(y)|+2|f(y)-f(z)|,\\ |f(y)-f(x)|+2|f(x)-f(z)|. \end{cases}$$

In view of Proposition 6.2, this implies that for $\frac{\alpha}{2} \in (0,\beta)$ we have three options (6.4), (6.5), (6.6) to estimate

$$|f(x)+f(y)-2f(z)|:$$

Firstly,

$$|x-z|^{\frac{\alpha}{2}}\ \left(I^{\beta-\frac{\alpha}{2}}|F|(x)+I^{\beta-\frac{\alpha}{2}}|F|(z)\right)+|y-z|^{\frac{\alpha}{2}}\ \left(I^{\beta-\frac{\alpha}{2}}|F|(y)+I^{\beta-\frac{\alpha}{2}}|F|(z)\right), \tag{6.4}$$

secondly,

$$|x-y|^{\frac{\alpha}{2}}\ \left(I^{\beta-\frac{\alpha}{2}}|F|(y)+I^{\beta-\frac{\alpha}{2}}|F|(x)\right)+|y-z|^{\frac{\alpha}{2}}\ \left(I^{\beta-\frac{\alpha}{2}}|F|(y)+I^{\beta-\frac{\alpha}{2}}|F|(z)\right), \tag{6.5}$$

or thirdly

$$|x-y|^{\frac{\alpha}{2}}\ \left(I^{\beta-\frac{\alpha}{2}}|F|(y)+I^{\beta-\frac{\alpha}{2}}|F|(x)\right)+|x-z|^{\frac{\alpha}{2}}\ \left(I^{\beta-\frac{\alpha}{2}}|F|(x)+I^{\beta-\frac{\alpha}{2}}|F|(z)\right). \tag{6.6}$$

We now consider the cases of Proposition 6.1:

Case 1: $|x-y| \le \frac{1}{2}|x-z|$ or $|x-y| \le \frac{1}{2}|y-z|$,
Case 2: $2|x-y| \ge \max\{|x-z|,|x-z|\}$ and $|x-z| \le |y-z|$,
Case 3: $2|x-y| \ge \max\{|x-z|,|x-z|\}$ and $|x-z| > |y-z|$,

In Case 1, since then $|x-z| \approx |y-z|$, we have for $\gamma_1, \gamma_2 \in [0,1]$,

$$\begin{aligned}
&|f(x)+f(y)-2f(z)|\,||x-z|^{\beta-n}-|y-z|^{\beta-n}|\\
&\overset{(6.5)}{\precsim} |x-y|^{\frac{\alpha}{2}}\left(I^{\beta-\frac{\alpha}{2}}|F|(y)+I^{\beta-\frac{\alpha}{2}}|F|(x)\right)\left||x-z|^{\beta-n}-|y-z|^{\beta-n}\right|\\
&\quad+|y-z|^{\frac{\alpha}{2}}\ \left(I^{\beta-\frac{\alpha}{2}}|F|(y)+I^{\beta-\frac{\alpha}{2}}|F|(z)\right)\left||x-z|^{\beta-n}-|y-z|^{\beta-n}\right|\\
&\precsim |x-y|^{\frac{\alpha}{2}}\left(I^{\beta-\frac{\alpha}{2}}|F|(y)+I^{\beta-\frac{\alpha}{2}}|F|(x)\right)|y-z|^{\beta-n-\gamma_1}|x-y|^{\gamma_1}\\
&\quad+|y-z|^{\frac{\alpha}{2}}\ \left(I^{\beta-\frac{\alpha}{2}}|F|(y)+I^{\beta-\frac{\alpha}{2}}|F|(z)\right)|y-z|^{\beta-n-\gamma_2}|x-y|^{\gamma_2}\\
&=\left(I^{\beta-\frac{\alpha}{2}}|F|(y)+I^{\beta-\frac{\alpha}{2}}|F|(x)\right)|y-z|^{\beta-n-\gamma_1}|x-y|^{\gamma_1+\frac{\alpha}{2}}\\
&\quad+\left(I^{\beta-\frac{\alpha}{2}}|F|(y)+I^{\beta-\frac{\alpha}{2}}|F|(z)\right)|y-z|^{\beta-n-\gamma_2+\frac{\alpha}{2}}|x-y|^{\gamma_2}.
\end{aligned}$$

Now we choose $\gamma_1 := \frac{\alpha}{2}+\varepsilon$, $\gamma_2 = \alpha+\varepsilon$, which is admissible by the conditions on ε, and $\beta-\frac{\alpha}{2}-\varepsilon > 0$.

$$\begin{aligned}
&|f(x)+f(y)-2f(z)|\left||x-z|^{\beta-n}-|y-z|^{\beta-n}\right|\\
&\quad\precsim \left(I^{\beta-\frac{\alpha}{2}}|F|(y)+I^{\beta-\frac{\alpha}{2}}|F|(x)+I^{\beta-\frac{\alpha}{2}}|F|(z)\right)|y-z|^{\beta-\frac{\alpha}{2}-\varepsilon-n}\ |x-y|^{\alpha+\varepsilon}
\end{aligned}$$

Thus, in this case the kernel is of the form (6.1).

Next we have in Case 2, for any $\gamma_1, \gamma_2 > 0$, later choosing $\gamma_1 := \frac{\alpha}{2} + \varepsilon$, and $\gamma_2 := \alpha + \varepsilon$,

$$\begin{aligned}
&|f(x) + f(y) - 2f(z)| \left| |x-z|^{\beta-n} - |y-z|^{\beta-n} \right| \\
&\overset{(6.5)}{\precsim} |x-y|^{\frac{\alpha}{2}} \left(I^{\beta-\frac{\alpha}{2}}|F|(y) + I^{\beta-\frac{\alpha}{2}}|F|(x) \right) |y-z|^{\beta-n} \\
&\quad + |y-z|^{\frac{\alpha}{2}} \left(I^{\beta-\frac{\alpha}{2}}|F|(y) + I^{\beta-\frac{\alpha}{2}}|F|(z) \right) |y-z|^{\beta-n} \\
&= \left(I^{\beta-\frac{\alpha}{2}}|F|(y) + I^{\beta-\frac{\alpha}{2}}|F|(x) \right) |x-y|^{\frac{\alpha}{2}+\gamma_1} |y-z|^{\beta-\gamma_1-n} \left(\frac{|y-z|}{|x-y|} \right)^{\gamma_1} \\
&\quad + \left(I^{\beta-\frac{\alpha}{2}}|F|(y) + I^{\beta-\frac{\alpha}{2}}|F|(z) \right) |x-y|^{\gamma_2} |y-z|^{\beta-n+\frac{\alpha}{2}-\gamma_2} \left(\frac{|y-z|}{|x-y|} \right)^{\gamma_1+(\gamma_2-\gamma_1)} \\
&\overset{\gamma_1<\gamma_2}{\precsim} \left(I^{\beta-\frac{\alpha}{2}}|F|(y) + I^{\beta-\frac{\alpha}{2}}|F|(x) \right) |x-y|^{\frac{\alpha}{2}+\gamma_1} |y-z|^{\beta-\gamma_1-n} \left(\frac{|y-z|}{|x-y|} \right)^{\gamma_1} \\
&\quad + \left(I^{\beta-\frac{\alpha}{2}}|F|(y) + I^{\beta-\frac{\alpha}{2}}|F|(z) \right) |x-y|^{\gamma_2} |y-z|^{\beta-n+\frac{\alpha}{2}-\gamma_2} \left(\frac{|y-z|}{|x-y|} \right)^{\gamma_1},
\end{aligned}$$

Since we are in Case 2, the kernel can be written as in (6.2). By an analogous argument from Case 3 we obtain an estimate with (6.3) $\square$

Proposition 6.4. *Let* $F, G, H : \mathbb{R}^n \to \mathbb{R}_+$, $\alpha \in (0,n)$, $s, \beta \in (0,1)$, $s + \alpha < \beta$, *and consider*

$$I := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (F(x) + F(y))\,(G(z) + G(x) + G(y))\,|x-y|^{\alpha-n}\, H(z)\, k_{s,\beta}(x,y,z)\,dx\,dy\,dz,$$

where $k_{s,\beta}(x,y,z)$ *is of the form* (6.1), (6.2), *or* (6.3). *Then*

$$I \le \int_{\mathbb{R}^n} G\ H\ I^{s+\alpha}F + \int_{\mathbb{R}^n} F\ G\ I^{\alpha+s}H + \int_{\mathbb{R}^n} F\ I^{\alpha}G\ I^{s}H + \int_{\mathbb{R}^n} G\ I^{\alpha}F\ I^{s}H.$$

Proof. We are going to show that

$$\begin{aligned}
I \le& \int_{\mathbb{R}^n} I^{\alpha}F\ I^{s}(GH) + \int_{\mathbb{R}^n} I^{\alpha}(FG) I^{s}H + \int_{\mathbb{R}^n} F\ I^{\alpha}G\ I^{s}H \\
&+ \int_{\mathbb{R}^n} G\ I^{\alpha}F\ I^{s}H + \int_{\mathbb{R}^n} F\ I^{s+\alpha}(GH) + \int_{\mathbb{R}^n} FG\ I^{s+\alpha}H,
\end{aligned}$$

which, by integration by parts, simplifies to the claim.

We have to consider only products of the following form, the other cases follow from symmetric considerations.

$$F(x)\ G(z)\ H(z), \tag{6.7}$$

$$F(x)\ G(x)\ H(z), \tag{6.8}$$

$$F(y)\ G(x)\ H(z). \tag{6.9}$$

In the case of (6.1), (6.2), where we have

$$k_{s,\beta}(x,y,z) \precsim |y-z|^{s-n},$$

we have for (6.7),

$$\begin{aligned}
&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} F(x)\ G(z)\ |x-y|^{\alpha-n}\ H(z)\ k_{s,\beta}(x,y,z)\,dx\,dy\,dz\\
&\quad\precsim \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} F(x)\ |x-y|^{\alpha-n}\ dy\ H(z)\ G(z)\ |y-z|^{s-n}\,dx\,dy\,dz\\
&\quad\overset{(2.2)}{\approx} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} I^\alpha F(y)\ G(z)\ H(z)\ |y-z|^{s-n}\ dx\ dz\\
&\quad\overset{(2.2)}{\approx} \int_{\mathbb{R}^n} I^\alpha F(y)\ I^s(GH)(z)\ dz.
\end{aligned}$$

Similarly, for (6.8),

$$\begin{aligned}
&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} F(x)\ G(x)\ |x-y|^{\alpha-n}\ H(z)\ k_{s,\beta}(x,y,z)\,dx\,dy\,dz\\
&\quad\precsim \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} F(x)\ G(x)\ |x-y|^{\alpha-n}\ dx\ H(z)\ |y-z|^{s-n}dy\,dz\\
&\quad\overset{(2.2)}{\approx} \int_{\mathbb{R}^n} I^\alpha(FG)(y)\ I^s H(y)\ dy.
\end{aligned}$$

For (6.9),

$$\begin{aligned}
&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} F(y)\ G(x)\ |x-y|^{\alpha-n}\ H(z)\ k_{s,\beta}(x,y,z)\,dx\,dy\,dz\\
&\quad\precsim \int_{\mathbb{R}^n} F(y)\ \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} G(x)\ |x-y|^{\alpha-n}\ dx\ H(z)\ |y-z|^{s-n}\ dy\ dz\\
&\quad\overset{(2.2)}{\approx} \int_{\mathbb{R}^n} F(y)\ I^\alpha G(y)\ I^s H(y)\ dz.
\end{aligned}$$

In the case of (6.3), that is

$$k_{s,\beta}(y,x,z) = \left(\frac{|x-z|}{|x-y|}\right)^{\beta-s} |x-z|^{s-n}\chi_{\{|x-z|<2|x-y|\}},$$

we have for (6.7),

$$\begin{aligned}
&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} F(x)\ G(z)\ |x-y|^{\alpha-n}\ H(z)\ k_{s,\beta}(x,y,z)\,dx\,dy\,dz\\
&\quad\precsim \int_{\mathbb{R}^n} F(x)\int_{\mathbb{R}^n}\int_{\{|x-y|\succsim|x-z|\}} |x-y|^{s+\alpha-\beta-n}\ dy\quad H(z)\ G(z)\ |x-z|^{\beta-n}\ dz\ dx\\
&\quad\overset{s+\alpha<\beta}{\approx} \int_{\mathbb{R}^n} F(x)\int_{\mathbb{R}^n} |x-z|^{s+\alpha-\beta-n}\ H(z)\ G(z)\ |x-z|^{\beta-n}\ dz\ dx\\
&\quad\overset{(2.2)}{\approx} \int_{\mathbb{R}^n} F(x)\ I^{s+\alpha}(HG)(x)\ dx.
\end{aligned}$$

Similarly, for (6.8),

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} F(x)\ G(x)\ |x-y|^{\alpha-n}\ H(z)\ k_{s,\beta}(x,y,z)\,dx\,dy\,dz$$
$$\precsim \int_{\mathbb{R}^n} F(x)\ G(x)\ I^{s+\alpha}H(x)\ dy.$$

Lastly, for (6.9),

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} F(y)\ G(x)\ |x-y|^{\alpha-n}\ H(z)\ k_{s,\beta}(x,y,z)\,dx\,dy\,dz$$
$$\precsim \int_{\mathbb{R}^n} G(x)\ \int_{\mathbb{R}^n} F(y)\ |x-y|^{\alpha-n}\ \int_{\mathbb{R}^n} H(z)\ |x-z|^{s-n}\ dz\ dy\ dx$$
$$\overset{(2.2)}{\approx} \int_{\mathbb{R}^n} G(x)\ I^{\alpha}F(x)\ I^{s}H(x)\ dx.$$

This concludes the proof of Proposition 6.4. □

6.2. *The Compensation Estimates: Proof of Theorem 1.4*

Theorem 6.5 follows from Lemma 6.5 and Lemma 6.6.

Lemma 6.5. *Fix $s \in (0,1)$. For all $t < s$ large enough, let*

$$\Upsilon_1(z) := \int_{B_\rho}\int_{B_\rho} \frac{|f(x)-f(y)|^{p_s-1}\ |\Gamma(x,y,z)|}{|x-y|^{n+sp_s}}\ dx\ dy, \tag{6.10}$$

where

$$\Gamma(x,y,z) = |g(x)+g(y)-2g(z)|\ \big||x-z|^{t-n}-|y-z|^{t-n}\big|.$$

Then we have for any $L \in \mathbb{N}$,

$$\|\Upsilon_1\|_{p'_t} \precsim [f]^{p_s-1}_{s,p_s,B_{2^L\rho}}\ [g]_{s,p_s,B_{2^L\rho}} + \sum_{k=1}^{\infty} 2^{-\sigma(L+l)}[f]^{p_s-1}_{s,p_s,B_{2^{L+l}\rho}}\ [g]_{s,p_s,B_{2^{L+l}\rho}}.$$

Proof. Let $F := |(-\Delta)^{\frac{t}{2}} f|$, $G := |(-\Delta)^{\frac{t}{2}} f|$ both of which by Theorem 1.6 satisfy

$$\|F\|_{p_t} \precsim [f]_{s,p_s,\mathbb{R}^n}, \quad \|G\|_{p_t} \precsim [g]_{s,p_s,\mathbb{R}^n}. \tag{6.11}$$

By Proposition 6.2, for any small $\delta > 0$,

$$|f(x)-f(y)|^{p_s-1} \precsim |x-y|^{(t-\delta)(p_s-1)}\ \left((I^\delta F)^{p_s-1}(x) + (I^\delta F)^{p_s-1}(y)\right),$$

and Proposition 6.3, for $\varepsilon < t - \frac{s}{2}$,

$$\Gamma(x,y,z) \precsim \left(I^{t-\frac{s}{2}}G(y) + I^{t-\frac{s}{2}}G(x) + I^{t-\frac{s}{2}}G(z)\right)\ |x-y|^{s+\varepsilon}\ k_{t-\frac{s}{2}-\varepsilon,t}(x,y,z).$$

Consequently, for some $\varphi \in C_0^\infty(\mathbb{R}^n)$, $\|\varphi\|_{p_t} \leq 1$

$$\|\Upsilon_1\|_{p_t'} \precsim \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\Theta(x,y,z)}{|x-y|^{n+(p_s-1)(s-t+\delta)-\varepsilon}}\,dx\,dy\,dz,$$

where $\Theta(x,y,z)$ is composed by the following terms, using also symmetry of x and y,

$$k_{t-\frac{s}{2}-\varepsilon,t}(x,y,z)\ |\varphi|(z)\ I^{t-\frac{s}{2}}G(x)\ \chi_{B_\rho}(x)(I^\delta F)^{p_s-1}(x) \tag{6.12}$$

$$k_{t-\frac{s}{2}-\varepsilon,t}(x,y,z)\ |\varphi|(z)\ I^{t-\frac{s}{2}}G(x)\ \chi_{B_\rho}(y)(I^\delta F)^{p_s-1}(y) \tag{6.13}$$

$$k_{t-\frac{s}{2}-\varepsilon,t}(x,y,z)\ |\varphi|(z)\ I^{t-\frac{s}{2}}G(z)\ \chi_{B_\rho}(x)(I^\delta F)^{p_s-1}(x) \tag{6.14}$$

We can choose δ small enough and t close enough to s so that an admissible $\varepsilon > 0$ guarantees that

$$\alpha := \varepsilon - (s-t+\delta)(p_s-1) > 0.$$

Now the conditions for Proposition 6.4 are satisfied, since always

$$t - \frac{s}{2} - \varepsilon + \alpha < t.$$

Let

$$\begin{aligned}\tilde{G} &:= I^{t-\frac{s}{2}}G \in L^{2\frac{n}{s}}\\ \tilde{F} &:= \chi_{B_\rho}(I^\delta F)^{p_s-1} \in L^{\frac{sn}{(t-\delta)(n-s)}} \subset L^1_{loc}.\end{aligned}$$

We now apply Proposition 6.4,

$$\begin{aligned}&\leq \int_{\mathbb{R}^n} \tilde{G}\ \varphi\ I^{t-\frac{s}{2}-\varepsilon+\alpha}\tilde{F} + \int_{\mathbb{R}^n} \tilde{F}\ \tilde{G}\ I^{t-\frac{s}{2}-\varepsilon+\alpha}\varphi\\ &\quad + \int_{\mathbb{R}^n} \tilde{F}\ I^\alpha \tilde{G}\ I^{t-\frac{s}{2}-\varepsilon}\varphi + \int_{\mathbb{R}^n} \tilde{G}\ I^\alpha \tilde{F}\ I^{t-\frac{s}{2}-\varepsilon}\varphi.\end{aligned}$$

First of all, these integrals make sense: Possibly using partial integration,

$$\int (I^\gamma f)\ g = \int f\ I^\gamma g,$$

one checks that by Hölder and classical Sobolev inequality, Theorem 1.5, and then (6.11),

$$\int \Upsilon_1\ \varphi \precsim \|F\|_{p_t}^{p_s-1}\ \|G\|_{p_t}\ \ \|\varphi\|_{p_t} \precsim [f]^{p_s-1}_{p_s,s,\mathbb{R}^n}\ [g]_{p_s,s,\mathbb{R}^n}.$$

To localize this argument note that $\tilde{F}$ has a cutoff function χ_{B_ρ}. Then we can apply Proposition B.4, and several times Proposition B.3, and finally Lemma C1, to obtain the claim. □

Lemma 6.6.

$$\Upsilon_2 := \int_{B_\rho} \int_{B_\rho} \frac{|f(x) - f(y)|^{p_s - 1}\ |\Gamma(x, y)|}{|x - y|^{n+sp}}\ dx\ dy, \tag{6.15}$$

where

$$\Gamma(x, y) = I^t(g(-\Delta)^{\frac{t}{2}} h)(x) - I^t(g(-\Delta)^{\frac{t}{2}} h)(y) - \frac{1}{2}(h(x) - h(y))(g(x) + g(y))$$

Then we have

$$\Upsilon_2 \precsim \|(-\Delta)^{\frac{t}{2}} g\|_{p_t} [f]^{p_s - 1}_{s, p_s, B_{2L_\rho}, s, p_s}\ [h]_{s, p_s, B_{2L_\rho}, s, p_s}$$
$$+ \|(-\Delta)^{\frac{t}{2}} g\|_{p_t} \sum_{k=1}^{\infty} 2^{-\sigma(L+l)} [f]^{p_s - 1}_{s, p_s, B_{2L+l_\rho}, s, p_s}\ [h]_{s, p_s, B_{2L+l_\rho}, s, p_s}$$

Proof. Let $F := |(-\Delta)^{\frac{t}{2}} f|$, $G := |(-\Delta)^{\frac{t}{2}} g|$, $H := |(-\Delta)^{\frac{t}{2}} h|$.
To prove (6.15), first we observe,

$$\begin{aligned}
\Gamma(x, y) &= I^t(gH)(x) - I^t(gH)(y) - \frac{1}{2}(I^t H(x) - I^t H(y))(g(x) + g(y)) \\
&= \int_{\mathbb{R}^n} (|x - z|^{t-n} - |y - z|^{t-n})\ g(z)\ H(z)\ dz \\
&\quad - \frac{1}{2} \int_{\mathbb{R}^n} (|x - z|^{t-n} - |y - z|^{t-n})\ H(z)(g(x) + g(y))\ dz \\
&= -\frac{1}{2} \int_{\mathbb{R}^n} (|x - z|^{t-n} - |y - z|^{t-n})\ H(z)\ (g(x) + g(y) - 2g(z))\ dz.
\end{aligned}$$

In view of Proposition 6.3, for $t < s$ close enough to s, and $\varepsilon < t - \frac{s}{2} < 1$ small enough

$$\begin{aligned}
|\Gamma(x, y)| &\precsim \int_{\mathbb{R}^n} \| x - z|^{t-n} - |y - z|^{t-n}|\ |H(z)|\ |g(x) + g(y) - 2g(z)|\ dz \\
&\precsim \int_{\mathbb{R}^n} H(z)\ \left(I^{t - \frac{s}{2}} G(x) + I^{t - \frac{s}{2}} G(y) + I^{t - \frac{s}{2}} G(z)\right) |x - y|^{s+\varepsilon}\ k_{t - \frac{s}{2} - \varepsilon}(x, y, z)\ dz
\end{aligned}$$

Before we estimate Υ_2 we also need by Proposition 6.2, which ensures, for $\delta > 0$

$$|f(x) - f(y)|^{p_s - 1} \precsim |y - z|^{(t-\delta)(p_s - 1)} \left((I^\delta F)^{p_s - 1}(x) + (I^\delta F)^{p_s - 1}(y)\right).$$

So, all in all for Υ_2, we have to estimate

$$\Upsilon_2 \le \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Theta(x, y, z)\ |x - y|^{-n-(s-t+\delta)(p_s - 1)+\varepsilon} dz\ dx\ dy.$$

Here $\Theta(x, y, z)$ is composed by the following terms, using also symmetry of x and y,

$$k_{t - \frac{s}{2} - \varepsilon}(x, y, z)\ H(z)\ \chi_{B_\rho}(x) I^{t - \frac{s}{2}} G(x)\ \chi_{B_\rho}(x) (I^\delta F)^{p_s - 1}(x) \tag{6.16}$$

$$k_{t - \frac{s}{2} - \varepsilon}(x, y, z)\ H(z)\ \chi_{B_\rho}(x) I^{t - \frac{s}{2}} G(x)\ \chi_{B_\rho}(y) (I^\delta F)^{p_s - 1}(y) \tag{6.17}$$

$$k_{t-\frac{s}{2}-\varepsilon}(x,y,z)\ H(z)\ I^{t-\frac{s}{2}}G(z)\ \chi_{B_\rho}(x)(I^\delta F)^{p_s-1}(x) \tag{6.18}$$

This is exactly the same term as in the proof of Lemma 6.5, and we conclude the same way. □

Appendix A. Three-Term-Commutator Estimates

Let for $\alpha > 0$ the three term commutator given as

$$H_\alpha(a,b) := (-\Delta)^{\frac{\alpha}{2}}(ab) - b(-\Delta)^{\frac{\alpha}{2}}a - a(-\Delta)^{\frac{\alpha}{2}}b.$$

A version similar to H was first was introduced (to the best of our knowledge) in the pioneering [15], see Theorem A.1. They treated these commutators with the powerful tool of Littlewood-Paley decomposition. A more elementary approach, but less effective for limit estimates in Hardy-space and BMO was introduced in [37]. The following estimate can be deduced from both arguments, see also [7, Lemma A.5, 16].

Theorem A.1. *For any small $\varepsilon \geq 0$,*

$$\|(-\Delta)^{\frac{\varepsilon}{2}}H_\alpha(a,b)\|_p \precsim \|(-\Delta)^{\frac{\alpha}{2}}a\|_{p_1}\|(-\Delta)^{\frac{\alpha}{2}}b\|_{p_2},$$

where for $p \in (1,\infty)$ $p_1, p_2 \in (1, \frac{n}{\alpha}]$,

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{\alpha-\varepsilon}{n}.$$

If $\operatorname{supp} a \subset B_{2^K}$, *then we have*

$$\|(-\Delta)^{\frac{\varepsilon}{2}}H_\alpha(a,b)\|_p \precsim \|(-\Delta)^{\frac{\alpha}{2}}a\|_{p_1}\left(\|(-\Delta)^{\frac{\alpha}{2}}b\|_{p_2,B_{2^{K+L}}} + \sum_{k=1}^{\infty} 2^{-\sigma(L+k)}\|(-\Delta)^{\frac{\alpha}{2}}b\|_{p_2,B_{2^{K+L+k}}}\right).$$

Appendix B. Localization Arguments

We collect here some results which are related to localization. The following Lemma was stated and proved in this way in the appendix of [8], but of course these kind of results have been used throughout the literature.

Lemma B.1. *Let $s \in (-n,n)$, and if $s > 0$, and T^s defined as follows.*

- *if $s > 0$, $T^s = \nabla^s$ or $T^s = (-\Delta)^{\frac{s}{2}}$*
- *if $s = 0$, $T^0 = \mathcal{R}_\alpha$, for any $\alpha \in \{1,\dots,n\}$,*
- *and if $s < 0$, $T^s = I^s$.*

Then, $l \geq k+1$, for any f,

$$\|\chi_{A_{2^l}}T^s[\chi_{B_{2^k}}f]\|_\infty \precsim (2^k)^{-n-s}\|\chi_{B_{2^k}}f\|_1$$

and

$$\|\chi_{B_{2^k}}T^s[\chi_{A_{2^l}}f]\|_\infty \precsim (2^l)^{-n-s}\|\chi_{A_{2^l}}f\|_1$$

In a similar fashion we also have

Proposition B.2. *For any p and, $t \in (0,1)$, small $\delta \geq 0$. Let $\varphi \in C_0^\infty(B_{2^K})$, for any $L > 2$,*

$$\|(-\Delta)^{\frac{\delta}{2}}(\eta_{A_{2^{K+L}}}(-\Delta)^{\frac{t}{2}}\varphi)\|_{\frac{pn}{n+\delta p}} \precsim 2^{-L(\frac{n}{p'}+t)}\|(-\Delta)^{\frac{t}{2}}\varphi\|_p.$$

$$\|(-\Delta)^{\frac{t+\delta}{2}}(\eta_{A_{2^{K+L}}}I^t\varphi)\|_{\frac{pn}{n+\delta p}} \precsim 2^{-L\frac{n}{p'}}\|\varphi\|_p$$

Proposition B.3. *Let $s \in (0,n)$, $p \in (1,\frac{n}{s})$. Then for some $\sigma > 0$, for any $L \in \mathbb{N}$*

$$\|I^s f\|_{\frac{np}{n-sp},B_\rho} \precsim \|f\|_{p,B_{2^L\rho}} + \sum_{l=1}^{\infty} 2^{-\sigma(L+l)}\ \|f\|_{p,B_{2^{L+l}\rho}}$$

Proposition B.4. *Let $s_1, s_2, s_3 \in [0,n)$ and $p_1, p_2, p_3 \in (1,\infty)$ so that*

$$p_i^* := \frac{np_i}{n - s_i p_i} \in (1,\infty).$$

If moreover

$$\sum_i \frac{1}{p_i} - \sum_i \frac{s_i}{n} = 1,$$

then we have the following pseudo-local behavior for any $L \in \mathbb{N}$:

$$\begin{aligned}&\int_{\mathbb{R}^n} I^{s_1}(\chi_{B_\rho} f_1)\ I^{s_2}f_2\ I^{s_3}f_3\\ &\quad\precsim \|f_1\|_{p_1,B_{2^L\rho}}\ \|f_2\|_{p_2,B_{2^L\rho}}\ \|f_3\|_{p_3,B_{2^L\rho}}\\ &\qquad + \sum_{l=1}^{\infty} 2^{-(L+t)\sigma}\ \|f_1\|_{p_1,B_{2^{L+l}\rho}}\ \|f_2\|_{p_2,B_{2^{L+l}\rho}}\|f_3\|_{p_3,B_{2^{L+l}\rho}}\end{aligned}$$

Appendix C. Localized Sobolev Inequality

We will also need a localized version of the Sobolev inequality from Theorem 1.6.

Lemma C1. *Given $0 < t < s < 1$, then for any $L \in \mathbb{Z}$, $K \in \mathbb{N}$, setting $p_s = \frac{n}{s}$, $p_t = \frac{n}{t}$,*

$$\|\chi_{B_{2^L}}(-\Delta)^{\frac{t}{2}}f\|_{p_t} \precsim [f]_{s,p_s,B_{2^{L+K}}} + \sum_{k=1}^{\infty} 2^{-\sigma(K+k)}[f]_{s,p_s,B_{2^{L+K+k}}}.$$

Lemma C1 follows from Theorem 1.6 via a cutoff argument: since the fractional Laplacian of a constant is zero,

$$(-\Delta)^{\frac{t}{2}}f = (-\Delta)^{\frac{t}{2}}(\eta_{B_{2^{L+K}}}(f-(f)_{B_{2^{L+K}}})) + (-\Delta)^{\frac{t}{2}}((1-\eta_{B_{2^{L+K}}})(f-(f)_{B_{2^{L+K}}}))$$

For the first term, one uses Sobolev inequality, Theorem 1.6. The second term can be estimated using the disjoint support of $(1-\eta_{B_{2^{L+K}}})$ and $\chi_{B_{2^L}}$ via Lemma B.1. Since these are analogous arguments which appear in similar fashion in the literature

[8, 14, 15, 36], we leave the details as an exercise. More details can also be found in the arxiv-version, [39].

Appendix D. Some Estimates with the Slobodeckij/Gagliardo-Seminorm

In this section we state some estimates related to Gagliardo-seminorm. Firstly, a simple observation which just follows from the definition of $[u]_{s,p}$.

Proposition D.1. *Let* $0 < \rho < R$, $p \in (1, \infty)$, $s \in (0, 1)$. *Then*

$$\int_{B_R} \int_{B_R \backslash B_\rho} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} dx\ dy \leq [u]^p_{s,p,B_R} - [u]^p_{s,p,B_\rho}.$$

The next estimate follows from Lipschitz estimates of the cutoff functions η from Section 2, Jensen's inequality, and Proposition D.1.

Proposition D.2. *Let* $s \in (0, 1)$, $p \in (1, \infty)$. *For any* $L > K \in \mathbb{Z}$

$$\int_{B_{2L}} \int_{B_{2L}} \frac{|\eta_{B_{2K}}(x) - \eta_{B_{2K}}(y)|^p\ |u(y) - (u)_{B_{2K}}|^p}{|x - y|^{n+sp}} dx\, dy \precsim [u]^p_{s,p,B_{2K+2}} + ([u]^p_{s,p,B_{2L}} - [u]^p_{s,p,B_{2K}}).$$

The above implies also

Proposition D.3. *Let*

$$\psi(x) := \eta_{B_{2K}}(x)(u(x) - (u)_{B_{2K}}),$$

then

$$[\psi]_{s,p,\mathbb{R}^n} \precsim [u]_{s,p,B_{2K+1}}.$$

Proposition D.4. *For all small* $\delta > 0$

$$\|T_{s+\delta,B_\rho} u^i(z)\|_{\frac{n}{n-s-\delta}} \precsim [u]^{p-1}_{s,p_s,B_\rho}.$$

Proof. Pick $f \in \mathcal{S}(\mathbb{R}^n)$, $\|f\|_{\frac{n}{s+\delta}} \leq 1$ such that

$$\begin{aligned}
\|T_{s+\delta,B_\rho} u^i(z)\|_{\frac{n}{n-s-\delta}} &\precsim \int_{\mathbb{R}^n} T_{s+\delta,B_\rho} u^i(z)\ f(z)\ dz \\
&\precsim \int_{B_\rho} \int_{B_\rho} \frac{|u(x) - u(y)|^{p-1} |I^{s+\delta} f(x) - I^{s+\delta} f(y)|}{|x - y|^{n+sp}}\ dx\ dy \\
&\precsim [u]^{p_s - 1}_{s,p_s,B_\rho} \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|I^{s+\delta} f(x) - I^{s+\delta} f(y)|^{p_s}}{|x - y|^{n+sp_s}}\ dx\ dy \right)^{\frac{1}{p_s}} \\
&\precsim [u]^{p_s - 1}_{s,p_s,B_\rho} \|f\|_{\frac{n}{s+\delta}}.
\end{aligned}$$

The second line comes from the same arguments that lead to (3.5) in the outline of the proof. The third line is Hölder's inequality. The last estimate is Sobolev's inequality. □

Proposition D.5. *There exists a constant $\sigma > 0$, such that the following holds: Let $\varphi \in C_0^\infty(B_R)$. Then for any $L \geq 2$*

$$\int_{\mathbb{R}^n \setminus B_{2^L R}} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^{p-1} |\varphi(x)|}{|x - y|^{n+sp}} \, dx \, dy \precsim [\varphi]_{s,p,B_R} \sum_{l=1}^{\infty} 2^{-\sigma(L+l)} [u]^{p-1}_{s,p,B_{2^{L+l+1}R}}.$$

The proof goes as follows: By the support of φ, we need to estimates the sum for $l \in \mathbb{N}$ of

$$\int_{\mathbb{R}^n \setminus B_{2^{L+l-1}R}} \int_{B_R} \frac{|u(x) - u(y)|^{p-1} |\varphi(x)|}{|x - y|^{n+sp}} \, dx \, dy.$$

Notice that now $|x - y| \succsim 2^{L+l} R$.

Firstly one observes that $|x - y| \geq 2^L R$ by the support of φ. Now one uses Hölder and Sobolev inequalities, tracing the dependence of the constant on $L + l$.

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