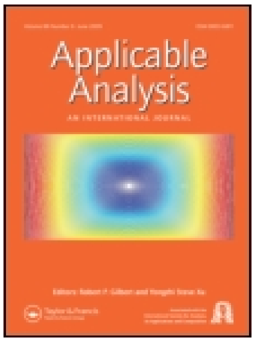

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Applicable Analysis

ISSN: 0003-6811 (Print) 1563-504X (Online) Journal homepage: https://www.tandfonline.com/loi/gapa20

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Applicable Analysis
Vol. 89, No. 3, March 2010, 409–414

Stability of equilibrium solution to inhomogeneous heat equation under a 3-point boundary condition

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Communicated by P. Broadbridge

(*Received 1 July 2009; final version received 29 November 2009*)

We consider a one-dimensional heat equation with inhomogeneous term, satisfying three-point boundary conditions, such that the temperature at the end is controlled by a sensor at the point η. We show that the integral solution, in the space of continuous functions satisfying the boundary values, converges to the equilibrium solution. This answers a question posed for nonlinear Laplacians, but in the linear case only.

Keywords: three-point boundary value problems; heat equation; asymptotic stability

AMS Subject Classifications: 34B10; 47D06

1. Introduction

In [1], the author considers the Cauchy problem on $[0,\infty)\times[0,1]$,

$$u_t(t,x) = (g(u_x))_x(t,x) - f(x), \tag{1}$$

$$u(t,0) = 0, \tag{2}$$

$$u(t,\eta) = \beta u(t,1), \tag{3}$$

$$u(0,x) = u_0(x), \tag{4}$$

where $\eta\in(0,1)$ and $\beta>1$ are given, along with f and g. It is supposed that $g:(a,b)\to\mathbb{R}$ is an increasing homeomorphism and $a<0<b$.

It is shown that we have an integral solution to the Cauchy problem $\mathrm{d}u/\mathrm{d}t = Au - f$ with initial value u_0, in the space of continuous functions, where A is the nonlinear Laplacian $(g(u_x))_x$ subject to the boundary conditions. The question is asked; does the solution converge to the equilibrium solution, $A^{-1}f$? In this note we show that this holds if $g:\mathbb{R}\to\mathbb{R}$ is linear, i.e. for some $k\in\mathbb{R}$, $g(x)=kx$, so that (1)

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ISSN 0003–6811 print/ISSN 1563–504X online

DOI: 10.1080/00036810903569457
http://www.informaworld.com

becomes $u_t(t,x)=ku_{xx}(t,x)-f(x)$, or after an adjustment, replacing t by $\tau=kt$ and f by f/k, we assume that we have

$$u_t(t,x) = u_{xx}(t,x) - f(x),$$

together with Equations (2)–(4). Note that the equilibrium solution has been investigated for the linear case in [2], as well as [1]. Unfortunately, the Sobolev space setting of Guidotti and Merino [3] seems to be unavailable, and we rely on the space of continuous functions to describe our equations. The boundary conditions in [3] included $u'(0)=0$. The paper [3] models the usage of a thermostat, and a nonlinear problem based on [3] was studied in the papers [4,5]. It should be interesting to get stability for the situation in which $f(x)$ is replaced by $f(u(x))$.

The convergence of the solution to the inhomogeneous heat equation

$$u_t(t,x) = (g(u_x))_x(t,x) - f(x)$$

under other boundary conditions, such as Dirichlet, and Neumann, is well known, and the interested reader may consult and follow up [6, Ch 10.1] and [7, Ch 3.5] and the commentaries on these sections.

2. Preliminaries

Suppose $\beta>1$ and $\eta\in(0,1)$ are given. Let X denote the Banach space of continuous functions $u:[0,1]\to\mathbb{C}$, satisfying $u(0)=0$ and $u(\eta)=\beta u(1)$, under the sup norm. We define a linear operator in X. Let $D(L)$ consist of $u\in X$ which have first and second continuous derivatives on $[0,1]$, i.e. one-sided derivatives at the endpoints. For $u\in D(L)$ let $Lu=u_{xx}$.

Lemma 1 *Given $\beta>1$ and $\eta\in(0,1)$, the equation*

$$\sin(\eta z) = \beta\sin(z) \tag{5}$$

in the complex variable z has only real solutions.

Proof (a) Suppose $z=iy$ is a purely imaginary solution to (5). The identity

$$\sin(x+iy) = \sin(x)\cosh(y) + i\cos(x)\sinh(y) \tag{6}$$

gives $i\sinh(\eta y)=\beta i\sinh(y)$ and $y=0$.

(b) Now we suppose $z=a+ib$, a and b are real and $ab\neq 0$. Define $z(t)=\sin(t(a+ib))$ for $t\in[0,1]$. We claim that if $ab>0$ then $\arg(z(t))$ is strictly decreasing on $(0,1)$, while if $ab<0$ then it is strictly increasing. Suppose $ab>0$. Write $z(t)=x(t)+iy(t)$, x and y real; we claim that

$$\frac{\mathrm{d}}{\mathrm{d}t}\arg(z(t)) = \frac{x\dot{y}-y\dot{x}}{x^2+y^2} < 0. \tag{7}$$

We want the numerator to be negative, i.e.

$$\begin{aligned}&\sin(ta)\cos(tb)[\cos(ta)b\cosh(tb) - a\sin(ta)\sinh(tb)]\\ &\quad - \cos(ta)\sinh(tb)[\sin(ta)b\sinh(tb) + a\cos(ta)\cosh(tb)] < 0.\end{aligned} \tag{8}$$

Simplify this to give

$$\frac{\sin(2ta)}{2ta} < \frac{\sinh(2tb)}{2tb}, \tag{9}$$

which holds because the LHS is less than 1 and the RHS is greater than 1. This proves the claim for $ab > 0$. Suppose instead that $ab < 0$. Then (9) holds, so that (8) holds with the inequality reversed, and hence $\arg(z(t))$ is strictly increasing.

Suppose $a \neq 0$, $b \neq 0$, a and b are real and

$$z(t) = \sin(t(a + ib)) \tag{10}$$

for $t \in (0, 1)$. We claim that $z(t) \neq \beta z(1)$ for all $t \in (0, 1)$. The curve $t \mapsto z(t)$ gives the solution to the initial value problem

$$\frac{\mathrm{d}}{\mathrm{d}t} z(t) = (a + ib)\sqrt{1 - z(t)^2}, \quad z(0) = 0, \tag{11}$$

where we choose $\sqrt{1} = 1$, as we see by substituting (10) into (11). We have the RHS single valued on the cut plane given by a cut between -1 and 1, and we check that the solution does not cross the real axis between -1 and 1 for $t > 0$. Suppose $y(t) = 0$, then $\cos(ta) = 0$, so $\sin(ta) = \pm 1$ and $\cosh(tb) > 1$, giving $|z(t)| > 1$. Thus we have the uniqueness of solutions of (11), and, in particular, the forward orbit does not intersect itself. Assume that $ab > 0$. Then we have a forward orbit spiralling clockwise out from the origin, so that if $\arg(z(t)$ decreases by 2π, then $\mathrm{mod}(z(t))$ increases, so we cannot have $z(t_0) = \beta z(t_0)$ for $\beta \geq 1$ and $t_0 < 1$. ■

LEMMA 2 *Suppose $\beta > 1$. The eigenvalues of L consist of a sequence $\langle \lambda_n \rangle_{n=0}^{\infty}$ with $\lambda_n = -k_n^2$ and*

$$k_n \in (\pi/2 + n\pi, \pi/2 + (n + 1)\pi), \tag{12}$$

with eigenvectors $u_n = x \mapsto \sin(k_n x)$.

Proof (a) We claim that for each $n = 0, 1, \ldots$ there is a unique $k_n \in (\pi/2 + n\pi, \pi/2 + (n + 1)\pi)$ with

$$\sin(\eta k_n) = \beta \sin(k_n). \tag{13}$$

Now $k \mapsto \beta \sin(k)$ takes values β and $-\beta$ at the two endpoints of $(\pi/2 + n\pi, \pi/2 + (n + 1)\pi)$, whereas $k \mapsto \sin(\eta k)$ has values in $[0, 1]$, so there does exist k_n satisfying (13).

Suppose there are two or more solutions of (13), then the slope of $k \mapsto \sin(\eta k)$ at some point q with

$$\sin(\eta q) = \beta \sin(q) \tag{14}$$

is in absolute value at least as big as that of $k \mapsto \beta \sin(k)$, i.e.

$$|\beta \cos(q)| \leq |\eta \cos(\eta q)|. \tag{15}$$

Thus

$$\cos^2(\eta q) \geq \eta^2 \cos^2(\eta q) \geq \beta^2 \cos^2(q) = \beta^2 - \beta^2 \sin^2(q) = \beta^2 - \sin^2(\eta q) \tag{16}$$

by (14), giving $1 \geq \beta^2$, contradicting $\beta > 1$.

(b) One checks that for each $n=0,1,\ldots,$ with $\lambda_n = -k_n^2$, and $u_n(x)=\sin(k_n x)$, we have

$$Lu_n = \lambda_n u_n. \tag{17}$$

(c) Suppose $Lu=\lambda u$; $\lambda \in \mathbb{C}$, and $u \neq 0$. We show that $\lambda = -k_n^2$ for some n, and $u=u_n$. Now $\lambda \neq 0$, and we let $\lambda = -k^2$. Since $u_{xx}=-k^2 u$, we have

$$u(x) = A\sin(kx) + B\cos(kx) \tag{18}$$

for some A, B. Since $u(0)=0$, $B=0$, and then $u(\eta)=\beta u(1)$, which gives

$$\sin(\eta k) = \beta\sin(k). \tag{19}$$

By Lemma 1, $k \in \mathbb{R}$. Hence all eigenvectors of L are real, nonzero and (19) holds with eigenvector $x \mapsto \sin(kx)$. Hence by (a), $k=k_n$ for some $n=0,1\ldots,$ and $\lambda = -k_n^2$. ■

LEMMA 3 *Let $\sigma \in \mathbb{C}$ be not an eigenvalue of L. Then $L-\sigma I$ is surjective and has continuous inverse.*

Proof Note that L is surjective, with continuous single-valued inverse which is compact. Since $L-\sigma I$ is one to one, if $f \in X$ is given, then $Lu-\sigma u=f$ iff $u-\sigma L^{-1}u=L^{-1}f$, and $I-\sigma L^{-1}$ is one to one, so is open and surjective by the invariance of domain. Hence $L-\sigma I$ is surjective, and bounded by the closed graph theorem. ■

THEOREM 1 [8] *Let T be a positive C_0 semigroup in a Banach lattice, with generator B. Then $s(B)=\omega_1(B)$.*

In this result the only condition on B is that it is the generator of a positive C_0 semigroup in a Banach lattice. We recall that T is called positive when for each $t \geq 0$, $T(t)$ maps the positive cone of the Banach lattice to itself. We recall [8, page 8] that $s(B):=\sup\{\mathrm{Re}(\lambda):\lambda \in \sigma(B)\}$ in general, and hence $s(L) = -k_0^2 < 0$ in this article. Also,

$$\omega_1(T) = \inf\{\omega \in \mathbb{R} : \text{ there exists } M > 0, \|T(t)x\| \leq Me^{\omega t}\|x\|_{D(B)} \text{ for all } x \text{ in } D(B), t \geq 0\}. \tag{20}$$

Here $\|x\|_{D(B)} := \|x\| + \|Bx\|$. Note that by [1, Theorem 12], L is an m-dissipative operator in X, and hence is the generator of a C_0 semigroup in a Banach lattice. We check that the semigroup is positive. The resolvent $J_n=(I-nL)^{-1}$, n a positive integer, is positive since if $u-n^{-1}Lu=v$, and $v \geq 0$, then $u \geq 0$, else u would be minimized at x_0 with $u(x_0)<0$, and then $Lu(x_0) \geq 0$ because the three-point boundary condition implies that $x_0<1$, and then $v(x_0)<0$, contradicting $v \geq 0$. Hence the semigroup is positive, being given, for $x \in X$ and $t \geq 0$, by

$$T(t)x = \lim_{n\to\infty} \exp(-nt)\exp(ntJ_n),$$

this exponential being defined via the power series and hence mapping the positive cone of X to itself.

COROLLARY 1 *The semigroup T generated by the operator L in X has the property that for all $x \in X$, $T(t)x \to 0$ as $t \to \infty$.*

Proof We check [1] that T is positive. By Theorem 1, there is M such that for all $x \in D(L)$,

$$\|T(t)x\| \leq Me^{-k_0^2 t/2}\|x\|_{D(L)}, \tag{21}$$

and $T(t)x \to 0$. Hence for all $x \in \mathrm{cl}(D(L)) = X$, since T is nonexpansive, $T(t)x \to 0$ as $t \to \infty$. ■

We consider the Cauchy problem: given $u_0 : [0,1] \to \mathbb{R}$, find $u(t,x)$ for $x \in [0,1]$ and $t \geq 0$, satisfying

$$\frac{\mathrm{d}u}{\mathrm{d}t} = L(u) - f, \quad u(0) = u_0. \tag{22}$$

By [1], L is m-dissipative in X. By [9] there is a unique integral solution of the Cauchy problem (22) if $u_0 \in X$.

Theorem 2 *Suppose* β, η, X *and* L *are as specified in Section* 2. *Let* u_0 *and* f *be in* X. *Then the integral solution to* (22) *converges to* $L^{-1}f$ *as* $t \to \infty$.

Proof Let $w_0 = L^{-1}f$. We know that L generates a nonexpansive semigroup T since L is m-dissipative. Let $u(t) = T(t)(u_0 - w_0) + w_0$ for $t \geq 0$. Suppose first that $u_0 \in D(L)$. Then u is C^1 and $u'(t) = Lu(t) - f$, since $u_0 - w_0 \in D(L)$; see [8, p. 3] on classical solutions. Then $u(t)$ is an integral solution by [9, Theorem 5.5]. Then for general u_0 in X we have $u(t)$ the integral solution, by continuity. From the Corollary we have $u(t) \to w_0$.

Remark $(t,x) \mapsto (T(t)(u_0 - w_0))(x)$ is a distributional solution of the heat equation and by hypoellipticity [10] it is C^∞ on $(0,\infty) \times (0,1)$. Hence the solution $u(t) = T(t)(u_0 - w_0) + w_0$ is as smooth as $L^{-1}f$. From the boundary conditions, $u(t)$ is smooth on the boundary $x = 1$ for $t > 0$.

Remark The question arises as to whether the condition $\beta > 1$ is necessary for this article, or whether $\beta > \eta$ suffices. In [11] it is shown that the condition $\beta > 1$ is necessary for their results. We note that a different case $\beta < \eta$ has been discussed in [12], and the integral operator is then negative. Lemmas 1 and 2 use $\beta > 1$, but Corollary 1 may go through without their detailed conclusions, because we merely used $s(L) < 0$ when applying Theorem 1. However, for $\eta < \beta < 1$, we do not apply the theory of integral solutions, because we can show that we do not have $L - \omega I$ dissipative for any ω, and integral solutions concern such operators.

Proposition 1 *Suppose* $\beta \in (0,1)$, $\eta \in (0,1)$, $\omega > 0$, $a < 0 < b$ *and* $g : (a,b) \to \mathbb{R}$ *is an increasing homeomorphism, and is* C^1. *Then* $L - \omega I$ *is not dissipative in* $C([0,1])$.

Proof Let

$$u(x) = \begin{cases} \epsilon\left(1 + \frac{(\beta^{-1}-1)(x-\eta)^4}{(1-\eta)^4}\right) & x \geq \eta \\ \epsilon\left(1 - \frac{(x-\eta)^4}{\eta^4}\right) & x \leq \eta. \end{cases} \tag{23}$$

If $\epsilon > 0$ is small, then $u \in D(L)$, and we check that $u - \lambda(L - \omega I)u$ attains its maximum value at 1 for small $\lambda > 0$, but is less than u there. ■

Remark On the other hand, we can still ask about other notions of solution of the Cauchy problem for $\eta < \beta < 1$, in case $g(x) = x$, and we can ask if the corresponding version of Theorem 2 will hold. But this study is not in the scope of this article. In fact, L is the generator of a positive C_0 semigroup.

Acknowledgements

Thanks to Jeff Webb, Chaitan Gupta and the referees for their advice.

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