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On Green's functions in generalized axially symmetric potential theory

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ABSTRACT

Fundamental solutions and Green's functions of the operator $\partial_t^2 + (1 + 2\alpha)t^{-1}\partial_t + \Delta_n$, $\alpha \in \mathbb{C}$, are calculated in the half-space $t > 0$.

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1. Introduction and notation

The goal of this paper consists in deriving, in the framework of Schwartz' distribution theory [1], fundamental solutions and Green's functions of the operator of 'generalized axially symmetric potential theory' (GASPT), i.e. of

$$P_\alpha(\partial) = \partial_t^2 + (1 + 2\alpha)t^{-1}\partial_t + \Delta_n, \quad \partial = (\partial_t, \partial_1, \dots, \partial_n), \quad \Delta_n = \partial_1^2 + \dots + \partial_n^2, \quad \alpha \in \mathbb{C}. \quad (1)$$

Fundamental solutions of $P_\alpha(\partial)$ were presented first by Weinstein (see [2,3]). His method of derivation was based on classical analysis and did not involve distribution theory, which at that time was not yet state of the art. Weinstein's method ran along the following five steps:

- assume first that $1 + 2\alpha$ is a natural number;
- use the known fundamental solution of the Laplacean operator in $2 + 2\alpha + n$ variables;
- introduce polar coordinates with respect to the first $2 + 2\alpha$ variables;
- integrate with respect to the sphere $\mathbb{S}^{1+2\alpha}$;
- replace $1 + 2\alpha \in \mathbb{N}$ by $1 + 2\alpha \in \mathbb{C}$.

Finally, Weinstein checked the 'nature of singularity' of the function found by the procedure in (a) to (e). The resulting fundamental solutions are expressed by definite integrals, and as customary until 1951, they are defined only up to multiplicative constants.

However ingenious Weinstein's approach may be, it does not seem satisfactory from the viewpoint of modern analysis. In the literature, there are two further treatments of the GASPT operator

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we know of. In [4, Ch. VIII: Degenerate elliptic operators], a formulation of existence and uniqueness results for a class of operators including the one in GASPT is given. However, the approach is based on Sobolev spaces and Green’s functions are not expressed by special functions but only as definite integrals.

A different attempt at constructing fundamental solutions of $P_\alpha(\partial)$ in a distributionally correct way is contained in the studies [5,6]. Therein, fundamental solutions are set up as infinite series motivated by the derivation of the fundamental solution of the EPD-operator in [7] (see also [8]). However, no effort is made of deriving uniqueness results or Green’s functions; furthermore, the result in the case $n = 1$ [6, Theorem 3.4, Equation (3.27), p.507] seems to be incorrect.

For the reasons explained above, we have taken up anew the study of fundamental solutions and Green’s functions of the operator in (1). In Definition 2.1, we define the notions of temperate fundamental solutions and Green’s functions of the Dirichlet problem and the Neumann problem in the half-space

$$H = \{(t, x) \in \mathbb{R}^{n+1}; t > 0, x \in \mathbb{R}^n\}$$

for the singular operator $P_\alpha(\partial)$ in (1). The uniqueness of Green’s functions is investigated in Proposition 2.2, and we represent Green’s functions and temperate fundamental solutions of $P_\alpha(\partial)$ by hypergeometric functions in Theorem 2.3. In the case of even n , we represent these fundamental solutions by elementary transcendental functions in Corollary 2.4. Of course, some of our formulas can be found already in [2–6] (see the remarks following Theorem 2.3).

We derive our results by employing the partial Fourier transform with respect to the x -variables and by using suitable identities for the hypergeometric function. We also make use of the theory of distribution-valued analytic functions as expounded in [9].

Let us introduce some notation. Besides the spaces $\mathcal{D}'(U)$, $U \subset \mathbb{R}^n$ open, and $\mathcal{S}'(\mathbb{R}^n)$ of distributions and temperate distributions, respectively, we also use the space

$$\mathcal{S}'(H) = \{T \in \mathcal{D}'(H); \exists T_1 \in \mathcal{S}'(\mathbb{R}^{n+1}) : T = T_1|_H\}$$

of temperate distributions on the half-space H defined above. Note that the partial Fourier transform

$$\mathcal{F}_x : \mathcal{S}'(\mathbb{R}^{n+1}) \longrightarrow \mathcal{S}'(\mathbb{R}^{n+1}),$$

which is extended by continuity from

$$(\mathcal{F}_x \phi)(t, \xi) = \int_{\mathbb{R}^n} \phi(t, x) e^{-ix\xi} dx, \quad \phi \in \mathcal{S}(\mathbb{R}^{n+1}),$$

yields also an isomorphism on $\mathcal{S}'(H)$. The Heaviside function is denoted by Y , and we write $\delta(t - \tau) \in \mathcal{S}'(\mathbb{R}_t^1)$, $\tau > 0$, for the delta distribution with support in τ , i.e. for the derivative of $Y(t - \tau)$.

2. Temperate fundamental solutions and Green’s functions in GASPT

As mentioned already in the introduction, the operator $P_\alpha(\partial) = \partial_t^2 + ((1 + 2\alpha)/t) \partial_t + \Delta_n$ arises in the so-called GASPT (see [2] for historical remarks and connections to physics). Let us first introduce the notions of temperate fundamental solution and Green’s functions for $P_\alpha(\partial)$.

Definition 2.1: Set $H = (0, \infty) \times \mathbb{R}^n$ and fix $\tau > 0$ and $\alpha \in \mathbb{C}$.

- (a) $E \in \mathcal{S}'(H)$ is called *temperate fundamental solution* of $P_\alpha(\partial)$ if and only if $P_\alpha(\partial)E = \delta(t - \tau) \otimes \delta(x)$ holds in H .
- (b) $E \in \mathcal{S}'(H)$ is called *Green’s function of the Dirichlet problem* for $P_\alpha(\partial)$ if and only if E is a temperate fundamental solution of $P_\alpha(\partial)$ that satisfies $\lim_{t \rightarrow \infty} E(t, x) = \lim_{t \searrow 0} E(t, x) = 0$ in $\mathcal{S}'(\mathbb{R}_x^n)$.

(c) $E \in \mathcal{S}'(H)$ is called *Green's function of the Neumann problem* for $P_\alpha(\partial)$ if and only if E is a temperate fundamental solution of $P_\alpha(\partial)$ that satisfies $\lim_{t \rightarrow \infty} E(t, x) = \lim_{t \searrow 0} (\partial_t E)(t, x) = 0$ in $\mathcal{S}'(\mathbb{R}_x^n)$.

Remark 2.1: Note that a fundamental solution E of $P_\alpha(\partial)$ is C^∞ in $H \setminus \{(\tau, 0)\}$ due to [10, Theorem 13.4.1, p.191]. Hence we can fix t in $E(t, x)$ for $t \neq \tau$. For example, the hypothesis $\lim_{t \rightarrow \infty} E(t, x) = 0$ in $\mathcal{S}'(\mathbb{R}^n)$ in Definition 2.1 then means that $E(t, x)$ belongs, for fixed large t , to $\mathcal{S}'(\mathbb{R}_x^n)$ and converges therein to 0 if $t \rightarrow \infty$.

The next proposition will show that the Green functions of the Dirichlet problem and the Neumann problem, respectively, for $P_\alpha(\partial)$ are uniquely determined in those cases where they exist.

Proposition 2.2: Fix $\alpha \in \mathbb{C}$ and let $T \in \mathcal{S}'(H)$ fulfill $P_\alpha(\partial)T = 0$ in H and $\lim_{t \rightarrow \infty} T(t, x) = 0$ in $\mathcal{S}'(\mathbb{R}^n)$. If, additionally, either $\lim_{t \searrow 0} T(t, x) = 0$ or $\lim_{t \searrow 0} (\partial_t T)(t, x) = 0$ hold in $\mathcal{S}'(\mathbb{R}^n)$, then T vanishes identically.

Proof: The partial Fourier transform $U = \mathcal{F}_x T$ of T satisfies the ‘ordinary’ differential equation

$$\left(\partial_t^2 + \frac{1 + 2\alpha}{t} \partial_t - |x|^2 \right) U = 0 \quad \text{in } H. \tag{2}$$

For $x \neq 0$, let $V(t, x) = U(t/|x|, x)$. Then $V \in \mathcal{D}'(H_0)$ where $H_0 = \{(t, x) \in H; x \neq 0\}$. Since V fulfills $(\partial_t^2 + ((1 + 2\alpha)/t) \partial_t - 1)V = 0$ in H_0 , we conclude that

$$V = t^{-\alpha} I_\alpha(t) \otimes V_1(x) + t^{-\alpha} K_\alpha(t) \otimes V_2(x), \quad V_1, V_2 \in \mathcal{D}'(\mathbb{R}^n \setminus \{0\}).$$

Therefore, $U = t^{-\alpha} I_\alpha(t|x|)W_1(x) + t^{-\alpha} K_\alpha(t|x|)W_2(x)$ holds in H_0 for $W_j(x) = |x|^{-\alpha} V_j(x) \in \mathcal{D}'(\mathbb{R}^n \setminus \{0\}), j = 1, 2$.

Let us use now the boundary conditions for T . The assumption $\lim_{t \rightarrow \infty} T(t, x) = 0$ in $\mathcal{S}'(\mathbb{R}^n)$ implies $\lim_{t \rightarrow \infty} U(t, x) = 0$ in $\mathcal{S}'(\mathbb{R}^n)$, and therefore W_1 vanishes and $U = t^{-\alpha} K_\alpha(t|x|)W_2(x)$ holds in H_0 . On the other hand, either of the limits $\lim_{t \searrow 0} U(t, x) = 0$ or $\lim_{t \searrow 0} (\partial_t U)(t, x) = 0$ implies $W_2 = 0$. Hence $U|_{H_0}$ vanishes and $\text{supp } U \subset (0, \infty) \times \{0\} \subset H$, i.e.

$$U = \sum_{|\beta| \leq m} f_\beta(t) \otimes \partial^\beta \delta(x), \quad m \in \mathbb{N}_0, \quad f_\beta \in \mathcal{D}'((0, \infty)), \quad \beta \in \mathbb{N}_0^n.$$

(Note that U is a distribution of finite order due to $U \in \mathcal{S}'(H)$.)

Let us assume that $\beta \in \mathbb{N}_0^n$ is such that $|\beta| = m$ and that f_β does not vanish identically. Then (2) implies that $(\partial_t^2 + (1 + 2\alpha)t^{-1} \partial_t) f_\beta = 0$ and hence $f_\beta = C_1 + C_2 t^{-2\alpha}$ for $\alpha \in \mathbb{C} \setminus \{0\}$ or $f_\beta = C_1 + C_2 \log t$ if $\alpha = 0$. In both cases, the conditions $\lim_{t \rightarrow \infty} U(t, x) = \lim_{t \searrow 0} U(t, x) = 0$ or $\lim_{t \rightarrow \infty} U(t, x) = \lim_{t \searrow 0} (\partial_t U)(t, x) = 0$ then imply that f_β vanishes and that leads to a contradiction. Therefore $U = 0$ and thus also $T = 0$ and the proof is complete. ■

Theorem 2.3: As before, set $H = (0, \infty) \times \mathbb{R}^n$. Let $\alpha \in \mathbb{C} \setminus (-(n/2) - \mathbb{N}_0)$, $\tau > 0$ and $(t, x) \in H \setminus \{(\tau, 0)\}$ and set

$$z = \frac{\tau^2 + t^2 + |x|^2}{2\tau t}, \quad r = \sqrt{(t - \tau)^2 + |x|^2}. \tag{3}$$

The functions

$$\begin{aligned} E_{\alpha, \tau}^N(t, x) &= -(2\pi)^{-(n+1)/2} e^{-i(n-1)\pi/2} \frac{\tau^{-n/2+1+\alpha}}{t^{n/2+\alpha}} (z^2 - 1)^{-(n-1)/4} Q_{-1/2+\alpha}^{(n-1)/2}(z) \\ &= -\frac{1}{2\pi^{n/2}} \frac{\Gamma(\frac{n}{2} + \alpha)}{\Gamma(1 + \alpha)} \frac{\tau^{1+2\alpha}}{r^{n+2\alpha}} {}_2F_1\left(\frac{n}{2} + \alpha, \frac{1}{2} + \alpha; 1 + 2\alpha; -\frac{4\tau t}{r^2}\right) \end{aligned} \tag{4}$$

are C^∞ in $H \setminus \{(\tau, 0)\}$ and locally integrable in H . For even n , the mapping $\alpha \mapsto E_{\alpha,\tau}^N$ extends to an entire function

$$\mathbb{C} \longrightarrow \mathcal{S}'(H) : \alpha \longmapsto E_{\alpha,\tau}^N;$$

for odd n , this mapping is meromorphic on \mathbb{C} with simple poles in the set $-n/2 - \mathbb{N}_0$. Furthermore, we set $E_{\alpha,\tau}^D(t, x) = (\tau/t)^{2\alpha} E_{-\alpha,\tau}^N(t, x)$ for $\alpha \in \mathbb{C}$ if n is even and for $\alpha \in \mathbb{C} \setminus ((n/2) + \mathbb{N}_0)$ if n is odd, respectively.

For those $\alpha \in \mathbb{C}$ for which $E_{\alpha,\tau}^N, E_{\alpha,\tau}^D$, respectively, are defined, they are temperate fundamental solutions of

$$P_\alpha(\partial) = \partial_t^2 + \frac{1+2\alpha}{t} \partial_t + \Delta_n.$$

Furthermore, $E_{\alpha,\tau}^N$ is the uniquely determined Green function of the Neumann problem for $P_\alpha(\partial)$ if $\operatorname{Re} \alpha > -n/2$, and $E_{\alpha,\tau}^D$ is the uniquely determined Green function of the Dirichlet problem for $P_\alpha(\partial)$ if $\operatorname{Re} \alpha < 0$.

(In (4) Q_ν^μ denotes an associated Legendre function and ${}_2F_1$ denotes Gauß' hypergeometric function. If α is a negative entire number not belonging to $-n/2 - \mathbb{N}_0$, then $\Gamma(\alpha + 1)^{-1} {}_2F_1(\dots)$ in (4) has to be interpreted as a limit.)

Proof: (a) Let us first assume $\operatorname{Re} \alpha > -n/2$ and represent $E_{\alpha,\tau}^N$ by a partial Fourier transform with respect to x . From

$$\left(\partial_t^2 + \frac{1+2\alpha}{t} \partial_t + \Delta_n \right) E_{\alpha,\tau}^N = \delta(t - \tau) \otimes \delta(x) \quad \text{and} \quad S_{\alpha,\tau} = \mathcal{F}_x(E_{\alpha,\tau}^N),$$

we obtain

$$\left(\partial_t^2 + \frac{1+2\alpha}{t} \partial_t - |x|^2 \right) S_{\alpha,\tau} = \delta(t - \tau).$$

From $S_{\alpha,\tau} \in \mathcal{S}'(H)$ and $\lim_{t \searrow 0} (\partial_t S_{\alpha,\tau})(t, x) = 0$ by the Neumann boundary condition, we infer, for $x \neq 0$ fixed, that

$$S_{\alpha,\tau}(t, x) = \begin{cases} C_1(x) t^{-\alpha} K_\alpha(t|x|) : t \geq \tau, \\ C_2(x) t^{-\alpha} I_\alpha(t|x|) : 0 < t \leq \tau \end{cases}$$

with the jump conditions

$$\begin{aligned} C_1(x) \tau^{-\alpha} K_\alpha(\tau|x|) - C_2(x) \tau^{-\alpha} I_\alpha(\tau|x|) &= 0, \\ C_1(x) \partial_\tau (\tau^{-\alpha} K_\alpha(\tau|x|)) - C_2(x) \partial_\tau (\tau^{-\alpha} I_\alpha(\tau|x|)) &= 1. \end{aligned}$$

The ‘Wronskian’ determinant

$$W(\tau, x) = \det \begin{pmatrix} \tau^{-\alpha} K_\alpha(\tau|x|) & \tau^{-\alpha} I_\alpha(\tau|x|) \\ \partial_\tau (\tau^{-\alpha} K_\alpha(\tau|x|)) & \partial_\tau (\tau^{-\alpha} I_\alpha(\tau|x|)) \end{pmatrix}$$

of this linear system of equations fulfills $W(\tau, x) = D(x) \tau^{-1-2\alpha}$ (see [11, A, 17.1, p.72]), and employing the series expansions of K_α and I_α yields $D = 1$. Thus $C_1 = -\tau^{1+\alpha} I_\alpha(\tau|x|)$, $C_2 = -\tau^{1+\alpha} K_\alpha(\tau|x|)$ and

$$S_{\alpha,\tau}(t, x) = -\tau^{1+\alpha} t^{-\alpha} [Y(t - \tau) I_\alpha(\tau|x|) K_\alpha(t|x|) + Y(\tau - t) Y(t) K_\alpha(\tau|x|) I_\alpha(t|x|)].$$

The inequalities

$$|K_\alpha(u)| \leq C \min\{1, u\}^{-|\operatorname{Re} \alpha|} (1 + \log^2 u) e^{-u}, \quad |I_\alpha(u)| \leq C \min\{1, u\}^{\operatorname{Re} \alpha} e^u, \quad u > 0,$$

imply that $S_{\alpha,\tau} \in \mathcal{S}'(H)$ and $S_{\alpha,\tau}(t, x) \in L^1(\mathbb{R}_x^n)$ for fixed positive $t \neq \tau$ due to the hypothesis $\operatorname{Re} \alpha > -n/2$. These inequalities also imply that the limits $\lim_{t \searrow 0} (\partial_t S_{\alpha,\tau})(t, x) = 0$ and

$\lim_{t \rightarrow \infty} S_{\alpha, \tau}(t, x) = 0$ hold in $L^1(\mathbb{R}_x^n) \subset \mathcal{S}'(\mathbb{R}_x^n)$ by Lebesgue's theorem on dominated convergence. Hence $E_{\alpha, \tau}^N = \mathcal{F}_x^{-1}(S_{\alpha, \tau})$ is indeed the Green function of the Neumann problem for $P_\alpha(\partial)$ and $\lim_{t \searrow 0} (\partial_t E_{\alpha, \tau}^N)(t, x) = 0$ and $\lim_{t \rightarrow \infty} E_{\alpha, \tau}^N(t, x) = 0$ hold even uniformly in x .

(b) In order to calculate $E_{\alpha, \tau}^N$ for $\operatorname{Re} \alpha > -n/2$, we apply the classical Poisson–Bochner formula (see [[1, (VII, 7; 22), p.259], [12, Satz 56, p.186], [8, (1.1)]]). For $0 < t < \tau$, Equation 6.578.11 in [13] then implies

$$\begin{aligned} E_{\alpha, \tau}^N(t, x) &= -(2\pi)^{-n/2} \frac{\tau^{1+\alpha}}{t^\alpha} |x|^{-n/2+1} \int_0^\infty \rho^{n/2} K_\alpha(\tau\rho) I_\alpha(t\rho) J_{n/2-1}(|x|\rho) \, d\rho \\ &= -(2\pi)^{-(n+1)/2} e^{-i(n-1)\pi/2} \frac{\tau^{-n/2+1+\alpha}}{t^{n/2+\alpha}} (z^2 - 1)^{-(n-1)/4} Q_{-1/2+\alpha}^{(n-1)/2}(z) \end{aligned} \tag{5}$$

with z as in (3). Equation (5) also holds for $t > \tau$, either by the real analyticity of $E_{\alpha, \tau}^N$ in $H \setminus \{(\tau, 0)\}$, or by using Equation 6.578.11 in [13] again with t and τ interchanged. Eventually, we employ formula [14, 7.3.1.72] for $Q_{-1/2+\alpha}^{(n-1)/2}$ in order to derive the representation in (4) of $E_{\alpha, \tau}^N$ by the hypergeometric function. (Note that $S_{\alpha, \tau}$ and hence also $E_{\alpha, \tau}^N$ are continuous functions of t with values in $\mathcal{S}'(\mathbb{R}_x^n)$, and hence $E_{\alpha, \tau}^N$ is already determined by its restriction to $t \neq \tau$.)

(c) Let us next investigate the analytic continuation of $E_{\alpha, \tau}^N$ with respect to α . If $\alpha \in \mathbb{C} \setminus (-n/2 - \mathbb{N}_0)$, then formula (4) yields

$$\lim_{t \searrow 0} E_{\alpha, \tau}^N(t, x) = -\frac{1}{2\pi^{n/2}} \frac{\Gamma(\frac{n}{2} + \alpha)}{\Gamma(1 + \alpha)} \tau^{1+2\alpha} (\tau^2 + |x|^2)^{-n/2-\alpha}.$$

Similarly, for $(t, x) \rightarrow \infty$, we have $\frac{t}{\tau^2} \rightarrow 0$ and hence $E_{\alpha, \tau}^N$ converges to 0 if $\operatorname{Re} \alpha > -n/2$ and else grows like a multiple $r^{-n-2\alpha}$.

In order to analyze the behavior of $E_{\alpha, \tau}^N(t, x)$ near $(\tau, 0)$, we employ Equation 9.131.1 in [13]. This furnishes

$$\begin{aligned} E_{\alpha, \tau}^N &= -\frac{1}{2\pi^{n/2}} \frac{\Gamma(\frac{n}{2} + \alpha)}{\Gamma(1 + \alpha)} \frac{\tau^{1+2\alpha}}{r^{n-1}[(t + \tau)^2 + |x|^2]^{1/2+\alpha}} \\ &\quad \times {}_2F_1\left(\frac{1}{2} + \alpha, 1 - \frac{n}{2} + \alpha; 1 + 2\alpha; \frac{4\tau t}{(t + \tau)^2 + |x|^2}\right). \end{aligned} \tag{6}$$

Formula (6) clearly implies, for each $\alpha \in \mathbb{C} \setminus (-n/2 - \mathbb{N}_0)$, that $E_{\alpha, \tau}^N$ is well defined and depends \mathcal{C}^∞ on $(t, x) \in H \setminus \{(\tau, 0)\}$. Furthermore, if $(t, x) \rightarrow (\tau, 0)$, then $4\tau t / ((t + \tau)^2 + |x|^2)$ converges to 1 from below and Equation 9.122.1 in [13] yields that

$$\lim_{u \nearrow 1} \frac{\Gamma(\frac{n}{2} + \alpha)}{\Gamma(1 + \alpha)} {}_2F_1\left(\frac{1}{2} + \alpha, 1 - \frac{n}{2} + \alpha; 1 + 2\alpha; u\right) = \frac{2^{2\alpha} \Gamma(\frac{n-1}{2})}{\sqrt{\pi}}$$

if $n > 1$. Hence formula (6) shows that $E_{\alpha, \tau}^N(t, x)$ is bounded by a constant multiple of $[(t - \tau)^2 + |x|^2]^{(1-n)/2}$ near $(\tau, 0)$ for $n > 1$. If $n = 1$, we use [14, 7.3.1.30] and obtain that $E_{\alpha, \tau}^N$ grows like $(4\pi)^{-1} \log[(t - \tau)^2 + |x|^2]$ near $(\tau, 0)$. In particular, we see that $E_{\alpha, \tau}^N$ is locally integrable, depending holomorphically in $\mathcal{S}'(H)$ on $\alpha \in \mathbb{C} \setminus (-n/2 - \mathbb{N}_0)$, and by analytic continuation, we conclude that $E_{\alpha, \tau}^N$ is a temperate fundamental solution of $P_\alpha(\partial)$ for such α .

(d) Let us consider now the behavior of $E_{\alpha, \tau}^N$ if α converges to $-n/2 - k$, $k \in \mathbb{N}_0$. If n is even, then $\Gamma(\alpha + n/2) / \Gamma(\alpha + 1)$ is holomorphic, and hence $E_{\alpha, \tau}^N$ is an entire function of α . In contrast, if n is odd, then $E_{\alpha, \tau}^N$ has simple poles at $-n/2 - k$, $k \in \mathbb{N}_0$. In fact, [13, Equation 9.134.1] yields the

representation

$$E_{\alpha,\tau}^N = -\frac{1}{2\pi^{n/2}} \frac{\Gamma((n/2) + \alpha)}{\Gamma(1 + \alpha)} \frac{\tau^{1+2\alpha}}{(2\tau tz)^{n/2+\alpha}} {}_2F_1\left(\frac{n}{4} + \frac{\alpha}{2}, \frac{n}{4} + \frac{1 + \alpha}{2}; 1 + \alpha; z^{-2}\right).$$

Due to $\text{Res}_{u=-k}\Gamma(u) = (-1)^k/k!$, this implies

$$\text{Res}_{\alpha=-n/2-k} E_{\alpha,\tau}^N = \frac{(-1)^{(n+1)/2}\Gamma(\frac{n}{2} + k)}{2\pi^{n/2+1}\tau^{n+2k-1}k!} (2\tau tz)^k {}_2F_1\left(-\frac{k}{2}, \frac{1-k}{2}; 1-k-\frac{n}{2}; z^{-2}\right) \tag{7}$$

upon using the complement formula of the gamma function. Note that the residue $R = \text{Res}_{\alpha=-n/2-k} E_{\alpha,\tau}^N$ is a polynomial in x since the hypergeometric series in (7) terminates, and that $P_{-n/2-k}(\partial)R = 0$.

(e) Let us finally discuss $E_{\alpha,\tau}^D = (\tau/t)^{2\alpha} E_{-\alpha,\tau}^N$. Clearly, $E_{\alpha,\tau}^D \in S'(H)$. From the equation $P_\alpha(\partial)t^{-2\alpha} = 0$, we infer

$$\begin{aligned} P_\alpha(\partial)E_{\alpha,\tau}^D &= 2 \partial_t \left(\frac{\tau}{t}\right)^{2\alpha} \cdot \partial_t E_{-\alpha,\tau}^N + \left(\frac{\tau}{t}\right)^{2\alpha} \cdot P_\alpha(\partial)E_{-\alpha,\tau}^N \\ &= -4\alpha\tau^{2\alpha} t^{-2\alpha-1} \partial_t E_{-\alpha,\tau}^N + \left(\frac{\tau}{t}\right)^{2\alpha} \cdot \left[P_{-\alpha}(\partial) + \frac{4\alpha}{t} \partial_t \right] E_{-\alpha,\tau}^N \\ &= \left(\frac{\tau}{t}\right)^{2\alpha} \delta(t - \tau) \otimes \delta(x) = \delta(t - \tau) \otimes \delta(x). \end{aligned}$$

Hence $E_{\alpha,\tau}^D$ is a temperate fundamental solution of $P_\alpha(\partial)$ for each $\alpha \in \mathbb{C} \setminus (n/2 + \mathbb{N}_0)$. Furthermore, (4) shows that $\lim_{t \searrow 0} E_{\alpha,\tau}^D(t, x) = 0$ and $\lim_{t \rightarrow \infty} E_{\alpha,\tau}^D(t, x) = 0$ hold uniformly with respect to $x \in \mathbb{R}^n$ if $\text{Re } \alpha < 0$. Thus $E_{\alpha,\tau}^D$ is the Green function of the Dirichlet problem for $P_\alpha(\partial)$ if $\text{Re } \alpha < 0$. This completes the proof. ■

Remark 2.2: (1) By analyzing the partial Fourier transform $\mathcal{F}_x E$ similarly as in the proof of Proposition 2.2, one readily sees that Green's functions E of the Neumann problem and the Dirichlet problem, respectively, for $P_\alpha(\partial)$ can exist only if $\text{Re } \alpha > -n/2$ and $\text{Re } \alpha < 0$, respectively.

(2) The Green function $E_{\alpha,\tau}^D$ of the Dirichlet problem for $P_\alpha(\partial)$ could, albeit more laboriously, also be derived by the partial Fourier transform. Setting $S_{\alpha,\tau}^N = \mathcal{F}_x(E_{\alpha,\tau}^N)$ and $S_{\alpha,\tau}^D = \mathcal{F}_x(E_{\alpha,\tau}^D)$ yields, first for $-n/2 < \text{Re } \alpha < 0$, the equation

$$S_{\alpha,\tau}^D = S_{\alpha,\tau}^N - \frac{2 \sin(\alpha\pi)}{\pi} \cdot \frac{\tau^{1+\alpha}}{t^\alpha} \cdot K_\alpha(\tau|x|)K_\alpha(t|x|).$$

Since $\alpha \mapsto K_\alpha(\tau|x|)K_\alpha(t|x|) \in S'(H)$ is meromorphic with simple poles in $\pm(n/2 + k)$, $k \in \mathbb{N}_0$, we can conclude from this that $E_{\alpha,\tau}^N = E_{\alpha,\tau}^D$ if and only if α is entire and $[n \text{ is odd or } |\alpha| < n/2]$.

(3) Let us point out that the finite parts $\text{Pf}_{\alpha=-n/2-k} E_{\alpha,\tau}^N$ and $\text{Pf}_{\alpha=n/2+k} E_{\alpha,\tau}^D$, n odd, $k \in \mathbb{N}_0$, respectively, are not, in general, temperate fundamental solutions of $P_{-n/2-k}(\partial)$ and of $P_{n/2+k}(\partial)$, respectively. In fact, if, e.g. $R = \text{Res}_{\alpha=-n/2-k} E_{\alpha,\tau}^N$, then

$$\begin{aligned} &P_{-n/2-k}(\partial) \text{Pf}_{\alpha=-n/2-k} E_{\alpha,\tau}^N \\ &= \lim_{\alpha \rightarrow -n/2-k} (P_{-n/2-k}(\partial) - P_\alpha(\partial) + P_\alpha(\partial)) \left(E_{\alpha,\tau}^N - \frac{R}{\frac{n}{2} + k + \alpha} \right) \\ &= \lim_{\alpha \rightarrow -n/2-k} \left(-\frac{n + 2k + 2\alpha}{t} \partial_t E_{\alpha,\tau}^N + \delta(t - \tau) \otimes \delta(x) \right) \\ &= -\frac{2}{t} \partial_t R + \delta(t - \tau) \otimes \delta(x). \end{aligned}$$

Hence $\text{Pf}_{\alpha=-n/2-k} E_{\alpha,\tau}^N$ is a temperate fundamental solution of $P_{-n/2-k}(\partial)$ if and only if R is constant with respect to t .

For example, if $k = 0$, then this is the case (see (7)) and therefore $\text{Pf}_{\alpha=-n/2} E_{\alpha,\tau}^N$ and $(\tau/t)^n \text{Pf}_{\alpha=-n/2} E_{\alpha,\tau}^N$, respectively, are temperate fundamental solutions of $P_{-n/2}(\partial)$ and of $P_{n/2}(\partial)$, respectively. For example, if $n = 1$, we obtain from [14, 7.3.1.30] that

$$T_1 = \text{Pf}_{\alpha=-1/2} E_{\alpha,\tau}^N = \frac{1}{4\pi} [\log(r^2) + \log((t + \tau)^2 + x^2)] - \frac{1}{2\pi} \log(4\tau^2),$$

which of course fulfills the two-dimensional Laplace equation $(\partial_t^2 + \partial_x^2)T_1 = \delta(t - \tau) \otimes \delta(x)$ in $H = (0, \infty) \times \mathbb{R}$. Similarly, $T_2 = (\tau/t)T_1$ fulfills $(\partial_t^2 + 2t^{-1}\partial_t + \partial_x^2)T_2 = \delta(t - \tau) \otimes \delta(x)$ in H .

(4) Let us now refer to the literature. The relation $E = (\tau/t)^{2\alpha}F$ connecting two fundamental solutions E of $P_\alpha(\partial)$ and F of $P_{-\alpha}(\partial)$, respectively, can be found in [2, (2.11), p.106], where it is traced back to G. Darboux. Furthermore, the Green function $E_{\alpha,\tau}^N$ of the Neumann problem is given (up to a multiplicative constant) in the form of Euler's definite integral of the hypergeometric function in [2, (3.4), p.108], and some hints regarding uniqueness are also given at the bottom of page 108. The Green function $E_{\alpha,\tau}^D$ of the Dirichlet problem appears in [2, (4.1), p.109] and is referred to M. Olevskii. Green's functions for $P_\alpha(\partial)$ also appear in [4, (8.4), p.217, and Theorem 8.2, p.219].

As discussed in the introduction, the paper [6] already contains some of the above results, albeit in a less systematic way. First note that the notation in [6] slightly differs from ours: there, s, s_0, α, Q_α are written for our $t, \tau, \alpha + \frac{1}{2}, P_{1/2+\alpha}$. In [6, Theorem 3.1, Equation (3.2), p.503], in the case of even n , a fundamental solution ' F_α^{ev} ' of Q_α is given by a hypergeometric function, which is verified by termwise differentiation of the series expansion. In our notation, ' F_α^{ev} ' corresponds to the fundamental solution $\frac{1}{2}(E_{\alpha,\tau}^N + E_{\alpha,\tau}^D)$ and Equation (3.2) in [6] follows from formula (4) by using [13, 9.132.2]:

$$\begin{aligned} \frac{1}{2}(E_{\alpha,\tau}^N + E_{\alpha,\tau}^D) &= -\frac{1}{4\pi^{n/2}} \frac{\Gamma(\frac{n}{2} + \alpha)}{\Gamma(1 + \alpha)} \frac{\tau^{1+2\alpha}}{r^{n+2\alpha}} {}_2F_1\left(\frac{n}{2} + \alpha, \frac{1}{2} + \alpha; 1 + 2\alpha; -\frac{4\tau t}{r^2}\right) \\ &\quad - \frac{1}{4\pi^{n/2}} \frac{\Gamma(\frac{n}{2} - \alpha)}{\Gamma(1 - \alpha)} \frac{\tau}{t^{2\alpha} r^{n-2\alpha}} {}_2F_1\left(\frac{n}{2} - \alpha, \frac{1}{2} - \alpha; 1 - 2\alpha; -\frac{4\tau t}{r^2}\right) \\ &= -\frac{\Gamma(\frac{n-1}{2})}{4\pi^{(n+1)/2}} \left(\frac{\tau}{t}\right)^{1/2+\alpha} r^{1-n} {}_2F_1\left(\frac{1}{2} + \alpha, \frac{1}{2} - \alpha; \frac{3-n}{2}; -\frac{r^2}{4\tau t}\right). \end{aligned}$$

In [6, Theorems 3.2, 3.3], the case of odd $n \geq 3$ is treated and the representation of ' F_α^{od} ' in [6, Theorem 3.3, Equation (3.21), p.506] corresponds to the one of $E_{\alpha,\tau}^N$ in (4) above. (Note that the values $\alpha \in \{-\frac{1}{2}, -1, -\frac{3}{2}, \dots\}$ are excluded in [6, Theorem 3.3, p.506] although $E_{\alpha,\tau}^N$ has poles only for $\alpha \in \{-(n/2), -(n/2) - 1, \dots\}$ and is a fundamental solution of $P_\alpha(\partial)$ for all other complex values of α .)

Finally, in [6, Theorem 3.4], the case $n = 1$ is considered. We observe that the fundamental solution ' F_α^1 ' in [6, Theorem 3.4, Equation (3.27), p.507] does not seem to be correct. For example, for $P_0(\partial) = \partial_t^2 + t^{-1}\partial_t + \partial_x^2$, Equation (4) in Theorem 2.3 yields

$$E_{0,\tau}^N = E_{0,\tau}^D = -\frac{\tau}{\pi\sqrt{(\tau+t)^2+x^2}} \mathbf{K}\left(2\sqrt{\frac{\tau t}{(\tau+t)^2+x^2}}\right), \tag{8}$$

whereas ' $F_{1/2}^1$ ' in [6, Theorem 3.4, Equation (3.27), p.507] would furnish the function

$$f(t, x) = -\frac{\tau}{\pi\sqrt{(\tau+t)^2+x^2}} \mathbf{E}\left(2\sqrt{\frac{\tau t}{(\tau+t)^2+x^2}}\right).$$

However, f cannot be a fundamental solution of $P_0(\partial)$ since it is finite at $(\tau, 0)$ due to $\mathbf{E}(1) = 1$. (The letters \mathbf{K}, \mathbf{E} denote, as usually, complete elliptic integrals.)

Note that the fundamental solution $E_{0,\tau}^N = E_{0,\tau}^D$ for $n = 1$ in (8) coincides, up to a multiplicative constant, with the expressions given in [15, Equation (5.35), p.149],[16, Equation (2.2.16), p.9],[17, p.1655]. For $n = 1, \alpha = 1$, see [18, Equation (17), p.146].

Let us finally express $E_{\alpha,\tau}^N$ by elementary transcendental functions if the dimension n is even. That this is impossible in the case of odd dimensions is plainly shown by the example in (8).

Corollary 2.4: *Let $f_n(\alpha, \tau, t, z), \alpha \in \mathbb{C}, \tau > 0, t > 0, z > 1, n \in \mathbb{N}_0$, be the function given by Equation (5), which represents $E_{\alpha,\tau}^N(t, x)$ according to Theorem 2.3 for $(t, x) \in H = (0, \infty) \times \mathbb{R}^n, n \in \mathbb{N}$. Then the recursion formula*

$$f_{n+2} = -\frac{1}{2\pi \tau t} \cdot \frac{\partial f_n}{\partial z}, \quad n \in \mathbb{N}_0, \tag{9}$$

holds. Furthermore, for even $n = 2m, m \in \mathbb{N}_0, \alpha \in \mathbb{C} \setminus \{0\}$, we have

$$f_{2m}(\alpha, \tau, t, z) = \frac{(-1)^{m-1} \tau^{1-m+\alpha}}{2^{1+\alpha} (2\pi)^m \alpha t^{m+\alpha}} \left(\frac{d}{dz}\right)^m (\sqrt{z+1} - \sqrt{z-1})^{2\alpha}. \tag{10}$$

In particular, with the notation $r = \sqrt{(t - \tau)^2 + |x|^2}$ and $s = \sqrt{(t + \tau)^2 + |x|^2}$, we obtain (if $\alpha \in \mathbb{C}$ and $(t, x) \in H \setminus \{(\tau, 0)\}$)

$$\begin{aligned} \text{for } n = 2, E_{\alpha,\tau}^N(t, x) &= -\frac{\tau(s-r)^{2\alpha}}{2^{1+2\alpha} \pi t^{2\alpha} r s}, \\ \text{and for } n = 4, E_{\alpha,\tau}^N(t, x) &= -\frac{\tau(s-r)^{2\alpha}}{2^{2+2\alpha} \pi^2 t^{2\alpha} (rs)^3} (r^2 + 2\alpha r s + s^2). \end{aligned} \tag{11}$$

More generally, for even $n = 2m, m \in \mathbb{N}, \alpha \in \mathbb{C}$ and $(t, x) \in H \setminus \{(\tau, 0)\}$, we have

$$\begin{aligned} E_{\alpha,\tau}^N(t, x) &= -\frac{(m-1)! \tau(s-r)^{2\alpha}}{2^{n-1+2\alpha} \pi^m t^{2\alpha} (rs)^{n-1}} \\ &\times \sum_{j=0}^{m-1} \binom{m-1-\alpha}{j} \binom{m-1+\alpha}{m-1-j} \binom{m-1}{j}^{-1} (s-r)^{2j} (s+r)^{2(m-1-j)}. \end{aligned} \tag{12}$$

Proof: (a) The integral representation in [13, Equation 8.712] for the associated Legendre function Q_ν^μ implies that

$$\frac{d}{dz} [(z^2 - 1)^{-\mu/2} Q_\nu^\mu(z)] = (z^2 - 1)^{-(\mu+1)/2} Q_\nu^{\mu+1}(z), \quad z > 1.$$

From this and the representation of f_n in (5), we infer that

$$\begin{aligned} f_{n+2}(\alpha, \tau, t, z) &= -(2\pi)^{-(n+3)/2} e^{-i(n+1)\pi/2} \frac{\tau^{-n/2+\alpha}}{t^{n/2+1+\alpha}} (z^2 - 1)^{-(n+1)/4} Q_{-1/2+\alpha}^{(n+1)/2}(z) \\ &= \left(-\frac{1}{2\pi \tau t} \cdot \frac{\partial}{\partial z}\right) f_n(\alpha, \tau, t, z) \end{aligned}$$

holds for $n \in \mathbb{N}_0$, and this is the recursion relation (9).

According to [13, Equation 8.777.2], we have

$$(z^2 - 1)^{1/4} Q_{-1/2+\alpha}^{-1/2}(z) = \frac{-i\sqrt{\pi}}{\sqrt{2}\alpha} (z + \sqrt{z^2 - 1})^{-\alpha}, \quad z > 1,$$

and hence

$$f_0 = -\frac{i}{\sqrt{2\pi}} \frac{\tau^{1+\alpha}}{t^\alpha} (z^2 - 1)^{1/4} Q_{-1/2+\alpha}^{-1/2}(z) = -\frac{\tau^{1+\alpha}}{2^{1+\alpha}\alpha t^\alpha} (\sqrt{z+1} - \sqrt{z-1})^{2\alpha}.$$

Together with (9), this implies formula (10) for f_{2m} , $m \in \mathbb{N}_0$. The equations in (11) follow from (10) taking into account that $z + 1 = s^2/(2\tau t)$ and $z - 1 = r^2/(2\tau t)$.

(b) Obviously, the general formula for $n = 2m$ in (12) could be proven by induction over m by employing the recursion formula (9). We prefer to give a direct proof based on one of Kummer's transformation formulas for the hypergeometric function.

Let us apply [13, Equation 9.134.3] to formula (6). If we set $4\zeta/(1 + \zeta)^2 = 4\tau t/s^2$ and assume $|\zeta| \leq 1$, we obtain $\zeta = (s - r)/(s + r)$ and $1 + \zeta = s(s - r)/(2\tau t)$ and hence

$$E_{\alpha,\tau}^N(t, x) = -\frac{1}{2^{3-n+2\alpha}\pi^{n/2}} \frac{\Gamma(\frac{n}{2} + \alpha)}{\Gamma(1 + \alpha)} \frac{\tau^{n-1}(s - r)^{2-n+2\alpha}}{t^{2-n+2\alpha}(rs)^{n-1}} \times {}_2F_1\left(1 - \frac{n}{2} + \alpha, 1 - \frac{n}{2}; 1 + \alpha; \left(\frac{s - r}{s + r}\right)^2\right). \tag{13}$$

If $n = 2m$ is even, then the hypergeometric series in (13) terminates, and it readily yields the finite sum in Equation (12). This completes the proof. ■

Disclosure statement

No potential conflict of interest was reported by the authors.

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