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Exploring The Understanding Of Whole Number Concepts And Operations: A Case Study Analysis Of Prospective Elementary School Teachers

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EXPLORING THE UNDERSTANDING OF WHOLE NUMBER
CONCEPTS AND OPERATIONS:
A CASE STUDY ANALYSIS OF PROSPECTIVE
ELEMENTARY SCHOOL TEACHERS

by

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ABSTRACT

This research project aimed to extend the research literature by providing greater insight into the way *individual* prospective teachers develop their conceptual understanding of whole number concepts and operations in a social context. In this qualitative study, a case study analysis provided the opportunity for careful exploration of the manner in which prospective teachers' understanding changed and the ways two selected participants reorganized their mathematical thinking within a classroom teaching experiment. While previous research efforts insisted on creating a dichotomy of choosing the individual *or* the collective understanding, through the utilization of the emergent perspective *both* the individual and the social aspects were considered. Specifically, using the emergent perspective as a theoretical framework, this research endeavor has outlined the mathematical conceptions and activities of individual prospective teachers and thus has provided the psychological perspective correlate to the social perspective's classroom mathematical practices.

As the research participants progressed through an instructional sequence taught entirely in base-8, a case study approach was used to select and analyze two individuals. In order to gain a more thorough understanding of the individual perspective, this research endeavor focused on whether teachers with varying initial content knowledge developed differently through this instructional sequence. The first participant initially demonstrated "Low-Content" knowledge according to the CKT-M instrument database questions which measure content knowledge for teaching mathematics. She developed a greater understanding of place value concepts and was able to apply this new knowledge to gain a deeper sense of the rationale behind counting strategies and addition and subtraction operations. She did not demonstrate the ability to

consistently make sense of multiplication and division strategies. She participated in the classroom argumentation primarily by providing *claims* and *data* as she illustrated the way she would use different procedures to solve addition and subtraction problems.

The second participant illustrated "High-Content" knowledge based on the CKT-M instrument. She already possessed a solid foundation in understanding place value concepts and throughout the instructional sequence developed various ways to connect and build on her initial understanding through the synthesis of multiple pedagogical content tools. She demonstrated conceptual understanding of counting strategies, and all four whole number operations. Furthermore, by exploring various ways that other prospective teachers solved the problems, she also presented a greater pedagogical perspective in how other prospective teachers think mathematically. This prospective teacher showed a shift in her participation in classroom argumentation as she began by providing *claims* and *data* at the outset of the instructional sequence. Later on, she predominantly provided the *warrants* and *backings* to integrate the mathematical concepts and pedagogical tools used to develop greater understanding of whole number operations. These results indicate the findings based on the individual case-study analysis of prospective elementary school teachers and the cross-case analysis that ensued. The researcher contends that through the synthesis of the findings of this project along with current relevant research efforts, teacher educators and educational policy makers can revisit and possibly revise instructional practices and sequences in order to develop teachers with greater conceptual understanding of concepts vital to elementary mathematics.

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LIST OF ACRONYMS/ABBREVIATIONS

CGI   Cognitively Guided Instruction

CKT-M   Content Knowledge for Teaching - Mathematics

CMP   Classroom Mathematical Practices

CTE   Classroom Teaching Experiment

HLT   Hypothetical Learning Trajectory

IRB   Institutional Review Board

MKT   Mathematics Knowledge for Teaching

NCTM   National Council of Teachers of Mathematics

NMAP   National Mathematics Advisory Panel

RME   Realistic Mathematics Education

CHAPTER 1: INTRODUCTION

In an ever increasing technological world, the emphasis on learning and understanding mathematics remains paramount. Daily operations and equipments in schools, homes, and workplaces all around the world rely heavily on mathematical notions. As for the future, mathematics serves as a gateway towards educational opportunities and discoveries in science, computer technologies, entertainment, and communication (National Research Council, 2001). The key to prolonged success of any society rests with its educational opportunities; and elementary schools are at the forefront of preparing the minds of our children for the future. In Principles and Standards for School Mathematics (NCTM, 2000), the National Council of Teachers of Mathematics (NCTM) highlighted the significance of critical educational reforms and specified fluency with numbers and operations along with number sense as major themes of elementary education.

Statement of the Problem

Throughout the history of mathematics education reform movements, one aspect has remained constant. Insightful programs can succeed – often times with dramatic results –when efforts are made to assist teachers and administrators assume and carry out their new roles (Darling-Hammond, 1997; NCTM, 2003). The National Mathematics Advisory Panel (NMAP) published its findings in 2008 and recognized the central role of mathematics teachers' knowledge. Specifically, mathematics teachers need to know the mathematics content in detail and from a more advanced perspective than their students (National Mathematics Advisory

Panel, 2008). In *How people Learn –Brain, Mind, Experience, and School* (Bransford, Brown &

Cocking, 1999), Bransford and his colleagues address one of the problems facing our elementary

school teachers:

> *"If teachers are to prepare an ever more diverse group of students for much more challenging work—for framing problems; finding, integrating and synthesizing information; creating new solutions; learning on their own; and working cooperatively - they will need substantially more knowledge and radically different skills than most now have and most schools of education now develop"* (p. 233).

Since U.S. teachers are entering the classrooms without a profound knowledge of the

precise mathematics they are/will be teaching (Ma, 1999), efforts are needed to insure that

prospective teachers gain a deeper, conceptual understanding as a part of their educational

training. Whole number concepts and operations form the foundational understanding in

elementary grades which are vital to further development of fractions and geometry topics and

eventually leading to algebraic notions. It is precisely this foundational understanding that needs

to have deep, firm roots in order to eventually guide our students to the algebraic gateways that

are essential in attaining academic and financial success (National Mathematics Advisory Panel,

2008).

To compound the issue of the dire need for highly qualified teachers who possess this

conceptual understanding, current projections indicate that "the total elementary and secondary

enrollment is projected to increase an additional 10 percent between 2005 and 2017" (National Center for

Education Statistics, 2008, p. 5). In order to meet the increase from 55.2 million to the 60.4 million

students projected to be enrolled in PK-12 by 2017, the United States will need to hire an additional

two million teachers to account for the rising student enrollment, teacher attrition, and the

growing group of retiring teachers. Furthermore, current and prospective teachers will "need to

be prepared to teach an increasingly diverse group of learners to ever-higher standards of academic achievement" (Darling-Hammond, 1997, p. 162). Teachers need to understand the big ideas of mathematics and have the ability and background to represent mathematics as a connected endeavor in a coherent fashion (Schifter, 1993). Planned inquiry and examination of the notion of place value and whole number concepts and operations help to develop insights for prospective teachers both to enhance their students' conceptual understanding and simultaneously construct a deeper content knowledge base for the teachers.

Significance of Study

A prospective elementary teacher was provided the following word problem along with a fictitious student's work. The prospective elementary teacher was asked to explain whether the student was correct and was expected to provide a rationale for her choice. Upon examining the word problem and the provided student's work, note how the prospective teacher was able to unpack the student's thinking relative to place value concepts.

"Word Problem:"

There were 312 marbles in a toy store. 165 marbles were sold. How many marbles were left?

Student's solution:

$$
\begin{array}{ccc}
 & \overset{11}{\cancel{1}} & \overset{12}{\cancel{2}} \\
- \quad 3 & & \\
2 & 7 & \\
\cancel{1} & \cancel{6} & 5 \\
\hline
1 & 4 & 7
\end{array}
$$

Upon examining this fictitious student's work, the prospective teacher reflected:

The student knew that he couldn't take 5 from 2; therefore he took a 10 from the 6 which made the 6 a 7 and the 2 a 12. Then he knew he couldn't take a 7 from 1, therefore he took a 10 from the hundreds column and made the 1 in the hundreds column a 2 and the 1 in the tens column an 11. Lastly, he subtracted 5 from 12, the 7 from the 11 and the 2 from the 3 to get the answer.

This example illustrated the prospective teacher's limited understanding in making sense of the student's work. While the prospective teacher demonstrated that she understood the manner in which the student related the numbers according to place value, she did not fully grasp his solution. This student did not "take a 10 from the 6" as the prospective teacher reflected, but rather he realized that by making the 2 into a 12 he had added 10 ones. To compensate for adding 10 ones, he added one group of tens to the 6, therefore replacing the 6 by a 7. This student continued to reason according to place value by making the 1 into an 11 and *equally compensated* by adding one group of hundreds to the 1 making it into a 2. Next, he proceeded to subtract each of the columns according to place value and arrived at his answer of 147 marbles. Through a solid foundation of whole number concepts and operations, prospective teachers such as the one described above will be much better prepared to teach mathematics in a meaningful way and have the conceptual understanding to relate to students' thought processes more effectively.

The primary purpose of this study was to highlight the individual understanding of prospective elementary teachers' development of whole number concepts and operations. Current efforts in the reform of education and teacher preparation emphasize the significance of teacher knowledge (NCTM, 2000; National Mathematics Advisory Panel, 2008). Liping Ma (1999) writes: "While we want to work on improving students' mathematics education, we also

need to improve their teachers' knowledge of school mathematics. The quality of teacher subject matter knowledge directly affects student learning-and it can be immediately addressed." (p. 144). In fact, teachers are viewed as the key figures in the implementation of the standards and guidelines set forth by NCTM and the National Research Council (National Research Council, 2001).

Another significant component towards the improvement of understanding in teachers deals with the social context in which learning occurs. Cochran, DeRuiter & King (1993) infer that teaching for understanding and teachers' abilities are enhanced if they are acquired in contexts that resemble those in which they will be using their knowledge – specifically a classroom context. In fact, prospective and in-service teachers can gain valuable insights into the learning and teaching of mathematics in an inquiry-laden classroom environment (Carpenter, Franke, Jacobs, Fennema & Empson, 1998; Kazemi, 1999).

In this qualitative study, a design-based research project was undertaken as the framework to examine prospective teachers' understanding of whole number concepts and operations. The goal of design-based research is to lead to an eventual educational theory (Cobb, 2003) while going through the iterative process in a social context that integrates the analysis of students' learning within a collective classroom setting. Classroom teaching experiments allow researchers to carefully analyze the manner in which student understanding changes and the ways students reorganize their mathematical thinking (Steffe & Thompson, 2000). Typically, the goals of design-based research projects involve: (1) Analyzing the relationship between the instructional design and student learning, and (2) Analyzing the collective classroom dynamic in relation to individual student's developing understanding. As for the participants, this type of

classroom inquiry promotes opportunities to become flexible learners and allows for the

development of conceptual knowledge with depth including a connected web of understanding

(Hiebert & Carpenter, 1992). The aforementioned classroom teaching experiment (Cobb, 2000;

Steffe & Thompson, 2000) took place in an undergraduate mathematics content course intended

for prospective elementary school teachers. Due to the cyclic nature and design of classroom

teaching experiments, this iteration continued along the line of a previous classroom teaching

experiment (Andreasen, 2006) conducted two years earlier in a similar setting.

In order to collect and analyze the data, a framework was needed that examined both the

learning and development of the individual as well as the collective in the social context. While

various previous research efforts insisted on creating a dichotomy of choosing the individual OR

the collective understanding, the emergent perspective insists that no such division is needed. In

fact, learning takes places simultaneously in both contexts while neither the individual

perspective nor the social aspect takes primacy over the other (Cobb & Stephan, 2003). The

social and the individual are forever linked; and yet each needs to be carefully analyzed to create

the full picture. Furthermore, teachers need to understand the big ideas of mathematics and have

the ability and background to represent it as a connected endeavor in a coherent fashion

(Schifter, 1993, 1998). Prospective teachers in this classroom teaching experiment were viewed

as reorganizing their thinking as they participated and contributed to their context both socially

and mathematically.

Previous research efforts have illustrated that children tend to progress through several

developmental phases in learning about whole numbers and place value concepts (Cobb &

Wheatley, 1988; Steffe, 1983). In fact, recent research endeavors have demonstrated that

prospective teachers followed developmental stages similar to children if placed in a context to examine whole number concepts and operations from a different perspective (Andreasen, 2006; Roy, 2008, Tobias, 2009, Wheeldon, 2008). McClain (2003) introduced the notion of using an alternative base with prospective elementary teachers in order to develop instructional tasks and goals to foster understanding. For the purposes of this research study, the particular alternative base selected was base-8 in continuation of previous research efforts (Andreasen, 2006; McClain, 2003, Roy, 2008). Base-8 was quite suitable since the values in this particular base do not immediately reach its "maximum place value" or "ten" – as opposed to a base system such as base-4. Since both base-8 and base-10 are even, they share common base system characteristics and eventually lead to common strategies used to solve whole number operations. Unlike children who enter elementary classrooms without significant experience with our base-10 number system, prospective elementary teachers have experienced the traditional base-10 system for a number of years. In a social context using an instructional sequence in base-8, prospective teachers had the opportunity to reexamine their thinking and re-evaluate their conceptual understanding of whole number concepts and operations.

Research Focus

Teachers continue to be the central figures in guiding students in a community of practice. Teacher understanding of the inner relationships between classroom norms and mathematical learning is critical for designing an appropriate learning environment. Prospective teachers develop flexible understanding of how and when to use their content knowledge if they learn how to extract ideas from their learning and teacher preparation programs. With whole number concepts and operations at the core of elementary school mathematics, prospective

teachers' development of these notions should be studied further. In particular, this research study explored the following question:

> ➢ In what way does the conceptual understanding of individual prospective teachers develop during an instructional unit on whole number concepts and operations situated in base-8?

Summary

Teaching for understanding remains essential as it relates to student achievement and progress. The fundamental aspects of the understanding of whole number concepts and operations have been well cited in the research literature. While many research studies have contributed greatly towards understanding children's development of whole number concepts and operations, significantly less research has been done with respect to prospective teachers' development of the same topics. In particular, this research intended to contribute to the study of prospective teachers in the social context and expand on their individual understanding of whole number concepts and operations within this collective setting.

In the next chapter, a review of the relevant literature was provided that addressed prospective teachers as well as children's development and understanding of whole number concepts and operations. Furthermore, teacher knowledge and the manner in which the particular instructional sequence of this research effort affected prospective teachers' development were discussed. The third chapter focused primarily on methodological aspects of this research including a case study methodology that relied on the interpretive framework based on the emergent perspective (Cobb & Yackel, 1996). The fourth and fifth chapters involved the case

study analysis of the individual prospective teachers and their mathematical activities with a focus on the way each individual developed her conceptual understanding of whole number concepts and operations. The sixth chapter involved a cross-case analysis of the prospective teachers while comparing and contrasting their development through the instructional sequence. Finally, Chapter seven included implications for future research, limitations of this study and some concluding remarks.

CHAPTER 2: LITERATURE REVIEW

In order to accurately portray the developments in mathematics education relevant to the understanding of whole number concepts and operations, an overview of the literature was presented in the following manner. First, a historical perspective demonstrated the thoughts and theories that have led to our present notions. Next, this study specifically considered children's development of whole number concepts followed by operations. At this juncture, this study focused on prospective teachers and their knowledge base in relation to the topic at hand. This review of literature next included the preparation of prospective teachers in their teacher education programs and the manner in which they have been prepared to meet the needs of their future students. After this point, the proposed path that prospective teachers would take in their mathematics content course in order to learn whole number concepts and operations was examined by looking at a hypothetical learning trajectory (HLT). This chapter on the literature review aimed to provide the background research as well as set the stage for the design and methodology which was discussed in Chapter 3.

Historical Perspective

The goal of understanding students' learning of mathematical topics is not new to the field of education as efforts have been made since the days of Pestalozzi (1746-1827) and Montessori (1870-1953) during the last few centuries. Specifically, John Dewey advocated that the most effective learning tends to be self-directed, guided by theory, and ideally attached to experiences (Dewey, 1915). He emphasized that learning does not represent a set of disconnected events which take place in isolation, but rather as an integrated lifelong process.

Dewey provided a multi-step approach to the learning process by identifying an initial stage of experience, followed by a reflection phase. During this phase of reflection, the emphasis is on the synthesis of experience with theory as one revisits and generalizes based on the reflection. The postulation of the learning process concludes with a new generalization which may be verified and tested in the realm of practice guiding new learning cycles (Dewey, 1997).

Learning, according to Piaget, is understood as a process of conceptual growth which requires the formation and reorganization of concepts in the mind of the learner (Piaget, 1965). This knowledge may not be communicated but rather is constructed and continuously reconstructed by individuals through an active process of "doing" mathematics. Empirical evidence has repeatedly supported this argument which favors learning and teaching mathematics for understanding (Polya, 1957, 1985). In the specific domain of arithmetic and understanding whole number concepts and operations, Brownell (1935) contended, "If one is to be successful in quantitative thinking one needs a fundamental of meanings, not a myriad of 'automatic responses' " (p. 10). If this notion is to be "meaningful", the instruction needs to emphasize the teaching of arithmetical meanings and making these notions sensible to children through the development of mathematical relationship (Brownell, 1947, Buswell, 1951, NCTM, 1970, 2003). In the 10th yearbook published by NCTM, Brownell addressed whole number concepts and operations by recommending an intelligent grasp of these topics for children in order to deal with proper comprehension of the mathematics as well as their practical significance.

Upon the discussion of the critical nature of learning whole number concepts and operations, the next natural question should address who will be responsible for teaching these

significant notions and how should they proceed? Clearly, since elementary school children

spend a majority of their time dealing with such topics, elementary school teachers carry the vast

majority of the responsibility for establishing and developing these topics in children. In the

words of Bruner (1962), "When we try to get a child to understand a concept…the first and most

important condition, obviously is that the expositors themselves understand it" (p. 105).

Therefore, current and future elementary school teachers should have the foundational

understanding and appropriate education to accomplish this vital task. Furthermore, the teaching

and learning of mathematics for understanding has and continues to be consistent with the

recommendations of NCTM (1989, 2000) and the National Research Council (1996, 2001).

Historically, the reform programs and movements that were implemented through careful

deliberation and by people who understood the intricacies of learning in children and prospective

teachers have had "often dramatic results" (NCTM, 1970, p. 627).

As the primary focus of this research, this study intentionally examined the conceptual

understanding of prospective teachers within the domain of whole number concepts and

operations. As previous research in this topic has indicated (Andreasen, 2006; Roy, 2008),

prospective teachers tend to progress through levels of development very similar to the stages

that children experience in learning whole number notions. Hence, this review of literature shall

next describe the ways that children typically develop their thinking in order to inform the

understanding of prospective teachers.

Children's Development of Whole Number Concepts

From an analysis perspective, it should not be difficult to fathom the reasons behind

children's struggles with the notion of number. After all, in the base-ten system, the value of a

digit is comprised of a dual meaning: the value of the digit AND its position within the numeral. That is to say a single numerical symbol can simultaneously represent different notions depending on its placement within a number. The compactness and sophistication of this number system are intertwined with its inherent initial complicated nature.

When children begin to learn to count, they do not attend to a difference between a single-digit and a multi-digit number. From a child's perspective, the number 10 follows the number 9 very much in the same way that the number 9 followed the number 8. The numbers aforementioned simply represent a number of items. In fact, research has illustrated that children observe that 10 only differs from its precedent number 9 in that 10 represents one additional item (Fuson et al., 1997; Steffe & Cobb, 1988). Needless to say, the number ten, and for that matter the naming of *ten*, inherently does not serve notice that *nine* and *ten* represent different numbers of digits (Thanheiser, 2005). Furthermore, children do not perceive multi-digit numbers as a partitioning of parts but as a collection of objects. Ross (1990) reported on students in grades 3-5 and their construction of meaning for the individual digits in a multi-digit numeral by matching the digits to quantities in a collection of objects. With such tasks, results from 4th and 5th graders indicated that no more than half of the 71 students in the study demonstrated an understanding that the '5' in '25' represented five of the objects and the '2' the remaining 20 objects (Kamii, 1986; Ross, 1990).

In fact, a significant change in understanding occurs precisely when children can simultaneously recognize single objects as units - iterable in nature - as well as a single, countable quantity (Fuson et al., 1997; Reys et al., 1995; Steffe & Cobb, 1988). Fuson (1997) distinguishes between a *unitary conceptual structure* and a *multiunit conceptual structure*. The

unitary conceptual structure refers to a group of single objects, where the child may count by ones or tens, however; the ten represents a shortcut to counting the single units. In this case, the "number of tens" or perhaps the "number of hundreds" is not being counted (Thanheiser, 2005, 2009). Conversely, in the *multiunit conceptual structure,* this notion of "number of tens" or "number of hundreds" represents

> …a collection of entities (such as counting "one hundred, two hundred, three hundred," in which the referent for each "hundred" is a collection of 100 entities of some kind) or a collection of collections of objects (as in the cardinal reference of "seven hundred" to a collection of seven collections of 100 entities). (Fuson, 1990, p. 273)

The primary distinction between the two aforementioned levels is that children who are unable to form units of units observe the number to be a collection of units to assist them in counting. On the other hand, children who can employ the *multiunit conceptual structure* tend to form units of units in order to observe the number of sets of tens, hundreds, thousands, etc. Figure 1, on the following page, illustrates the students' abilities to "flexibly see ten as both 1 ten and 10 ones – that is, to be aware that ten can refer to 10 ones and 1 ten simultaneously and be able to choose the more suitable interpretation, as 1 ten or 10 ones, for a given situation." (Thanheiser, 2009, p. 253).

Image of ten as representing 10 ones and 1 ten simultaneously.

Image of ten with attention to 1 ten.

Image of ten with attention to 10 ones.
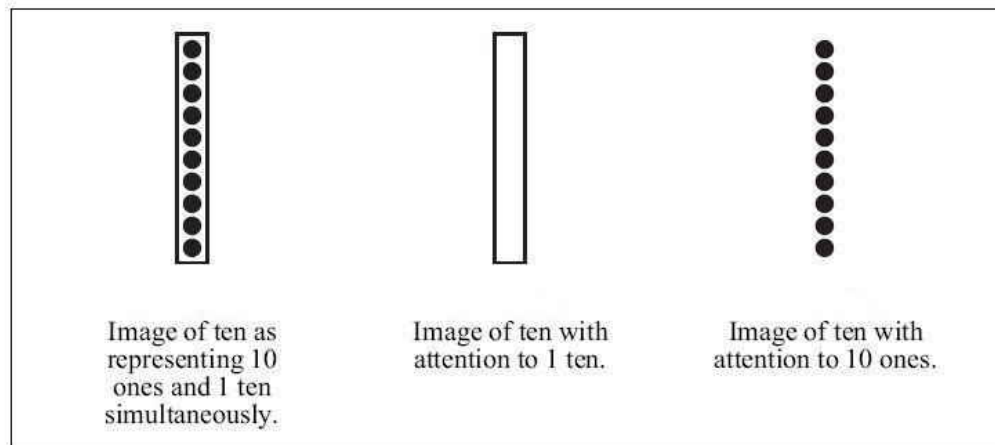
Figure 1: Flexible Understanding of Ten as Both 1 Ten and Ten Ones (Thanheiser, 2009)

According to both Fuson (1997) and Thanheiser (2009), in order to have the most sophisticated conceptual understanding of two-digit numbers, students need to possess the flexibility to realize that ten could concurrently represent ten ones and one ten. In Figure 2, a flexible understanding of ten enables students to consider a number such as 37 – as demonstrated below on the left – and see 1 ten and 10 ones simultaneously in the number. However many times - as shown in Figure 2 on the right - while students are aware that "ones and tens are different unit types, they do not see that 10 ones make 1 ten." (Thanheiser, 2009, p. 253)

Regrouped 37 with flexible image of ten.

Regrouped 37 with focus on 1 ten.
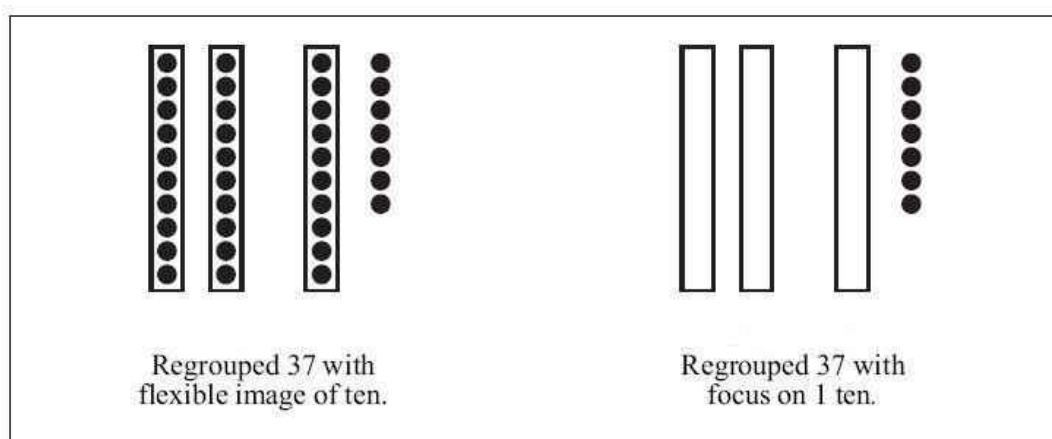
Figure 2: Students' Possible Conceptions of the Number 37 (Thanheiser, 2009)

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In examining children's development of units, there are three major conceptions of *Ten*:

Ten as a numerical composite:

Ten would represent a collection of objects that in itself does not yet comprise a unit. This conception relies on the single pieces or elements of the composite rather than the whole composite. According to Cobb and Wheatley (1988), "Children for whom ten is a numerical composite are yet to construct as *(sic)* a unit of any kind. There are either ten ones or a single entity sometimes called 'ten' but not both simultaneously" (p. 5)

 Ten as an abstract composite unit:

Ten in this aspect does represent a composite unit and yet simultaneously maintains a sense of "ten"ness. Even though the recognition is made that ten represents 10 one's, the key aspect in this stage lies in the fact that children still do not increment by 10 when they count by 10's. For example, in the sequence of 10, 20, 30 objects, the 30 would represent 20 more single objects than 10 and not two more tens. While children at this stage will recognize that 24 objects can be grouped into two groups of ten and 4 ones, they struggle with sharing these 24 objects among 3 friends. In other words, once the child has thought of ten as a composite unit, s/he experiences a multitude of difficulties in reimagining the ten as ten single objects (Cobb &Wheatley, 1987).

Ten as an iterable unit:

During this conception of ten, children do view ten as a group of objects and concurrently can extract and deal with the single units. Here ten can be a single unit of 10 AND a collection of 10 single objects. According to Steffe and Cobb (1988):

The construction of the iterable unit of ten was required before the children could understand the positional principle of the numeration system. We were surprised at how difficult it was for them to understand that each decade comprises a number sequence of numerosity then and also that a counting by ten act could increment by ten more ones. (p. 8)

Over the course of the past few decades, researchers have identified significant stages in the early number development of children (Fosnot & Dolk, 2001; Kamii, 1985; National Research Council, 2001). Some of these stages include the notion of place value and unitizing as already mentioned. Other important stages include the notion of cardinality, one-to-one correspondence, and hierarchical inclusion. According to the National Research Council (2001), *cardinality* refers to the notion that the last counting word spoken will indicate how many objects are in the set as a whole. *One-to-one correspondence* implies that there must be such a 1:1 relationship between objects counted and the counting words. *Hierarchical Inclusion* refers to the notion that whole numbers will increase by precisely 1 every time in the progression.

Furthermore, Fuson and colleagues discuss children's numerical associations between numbers spoken, numbers written, and numerical quantities. Such associations often illustrate strategies such as counting in a disorganized fashion, counting on fingers, noticing and organizing patterns in numbers, and connecting numbers and words as well as experiences with manipulatives. The progression that many children follow observes a trend from using fingers to connect a quantity of objects with numbers, onto counting with a one-to-one correspondence, and resorting to various counting strategies upon gaining proficiency with previous strategies (Fuson et al., 1997, 2001). Such approaches lead to skip counting, counting forwards and backwards, as well as to addition and subtraction strategies (Fosnot & Dolk, 2001).

Children's Development of Whole Number Operations

Upon becoming more proficient with counting numbers, children are asked to examine whole numbers in meaningful contexts such as story problems. In such "word problems", children are often provided single-digit whole numbers and asked to add or subtract them in order to arrive at an unknown quantity. Contrary to popular belief and the apparent disdain of word problems by older students, children actually excel in solving story problems as the context adds meaning and allows for the development of strategies to solve such problems (Burns, 1992). Initially, children tend to model or represent story problems using a drawing or some manipulatives (Fuson, 1992). Some children tend to count by ones, whereas others with a more developed understanding of whole number concepts may directly model the problem to eventually arrive at a recognized operation (Cobb & Wheatley, 1988; Steffe, 1983).

Many young children come to elementary schools with the ability to solve single digit addition problems. As Fuson (1992) and Steffe (1983) have indicated, children who did not possess an understanding of the place value notion tended to struggle at this stage and beyond. In the next few sections, some of children's strategies and invented algorithms will be presented to provide insight into the mathematical understanding that is needed to use each of these approaches. Andreasen (2006) and Roy (2008) have shown that placed in a new context, prospective teachers followed a similar progression to children in developing understanding of whole number concepts and operations. Hence, this research project benefited from having an understanding of the path children follow to gain proficiency with whole number operations in order to inform the path that prospective teachers ultimately followed.

Research supports that allowing children to develop strategies that are "new" (to them) deepens students' conceptual understanding and eventually leads to further understanding of traditional algorithms (Caroll & Porter, 1997; National Research Council, 2001). Children typically use flexible strategies to solve (story) addition and subtraction problems (Russell, 2000; Schifter, 1998). In fact, students tend to modify their approach and become more efficient at these strategies as they progress through the whole number operations portion of the curriculum. As stated earlier, while exploring whole number concepts and operations, prospective teachers in earlier studies modeled addition and subtraction strategies commonly illustrated by children (Andreasen, 2006; Roy, 2008). Each of the strategies stated in Table 1 are demonstrated in the upcoming section, note the significant role of understanding place value concepts in developing proficiency with whole number concepts and operations.

Table 1: Children's Addition and Subtraction Strategies

	Strategy 1	**Strategy 2**	**Strategy 3**	**Strategy 4**
Addition	Counting-on	Near doubling	Adding to ten	Adding one-less-than-ten
Subtraction	Front-end subtraction	Compensation	Taking extra and adding back	Subtracting by equal additions

Addition Strategy 1: Counting-on

Elementary school children often times use the counting on strategy when one addend is small. For instance, in solving $7 + 4$, children begin with 7 then counts on four more: 8, 9, 10, 11. Children typically begin with the larger addend and then *count on* as many numbers as needed according to the other addend. In a similar scenario given $3 + 14$, children may start by counting on 14 from 3, or in some cases start with the larger addend 14 and count on 15, 16, 17 even though the larger addend is the second addend in the expression.

Addition Strategy 2: Near doubling

Another strategy used by children in solving problems of the type $7 + 8$ is called near-doubling. Some children will use the addition fact of $7 + 7 = 14$, and then simply add 1 to finalize their solution. This method suggests a "double plus 1" approach. Other children will take a similar problem, $8 + 7$ and use the addition fact of $8 + 8 = 16$, and then subtract 1 to finalize their solution. This method utilizes a "double minus 1" approach in solving whole number addition problems.

Addition Strategy 3: Adding to ten

When adding pairs of numbers whose total is greater than ten, many children use *adding to ten* strategies. Taking the previous example of $8 + 7$, children will think through the transitory stage of "what do I need to add to 8 to make a 10?" Next, they regroup 2 of the 7 objects and group them with the 8. The next question to be answered involves "ten and 5 more makes how much?" In Figure 3 on the next page, note the way that children's mental process of *adding to ten* may be recorded visually.

8 + 7   Adding to Ten   10 + 5 = 15
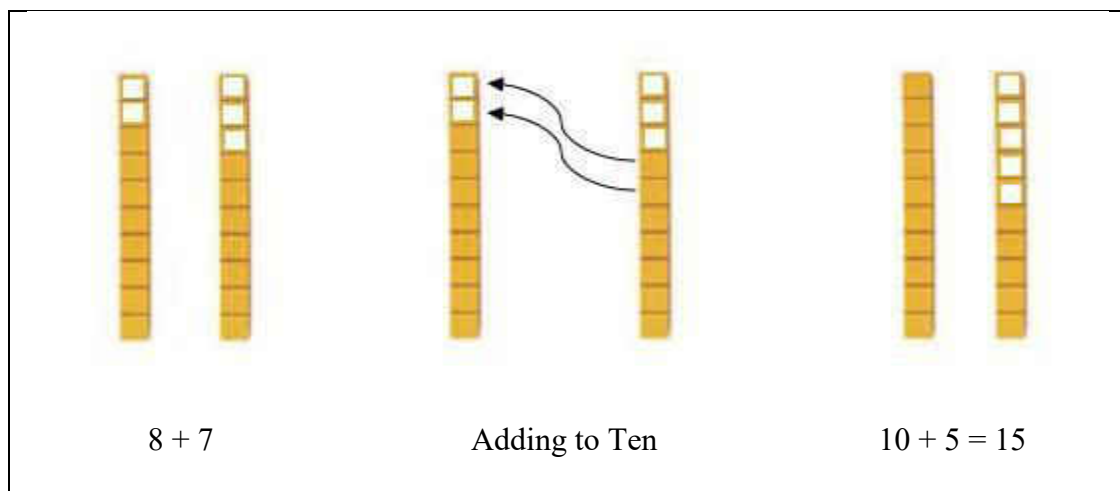
Figure 3: Model of Adding to Ten Strategy in Solving 8 + 7

Addition Strategy 4: Adding one-less-than-ten

This particular strategy is used by children in the particular instances when adding nine as one of the addends. The one-less-than-ten strategy becomes very efficient when children are asked to solve problems such as 6 + 9. More advanced children solve this problem by performing 6 + 10 = 16. Next, knowing that 9 is *one-less-than-ten*, then the result should be one less than 16. In other words, through familiarity with adding by place value, children can quickly arrive at the result of 6 + 9 = 16 − 1 = 15. This particular strategy can be extended to multi-digit addition problems in scenarios when one of the addends has a 9 such as an addend of 39 or 59.

Similar to addition strategies discussed above, many subtraction strategies also require a solid foundation of place value understanding. Children need to realize that a number such as 23 is not simply a 2 and 3. According to Cobb and Wheatley (1988), children arrive at the understanding of conservation of number as well as seeing "ten" as an abstract notion in addition to an iterable unit. A few of the subtraction strategies consistently used by children have been illustrated in the following section.

Subtraction Strategy 1: Front-end subtraction

In solving a problem such as 62 – 38, many children who have not yet been taught standard algorithms for multidigit subtraction begin by taking away the ten first. In other words, their thought process begins at the "front" of the number and can be modeled by 62 – 30 = 32. Next, children using this subtraction strategy will subtract 8 from the resultant number. By performing 32 – 8 = 24, these children have not followed the traditional subtraction algorithm which begins with subtracting the ones unit first. Using this technique, children need to have an understanding of decomposing numbers according to place value to arrive at 62 – 38 equals 24.

Subtraction Strategy 2: Compensation

Continuing with the subtraction problem 62 – 38, some children will compensate by adding what it takes to make the bottom number a multiple of ten. In this case, children would add 2 to both numbers and instead solve the problem 64 – 40 (compensating by 2). The new problem of 64 – 40 appears to be much easier for children and can be solved using a variety of methods to arrive at the result of 24. Through the use of compensation strategies, children have once again solved the problem 62 – 38 to get 24.

Subtraction Strategy 3: Taking extra and adding back

In a somewhat different approach from the compensation strategy, some children only change one of the numbers and then adjust to make it model the original problem. In the case of the same example 62 – 38, some children might take an extra two away so that the problem could be rewritten as 62 – 40 = 22. Then, knowing that they have taken an extra two away, they will add the amount 2 back to the result. Therefore, 2 must be added to 22 to get 24 as the final answer. Note that this strategy is different from compensation as children only adjusted one of

the two original numbers to perform this subtraction problem in a manner that made sense to them individually.

Subtraction Strategy 4: Using place value understanding to subtract by equal additions

A related - yet distinct – approach from the compensation strategy discussed earlier involves using place value understanding. Children who possess an understanding of 1 group of *tens* simultaneously equaling 10 *ones* use the ideas of decomposing numbers and an algorithmic approach to subtract by equal addition. In this approach, children will add the same amount but add it in different place values – such as 10 *ones* in the ones column and 1 *ten* in the tens column - to solve a desired subtraction problem. Figure 4 revisits the same subtraction problem 62 – 38 in order to demonstrate subtracting by equal addition. Note the way children will change the 2 in 62 and the 3 in 38 by adding the same *amount* to both numbers.
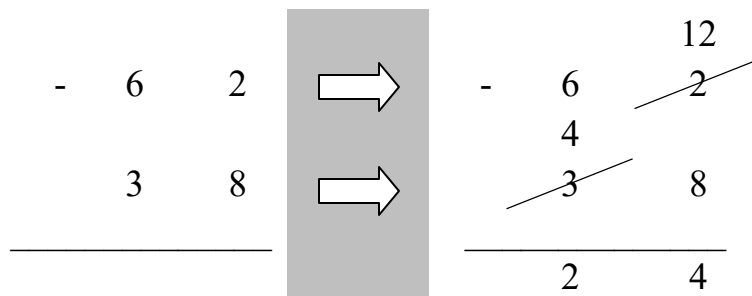
Figure 4: Strategy of Subtracting by Equal Addition

Children's thinking in this example illustrated their thought process and approach to subtracting 8 (the ones place value in 38) from the 2 (the ones place in 62). By adding ten to the ones place in 62 (making it 12), it would make it easier for children to subtract. But by adding ten to the 62, the 38 needs to have ten added to it as well in order to keep the difference the same. As a result, children will add not 10 ones, but the equivalent 1 tens to account for this approach.

In an illustration of understanding place value and decomposition of whole numbers, children add the ten by adding 1 to the tens digit of the number being subtracted (making the 3 in 38 into a 4 to make 48). Finally, children proceed by subtracting according to place value. In the ones place, 8 from 12 is 4, and in the tens place, 4 from 6 yields 2. Using this complex and advanced strategy that required conceptual understanding of whole number concepts and operations, children solved 62 – 38 = 24.

At this point, this research study addresses children's development of multiplication and division concepts and the associated strategies. Children typically begin by directly modeling multiplication and division problems (Carpenter, et al., 1996). According to Fosnot and Dolk (2001), elementary school children tend to use groups of objects to extend their ideas of addition and subtraction in order to model multiplication and division problems. As outlined by Steffe (1988), notions of multiplicative reasoning begin to take form when children have realized the conceptual meaning of the number of groups and the number of objects in each group. Gradually, similar to addition and subtraction, children gain increasing efficiency in solving multiplication and division problems and experiment with using iterable units (Cobb & Wheatley, 1988). The next section will address some common multiplication and division strategies typically observed.

In *Adding It Up* (National Research Council, 2001), various whole number concepts and operations are discussed. While exploring whole number concepts and operations, prospective teachers in studies by Andreasen (2006) and Roy (2008) modeled multiplication and division strategies commonly illustrated by children as described in *Adding It Up*. Each of the strategies stated in Table 2 will be demonstrated in the upcoming section. Note the significant role of

understanding counting strategies as well as addition and subtraction strategies in children's multiplication and division approaches.

Table 2: Children's Multiplication and Division Strategies

	Strategy 1	**Strategy 2**	**Strategy 3**	**Strategy 4**
Multiplication	Repeated addition	Skip counting	Using known number facts	Products with nine as one factor
Division	Repeated subtraction	Skip counting	Using known number facts	Division with no regrouping

Frequently, children's strategies in solving multiplication and division problems neglect the actual context of the problem (Fosnot & Dolk, 2001; Russell, 2000). A few examples of multiplication and division strategies should provide a better grasp of children's approaches to solving problems.

Multiplication Strategy 1: Repeated addition

In solving a rather elementary multiplication problem such as 6×7, often times story problems are used to model this scenario. For instance, children may be asked to solve the following problem: "A box of greeting cards included 7 cards in each packet. How many total cards would there be in 6 boxes of greeting cards?" Some children model a repeated addition strategy to solve this problem by taking $7 + 7 = 14$ and then add 7 onto the previous sum repeatedly until they have accounted for all the cards in the 6 boxes. Therefore, these children

would illustrate their mental process by writing: $7 + 7 = 14$; $14 + 7 = 21$; $21 + 7 = 28$; $28 + 7 = 35$; and finally $35 + 7 = 42$. Using repeated addition, children would arrive at $6 \times 7 = 42$. While effective, this strategy typically proves itself quite inefficient to children as the number of groups and/or the number of objects in each group increase.

Multiplication Strategy 2: Skip counting

Continuing with the same example of 6×7, children keep track of every time they count a group of 7 by using a finger or a similar technique of maintaining the number of groups counted. Children modeling a skip counting strategy for multiplication would count 7, 14, 21, 28, 35, and finally 42. This approach of solving multiplication problems does presuppose knowledge of the multiples of numbers – 7 in this case.

Multiplication Strategy 3: Using known number facts

A modification of the previously mentioned strategies involves using number facts to solve multiplication problems such as 6×7. Children may approach 6 groups of 7 objects by thinking: $7 + 7 = 14$; $14 + 14 = 28$. So far, these children have accounted for 4 groups of 7 objects; and what remains to be done is to add fourteen (two additional groups of 7) to their result to finalize this problem. Thus, the next step would involve $28 + 14 = 42$. Such strategies using number facts can resemble doubling strategies – often seen in children - with the number of groups of an object.

Multiplication Strategy 4: Products with nine as one factor

Children who have become adept with multiplicative reasoning and possess a good conceptual understanding of whole number concepts and multiplication utilize this particular strategy in solving problems when nine is one of the factors. For instance, asked to solve the

problem 9×6, some children use multiplicative reasoning to state that $10 \times 6 = 60$. Next, they

will reason that they have taken one too many groups of 6 since 9 is 1 less than 10. As a result,

they will subtract 1 group of 6 from 60 to attain 54. In short, $9 \times 6 = (10 \times 6) - (1 \times 6) = 60 - 6 =$

54. This example highlights how important it is for children to have a solid concept of ten as an

iterable unit. This particular strategy illustrates a solid foundation of the notion of multiplication

and naturally leads to the distributive property of multiplication over addition and subtraction.

　According to Baek (1998), as children's multiplicative reasoning develops through their

educational experiences, they become more efficient with multiplication strategies as well as

partitioning and compensation strategies. Furthermore, as Fuson (1990) pointed out

"understanding operations on multidigit numbers requires understanding how to compose and

decompose multidigit numbers into these multiunits in order to carry out the various operations"

(p. 273). Here, multiunits refer to larger units that are made up of multiple smaller units. Aspects

of decomposing a number in multiplication and incorporating partitive strategies lend themselves

very well to children's strategies in solving division problems. Next, a few of these division

strategies will be discussed in greater detail.

Division Strategy 1: Repeated subtraction

　Similar to the way children's strategies in subtraction followed their addition strategies,

division strategies naturally arise out of multiplicative reasoning. When presented with a

problem such as $42 \div 7$, children use their understanding of multiplication to repeatedly subtract

sets of 7 from 42 to exhaust the number 42. Specifically, a student may model their repeated

subtraction strategy by reasoning $42 - 7 = 35$; $35 - 7 = 28$; $28 - 7 = 21$; $21 - 7 = 14$; $14 - 7 = 7$;

and finally get to $7 - 7 = 0$. Since they subtracted 7 six (6) times, they conclude that $42 \div 7 = 6$. This method is analogous to children's use of repeated addition as a multiplicative strategy.

Division Strategy 2: Skip counting

Modeling skip counting for multiplication, keeping track of how many times one counts the group is another common strategy employed by elementary school children. Revisiting $42 \div 7$, children may use their fingers or tally marks to keep track of how many times they count sets of 7. Their thought process may proceed as 7, 14, 21, 28, 35, 42 – therefore 6 would be their resulting answer. The intricate nature of multiplication and division is illustrated in this strategy as children can demonstrate proficiency and efficiency of one operation to aid in developing another operation – namely, division.

Division Strategy 3: Using known number facts

Some children's ability to use number facts has been illustrated in solving multiplication problems. Similarly, children asked to solve the problem $42 \div 7$ may begin to reason through a collection of number facts at their disposal. One way of reasoning about the solution to $42 \div 7$ using known number facts entails the following strategy by a fictitious child: "10 sevens would be 70. Since 42 is much less than 70, the answer has to be much less than 10. Trying 5 sevens that would be 35, so we need 1 more seven(s). The answer would be 5 plus 1 more, so 6. So, $42 \div 7$ would equal 6." In this manner, children solve division problems by using known number facts along with reasoning strategies.

Division Strategy 4: Division with no regrouping

Presented with a scenario to reason in the context of story problems, children can combine their reasoning strategies and knowledge of number systems to perform division

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without regrouping. Consider the example, "Neeka wishes to make a book using the 54 stickers she owns. She plans to put exactly 3 stickers per page in this book. How many pages does she need for her book?" The division problem 54 ÷ 3 can be done with no regrouping as illustrated through the following approach:

$$
\begin{array}{r|l}
3\overline{)54} & \\
30 & 10\times3 \\
\hline
24 & \\
24 & 8\times3 \\
\hline
\end{array}
$$

Figure 5: Example Illustrating Division with No Regrouping

The strategy above can be described through the following fictitious dialogue: "She needs at least 10 pages since 10 pages would represent (10 × 3) thirty (30) stickers. She will not need 20 pages since 20 pages would represent (20 × 3 = 60) sixty stickers which she does not have. After placing 30 of the stickers, she would have 24 stickers left; and 24 stickers would be 8 more pages since 8 × 3 = 24. Neeka will need (10 + 8) eighteen pages for her book."

By the time elementary school children get to division, some have the ability to reason using their conceptual understanding of numbers, place value as well as knowledge of addition, subtraction and multiplication strategies. In *Adding It Up* (National Research Council, 2001), procedural fluency has been defined as "skills in carrying out procedures flexibly, accurately, efficiently and appropriately" (p. 5). Many researchers have argued that it can be very difficult to establish procedural fluency with multidigit multiplication and division without the culmination of the aforementioned understanding (Hiebert & Carpenter, 1992; Hiebert & Wearne, 1996).

Prospective Teacher Knowledge

"Like any complex task, effective mathematics teaching must be learned.
Teachers need a special kind of knowledge. To teach mathematics well,
they must themselves be proficient in mathematics, at a much deeper level
than their students. They also must understand how students develop
mathematical proficiency, and they must have a repertoire of teaching
practices that can promote proficiency."

(National Research Council, 2001, p.31)

Over the course of the past two decades, the plethora of research on teaching has been
very clear on the importance of teacher knowledge and understanding of students' ways of
thinking related to specific topics (Ball, 1988, 1991; Ball & Bass, 2000; Davis, Maher, &
Noddings, 1990; Even, Tirosh, & Moskovitz, 1996). Several studies have suggested that
teachers' knowledge of mathematics and students can greatly influence teachers' practice (Even,
1993; Even & Tirosh, 1995; Hill, Rowan, & Ball, 2005; Stein, Baxter, & Leinhardt, 1990). Stein,
Baxter, and Leinhardt (1990) suggest that the depth of a teacher's knowledge may indeed
influence the presentation of subject matter as well as whether the teacher lays the appropriate
foundation for future learning. Furthermore, they contend that the teacher's subject matter
knowledge will allow him/her to make appropriate connections with other mathematical ideas.

In 1996, the National Commission on Teaching and America's Future (1996) issued a
report called *What Matters Most: Teaching for America's Future* recommending specific steps
towards the improvement of the schools in the United States. Ball, Bass, and Hill (2004)
summarized this report:

"What teachers know and can do is the most important influence on what students learn, the report argues that teachers' knowledge affects students' opportunities to learn and learning. Teachers must know the content thoroughly in order to be able to present it clearly, to make the ideas accessible to a wide variety of students, and to engage students in challenging work."

Historically, researchers tried to correlate the number of mathematics courses taken or scores on standardized tests as a proxy measure of teacher knowledge (Begle, 1979). However, as Carpenter, et al. (1998) describe, these measures do not account for some of the complexity inherent in the teaching of mathematics. In addition, these measures do not clearly describe the nature of the relationship between student learning and teacher knowledge. Due to the immense implications on teacher education and student achievement, more recent studies are beginning to shed a more clear light on teacher knowledge. Hill, Rowan, and Ball (2005) have done significant research in identifying elementary teachers' mathematical knowledge utilizing a researcher-constructed measure of mathematical knowledge for teaching and have found that it correlates with student achievement. Conner (2007) has discussed a correlation in the ability to facilitate classroom discussion in relation to the teacher's subject matter knowledge. Shulman (1986) has famously stated that a teacher must "not only understand *that* [italics added] something is; the teacher must further understand *why* [italics added] it is so" (p. 9) These studies taken in unison suggest emphatically that a teacher's subject matter knowledge does influence his/her practice including how and how much content is presented. In addition, the type of questions asked during class, the activities designed and selected in lesson planning as well as the ability to set the foundation for connections among mathematical ideas and representations all rely heavily on a teacher's subject matter knowledge.

In order to further explain teachers' knowledge and understanding, Carpenter et al. (1996) reported on a critical program called Cognitively Guided Instruction (CGI). This program focused on understanding of children's thinking and assisted in providing teachers a better knowledge base for mathematics. CGI helped teachers formulate models of children's thinking in very specific content domains including whole number operations, fractions, measurement, and geometry. They discovered that when in-service teachers participated in a CGI program, their beliefs and instruction saw noticeable changes. These changes ranged from the mathematical procedures including problem solving skills used with children in order to build on their own mathematical thoughts to the encouragement and enhancement of the communication of mathematical skills between teachers and students. The significant result of the CGI programs noted that changes in instruction will directly increase student achievement. The in-service teachers in these programs were able to extend their knowledge directly to their own classrooms and draw upon their own experiences to inform their teaching. In programs such as CGI, increase in teachers' awareness - along with subsequent changes in instruction - have been shown to increase student achievement (Carpenter et al., 1996).

Studies that show Prospective Teacher Prior Knowledge

Even though NCTM has declared number sense a major theme of the Principles and Standards for School Mathematics (NCTM, 2000), studies on prospective teachers show a lack of understanding on various aspects of number. Next, this study explores some efforts that highlight the issues related to the content knowledge of prospective elementary teachers.

Zazkis and Khoury (1993) illustrated that some prospective elementary teachers failed to understand the foundational structure of our base-ten number system. They arrived at the

conclusion that these teachers "demonstrated insufficient understanding of the structure of our number system" (p. 50). Specifically, they showed that 47 of the 59 students (80%) failed to incorporate the fact that in a multi-digit number as one moves from right to left, each digit is ten times the value of the previous digit. Subsequently, they suggested that "further research in interview settings on similar tasks would enhance our understanding of the conceptual base of students' knowledge" of number representation (p. 50).

Ross (2001) demonstrated that prospective elementary teachers failed to see number in terms of the appropriate unit types. She studied the understanding and perceived meanings of the digits in a two-digit number. In working with 85 prospective teachers in their first mathematics content course, she wished to inquire about their understanding of place value meaning. Using an arrangement of objects – pennies in this case – Ross provided a collection of 25 pennies and stated that they totaled 25 in all. She asked the prospective teachers questions regarding the meanings of the 2 and 5 in 25. Alarmingly, only 53% of the participants could identify that the 2 in 25 illustrated 20. Since Sowder et al. (1998) illustrated that teachers tend to teach the topics that they feel comfortable teaching, what can be said of prospective elementary students' present and future hopes of engaging in meaningful instruction in the context of whole numbers and operations if they lack an understanding of place value concepts?

In regards to content knowledge, Ball (1988, 1989), Ma (1999), Menon (2004), and Thanheiser (2005, 2009) have clearly shown that some prospective elementary teachers are unable to explain algorithms and procedures with true meaning and are hence unable to understand alternative student perspectives. Prospective elementary teachers, in this sense, are different from children who learn whole number concepts and operations since these prospective

teachers already know that certain algorithms and procedures in fact work. In order to explicate *how* and *why* the algorithms work, prospective teachers need to have the content knowledge that allows them to explore these notions further and in greater detail.

Specifically, over the course of the past two decades, Ball and her colleagues have contributed a great deal to the existing research regarding prospective teacher knowledge. Ball (1991) has written about knowledge *of* mathematics as well as knowledge *about* mathematics. With respect to the former, knowledge *of* mathematics, she defines it as the understanding of particular topics, procedures, and concepts including the inter- and intra-relationships among them. As for knowledge *about* mathematics, her definition stipulates that this type of knowledge

> "Includes understanding about the nature of mathematical knowledge and activity: what is entailed in doing mathematics and how truth is established in the domain. What counts as a solution in mathematics? How are solutions justified and conjectures disproved? …
> Knowledge about mathematics entails understanding the role of mathematical tools and accepted knowledge in pursuit of new ideas, generalizations, and procedures."
>
> > (Ball, 1991, p. 7)

To see how the knowledge *of* and *about* mathematics impacts prospective teachers' content knowledge, Ball (1988) asked a group of prospective teachers what they noticed about a student's incorrect procedure for whole number multiplication. Displaying operational knowledge of multiplication, these prospective teachers noticed the mistakes made by the student. Ball reported that only 5 of the 19 prospective teachers could speak explicitly regarding "place value and numeration that underlie the multiplication algorithm. The others gave answers that were ambiguous or focused exclusively on the procedure"(p. 51). In another instance during the same research, when asked to evaluate a 2nd grader's solution to a subtraction problem

involving regrouping, all 19 prospective teachers commented on the importance of the places of the digits and the language of tens and ones. Ball discovered that even though the prospective teachers could perform and identify the computations, they were not able to fully explain the foundational concepts involved in whole number operations. She summarized her findings by stating that the analysis of prospective teachers' knowledge "highlights the dangers of assuming that they have explicit and connected understandings of basic mathematical ideas [such as number], even when they are able to operate with them" (p. 59). These alarming results are indicative of the point made by Shulman (1986) regarding teachers needing to know not only whether something *is*, but also *why* it is so.

In extending the work of Ball (1988) and Cobb (1988, 1995, 1997) related to the meanings and conceptual understanding required to fully comprehend elementary mathematics, Ma (1999) examined the procedural understanding of teachers in the United States and China. As stated earlier, whole number concepts and operations form the foundational understanding in elementary grades which are vital to further development of fractions and geometry topics and eventually to algebraic notions. In comparing the teachers in China with their counterparts in the United States, Ma discovered that the Chinese teachers had more conceptual understanding and less of a propensity towards a strictly procedural explanation of whole number operations. She found that only 14% of the teachers studied in China would be classified as procedurally oriented, whereas the primary focus of most U.S. teachers was to teach mastery of the algorithms involved in whole number operations (Ma, 1999). It is noteworthy that Ma even noticed a difference in the language used in mathematics classrooms. Chinese teachers for the most part used the word *decompose* instead of the word *borrow* in the context of subtraction. For instance,

Ma mentions that "With the concept of 'decomposing 1 ten into 10 ones', the conceptually directed Chinese teachers had actually explained both the 'taking' and the 'changing' steps in the algorithm. However, many of them further discussed the 'changing' steps in the procedure" (p. 10). In the concluding section of her book, *Knowing and Teaching Elementary Mathematics*, Ma reiterates the significance of teachers' subject matter knowledge: "The quality of teacher subject matter knowledge directly affects student learning-and it can be immediately addressed."(p. 144)

Menon (2004) continues the discussion on knowledge that prospective teachers often lack. He conducted a study on prospective teachers during their mathematics methods course following the mandatory mathematics content course. His findings included prospective teachers relying very heavily on algorithms and simultaneously being unable to use efficient strategies to approach whole number problems. For instance, when presented with the addition problem $45 + 32 = 77$, and then asked to compute a different- albeit related - problem $46 + 32 = ?$, the prospective teachers failed to use the relationship between the two problems and simply performed the calculation to attain the answer (Menon, 2004). Since prospective teachers are expected to develop appropriate and efficient strategies in elementary school children, how can they do so when they are unable to demonstrate that skill themselves? Menon wrote "if these future K-8 mathematics teachers seem to rely on learned procedures, without the profound understanding of fundamental mathematics suggested by Ma (1999), as shown by some of their explanations to the number sense test items, how well equipped will they be to teach conceptually?" (p. 57).

Thanheiser (2005) developed a framework for analyzing prospective elementary teachers' conceptions about multi-digit whole numbers. Through her research and framework,

building on the work of Kamii (1994) and Fuson et.al (1997), she described prospective teachers as falling into 4 different, broad conceptions in relation to multi-digit whole numbers. In decreasing orders of sophistication, she identified prospective teachers as having a Reference-Units conception, Groups-of-Ones conception, Concatenated-Digits Plus conception, or Concatenated-Digits conception (Thanheiser, 2005). Thanheiser discovered that prospective elementary teachers "who held either a reference-units or a groups-of-ones conception in the context of the standard algorithms were generally able to give correct answers in the additional contexts", while those prospective teachers who held "either a concatenated-digits-plus or a concatenated-digits-only conception in the context of the standard algorithms were less likely to give correct answers across the additional contexts." (p. 172) Similar to previous works cited, this research indicates some distinctions between prospective teachers' levels of understanding and the need for future research in this area.

In this section, this study noted the significance of prospective teachers' content knowledge and the need for "profound understanding" as referred to by Ma and others. In the next section, this study connects the content knowledge of prospective teachers with the role that teacher education programs play in effectively preparing prospective elementary teachers to teach whole number concepts and operations.

Development of Prospective Teacher Knowledge

Despite the crucial role that teacher content knowledge plays in teaching, "the subject matter knowledge of prospective teachers rarely figures prominently in preparing teachers". (Ball, 1988, p. 42) Often times, new teachers enter the classroom lacking confidence in their

content knowledge of the mathematics they will be teaching. Ball (1993) stated "we need to know more about these learners and develop strategies for working with what they bring with them (p. 40). According to the National Math Panel Report (2001), as a part of their teacher preparation, teachers must be given ample opportunities to learn mathematics for teaching and know in detail and from a more advanced perspective the mathematical content they will teach.

Tirosh (2000) stated that "a major goal in teacher education programs should be to promote development of prospective teachers' knowledge of common ways children think about the mathematics topics the teacher will learn" (p. 5). As illustrated earlier in the discussion regarding Cognitively Guided Instruction (Carpenter et al., 1996), these programs provide teachers with opportunities to consider what their future students know and the manner in which to approach solutions to problems. Teacher preparation programs should adequately provide instances when the prospective teacher has the opportunity to examine common misconceptions in order to challenge students' thinking (Ashlock, 2002). Furthermore, teacher education should consistently provide avenues for future teachers to engage in more advanced discussions regarding what they themselves know and additional knowledge about elementary students' conceptions. (Bransford, Brown & Cocking, 1999)

Even and Tirosh (1995) contend that prospective teachers need knowledge about students' conceptions of the mathematics they are learning. If prospective teachers do not gain this knowledge during their teacher preparation programs, how can they be expected to be adequately prepared once they start their teaching career? Specifically, they mention that a teacher's decision regarding student responses – correct or incorrect – relies on the teacher's adequate preparation including his/her content knowledge. During teacher preparation, teachers

should have the chance to "take account of common students' conceptions and ways of thinking related to specific mathematical topics ('knowing that'). S/he should be able to understand the reasoning behind students' conceptions and anticipate sources for common mistakes ('knowing why')" (Even & Tirosh, 1995, p. 13).

In fact, in order to keep pace with the recommendations of the National Council of Teachers of Mathematics, prospective elementary teachers need to be prepared properly to engage in rich, meaningful discussions regarding topics such as whole number concepts and operations. Research supports that prospective teachers must be taught in the same manner in which they will one day be teaching (Cuban, 1993; Lortie, 1975). As a part of teaching, teachers need to create some cognitive dissonance in order "to promote disequilibrium so the students would reconsider the issue" (Simon, 1995, p. 129). Therefore, teacher educators must engage prospective teachers in precisely such opportunities during their preparation programs. According to Lerman (2000, 2001), learning is achieved through cognitive conflict, which can be brought about in situations that prospective teachers will encounter. A significant aspect of teacher preparation should allow prospective teacher to develop "a new ear, one that is attuned to the mathematical ideas of one's own students" (Schifter, 1998, p. 79). As Andreasen (2006) mentioned - specifically in the domain of whole number concepts and operations – teacher preparation programs can aid future teachers to begin their careers with a foundation relying on content knowledge and fostering pedagogical considerations.

In this section, this study focused on prospective teacher preparation in order to highlight the significance of teacher experiences in manners that will promote deeper understanding and

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ultimately lead to higher student comprehension and achievement. In the following sections, this study further examines the path that learners will embark upon to achieve this realization.

Hypothetical Learning Trajectory

As a researcher, one envisions a path to build on students' notions in order to support the construction of their reasoning along the instructional sequence. Simon (1995) analyzed his own role as a teacher and researcher as he was attempting to influence his students' progression of mathematical concepts. This notion, first referred to as a hypothetical learning trajectory (HLT) by Simon, tries to imagine how students will engage in the activities and anticipates their lines of argumentation in the activities as this discourse occurs within the classroom dynamic. In describing this process, Simon proclaims: "The consideration of the learning goal, the learning activities, and the thinking and learning in which the students might engage make up the hypothetical learning trajectory…" (p. 133).

Even though several aspects and/or interpretations of the HLT exist in the literature (Gravemeijer et.al, 2003; Simon, 1995; Simon & Tzur, 2004) most experts agree that an HLT consists of the following three components: (1) the desired learning goals of instruction, (2) the instructional sequence of tasks that will be undertaken in order to support the learning goals and (3) The progression of students' development as they journey through the designed instructional sequence. At the outset, the HLT provides a framework for the instructional tasks with stated expectations that the researcher imagines the class to embark upon during the course of instruction.

At this point, it is worth noting that the hypothetical learning trajectory is in fact vastly different from a lesson plan. According to Gravemeijer et al. (2003) the distinguishing

characteristics of the HLT juxtaposed with a traditional lesson plan include the following aspects: (1) a learning trajectory possesses a socially situated nature in that it proposes the anticipated path of a particular group in a specific social context, (2) instead of a single-shot approach to a traditional lesson plan, the HLT entails an iterative cycle of planning. That is the HLT will be revisited and the hypothetical will be revised in order to arrive at an actualized learning trajectory, (3) primary focus of the HLT is on the mathematical construction of the students not the content covered, and (4) the HLT presents an opportunity for the teacher to develop a grounded theory to explain the manner in which the instructional tasks interplay in a given social environment (Cobb, 2002; Simon, 1995; Simon & Tzur, 2004). In fact, teachers may choose to adapt all or specific parts of the instructional sequence to fit their own classroom goals based on the HLT. A cyclical process thus begins with the constant revision and modification of the HLT and the instructional sequence throughout the course of the experiment in order to arrive at the actualized learning trajectory (Simon & Tzur, 1995).

The specific aspects of the HLT regarding this study of prospective teachers' conceptual understanding of whole number concepts and operations are described in detail during the third chapter on methodology. At that point, the significance of the HLT is discussed including the specific sequencing of the instructional activities in order to bring about the desired learning goals.

CHAPTER 3: METHODOLOGY

The primary purpose of this study was to examine the conceptual understanding of whole number concepts and operations of individual prospective elementary school teachers within the collective classroom setting. Specifically, this study intended to explore the following research question:

> ➤ In what way does the conceptual understanding of individual prospective teachers develop during an instructional unit on whole number concepts and operations situated in base-8?

In this chapter, this study will specify the precise procedures used during this case study analysis in order to address the aforementioned research question. The chapter consists of the following sections: (1) research design, (2) research setting, (3) selection of participants, (4) data collection procedures, (5) data analysis procedures, (6) trustworthiness, and (7) selection of individual cases.

Research Design

Overview and Justification of Research Design

This research was designed as a collective case study as described by Merriam (1998) in order to contribute to the understanding of how individual prospective elementary school teachers develop an understanding of whole number concepts and operations within a collective classroom setting (Denzin, 2000; Stake, 2006). According to Merriam (1998), "case study design is employed to gain an in-depth understanding of the situation and meaning for those involved" (p. 19). Yin (2003) contends that case studies provide "an empirical inquiry that investigates a contemporary phenomenon within its real life context, especially when the boundaries between phenomenon and context are not clearly evident" (p. 13). In particular, case studies are deemed appropriate when the research wishes to focus on understanding individual participants within a complex, real-life social context such as a constructivist classroom (Merriam, 1998; Stake, 2006, Yin, 1994, 2003).

A qualitative research design methodology was employed in order to illuminate an individual's thinking as it developed along, contributed to, and interacted with the classroom mathematical practices that evolved within the social nature of the class. "Qualitative research is an inquiry process of understanding" (Creswell, 1998, p. 15) and this design was selected in the natural setting of a classroom focused on conceptual understanding as a means to gain valuable insight into the understanding of the participants. Case studies such as the task undertaken during this research project enable researchers to follow and document students' thinking and demonstrate the manner in which they make sense of mathematics. Romberg and Carpenter

(1986) emphasized the need for case study research by stating "The kind of teaching study that needs to be done would bring together both notions about the classroom, the teacher, and the student's role in that environment, and how individuals construct knowledge….Dynamic models are needed that capture the way meaning is constructed in classroom settings on specific mathematical tasks" (pp. 868-869).

Following the works of Cobb and his colleagues, this qualitative analysis examined individual prospective teachers as they developed conceptual understanding of whole numbers and operations using an instructional sequence taught entirely in a base-8 setting. Previous and ongoing research efforts have documented and continue to develop the collective aspects of prospective elementary teachers' understanding within the social context (Andreasen, 2006; Roy, 2008; Tobias, 2009; Wheeldon, 2008). However, as Stephan et al. (2003) point out, "The mathematical practice analysis and complementary case studies serve a more complete picture of the learning" (p. 68) and indeed there is a need for additional elaboration using case studies to further describe prospective teachers' understanding. The collective research analysis has focused on the development of the social norms and the development of the classroom mathematics practices (Andreasen, 2006; Dixon, Andreasen & Stephan, in press; Roy, 2008; Tobias, 2009; Wheeldon, 2008). At this point, the individual perspective *related to the mathematical conceptions and activities of prospective teachers* needed to be analyzed in order to better grasp the development and contribution of the psychological aspects of the emergent perspective (Cobb, 2002). This research project focused on the way prospective teachers interacted within the social dynamic and used the emergent perspective as a lens to explore the way "(1) students' learning occurred as they participated in these emerging practices, and (2) the

mathematical practices emerged as students, often the target students, contributed to them" (Stephan, Bowers & Cobb, 2003, p. 99).

The current research study focused on a collective case study on a single class of prospective elementary teachers, but as Yin (1994) asserts "Even a single-case study can often be used to pursue an explanatory, and not merely exploratory (or descriptive), purpose" (p. 5). Collective case studies can elaborate on the intricate aspects of the complexities of a phenomenon of interest. This case study design was deemed appropriate since the aspects of explanation and justification – the *how* and the *why* inquiries– of prospective teachers' understanding of whole number concepts and operations were central to this study and ought not be separated from their context within the study (Yin, 2003). Through the selection of multiple cases, this collective case study intended to increase the applicability of the findings (Merriam, 1998; Yin, 2003) as well as to provide the desired breadth in illustrating the ways in which prospective teachers with different content knowledge developed their conceptual understanding.

Research Setting

Participants

The research study took place at a major public, urban university in the southeastern United States. The research participants were primarily prospective elementary teachers or prospective teachers of exceptional education. All 32 participants were female students and were classified with at least sophomore standing. In particular, this mathematics content course was comprised of 18 sophomores, 8 juniors and 6 seniors. The university's Institutional Review Board (IRB) approved all aspects of this study (See Appendix A). Also, every student who participated in the study agreed and signed an informed consent letter (See Appendix B).

This study was conducted during the spring semester of 2007 in a four credit hour semester-long undergraduate elementary mathematics content course. This particular course served as the prerequisite to the mathematics method course that the elementary education undergraduate students took prior to their first internship. The exceptional education majors took this mathematics content course as their only required mathematics education class in their program. During the spring term of 2007, this course convened twice per week and each class session was scheduled to last 110 minutes. The prospective teachers began the course discussing problem solving activities for two class sessions and the whole number concepts and operations instructional sequence was conducted over a 10-session period.

These prospective teachers participated in classroom discussions using a problem-based curriculum that required prospective teachers to work on mathematical problems first individually or in small groups. Following the initial exploration of the problems at the individual or small group level, the prospective teachers took part in whole-class discussions. In order to facilitate small group interaction and discussion, prospective teachers were situated in tables of at least four and no more than six participants per table. The specific individuals who were selected for case study analysis are discussed in much greater detail in the section labeled selection of participants.

Research Team

The research team for this classroom teaching experiment consisted of eight members. These eight members included the course instructor, a mathematics education faculty member, and six mathematics education doctoral students. The course instructor was an associate professor in mathematics education with significant background in teaching constructively and

had taught this particular content course on multiple occasions at the same university prior to this research project. The mathematics education faculty member was a visiting assistant professor with background in design research and particular insight into the research on whole number concepts and operations as a result of her own dissertation research. Among the six doctoral students, three included individuals who were particularly informed with the research literature and topics involved in this project as this research provided data for their own dissertation projects. Research team members played an essential part in observing each class session as well as reflecting individually and collectively during weekly research meetings.

Hypothetical Learning Trajectory (HLT)

According to the works of Simon (1993, 1995), the hypothetical learning trajectory (HLT) will serve as "the teacher's prediction as to the path by which learning might proceed" (Simon, 1995, p. 35). Even though previous research and familiarity with research participants can illuminate the expected path, the actual trajectory or path is nearly impossible to anticipate completely in advance. Every classroom community maintains its own identity and each individual possesses her own unique learning style given the context. Andreasen (2006) paved the way for the anticipated learning trajectory in the collective setting as it predicts the main ideas involved in prospective teachers' progression towards proficiency with whole number concepts and operations.

As with previous studies (Andreasen, 2006; McClain, 2003; Roy 2008; Tobias, 2009; Wheeldon, 2008), this research effort used children's progression to guide prospective teachers' development of whole number concepts and operations due in large part to the limitation of research with prospective elementary teachers' development in this context. In the past,

classroom instruction initiated with learning goals of counting, unitizing and the ability to decompose and compose numbers in a flexible fashion. The development of these learning goals contributed to the synthesis of meaning and approaches in computational fluency with addition, subtraction, multiplication and division. This current research initially utilized the HLT – provided in Table 3 – in accordance with the findings of Andreasen (2006). Note the manner in which for each phase of the instructional sequence, the learning goals were clearly defined followed by the tasks and tools used to support instruction.

Table 3: Initial Hypothetical Learning Trajectory, Andreasen (2006)

HLT phase	Learning goal	Supporting tasks for instructional sequence	Supporting tools
Phase One	Count and unitize objects efficiently	Counting strategies and representations, 10 frames	Snap cubes, 10 frames, and open number lines
Phase Two	Flexible representations of numbers	Candy factory scenario involving estimating, packing and unpacking candy, and inventory forms	Pictorial representations of boxes, rolls, and pieces and inventory forms
Phase Three	Operational fluency	Candy factory transactions, inventory forms, 10 frames, dot arrays, and context-based problems	Pictorial representations of boxes, rolls, and pieces; inventory forms; dot arrays; snap cubes; and open number lines

As defined by Andreasen (2006), the hypothetical learning trajectory included three phases initiating with the learning goal of counting and unitizing in phase one, flexible representation of numbers in phase two and concluding with operational fluency in the third and final phase.

The learning trajectory used was considered hypothetical since the instructional sequence

was revised upon documenting and analyzing the way the class progressed through the content

discussed. While the previous HLT by Andreasen (2006) informed the instructional sequence of

this research study, the actualized learning trajectory was finalized by Roy (2008) including all

phases, learning goals and supporting tasks and tools of instruction. Note the three phases in the

actualized learning trajectory of this research project – illustrated in Table 4 – and the supporting

tasks and tools involved in the instructional sequence.

Table 4: Actualized Learning Trajectory, Roy (2008)

HLT phase	Learning goal	Supporting tasks for instructional sequence	Supporting tools
Phase One	Counting	Counting; Skip Counting; Open number line Problems; Counting Problem Set #1; Counting Problem Set # 2	Double 10-Frames, and Open number line
Phase Two	Unitizing, Flexible representations of numbers	Estimating with Snap Cubes; Candy Shop 1; **Candy Shop Exercise 1 (Exit Question); Candy Shop Exercise #2;** Torn Forms	Snap Cubes; Boxes, rolls and pieces; Inventory forms
Phase Three	Invented Computational Strategies	Candy Shop 2; Candy Shop Inventory; Candy Shop Addition and Subtraction; Inventory Forms for addition and subtraction (in context); Inventory Forms for addition and subtraction (out of context); 10 Dot Frames; Broken Machine; Multiplication Scenario, Multiplication Word Problems; Division Word Problems	Boxes, rolls, and pieces; Inventory forms; dot arrays; and Open number line

Phase one focused on the learning goal of counting, phase two's learning goal involved unitizing

and flexibly representing numbers and the third phase was concerned with the invented

computational strategies of the prospective teachers as they progressed through the instructional

sequence on whole number concepts and operations taught entirely in base-8.Instructional Sequence.

This research study relied heavily on previous research efforts on prospective teachers' development of whole number concepts and operations as described in the literature review section. The hypothetical learning trajectory predicted the anticipated path along which the participants would progress. At this juncture, findings of previous research efforts on this target audience regarding their experiences with whole number notions will be presented. While knowledge of children's learning and understanding of whole number concepts and operations proved quite valuable, prospective teachers presented a different challenge than children. Prospective teachers in education programs have had a lifetime of learning prior to entering the classroom. Hence, they could not be expected to simply forgo their previous notions familiar to them regarding whole number concepts and operations. Hopkins and Cady (2007) insisted that prospective teachers' learning of whole number concepts and operations may be masked by their familiarity with base-10. How could a realistic situation be provided as a setting for prospective teachers to examine their understanding of these notions without the familiarity getting in the way of exploration?

Gravemeijer (2004) discussed the notion of Realistic Mathematics Education (RME) by situating students in a realistic environment in order for the "reinvention" of the concepts to occur in a similar setting to where the notion was developed. The course used for this research project was designed in accordance with the notions of RME at the Freudenthal Institute in the Netherlands (Freudenthal, 1993). The guidelines in place pertained to three principles consisting of guided reinvention, didactical phenomenology, and emergent models (Gravemeijer, 2004).

Briefly, guided reinvention placed prospective teachers in a context that had to make realistic mathematical sense to them in order to examine and reinvent the mathematics for themselves. Didactical phenomenology refers to the instructional activities and the precise content and sequence in which they will be discussed with the prospective teachers. Finally, emergent models deal with the examination and potential evolution of models utilized by prospective teachers over time.

Following the recommendations of RME, all instruction during the course of this study occurred in base-8 which provided a realistic setting and yet allowed for perspective teachers to "reinvent" their notions of whole number concepts and operations. Dealing with a different base system undoubtedly caused initial discomfort and lack of familiarity which ultimately allowed for deep exploration of an adult-learner's long held perceptions regarding the research topic. Hart (2004) in working with undergraduate students stipulated that novice learners – as prospective teachers would be in a base-8 setting – must go through an unstable transition time in order to become expert in a setting. In addition, research has long shown that optimal learning takes place on occasions when students are asked to defend their ideas and make sense of the mathematics they are learning (Ball, 1991, 1993; Tall, 1992). As prospective teachers progressed through this instructional sequence taught entirely in base-8, they experienced some cognitive dissonance and benefited from making their own conjectures. They discussed and reflected on their own manifestations of whole number notions and operations and were asked to examine differing opinions during the course of instruction (Cobb, 1999, 2002; Cobb & Wheatley, 1988; Cobb, et al., 2001; Schoenfeld, 2002; Sfard & Kieran, 2001).

Consistent with the notions of RME, this instructional sequence was placed in the context of the "Candy Shop" (Bowers, 1996; Bowers, Cobb & McClain, 1999). In investigating children's development of whole number concepts and operations in the "Candy Shop" setting, Bowers and colleagues discovered that children improved their understanding of place value notions and number concepts,and enhanced their notions of algorithms. Children in these studies gradually built on place value concepts by composing and decomposing numbers as well as the exploration of addition and subtraction strategies in and out of the context of the "Candy Shop" (Bowers, 1996; McClain, 2003).

In order to guide prospective teachers towards the stated goals of this research study, the HLT provided the foundation for the prospective teachers to interact with their content knowledge of whole number concepts and operations. The HLT which guided our instructional sequence was comprised of three phases. During the initial phase of the instructional sequence, the activities emphasized counting and unitizing. Phase two required prospective teachers to represent numbers flexibly by composing, decomposing and sometimes even re-composing numbers. Phase three in this instructional sequence emphasized fluency with whole number procedures and operations as well as the development and examination of accurate and efficient algorithms for the addition, subtraction, multiplication and division of whole numbers.

Instructional Tasks

As we discussed in the previous section, the instructional sequence was designed and implemented entirely in base-8. The instructional tasks included a variety of problems and scenarios in order to provide prospective teachers the opportunity to explore their understanding of whole number concepts and operations – See Appendix C. In order to distinguish between

nomenclatures of numbers in base-8, particularly due to the familiarity of number names in base 10, a specific convention was adopted during the course of instruction. As the numerals 0, 1, 2, 3, 4, 5, 6, 7 comprise the digits in "Eight World", the next number in the sequence was referred to as 10 (Read, one-ee-zero). For instance, the number following 10 would be 11 (one-ee-one) and the next number after 17 (one-ee-seven) would be 20 (two-ee-zero). Table 5 provided below illustrates the progression of numbers in base-8. Another significant number name that arose in the classroom discussions included the number that would follow 77 (seven-ee-seven) in the sequence. Eventually after a rich conversation that foreshadowed the discussion on place value, this number was written as 100 and referred to as one-hundree.

Table 5: Base-8 Numbers Chart

0	1	2	3	4	5	6	7
10	11	12	13	14	15	16	17
20	21	22	23	24	25	26	27
30	31	32	33	34	35	36	37
40	41	42	43	44	45	46	47
50	51	52	53	54	55	56	57
60	61	62	63	64	65	66	67
70	71	72	73	74	75	76	77

As the prospective teachers progressed through the instructional sequence, they were asked to skip count forwards and backwards by specific numbers. For instance, the course instructor would ask the prospective teachers to begin on the number 5 and skip count by 2's.

Not only did prospective teachers realistically gain practice in mentally counting numbers, but they also encountered benchmark numbers in the sequence of base-8 numbers. Purposefully, the course instructor would lead discussions to arrive at anticipated numbers such as the number that would follow seven or seven-ee-seven.

For the first few class sessions of this instructional sequence, in the beginning of each meeting, the instructor would acclimate prospective teachers with base-8 by asking them to skip count. The next instructional task introduced incorporated the pedagogical content tool called "Double 10-frames". By asking prospective teachers to think in terms of base-8 using the Double 10-frames, the instructor explored the manner in which the prospective teachers would combine numbers mentally. Figure 6 below illustrated an example involving 2 and 6 dots on the Double 10-frames. Next, prospective teachers were asked to mentally combine the total number of dots represented on the Double 10-frames and describe how they arrived at their answer.
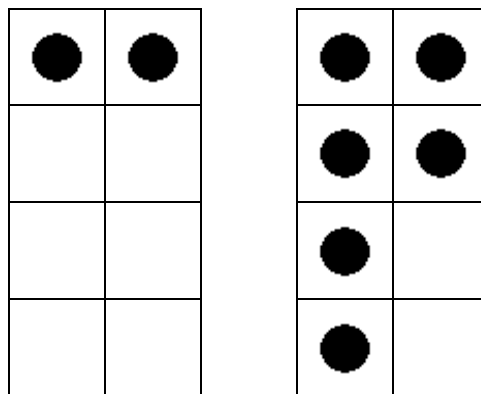

Figure 6: Double 10-Frames Representing 2 and 6 for a Total of 10.

Using the overhead projector, the instructor displayed combinations of dots on the Double 10-frames, but only allowed approximately 2-3 seconds before asking "How many dots

did you see?" These questions allowed prospective teachers to think in terms of base-8 efficiently and at the same time led to class discussions which illustrated the unique, creative ways that different prospective teachers would combine numbers mentally.

Consistent with Realistic Mathematics Education's third principle of emergent modeling, teachers often use graphs, diagrams or notations to record students' thinking. *Pedagogical content tools* represent devices used to record and eventually connect students' thinking as they explore the mathematics at hand. According to Gravemeijer (2004), as students model their informal activities, "The aim is that the model with which the students model their own informal activity gradually develops into a model for more formal mathematical reasoning" (p. 117). During this research study, the instructor employed a pedagogical content tool classified as a *transformational record.* Such diagrams or graphical representations are considered *transformational* since they initially reflect students' thinking with the hope that eventually students use them to answer new problems (Rasmussen & Marrongelle, 2006).

In order to record prospective teachers' responses, the instructor for this research study introduced a transformational record called an *open number line*. This transformational record served multiple purposes including (1) documenting prospective teachers' thinking, and (2) providing prospective teachers a tool to be utilized in representing and solving further problems. The open number line depicted a blank number line without any particular numbers initially marked. Upon analysis of prospective teachers' use of the *open number line* as a transformational record in this particular research study, Roy (2008) indicated, "the numbers are not placed on the line in predetermined locations but are added to the line to represent the mathematical moves in the given solution to the particular problem" (p. 98).

The instructor's use of the open number line became more evident as prospective teachers next encountered addition and subtraction problems in the context of the candy shop. The instructor modeled the open number line by illustrating the scenario, "There are 30 candies in the candy shop. I make 13 more candies. How many candies are there now?" In Figure 7, note the way the instructor recorded the mathematical moves through the use of the open number line.
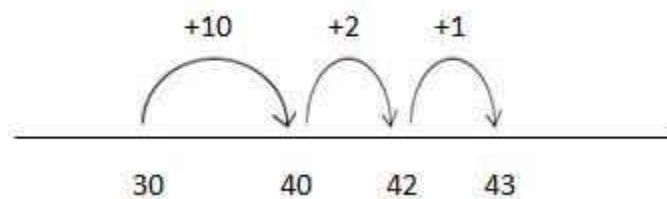
Figure 7: Instructor's Use of the Open Number Line to Record Students' Thinking

The open number line served two purposes in that it allowed for the accurate depiction of the mathematical moves of adding one-ee-zero first, followed by adding two and then one candy to arrive at the total four-ee-three candies. It must be stated that the jumps indicated are not proportional. Note that this system of documenting prospective teachers' thought processes would be modified in its role as a pedagogical content tool as it lent itself to solving future addition and subtraction problems.

As the instructional sequence moved from addition and subtraction towards dealing with multiplication, prospective teachers were also presented pictorially with boxes, rolls, and pieces of candy within the candy shop scenario. In this scenario, boxes contained 100 (one-hundree) pieces, and rolls contained 10 (one-ee-zero) individual pieces of candy. Figure 8 illustrates the images that the prospective teachers used during the course of the instructional sequence.
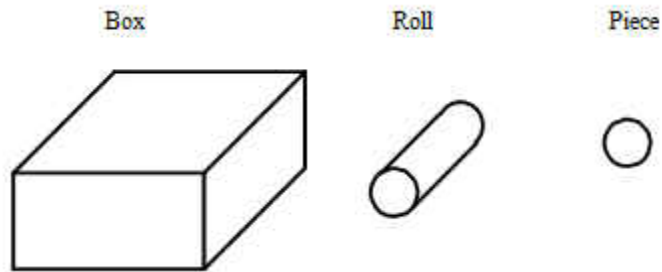
Figure 8: Boxes, Rolls, and Pieces as Used in the Candy Shop

The prospective teachers also used their own sketches of the boxes, rolls and pieces to illustrate their thinking throughout the course of the instructional sequence. As a box contained 10 rolls and a roll contained 10 pieces, this multiplicative relationship allowed for prospective teachers to begin unitizing (Cobb & Wheatley, 1988). Thus far in the first two phases of the instructional sequence, prospective teachers used various instructional tasks towards the learning goals of counting and unitizing.

Also in the second phase of the instructional sequence, one of the intended focuses revolved around the accomplishment of the learning goal of decomposing and composing numbers. As prospective teachers began to use boxes, rolls, and pieces to "put-together" and "break-open" packages as they deemed appropriate, the need for a more efficient method to record the amounts emerged. The instructor introduced the second transformational record – open number line being the first – called an Inventory Form. Note in Figure 9, the way that the Inventory Form purposefully has separated the number of boxes, rolls, and pieces.

Boxes	Rolls	Pieces
2	3	1

Figure 9: Inventory Form Illustrating 231 Candies

Up until this part of the instructional sequence, the prospective teachers explored counting, unitizing as well as the composition and decomposition of numbers in the context of the candy shop. During the third and final phase of the HLT, the candy shop provided the setting for students to engage in meaningful ways to "re-invent" whole number operations. The instructor once again used images of boxes, rolls and pieces – illustrated in Figure 8 earlier - in order to show *combinations* of boxes, rolls, and pieces on the overhead projector. Next, she asked the prospective teachers to *give the total* given the two configurations, or given the total amount of candies to *provide the missing amount of candy*. Such tasks required the prospective teachers to efficiently compute addition and subtraction problems in the absence of preconceived algorithms. These tasks along with subsequent ones, allowed prospective teachers to examine ways of performing whole number operations in and out of the context of the candy shop.

Towards the end of the third phase, tasks were provided to encourage multiplication ideas in the context of the candy shop through the use of a broken machine. In this scenario, instead of 10 representing the number of objects packaged together, the broken machine would place different amounts such as 7 sticks of gum in a pack. In this instance, prospective teachers had the opportunity to reason and use their knowledge of unitizing and addition to construct their own notions of multiplication. In the various problems provided, the broken machine placed different numbers of sticks of gum in a pack – for instance 6, 17, 16, and 22 - in order to foster multiplication strategies.

The broken machine was followed by the "Egg Carton Scenario" in the instructional sequence. This instructional task presented prospective teachers with egg cartons in various dimensions. The prospective teachers were provided the following multiplication scenario: "A

marketing team has created three new prototypes for an egg carton. How many eggs would fit in an (a) 5 by 6 (b) 6 by 12 (c) 3 by 16 egg carton? Explain and justify." This multiplication scenario was utilized to support computational strategies using algorithms and visual reasoning.

Finally, the instructional sequence concluded with story problems that fostered prospective teachers' development of algorithms in accurate, flexible and efficient manners. In summary, the various instructional tasks provided throughout the three phases of the instructional sequence provided the opportunity to re-invent whole number concepts and operations in base-8 while developing a deeper understanding in a realistic setting.

Interpretive Framework

This research effort was designed as a collective case study in which prospective elementary teachers developed their understanding of whole number concepts and operations in the social context of a classroom community. This qualitative research documented the development of individual prospective teachers in an undergraduate mathematics content course through the examination of their mathematical conceptions and activities. Student learning was explored using the "emergent perspective" as individuals interacted with their peers through classroom discussions (Cobb, 2000; Cobb & Yackel, 1996; Yackel, & Cobb, 1996). The emergent perspective served as a theory of learning that incorporated both the social and individual dimensions without either taking primacy over the other (Cobb & Yackel, 1996).

Perhaps the single most important aspect of this theory resides in its reflexive nature. The social and psychological perspectives are interrelated and "the existence of one depends on the existence of the other" (Stephan, 2003, p. 28). According to Cobb (2000) "A basic assumption of the emergent perspective is…that neither individual prospective teachers'

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activities nor classroom mathematical practices can be accounted for adequately except in relation to the other" (p. 310). Cobb and Yackel (1996) provided the interpretive framework - emphasizing the emergent perspective. In Table 6, note the social and psychological perspectives and in particular the mathematical conceptions and activity in the psychological perspective.

Table 6: Interpretive Framework (Cobb & Yackel, 1996)

Social Perspective	Psychological Perspective
Social Norms	Beliefs about one's role, other's role, and general nature of mathematical activity in school
Sociomathematical Norms	Mathematical beliefs and values
Classroom Practices	Mathematical conceptions and activity

In recent years, as more research projects have implemented the emergent perspective, the social aspects of this framework have been highlighted in the context of prospective elementary teachers (Andreasen, 2006; Roy, 2008; Tobias, 2009; Wheeldon, 2008). Further examination of the psychological perspective would provide a much more comprehensive picture of prospective teachers' understanding of whole number concepts and operations. While mathematical beliefs and beliefs regarding roles constitute an important part of the interpretive framework, this current research project intended to highlight the psychological perspective by *focusing on the mathematical conceptions and activity of individual prospective teachers* as they interact with the social correlate of classroom mathematics practices.

As such, both the social aspects of learning as well as the concurrent individual component must be considered and discussed simultaneously. Stephan (2003) elaborated on the interrelation of the social and psychological perspectives by stating "the existence of one depends on the existence of the other" (p. 28). Yackel and colleagues (Yackel & Cobb, 1996)

maintain that individual student's mathematical activity and the classroom micro-culture are related in a reflexive fashion. The interpretive framework - illustrated earlier in Table 6 - concisely related the role and interrelation of the social and individual components.

According to Cobb and Yackel (1996), classroom mathematical practices represent the third and final aspect of the social perspective. Roy (2008) extended previous research efforts by focusing on the classroom mathematical practices developed by prospective teachers in learning whole number concepts and operations through an instructional sequence situated in base-8 (Andreasen, 2006; Bowers, Cobb & McClain, 1999). In order to contribute to the existing literature, this research effort was particularly interested in exploring the development of individual prospective teachers' understanding of whole number concepts and operations – through the analysis of mathematical conceptions and activities - as it occurred within the social classroom dynamic.

Selection of Participants

Sampling Procedure

During this research project, the research question involved examining the conceptual development of prospective teachers in understanding whole numbers concepts and operations as it occurred during an instructional unit in base-8. Research efforts have discussed prospective teachers' knowledge and how it related to student thinking and achievement (Hill, Rowan & Ball, 2005; Lassak, 2001; McAdam, 2000; Sfard & Kieran, 2001). In the context of whole number concepts and operations, this research project intended to further elaborate on *individual* teachers' development in the instructional unit taught in base-8. This project aimed to investigate

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if and *how* an individual prospective teacher with a strong initial content knowledge would develop differently than one with a weaker initial content knowledge. Due to the qualitative nature of this project, this researcher chose a purposive sampling strategy in order to use a criterion-based selection process to choose the target participants for the case study. In purposive sampling, the primary goal is not one of generalizability, but rather the understanding of a concept or topic in detail. Maxwell (1996) stated that "selecting those times, settings, and individuals that can provide you with the information that you need in order to answer your research questions is the most important consideration in qualitative sampling decisions" (p. 79). As such the selected participants represent a bounded system – by time, place and context (Creswell, 1998; Miles & Huberman, 1994; Stake, 2003). In order to gain a better sense of the initial content knowledge of these prospective teachers, this research study will next discuss a measure intended to identify the content knowledge for teaching.

Content Knowledge for Teaching – Mathematics (CKT-M)

As we discussed in the literature review chapter, mathematical knowledge for teaching serves as one of the distinguishing characteristics of teaching mathematics professionally compared to mathematical knowledge needed for various other occupations (Ball, Hill & Bass, 2005). The Content Knowledge for Teaching – Mathematics (CKT-M) Measures represents a statistically reliable, verified instrument that represents the mathematical knowledge necessary to teach elementary mathematics including the specific role that this content plays in children's learning (Hill, Ball & Schilling, 2008).

The database of items from this instrument contains questions related to two types of knowledge: common knowledge of content and specialized knowledge of content. The common

content items deal with knowledge of mathematics including computing, making accurate statements and the ability to correctly solve problems. The specialized knowledge of content items deal with operations, the ability to provide alternative representations as well as evaluating inventive student solutions (Ball, Hill, & Bass, 2005). After members of the research team including the course instructor had reviewed the items, 25 items were selected and administered to the 32 prospective teachers in the class. Nine of the items selected were common content knowledge items and another sixteen were specialized content knowledge items. While the specific contents of the CKT-M instrument may not be included as a part of this study, in order to provide the reader with insight into this instrument, an example –released by the authors – has been provided in Figure 10.

Imagine that you are working with your class on multiplying large numbers. Among your students' papers, you notice that some have displayed their work in the following ways:

Student A	Student B	Student C
35	35	35
× 25	× 25	× 25
125	175	25
⏊ 75	⏊ 700	150
875	875	100
		+ 600
		875

Which of these students is using a method that could be used to multiply any two whole numbers?

	Method would work for all whole numbers	Method would NOT work for all whole numbers	I'm not sure
a) Method A	1	2	3
b) Method B	1	2	3
c) Method C	1	2	3

Figure 10: Example of Specialized Content Knowledge Item
(Ball, Hill, & Bass, 2005)

The 25 items from the CKT-M Measures instrument were selected as pre- and post-test items in order to help identify the impact of the instructional unit taught entirely in base-8 on the content knowledge of prospective elementary teachers.

As described earlier, in order to gain a more thorough understanding of the individual perspective, this research endeavor focused on whether teachers with varying initial content knowledge developed differently through this instructional sequence. Primarily, this researcher was interested in being able to identify at the outset - through an objective instrument - those particular participants who had "Low-Content" knowledge for teaching versus others who

displayed "High-Content" knowledge for teaching in whole number concepts and operations.

The results of the CKT-M Pre-test scores for all 32 classroom participants (Mean=13.1, Standard Deviation=2.4) have been provided using a box and whiskers plot in Figure 11.
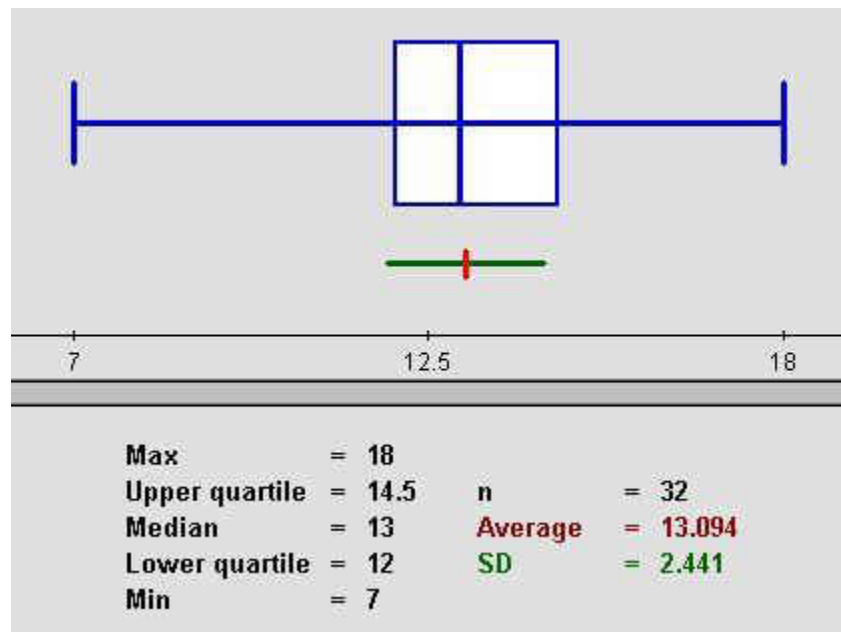
Figure 11: Pre-test CKT-M Scores

The scores on the Pre-test CKT-M for the research participants ranged from a lowest score of 7 to a highest score of 18 (out of a possible 25.) The median score was 13, the lower quartile was 12, and the upper quartile was 14.5. It is worth noting that the prospective teachers in this research study did not illustrate a large dispersion in their CKT-M scores as 17 of the 32 participants fell in between the scores of 12 and 14, inclusive. In order to differentiate among the participants, the researcher considered individuals with CKT-M scores that were below the lower quartile – here forth referred to as "Low-Content" – and conversely individuals with CKT-M

scores above the upper quartile – referred to as "High-Content.". This designation will be

discussed again in the section related to the selection of individual cases for this research project.

Selected Research Participants

As stated previously, the primary purpose of this study involved exploring the

development of individual prospective elementary teachers through the instructional sequence in

base-8. It was of particular interest to the researcher to examine how individuals with *different*

incoming content knowledge would develop their understanding of whole number concepts and

operations and whether they would participate differently in the classroom micro-culture. After

all, these prospective teachers would one day be *individually* teaching in their own classrooms

and thus this research effort wanted to explore their *individual development* through the

instructional sequence. To this purpose, the researcher considered individuals with CKT-M

scores that were below the lower quartile – referred to as "Low-Content" – and conversely

individuals with CKT-M scores above the upper quartile – referred to as "High-Content."

In order to examine individuals with *different initial content knowledge*, four total

prospective teachers were selected as research participants. While a case study involving one

prospective teacher's development would have examined that one particular individual in great

detail, this researcher made the conscious choice of wanting to be able to gather more

information in order to compare and contrast different individuals. By selecting multiple

individuals for the purposes of analysis, this researcher was able to explore multiple individuals'

development while still maintaining a reasonable number of cases. One of these individuals –

referred to as Cordelia – scored below the lower quartile of 12. The two individual - referred to

as Claudia – scored above the upper quartile of 14.5. To reiterate, this researcher wanted the

opportunity to examine multiple individuals who displayed *different incoming content knowledge* in order to gain greater insight into prospective teachers' development of whole number concepts and operations.

In summary, the initial content knowledge served as the *primary* determinant factor for the selection of the research participants. In order to explore cases that involved prospective teachers with *potentially different experiences* prior to this instructional sequence, the researcher also considered the following two factors: varying number of mathematics courses taken during undergraduate university coursework, and the individual's reflection of her experiences with mathematics. The number of undergraduate mathematics courses could potentially be linked to the prospective teacher's content knowledge – albeit in a context different than whole number concepts and operations. Lastly, using the "My Experience with Mathematics" paper submitted by each student in the class, the researcher explored what each prospective teacher shared regarding her prior learning and teaching of mathematics.

Data Collection Procedures

The goal of data collection procedures was to provide an insider's perspective to the individual and shared experiences of the research participants (Stake, 2006). The data for this study were collected from multiple sources in adherence to Patton's (1990) suggestions:

> Multiple sources of information are sought and used because no single source of information can be trusted to provide a comprehensive perspective…By using a combination of observations, interviewing, and document analysis, the fieldworker is able to use different data sources to validate and cross-check findings. (p. 244)

A wide variety of data were collected during the course of this classroom teaching experiment including recordings (video and written forms) of whole class sessions, and small

group discussions, prospective teacher artifacts, and video and written recordings of individual as well as focus group interviews (Cobb, et al., 2001). Through the synthesis of individual responses along with the individual contributions toward class discussions, the researcher could understand the data in a more thorough fashion. The analysis also included interviews and written work in order to illuminate the nature of prospective teachers' individual understanding and standards for what constitutes as an acceptable explanation and justification. In order to answer the research questions put forth using a case study methodology, this researcher needed to have multiple sources of data for triangulation purposes (Creswell, 1998; Merriam, 1998).

As previously discussed during the sample selection section, data were also collected through the administration of the Content Knowledge for Teaching Mathematics (CKT-M) Measures database (Hill, et al., 2005) before and after the instructional unit on whole number concepts and operations. Lastly, all members of the research team took field notes during each class session. Often times, these field notes were discussed during the weekly research meetings and they contributed quite heavily towards instructional decisions and future planning.

Videotaped data were collected to capture the whole-class discussions through the use of 3 concurrent video cameras from different areas in the classroom. One camera was placed in the center back of the classroom and focused on whole class discussion and followed any individual student who spoke as a part of the whole class conversation. Another camera was situated towards the back right of the classroom and was designed to focus on the work done on the board and/or work displayed on the screen using the overhead document camera. A third camera was positioned in the front left of the room and was focused on the instructor and individual

prospective teachers as they engaged in conversations as well as any questions or challenges that were raised during class discussion.

Throughout this research effort, a large volume of data including personal statements and prospective teacher artifacts were collected from each of the research participants. Prior to the instructional sequence, the instructor asked each person to submit a one-page "My Experience with Mathematics" entry. This personal statement provided a glimpse of each prospective teacher's background and experiences in mathematics. Often times, these writings included the challenges and successes that the individual had shared in learning and teaching mathematics. Upon the start of the instructional sequence, the researcher with the assistance of the research team, collected any artifacts produced by the prospective teachers. These artifacts included all notes, problems explored during classroom discussions, homework assignments, and tests. Each of these prospective teacher artifacts were closely analyzed in order to gain a better understanding of the development of whole number concepts and operations by each individual.

All four individual prospective teachers who were selected as research participants were interviewed individually at the beginning of the instructional sequence as well as at the end of the unit on whole number concepts and operations. These individual interviews were all conducted by this researcher and lasted approximately 45 minutes. Each interview was videotaped and transcribed for later analysis. During these interviews, the researcher began with a set of questions to be addressed by each participant. The primary purpose of these interview questions involved gaining insight into each prospective teacher's knowledge of place value concepts and operations prior to the instructional sequence. These interviews also served as a baseline of illustrated understanding of the mathematical topics of interest. Due to time

constraints and the responses provided by a participant, not all questions were asked of all individuals. A copy of the questions used to conduct the interviews was included in Appendix D. Furthermore, the researcher exercised opportunities to inquire more deeply about an individual's mathematical conceptions by asking follow-up questions.

In addition, a focus group interview of the research participants was conducted at the end of the instructional unit to reflect on the individual and shared experiences on base-8 as well as any implications on whole number concepts and operations in base 10. This focus group interview was conducted by a different member of the research team and lasted approximately 1 hour. All focus group sessions were videotaped and transcribed for purposes of analysis.

Data Analysis Procedures

The analysis for this research was conducted using the transcriptions of the classroom discussions, the audiotapes of the small group interactions, the videotape of each individual pre- and post-interview, the videotape of the focus group for the case study participants, and all additional documents collected from each participant including homework, tests and classroom activities. Furthermore, field notes and research team members' notes from weekly meetings were collected and analyzed. The analysis of the development of prospective teachers' understanding of whole number concepts and operations was divided into two portions: the individual development and the participation in the classroom mathematical practices.

Analysis of the Individual Development

The individual development was analyzed by systematically examining each prospective teacher prior to the instructional sequence, during the instructional sequence and following the

instructional sequence. In examining the data prior to the instructional sequence, this researcher analyzed the personal statement called "My Experience with Mathematics" as well as the videotaped and transcribed personal interview which took place before the beginning of the unit on whole numbers concepts and operations. During the instructional sequence, all prospective teacher artifacts including notes, homework, class activities and tests were analyzed. This analysis was divided into sections which highlighted each individual's development of understanding of the main concepts within the instructional sequence. As illustrated in the chapters 4 and 5, each participant's development was explored for conceptual understanding of place value and counting strategies, addition and subtraction strategies, and finally multiplication and division strategies.

Following the instructional sequence, the individual post-interview and the focus group including all research participants were closely analyzed. During the post-interview, individuals revisited some of the problems they had been asked to solve during the pre-interview as well as follow up questions and reactions to the instructional sequence. This interview focused on aspects of the individual's experiences through the instructional sequence such as:

➢ Changes in understanding of whole number concepts and operations
➢ Roles and responsibilities as a participant in classroom discussions
➢ Comparing and contrasting base-8 and base-10

Analysis of Participation in Classroom Mathematical Practices

Roy (2008) identified the classroom mathematical practices through analyzing the classroom argumentation in the same environment with the same group of prospective teachers as this research effort. This researcher played an active role in the aforementioned analysis of the classroom argumentation by Roy prior to the examination of the psychological aspects of the

71

interpretive framework. The social aspects of argumentation indicated how and to what extent each individual contributed to and on occasion steered small group and whole class discussions.

<div align="center">Model for Analysis of Classroom Argumentation</div>

Toulmin (1969) in his book, *The Uses of Argument,* illustrated the concepts of *claims, data, warrants* and *backings* which have been prevalent since their publication in various social science research settings as well as mathematics education research. Toulmin's model of argumentation was briefly discussed. Next, the degree to which each prospective teacher's conceptual understanding of whole number concepts and operations was aligned with her support for argumentation was analyzed. At that point, any similarities or differences among the prospective teachers were described in the cross-case analysis in conjunction with the description provided by Yin (2003). Toulmin's argumentation model "allows one to reconstruct an argument as well as the specific parts of that argument as they emerge…..How an argument is developed, further elaborated, or restructured is indeed socially motivated as the individual attempts to help others 'see' her point of view" (Whitenack & Knipping, 2002, p. 442).

According to Toulmin (1969), in the course of argumentation, a *claim* is a "conclusion whose merits we are seeking to establish" and *data* are the "facts we appeal to as a foundation for the claim" (p. 97). *Warrants* are "general, hypothetical statements, which can act as bridges, and authorize the sort of step to which our particular argument commits us" (p. 98). Quite frequently, data will be declared or stated rather explicitly, whereas warrants are largely left implicit unless specifically asked for or challenged. Finally, the *backing* of an argument is often times not specified. "Backing is other assurances without which the warrants themselves would possess neither authority nor currency" (p. 103). Since the criteria that researchers use in order to

gauge the validity of an argument varies (Cobb, et al., 2001; Whitenack & Knipping, 2002; Yackel, 2002), this researcher has provided an example of a classroom argumentation in the next section.

<div align="center">Example of Classroom Argumentation</div>

Whitenack and Knipping (2002) coordinated Toulmin's argumentation and the instructional theory of RME to analyze the discussion of a second-grade classroom in the context of whole numbers and operations. They maintained that prospective teachers engage in mathematical reasoning when their ideas, explanations and justifications are at the center of classroom discussions.

During this example, a 2nd grade student was asked to explain her solution given the problem $23 - 16 =$ ____. The student proceeded to explain that $23 - 16 = 7$ because $16 + 7$ is 23. In this example, the mathematical *claim* was $(23 - 16 = 7)$. When the student attempted to support or "ground" her claim by stating $(16 + 7 = 23)$, this was the *data* in the argument. Once a student asked her to explain how her data supported her claim, she explained further by saying that "she added 4 to get to 20 and 3 more to get to 23." This additional information provided in order to further support her data qualified as the *warrant* in this argument. According to Whitenack and Knipping, the warrant "serves as a bridge between the conclusion (claim) and its data and grounds the ensuing inferences." (p. 443). The backing in this instance was the "bottom line of sorts" or a general expression that all accepted without question. Hence, the student provided the *backing* – upon request – by stating "because $3 + 4$ is 7, the answer is 7" (Whitenack & Knipping, 2002).

Often times, a graphic representation has been used to outline the argumentation in a rather succinct fashion. In Figure 12, Whitenack and Knipping (2002) illustrated and summarized the previous argumentation regarding the initial problem of 23 − 16 = ____?
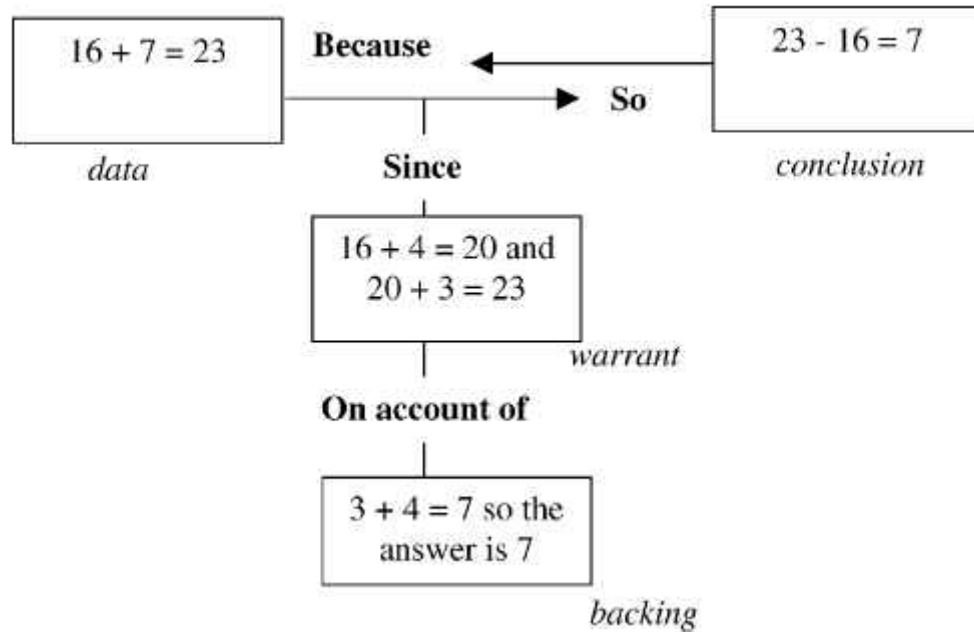

Figure 12: An Illustration of Toulmin's Argumentation

(Whitenack & Knipping, 2002, p.443)

As a part of this case study analysis, the researcher demonstrated the manner in which a particular participant provided *claims, data, warrants,* and *backings* to facilitate and at times lead whole classroom discussion. Furthermore, the analysis using Toulmin's argumentation allowed for collaboration between the social and the individual aspects of the emergent perspective to describe the individuals in the case study and their participation in classroom discussions. By illustrating the importance of the individual prospective teacher's development as it contributed

and sometimes pushed classroom argumentation, this research effort has focused on highlighting the significance of the psychological perspectives of the emergent perspective, and extending the research on the impact of individuals within the social classroom setting. The research analysis of prospective teachers' mathematical conceptions and activity, along with the classroom mathematical practices and social norms established by Roy (2008) and Andreasen (2006), respectively, would provide a much more detailed account of prospective teachers' understanding of whole number concepts and operations.

Analyzing the Individual Activity within the Classroom Dynamic

The transcripts of the classroom argumentation were previously analyzed using Toulmin's (1969) method in which the researchers identified *claims, data, warrants,* and *backings*. In order to document and analyze the individual's activity within the classroom dynamic, Glaser and Strauss's (1967) constant comparative method was employed to identify any existing patterns in individual's contributions to these *claims, data, warrants,* and *backings*. This method was also used to synthesize the *manners in which* these prospective teachers developed an understanding of whole number concepts and operations.

As witnessed earlier through the Whitenack and Knipping illustration (Figure 12), *claims, data, warrants,* and *backings* provide the different ways that an individual participated in and contributed to the argumentation. Each individual's participation was identified as providing *minimal, some, moderate* or *extensive* support in classroom argumentation as it led to the establishment of the classroom mathematical practices becoming *taken-as-shared*.

Minimal support indicated none or virtually nonexistent participation in the classroom discussion. On occasion, the prospective teacher may provide a claim or data without

connections to other argumentation. *Some* support signified an increase from the previous

category, however; this classification was mostly initiated by the instructor. The prospective

teachers' contributions moved the discussion along without providing a "challenge or push" to

the classroom argumentation. *Moderate* support indicated a steady back and forth participation

by the individual in classroom discussions. This classification of support for argumentation was

often initiated by the student and or was in response to other students' contributions. E*xtensive*

support signified a continuous, insightful contribution to the classroom dynamic. This particular

classification was primarily reserved for instances when the prospective teachers' contributions

would "challenge or push" the thinking of the entire class in a meaningful way. Note each

individual prospective teachers' *claims, data, warrants,* and *backings* have been identified per

each of the classroom mathematical practices that were established as a part of this instructional

sequence.

<u>**Guidelines for Individual Participation in Argumentation**</u>

N/A	Not Applicable
◎	No or Minimal Support of Argumentation
◉	Some Support of Argumentation
◉	Moderate Support of Argumentation
●	Extensive Support of Argumentation

These classifications are illustrated in more detail in the summary section of each case analysis.

Lastly, this researcher also wished to further explore any similarities and differences

between the prospective teachers who participated in this research study through a cross-case

analysis (Stake, 2006; Yin, 2004). The individuals' development of place value concepts,

counting strategies, addition and subtraction strategies, as well as multiplication and division strategies were analyzed side-by-side in order to provide trends in their understanding. Also, the cross-case analysis examined the prospective teachers' participation in each of the classroom mathematical practices in order to identify any similarities or differences in the way each participated in providing *claims, data, warrants,* and *backings*. Details of these analyses have been included in chapters 4-6.

<u>Trustworthiness</u>

In conducting qualitative research, the researcher needed to provide the burden of proof on issues of trustworthiness. Lincoln and Guba (1986) emphasized the importance of a researcher taking precautionary steps along the various stages of a research endeavor to ensure the trustworthiness of the findings. Throughout this study, the design and implementation of the project allowed for a prolonged engagement and persistent observation of the research setting. To begin with, the researcher and the research team maintained extensive and meaningful communication with the prospective elementary teachers in the mathematics classroom. The researcher - along with at least 5 other members of the research team - remained actively involved in every single class session during the 10-day duration of the instructional unit on whole number concepts and operations. Furthermore, the researcher continuously collaborated with research team members before, during and after the selection of the participants who were chosen for the case study analysis. Members of the research team were in accord regarding which participants would be able to provide the desired research and accomplish the intentions of this project.

Another strategy used in order to ensure trustworthiness involved triangulation. As Stake (2006) stated "triangulation is an effort to see if what we are observing and reporting carries the same meaning when found under different circumstances" (p. 113). Written artifacts, videotapes, audiotapes, interviews and other artifacts such as tests and assignments granted avenues for the exploration, clarification and verification of results that were examined during analysis. Triangulation was used by all members of the research team as the learning goals of each classroom session, and all instructional tasks were discussed (Lincoln & Guba, 1986; Stake, 2006). Next, they reflected on potential courses of action after each class and during research meetings. Furthermore, triangulation was used to compare the classroom observations with the written artifacts, and to compare interview and focus group responses with exercises, assignments and other assessments. All data that required coding were independently analyzed and then discussed at length via cross checking with another member of the research team to verify each of the *claims, data, warrants*, and *backings* that had been transcribed.

Upon the completion of the analysis stage of this research project, a rich and extensive narrative will be provided so that interested parties can examine the transferability of the findings and to draw conclusions relevant to their area of interest.

Selection of Individual Cases

The researcher conducted interviews prior to and then again after the instructional sequence, collected student artifacts and examined contributions to classroom discussion during the instructional sequence as well analyze a focus group following the completion of the instructional sequence. In order to provide a full, rich story for each case study participant, the

researcher made a conscious choice of writing about one individual from each classification of initial content knowledge. Specifically, the analysis of the individual cases for this research effort focused on Cordelia and Claudia's development of whole number concepts and operations through the instructional sequence. Following the individual analysis, a cross-case analysis of these two individuals was done to synthesize the cases and to build a more complete picture of individual prospective teachers' understanding of whole number concepts and operations.

CHAPTER 4: THE CASE OF CORDELIA

Cordelia had taken no previous undergraduate mathematics or mathematics education content courses prior to the course used for this research. In the beginning of the course, all prospective teachers submitted a one-page "My Experience with Mathematics" entry. Cordelia gave an insight into her motivations of wanting to know the reason behind mathematical notions and her desire to understand mathematics at a deeper conceptual level.

> *I'd like to think of myself as good at basic math… I'd get frustrated*
> *when my teachers couldn't answer 'why?' I didn't like being told*
> *'because that is how it is done.' I always thought there is a reason why*
> *things are done a certain way and that my teacher should know why.*
> *…I hope to do well in this class and re-learn things in a way better to*
> *teach my future students.*
>
> *Cordelia, January 2007*

On the CKT-M Pre-test instrument, Cordelia had a score of 10 which placed her in the "Low-Content" category of mathematical content knowledge at the outset of this research.

Individual Development

In order to specifically illustrate the development of each individual through the instructional sequence on whole number concepts and operations, this researcher decided to examine the individual's development on several levels. First, this analysis focused on what was known about each individual prior to the beginning of the instructional sequence. Next, each participant's *individual* artifacts – outside of her contributions to group and classroom discussions – were analyzed. The third level of analysis occurred based on what the individual

revealed during the post-interview as well as her participation in a focus group involving all research participants. The fourth level of analysis involved each individual's participation in the social situation of the classroom.

Prior to Instructional Sequence

During the pre-interview conducted prior to the beginning of the instructional unit on whole number concepts and operations situated in base-8, the researcher (who also served as the interviewer) asked Cordelia some questions related to place value, whole number operations and her role in the classroom. These pre-interview questions were all in the traditional base-10 system since the participants had not been introduced yet to the base-8 system in the instructional sequence. When asked the question, "Write the numbers 1 through 32 in a way that is meaningful to you." Cordelia's response was written all the way across the page and appeared as:

$$1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12\ 13\ 14\ 15\ 16\ 17\ 18\ 19\ 20\ 21$$
$$22\ 23\ 24\ 25\ 26\ 27\ 28\ 29\ 30\ 31\ 32$$

Here, it should be noted that no breaks were made with respect to tens and ones by Cordelia and the groupings of the numbers seemed to be determined by the space on the page rather than any particular notion of place value. As the follow up question, the researcher asked:

Interviewer: Which numbers are comparable?
Cordelia: The numbers 1 through 9. And then 10 through 32.
Interviewer: How is that?
Cordelia: Well, 1 through 9 have a single symbol and 10 through 32 are two numbers put together.

Notice again that while Cordelia did distinguish between single digit and multidigit numbers, she did not attach a definite significance to 10 or place value. Next, the interviewer wanted to

examine whether Cordelia would connect the next place value of 100 involved in a number such

as 102.

Interviewer:	Would you write the number 1 through 102. Feel free to skip the numbers as you see fit.
Cordelia:	Okay (wrote the following)
	1 2 3 4 5 6 7 8 9 10
	11…
	21 …
	31…
	41…
	51…
	61…
	71…
	81…
	91…
	101 102
Interviewer:	How come you decided to write the numbers that way?
Cordelia:	They all line up in a row. 11, 21, 31, etc. all end in 1. It looks better.

At the outset, it appeared that Cordelia's writing of the number 1 through 32 was completely

unrelated to place value. However, using the follow-up question of writing the number 1 through

102, Cordelia revealed that she did use place value to group some if not all numbers.

The next question on the interview involved the use of Base Ten Blocks (See Figure 13).

These blocks are manipulatives that are commonly used to model numbers and number

relationships. Students typically use them at the elementary and middle school levels.
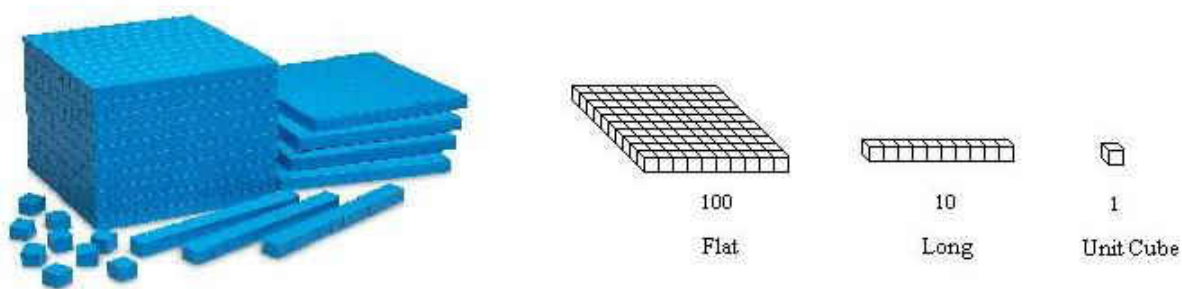
Figure 13: Configuration of Base-Ten Blocks

In order to gain insight into her ability to compose and decompose numbers, Cordelia was given 3 Flats, 5 Longs, and 2 Unit Cubes and asked to represent the number 254. Purposefully, the interviewer arranged for an insufficient number of Unit Cubes to explore Cordelia's notions of composition and decomposition of numbers.

Interviewer: Using the Base Ten Blocks provided (3 Flats, 5 Longs, 2 Unit Cubes), how would you represent the number 254?
Cordelia: I couldn't do that…unless I broke one of the pieces (laughs)
Interviewer: You can use the Base Ten Blocks in any way you wish.
Cordelia: (Playing around with the Base Ten Blocks) I could do it if each 10 block could be 20. I could do it with two's. I could redefine the blocks.
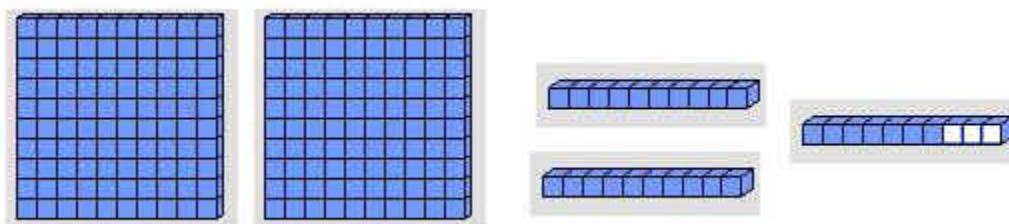

Figure 14: Cordelia's Illustration of 254 using Base-Ten Blocks

In this case, Cordelia observed that she would need to break one of the pieces – namely, the Flat (100 block) – to be able to represent 254 using Base Ten Blocks. Instead, she arrived at an alternate solution of reassigning each block to represent two units. In this fashion, the two Longs ($2 \times 20 = 40$) and the seven Unit Cubes ($7 \times 2 = 14$) would represent 54. It must be noted that the blocks that were not counted have been represented by the "blank" or white Unit Cubes in the figure on the previous page.

Cordelia's creativity in reassigning each Unit Cube does provide a way of getting 54, however; she had a misconception in that she still claimed to need two Flats. Recall, that she assigned each Unit Cube to have a value of two, and therefore the two Flats would represent 2×200 or 400. As a result, using the base ten blocks, Cordelia illustrated the number 454 instead of 254. In fact, she did demonstrate an ability to think about the problem posed to her using an alternative strategy. Curiously, even though she contemplated "breaking" or decomposing one of the blocks; she did not carry out this plan and decided on the strategy to "redefine the blocks".

The last question during the pre-interview involved a student's work which illustrated the manner in which the student had performed a multidigit multiplication problem.

> *"Some fifth-grade teachers noticed that several of their students were making the same mistake in multiplying large numbers. In trying to calculate*

$$
\begin{array}{ccc}
1 & 2 & 3 \\
\times \quad 6 & 4 & 5 \\
\hline
\end{array}
$$

> *The students seemed to be multiplying incorrectly. They were doing this:*

$$
\begin{array}{cccc}
 & 1 & 2 & 3 \\
\times & 6 & 4 & 5 \\
\hline
 & 6 & 1 & 5 \\
 & 4 & 9 & 2 \\
 & 7 & 3 & 8 \\
\hline
1 & 8 & 4 & 5 \\
\end{array}
$$

In the conversation that ensued, note the way that Cordelia made sense of the student's work.

Specifically, observe her notion of place value as she mentioned the word " place" to describe

the student's mathematical move as well as her suggestion of the way she would illustrate this

multiplication problem with her potential students. After Cordelia read through this problem, the

interview continued:

Interviewer: What do you think the student did in this problem?
Cordelia: They came up with the right number but in the wrong place…They didn't hold the *places*. It should be

$$
\begin{array}{r}
1\ \ 2\ \ 3 \\
\times\ \ 6\ \ 4\ \ 5 \\
\hline
6\ \ 1\ \ 5 \\
4\ \ 9\ \ 2\ \ 0 \\
7\ \ 3\ \ 8\ \ 0\ \ 0 \\
\hline
7\ \ 9,\ \ 3\ \ 3\ \ 5
\end{array}
$$

Cordelia: As a teacher, I would do

$$
\begin{array}{r}
6\ \ 4\ \ 5 \\
\times\ \ 1\ \ 2\ \ 3 \\
\hline
\end{array}
$$

Interviewer: Why would you write them that way?
Cordelia: Because students know their multiplication for 1, 2, and 3. They could do 3 times 5, or 2 times 4, or 1 times 6. Because it is just smaller numbers, it seemed less intimidating.

The previous episode illustrated that Cordelia did possess some knowledge of multiplication in

using a zero to hold the place value in the traditional multiplication algorithm. She even

suggested a method for becoming more efficient with whole number operations, while pointing out the idea of place value as the misconception the student in the example cited.

During the Instructional Sequence

As described in chapter 3, this analysis will explore each prospective teacher's development as she moved through the instructional sequence. Following the pre-interview, the focus of the analysis shifted to some of Cordelia's individual activity during the course of the instructional sequence specifically through analyzing homework assignments and items on a test. From this point on, problems listed and language used should be strictly in base-8 unless specifically noted otherwise.

After several class sessions dedicated to counting and skip counting, the prospective teachers were introduced to the open number line in order to record their thinking and discuss various strategies used to solve problems.

<u>Place Value and Counting Strategies</u>

After two weeks, the first homework examined the prospective teachers' abilities to individually solve problems related to counting and reasoning with addition and subtraction. Given the problem, "For Valentine's Day, Victor bought 57 heart-shaped chocolates. After purchasing some more, he had a total of 243 heart-shaped chocolates. How many more chocolates did Victor buy?" Note the manner in which Cordelia illustrated two methods of counting up from 57 (five-ee-seven) to 243 (two-hundree-four-ee-three) and then counting down from 243 to 57.
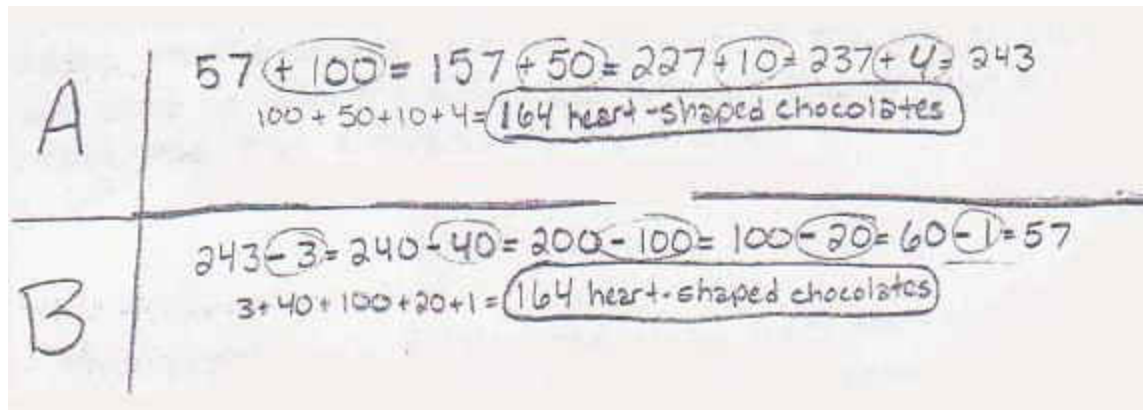
Figure 15: Cordelia's Illustrated Counting Strategy with Addition and Subtraction

As it can be observed, after only two weeks of "living in Eight-world", Cordelia demonstrated the strategy of starting with 57 and counting up to get to the desired result of 243. She used the numbers 100 (one hundree), 50 (five-ee-zero) and 10 (one-ee-zero) – using convenient numbers through understanding place value – to get to 237. After which using counting strategies, she counted up 4 more to arrive at 243. She recognized that from 57 to get to 243, she had to think of $100 + 50 + 10 + 4 = 164$ to find the solution to this particular problem.

When asked to use two different strategies, she used what was labeled as method B in Figure 15. This time, instead of counting up to 243, she conveniently counted down 3, then 40, then 100, then 20 and finally 1 to get to 57. It must be noted that she clearly illustrated an understanding of place value using those specific numbers - 3, 40, 100, 20 - to subtract what would be convenient in order to solve this problem. The final remark on Cordelia's solution to this particular problem is related to her incorrect use of the equal sign. She connected all her moves with equal signs implying that $57 + 100 = 157 + 50 = 227 + 10 = 237 + 4 = 243$. That would mean that in her solution A, $57 + 100 = 243$; or that in solution B, $243 - 3 = 57$. Even though the aforementioned misconception of the use of the equal sign represened a significant

one, it did not interfere with her illustration of understanding how to count and perform addition and subtraction in base-8. The misconception was addressed and later on in the instructional sequence, Cordelia corrected this misconception. Overall, she illustrated a good understanding of place value and initial counting strategies.

<div align="center">Addition and Subtraction</div>

On another problem in the same time frame, the question was asked:

"A student was given the following problem to solve in class: How many more stickers do you have to add to 47 stickers to get a total of 135?

To make the above problem a little easier for them to solve using the number line, they jumped 3 to go from 135 to 140. Then they jumped 100 to get from 140 to 40. Finally, they jumped 7 to go from 40 to 47. Since the student did that, they came up with the following solution:

$$3 + 100 + 7 = 112 \text{ spaces} \qquad\qquad \text{Answer: 112 stickers}$$

Is the student correct? If so, explain why? If not, explain what the student did incorrectly?"

Cordelia's solution to this problem is provided including her illustration of the use of a number line as well as her ability to look at another student's solution in the context of addition and subtraction. Note the manner in which Cordelia used the open number line and the way she chose to skip counted as demonstrated in Figure 16.
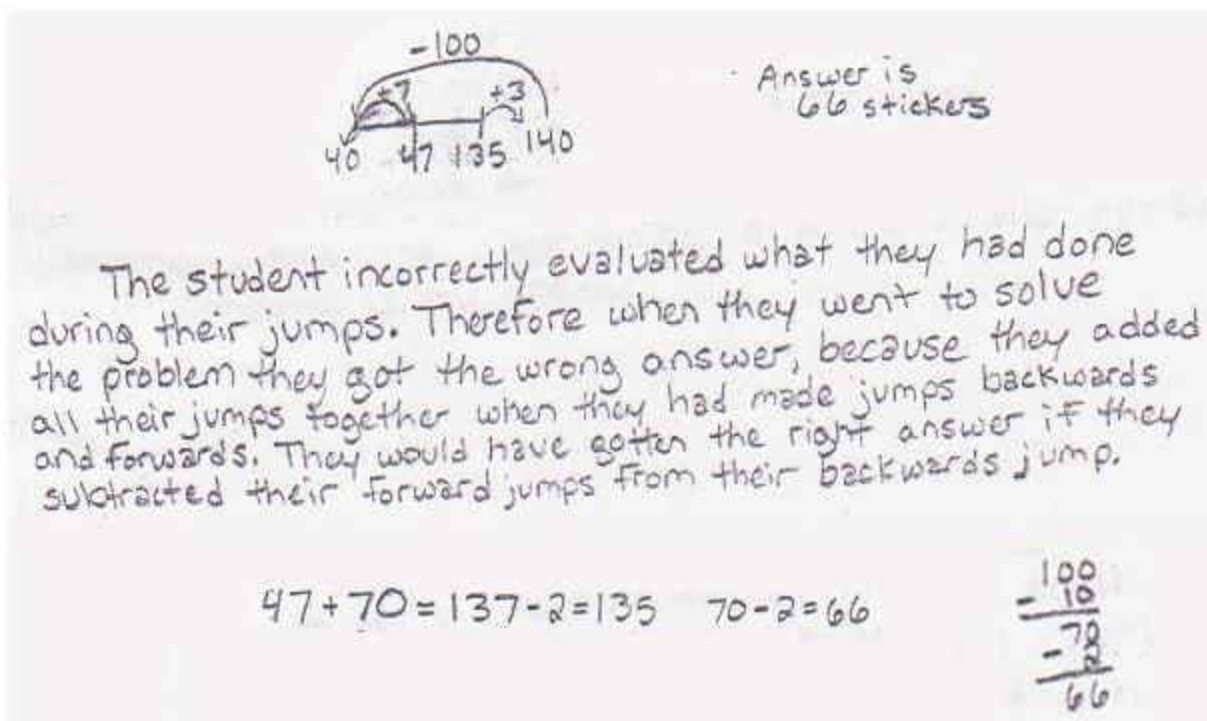
Figure 16: Cordelia's Solution Illustrating the Open Number Line

Cordelia illustrated a very good understanding of the open number line as well as providing some insight into her way of thinking. She demonstrated correctly being able to skip forward and backward and reflected her thinking using the open number line. She placed the distance traveled between each number on top of the arrow and reflected the direction of the move with a minus sign if moving backwards and a plus sign if moving forward. It must be noted that she checked the open number line approach through algorithmic approaches in order to verify her answer.

Over the next three weeks, the instructional tasks involved using the context of the candy shop to further highlight the significance of place value and explored invented strategies for addition and subtraction. Place value notions as well as the composition and decomposition of numbers according to place value were explored through the use of inventory forms. These

forms allowed for the separation of units into boxes, rolls and pieces. A piece referred to an individual piece of candy (one unit). A roll comprised of precisely onee-zero (10) pieces and a box contained one-hundree-zero (100) candies. Furthermore, various addition and subtraction strategies were examined as they came up during class discussions.

The second homework assignment was submitted two weeks after the data introduced in the example illustrated in Figure 16. On this assignment, the researcher continued to explore the development of Cordelia's understanding of whole number concepts and operations as she progressed through this instructional sequence entirely taught in base-8. Cordelia displayed a very good understanding of inventory forms as she grouped and regrouped objects with ease. Specifically, she decomposed and composed in various problems in the context of the candy shop without any difficulty. She illustrated her understanding of moving values across place value and explained and justified her thinking in verbal and written forms.

A problem on the second homework asked the prospective teachers to consider "a student did this…" scenario. Note the way that she wrote her procedure on the right hand side of Figure 17 to verify the solution. Also closely follow her explanation and justification provided afterwards.

2. A student did the following to solve the given story problem:

There were **312** candies in a Candy Shop. **165** candies melted. How many candies were left?

Is the student correct? Explain and justify the student's way of solving this problem.

Figure 17: Cordelia's Approach in Examining Another Student's Solution

Cordelia's approach in understanding another student's non-traditional subtraction algorithm provided further opportunities to gauge her development and understanding of whole number operations at this stage. She stated:

> "The student saw that if she adds 1 roll to the ones place in 312 she has to add one roll to the tens place in 165 because what you do to one number you have to do to the other. Then she added 1 box to the tens place in 312 and added a box to the 100 place in 165 because what she does to one number she has to do to the other. This gave her 3, 11, 12 minus 2, 7, 5 which gave her 125 which is correct. She was able to do this because 12 – 5 =5, 11 – 7 = 2 and 3 – 2 = 1."

While she realized what the student did and explained the process, she was *not* able to provide the *justification* for why this step would be mathematically correct. As for adding "1 roll to the ones place in 312", Cordelia did not demonstrate the thorough understanding that the student added one-ee-zero ones to the ones place in 312. In order to compensate equally by using "place value understanding to subtract by equal additions" the student then added an equal "amount" to 165 by adding 1 group of one-ee-zeros making the 6 into a 7. It is precisely the simultaneous realization that one-ee-zero ones and one group of one-ee-zeros are equivalent.

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Instead, Cordelia simply stated, "what you do to one number you have to do to the other." Cordelia's limited conceptual understanding is demonstrated again when she stated that the student, "added 1 box to the tens place in 312 and added a box to the 100 place in 165 because what she does to one number she has to do to the other." In this case, similar to the stated simultaneous realization that is needed for conceptual understanding, the student added one-ee-zero rolls to the "tens" place in 312 and added the equivalent "amount" by adding one box to 165 to change the 1 into a 2. The student – and in this case the prospective teacher Cordelia – should demonstrate conceptual understanding of *why* it is mathematically valid to proceed in the fashion illustrated above. Lastly, in the mind of this researcher, it seemed that she became convinced of whether the student was correct or not based on the traditional column subtraction algorithm that she has illustrated on the right hand side of the last figure provided. Even though the knowledge of traditional algorithms provided a mechanism for Cordelia to answer this question, the issue of giving credibility and proper worth to another student's work was an aspect of this instructional sequence which would be further examined as a part of the whole class discussions.

<div align="center">Multiplication and Division</div>

The last three days of instruction primarily focused on multiplication and division of whole numbers. One of the main distinctions that needed to be made in whole number multiplication involved the meaning of each of the numbers being multiplied. It was expected that prospective teachers demonstrate an understanding of which number in multiplication stood for the *groups of objects* and which one represented the *number of objects* in each group. The commonly used convention in United States schools signifies that in multiplying two numbers, the first represents the *number of groups* of objects and the second stands for the *number of*

objects in each group. Since teacher education programs ultimately prepare prospective teachers to teach in United States, this course also adhered to the same convention. In Figure 18, note Cordelia's depiction of this convention as the prospective teachers were asked to write their own story problems. In particular, observe the *number of groups* of objects and *number of objects* in each group she drew to model this multiplication problem.
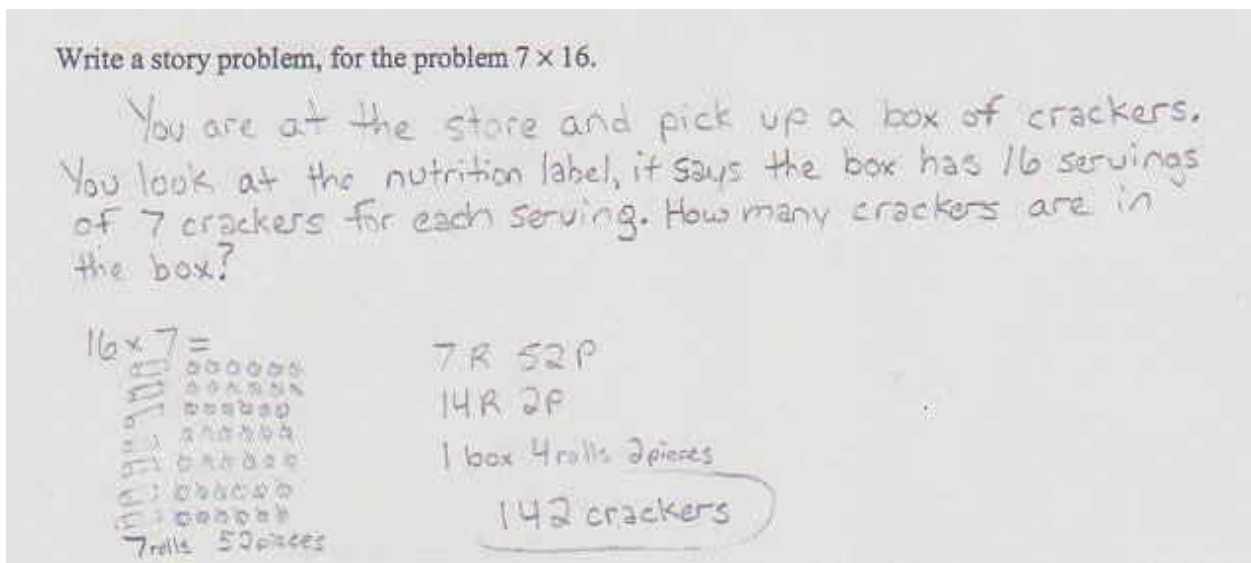

Figure 18: Cordelia's Initial Understanding of Multiplication (Groups of Objects)

Through this example, the researcher noticed that Cordelia's story problem posed issues regarding her use of the convention for 7 × 16. The 7 ought to have represented the number of groups (servings in this case) and the 16 should have represented the number of objects in each group (the number of crackers for each serving). However, in solving the problem, Cordelia wrote her story problem to represent 16 × 7 and yet drew 7 groups of objects each of which had 16 objects in each group. This example illustrated a lack of understanding of the commonly used convention of multiplication and it illustrated the manner in which Cordelia interchanged the meanings of the factors. Furthermore, her illustration did not model her story problem which

raised additional concerns. She did demonstrate an effective means of finding the total number of crackers by rewriting 7 rolls and 52 pieces into the 1 box, 4 rolls, and 2 pieces totaling 142 crackers.

After some classroom discussion and the opportunity to engage in discourse, prospective teachers were asked to solve the following homework problem individually: "Mrs. Wright wants to fill 5 bags so that each will contain 16 candies. How many candies should she use?" Cordelia depicted the multiplication problem of 5×16 pictorially as she drew the 5 bags and placed 16 candies in each bag. Furthermore, she calculated that there would be 5 rolls and 36 pieces of candy total. Analysis of this artifact indicated that she wrote *86* total pieces for her final answer to this multiplication problem. Despite correctly calculating the 5 rolls and 36 pieces of candy, Cordelia reverted back to base-10 since $50 + 36$ would be 86 in base-10. As the entire instructional sequence took place in base-8, the 5 rolls and 36 pieces should have yielded a total of one-hundred-six candies as the solution to 5×16.

Division represented the final part of the instructional sequence and due to time constraints only parts of two class periods were devoted to division and division examples. During whole class discussion, one student illustrated and explained the partial quotients algorithm and justified her solution. In the partial quotient algorithm, a student typically uses a successive approximation method by using convenient multiples of the divisor to get closer, and closer to the desired number. While the majority of the class followed this explanation, the approach to division and student responses were decidedly different in part due to a lack of time on this topic. In contrast to the class time spent on other operation strategies, due to time constraints, prospective teachers only had 1 ½ class sessions on the topic of division.

Cordelia was asked to solve the following division problem on the test:

> *Mary has 652 stickers that she wants to share with some friends in her class. If she gives each of her friends 17 stickers, how many friends can she share with? How many stickers will be left, if any?*

Illustrated by her solution provided in Figure 19, observe the manner in which Cordelia solved this division problem through repeated addition.

Figure 19: Cordelia's Solution to the Division Problem 652 ÷ 17

Examining Cordelia's solution to this division problem revealed that she did not feel comfortable with the one method that was explained and justified during class – the partial quotients method. Her strategy involved a repeated addition of 17 in order to use the resulting sum of 151 to get close to the desired total of 652. The inability to divide in the traditional sense was coupled in this case with an inaccurate illustration of multiplication. Overall, it seemed that Cordelia's efforts – in particular with division – were primarily procedural and lacked the

conceptual understanding that connected whole number operations as a connected endeavor. According to Steffe et al. (1988), conceptual understanding allows for the synthesis of previously learned material to construct new meanings. While Cordelia demonstrated a greater understanding of *her own* ways of thinking, the times when other prospective teachers explained and justified their thinking *in a manner distinct from her approach* seemed to have little impact on her. The researcher was also reminded of a previous case when Cordelia could not make sense of another student's work – see Figure 17. During the course of the instructional sequence, all prospective teachers were expected to understand, question and make sense of another student's strategies. This aspect of Cordelia's development through the instructional sequence is revisited in the latter stages of analysis.

Following the Instructional Sequence

Upon the completion of the instructional sequence, the individual post-interview with Cordelia provided further insight for this research. When Cordelia was asked what changes had occurred through her experiences with the instructional sequence, she responded: "Now it's easier to understand exactly what I am doing - because of inventory forms, and knowing place value." In reference to methods and strategies that she experienced through the class discussions on whole number concepts and operations, she commented: "I already knew the strategies, now people are just labeling them. They are naming things that I have already seen."

Cordelia seemed particularly adamant regarding her perceived role in the classroom versus the role of other prospective teachers – specifically when addressing the notion of authority in the classroom. She mentioned:

Cordelia: If you can't tell me *why* [italics added] you are doing something or that you *can* [italics added] do something, it discredits what you are telling me."

When asked about her role as a member of the classroom social dynamic, she responded:

Cordelia: I had to know when to explain what I did to other people or when it is not time to explain it yet….I know what I did and know it is right. In the small groups, I would show what they did wrong. To the (whole) class, I would show what I did and just explain that.

A follow-up question during the post-interview involved Cordelia's impressions of what constitutes as an acceptable solution and who she considered the position of authority in the classroom.

Cordelia: As long as they (the other students) know what they are doing, then it doesn't matter who says it. It is like if you need to have your car fixed, then you don't take it to a veterinarian – you take it to a mechanic! It doesn't matter if you take your car to a Honda dealer or a Chevy dealer, because they all know how to fix cars. So if they know what they are doing, it really doesn't make a difference who says it is right.

After answering some questions regarding her own role as well as the role of the other prospective teachers, the interviewer asked her regarding whole number operations and her impressions of how well she had learned them. With respect to addition and subtraction, Cordelia felt quite confident and responded: "They are easy and I can understand exactly what I am doing because of place value." However, her impressions of multiplication and division were decidedly different. Referring to multiplication, she exclaimed: "I just didn't get multiplication. If I couldn't recall what 4×3 was, it meant that I didn't understand it." Further analysis of her participation in classroom discussions – presented in the next section – helped to clarify her interpretation of her lack of understanding.

Throughout the instructional sequence, the researcher had also noticed that the "pictures" that Cordelia would draw in solving operations problems did not always seem to correspond to the actual problem being solved. During the post-interview she was asked:

Interviewer: Do the pictures that you draw help you to solve the problems? Can you elaborate on how they represent your thought process?

Cordelia: No, because it is backwards! I am drawing pictures because I need to not because I need it to understand. They just make it more confusing because I already understand the math.

Cordelia was presented with the following multiplication problem (in base-10) as a part of the post-interview. She was asked: "A student has 23 books in his library where each book has 14 pages. How many total pages are there in all the books?" Note her illustration with the circles and dots in relation to the place value aspects of multiplication.
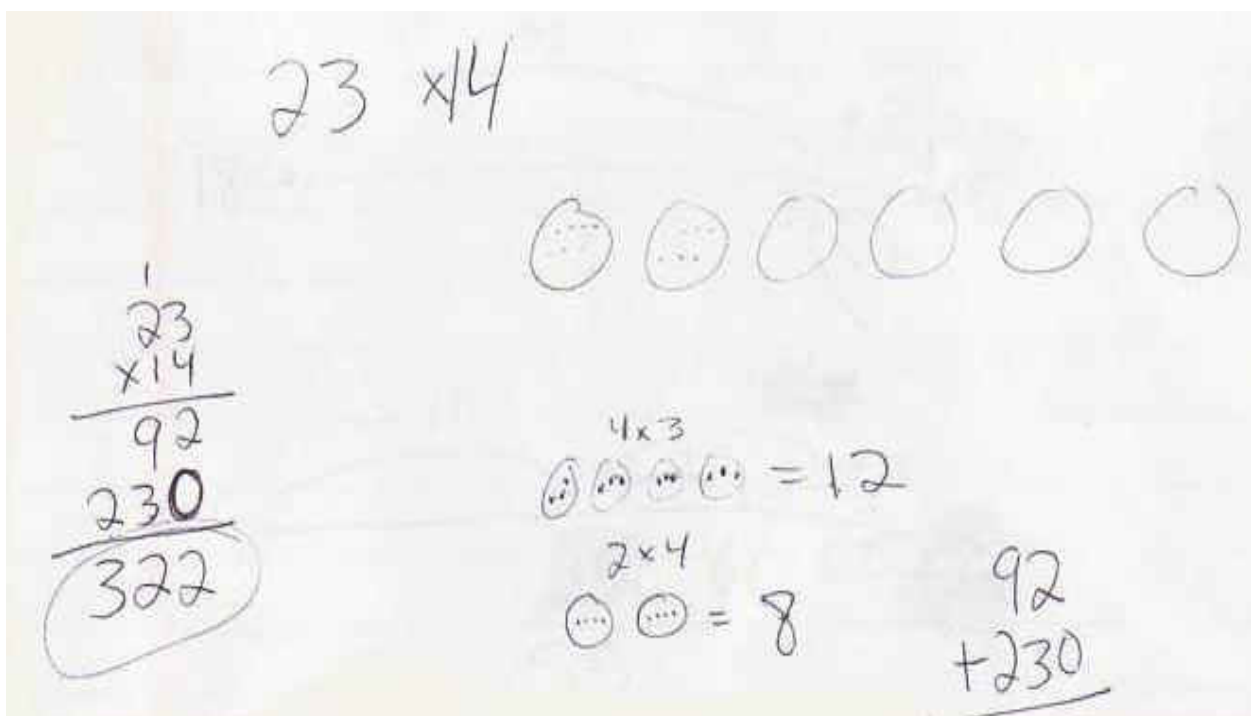
Figure 20: Cordelia's Solution to 23 × 14 (in Base-10)

It must be noted here that Cordelia went about solving this problem in the following order: First, she wrote 23 × 14. Secondly, she drew the 6 circles in one row. Next, she performed the multiplication algorithm seen on the left hand side. Finally, when asked how she did the algorithm, she wrote the 4 × 3 and the 2 × 4 portion (including the circles with dots in them). Here, her solution to 23 × 14 modeled what she had mentioned earlier regarding the use of pictures to "help" her perform whole number operations. Her algorithm by itself was correct in that she displayed good procedural understanding of *how* to solve this multi-digit multiplication problem.

Interviewer:	Could you describe what you did?
Cordelia:	I did 4 times 3 (drawing the 4 circles with 3 dots in each)… I multiplied the ones. Then, you multiply 2 times 4 to get 8.
Interviewer:	What did you do next?
Cordelia:	I just added 12 and 8 to get the 92.
Interviewer:	How is that?
Cordelia:	That's what you do. I can't explain it.

Here, Cordelia began by explaining her procedure in the same fashion that she had experienced during class. However, she reached a point where she could not explain and justify her solution. As consistent with her previous work, she displayed a good procedural understanding of the mathematics involved, however she encountered more difficulty in describing the *how* and *why* which required more conceptual understanding.

Cordelia was also asked regarding the ways that whole number operations compared in base-8 versus base-10.

Interviewer:	How well do you think you understood addition and subtraction in base-8?
Cordelia:	Fine. They work out just like they do in base-10.
Interviewer:	What about multiplication and division?
Cordelia:	In base-8, you can't multiply and divide the same way. I just didn't get division when we did it class – but I know I can do it.

The researcher gained valuable insight into Cordelia's understanding of whole number concepts and operations through the examination of her individual work by analyzing her written artifacts as well as interview questions included in the pre-interview, post-interview and the focus group that ensued after the completion of the instructional sequence. In order to fully understand the ways that Cordelia developed conceptual understanding through the instructional sequence situated entirely in base-8, this researcher needed to analyze her participation in the classroom community to see how she contributed to the social aspects of the classroom and reflexively the manner in which the group discussions may have influenced her development.

Cordelia's Participation in Taken-as-Shared Practices

This study placed each individual prospective teacher in the social setting of a classroom. As such, both the social aspects of learning as well as the concurrent individual component must be considered and discussed together. As stated earlier, Yackel and colleagues (Yackel & Cobb, 1996) maintain that individual student's mathematical activity and the classroom micro-culture are related in a reflexive fashion. The interpretive framework used for this study was previously discussed (See Table 6) and concisely related the social and individual components. Stephan (2003) elaborated on the interrelation of the social and psychological perspectives by stating "the existence of one depends on the existence of the other" (p. 28).

Roy (2008) extended previous research efforts by focusing on the classroom mathematical practices developed by prospective teachers in learning whole number concepts and operations through an instructional sequence situated entirely in base-8 (Andreasen, 2006; Bowers, Cobb & McClain, 1999). Roy identified the particular classroom mathematical practices

that occurred during this research effort in the same environment with the same prospective

teachers. Using Andreasen's (2006) initial instructional sequence and revised for the sake of this

research effort, Roy concluded

> "The instructional tasks supported the following taken-as-shared classroom mathematical practices in which prospective teachers (illustrated):
> (a) Developing small number relationships using Double 10-Frames,
> (b) Developing two-digit thinking strategies using the open number line,
> (c) Flexibly representing equivalent quantities using pictures or Inventory Forms,
> (d) Developing addition and subtraction strategies using pictures or an Inventory Form." (Roy, 2008, p. 136)

This current research effort now focused on analyzing Cordelia's specific participation in the

classroom mathematical practices outlined above. In particular, this analysis intended to identify

her development of conceptual understanding of whole number concepts and operations as it

occurred through interacting with classmates and engaging in classroom discussions.

As a part of this case study analysis, the researcher demonstrated the manner in which a

particular participant provided *claims, data, warrants,* and *backings* to facilitate and at times led

whole classroom discussion. Furthermore, the analysis using Toulmin's argumentation (1969)

allowed for collaboration between the social and the individual aspects to describe the

individuals in the case study and their participation in classroom discussions.

In the following sections, the researcher explored the specific ways that Cordelia took

part in the classroom discussion. Her participation in the classroom argumentation - as defined in

the previous chapter – involved providing *claims, data, warrants,* and *backings* to discuss the

topics in the instructional sequence. Cordelia's participation has been discussed as it occurred

within each of the four established classroom mathematical practices. The first section focused

on the development of number relationships using Double 10-Frames.

Developing Number Relationships using Double 10-Frames

As a part of the instructional sequence on Day 1, the instructor introduced Double 10-frames in order for prospective teachers to make sense of counting in base-8. In one particular example, the instructor used the overhead projector to flash the following Double 10-Frame for one second:
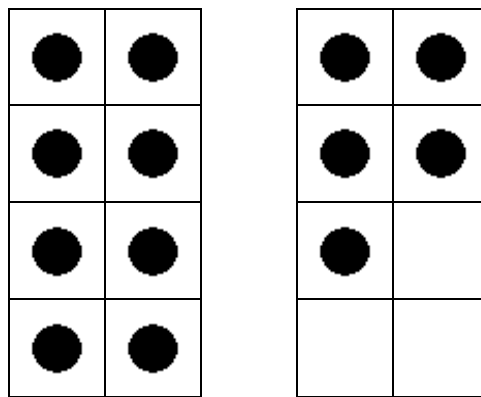

Figure 21: Double 10-Frame Illustrating 10 and 5

The transcript below indicated the conversation that took place involving Cordelia's participation within the classroom dynamic. The type of support - *Claim, Data, Warrant, Backing* - has been indicated in *(italics)*.

Instructor:	Okay here we go, ready? (Teacher flashes 10 and 5). (After pausing to watch students' reactions, she says) This is what I am looking at. (Teacher gestures at counting in the air)
Instructor:	Alright Cordelia, How did you get it?
Cordelia:	**Well you have a whole one full so that is one-ee-zero and then you have a half so that is plus four, one-ee-four, and plus one, one-ee-five**. (*Data/Claim*)
Instructor:	How many of you did it just like that?
Student:	Full, 4, 1

> Instructor: Full, 4, 1 is another way to describe it. How many of you counted by ones? Took a while, eventually those counting by ones strategies are not efficient enough to keep up and then you start working on developing other strategies.

In the episode described above, Cordelia provided a *claim* of "one-ee-five" in response to the teachers' question. She also provided the *data* to explain the manner in which she arrived at her *claim*. By participating in the social aspect of the classroom, other prospective teachers also share in the counting strategy of using one-ee-zero's and 4's with Double 10-frames. Simultaneously, Cordelia's method was incorporated into the larger class discussion and she had the opportunity to see how other prospective teachers responded to her method. In future episodes, when a student counted by one-ee-zero's and/or 4's, she did not need to provide an explanation since the idea of using small number relationships in Double 10-frames had become taken-as-shared. Through this episode, Cordelia displayed her support for argumentation and a contribution to the taken-as-shared notion of developing small number relationships using Double 10-Frames.

Two-Digit Thinking Strategies Using the Open Number Line

On Day 2, prospective teachers were presented with the following problem:

> "There were 62 children in the band. 36 were boys and the rest were girls. How many girls were in the band?"

The following represented the exchange between Cordelia and the teacher in solving this problem.

> Cordelia: Can you write the numbers on the board?
> Instructor: Sure. Sixee-two children, three-ee-six boys. How many girls? How can I start without setting these up in columns? How can I start? Cordelia.
> Cordelia: I came up with two-ee-…(*Claim*)

| Instructor: | I am not asking for the answer. How can I start? |
| Cordelia: | Well, like take six-ee-two and three-ee-six and make one-ee-zeros; six one-ee-zeros minus three one-ee-zeros is three-ee-zero and then you still have the two and the six. (*Data*) |

In Figure 22 below, the researcher represented Cordelia's mathematical moves as recorded by the instructor on the board. Note the manner in which Cordelia decomposed the number three-ee-six.
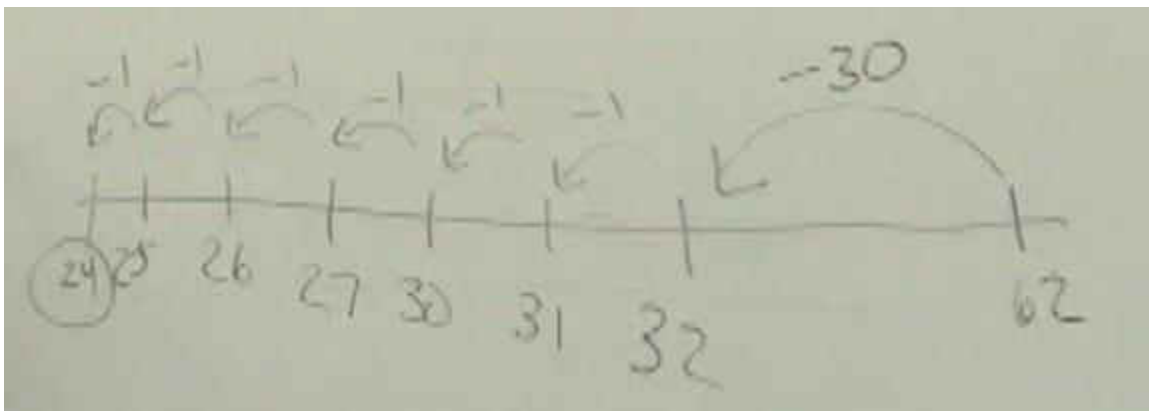

Figure 22: Cordelia's Method of Solving 62 - 36

Instructor:	Oh, I think I see what you are talking about. Okay, you started with your six-ee-two. Which side do I put six-ee-two? Which side, here or here?
(Laughing)	
Instructor:	We'll go over here. You said, you took away three oneee-zeros, all at once? Or one-ee-zero minus one-ee-zero minus one-ee-zero?
Cordelia:	All at once. (*Data*)
Instructor:	So you took away three-ee-zero. And then, then you said…what did you have left after you did that?
Cordelia:	Three-ee-two. (*Data*)
Instructor:	Three-ee-two. And then what did you do?
Cordelia:	And then minused six. (*Data*)
Instructor:	Why?
Cordelia:	Because the original number had six. (*Warrant*)
Instructor:	Okay, you started with sixee-two then you took away three-ee-zero.
Cordelia:	But I still had to take away six. (*Data*)
Instructor:	How did you do that? Did you do it all at once?
Cordelia:	I counted by ones. (*Data*)
Instructor:	Okay. So you said minus one is

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Cordelia:	Three-ee-one. (*Data*)
Instructor:	Minus one is
Cordelia:	Three-ee-zero, minus one is two-ee-seven, minus one is two-ee-six, minus one is two-ee-five…(*Data*)
Instructor:	How many have we done?
Cordelia:	Five. (*Data*)
Instructor:	Okay, so.
Cordelia:	minus one is two-ee-four. (*Data*)
Instructor:	So where do we find the answer in all this?
Cordelia:	At the left end. (*Data*)
Instructor:	Because we start with six-ee-two and we took away
Cordelia:	Three-ee-six. (*Data*)
Instructor:	Three-ee-six to give us two-ee-four, but what's the answer?
Cordelia:	Two-ee-four girls. (*Claim*)
Instructor:	Two-ee-four girls. That's important, I'm not going to stress it too much in here, but it's important I give you a word problem that has a context; your answer should be within that context. You should stress that when you are teaching; I'm not going to stress it that much in here. Questions for Cordelia? Who solved it just like her? Raise your hands. Okay, interesting not just one table's worth, but a sprinkling around the room. Who has got questions for Cordelia? Who solved it differently than Cordelia?
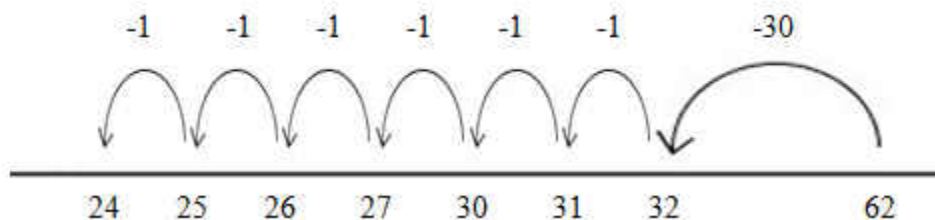
Figure 23: Cordelia's Participation in Using the Open Number Line

This episode illustrated by the transcript and Figure 23 above represented one of the first occurrences of a *warrant* as a part of the classroom argumentation. Cordelia was able to explain not only what she did and how she did that, but also provided the justification behind *why* she chose to subtract six. This example further illustrated that Cordelia managed to decompose the number three-ee-six according to place value into three-ee-zero (Three one-ee-zero's) as well as

six (six one's). As suggested in the literature review, Cordelia has demonstrated the ability to compose and decompose numbers and in addition uses one and one-ee-zero as iterable units. Lastly, the instructor pointed out the significance of situating the numerical answer within the context of the problem as stated. Cordelia was able to provide the result of two-ee-four, but added to it by suggesting that it stood for two-ee-four girls.

Her participation in the classroom discussion through this example signified her support for argumentation and her contribution towards establishing 2-Digit thinking strategies using the open number line becoming *taken-as-shared*.

Flexibly Representing Equivalent Quantities

In order to accomplish the learning goal of flexibly representing numbers, the instructional sequence utilized specific tasks in order to provide for exploration of the mathematics involved. During Day 3, the instructor presented prospective teachers with Mrs. Wright's Candy Shop. In this setting – in accordance to previous research and Realistic Mathematics Education – candy was packaged into boxes, rolls, and pieces as seen in Figure 24.
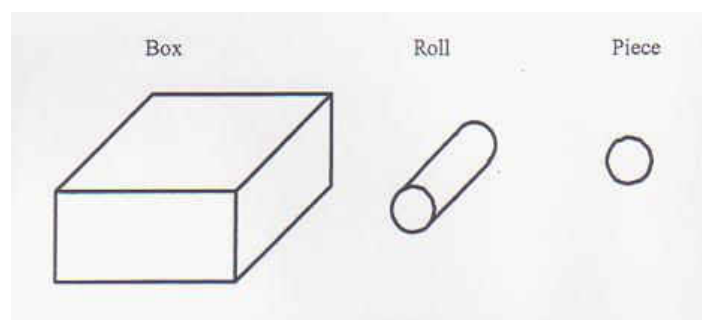
Figure 24: Boxes, Rolls, and Pieces in the Candy Shop

Through this systematic "packaging" of numbers, classroom discussion led to the following conclusions. Individual pieces of candy are packaged into rolls, and rolls of candy can be packaged into boxes of candy. Specifically, 10 pieces would comprise 1 roll, 10 rolls would equal 1 box of candy. Prospective teachers involved in this research project would solve activities in order to gain further experience with unitizing. As discussed earlier in the literature review, Cobb and Wheatley (1988) emphasized the significance of prospective teachers' simultaneous realization that 10 represented one unit as well as 10 individual units.

As the amount of candy increased, the need arose to have an efficient way of representing the number of candies without drawing the boxes, rolls, and pieces every time. The instructor presented a method of recording the number of candies in boxes, rolls and pieces called an Inventory Form, illustrated in the Figure 25 below.

Boxes	Rolls	Pieces

Figure 25: Inventory Form of Recording Boxes, Rolls, and Pieces

Through the use of the boxes, rolls, and pieces as well as Inventory Forms, the instructor provided the setting for prospective teachers to experience unitizing in base-8 since the notion of unitizing remains the core notion in the ability to flexibly represent numbers. The two aforementioned pedagogical content tools and their use were illustrated in the following episode involving Cordelia.

Instructor: Mrs. Wright might come into the candy shop and find different situations, or different amounts of candy on different tables. What the people working in her factory don't do is put them in nice careful ways. So I am

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curious if she came into the candy factory at 5 o'clock and she (noticed) what is represented in number one on page four?

Figure 26 reflected the problem being discussed in this episode.
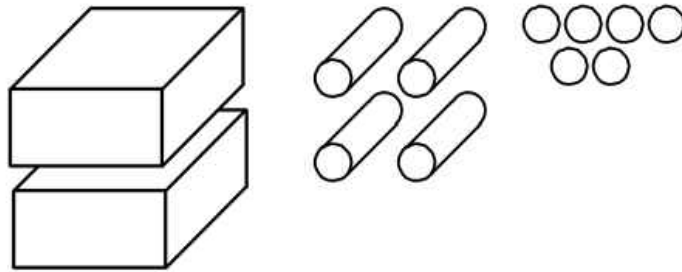
Figure 26: Candy Shop Example (2 Boxes, 4 Rolls, 6 Pieces)

Instructor:	What is represented?
Class:	Two
Instructor:	Two boxes
Class:	Four rolls
Instructor:	Four rolls
Class:	Six pieces.
Instructor:	And six pieces. Okay, thank you….. So sometimes it is good to have all the candy packaged as much as possible, sometimes it is good to have it completely unpackaged, and sometimes it is good to have places in between. So your problems may represent that. Who is ready to share drawings of how they packaged their candy – of how they found the candy? Cordelia, okay come on up.
Cordelia:	Let's see, so I decided I should know how many there were before I started drawing any pictures. So I counted everything, and I came up with two-hundree-four-ee-six, because I can't like … (*Claim*)
Cordelia:	Well, I figured that before I try and repackage everything that I should know how many I have all together. So I counted the boxes, one-ee-zero, two-ee-zero, two hundree, and then four-ee. I came up with two-hundree-four-ee-six. This is my rolls from a box. Okay, that is two-hundree-four-ee-six because… (*Data*)
Cordelia:	This is two-hundree-four-ee-six because I have got one-hundree here, one-hundree-zero in each of these rows and four-ee-zero and six single pieces. And then I drew it like this. So then I got one-hundree-zero, and then one-ee-zero rolls of one-ee-zero, and four-ee-six individual pieces. (*Data*)

Figure 27 illustrated Cordelia's recording of 246 involving boxes, rolls and pieces. Note the way that Cordelia arranged the rolls into groups of one-ee-zeros.
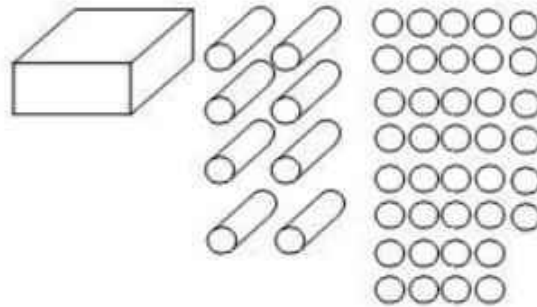
Figure 27: Cordelia's Recording of 246 using Boxes, Rolls, and Pieces

This episode was followed by a classroom discussion involving another student making sense of Cordelia's solution. After this explanation, the instructor asked:

Instructor:	Is that what you did Cordelia?
Cordelia:	Hmmm (Thinking).
Instructor:	You may have done it in another order
Cordelia:	I didn't think of opening rolls, I just took the **total number** figured out what I would need to get that number. (*Claim/Data*)
Instructor:	So you went from the picture that was given to show two-hundree-four-ee-six.
Cordelia:	Right!
Instructor:	And then you made it into candies.
Cordelia:	And then, I figured how would I pack it? (*Data*)
Instructor:	So, you packed it instead of unpacking it?
Cordelia:	Right! (*Claim*)
Instructor:	Do you see the difference between that? Can someone explain the difference?

During this particular episode, Cordelia participated in the *taken-as-shared* practice of representing equivalent quantities by using a total number approach. While the majority of prospective teachers "unpacked" the boxes and rolls provided, Cordelia considered the total number of candies and then represented that total number using boxes, rolls and pieces by "packaging" the candies in a different way than had previously been discussed.

The researcher also was given another glimpse of Cordelia's tendency not to use pictures to assist her in *solving* the problem. As was the case in the earlier *taken-as-shared* practice, Cordelia seemed to once again draw the picture of boxes, rolls, and pieces *after* she had already decided on her solution. This approach – while significantly different from her classroom counterparts – remained consistent with Cordelia's previous notions of the role of pictures. Earlier during the individual analysis, this researcher indicated that Cordelia had mentioned that she only drew pictures "because she knew she had to" and did not use them to assist her in solving mathematical problems.

During Day 4, Cordelia's participation in *taken-as-shared* classroom mathematical practices also came to the forefront. Using Inventory Forms (Figure 28), the instructor asked the prospective teachers to identify 2 equivalent representations of the 457 candies in order to explore prospective teachers' development of place value leading to whole number operations.

Boxes	Rolls	Pieces
4	5	7

Figure 28: Inventory Form Representing 457

After one student presented her solution, the instructor cultivated the sociomathematical norm of a different solution as presented in the following discussion:

Instructor:	Who got it in another way? Cordelia.
Cordelia:	I did, I wrote the boxes, and then I…
Instructor:	So how many were there?
Cordelia:	Four, and then I took apart one roll, so that would be four
Instructor:	Four?
Cordelia:	Rolls, and then I got more pieces, and I got one-ee-seven pieces.

One of the ways that an idea was considered *taken-as-shared* occurred when prospective teachers no longer questioned the reasoning behind an explanation. In this episode, when

Cordelia suggested getting "more pieces" and arriving at "one-ee-seven" pieces; the classroom community did not require nor need justification for the manner in which she proceeded. In the past, prospective teachers would have asked for and provided a *warrant* for such explanations. Citing the way arguments evolve and the "change in function", Roy (2008) identified various classroom mathematical practices including the use of Inventory Forms to flexibly represent equivalent quantities. As a result, Cordelia's contributions including the solution to representing 457 through the use of Inventory Forms demonstrated her participation in establishing this particular classroom mathematical practice.

Developing Addition and Subtraction Strategies

Through the course of the instructional sequence, prospective teachers progressively built on their knowledge to assist them in solving new problems. After having gained experience with boxes, rolls, and pieces as well as Inventory Forms, prospective teachers illustrated ways in which reasoning with these pedagogical content tools allowed for them to arrive at addition and subtraction strategies.

On Day 6 of the instructional sequence, the prospective teachers continued to explore strategies including the traditional addition and subtraction algorithms as well as column addition. After examining a few examples of non-traditional strategies including subtracting left to right, the instructor posed the following question to the prospective teachers. "How would you do 500 minus 243?"

The following figure (Figure 29) represented Cordelia's solution and the transcriptions that

follow illustrated her contribution towards the classroom mathematical practice involving

addition and subtraction strategies:

```
      5  0  0
  -   2  4  3
      3  0  0
      -  4  0
         -  3
      2  3  5
```

Figure 29: Cordelia's Strategy in Solving 500 - 243

Instructor:	Who did it differently? …So who is going to share how they did it next? Cordelia.
Cordelia:	Well, I did it…, I didn't change my place values before I started so two-hundree-zero. (*Data*)
Instructor:	Two-hundree
Cordelia:	Two-hundree from five-hundree is three-hundree, and then four from no-ee-zero, I don't know how to say this…I am short four to make one-ee-zero so minus forty and then three from nothing. I am short three, so minus three. (*Data*)
Class:	Five (*Student Challenge*)
Cordelia:	It's not five because I did that and that's how I got the wrong answer, but it is not five because it (pointing to the zero in the one's place in 500) is not one-ee-zero. It's just one, so I am like short three. (*Warrant*)
Cordelia:	I mean, it's just zero, there is nothing so I am short three because if you draw out a number line (draws a number line with 3 on the left and 0 on the right – motioning the distance between them). If this is zero and you're here (pointing to 3), you're short only three regardless. (*Warrant*)
Cordelia:	Yeah, so now I am short three. So three-hundree minus four-ee-zero is two- hundree-four-ee-zero, and then (pointing to 240) minus three is… (*Data*)
Class:	Two hundree three-ee-five.
Cordelia:	Two-hundree-three-ee-five. (*Claim*)

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Through the episode described above, Cordelia played a very active role in establishing the classroom mathematical practice of developing addition and subtracting strategies. She relied on her understanding of place value as well as reasoning strategies with Inventory Forms in order to demonstrate, explain and justify the argumentation mentioned. Cordelia provided repeated *data* and perhaps more importantly in this case, the *warrant*s necessary to justify why her solution was mathematically valid. Through actively participating in the classroom dynamic, Cordelia helped to lead and assisted in the establishment of addition and subtraction strategies as a *taken-as-shared* mathematical practice.

Summary

Through the course of the instructional sequence, Cordelia provided numerous instances when she demonstrated a more developed understanding of whole number concepts and operations. In Table 7, the chronological summary is intended to illustrate *if* and *when* Cordelia showed a conceptual understanding of the topics at hand beyond merely performing procedures to arrive at an answer.

Table 7: Cordelia's *Demonstrated* Occurrences of Conceptual Understanding

	Pre-Interview	Student Artifacts	Small Group & Whole Class Discussions	Post-Interview	Focus Group Interview
Place Value		∗	∗	∗	∗
Counting Strategies		∗	∗	N/A	
Addition & Subtraction	∗	∗	∗	∗	∗
Multiplication & Division					

In the case of Cordelia, through the base-8 instructional unit, she displayed a qualitatively significant change in her understanding of place value concepts as well as counting strategies utilized by her in solving and explaining problems. This development represented a noted departure from her initial understanding demonstrated in the Pre-interview and at the outset of the instructional sequence.

As for addition and subtraction, Cordelia showed early on that she came in with a good understanding of these notions and her ability to explain and justify these procedures only became enhanced through the instructional sequence. While she began to consider and use some strategies demonstrated by other prospective teachers and established through the classroom discourse, Cordelia remained rather insistent on using her own methods. She illustrated a very good understanding of how to use the open number line as well as boxes, rolls, and pieces to describe and validate her thinking strategies and assisted in moving the classroom dialogue towards greater understanding of the mathematics involved with these topics.

While Cordelia was able to demonstrate procedurally the way that she would solve various multiplication problems and - on some occasions - division problems, she was not able to provide the reasoning and justification needed to illustrate conceptual understanding of these topics. However, as shown in much of her individual work, Cordelia was not able to make sense of other prospective teachers' work or processes that required a greater conceptual understanding of multiplication and division.

In order to summarize Cordelia's participation as one individual in the whole class discussions, her role and contributions towards establishing classroom mathematical practices were discussed. In Table 8 provided on the next page, Cordelia's participation has been broken

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down and categorized according to Toulmin's argumentation analysis. As witnessed earlier,

claims, data, warrants, and *backings* provide the different ways that an individual participated in

and contributed to the argumentation. Each individual's participation was identified as providing

minimal, some, moderate or *extensive* support in classroom argumentation as it led to the

establishment of the classroom mathematical practices being *taken-as-shared*. The four

classroom mathematical practices included (a) Developing number relationships using Double

10-frames (b) 2-digit thinking strategies using the open number line (c) flexibly representing

equivalent quantities and (d) developing addition and subtraction strategies. Note Cordelia's

claims, data, warrants, and *backings* have been identified per each of the classroom

mathematical practices that were established as a part of this instructional sequence.

Guidelines for Individual Participation in Argumentation

N/A	Not Applicable
◎	No or Minimal Support of Argumentation
⊙	Some Support of Argumentation
◉	Moderate Support of Argumentation
●	Extensive Support of Argumentation

Table 8: Summary of Coredelia's Participation in Establishing Classroom Mathematical Practices

	Claims	*Data*	*Warrants*	*Backings*
➢ Developing Number Relationships using Double 10-frames	◉	◉	◎	◎
➢ 2-Digit Thinking Strategies Using the Open Number Line	◉	●	◉	◎
➢ Flexibly Representing Equivalent Quantities	◉	◉	⊙	◎
➢ Developing Addition and Subtraction Strategies	◉	◉	◉	◎

Looking at the classroom mathematical practices in the order in which they became

established, notice that Cordelia mainly provided the statement (often times the answer) as well

as how she went through the steps it took to solve the problem. Therefore as illustrated in Table

8 above, she produced moderate support of argumentation mostly in the form of *claims*, and

data. Since the *warrants* involved the justification of *why* certain statements were

mathematically valid, a procedural explanation by itself did not merit proof of conceptual

understanding in the form of a *warrant*. *Backings* tended to synthesize the episode by connecting the various pieces of information provided and often times linking back other mathematical notions discussed.

While Cordelia developed and discussed the manner in which she provided her answers, she could not explicitly contribute towards the reasons why her steps worked or would not work in other scenarios. The trend observed by this researcher involved a decrease in participation in the classroom argumentation as each episode shifted from *claims* and *data* to *warrants* and *backings*. The only exception seemed to be in the second classroom mathematical practice involving 2-digit thinking strategies using the open number line. In this particular case, Cordelia provided a preponderance of evidence and participated extensively in the classroom discussions by explaining her thinking quite frequently. In addition, in multiple episodes, she suggested procedures that she thought would work and shared those with the rest of the class. Often times, these suggested procedures involved an attempt to *import* a method or algorithm from base-10 to base-8. In accordance to the classroom social and sociomathematical norms, solutions were deemed acceptable if and only if they could be explained and justified thoroughly. Therefore, many times her suggestions would open up the discussion that would prove fruitful in leading towards conceptual understanding. Almost always, other prospective teachers ended up providing the rationale behind such procedures and thus Cordelia's participation was vital towards the establishment of classroom mathematical practices albeit in a limited sense.

CHAPTER 5: THE CASE OF CLAUDIA

Claudia had previously taken two mathematics courses at the university level: a college algebra course and a trigonometry course. Both of these courses had been taken between 2-3 semesters prior to the beginning of this research project. She quickly expressed that math had always been her favorite subject – primarily due to the fact that her teachers could explain why something was right or wrong:

> *Every time I turned in my five paragraph essay and the teacher deducted on grammar and sentence structure they could never tell me why it was wrong they just told me it was. Sometimes I wondered if they really even knew why themselves. My math teachers were always different. If I asked them why I got a problem wrong they could always prove it to me.*
> *Before changing my major to elementary education I almost changed it to math education. I am very excited about this course. I hope in the future I can allow my students to also have a positive experience in mathematics and I will never tell them they are not capable of doing something.*
>
> *Claudia, January 2007*

Claudia scored an 18 on the CKT-M Pre-test instrument which placed her in the Upper Quartile and in the "High-Content" category of mathematical content knowledge at the outset of this research. This score matched the highest score attained during the Pre-test by any student during this research experiment.

<u>Individual Development</u>

In order to illustrate the development of this individual through the instructional sequence on whole number concepts and operations, this researcher once again has decided to examine

Claudia's individual progression at several stages. First, this analysis focused on what was known about this individual prior to the beginning of the instructional sequence. These data primarily stemmed from individual interviews conducted and videotaped immediately before the first day of instruction in base-8. Next, Claudia's *individual* artifacts – outside of her contributions to group and classroom discussions – were analyzed. These artifacts included collected individual papers during class, homework assignments, as well as tests.

The third level of analysis occurred based on what Claudia shared during the post-interview after the conclusion of the instructional unit as well as her comments as a part of a focus group conducted one month after the post-interview. The fourth level of analysis involved her participation in the social situation of the classroom. Similar to the previous case involving Cordelia, videotaped recordings of each day of the instructional sequence were transcribed and analyzed for Claudia's contribution to general classroom discussions and specifically the manner in which she participated in the *taken-as-shared* practices.

Prior to Instructional Sequence

During the pre-interview conducted prior to the beginning of the instructional unit on whole number concepts and operations situated in base-8, the researcher (who also served as the interviewer) asked Claudia some questions related to place value, whole number operations and her role in the classroom. These pre-interview questions were all in the traditional base-10 system since the participants had not been introduced yet to the base-8 system as a part of the instructional sequence. When given the task, "Write the numbers 1 through 31 in any order or pattern that is meaningful to you." Note the manner in which she started and ended each sequence of numbers.

Claudia:	1 2 3 4 5 6 7 8 9 10
	11 12 13 14 15 16 17 18 19 20
	21 22 23 24 25 26 27 28 29 30
	31
Interviewer:	Would you describe to me how you wrote these numbers?
Claudia:	I started at 1, and then after 10, I went to the next line.
Interviewer:	Why would you do that after 10?
Claudia:	I did it to keep them in order, because they are both …they are pretty much going up by 10. One plus 10 is 11, and so on.

In order to pursue Claudia's way of thinking, the interviewer asked her to continue on to 101 and then 1003. Note the way that she used the dotted line to indicate the numbers that she did not write down, but would be included in the sequence.

Interviewer:	Can you write down 1 through 101.
Claudia:	1………………10
	11…….............
	21…………….
	31…………….
	…
	…
	101
Interviewer:	How come you decided to write them in this fashion?
Claudia:	Distance 10 is an easy one to follow.
Interviewer:	Okay. Would you write down the numbers through 1003?
Claudia:	1…………………
	…
	21
	…
	…
	1001 1002 1003
Claudia:	(*Without being prompted*) I could have written them differently!
	1………………100
	101……………200
	201……………300
	301……………400
	... .
	… .
	1,001 1,100
Interviewer:	What did you choose to do this time?
Claudia:	**Zeros are like place holders…Patterns would start over**.

Already, Claudia has started to illustrate understanding of the notion of place value and mention language - "place holders – that are related to the concept of place value. Furthermore, she showed that even prior to the instructional sequence she had a sense of notions such as distance, and patterns in numbers in relation to the concept of place value.

The next question on the interview involved the use of Base Ten Blocks. A picture of Base Ten Blocks was previously provided in Figure 12. Claudia indicated that she had used these manipulatives in the past. The interviewer wanted to explore her understanding of place value and whole number concepts through a series of questions involving the use of Base Ten Blocks. As the interviewer prepared to start the questions, he noticed that Claudia was intentionally putting some of the blocks together – namely the tens all standing up. As a result, the interviewer wished to explore her reasons for grouping blocks.

| Interviewer: | How would you show 41? |
| Claudia: | 4 tens and then plus 1 (squeezing the 4 tens shown below together) |

In Figure 30, notice the manner in which she grouped and held the tens together.
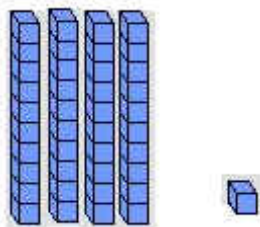
Figure 30: Claudia's Use of Base-Ten Blocks to Illustrate 41

Interviewer:	You put the 4 tens together.
Claudia:	Yeah, because they go together
Interviewer:	Tell me what you mean – mathematically.
Claudia:	I have 10 here (holding up one ten block), and 10 here,…
	I guess it goes back to what I was doing before (pointing to the way she grouped the numbers that she wrote 1 through 31). **I am just thinking in tens**.

Here, Claudia illustrated signs of distinguishing between numbers according to their place value.

Furthermore, she displayed the first signs of using ten as an iterable unit. The interviewer wanted

to further examine Claudia's ability to use Base Ten Blocks as an insight of how she perceived

whole numbers including ways to compose and decompose numbers.

Consistent with the last prospective teacher interviewed, Claudia was given Base Ten Blocks

(3 Flats, 5 Longs, and 2 Unit Cubes.)

Interviewer: (Using Base Ten Blocks) How would you show 254?
Claudia: (Thinks as she is moving some blocks around) Could you make an equation?
Interviewer: Tell me what you mean.
Claudia: You could make this 300 (Three 100-blocks) and we want 54 so we would put a minus sign and subtract from 300. I would have…I could take away 50 and then add the 2. (See Figure 31 below)

Figure 31: Claudia's First Solution in Manipulating Base-Ten Blocks

Claudia: Oh we need 254.
Interviewer: Yes. Use the Base Ten Blocks in any manner you think would help you.
Claudia: Okay, I see. I could take two pieces of this (puts thumb over two pieces in a 10-block) and put it here (next to the two single unit blocks in figure above).

Here, Claudia demonstrated algebraic thinking in initially using an equation to assist her

in thinking through this problem. Next, she showed that she could decompose a 10-block into

single unit blocks in order to arrive at the number she needs. Having observed Claudia solve this

problem, the researcher – who served as the interviewer as well – asked her to show another way of solving the same problem.

Claudia: Another way would be … (See Figure 32).
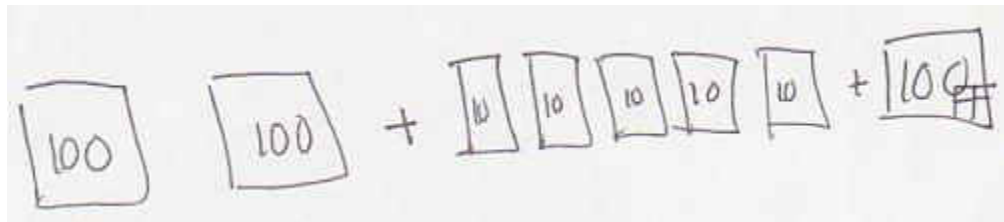
Figure 32: Claudia's Alternative Solution A to Represent 254

Claudia: Or you would need the two hundred fifty and 4 tenths of this single piece (See Figure 33).
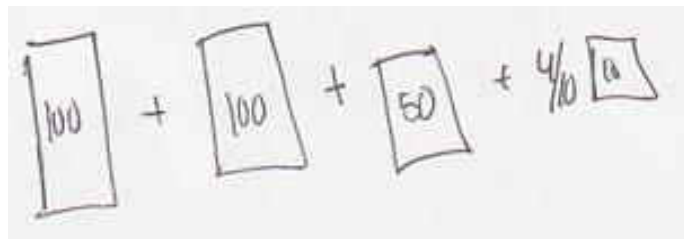
Figure 33: Claudia's Alternative Solution B to Represent 254

In trying to clarify what Claudia had just mentioned, the interviewed followed up on the idea of "tenths".

Interviewer: What do you mean by a tenth? Can you use the example of the number 21.
Claudia: **Twenty one would be two ten's and a tenth of a ten which is one**.

Claudia displayed her ability in using Base Ten Blocks to flexibly represent numbers in a variety of equivalent ways. This point was highlighted by her explanation that "a tenth of a ten which is one". As indicated by the previously mentioned research of Steffe (1988), the concurrent understanding of ten as ten ones as well as one iterable unit of ten represented a key aspect of her understanding of place value and whole number concepts. Through her choices, she

demonstrated that she could decompose bigger units as in the case of blocks of 10 and 100 in order to accomplish the task at hand. This researcher also became intrigued by Claudia's depiction of the 4 single unit blocks as a part of the 100 block as represented in Figure 30. It seemed that numerically, conceptually and visually Claudia maintained a solid understanding of the relationship between one's, ten's and hundred's.

The last question during the pre-interview – as with all research participants - involved examining a fictitious student's work.

"*Some fifth-grade teachers noticed that several of their students were making the same mistake in multiplying large numbers. In trying to calculate*

$$
\begin{array}{r}
1\ \ 2\ \ 3 \\
\times\ \ 6\ \ 4\ \ 5 \\
\hline
\end{array}
$$

The students seemed to be multiplying incorrectly. They were doing this:

$$
\begin{array}{r}
1\ \ 2\ \ 3 \\
\times\ \ 6\ \ 4\ \ 5 \\
\hline
6\ \ 1\ \ 5 \\
4\ \ 9\ \ 2 \\
7\ \ 3\ \ 8 \\
\hline
1\ \ 8\ \ 4\ \ 5
\end{array}
$$

What is the students' misconception? How would you approach this misconception with students? (Ma, 1999)

Claudia began to write out the problem for herself and it must be noted that immediately upon writing the multiplication problem of 123 × 645, she first wrote in the zeros as it may be seen in Figure 34. Note the particular language that she used with the emphasis on place value understanding as well as reasoning strategies involving place values used to estimate the solution to this multiplication problem.

Figure 34: Claudia's Solution to the Multiplication Problem 123 × 645

Interviewer: From what you can gather, what did the student do?

Claudia: They did not put the zero. They forgot to hold the value. They are not thinking of the 4 (pointing to the 4 in 645) as a 40. They are just thinking of it as a 4.

Interviewer: Okay. What should the student have done?

Claudia: Thinking of the 6 as a 600. This is a 40 and not a 4.
When I multiply, I add these place holders. I put a line through them so I don't confuse them with something else. **By putting these place holders or zeros, you are thinking of the 6 as a 600. By putting this one zero, you are thinking of this as a 40 and not just 4.**

Through listening to Claudia's explanation, the researcher came to a few realizations regarding her level of understanding multiplication. At the outset, it was clear that she possessed a solid understanding of the procedures involved in solving a multiplication problem such as 123 × 645. Secondly, she was able to justify her own actions and deciphered where the student had gone wrong by paying particular attention to the misconception of treating the 40 as a 4 and the 600 as a 6. The ensuing comments related to the zeros placed to avoid confusion further reinforce her conceptual and procedural understanding of place-value notions. Overall, through this examination of another student's work, Claudia illustrated a deep understanding of whole number concepts – specifically place value – and multiplication.

During the Instructional Sequence

<u>Place Value and Counting Strategies</u>

Analysis of Claudia's individual activity - specifically homework assignments, handouts and items on a test – represented the next aspect of exploring her development during the course of the instructional sequence. From this point on, problems listed and language used must be treated strictly as having occurred in base-8 unless specifically noted otherwise. After several class sessions dedicated to counting and skip counting, the prospective teachers were introduced to the open number line in order to record their thinking and discuss various strategies used to solve problems. After two weeks, the first homework examined the student's abilities to individually solve problems related to counting and reasoning with addition and subtraction.

In the following homework problem, Claudia utilized an open number line to solve the problem. Consider the following scenario:

"For Valentine's Day, Victory bought 57 heart-shaped chocolates. After purchasing some more, he had a total of 243 heart-shaped chocolates. How many more chocolates did Victor buy?"

In Figure 35 – on the next page - note the way Claudia *counted down* 57 chocolates by 1's and 10's from the initial value of 243 chocolates and recorded her thinking using the pedagogical content tool of the open number line.
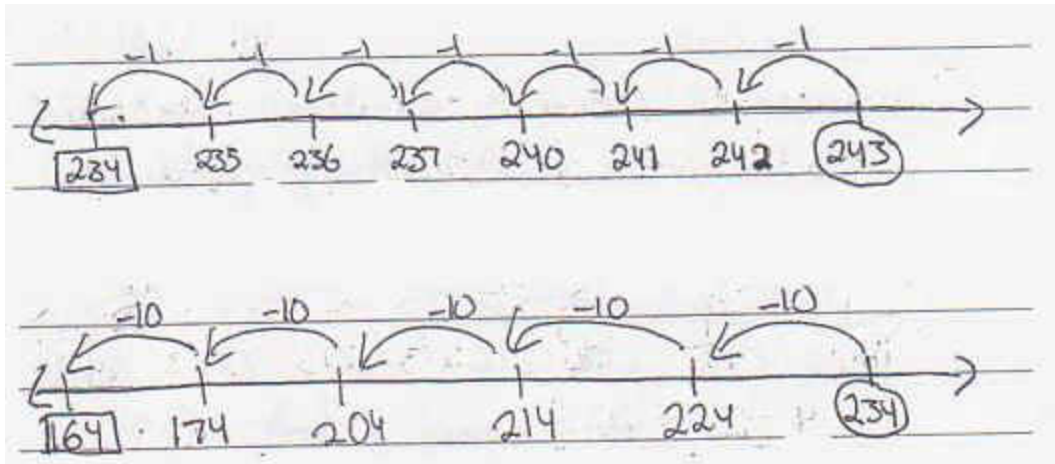
Figure 35: Claudia's Use of the Open Number Line to Solve 243 – 57 = ?

In her written work, she described that solving this problem was represented by the equation "243 – 57 = ?". Next, note that she decomposed the 57 chocolates to be taken away according to place value by first "taking away 7 chocolates from 243 by counting back by 1". After she took away all the single chocolates, she proceeded to "take away the remaining 50 by taking away 10 at a time." As a result, she concluded that by taking away 57 chocolates from the total of 243; her answer would be that "Victor bought 164 more chocolates."

The instructions for this problem involved showing two *different* ways to solve this particular question. In this solution, note the manner in which she started off with the 57 chocolates and *counted up* first by 1's and then by 100's and 10's to arrive at the total of 243 chocolates.
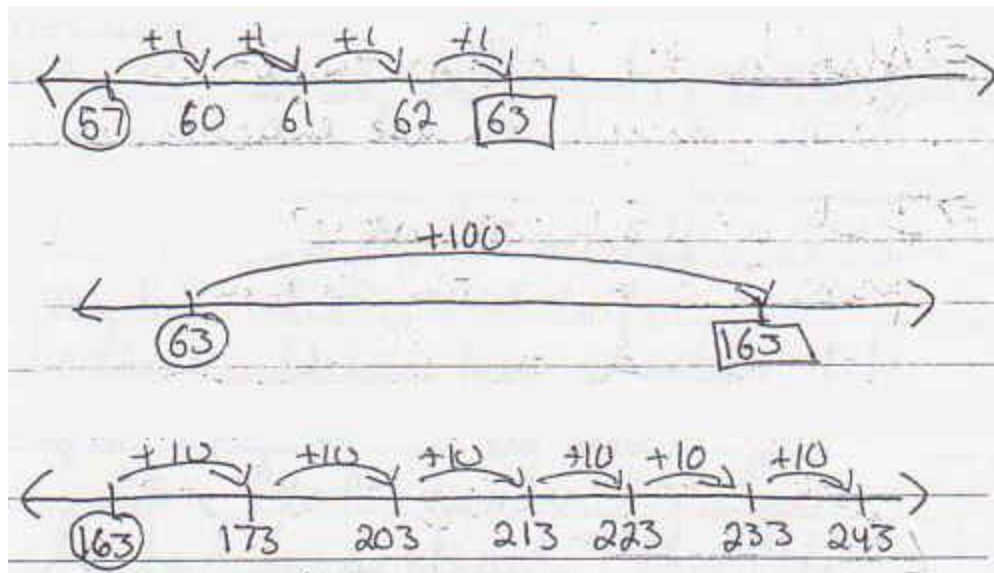
Figure 36: Claudia's Use of the Open Number Line to Solve 57 + ? = 243

This time, Claudia decided to add "1 until we get a number with a 3 as the last digit."

Next, she added the largest number by place value – 100 candies – that she could without going

over the desired amount. She stated: "We can't continue to add 100 because we will go farther

than our goal so we will continue to count up by 10 until we reach our goal." Finally, having

counted up to 243, Claudia concluded that she took 164 "jumps" and therefore using this strategy

again she maintained that "Victor bought 164 more chocolates." Through *counting up* and *take*

away strategies illustrated by the open number line, Claudia illustrated great understanding in

using place value concepts in a flexible fashion to assist her in solving this problem.

<u>Addition and Subtraction</u>

Over the next three weeks, the instructional tasks involved using the context of the candy

shop to further highlight the significance of place value and exploring invented strategies for

addition and subtraction. Place value notions as well as the composition and decomposition of

numbers according to place value were explored through the use of inventory forms. These

forms allowed for the separation of units into boxes, rolls and pieces as previously discussed. The researcher continued to explore the development of Claudia's understanding as she progressed through this instructional sequence entirely taught in base-8.

In the following problem from the second homework, the prospective teachers were presented with the following scenario:

"Mrs. Wright found the following two inventory forms in the Candy Shop.

Form A				Form B		
Boxes	Rolls	Pieces		Boxes	Rolls	Pieces
2	14	43		1	17	56

Which form represents more candy? How many more pieces of candy does that form have?"

Table 9 below illustrated Claudia's solution as she set out to "find the least amount of boxes, rolls, and pieces we need in order to package all pieces of candy." Note the manner in which she used her knowledge that 10 pieces equaled 1 roll, and 10 rolls equaled 1 box to repackage the candy in accordance to place value.

Table 9: Claudia's Solution Using "Least Amount" of Boxes, Rolls, and Pieces

	Form A				Form B		
	Boxes	Rolls	Pieces		Boxes	Rolls	Pieces
Step 1	2	14+4	3		1	17+5	6
Step 2	2	20	3		1	24	6
Step 3	2+2	0	3		1+2	4	6
Step 4	4	0	3		3	4	6

In analyzing her step-by-step approach, she systematically converted objects from pieces to rolls and eventually from rolls to boxes in a right-to-left fashion. Her intended goal of having the "least amount" of boxes, rolls and pieces culminated in getting a single digit in each place value. Claudia wrote "Now that Form A and B are in a standard unit, we can now compare them" and concluded that Form A with 403 had more pieces of candy than Form B representing 346 pieces of candy.

The next aspect of this problem involved finding how much more candy one form contained in comparison to the other. In Table 10, note the manner in which she proceeded to find the difference between these two inventory forms and the strategies necessary to perform this operation.

Table 10: Claudia's Addition and Subtraction Strategies with Boxes, Rolls, and Pieces

Procedure Performed	Claudia's Explanation
Form A / Form B — Boxes: 4 / 3, Rolls: 0 / 4, Pieces: 3 / 6	"Since there are not enough pieces and rolls in Form A to take away the pieces and rolls in Form B, we will **repackage** the pieces of candy in Form A."
Form A — Boxes: 4, 3, 3; Rolls: 0, 10, 7; Pieces: 3, 3, 13 (with 4, 10, and 3 crossed out)	"We knew we had to **unpackage** that many boxes and rolls in order to have more rolls and pieces and boxes than Form B."
Form A / Form B — Boxes: 3 / 3, Rolls: 7 / 4, Pieces: 13 / 6	"Now, we can subtract. There are <u>35 more pieces</u> of candy in Form A than Form B."

Claudia demonstrated a very thorough reasoning strategy involving boxes, rolls and pieces to move objects across "place-value". In the homework problem above, she illustrated that she could repackage and unpackage – in other words, compose and decompose – flexibly in order to find the difference between the two inventory forms provided. Her strategy required not only knowledge of place value concepts, but also an efficient strategy that would allow her to effectively solve this problem. Through this example, Claudia displayed a developed and conceptual understanding of place value notions to explain and justify her solution in a coherent and efficient fashion.

<div align="center">Multiplication and Division</div>

As indicated in the last chapter, the last three days of instruction primarily entailed discussion of topics related to multiplication and division of whole numbers. One distinction previously discussed involved the ability for prospective teachers to understand the meaning of each of the two numbers that were being multiplied. Prospective teachers were expected to develop the understanding of the number in multiplication which stood for the *groups of objects* and the one that represented the *number of objects* in each group. This notion was initially introduced through a problem that asked the prospective teachers to "Write a story problem for 7 × 16. Explain and justify a solution to your problem."

In the following illustration (Figure 37 on the next page), note the language and also the model that Claudia used to represent this multiplication problem. Also observe the units she used in the phrasing of her question to model this problem.
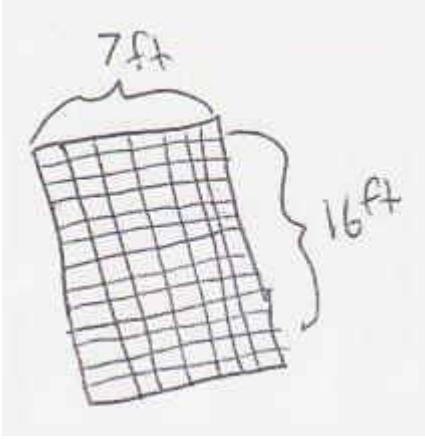
Claudia's Illustration of 7 × 16	Claudia's Story Problem
	"Tony wanted to put a fence around his yard. The width of Tony's yard was 7 ft and the length was 16 ft. Before putting in the fence, he wanted to put new grass. How much feet of grass would he need to fill his yard?"

Figure 37: Claudia's Model for a Story Problem Representing 7 × 16

It is noteworthy to report that the vast majority of prospective teachers represented this problem using a "7 groups of 16" approach to start the topic of multiplication. Claudia, however; chose a story problem that involved an area model; and customary to her development in the instructional sequence reflected a different initial understanding of multiplication than the majority of her fellow prospective teachers. Notice that in the context of multiplication, in her model she displayed 7 columns and 16 rows to indicate the product of 7 times 16. Claudia provided more insight into her thinking as she illustrated 7 as 7 one's and the 16 as 16 one's – due to a perceived oversight on her part as she only drew 15 rows. While Claudia illustrated the area model, the language – in particular, the units – she used in her question do not reflect that of area. In other words, her question should have read "How many *square* feet of grass would he need to fill his yard?"

This approach led the researcher to believe that perhaps Claudia was on the verge of decomposing the numbers 7 and 16 in order to multiply them. This supposition was explored further in the next example to follow.

As the last question on the second homework, Claudia was asked to solve: "Mrs. Wright wants to fill 5 bags so that each will contain 16 candies. How many candies should she use?" Note in Figure 38, Claudia's decomposition of 16 according to place value and her strategy in solving 5 × 16.
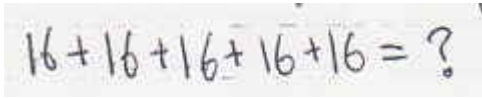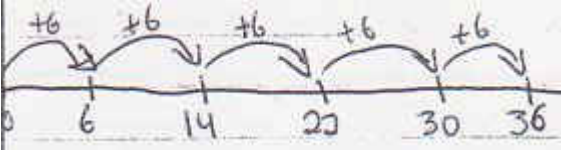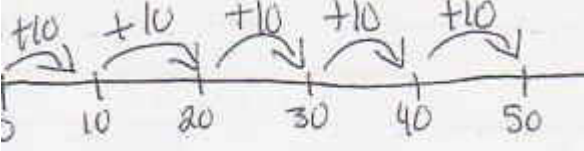
Claudia's Illustration of 5 × 16	Claudia's Explanation
16 + 16 + 16 + 16 + 16 = ?	"Since there are 5 bags and each contains 16 candies, we are looking for the **sum of 16 plus itself 5 times**."
	"Counting up by 6, 5 times we get the answer to be **36**."
	"Continue by counting up by 10, 5 times we get the answer to be **50.**"
36 + 56 = 106	"We use **106** candies**"**

Figure 38: Claudia's Solution to 5 × 16

Claudia revealed an efficient strategy in solving 5 × 16 by decomposing 16 into 6 and 10, followed by taking 5 six's and adding it to 5 one-ee-zero's. She added a note – including her illustration shown in Figure 39 on the next page - stating "we were able to break the problem apart in this matter because of the *distributive property*."

$$5 \cdot 16 = 5 \, (6 + 10) = 5 \cdot 6 + 5 \cdot 10 = 106$$

Figure 39: Claudia's Initial Illustration of the Distributive Property

In a noteworthy illustration of a student who connected her knowledge of place value, whole number concepts and operations; Claudia managed to develop the underlying notions of the distributive property well before these ideas had been discussed during classroom conversation. It is worth noting that it was not clear whether she developed this notion during the instructional sequence or perhaps she came into the class with the distributive property firmly in place. Given her collegiate mathematics background, it would not be unlikely that she had seen the distributive property within the last two years.

The next example of her development of whole number operations involved the array model - another commonly used model in teaching and learning multiplication. As Claudia developed her understanding of multiplication, one of the models that she had not experienced prior to this instructional sequence involved the array model. In her notes, she reflected that she was not familiar with this model of multiplication. As a result, the array model provided a chance to witness Claudia's development in the context of an unfamiliar notion. During class discussion, prospective teachers were asked to individually work out some problems that used egg cartons within the candy shop scenario. Egg cartons provided the chance to examine various dimensions involving the multiplication of two numbers and permitted prospective teachers to use pictures and invented approaches and algorithms to support computational strategies. Claudia's exploration with egg cartons is highlighted in the example provided in Figure 40. Note the

manner in which she combines numbers initially and then the doubling strategy used to

efficiently arrive at her solution.
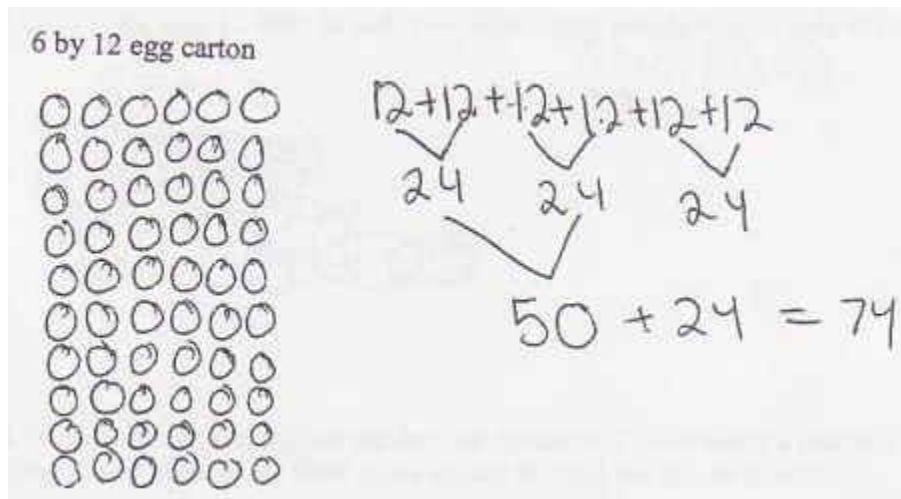
Figure 40: Claudia's Solution Using a 6 by 12 Egg Carton

Typically in mathematics – particularly secondary level mathematics – 6 × 12 would be

viewed as 6 rows and 12 columns akin to the set up of matrices where $a \times b$ would indicate a

matrix with a rows and b columns. In elementary schools, this is commonly referred to as 6 rows

with 12 objects in each row. In this example, Claudia actually demonstrated 12 rows with 6

objects in each row. Her exploration of the array model and the picture of the "eggs" in the

figure above permitted her to estimate an answer as she indicated by "around one-hundree" on

her paper. To elaborate on her solution, the researcher needed to carefully consider her approach.

Claudia used an approach similar to "6 groups of 12" as indicated by her writing 12 plus itself 6

times. Next, she quickly used a doubling strategy since she knew 12 + 12 = 24. This process was

repeated three times, followed by combining 24 + 24 = 50. Finally, she combined 50 + 24 = 74

to arrive at her solution. While Claudia's approach was certainly viable, it seemed to lack some

efficiency – particularly involving cases dealing with larger numbers. Analysis of Claudia's

work did suggest, however; that even in relatively new settings she was able to use the knowledge that she had gained in her development of multiplication through this instructional unit situated entirely in base-8. This observation reiterated the findings of Steffe et al. (1988) regarding the abilities of students with conceptual understanding to construct new meanings or deal effectively in new settings based on the knowledge gained through previous experience.

To this point, the analysis of Claudia's development has focused on her advancing knowledge base related to place value, counting strategies, as well as addition, subtraction and multiplication strategies. Due to time constraints, prospective teachers spent approximately 1 ½ class sessions addressing the concept of whole number division and related strategies. While Claudia illustrated significant understanding of division, her division strategies will be discussed later on in this report and in relation to other participants. The majority of time and subsequent data on division occurred within the large class discussion and the test that followed. Claudia's division strategies and contribution took place within the social environment of the large class discussion and as such shall be discussed in the section involving *taken-as-shared* classroom mathematical practices.

Following the Instructional Sequence

Individual post-interviews were conducted upon the completion of the instructional sequence by the researcher himself. In addition, a focus group interview was conducted by a different member of the research team after the individual post-interviews. Claudia – along with other research participant – shared thoughts and reflections on their experiences with the instructional sequence on whole number concepts and operations.

During the focus group, when Claudia was asked what changes had occurred through her experiences with the instructional sequence, she responded: "We were given an opportunity to understand the math that we teach to kids." She also reflected on her perceived role in the classroom versus the role of other prospective teachers when she explained: "I have a responsibility to listen to what other people are saying and even though sometimes it is not a new way of doing it, seeing how other people think about it can help to understand it better." Also during the focus group, Claudia along with Cordelia discussed their perceptions of the authority in the classroom and how they felt about parts of the instructional sequence. Due to the comparative aspects of this conversation, that discussion has been placed in the Cross-Case chapter that will follow this one.

In examining Claudia's development through the instructional sequence, the researcher asked her to describe her thinking during the post-interview by solving a few problems dealing with whole number concepts and operations. As discussed earlier in this research effort, prospective teachers were placed in *Eight-world* and the instructional sequence was taught entirely in base-8 in order to allow for exploration of whole number concepts and operations. Consistent with the research question for this study, the researcher wanted to see the way Claudia's understanding developed through the instructional sequence taught in base-8 and the effect that it may have had on her overall understanding of whole number concepts and operations.

The following examples represented some of her responses in the post-interview. The first question involved the notion of addition (in base-10) when Claudia was asked to add 18 (eighteen) plus 45 (forty-five). In Figure 41 on the next page, note not only *that* she

demonstrated an understanding of whole number concepts and operations, but also the *various ways* she was able to explain her thinking.
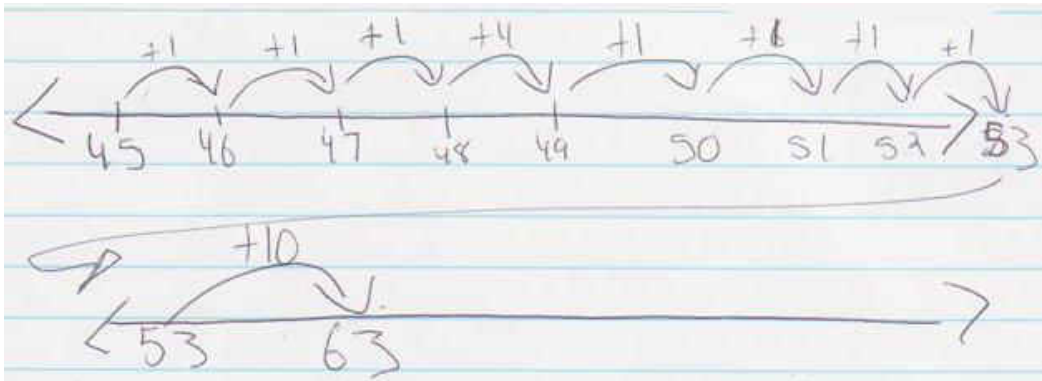
Figure 41: Claudia's Illustration of 18 + 45 (Open Number Line)

In solving the addition problem 18 + 45, Claudia illustrated a few aspects that demonstrated her development in efficiency and the ability to explain her understanding of whole number addition. Note that even though the problem was presented as 18 plus 45, she actually added 18 to the bigger number 45, "Since there wouldn't be as many numbers to count up". Also, consistent with what she demonstrated through the instructional sequence, she added 18 by decomposing the number according to place value. Specifically, she broke 18 into 8 one's and 1 group of ten. Next, she added each of the one's as indicated by the open number line in Figure 41, followed by the addition of ten to arrive at her solution of 63. Here, she demonstrated the use of a pedagogical content tool explored through the instructional sequence in base-8 to assist her in solving and explaining her thinking in base-10.

After the completion of this method, Claudia continued on to further illustrate her development of whole number concepts and operations when she demonstrated other ways of solving the same problem. In Figure 42, note the way she uses both a traditional algorithm as well as column addition to describe her solution to 18 + 45.
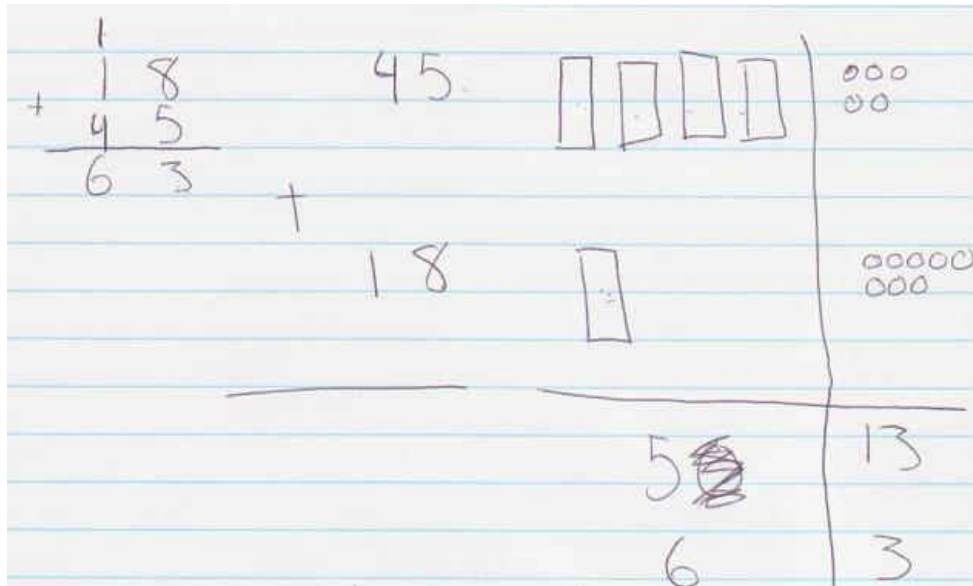
Figure 42: Claudia's Illustration of 18 + 45 (Column Addition)

In this example, Claudia initially displayed - Figure 42, upper left - a method of solving this problem by aligning the columns according to place value and then potentially used the conceptual approach of regrouping the ones into one ten in her inscription. The researcher, here stated potentially since based on her inscription alone it would be presumptive to conclude definitively without further evidence. Claudia did not comment on this part of the problem during the post-interview. However, she continued on and pictorially represented the quantities 45 and 18 - Figure 42, right hand side – in terms of rolls and pieces similar to her experiences in base-8. Observe that she utilized a strategy explored through the instructional sequence in base-8 to describe her thinking in base-10. Next, she combined the numbers according to place value as she added the one's place (5 + 8 = 13) and combined the numbers in the ten's place (4 + 1 = 5). Finally, she regrouped the 13 ones as 3 ones and a ten, with the ten being added to the other five to reach her answer of 63. The researcher was keenly interested in the way that Claudia initially wrote not 5 but 50. Claudia exclaimed: "I realized the 5 is not a 5 but 5 tens which is 50, but I am

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just writing down how many tens I have so it should be 5." She displayed a thorough understanding of the procedure - which illustrated her mastery of whole number addition. Just as importantly – *and critical for any prospective teacher* – Claudia was able to explain and justify her thinking verbally, symbolically and pictorially. The researcher will elaborate on this issue further in the conclusion section of this study.

Another example of Claudia's development following the instructional sequence was addressed in the context of a multiplication problem presented during the post-interview. She was presented with the following problem in base-10, "A student has 23 books in his library where each book has 14 pages. How many total pages are there in all the books?" Note Claudia's solution - in Figure 43 - with particular attention to her use of the area model as well as the decomposition of the number 23 and 14 according to place value.
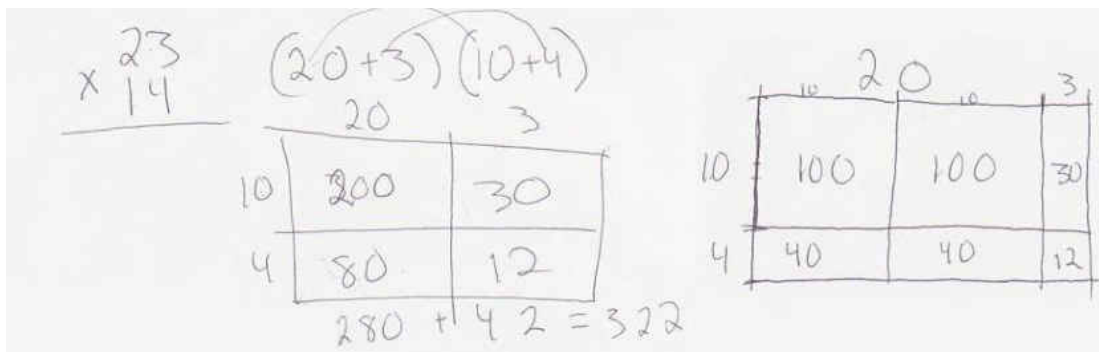
Figure 43: Claudia's Solution to 23 × 14 (Area Model)

Consistent with what she had demonstrated during the instructional sequence, Claudia illustrated her preference towards the area model. She began by writing 23 times 14 – Upper left in Figure 43 – but quickly decomposed the numbers 23 into (20 + 3) and 14 into (10 + 4). By "breaking the numbers down", she displayed her understanding of the area model and an

efficient way of illustrating these partial products both pictorially and numerically. Note that upon getting each of the four partial products - $(10 \times 20 = 200)$, $(10 \times 3 = 30)$, $(4 \times 20 = 80)$, $(4 \times 3 = 12)$ – Claudia added 280 plus 42 to finalize her solution of 322. In another instance of her development following the instructional sequence, Claudia mentioned that if she were teaching she might illustrate the problem using the picture on the right of Figure 43. The further decomposition of 20 into two groups of 10 illustrated her understanding of ten as an iterable unit as well as emphasizing the way children might use their knowledge of multiplication facts. Claudia's growing understanding of whole number concepts and operations following the instructional unit seemed to have enhanced her ability to address the needs of her future students. This aspect will be addressed more in the cross-case analysis and conclusion.

Claudia's Participation in Taken-as-Shared Practices

As described in the previous chapter, this study placed each individual prospective teacher in the social setting of a classroom. As such, both the social aspects of learning as well as the concurrent individual component were considered and discussed in conjunction. In the first part of this chapter, Claudia's individual development was analyzed by looking at artifacts and interviews which illustrated her progression through the instructional sequence. At this point, the focus will shift to Claudia's involvement in the classroom as a participant in discussions. Her contributions towards the classroom mathematical practices will be discussed in detail with particular attention to the episodes that influenced the *taken-as-shared* practices.

Roy (2008) identified the particular classroom mathematical practices and specified the *taken-as-shared* practices for this instructional sequence to include

➢ Developing small number relationships using Double 10-Frames,
➢ Developing two-digit thinking strategies using the open number line,
➢ Flexibly representing equivalent quantities using pictures or Inventory Forms,
➢ Developing addition and subtraction strategies using pictures or an Inventory Form.

As a part of this case study analysis, the researcher will demonstrate the manner in which Claudia provided *claims, data, warrants,* and *backings* to facilitate and at times lead whole classroom discussion. The analysis using Toulmin's argumentation (1969) allowed for collaboration between the social and the individual aspects to describe Claudia's participation in classroom argumentation.

Developing Number Relationships using Double 10-frames

Starting on Day 1 of the instructional sequence, Claudia asserted herself as one of the few individuals who spoke out. In the following episode, two prospective teachers - Olympia and Kassie - had asked the instructor to further discuss patterns in base-8. As the whole class discussion continued, note Claudia's role in providing an answer which identified an existing pattern.

Olympia:	I don't know if it is just me, but in counting and skip counting in base-8, I always try to look for patterns. Is that correct or not? Should I look for patterns like in base-10?
Instructor:	First, what does she mean look for patterns? Can anyone help us?
Kassie:	I would say we are already finding patterns. The way we look for patterns in base-10 counting by 3's. Do you know what I am saying, 3, 6, 9, but in base-8 it is not going to be the same as it is in base-10. I started to do that and I was off because I was going in base-10 multiples.
Instructor:	You were trying to be binumeral…What number could we count by so there would be (a pattern)?
Claudia:	Four *(Claim)*
Instructor:	What?

Claudia:	Would it be four?
Instructor:	Why? What do you guys think? Response? What did you suggest?
Claudia:	**Count by 4's**. *(Claim)*
Kassie:	Yeah, that will work.
Kristy:	Well, because each time it will be four and then one-ee-zero and then one-ee-five and then two-ee-zero. *(Data)*
Instructor:	Wait, one-ee-five?
Kristy:	One-ee-four, and then two-ee-zero. *(Claim)*
Instructor:	Would it be? Let's try it. But let's start with two-ee-three. We are going to count by four's.

In this episode, Claudia served as a facilitator in suggesting a key aspect of the base-8 number system. Initially, Claudia tended to make statements (*claims*) and then other prospective teachers along with the instructor would move the discussion forward. Her role would expand in the near future as she would gradually move from making strictly *claims* to also providing *data* and *warrants* through the next class meeting and beyond.

On Day 2 of the instructional sequence, Kassie – the individual involved in the previous episode – inquired about a pattern that she thought to be perhaps coincidental. During the following whole class discussion, observe the role of Claudia as she explained the "why" behind such a coincidence.

Kassie:	I am just saying, is it a coincidence?
Instructor:	Is what a coincidence?
Kassie:	Well, I am going back to base-10. Twelve and thirty-one is forty-three, right? Then you're saying that one-ee-two plus three-ee-one …You see what I am saying? They are separate worlds but they are not, are they the same? *(Claim)*

At this point, the class began to explore counting using fingers. Soon thereafter, Claudia revisited Kassie's statement.

Claudia:	I was just going back to the whole coincidence statement. I think the reason why this worked is because all the digits are less than eight. *(Warrant)*
Instructor:	Were less than?

Claudia:	Eight.
Instructor:	Were less than? You are saying something I don't understand.
Claudia:	One-ee-zero. (*Warrant* continued)
Instructor:	Okay.
Claudia:	Because it is one, two, three and one. So if for example, if it had been one-ee-two plus four-ee-one. No sorry, **if it was one-ee-two plus three-ee-seven then it would not have worked**. I don't know… (*Backing*)
Instructor:	Shall we do that problem? Okay.

Even though Claudia was not completely convinced of her statement, she was able to

provide the reason why Kassie's example worked. Furthermore, she led the classroom discussion

directed by that example in order to further explore the point involving the so-called coincidence

between base-8 and base-10. Notice that the specific example she provided illustrated her

realization that in order for the two bases to be different, the result had to cross the one-ee-zero

or place value. Claudia's backing was no small declaration as the majority of the class at that

point was not at the conceptual level of understanding place value and the impact it would have

on counting and combining numbers. This example demonstrated the way that the understanding

of an individual – Claudia in this case – can be *ahead* of the collective, social understanding. In

addition, her comments exhibited the significance of having individuals to continuously push the

conversation towards the exploration of critical mathematical concepts.

Two-Digit Thinking Strategies Using the Open Number Line

Claudia continued to progress throughout the instructional sequence and regularly

contributed to classroom discussions. She raised many valuable points and elicited clarifications

by raising issues including the role of the teacher, defining explanation and justification during

classroom argumentation, as well as maintaining what might be appropriate to show elementary

school age children. While many of these instances could have been documented in this section, the researcher made a conscious choice to include such comments in the synthesis and cross-analysis in order to compare and contrast both prospective teachers in this study. In this section, the study will adhere closely to the discussions which led to the establishment of the classroom mathematical practices.

During Day 5 of the instructional sequence, the instructor began class by discussing a problem from the recently returned homework assignment which caused difficulty for many prospective teachers. This episode began by looking at the instructor's intentions for asking the specific problem that follows and the manner in which Claudia accepted the leading responsibility and carried the conversation forward. Note the specific way that Claudia provided the *claim, data, warrant* and *backing* all within one episode of classroom argumentation involving making sense of a fictitious student's work. The original problem has been stated first followed by Claudia's explanation.

> "A student was given the following problem to solve in class: How many more stickers do you have to add to 47 stickers to get a total of 135?
>
> To make the above problem a little easier for them to solve using the number line, they jumped 3 to go from 135 to 140. Then they jumped 100 to get from 140 to 40. Finally, they jumped 7 to go from 40 to 47. Since the student did that, they came up with the following solution:
>
> $$3 + 100 + 7 = 112 \qquad \text{Answer: 112 stickers}$$
>
> Is the student correct? If so explain why? If not, explain what the student did incorrectly?"
>
> Claudia: So the student will be incorrect. *(Claim)*
> I put that the student is incorrect because he just added all the numbers together that he used instead of maybe subtracting some of the numbers he should have from it. *(Data)*

	The first thing he did was go up from one-three-ee-five, he jumped up to one-forty, or one-fouree-zero, sorry…
Instructor:	How do we say that number?
Claudia:	One-hundree-fouree-zero.

Okay. This is okay to do if at the end he subtracted the three that he had added, but he didn't do that. So in the second step he went down to…
to four-ee-zero, (*Claim*)
and that would have been okay cause he subtracted one-hundree. (*Data*)

But the next thing he did was incorrect, because he went too far (*Warrant*)

because we were trying to get to four-ee-seven, and if he would have gone from four-ee-zero and then just subtracted seven from the one-hundree that he had gone to, it would have been okay.
But instead he just added another seven because he added all the numbers he used and he actually would have gone down to three-ee-one instead of going to four-ee-seven which was his goal. (*Backing*)

So that's what he did wrong there. I am looking at puzzled faces.

Claudia used the open number line – Figure 43 – and illustrated *the student's solution*.
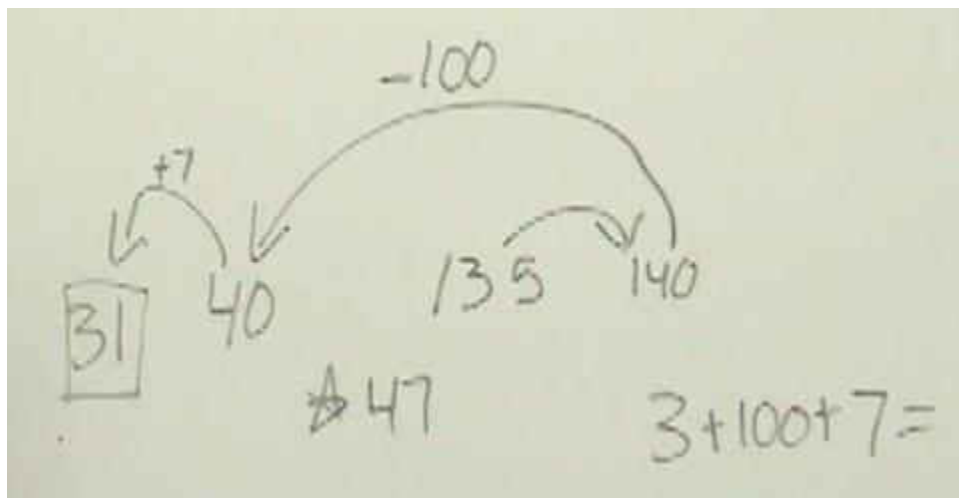
Figure 44: Claudia's Illustration of a Student's Misconception (47 + ? = 135)

The significant difference between Claudia's explanation and the majority of other prospective teachers' responses involved her discussion of *what the student actually did*, as opposed to *what the student should have done* to be correct. Not only did she illustrate that she

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knew how to do the problem correctly herself, but she also demonstrated and led the whole class discussion in realizing what a student's misconception may have been and how to overcome that issue. During this process, Claudia provided the statement of what she believed the student did (*Claim*), how he went about his procedure (*Data*) and why he was incorrect (*Warrant*). Finally, she was able to thoroughly synthesize the various pieces of argumentation (*Backing*) to describe all aspects of the student's approach, his proposed solution and the eventual misconception.

Flexibly Representing Equivalent Quantities

While working within the candy shop, prospective teachers explored their understanding of how to exchange pieces of candy packaged in boxes, rolls and pieces in the inventory forms. Recognizing how to manipulate specific quantities typically involved exchanging one-ee-zero rolls of candy for a box, and one-ee-zero pieces for a roll. Claudia's role within the classroom dynamic had gradually shifted from mainly providing statements (*Claims)* and descriptions of *how* to do things (*Data*) into one of primarily describing *why* things happen as well as the way to connect across examples (*Warrants* and *Backings*).

In the next few instances of classroom argumentation, the researcher will provide situations where Claudia's participation in the classroom discussions either directly led to or ultimately helped to establish the *taken-as-shared* notion of flexibly representing equivalent quantities. Consider the following case when Cordelia was describing her strategy of solving the problem 167 – 52. Specifically, note Claudia's role as she summarized and highlighted the mathematically significant aspects of this particular episode.

> Cordelia: If I took one-hundree-six-ee-seven pieces and minused five-ee-two pieces; two from seven is five. Six from five is one – well, (then the result is) one-hundree-one-ee-five pieces.

```
    1  6  7
-      5  1
───────────
    1  1  5
```

If I have one-hundree-six-ee-two and then I minus five-ee-seven. Six becomes a five, this becomes a one-ee-two. Seven, one-ee-zero, one-ee-one, one-ee-two (counting up from seven to one-ee-two) – three. Five from five is zero and that leaves one-hundree-three pieces. (*Data/Claim*)

```
        5  12
    1   6̸  2̸
-       5   7
──────────────
    1   0   3
```

Student:	I don't know why you just did that
Cordelia:	Because I think that is what we are being set up to do, so you can look at this and be able to do this. See the same thing.
Instructor:	Claudia?
Claudia:	I think what she is saying is that you can borrow or break up… (*Claim*)
Cordelia:	Yeah!
Claudia:	You can break up the middle one which would be the one-ee-zero's place. You can break that up, so that the one's place can have an amount that you can subtract. Always remembering that it is one-ee-two to understand... And that the six in one-hundree-six-ee-two is a six-ee-zero. (*Warrant*)

As it can be observed, Cordelia started to show the class how she would do the problem and its connection with base-10. However, Claudia's role entailed relating the procedure along with the justification of *how* and *why* that process was mathematically valid. It was in fact Claudia who was able to move the discussion forward by explaining in such a manner that the student who asked the question understood.

Shortly after the aforementioned episode, prospective teachers were discussing the validity of "borrowing" or "carrying" as well as using a more mathematically correct term to describe that action. In the following mini-episode, note Claudia's role in providing the justification by describing the mathematical reasoning behind the action.

Claudia: I think you could also just remind them that in one box there is one-ee-zero rolls and one-hundree pieces. And just remind them that when they are going to transfer, that there are one-ee-zero pieces there (pointing to the 1 in 162 in the previous episode) and not just one. (*Warrant*)

Instructor: And what you are doing is providing justification too. And we explained what we did and we are providing the justification for why.

As in the previous episode, Claudia at this point in the instructional sequence has established a role of mostly providing the justifications in classroom argumentation. Here, she summed up the actual meaning of each digit – according to place value - in a three-digit number by flexibly describing the equivalent quantities described.

Towards the end of this day in the instructional sequence, the instructor and the class were still discussing the problem $162 – 57 = 103$ by exchanging equivalent quantities using boxes, rolls and pieces. This discussion had helped to explain and justify the ability to move across place value while keeping the quantity the same. Inventory forms and their constituent parts – boxes, rolls, and pieces – served as a pedagogical content tool to facilitate the discussion which took place within the social collective of the classroom. Note another of Claudia's contributions to the whole class discussion as she related inventory forms with the previous pedagogical content tool utilized in classroom discussions – the open number line.

Claudia: Isn't that kind of like what we do on the number line? Break it down, break it down into like one-ee-zeros. Then we subtracted one-ee-zeros, six-ee-zeros, …

Instructor: And we did that even before we ever learned how to arrange numbers like this. And it made sense to us - even when we are just learning how to count in base-8.

Again through this episode, Claudia demonstrated her perspective in understanding whole number concepts and operations while possessing the ability to connect various pedagogical elements presented in the instructional sequence.

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Developing Addition and Subtraction Strategies

The last classroom mathematical practice that by definition became *taken-as-shared* involved the various addition and subtraction strategies discussed and understood within the classroom dynamic. Thus far, we have noticed that as Claudia's conceptual understanding has progressed through the instructional sequence situated entirely in base-8, so has her role in classroom discussions. Claudia's role has also progressed from providing statements and answers towards mostly providing *warrants* and *backings*.

Up until the start of Day 7, the instructor would introduce tasks from the instructional sequence and the prospective teachers would explore their understanding, engage in small group discussion, and then share their approaches and strategies as a whole class. At the beginning of Day 7, the researcher encountered a remarkable shift in the way classroom discussions had taken place.

Observe Claudia's role in revisiting an approach brought up by another student from the previous class as well as the manner in which she introduced, reasoned and synthesized her thought process regarding whole number operations – See Figure 45.

Instructor:	(Before we) go on to what I had planned, two people have asked to and shared something with me and I have asked them to share with the class. So first Claudia…
Claudia:	I don't know if you remember - when we were talking about the partial sums and - Jane brought up the fact that it really didn't make a difference - because you will still to have to carry over. We concluded as a class that we would still never have to, but I found one where we would have to.
Instructor:	Tell us what you did again.

150

Claudia: (Writes 425 + 256 on the board). If you do it the partial sum way, four
              plus two - I mean four-hundree plus 2-hundree would be 6 hundree.
              Two-ee-zero plus five-ee-zero is 7-ee-zero. Five plus six would be one-
              ee-three. (*Data*)
              And then when we went to add this up, we said we would never
              have to carry over. (*Claim*)
              But this would be three (pointing to 5 + 6) and then seven plus one would
              be one-ee-zero. So we would still have to carry over. (*Warrant*)
              So Jeannie was right!
Class: (Applause)
Instructor: That's impressive!

$$
\begin{array}{r}
4\ \ 2\ \ 5 \\
+\ \ 2\ \ 5\ \ 6 \\
\hline
6\ \ 0\ \ 0 \\
7\ \ 0 \\
1\ \ 3 \\
\hline
\end{array}
$$

Figure 45: Claudia's Illustration of Partial Sums

Through the analysis of this episode, Claudia exhibited several components of her conceptual

understanding of whole number operations in addition to distinguishing her individual

development from the collective understanding of the class. She understood not only the place

value notion that was at the core of partial sums, but also demonstrated a deeper understanding

by creating a problem that would bring the case of needing to carry over place values to the

forefront. In particular, the instructor also recognized Claudia's advancement and her deep

understanding of the topics at hand. In this fashion, Claudia distinguished her development and

understanding from the majority of her peers in the social phenomenon of the classroom.

Summary

In the case involving Claudia, the researcher was often times caught off guard by the

relative advancement of her conceptual understanding even prior to the beginning of the

instructional sequence. As described in the beginning of this chapter, Claudia had long been

interested in knowing the reasons behind what she was learning and had been fortunate to work with some teachers in her past that had possessed the knowledge to guide her exploration of mathematical concepts. As illustrated in Table 11, Claudia entered this instructional sequence taught entirely in base-8 already with a solid notion of place value concepts and was proficient at explaining her strategies involving whole number operations.

Table 11: Claudia's *Demonstrated* Instances of Conceptual Understanding

	Pre-Interview	Student Artifacts	Small Group & Whole Class Discussions	Post-Interview	Focus Group Interview
Place Value	∗	∗	∗	∗	∗
Counting Strategies		∗	∗	∗	N/A
Addition & Subtraction	∗	∗	∗	∗	∗
Multiplication & Division		∗	∗	∗	∗

This researcher only discovered instances involving counting strategies as well as multiplication and division where Claudia's understanding seemed to reflect *how* to solve the problems without a solid foundation or the ability to connect it with underlying concepts. Note that even in those rare instances where she did not illustrate a conceptual understanding, by the conclusion of the instructional sequence, she had fully demonstrated the ability to explain and justify problems involving all whole number concepts and operations.

This researcher next explored Claudia's participation in argumentation. The same guidelines as illustrated below were used to demonstrate Claudia's participation and contributions in the establishment of classroom mathematical practices. Note the almost *upper*

triangular trend in Table 12 illustrating her shift away from *claims* and *data* towards almost

exclusively producing *warrants* and *backings* during whole class discussions.

Guidelines for Individual Participation in Argumentation:

N/A	Not Applicable
⊙	No or Minimal Support of Argumentation
⊙	Some Support of Argumentation
◉	Moderate Support of Argumentation
●	Extensive Support of Argumentation

Table 12: Summary of Claudia's Participation in Establishing Classroom Mathematical Practices

	Claims	*Data*	*Warrants*	*Backings*
➢ Developing Number Relationships using Double 10-frames	◉	●	◉	◉
➢ 2-Digit Thinking Strategies Using the Open Number Line	⊙	◉	●	●
➢ Flexibly Representing Equivalent Quantities	⊙	⊙	●	●
➢ Developing Addition and Subtraction Strategies	⊙	⊙	●	●

As the instructional sequenced transgressed, Claudia began to shift away from providing

solutions and explanations in the forms of *claims* and *data*. At roughly the half way point in the

unit involving whole number concepts and operations, Claudia's role in argumentation had

become one of providing justifications for her own and other students' strategies. In addition, she

became one of the voices that the instructor used in classroom discussions to push the thinking of other prospective teachers. Often times, Claudia would make connections that not only illustrated her understanding of say, inventory forms, but rather the way the use of this pedagogical tool was analogous to the previously introduced open number line. Claudia's presence and active participation in classroom discussions challenged and pushed the thinking of all prospective teachers and in particular assisted the development of her small group's conceptual understanding of whole number concepts and operations.

Furthermore, due to her consistent improvement and greater development of conceptual understanding, Claudia repeatedly was able to make sense of other prospective teachers' work. She demonstrated on several occasions that she assisted other prospective teachers in their understanding of their thinking and commented on the links between the strategies. The ability of a (prospective) teacher to simultaneously possess a deep conceptual understanding of the subject matter as well as rich pedagogical insight into the understanding of other prospective teachers often tends to separate good teachers from extraordinarily effective educators.

CHAPTER 6: CROSS-CASE SYNTHESIS

The researcher wished to explore the understanding of individuals who participated in this research project beyond the single cases described in the previous two chapters. This research endeavor intended to analyze the cross-case findings in order to point to trends or applications beyond those mentioned for this particular study. Furthermore, cross-case analyses have the potential to develop more powerful explanations in cases involving only the single case analyses (Miles & Huberman, 1994).

The cross-case analysis involving Cordelia and Claudia's understanding of whole number concepts and operations addresses a comparison of the way they progressed prior to and following the instructional sequence taught entirely in base-8. This comparison details the manner in which each displayed a conceptual understanding of place value notions, counting strategies, as well as operations involving addition, subtraction, multiplication and division. Each individual research participant was thoroughly analyzed for their conceptual understanding of the aforementioned topics and this research effort focused on whether this understanding was "interconnected and, hence meaningful knowledge" (Baroody, 2003, p. 11).

In addition, this researcher wished to compare and contrast some qualitative differences that became realized through the course of analyzing the various sources of data including the interviews, student artifacts, classroom discussions and focus group findings. To reiterate, the primary intention of this research endeavor revolved around individual prospective teachers' mathematical conceptions and activity as it related to the conceptual understanding of whole number concepts and operations.

<u>Comparison of Understanding Place Value and Counting Strategies</u>

To begin the discussion on the cross-case analysis, the researcher organized the findings regarding each individual's conceptual understanding of the mathematical notions included as a part of the instructional sequence into Table 13 presented below. These results compared Cordelia and Claudia's *demonstrated* conceptual understanding prior to – labeled "Pre" - and following – labeled "Post" - the instructional sequence in base-8.

Table 13: Comparison of Participants' *Demonstrated* Conceptual Understanding

	Cordelia		Claudia	
	Pre	Post	Pre	Post
Place Value		*	*	*
Counting Strategies		*		*
Addition & Subtraction	*	*	*	*
Multiplication & Division				*

As indicated by the presence of an asterisk (*) in Table 13, both Cordelia and Claudia illustrated a conceptual understanding of place value notions and developed their initial understanding through the course of this instructional sequence. Cordelia's development seemed more pronounced since there were no instances when she either mentioned or used the notion of place value prior to the instructional sequence. During the post-interview, however; Cordelia stated: "It's easier to understand exactly what I am doing in base-8, because we talk about it in place value so we are adding up the place value and not just numbers." Overall, following the instructional sequence, she used place value to explain and justify her strategies, and

demonstrated understanding through connecting place value to counting strategies and operations.

Claudia began with a sense of place value concepts even prior to the instructional sequence, however; her developed understanding of this notion led to her *invented* strategies involving counting and operations. It also became clear that Claudia realized the significance of place value concepts as she continuously referred back to them during classroom discussions, and following the completion of the instructional sequence in the post-interview and the focus group. Claudia's development in understanding place value concepts became more evident as will shortly be illustrated when she incorporated these notions into counting and operation strategies.

In Figures 46 and 47, the researcher juxtaposed Cordelia and Claudia's solutions to the problem "In what ways would you solve 18 + 45 = ?" as asked during the post-interview. Note the strategies used by each prospective teacher as well as how concise and mathematically correct each solution proved to be.
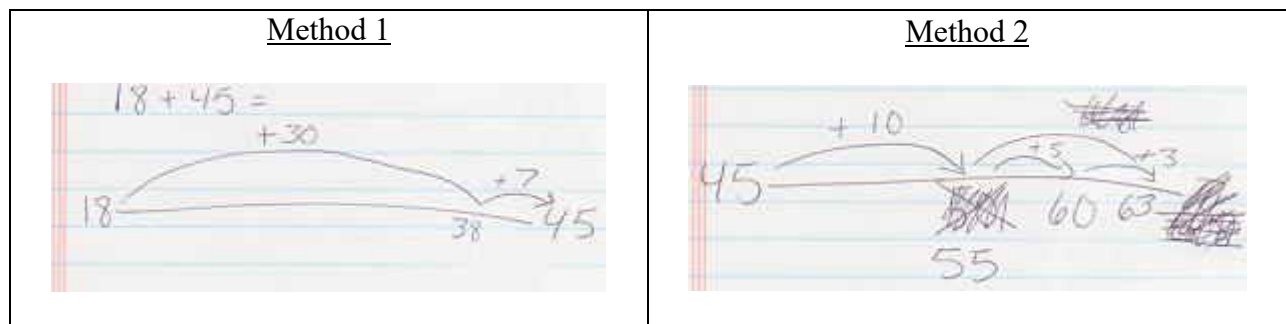
Method 1	Method 2

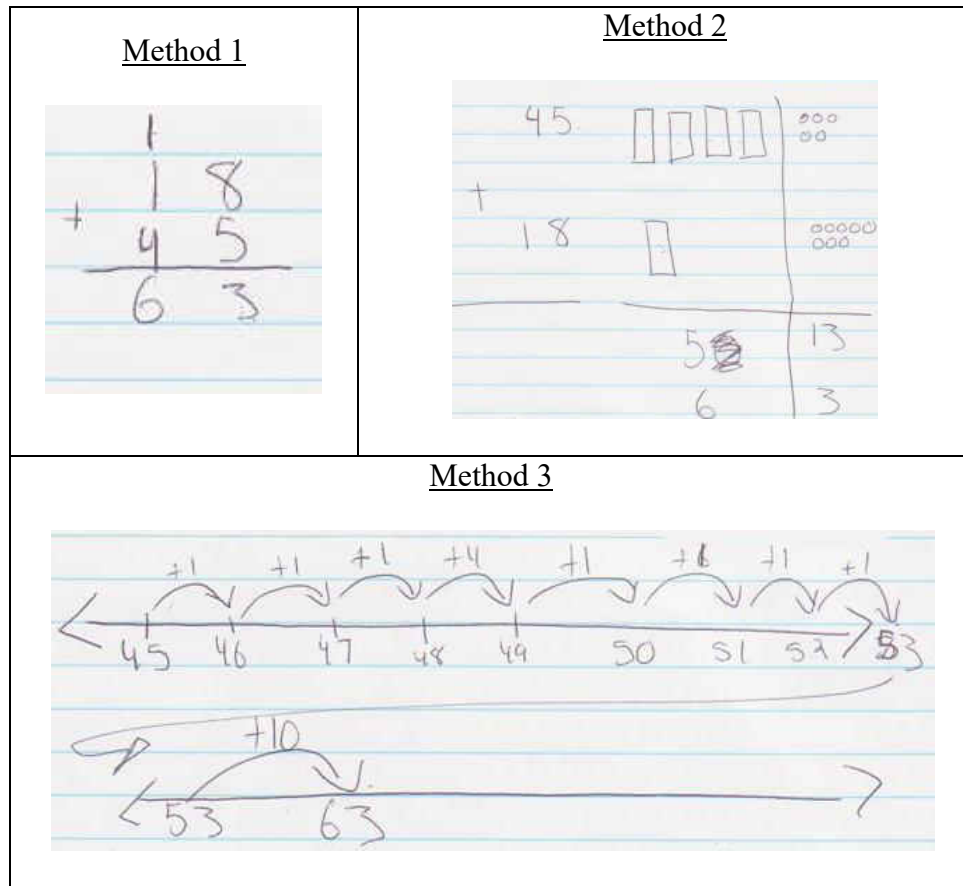**Figure 46: Cordelia's Methods for Solving 18 + 45 = ?**

Figure 47: Claudia's Methods for Solving 18 + 45 = ?

Cordelia presented her solution to 18 + 45 using an open number line as learned in class. However, in Cordelia's first method, she mistakenly tried to find the difference between 18 and 45. To compound her misconception, she also made an error by evaluating the distance between 18 and 38 to be +30 – instead of +20. Note that she did try to count up to 38 using units of ten. In her second method, Cordelia accurately performed the addition and in an efficient fashion began with the larger number 45. Next, she continued by adding 10 and commented that "another ten would be too much". She finished the problem by adding another 5 and 3. Through this example,

Cordelia demonstrated that she had gained a greater understanding of using the open number line and her counting strategies included counting by ten, by five and then three.

In assessing Claudia's methods, she illustrated *multiple* approaches to solving the same problem. She demonstrated that she could quickly and accurately perform the traditional algorithm commonly taught in U.S. schools. Next, she pictorially and numerically used the column addition strategy to solve the problem. Note that similar to Cordelia, Claudia also began this method by efficiently adding 18 to 45 and not vice versa. In her third method, Claudia showed the range of her development by illustrating her understanding of the open number line as demonstrated in class. Again, note that Claudia decomposed 18 into one's and ten's and exhibited the step-by-step approach accurately in Figure 47. Having analyzed both individuals through the entire instructional sequence, this researcher came to the following conclusion. While both individuals had demonstrated understanding of counting strategies and addition, this researcher felt that Claudia had developed a deeper and broader understanding and illustrated more ways to explain and justify her thinking.

Comparison of Understanding Multiplication

At this point in the cross-case analysis, the focus shifts to Cordelia and Claudia's conceptual understanding of multiplication and division. Roy (2008) concluded that in the context of multiplication and division, there was not sufficient evidence during the classroom argumentation for these operations to have become *taken-as-shared*. In order to compare and contrast the two research participants in this study, the researcher considered it appropriate to use the cross-analysis section to examine the manner in which each prospective teacher illustrated her understanding of problems involving these two operations.

During the post-interview, both individuals demonstrated their development of multiplication upon the conclusion of the instructional sequence when presented with the scenario: "A student has 23 books in his library where each book has 14 pages. How many total pages are there in all the books?" In Figure 48, note how each person solved the problem and the support and justification they provided during the interview process.
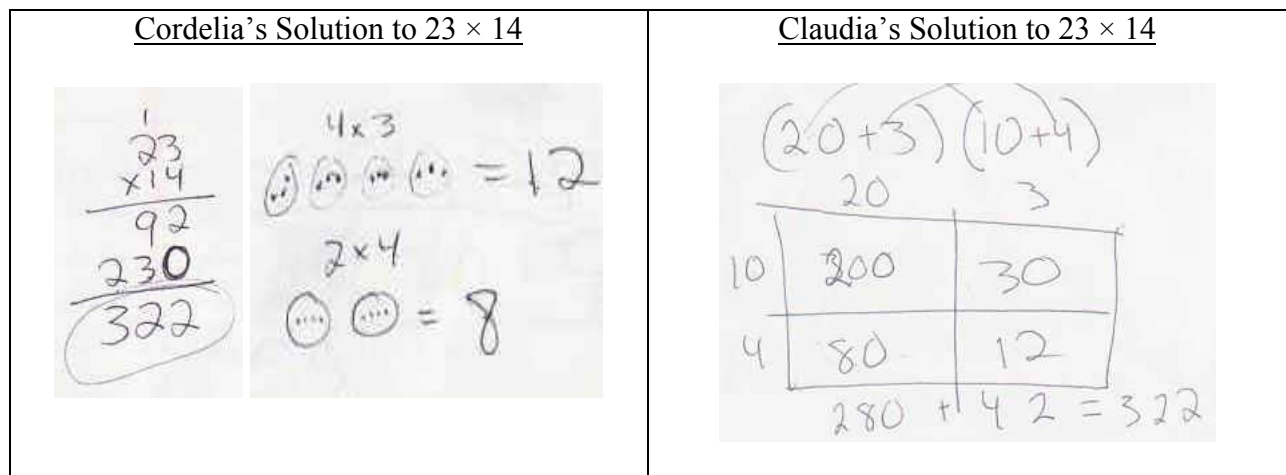
Cordelia's Solution to 23 × 14	Claudia's Solution to 23 × 14

Figure 48: Comparison of Cordelia and Claudia's Solutions to 23 × 14

Cordelia - in accord with her previous work throughout the class – demonstrated that she could multiply effectively using an algorithm commonly used in the schools. The researcher must emphasize at this point that this procedure came up during classroom discussion and due to a lack of meaning for the "1 that is *carried* above" did not constitute an acceptable solution as it could not be explained *and* justified. Cordelia attempted to provide the *how* and *why* using the drawings next to her procedure, but still failed to provide the justification for *why* she had written 2 × 4. When asked to explain, she responded: "I did 4 times 3 (drawing the 4 circles with 3 dots in each)…I multiplied the ones. Then, you multiply 2 times 4 to get 8. (Then) I just added 12 and 8 to get the 92. …That's what you do. I can't explain it." Evidently, she was unable to observe

the 2 in "2 times 4" as a 20. Subsequently, she "added 12 and 8 to get the 92" whereas the 8 should have been replaced by an 80 or 8 tens.

In contrast, Claudia decomposed the numbers 23 into (20 + 3) and 14 into (10 + 4). She displayed her understanding of the area model and an efficient way of illustrating these partial products both pictorially and numerically. Note that upon getting each of the four partial products - $(10 \times 20 = 200)$, $(10 \times 3 = 30)$, $(4 \times 20 = 80)$, $(4 \times 3 = 12)$ – Claudia added 280 plus 42 to finalize her solution of 322. In order to solve the problem in this fashion, Claudia illustrated that she effectively understood composition and decomposition of numbers according to place value. Moreover, she demonstrated conceptual understanding of the relationship between addition and multiplication involving "multiplying out" $(20 + 3)$ times $(10 + 4)$ and illustrated that she could connect her understanding of whole number concepts and operations with algebraic thinking. Once again, to this researcher, it seemed that while both individuals have the ability to perform whole number operations, Claudia illustrated a deeper and broader understanding which enabled her to explain and justify the mathematics involved.

Comparison of Understanding Division

The last topic included as a part of this instructional sequence involved division of whole numbers. In order to explore Cordelia's understanding of division and compare it with Claudia's development of this operation, an example from the test which followed the completion of the instructional sequence will be used. This particular example represented a division problem in base-8 and stated: "Mary has 652 stickers that she wants to share with some friends in her class. If she gives each of her friends 17 stickers, how many friends can she share with? How many stickers will be left, if any?" On the following page – in Figure 49 - Cordelia's solution to this

problem is juxtaposed with Claudia's to illustrate the way each approached this division problem. Compare the repeated addition strategy utilized by Cordelia compared to the partial quotients method used by Claudia as well as mistakes made by each student.
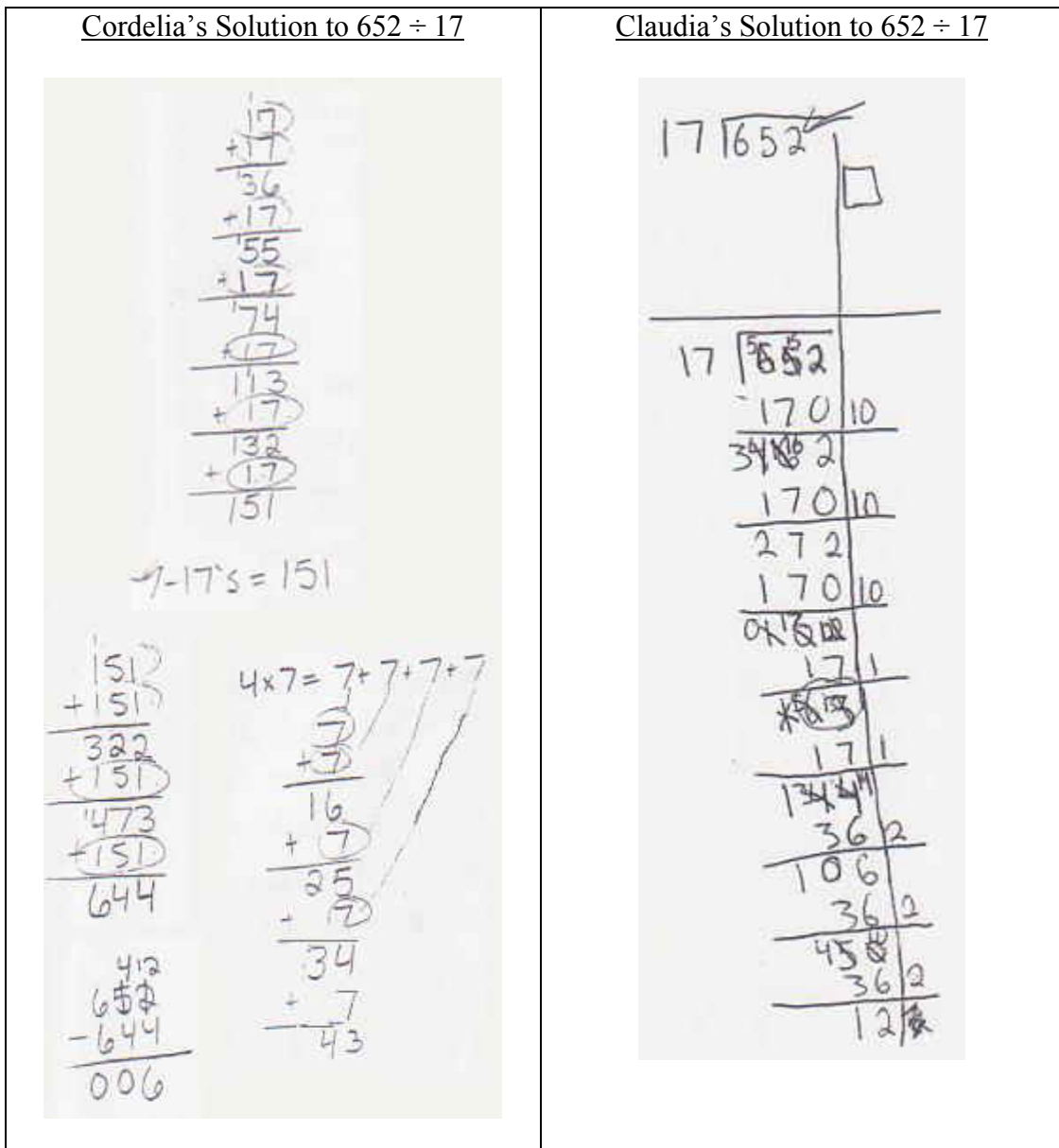
Cordelia's Solution to 652 ÷ 17	Claudia's Solution to 652 ÷ 17

Figure 49: Comparison of Cordelia and Claudia's Solutions to 652 ÷ 17

Cordelia's strategy involved a repeated addition of 17 in order to use the resulting sum of 151 to get close to the desired total of 652. The inability to divide in the traditional sense was coupled in this case by her inability to solve the problem correctly using multiplication. In contrast, Claudia's approach modeled the partial quotient process illustrated during whole class discussions. During the last day of the instructional sequence, the instructor noticed that Claudia had the partial quotients method demonstrated on her paper. The instructor asked Claudia to use the document camera to show her work as "A student did this…" which was a common approach to examine other students' ways of thinking. After a lengthy classroom argumentation episode, several prospective teachers collaboratively were able to explain and justify the method of partial quotients through analysis of Claudia's work.

In examining Claudia's solution to the same problem that Cordelia had solved, the researcher noticed that both prospective teachers made errors that resulted in deductions on the test. In fact, this question was worth 10 points, and both individuals received an 8 out of the possible 10 points. Without a qualitative analysis of these two prospective teachers' solutions, the quantitative results would imply that they displayed similar understanding. However, upon further review, it became evidently clear that Claudia chose the partial quotients method and demonstrated a conceptual understanding which required to her to systematically take away one-ee-zero and then single digit one-ee-seven's to bring the total as close to zero as possible. Due to an oversight, or perhaps even time constraints, Claudia made a subtraction error about half-way through this problem. In comparing and contrasting these two individuals' work, it was fairly clear that Cordelia did not possess the conceptual understanding to solve the problem using division strategies illustrated in class. Claudia, on the other hand, approached the problem and

provided a step-by-step explanation and justification of her division strategy on her paper, yet made what could be described as a procedural oversight and not a conceptual lack of understanding.

Comparison of Participation in Classroom Argumentation

During the summary section of Chapters 4 and 5, this research effort synthesized the manner in which each individual participated within the whole class dynamic and in the classroom argumentation. At this juncture, Cordelia and Claudia's participation should be examined side-by-side in order to understand *how* and *in what capacity* each was actively involved in establishing classroom mathematical practices as she progressed through the instructional sequence. In other words, this researcher wanted to know in what way each individual was able to contribute to the classroom discussions and the manner in which she helped to shape her own understanding concurrent with and as the social collective moved towards ideas being *taken-as-shared*.

In Table 14 provided on the next page, a summarized comparison of Cordelia (COR) and Claudia (CLA) has been demonstrated. Continuing with the same guidelines as were used in previous chapters, each individual's participation was identified as providing minimal, some, moderate or extensive support in classroom argumentation as it led to the establishment of the classroom mathematical practices being *taken-as-shared*. Note the manner in which *claims, data, warrants,* and *backings* have been compared and contrasted for each individual and broken down per each of the classroom mathematical practices that were established as a part of this instructional sequence.

Guidelines for Individual Participation in Argumentation

N/A	Not Applicable
◎	No or Minimal Support of Argumentation
◉	Some Support of Argumentation
◉ (ring)	Moderate Support of Argumentation
●	Extensive Support of Argumentation

Table 14: Comparison of the Participation of Cordelia (COR) and Claudia (CLA) in Establishing Classroom Mathematical Practices

	Claims		Data		Warrants		Backings	
	COR	*CLA*	*COR*	*CLA*	*COR*	*CLA*	*COR*	*CLA*
➤ Developing Number Relationships using Double 10-frames	Moderate	Extensive	Moderate	Moderate	No/Minimal	Moderate	No/Minimal	Moderate
➤ 2-Digit Thinking Strategies Using the Open Number Line	Moderate	Some	Extensive	Moderate	Moderate	Extensive	No/Minimal	Extensive
➤ Flexibly Representing Equivalent Quantities	Moderate	Some	Moderate	Some	Some	Extensive	No/Minimal	Extensive
➤ Developing Addition & Subtraction Strategies	Moderate	Some	Moderate	Some	Moderate	Extensive	No/Minimal	Extensive

In developing number relationships using Double 10-frames, both Cordelia and Claudia demonstrated moderate to extensive support in providing the *claims* and *data* in stating their thought process and how they went about arriving at that conclusion. However, analysis of the

data illustrated a drastic difference in that for this particular mathematical practice only Claudia was able to justify and connect the various concepts through *warrants* and *backings* provided in classroom argumentation.

In the second established classroom mathematical practice, Cordelia played a more active role in *initiating* the whole class discussion. She illustrated her participation by providing extensive *claims* and *data* in using the open number line to demonstrate 2-digit thinking strategies. While Cordelia was able to provide moderate justifications, it was still Claudia who provided more extensive *warrants* and *backings* to support and connect the mathematical concepts.

As for the third classroom mathematical practice – flexibly representing equivalent quantities – the researcher observed a definite trend through the analysis of the various data sources. While Cordelia provided moderate support throughout the establishment of this mathematical practice, her role seemed to be confined to the statements of *how* in the *claims* and *data* demonstrated her procedures entailed in describing the mathematics. Conversely, while by this point Claudia tended to no longer be initially involved in the discussion, repeatedly she would describe the justification and connect the concepts by providing *warrants* and *backings* through the various episodes.

In the final classroom mathematical practice of developing addition and subtraction strategies, the trend established and the roles demonstrated by both Cordelia and Claudia became even more pronounced. Cordelia maintained a very active role in showing and describing the manner in which she or other prospective teachers would perform some addition and subtraction procedures. Both Cordelia and Claudia were able to provide the mathematical reasoning to

justify and connect the strategies to already established mathematical examples and topics by

eliciting numerous *warrants*. Still, Claudia maintained her role in connecting it all together

through providing *backings* which synthesized what she and other prospective teachers had done

to establish addition and subtraction strategies.

CHAPTER 7: CONCLUSION

The primary purpose of this study involved the exploration of individual prospective teachers' development of a conceptual understanding of whole number concepts and operations. Current efforts in the reform of education and teacher preparation have outlined the significance of teacher knowledge (NCTM, 2000; National Mathematics Advisory Panel, 2008). Specifically, teachers are viewed as the key figures in the implementation of the standards and guidelines set forth by NCTM (National Research Council, 2001).

A significant component in the improvement of understanding in teachers has dealt with the social context in which learning occurs. Cochran, DeRuiter and King (1993) inferred that teaching for understanding and teachers' abilities are enhanced if they are acquired in contexts that resemble those in which they will be using their knowledge – specifically a classroom context. Prospective and in-service teachers have historically gained valuable insights into the learning and teaching of mathematics in an inquiry-based classroom environment (Carpenter, Franke, Jacobs, Fennema & Empson, 1998; Kazemi, 1999).

In this qualitative study, a case study analysis was undertaken in order to examine prospective teachers' understanding of whole number concepts and operations as it took place during a classroom teaching experiment. This research endeavor allowed this researcher to carefully analyze the manner in which prospective teachers' understanding changes and the ways prospective teachers reorganize their mathematical thinking within a collective classroom setting (Steffe & Thompson, 2000). As for the participants, this inquiry-based classroom promoted opportunities for these prospective teachers to become flexible learners and provided a window

for the development of conceptual knowledge with depth including a connected web of understanding (Hiebert & Carpenter, 1992).

While previous research efforts insisted on creating a dichotomy of choosing the individual *or* the collective understanding, through the utilization of the emergent perspective *both* the individual and the social aspects were simultaneously considered. In fact, prospective teachers' conceptual development took place in both contexts while neither the individual perspective nor the social aspect took primacy over the other (Cobb & Stephan, 2003). Each research participant had a chance to reorganize her thinking within the collective setting of the classroom and the analysis of each individual allowed for the exploration of aspects of her learning as it occurred independently as well as through these classroom interactions.

Recent research efforts had demonstrated that prospective teachers follow developmental stages similar to children if placed in a context to examine whole number concepts and operations from a different perspective (Andreasen, 2006, McClain, 2003, Roy, 2008). McClain (2003) introduced the notion of using an alternative base with prospective elementary teachers in order to develop instructional tasks and goals to foster understanding. For the purposes of this research study, in continuation of previous iterations of a similar study, base-8 served as the particular alternative base for the instructional sequence. The individuals analyzed through this research study also developed their understanding of whole number concepts and operations on a path similar to the one that children take. In accord with previous research, these prospective teachers also demonstrated counting by 1's and 10's, composing and decomposing of addends and minuends, and emphasized place value in developing an understanding of operations on their

path towards developing a conceptual understanding of whole number concept and operations (Andreasen, 2006, Baroody, 1993; Kamii, 1993, Roy, 2008).

In particular, two prospective teachers were selected who had initially demonstrated different incoming content knowledge for teaching according to the CKT-M instrument (Hill, Ball & Schilling, 2008). The first research participant, Cordelia, had scored a 10 (out of 25) on the CKT-M instrument prior to the beginning of the instructional sequence which placed her in the category of "Low-Content" knowledge as described in Chapter Three. Conversely, the other prospective teacher, Claudia, had a pre-test score of 18 (out of 25) on the CKT-M instrument which placed her in the "High-Content" knowledge category. The results from this case study analysis included both the individual development as well as their participation in the establishment of the classroom mathematical practices (Rasmussen & Stephan, 2008). Toulmin's (1969) argumentation model was used to identify *claims, data, warrants* and *backings* as each individual participated in and contributed to whole class discussions.

In the case of Cordelia, she did in fact improve through the course of the instructional sequence evidenced by a 7 point improvement (Post-test score: 17) on the CKT-M instrument. Analysis of her artifacts and classroom contributions illustrated that she typically explained the manner in which she provided her answers, but could not explicitly verify why the steps she took were valid. Her participation in the classroom argumentation was mostly confined to providing the *claims* and *data* rather than the *warrants* and *backings*. In other words, Cordelia displayed a good understanding of how to solve some of the problems but lacked the conceptual understanding to be able to justify the rationale behind all her mathematical moves.

As for Claudia, she began with a much more solid foundation as indicated by her Pre-test score of 18 on the CKT-M instrument. She also showed a marked increase in her conceptual understanding as she scored a 21 on the CKT-M Post-test instrument following the instructional sequence. In the case of Claudia, her development was illustrated more effectively through a qualitative analysis of her progress through the instructional sequence. She managed to refine her strategies in whole number concepts and operations and developed a greater ability to make sense of other prospective teachers' thinking. Claudia routinely made connections – through *warrants* and *backings* in classroom argumentation - that not only illustrated her understanding of pedagogical content tools such as the open number line and inventory forms, but she also had developed the knowledge to synthesize and connect the various mathematical notions discussed. Claudia's presence and active participation in classroom discussions challenged and pushed the thinking of all prospective teachers and in particular assisted the development of her small group's conceptual understanding of whole number concepts and operations.

Overall, both participants in this research effort developed their conceptual understanding of whole number concepts but in quite different ways. Cordelia gained more insight into the significance of place value notions and the manners in which this concept adds meaning to strategies in addition and subtraction. She also demonstrated an improved sense of understanding some of the reasoning behind her mathematical moves. Claudia - who already possessed much of the content knowledge prior to the instructional sequence - gained valuable insight into counting strategies as well as developing an improved understanding of multiplication and division models. Furthermore, by exploring various ways that other prospective teachers solved the

problems, she also gained a greater pedagogical perspective in how other people think mathematically.

Limitations

This study dealt with multiple possible limitations as the researcher designed, implemented and analyzed the research study. One limitation involved the use of the CKT-M instrument which was used as pre-test in order to differentiate between prospective teachers' initial content knowledge. The results of this measure did not indicate a great deal of dispersion in the scores of the prospective teachers' initial content knowledge for teaching. This issue resulted in a rather difficult task of definitively differentiating among the research participants. As illustrated in Figure 11, the lower quartile score was 12 and the upper quartile score was 14.5 which only separated the individuals by less than 3 questions on this instrument.

Next, the classroom argumentation analysis was performed by two members of the research team. While both individuals practiced the highest of ethical considerations and worked independently in the initial determination of *claims, data, warrants,* and *backings*; ideally a research team of multiple individuals could have analyzed the classroom argumentation data. The researcher realizes that in many projects a single researcher solely handles classroom argumentation guidelines, however; the participation of multiple research team members could provide for an extra measure of reliability.

In addition, there was not an opportunity to follow up with member checking with the research participants at the conclusion of the study. Sharing and discussing the research findings with the participants would have added further credibility and accuracy to the results of this research endeavor. The individual prospective teachers continued on to mathematics methods

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courses as well as internships which could have influenced their understanding of whole number concepts and operations. During and after such additional experiences with the mathematics content related to whole number concepts and operations, in this researcher's opinion, it would have become nearly impossible to detect how/when each prospective teacher developed their conceptual understanding. In the future, it would be advisable if member checking mechanisms could be planned into the course of the experiment in order to verify data and decrease potential incidents of incomplete or inaccurate observations through discussions with the research participants.

Implications

Following a thorough and intensive case study analysis, it must be noted that cases and theory do *not* and should *not* serve to *conclude* research efforts. Case studies such as the one undertaken by this researcher ultimately raise more questions and hopefully lead towards the integration of research efforts in order to arrive at a more thorough and coherent understanding. This study intends to serve any and all parties interested in teachers, teacher educators and in particular those involved in developing prospective teachers and their understanding. One of the goals of this research endeavor is to eventually serve as a step towards synthesizing the conceptual and pedagogical preparation of mathematics teachers. As such, this researcher has outlined a few implications for future research. The following points represent some of the questions that have occupied this researcher's thoughts during the various stages of this research endeavor.

At this point, various aspects of the theoretical framework outlined by Cobb and Yackel (1996) have been analyzed. One subsequent logical step would be an effort to synthesize the

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recent efforts of Andreasen (2006), Dixon, Andreasen & Stephan (in press), Roy (2008) and this research study in order to integrate the social and psychological perspectives. At this point, the social perspective of the emergent perspective in the context of whole number concepts and operations has been addressed. Specifically, the classroom mathematical practices have been identified for this instructional sequence as well as the analysis of the individual component comprising the mathematical conceptions and activities aspect of the psychological portion of the emergent perspective. To fully grasp the psychological perspective, future research projects need to examine the mathematical beliefs and values which serve as the correlate to the sociomathematical norms. Also, beliefs about own role and the role of others need to be addressed in order to thoroughly understand prospective teachers' motivations and perspectives. Such efforts would provide a much more coherent picture of the development of prospective teachers in the context of whole number concepts and operations.

A second suggestion - as a future area of research - would involve further analysis of the pedagogical content tools introduced and used throughout this instructional sequence. In particular, the open number line and the inventory forms served as vital tools in recording student thinking. But in a much more significant way, these tools allowed for prospective teachers to connect their understanding of specific mathematical notions and were applied in order to explore new situations. Specifically, how were the open number line and inventory forms used by prospective teachers in order to gain a better understanding of multiplication and division? Similar to the recent efforts of Tobias (2009) in the context of rational number understanding of prospective teachers, analyzing the tool use in the context of this instructional sequence would also serve as an important contribution to the on-going research in this field.

Based on the individual analyses performed, it became evident that those individuals who participated in small group discussions where at least one member had an initial "High-Content" knowledge seemed to progress in a qualitatively different way than other groups. It would be interesting to connect "group-work" research with the development of prospective teachers to examine whether in fact what transpired during this research study was an isolated occurrence or part of a greater trend. In reference to the current research project, was it that Claudia was able to provide the warrants and backings for other students' claims that enhanced her group's ability to move forward? Conversely, did other individuals in Cordelia's group make similar claims and yet the ideas could not be extended due to the absence of an individual to move the group forward? How would this research connect with the Zone of Proximal Development as it relates to the way a prospective teacher may learn with or without the presence of a guiding figure? If prospective teachers participated in heterogeneous groups in constructivist classrooms, how would individuals such as "Cordelia" be affected by the presence of a "Claudia" in their group? Lastly, as suggested by the Pirie and Kieran (1994) model of conceptual understanding, would the presence of prospective teachers with "High-Content" assist in providing the scaffolding from which "Low-Content" individuals would benefit?

For years, mathematics educators have discussed the need for a verifiable measure of prospective teacher understanding. While a case study and the subsequent findings should not be generalized in the broad sense of the word, the Content Knowledge for Teaching – Mathematics (CKT-M) instrument was used as a way to differentiate between research participants for this study. To this researcher, it was interesting that regardless of the way each individual developed her understanding of whole number concepts and operations, the CKT-M instrument and the

subsequent results did provide insight into student understanding. In particular, the CKT-M instrument did indicate that Cordelia had "Low-Content" knowledge coming into this instructional sequence. Even though she improved from a CKT-M score of 10 to 17 (out of 25), her scores could have been a factor in the analysis and perhaps even a predictor. Moreover, Claudia improved from a CKT-M score of 18 to 21 (out of 25). Would it be plausible to state that at least in this case, the CKT-M instrument helped to predict the way that the prospective teacher would develop her understanding of whole number concepts and operations? Perhaps future research efforts could identify whether there is a correlation between scores on the CKT-M instrument and the way individuals participate in classroom argumentation (See Table 14).

Another area of future research could involve questions similar to Conner's (2007) work with prospective science teachers. She explored how well the cases demonstrate the relationships between prospective teachers' content knowledge, their pedagogical content knowledge, their beliefs about teaching mathematics, as well as the connection to their students' emergent knowledge. Some aspects of this question were raised as this researcher listened to the participants during the focus group following the instructional sequence. Future studies could illustrate that a specific combination of the aforementioned criteria might affect the success of prospective teachers in their development. Such findings could greatly influence the way we conduct our teacher education programs and perhaps the individuals who represent promising teacher-candidates.

Lastly, as noted earlier, this research project analyzed the development of 4 prospective teachers and upon analysis concluded that the two individuals classified with "High-Content" knowledge developed in qualitatively similar fashion. Similarly, the two individuals classified

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with "Low-Content" knowledge also developed in qualitatively similar fashion. With respect to this particular research effort, how would these findings have differed had four other teachers been selected initially? What are some other factors that could be considered in the selection of research participants when exploring the development of prospective teachers' conceptual understanding?

<u>Summary</u>

This research effort explored the development of individual prospective teachers' conceptual understanding of whole number concepts and operations. As the research participants progressed through an instructional sequence taught entirely in base-8, a case study approach was used to identify and analyze two individuals. The first participant, named Cordelia throughout this research project, initially demonstrated "Low-Content" knowledge according to the CKT-M instrument database questions. She developed a greater understanding of place value concepts and was able to apply this new knowledge to gain a deeper sense of the rationale behind counting strategies and addition and subtraction operations. Cordelia did not demonstrate the ability to consistently make sense of multiplication and division strategies. She participated in the classroom argumentation primarily by providing *claims* and *data* as she illustrated the way she would use different procedures to solve addition and subtraction problems.

The second participant, Claudia, initially was classified as having "High-Content" knowledge based on the CKT-M instrument. She already possessed a solid foundation in understanding place value concepts and throughout the instructional sequence developed various ways to connect and build on her initial understanding through the synthesis of multiple

pedagogical content tools. She demonstrated conceptual understanding of counting strategies, and all four whole number operations. Furthermore, by exploring various ways that other prospective teachers solved the problems, she also presented a greater pedagogical perspective in how other prospective teachers think mathematically. Claudia showed a shift in her participation in classroom argumentation as she began by providing *claims* and *data* at the outset of the instructional sequence. Later on, she predominantly provided the *warrants* and *backings* to integrate the mathematical concepts and pedagogical tools used to develop greater understanding of whole number operations. These results indicate the findings based on the individual case-study analysis of prospective elementary school teachers and the cross-case analysis that ensued.

Furthermore, even though generalization is not an aspect of this qualitative research study, perhaps the instructional sequence and the HLT used for this project could be revisited to better explain the manner in which specific individuals develop in the collective setting of the classroom. For instance, would individuals similar to Cordelia benefit from an alternative instructional sequence with different learning goals and therefore a modified set of instructional tasks and activities? Could there be a need to have multiple content courses to address the concepts that require the additional time and discussion related to elementary mathematics from a conceptual understanding perspective? And what about individuals similar in content knowledge to Claudia? Could they benefit from a class that jointly discussed content and methods within the same time frame? Or perhaps these individuals with a deeper conceptual understanding could begin internships within the undergraduate classroom by serving as guides in assisting in the development of conceptual understanding of other prospective teachers?

In conclusion, this research project aimed to extend the research literature by providing greater insight into the way individual prospective teachers develop their conceptual understanding of whole number concepts and operations in a social context. Specifically, using the emergent perspective as a theoretical framework, this research endeavor has outlined the mathematical conceptions and activities of individual prospective teachers and thus has provided the psychological perspective correlate to the social perspective's classroom mathematical practices. The researcher hopes that through the synthesis of the findings of this project along with current relevant research efforts, teacher educators and educational policy makers can revisit and possibly revise instructional practices and sequences in order to develop teachers with greater conceptual understanding of concepts vital to elementary mathematics.

APPENDIX A: INSTITUTIONAL REVIEW BOARD FORMS

IRB Approval Letter

University of Central Florida

Office of Research & Commercialization

December 18, 2006

Juli Dixon, Ph.D.
University of Central Florida
Teaching and Learning Principles
ED 123F
Orlando, FL 32816-1250

Dear Dr. Dixon:

With reference to your protocol #06-4028 entitled, "Prospective Teachers' Development of Understanding of Mathematical Concepts during a Classroom Teaching Experiment," I am enclosing for your records the approved, expedited document of the UCFIRB Form you had submitted to our office. **This study was approved on 12/14/06. The expiration date for this study will be 12/13/2007.** Should there be a need to extend this study, a Continuing Review form must be submitted to the IRB Office for review by the Chairman or full IRB at least one month prior to the expiration date. This is the responsibility of the investigator.

Please be advised that this approval is given for one year. Should there be any addendums or administrative changes to the already approved protocol, they must also be submitted to the Board through use of the Addendum/Modification Request form. Changes should not be initiated until written IRB approval is received. Adverse events should be reported to the IRB as they occur.

Should you have any questions, please do not hesitate to call me at 407-823-2901.

Please accept our best wishes for the success of your endeavors.

Cordially,

Joanne Muratori
(FWA00000351 Exp. 5/13/07, IRB00001138)

Copies: IRB File

JM:jt

12201 Research Parkway • Suite 501 • Orlando, FL 32826-3246 • 407-823-3778 • Fax 407-823-3299
An Equal Opportunity and Affirmative Action Institution

IRB Committee Approval Form

THE UNIVERSITY OF CENTRAL FLORIDA

INSTITUTIONAL REVIEW BOARD (IRB)

IRB Committee Approval Form

PRINCIPAL INVESTIGATOR(S):   Juli Dixon, Ph.D.   #06-4028

PROJECT TITLE: Prospective Teachers' Development of Understanding of Mathematical Concepts during a Classroom Teaching Experiment

[X] New project submission   [ ] Resubmission of lapsed project #_____
[ ] Continuing review of lapsed project #   [ ] Continuing review of #
[ ] Study expires   [ ] Initial submission was approved by expedited review
[ ] Initial submission was approved by full board review but continuing review can be expedited
[ ] Suspension of enrollment email sent to PI, entered on spreadsheet, administration notified _____

Chair   **IRB Reviewers:**
[X] Expedited Approval

    Dated: 12/14/06
    Cite how qualifies for   Signed _Tracy L Dietz_
    expedited review: Dr. Tracy Dietz, Chair
    minimal risk and ±6, #7

                    Signed: _____
[ ] Exempt   Dr. Craig Van Slyke, Vice-Chair

    Dated: _____   Signed: _____
    Cite how qualifies for   Dr. Sophia Dziegielewski, Vice-Chair
    exempt status:
    minimal risk and _____

[X] Expiration   *Complete reverse side of expedited or exempt form*
    Date: 12/13/07   [ ] Waiver of documentation of consent approved
                       [ ] Waiver of consent approved
                       [ ] Waiver of HIPAA Authorization approved

NOTES FROM IRB CHAIR (IF APPLICABLE): _____
_____
_____
_____
_____
_____
_____

IRB Protocol Submission Form

UCF IRB Protocol Submission Form

☐ Initial ☐ Resubmission of IRB #_____

Please type this form using the Microsoft Word document. Expand as needed. Allow a minimum of 2-3 weeks for the approval process. A letter of approval will be mailed to you once approved. Information on this form must match information on the grant application, dissertation or thesis, consent forms or letters, and flyers for recruitment. **There are no deadlines for submission of minimal risk studies as they are reviewed at least weekly.** If it is deemed by the IRB that the study involves greater than minimal risk or extenuating factors, the complete IRB packet must be submitted by the 1st business day of the month for consideration at that monthly IRB meeting. At title note if investigator is Student, Master's Candidate or Doctoral Candidate.

1. **Title of Protocol:** Prospective Teachers' Development of Understanding of Mathematical Concepts during a Classroom Teaching Experiment

2. **Principal Investigator:** [List the faculty supervisor as both the Principal Investigator and the faculty supervisor if student(s) or staff members are doing the research. List student(s) as co-investigator(s).]

Signature: _Jul K of_
Name: Dr. Juli Dixon
Mr./Ms./Mrs./**Dr.** (choose one)
Employee ID or Student PID #: 0409053
Degree: Ph.D.
Title: Associate Professor

Department: Teaching and Learning Principles
College: Education
E-Mail: jkdixon@mail.ucf.edu
Telephone: (407) 823-4140
Facsimile: (407) 823-1776
Home Telephone:

Co-Investigator(s):

Signature:
Name:
Mr./Ms./Mrs./Dr. (choose one)
Employee ID or Student PID #:
Degree:
Title:
Department:
College: Education
E-Mail:
Telephone:
Facsimile:
Home Telephone:

Signature:
Name:
Mr./Ms./Mrs./Dr. (choose one)
Employee ID or Student PID #:
Degree:
Title:
Department:
College: Education
E-Mail:
Telephone:
Facsimile:
Home Telephone:

Signature:
Name:
Mr./Ms./Mrs./Dr. (choose one)
Employee ID or Student PID #:
Degree:
Title:
Department:
College:
E-Mail:
Telephone:
Facsimile:
Home Telephone:

Signature:
Name:
Mr./Ms./Mrs./Dr. (choose one)
Employee ID or Student PID #:
Degree:
Title:
Department:
College:
E-Mail:
Telephone:
Facsimile:
Home Telephone:

3. **Supervisor:**

Signature: Name:

4. Collaborating institution(s) and researcher(s) (identify the institution and its FWA number, if known. List the names of collaborating researchers and briefly describe their roles in the study. Provide contact information. If the collaborating institution receives federal funds and does not have a federalwide assurance, a completed UCF Individual Investigator Agreement is required prior to approval.)

N/A

5. Dates of proposed project (cannot be retroactive)   From: 01/08/2007   To: 01/08/2008

6. Source of funding for the project (project title, agency, account/proposal # or "Unfunded"):

Unfunded

7. Scientific purpose of the investigation (dissertation or thesis is not the scientific purpose):

The purpose of this study is to investigate prospective elementary teachers' reasoning about and understanding of number concepts and operations and geometry and measurement using instructional sequences related to these mathematical ideas.

8. Describe the research methodology in non-technical language (the UCF IRB needs to know what will be done with or to the research participants – include audio/video taping – explain the who, what, when, where, why and how of the procedures you wish to implement).

Participants will be undergraduate students enrolled in MAE 2801: Instructional Mathematics for Elementary School. Consent forms will be collected from each prospective participant prior to data collection.

At the beginning of the Spring 2007 semester, an elementary mathematics content knowledge test (see attached) will be administered to students enrolled in one section of MAE 2801. The test will be administered at the beginning and end of the semester in a pre- post- format. This test will not be used as a grade for the class. To insure anonymity on the test, students will be randomly assigned an ID number. The list of numbers given to students will be kept in a locked cabinet and destroyed upon completion of the post-test such that no identifying record will exist after the study is over to link a student with a particular ID number. A sample of students will also be interviewed several times throughout the semester to document their growth in elementary mathematics content knowledge (see attached sample interview questions). Interviews will be semi-structured since pre-selected questions will be asked of all students, however individual interviews will vary as a result of interviewee responses. Interviews will be audiotaped and videotaped. The instructor and students in MAE 2801 will be audiotaped and videotaped during each class session to capture instructor/student interactions as well as student/student interactions. A research team including students enrolled in doctoral programs in the College of Education and other faculty in the College of Education will observe each class and take field notes to further document these interactions. Artifacts including students' coursework will be collected to further inform the research study. Student coursework will be collected, photocopied, and originals returned to the student with the exception of the final exam. The research team will meet between class sessions. These meetings will be audiotaped.

9. **Describe the potential benefits and anticipated risks and the steps that will be taken to minimize risks and protect participants** (risks include physical, psychological, social or economic harm - if there are no direct benefits and/or no risks, state that).

There are no anticipated benefits or risks to participation.

10. **Describe how participants will be recruited, how many you hope to recruit, the age of participants, and proposed compensation** (if any). When recruiting college students, you should state here that "Participants will be 18 years of age or older" if you want to avoid the need for a parental consent form.

Initially, students enrolled in a 9:30 a.m. section of MAE 2801 will be asked to participate in the research study. If all students agree, then research will be conducted during this section. However, if not all students agree, then students in an 11:30 a.m. section of MAE 2801 will be asked to participate in the study. Those students that choose to participate in the study will remain in the section. Those students that choose to not participate in the study will be rescheduled in a concurrent offering of the same course. It is anticipated that all participants will be 18 years of age or older. If they are not, parental consent and child assent will be obtained prior to conducting research.

11. **Describe the informed consent process** (include a copy of the informed consent document – if a waiver of documentation of consent is requested to make the study completely anonymous, include a consent form or informational letter with no signature lines or reference to signing).

Informed consent will be obtained from the students prior to data collection via a signed letter. If any students are under the age of 18, their parent/guardian will be contacted to obtain consent and the student will give assent via signed letter. See attached for consent and assent letters.

12. **Describe any protected health information (PHI) you plan to obtain from a HIPAA-covered medical facility or UCF designated HIPAA component** (include the completed UCF HIPAA Authorization Form or the UCF HIPAA Waiver of Authorization Form giving the details of the planned use or disclosure of the PHI. See the UCF IRB Web page for HIPAA details and forms).

N/A

I approve this protocol for submission to the UCF IRB. Signature: _Noel O. Will_ _12/1/06_

                                                    Department Chair/Director   **Date**

Cooperating Department (if more than one Dept. involved)___: _____/_____

                                                    Department Chair/Director   **Date**

Note: If required signatures are missing, the form will be returned to the PI unprocessed.

APPENDIX B: STUDENT INFORMED CONSENT LETTER

IRB Student Consent Form

January 8, 2007 (Student Consent)

Dear Student:

I am conducting a study, the purpose of which is to investigate the ways in which prospective elementary school teachers understand number and operation and geometry and measurement concepts. I am asking you to participate in this study because you have been identified as a student in one of the elementary mathematics content courses at UCF. Researchers will observe and videotape class sessions of the Instructional Mathematics for Elementary School (MAE 2801). Selected groups may also be audiotaped during class discussions.

Selected students will be asked to participate in several interviews lasting no longer than 15 minutes each. You will not have to answer any question you do not wish to answer. The interviews will be conducted at your convenience on campus after we have received a copy of this signed consent from you. With your permission, we would like to audiotape and videotape these interviews.

Only the research team will have access to the audio and video tapes, which may be professionally transcribed, removing any identifiers during transcription. The tapes will then be kept in a locked file cabinet. After we have received a copy of this signed consent from you, you will be asked to complete a questionnaire about your mathematics content knowledge at the beginning and the end of the course. Your name will not appear on the questionnaires, but a unique code will be used for identification purposes. Only the researchers will have access to the identification codes which will be destroyed after the end of course questionnaire. Copies of your course assignments may be used as data for this study. Additionally, video tape segments may be used in presentations and/or publications related to this study. Your name will be kept confidential and will not be revealed in the final manuscript(s) or any related presentations.

There are no anticipated risks, compensation or other direct benefits to you as a participant in this study. You are free to withdraw your consent to participate and may discontinue your participation in the study at any time without consequence.

If you have any questions about this research project, please contact Dr. Juli K. Dixon at (407) 823-4140 or jkdixon@mail.ucf.edu. Questions or concerns about research participants' rights may be directed to the IRB Coordinator, Institutional Review Board (IRB), University of Central Florida (UCF), 12201 Research Parkway, Suite 501, Orlando, Florida 32826-3246. The phone number is (407) 823-2901 and the fax number is (407) 823-3299. The hours of operation are 8:00 am until 5:00 pm, Monday through Friday except on University of Central Florida official holidays.

Please sign and return one copy of this letter. A second copy is provided for your records. By signing this letter, you give me permission to videotape you and report your responses anonymously in the final manuscript(s) You also give me permission to use videotape segments as a part of related publications and/or presentations.

Sincerely,

Juli Dixon, Ph.D.

_____ I have read the procedure described above for this research study.

_____ I agree to participate in the research.

_____ I do not agree to participate in this research.

_____ I confirm that I am 18 years or older.

_____ / _____

Participant   Date

APPROVED BY
University of Central Florida
Institutional Review Board

12/14/06

CHAIRMAN

IRB Parent Consent Form

January 8, 2007 (Parental Consent)

Dear Parent/Guardian:

Your child has been asked to participate in a study to investigate the ways in which prospective elementary school teachers understand number and operation that is being conducted with the University of Central Florida, College of Education. Your child's identifying information has not been shared in any way with the researcher at this time. Your child was chosen because he/she meets the criteria for this study and you, as parent or guardian, are being offered the opportunity to have your child participate.

With your consent, your child will be videotaped and audiotaped in the MAE 2801 course in which he/she is enrolled. Additionally, he/she may be interviewed by a member of the research team. Selected students will be asked to participate in several interviews lasting no longer than 15 minutes each. Your child will not have to answer any question they do not wish to answer. The interviews will be conducted at your child's convenience on campus after we have received a copy of this signed consent from you. With your permission, we would like to audiotape and videotape these interviews. Only the research team will have access to the audio and video tapes, which may be professionally transcribed, removing any identifiers during transcription. The tapes will then be kept in a locked file cabinet. After we have received a copy of this signed consent from you, your child will be asked to complete a questionnaire about your mathematics content knowledge at the beginning and the end of the course. Your child's name will not appear on the questionnaires, but a unique code will be used for identification purposes. Only the researchers will have access to the identification codes which will be destroyed after the end of course questionnaire. Copies of your child's course assignments may be used as data for this study. Additionally, video tape segments may be used in presentations and/or publications related to this study. Your child's name will be kept confidential and will not be revealed in the final manuscript(s) or any related presentations.

Your child will be allowed the right to refuse to answer any questions that make him/her uncomfortable, and he/she may stop participating in this research at any time. Your child will be reminded of this immediately prior to the interview. I have attached a copy of the interview questions for your information.

You may contact me at 407-823-4140 or email at jkdixon@mail.ucf.edu, for any questions you have regarding the research procedures. Research at the University of Central Florida involving human participants is carried out under the oversight of the Institutional Review Board. Questions or concerns about research participants' rights may be directed to the IRB Coordinator, Institutional Review Board (IRB), University of Central Florida (UCF), 12201 Research Parkway, Suite 501, Orlando, Florida 32826-3246. The phone number is (407) 823-2901 and the fax number is (407) 823-3299. The hours of operation are 8:00 am until 5:00 pm, Monday through Friday except on University of Central Florida official holidays.

Sincerely,

Juli K. Dixon, Ph.D.

_____ I have read the procedure described on the previous page.

I voluntarily give my consent for my child, _____, to participate in Juli Dixon's study entitled, "Prospective Elementary School Teachers Understanding of Number Concepts and Operations."

_____ / _____
Parent/Guardian   Date

_____ / _____
2nd Parent/Guardian   Date

APPROVED BY
University of Central Florida
Institutional Review Board

CHAIRMAN
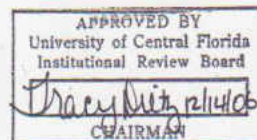
IRB Student Assent Form

ASSENT FORM

PROJECT: Prospective Teachers' Development of Understanding of Mathematical Concepts during a Classroom Teaching Experiment

RESEARCHER: Juli K. Dixon
CONTACT: Juli K. Dixon, 407-823-4140,

University of Central Florida, Department of Teaching and Learning Principles,
4000 Central Florida Blvd., Orlando, FL 32816

Please READ this explanation carefully, and ASK any QUESTIONS before signing.

You are being asked to participate in a research study. You will be asked to complete two brief surveys about number concepts and operations, one at the beginning and one at the end of your MAE 2801 course. You might also be asked to participate in an interview that will be no more than 15 minutes long. Your responses will be kept completely confidential, which means that your name will be separated from your answers. No one but me, Juli Dixon, and members of a research team will see your responses, so please try to answer honestly. The information will provide valuable knowledge about your understanding about numbers and operations. If you become uncomfortable at any time, please tell me immediately. Your participation in this project is voluntary, and YOU MAY STOP AT ANY TIME.

I volunteer to take part in this research study and know that I can quit any time I want to.

_____   _____
Signature of Student   Date

_____
Printed Name of Student

APPENDIX C: TASKS FROM INSTRUCTIONAL SEQUENCE

Sample Base-8 100's Chart

1							
				15			
		43					
						67	

Counting Problem Set # 1

1. There were 54 people in a movie theatre. 6 more entered the theatre. How many people watched the movie?

2. Marc has 12 marbles. He purchased 31 more at the store. How many marbles does he have in total?

3. There were 62 children in the band. 36 were boys and the rest were girls. How many girls were in the band?

4. There are 51 seagulls on the beach, 22 flew away. How many are still on the beach?

5. Greg has 45 playing cards. Melissa has 35 playing cards. How many cards do they have together?

6. There are 111 students at Jackson Elementary, 64 are on a field trip. How many students are still at school?

Counting Problem Set #2

1. Before lunch, you sold 37 cookies. After lunch you sold 45 cookies. How many cookies did you sell in the day?

2. Johnny bought 23 cookies. Steve bought some cookies, too. Johnny and Steve bought 52 cookies in all. How many cookies did Steve buy?

3. The local power company was buying cookies for their employees. They ordered some cookies on Monday and 243 cookies on Tuesday. They were billed for 422 cookies. How many did they order on Monday?

4. Mrs. Johnson brought 53 cookies for her class. She gave 25 cookies to Mr. Jones. How many cookies did Mrs. Johnson have left?

5. Jessica decided to share all her cookies with her friends. She gave 27 cookies to one friend and 43 cookies to another friend. How many cookies did she have to start with?

6. Susie had 154 crayons. She gave some crayons to her friend. She had 127 crayons left. How many did she give to her friend?

7. The Candy Shop made 237 cookies last night. The Cookie Company made 372 cookies last night. How many more cookies did the Cookie Company make than the Candy Shop?

Candy Shop 1

You own a candy shop in Base-8 World. Candy comes packaged in boxes, rolls, and individual pieces.
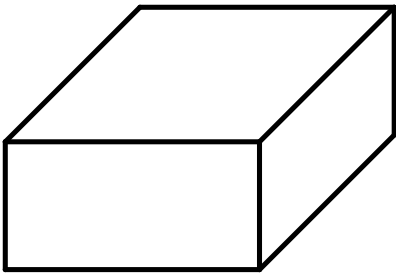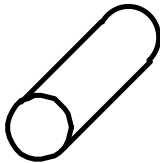
Box   Roll   Piece

There are 10 candy pieces in a roll and 10 rolls in a box.

Use this information to complete the following:

1. Show two different ways to represent the following:
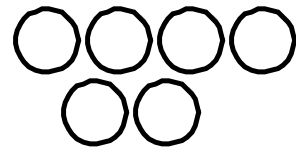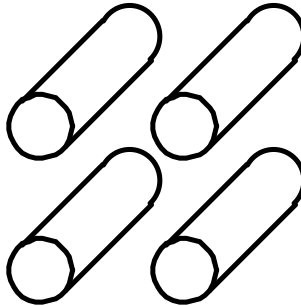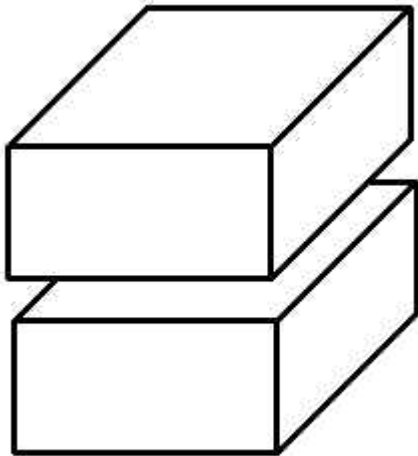
2.   Show two different ways to represent the following:
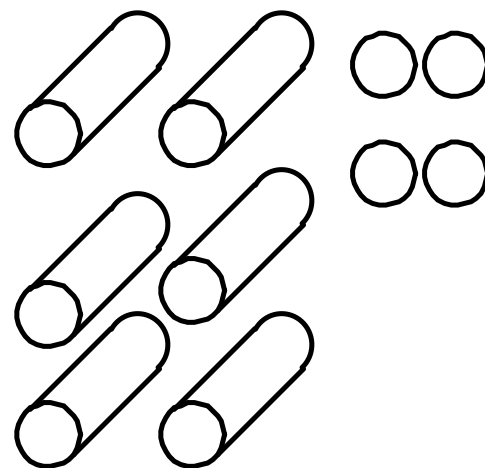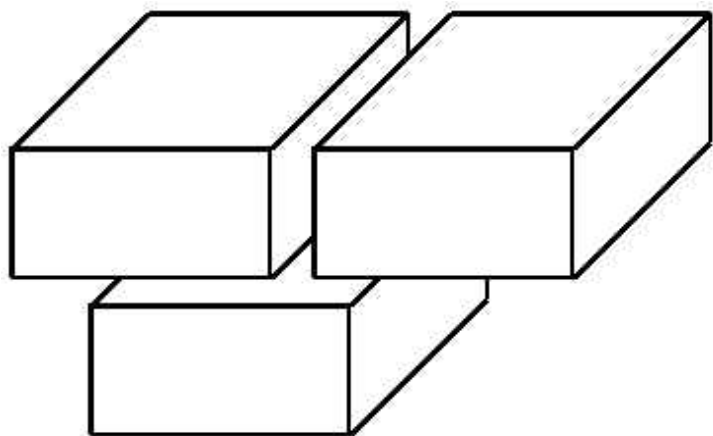
3. Show two different ways to represent the following:
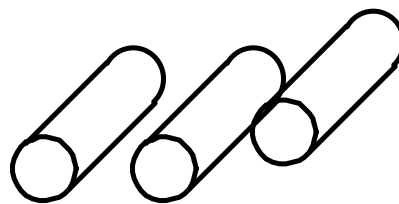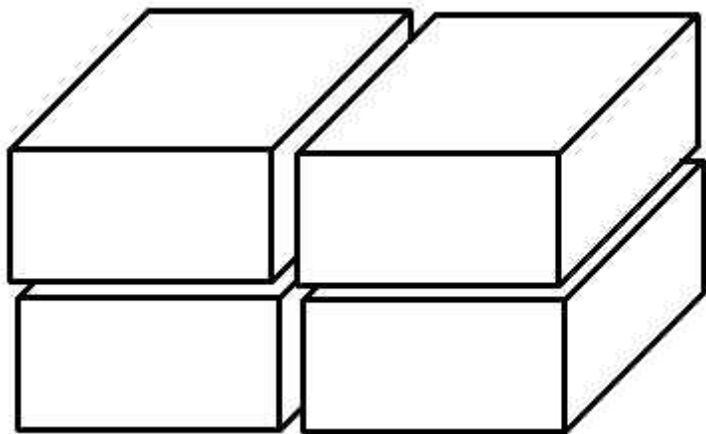
4. Show two different ways to represent 426 candies.

5. Show two different ways to represent 277 candies.

6. Show two different ways to represent 652 candies.

Torn Forms

Which forms represent the same quantities of candies? Write the quantities for each using single digits in each column.

Form A

B	R	P
1	36	7

Form B

B	R	P
0	57	37

Form C

B	R	P
3	27	37

Form D

B	R	P
0	0	347

Form E

B	R	P
2	6	107

Form F

B	R	P
0	27	57

Form G

B	R	P
2	33	77

Form H

B	R	P
2	10	47

Form I

B	R	P
2	17	77

Form J

B	R	P
2	32	107

Candy Shop Inventory

1. This many lemon candies are in the candy shop.
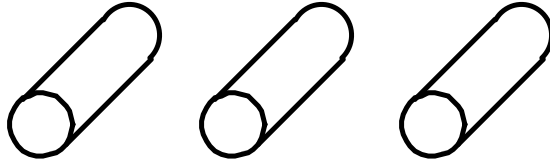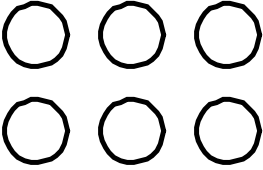
Mrs. Wright makes 23 more lemon candies. How many lemon candies are in the candy shop now?

2. This many chocolate candies are in the candy shop.
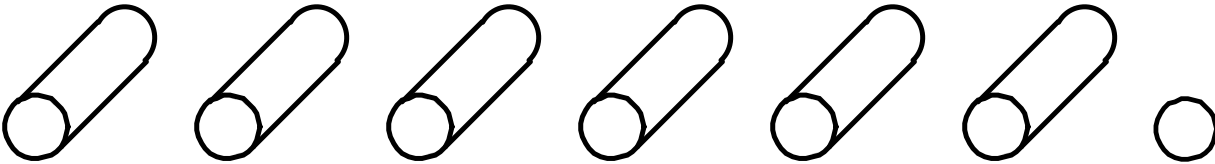
How would you unpack some of these candies so that you can sell 35 chocolate candies? How many chocolate candies will be left in the candy shop?

3. This many orange candies were in the candy shop.
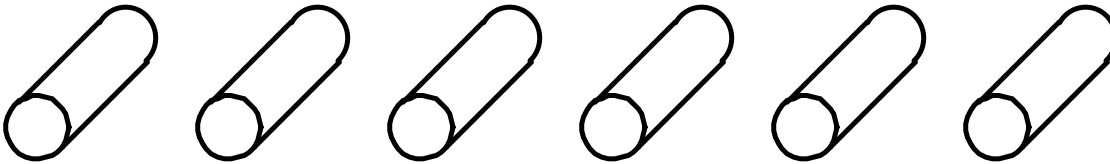
How would you unpack some of these candies so that you can sell 42 orange candies? How many orange candies will be left in the candy shop?

Candy Shop Addition and Subtraction

1.   There were 46 tangerine candies in the candy shop. Ms. Wright made 24 more tangerine candies. How many tangerine candies are in the shop now?

2.   There were 62 lemon candies in the candy shop. After a customer bought some there were only 25 lemon candies left in the shop. How many lemon candies did they buy?

3.   There were 34 grape candies in the candy shop. After Ms. Wright made some more there were 63 grape candies in the shop. How many more grape candies did she make?

4.   There were 53 orange candies in the candy shop. A customer buys 25 candies. How many orange candies are in the shop now?

Inventory Forms for Addition and Subtraction (In Context)

1. This many candies were in the store room.

Boxes	Rolls	Pieces
2	3	6

The factory makes 146 more candies. How many candies are in the store room now?

2. This many candies are in the store room.

Boxes	Rolls	Pieces
3	0	4

A customer orders 136 candies. How many candies will be left in the store room?

3. This many candies are in the store room.

Boxes	Rolls	Pieces
3	1	4

A customer orders 145 candies. How many candies will be left in the store room?

4. This many candies are in the store room.

Boxes	Rolls	Pieces
4	2	5

The factory makes 256 more candies. How many candies are in the store room now?

Inventory Forms for Addition and Subtraction (Out of Context)

Boxes	Rolls	Pieces
5	0	0
- 2	4	3

Boxes	Rolls	Pieces
5	1	1
- 2	6	3

Boxes	Rolls	Pieces
5	2	0
-2	5	3

Boxes	Rolls	Pieces
4	1	7
-2	5	3

Boxes	Rolls	Pieces
2	5	5
+2	5	3

Boxes	Rolls	Pieces
2	6	3
+2	4	6

Broken Machine

Ms. McLaughlin operates a machine that puts 10 sticks of gum in a pack.

    1. How many sticks of gum are in 5 packs?

The machine breaks down and now only puts 7 sticks in a pack.

    2. How many sticks of gum are in 5 packs?

    3. How many sticks of gum are in 7 packs?

After Ms. McLaughlin tries to fix the machine, she makes a mistake and now the machine only puts 6 sticks in a pack.

    4. How many sticks of gum will be in 5 packs?

    5. How many sticks of gum will be in 6 packs?

Mr. Strawberry has a machine for putting candies into bags. The machine is made to put 20 candies in each bag. The machine is not working correctly and is putting only 17 candies in each bag.

    6. How many candies does the machine use for 5 bags?

    7. How many candies does the machine use for 6 bags?

Mr. Strawberry tries to fix the machine but it gets worse. Now it puts only 16 candies in each bag.

8.   Now how many candies does the machine use for 6 bags?

9.   How many candies does the machine use for 3 bags?

Later that day, Mr. Strawberry tries to fix the machine yet again. Now it puts 22 candies in each bag.

10. How many candies does the machine use for 10 bags?

11. How many candies does the machine use for 6 bags?

Multiplication Scenario

A marketing team has created three new prototypes for an egg carton. Explain and justify how many eggs would fit in each carton?

a. 5 by 6 egg carton

b. 6 by 12 egg carton

c. 3 by 16 egg carton

Multiplication Word Problems

Solve each of the problem situations below. Draw pictures if that will help you understand the situation and solve the problem. Remember we are still in 8-world.

1. Kris has 7 packages of baseball cards. Each package contains 13 cards. How many cards does Kris have?

2. Margaret places 6 marbles in each of 7 cups. How many marbles did she place in cups?

3. There are 4 classes of fifth grade at Everett Elementary School. Each class contains 25 children. How many fifth graders are there at Everett Elementary?

4. Maria is putting 22 stickers on a page in rows of three. She has already made 4 rows. How many more rows will she make?

5. Brenda is putting 50 stickers on a page in rows of four. She has already made some rows and has 6 rows left to make. How many rows has she already made?

Division Word Problems

Solve each of the problem situations below. Draw pictures if that will help you understand the situation and solve the problem. Remember we are still in 8-world.

1. Katrina brings 52 marbles to school to give to her friends. She plans to give each of 10 friends the same number of marbles. How many marbles will each friend get? Will Katrina have any marbles left? If so, how many?

2. Jason has 43 pencils to share with some of his class. There are 5 students in his class that he would like to give his pencils to. How many pencils does each friend get?

3. Sarah has 125 candies. She wants to give each of her friends 12 candies. How many friends can she share with? Does she have any candies left for herself? If so, how many?

4. Micah has some friends he wants to share his stickers with. He has 236 stickers. How many friends can he share them with if he wants each friend to get 14 stickers? How many stickers, if any, does Micah have left?

Create Your Own Base-8 Problems

For this activity, you will be creating four word problems in base 8. Below, write one word problem for addition, one for subtraction, one for multiplication, and one for division. Once you have created these problems, solve them on a separate sheet of paper. After you have found the solutions, trade with someone else in the class and solve theirs.

1.

2.

3.

4.

APPENDIX D: INTERVIEW QUESTIONS

Sample Interview Questions

These questions will be given in a semi-structured interview that is scheduled with the students.

Questions 1 – 4: Given a collection of base ten blocks (with fewer than necessary unit blocks) (Ross, 1986)

1. Can you show me 254 with these blocks?

2. Can you show me 254 in a different way?

3. Can you show me 254 with the fewest number of blocks? How is this related to the number 254?

4. What does the 2 in 254 mean?

5. Some fifth-grade teachers noticed that several of their students were making the same mistake in multiplying large numbers. In trying to calculate

$$\begin{array}{r} 123 \\ \underline{\times\ 645} \end{array}$$

The students seemed to be multiplying incorrectly. They were doing this:

$$\begin{array}{r} 123 \\ \underline{\times\ 645} \\ 615 \\ 492 \\ \underline{738} \\ 1845 \end{array}$$

What is the students' misconception? How would you approach this misconception with students? (Ma, 1999)

For the following questions, students will be interviewed in class or immediately after class. The purpose will be to have them elaborate on their strategies for solving problems given in class. The questions will probe their thinking and be related to the problem. Examples of questions are given. These interviews will not be pre-scheduled.

1. How did you decide what to do to solve the problem?

2. Did you try any other strategies before you found the answer?

3. How do you know your answer is correct?

4. How would you explain your solution to a student?

5. Can you explain why your method works?

6. Will your strategy always work?

7. Can you solve the problem another way?

8. What would you do if the problem was [give revised problem]?

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