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Preservice Elementary Teachers' Diverlopment Of Rational Number Understanding Through The Social Perspective And The Relationship Among Social And Individual Environments

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PRESERVICE ELEMENTARY TEACHERS' DEVELOPMENT OF RATIONAL NUMBER UNDERSTANDING THROUGH THE SOCIAL PERSPECTIVE AND THE RELATIONSHIP AMONG SOCIAL AND INDIVIDUAL ENVIRONMENTS

by

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ABSTRACT

A classroom teaching experiment was conducted in a semester-long undergraduate mathematics content course for elementary education majors. Preservice elementary teachers' development of rational number understanding was documented through the social and psychological perspectives. In addition, social and sociomathematical norms were documented as part of the classroom structure.

A hypothetical learning trajectory and instructional sequence were created from a combination of previous research with children and adults. Transcripts from each class session were analyzed to determine the social and sociomathematical norms as well as the classroom mathematical practices. The social norms established included a) explaining and justifying solutions and solution processes, b) making sense of others' explanations and justifications, c) questioning others when misunderstandings occur, and d) helping others. The sociomathematical norms established included determining what constitutes a) an acceptable solution and b) a different solution. The classroom mathematical practices established included ideas related to a) defining fractions, b) defining the whole, c) partitioning, d) unitizing, e) finding equivalent fractions, f) comparing and ordering fractions, g) adding and subtracting fractions, and h) multiplying fractions.

The analysis of individual students' contributions included analyzing the transcripts to determine the ways in which individuals participated in the establishment of the practices. Individuals contributed to the practices by a) introducing ideas and b) sustaining ideas. The transcripts and student work samples were analyzed to determine

the ways in which the social classroom environment impacted student learning. Student learning was affected when a) ideas were rejected and b) ideas were accepted.

As a result of the data analysis, the hypothetical learning trajectory was refined to include four phases of learning instead of five. In addition, the instructional sequence was refined to include more focus on ratios. Two activities, the number line and between activities, were suggested to be deleted because they did not contribute to students' development.

This is dedicated to my family who has supported me throughout my whole educational process.

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LIST OF ACRONYMS/ABBREVIATIONS

IRB	Institutional Review Board
HLT	Hypothetical Learning Trajectory
RME	Realistic Mathematics Education

CHAPTER ONE: INTRODUCTION

The quality of mathematics instruction in United States' K-12 classrooms is the focus of many reform efforts (National Mathematics Advisory Panel, 2008; NCTM, 2000). Though studies show that teachers with a deep understanding of mathematics positively impact student achievement (Hill, Rowan, & Ball, 2005; Kaplan & Owings, 2000), little research documents how classroom teachers develop the knowledge base they need to be effective. The knowledge base of effective teachers, which includes using students' knowledge to inform instructional decisions, is beyond the experiences typically received in preservice teacher mathematics education classes (Darling-Hammond, Wei, Andree, Richardson, & Orphanos, 2009; NCTM, 2000). In order for teachers to assess students' knowledge accurately, they themselves need a deep understanding of the content. This is especially important for elementary teachers, as they typically do not have a substantive mathematics background.

One of the most difficult topics for elementary students to learn and teachers to teach is rational numbers (Behr, Wachsmuth, Post, & Lesh, 1984; Lamon, 1996; Mack, 1990, 1995; Ni & Zhou, 2005; Post, Wachsmuth, Lesh, & Behr, 1985). Rational numbers are important because they enhance students' abilities to solve real-world problem situations, are necessary for an increased mathematical understanding, and provide a foundation for algebraic thinking (Behr, Lesh, Post, & Silver, 1983; National Mathematics Advisory Panel, 2008). The difficulties students have with learning rational numbers include transferring whole number concepts incorrectly to rational numbers, such as $\frac{1}{6}$ is bigger than $\frac{1}{5}$ because six is bigger than five and that $\frac{1}{2}+\frac{3}{4}=\frac{4}{6}$ because 1

+ 3 = 4 and 2+ 4 = 6 (Behr et al., 1984; Ni & Zhou, 2005; Streefland, 1991). To help students overcome these difficulties, teachers themselves need a deep understanding of the topic. Unfortunately, studies show that preservice and inservice teachers harbor many of the misunderstandings that mathematics educators hope children do not develop (Ball, 1990b; Borko et al., 1992; Ma, 1999; Tirosh, 2000). For example, Ball (1990b) found that the majority of preservice elementary teachers could correctly solve a fraction division problem; however, these same preservice teachers could not present a correct model for the situation. Yet, little research documents how preservice and inservice teachers overcome these misconceptions. Exceptions include Wheeldon (2008).

Statement of the Problem

Teachers are entering the profession without a profound understanding of the mathematics they are to teach (Ma, 1999). Within the context of rational numbers, several studies document how preservice and inservice teachers' understandings are procedurally based and largely misunderstood (Ball, 1990a, 1990b; Borko et al., 1992; Ma, 1999; Tirosh, 2000). Though procedures may be sufficient for functioning from day-to-day, a teacher's knowledge should be different from the knowledge the everyday person needs to know.

Shulman (1986) introduced the term pedagogical content knowledge to refer to the knowledge base that teachers must have to be effective in promoting conceptual understandings within their own students. Pedagogical content knowledge includes a comprehension of the subject matter such that student understandings are accurately assessed, meaningful activities are presented, and student knowledge is built upon

(Shulman, 1986, 1987). Studies show that teachers who have this deep understanding are able to create a more productive mathematics learning environment (Sowder & Philipp, 1999).

Unfortunately, many preservice teachers believe that knowing only procedures is enough to constitute a deep understanding of mathematics (Ball, 1990a). To aid in children's development of a conceptual understanding of mathematics, teachers should understand rational numbers deeply enough to "be able to represent [them] appropriately and in multiple ways" (Ball, 1990a, p. 458). Thus, a mathematics content course within preservice teacher education programs should present more than just a reiteration of rules and procedures. However, research on the curriculum needed in teacher education programs to aid in preservice teachers' development of these understandings is limited.

Significance of Study

This study was a subsequent iteration of a previous study which sought to add to the limited research documenting preservice teachers' development of rational number understanding (Wheeldon, 2008). Using qualitative methods, data collected from a classroom teaching experiment were analyzed. The purpose of this study was to expand prior research (Wheeldon, 2008) by documenting the ways in which preservice elementary teachers developed an understanding of rational number concepts and operations as a collective group as well as to document the ways in which the social and individual aspects of a classroom impacted one another.

This study was conducted in a semester-long mathematics content course for preservice elementary teachers. The rational number unit was presented after a whole number unit in base 8 and constituted 10 days of instruction. The class met twice per week for 110 minutes per session.

This study incorporated a design-based research design, which "involves both developing instructional designs to support particular forms of learning and systematically studying those forms of learning within the context defined by the means of supporting them" (Cobb, 2003, p. 1). Previous research with children's and adults' understanding of rational number concepts and operations (Lamon, 1993; Mack, 1990, 1995; Wheeldon, 2008) informed a hypothetical learning trajectory (Simon, 1995). The hypothetical learning trajectory for this study then led to the development of an instructional sequence for rational number concepts and operations.

Research Focus

This study focused on the ways which the social context of the classroom and individual students' learning impacted one another. The emergent perspective (Cobb & Yackel, 1996), which coordinates both social and individual aspects of a classroom community, was the interpretive framework by which social activity was analyzed. The ways in which the social and individual environments impacted one another were analyzed through the constant comparative method (Glaser & Strauss, 1967).

The social context of the classroom, which includes a) classroom social norms, b) sociomathematical norms, and c) classroom mathematical practices, were determined through analysis of video-taped class sessions and field notes. The ways in which tools supported the classroom mathematical practices were also determined through whole-class discussions and student work samples. The interaction between social and individual environments, which includes a) individual students' contributions to the practices and b) students' knowledge reorganization, were examined through students' participation in classroom conversations and their coursework. Specifically, the research questions were:

1. In what ways do classroom mathematical practices develop related to rational numbers?
2. In what ways do the social and individual environments impact one another?
3. In what ways does the instructional sequence facilitate the development of preservice elementary teachers' rational number understanding?

Conclusion

Rational numbers were chosen as the focus for this study because they enhance students' mathematical understanding and are the basis for algebraic thinking. In addition, the research documenting the experiences preservice teachers need in order to develop a deep understanding of rational number concepts and operations is limited (Wheeldon, 2008).

Chapter two provides a synthesis of research, including an overview of the meanings of rational numbers. Next, rational number concepts are discussed. This is followed by a discussion of children's and adults' rational number learning and misconceptions. The chapter concludes with an overview of tools and hypothetical learning trajectories.

Chapter three includes the research methodology used for this study. Background on design-based research and classroom teaching experiments are presented. This is followed by an in-depth discussion of the hypothetical learning trajectory and instructional sequence used in this study. The chapter concludes with a discussion of the data that were collected and a summary of how the data were analyzed.

Chapter four includes the results from the social analysis. First the social and sociomathematical norms that were established are discussed. This is followed by an analysis of the classroom mathematical practices in terms of overarching topics. The ways in which the social and individual environments impacted one another are also illustrated through the discussion of the classroom mathematical practices.

Chapter five includes the ways in which the analysis led to the revisions of the instructional sequence and hypothetical learning trajectory. Implications for future research are presented.

CHAPTER TWO: LITERATURE REVIEW

Rational number concepts and operations are not easy to grasp. Traditionally, rational number curricula present topics in such a way that one only needs to memorize a procedure for how to solve a problem to be successful. Several research studies have illustrated that when children and adults are only presented with procedural rules, their knowledge of rational numbers is fragmented and incorrect (Ball, 1990a, 1990b; Behr et al., 1983; Behr et al., 1984; Erlwanger, 1973; Kajander, 2005; Kieren, 1976; Lamon, 1996; Mack, 1990, 1995). Thus, more research is needed to develop a comprehensive rational number curriculum that aids in children's and adults' conceptual understanding of the topic.

Though more research is needed with both children and adults, this study seeks to add to the research on preservice teachers' development of rational numbers. Research with preservice teachers is largely focused on fraction division as opposed to an entire rational number curriculum (Ball, 1990a, 1990b; Tirosh, 2000; Tirosh & Graeber, 1990), with the exception of Wheeldon (2008), which analyzed preservice teachers' learning and development of rational number concepts and operations. Since research with adults is limited, research with children's learning and understanding of rational numbers needs to be included in order to initially determine the ways in which rational number learning might progress.

The difficulties that arise with rational number learning and development, "result from deficiencies in the curricular experiences provided in school" (Behr, Harel, Post, &

Lesh, 1992, p. 300). Before these deficiencies can be eradicated, a more complete understanding of rational numbers is needed.

This chapter starts with an overview of rational numbers including a discussion of the five subconstructs. This is then followed by a discussion of the tasks needed to support the subconstructs. The next section is devoted to children's and adults' learning and understanding of rational number concepts and operations. This chapter concludes with discussions of curriculum implications for preservice teachers and hypothetical learning trajectories.

Overview of Rational Numbers

Vergnaud (1983) used the term conceptual field to describe "a set of problems and situations for the treatment of which concepts, procedures, and representations of different but narrowly interconnected types are necessary" (p. 127). He defined a conceptual field of multiplicative structures as "simple and multiple proportion problems, which include linear and n-linear functions, vector spaces, dimensional analysis, fraction, ratio, rate, rational number, and multiplication and division" (Vergnaud, 1983, p. 141). Though several of these structures are beyond the realm of this study, it is important to note that rational numbers fall under this category and alone encompass a variety of topics.

How rational numbers should be "understood lie in the *many related but only partially overlapping ideas that surround them"* (Ohlsson, 1988, p. 53). A complete understanding of each rational number component individually and how they interrelate is needed in order to have a comprehensive understanding of rational numbers (Kieren, 1976; Post, Cramer, Behr, Lesh, & Harel, 1993; Vergnaud, 1983). However, it is impossible to have a complete understanding of rational numbers because the definition for rational numbers and its components still remain ambiguous (Ohlsson, 1988).

Rational Number Subconstructs

Mathematically, the concept of rational numbers has been defined "as an equivalence class of ordered pairs of whole numbers" (Vergnaud, 1983, p. 160), where an ordered pair is written in the form a/b such as $\left\{\frac{1}{2},\frac{2}{4},\frac{3}{6}...\right\}$ (Kieren, 1976). Educationally, this definition is not sufficient. Research has shown that rational numbers are comprised of more than just equivalence classes (Behr et al., 1983; Kieren, 1976, 1980; Nesher, 1985; Ohlsson, 1988).

Kieren (1976) first introduced this idea when he proposed the following seven interpretations of rational numbers:

- "Rational numbers are fractions which can be compared, added, subtracted, etc.
- Rational numbers are decimal fractions which form a natural extension (via our numeration system) to the whole numbers.
- Rational numbers are equivalence classes of fractions. Thus, {1/2, 2/4, 3/6, …} and {2/3, 4/6, 6/9, …} are rational numbers.
- Rational numbers are numbers of the form p/q, where p, q are integers and $q \neq 0$. In this form, rational numbers are "ratio" numbers.
- Rational numbers are multiplicative operators (e.g., stretchers, shrinkers, etc.).
- Rational numbers are elements of an infinite ordered quotient field. They are numbers of the form $x = p/q$ where x satisfies the equation $qx = p$.
- Rational numbers are measures or points on a number line."

(p.102-103)

Later, Kieren (1980) used these seven interpretations to define the five meanings, or the subconstructs, of rational numbers to be part-whole, quotient, measure, ratio, and operator. This was then revised to the last four subconstructs as the part-whole interpretation of rational number is subsumed under the quotient and measure constructs (Kieren, 1993).

Behr, Lesh, Post, and Silver (1983) used Kieren's meanings of rational numbers as part of the theoretical foundation for the formulation of the Rational Number Project as well as to develop their own definition of rational numbers. Behr et al. defined rational numbers as a fractional measure or part-whole relation, a ratio, a rate, a quotient, a linear coordinate or measure, a decimal, and an operator. After conducting their study incorporating these eight meanings, Behr et al. concluded from their results that there are only five meanings or subconstructs of rational numbers, which agreed with Kieren's (1980) subconstructs. These subconstructs were again tested when Nesher (1985) proposed that in addition to the part-whole, quotient, operator, measure, and ratio interpretations, a rational number can also be defined as a probability.

Research is consistent in that the "quotient, ratio, operator, and some version of the part-whole relation are central concepts of rational numbers" (Ohlsson, 1988, p. 56). Though others have proposed different subconstructs of rational numbers (Nesher, 1985; Ohlsson, 1988; Vergnaud, 1983), Behr, Harel, Post, and Lesh (1992) note that "the part-whole, quotient, measure, operator, and ratio subconstructs have, to some extent, stood the test of time, and still suffice to clarify the meaning of rational number" (p. 298). Though these subconstructs seek to clarify rational numbers, a cohesive definition for rational numbers and for each of these subconstructs still does not exist. As a result, it is not clear what topics within rational numbers should be taught and how. Since the part-whole, quotient, ratio, operator, and measure subconstructs are used to define rational numbers (Behr et al., 1992), they were all included within this study.

Part-Whole

The part-whole subconstruct of rational numbers is defined as a representation describing how many equal sized pieces out of the total number of equal sized pieces there are in a specified whole (Lamon, 2005), which is written in the form of a/b (Behr et al., 1983). For example, $\frac{3}{4}$ means to take three pieces out of the four equal pieces it takes to make the whole. The part-whole meaning directly results from partitioning both continuous and discrete situations into equal-sized pieces and is instrumental in understanding all of the other rational number subconstructs (Behr et al., 1983). This subconstruct pertains to representations that are less than or equal to one as the definition implies that “the numerator of the fraction must be less or equal to the denominator” (Charalambous & Pitta-Pantazi, 2007, p. 296).

Charalambous and Pitta-Pantazi (2007) describe several underlying concepts associated with the part-whole meaning. These include, “(a) the parts, taken together, must exhaust the whole, (b) the more parts the whole is divided into, the smaller the produced parts become, and (c) the relationship between the parts and the whole is conserved, regardless of the size, shape, arrangement, or orientation of the equivalent parts” (p. 296). Underlying all of these concepts is the concept of keeping track of the unit (Simon, 1993), or defining the whole. An understanding of how 1/3 of one whole may not necessarily be equivalent to 1/3 of a different whole provides a foundation for all other rational number topics. An inability to define the whole confounds misunderstandings in later tasks, such as partitioning, and the tasks become difficult to accurately assess as a result (Moskal & Magone, 2002).

Though studies have documented that children come into the classroom with an informal understanding of the part-whole relationship (Mack, 1990, 1993), the part-whole subconstruct should not be the only meaning of rational numbers taught (Vanhille & Baroody, 2002). Students who are only exposed to the part-whole meaning will not adequately develop the other meanings of rational numbers.

Quotient

As with the part-whole subconstruct, the quotient meaning of rational numbers also derives from partitioning situations. Partitioning is "the major cognitive structure underlying the notion of quotient" (Kieren, 1976, p. 121). The quotient subconstruct is used in two forms. The quotient can refer to either the posed partitioning situation or the result. For example, when sharing three pizzas equally among four people, each person receives three-fourths of a pizza. It is important to note that $\frac{3}{4}$ represents both the problem and its solution, however, in the posed problem $\frac{3}{4}$ refers to 3 pizzas ÷ 4 people, and in the solution $\frac{3}{4}$ describes how much of a whole each person receives. Within the solution, the quotient and part-whole subconstructs intertwine, as $\frac{3}{4}$ refers to the three out of four equal pieces that everyone receives. Specific to the quotient subconstruct, the solution of $\frac{3}{4}$ also refers to the quantity each person gets. It is in this respect that the quotient and part-whole subconstructs are separated.

A rational number as a quotient also introduces and leads to an understanding of the meaning of mixed numbers, or fractions greater than one (Charalambous & Pitta-Pantazi, 2007). For example, when partitioning, or sharing five pizzas among three people, each person will get a whole pizza plus two-thirds of another pizza or $1\frac{2}{3}$ pizzas. Instead of sharing one whole pizza with everyone first, each pizza can be cut into thirds. Within this situation, the solution would be $\frac{5}{3}$, where each person receives five one-third size pieces of pizza. Students, who are only introduced to the part-whole meaning, will say that this is impossible because you cannot take five pieces out of three (Streefland, 1991). Having a partitioning situation with an answer greater than one will also lead to the discovery of converting mixed numbers into fractions greater than one, and vice versa.

A complete understanding of the quotient subconstruct also aids in students' conceptualization of unitizing processes. By conceptualizing the unit as a group of 3, a connection can be made from breaking a whole group into four equal parts to taking four of those equal parts to make the whole. This reciprocal understanding is needed to conceptualize other rational number subconstructs, such as ratios (Lamon, 1993).

Ratio

Ratios are defined as representing part-whole or part-part situations where the quantities are somehow related (Marshall, 1993; Van de Walle, Karp, & Bay-Williams, 2010). Though ratios can represent part-whole and part-part situations, it is "considered a comparative index rather than a number" (Behr et al., 1983, p. 95). When a ratio consists of two unrelated quantities, it is called a rate (Van de Walle et al., 2010). It is in this respect, that ratios are similar to partitioning problems, such as three pizzas for four people.

Ratios introduce what Noelting (1980b) describes as the *between* and *within* relationships of rational numbers. The between relationship "leads to the Common Denominator algorithm" (Noelting, 1980b, p. 338), because the relationship between two or more ratios is analyzed. Two ratios are found to be equivalent when the same nonzero number can be multiplied by both of the individual quantities in one ratio and the result is the other ratio. For example, three pizzas for nine people and six pizzas for eighteen people are equivalent ratios because three times two is six and nine times two is eighteen.

The within relationship analyzes the correlation of the individual quantities that comprise a single ratio. Two ratios that have the same multiplicative relationship within these quantities are equivalent. For example, three pizzas for nine people and six pizzas for eighteen people are equivalent ratios because three times three is nine and six times three is eighteen. The two relationships are illustrated below (See Figure 1).

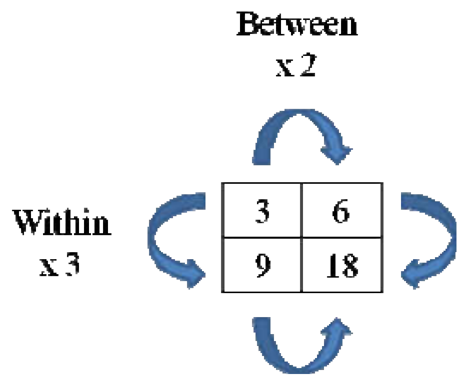


Figure 1: Between and Within Relationships

The within relationship is used to determine the relative magnitude or size of the ratio. This is especially useful in situations in which two or more rational numbers are to be compared to one another.

The between and within relationships reinforce the multiplicative nature of rational numbers. Unlike whole numbers, where discrete wholes can be added to generate another whole number, rational numbers require multiplication to generate another rational number. The cognitive shift required to move from additive thinking to this type of multiplicative reasoning, especially within the context of ratios, is a difficult transition to make. As a result, ratios are a not an easy topic to learn. Though ratios are initially difficult to comprehend, a "mathematics curriculum must not wait … to advance multiplicative concepts, such as ratio and proportion. These principles must be introduced early when considering additive situations" (Post et al., 1993, p. 331). Introducing ratios early will aid in the transition required to move from additive to multiplicative reasoning. Being able to reason multiplicatively will also aid in the development of learning the operator subconstruct.

Operator

The operator subconstruct is thought of as a function that acts on an object. Behr et al., (1983) note this as being a "function that transforms" (p. 96) where an object is "stretched or shrunk, contracted or expanded, enlarged or reduced, or multiplied and divided" (Lamon, 2005, p. 151). The operator subconstruct also leads to the notion of proportion, composition, and identity or inverse (Kieren, 1976). Within multiplication situations the operator subconstruct "leads naturally to the group properties" (Kieren, 1976, p. 117). In a "groups-of" situation, the first fraction acts on the second. For example, when taking three-fourths of a group of one-half, the three-fourths acts on the one-half, such that you start with one-half and then take three-fourths of it. Symbolically this is written as $\frac{3}{4}\times\frac{1}{2}$, and the solution of $\frac{3}{8}$ is the result of cutting $\frac{1}{2}$ into four equal pieces, then taking three of those pieces and describing them in terms of the original whole.

Incorporating the operator subconstruct within a rational number curriculum not only "highlights certain algebraic properties, particularly those relating the multiplicative inverse and identity elements, but also provides experience with the notion of composite functions in a fairly concrete way" (Kieren, 1993, pp. 59-60). Though important, the notion of an operator is another subconstruct that is difficult to comprehend because of its multiplicative nature (Kieren, 1976).

Measure

The measure subconstruct introduces fractions as a length or distance, such as three-fourths of a mile. Using fractions as measures introduces number lines to rational numbers, which “adds an attribute not present in region or set models particularly when a number line of more than one unit long is used” (Behr et al., 1983, p. 94). The rational number as a measure incorporates an iteration process such that a unit is partitioned into a composite set of individual equal measures (Kieren, 1980). This “notion of ‘flexible partitioning of the unit’ allows the algebraic notions of operation and equivalence to emerge” (Kieren, 1976, p. 124). For example, within an addition situation such as $\frac{1}{2}+\frac{1}{4}$, the process of adding these fractions as measures requires the coordination of two vectors of length $\frac{1}{2}$ and $\frac{1}{4}$. This is done such that the vectors are placed end-to-end and the place on the number line where the second vector ends must be “on an exact division of the unit” (Kieren, 1976, p. 124). This ‘division’ is then the solution to the problem. There are three underlying cognitive structures related to the rational number as a measure. In addition to unit concepts, equivalence and order relationships are introduced. When partitioning the same unit into thirds, and subsequently into sixths, one can see that a sixth is half of a third, thus every group of two-sixths is equivalent to a group of one-third. This process can show how one-sixth will come before one-third when ordering from least to greatest.

Each of the five subconstructs highlights a different approach to interpreting rational numbers. It is important to be able to distinguish and understand each subconstruct as they are needed to develop a complete understanding of rational numbers. The table below summarizes each of the five subconstructs and illustrates how $\frac{3}{4}$ is interpreted under every subconstruct.

Table 1: Meanings of Rational Numbers

Subconstruct	Definition	Example: ¾
Part-Whole	A number of specified pieces out of the total number of equal sized pieces it takes to make the whole	Three pieces out of the four equal pieces it takes to make the whole
Quotient	A specified number of objects divided into a set number of equal groups. OR The solution of how much of one object is in each group when the division is carried out	Problem: Three objects divided into four equal groups Solution: Each group receives three-fourths of one object
Ratio	A comparison of two distinct quantities	Three pizzas for every four people
Operator	A transformation function that alters the size of the original figure/object	Three-fourths of the original size of an object
Measure	A quantity that defines a distance	Three-fourths of a mile

How the Subconstructs are Intertwined

Though the subconstructs are currently defined as just described, ambiguity with how the subconstructs are related still exists. Behr, et. al. (1983) proposed the following diagram (See Figure 2) to describe how the five subconstructs are related and to which mathematical understandings they foster.

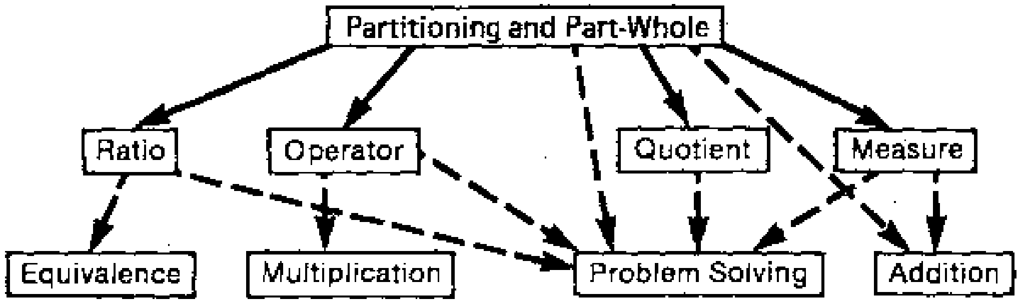


Figure 2: Relationship of Subconstructs

Within this diagram, the solid lines are established relationships, and the dashed lines represent hypothesized relationships (Behr et al., 1983). According to Behr et al., an understanding of the part-whole subconstruct and the act of partitioning together directly lead to the development of understanding each of the other four subconstructs. Thus, the ratio, operator, quotient, and measure subconstructs are subsets of partitioning and the part-whole subconstruct. Each of these subconstructs then lead naturally to varying tasks and concepts, such as multiplication and problem-solving.

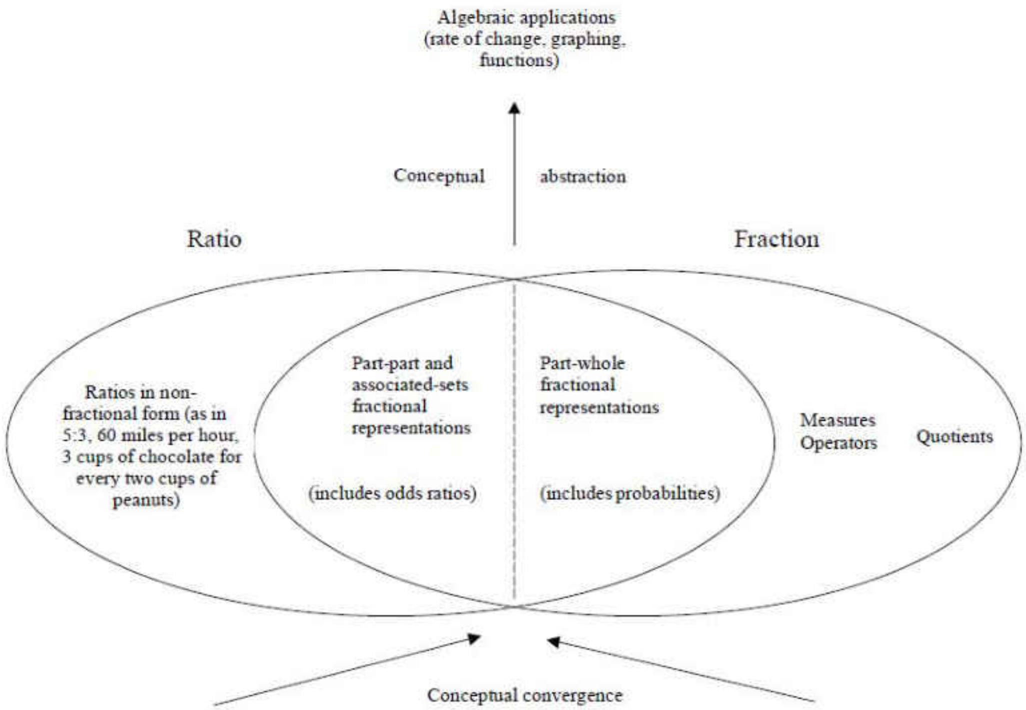
One of the big questions still surrounding these relationships is how ratios and fractions are related. Arguments have been made for ratios being a subset of fractions, fractions being a subset of ratios, fractions and ratios as distinct sets, fractions and ratios as identical sets, and fractions and ratios as overlapping sets (Clark, Berenson, & Cavey, 2003; Marshall, 1993; Van de Walle et al., 2010). Clark, Berenson, and Cavey illustrated that each of these five arguments have been used in mathematics at some point.

The first model, ratios as a subset of fractions, implies “that all ratios *can be* written as fractions” (Clark et al., 2003, p. 299). This was proven false with the example of a three-part ratio, which compares three quantities to one another. For example, a paint solution consisting of one part red, two parts blue, and five parts white or 1:2:5 is acceptable written in ratio form, however written in fraction form, 1/2/5 is not acceptable.

The second model, fractions as a subset of ratios, is the direct opposite of the first model. This model was shown to be not true with the example of a ratio comparing two distinct quantities, such as seeing 3 boys for every 4 girls. Within a fraction meaning, $\frac{3}{4}$ is an acceptable fraction, however 3 boys/4 girls do not exist as the quantities are not in part-whole form.

The third model, ratios and fractions are distinct sets, was eliminated because ratios can be written in the form a/b, just as fractions. The fourth model defined ratios and fractions as identical sets. This was also proven false when the ratio 12:0 was acceptable for a ratio however 12/0 is unacceptable as a fraction.

The fifth model, ratios and fractions as overlapping sets, was claimed to be the best model. Within this model, "some, but not all, ratios are fractions, and some, but not all, fractions are ratios" (Clark et al., 2003, p. 300). Clark, Berenson, and Cavey provided a figure to define the fifth model (See Figure 3). As seen in the model, the measure, operator, and quotient subconstructs are specific to fractions. A coordination of distinct quantities, including rates, is limited to ratios. Fractions and ratios intertwine when part-whole and part-part relationships are acceptable as both fractions and ratios, such as seen in odds and probabilities. The ability to conceptualize when ratios and fractions are distinct and related, is needed for formalizing algebraic concepts such as slope.



(Clark et al., 2003, p. 307)

Figure 3: Ratio and Fraction Relationships

<u>Rational Number Concepts</u>

Within rational numbers, there are several concepts to aid in students' developing an understanding of the rational number subconstructs (Lamon, 1996; Pothier & Sawada, 1983). These include partitioning, unitizing, comparing and ordering, fraction density and size, and fraction operations.

Partitioning

Partitioning involves the physical act of taking a specified quantity and dividing it equally into a set number of pieces, such as sharing 2 pizzas among 4 people. Mathematics educators and researchers agree that the natural link between whole number and fraction instruction is through fair-sharing and partitioning activities, such as the one just described (Lamon, 1993; Streefland, 1993). Kieren (1980) hypothesized and other researchers agree that “partitioning plays a similar role in the development of rational number that counting plays for whole number concepts and operations” (Carpenter, Fennema, & Romberg, 1993). Partitioning strategies simultaneously develop with the part-whole understanding of rational numbers. Together they “are basic to learning other subconstructs of rational numbers" (Behr et al., 1983, p. 100).

Pothier and Sawada (1983) proposed a five-level theory for children’s development of partitioning strategies. The five levels are sharing, algorithmic halving, eveness, oddness, and composition. Children in the first level are not interested in exact solutions, and though they start to develop partitioning by cutting in half, refer to sections as pieces as opposed to halves. When children move to the second level of algorithmic halving, children are able to partition area models based on powers of 2. Again, children are not concerned whether or not each section is equivalent to one another. Children, who have reached the eveness level, understand and are mindful of creating sections with equivalent areas. Within this level, children are also able to create an even number of partitions that is not a power of two, where the partitions are not obtained by just repeatedly cutting everything in half. In the next level, oddness, children move away from starting with halves when the amount cannot easily be attained by starting from a half. An example of this would include needing to cut something into ninths. Children at this level, partition by cutting one section at a time until the desired number of cuts is reached. Children who have reached the fifth level are able to use their prior partitioning strategies to find shortcuts to successfully cut area models into an odd composite number.

For example, if needing to cut a whole into nine pieces, children will first cut this whole into thirds, than cut each third into three, rather than cutting one-by-one to get up to nine pieces. Pothier and Sawada note that this final level is hypothetical as none of the first grade children within their study developed past the oddness level. Developing more efficient partitioning strategies is needed to conceptualize other concepts, such as unitizing.

Unitizing

As with whole numbers, unitizing also plays an important role in the acquisition of rational number understanding. Unitizing is the process of grouping the same whole amount in different ways (Lamon, 2005). For example, a group of $\frac{3}{4}$ can be grouped as one group of $\frac{3}{4}$ or three groups of $\frac{1}{4}$. Unitizing involves a three-step process. The first step is finding a unit fraction, or a fraction with a numerator of one. When presented with the fraction $\frac{5}{6}$, this can be broken up into five groups of $\frac{1}{6}$, thus $\frac{1}{6}$ is the resulting unit fraction. The second step involves continually iterating that unit fraction. Using $\frac{1}{6}$, this can be iterated to generate $\frac{2}{6}, \frac{3}{6}, \frac{4}{6}$, etc. The final step is developing a composite unit of one. When $\frac{1}{6}$ is iterated up to $\frac{6}{6}$, the whole unit of one is composed and can then be viewed as either one group of $\frac{6}{6}$ or six groups of $\frac{1}{6}$ (Lamon, 1996).

Lamon (1996) notes that it is more difficult to unitize in terms of the whole as opposed to unitizing down to unit fractions. However, students who can unitize flexibly, are able "to apply compositions, decompositions, and conversion principles on quantities" (Post et al., 1993, p. 331). When presented with a situation, such as nine slices of pepperoni make up three-fourths of a whole package of pepperoni, and asked to determine how many slices of pepperoni are in $1\frac{5}{6}$ packages, students first need to understand that the nine slices together compose a unit of three-fourths of a package of pepperoni. In order to solve the problem, the three-fourths then needs to be decomposed into three groups of $\frac{1}{4}$, thus the nine also needs to be decomposed into three groups. The result of three groups of three, or three slices of pepperoni in each $\frac{1}{4}$, is then iterated up to a whole package or one. This whole unit of one then needs to be converted from a composite unit of fourths to a composite unit of sixths in order to determine how many slices of pepperoni are in $1\frac{5}{6}$ packages.

Difficulties that people have with defining a unit within a multiplicative structure, result from the conceptual shifts necessary to understand that units are not necessarily comprised of discrete wholes (Mack, 1993). In the pepperoni situation, the whole was initially three-fourths of one. Rational numbers require that you define the unit or whole. A student who cannot define the whole or unit will not be able to fully understand rational number concepts (Moskal & Magone, 2002). When students have a conceptual understanding of unitizing, they will have a multitude of strategies at their disposal.

These include:

- "Using unitizing to reason up and down replaces the need for rules for generating equivalent fractions as well as for reducing or lowering fractions.
- Unitizing gives opportunity to reason about fractions even before one has the physical coordination to be able to draw fraction parts accurately.
- Reasoning up or down while coordinating size and number of pieces lays the groundwork for proportional reasoning.
- Unitizing appropriately emphasizes a fraction as a number; that is, the emphasis is on the same relative amount, regardless of the size of the chunks.
- Unitizing aids self-assessment. Without the use of rules, students can check to see whether they have produced equivalent fractions because the number of chunks multiplied by the number in each chunk never changes."

(Lamon, 2002, p. 82)

Thus, students should develop unitizing strategies early in the curriculum to aid in their development of other rational number concepts, such as equivalence.

Equivalence

Equivalence concepts are "one of the most important and abstract mathematical ideas that elementary school children ever encounter" (Ni, 2001, p. 400). Kamii and Clark (1995) note "researchers have generally viewed knowledge of equivalent fractions as the ability to call the same number by different names, the ability to ignore or imagine partition lines, and/or a manifestation of flexible thought" (p. 368-369). Ni (2001) found that equivalence understanding is learned differently depending on the rational number subconstruct focused upon in instruction. Thus, equivalence should be taught throughout a rational number unit instead of being treated as an isolated topic. However, it is still unclear when equivalence should be taught and how (Kamii & Clark, 1995).

Tarlow and Fosnot (2007) suggest that equivalence should be taught within the context of ratio and rate tables. Presenting equivalence within the context of ratios not only emphasizes the multiplicative and additive nature of equivalent fractions, but also provides students with a foundation to develop more complex ratio and proportionate thinking (Post et al., 1985), such as the between and within relationships of ratios (Noelting, 1980a, 1980b; Streefland, 1991), which were discussed earlier. The ability to flexibly think about equivalence situations, in this way, is needed to not only be successful with equivalence but also with comparing and ordering fractions (Post et al., 1985).

Comparing and Ordering

Ordering fractions is important for understanding fractions as quantities (Post et al., 1993), as ordering requires one to coordinate the relative and/or absolute size of two or more fractions in order to determine their order. When asked to compare and order a set of fractions, students are typically presented with the common denominator method, and asked to only use that method for determining the order. Instead Behr, Wachsmuth, Post, and Lesh (1984) suggest three different strategies to compare fractions including the application to ratios, reference point, and manipulative strategies.

The application to ratios strategy involves understanding the relative size of each fraction. For example, when comparing $\frac{5}{6}$ and $\frac{2}{3}$, "children should eventually become able to make a judgment based on the relation (ratio) between 5 and 6 and between 2 and 3. This judgment requires that they observe that 5/6 is *relatively* larger than 2/3 regardless of the common unit chosen." (Post et al., 1985, p. 21). This can be observed by noticing that each fraction is missing one piece. In $\frac{5}{6}$, a one-sixth size piece is missing, where $\frac{2}{3}$ is missing one one-third size piece. Since one-sixth is smaller than one-third, then $\frac{5}{6}$ is larger. In this sense, the application to ratios strategy incorporates the part-whole definition of fractions as well.

The reference point strategy requires students to refer to a benchmark fraction to compare two or more fractions. For example, 1/3 is less than 3/5 because 1/3 is less than 1/2, whereas 3/5 is more than 1/2. Within this strategy, the exact size of each fraction is not needed as one only needs to know if the fractions are greater or less than one-half.

The manipulative strategy is similar to finding a common denominator, only it is done with the aid of a manipulative. Post, Wachsmuth, Lesh, and Behr (1985) illustrate the ways in which students can use this strategy with two-color counters. If students are asked to compare $\frac{5}{6}$ and $\frac{2}{3}$, they can make a group of $\frac{5}{6}$ by laying out six counters where five are red and the sixth is yellow. Then, a second set can consist of two red counters and one yellow counter for $\frac{2}{3}$. The manipulative strategy would then require that each pile consist of the same number of counters. In this case, another group of $\frac{2}{3}$ would need to be added to the second pile so that each pile now has six. This would show that the common denominator is six. The first pile would still have five red counters, compared to the four red counters in the second pile. With both piles representing the original fractions, this would show that $\frac{5}{6}$ is larger. Post, Wachsmuth, Lesh, and Behr (1985) caution that students who use manipulatives should eventually develop strategies so that the manipulative is no longer needed.

Students, who do not conceptually understand the multiplicative nature of fractions, may overgeneralize their whole-number knowledge to comparing fractions incorrectly. Behr, Wachsmuth, Post, and Lesh (1984) discovered two incorrect strategies used by children. The additive strategy involved comparing fractions by adding the numerator and denominator together to create a new fraction. Within their study, one student said that "three-fourths equals seven-eighths because 'three plus four equals seven, and four plus four equals eight" (p. 331). A second incorrect strategy used is what Behr, Wachsmuth, Post, and Lesh (1984) define as a whole-number dominance strategy. This strategy involves comparing two fractions by comparing the numerators and denominators separately. A student, who uses this strategy to compare three-fourths and five-eighths, will say that three-fourths is smaller because three is less than five and four is less than eight.

Fraction Density and Size

Comparing and ordering fractions becomes difficult because fractions cannot be found using counting procedures, as with whole numbers. "The density of the rational numbers implies the counterintuitive notion that there is no 'next' fraction" (Post et al., 1985, p. 33). This notion introduces the idea that between any two given fractions are infinitely many fractions. Students who understand fraction density are able to develop estimation skills, which “are important in evaluating the reasonableness of results of computation involving fractions” (Sowder, Bezuk, & Sowder, 1993, p. 247).

Fraction Operations

Research suggests that algorithms for fraction operations should not be the means of instruction, but rather generated by students (Huinker, 1998). Instruction that focuses just on procedures sets students up for misunderstanding those procedures and misusing them (Erlwanger, 1973). Thus, instruction needs to allow students to develop their own procedures for fraction operations.

The difficulties students and teachers have with fraction operations results from remembering the procedures incorrectly (Ma, 1999), as well as incorrectly transferring whole number ideas to fraction operation concepts. One of the many misunderstandings both students and teachers have with fraction operations is the misconception that multiplication always makes bigger, and division always makes smaller (Fischbein, Deri, Nello, & Marino, 1985). With both multiplication and division, the answer to the problem may be smaller, bigger, or the same as one or both numbers in the problem.

Language is also another difficulty that must be overcome in order to understand operations with fractions (Anghileri, 1991; Kerslake, 1991). Children have a very limited understanding of the four operation symbols. To children, plus means and or add, minus means take-away, multiplication means times, and divide means share (Anghileri, 1991, p. 103). In the context of rational numbers, the operations, in many cases, take on alternative meanings. For example, when interpreting the situation three minus two, this can be stated as starting with three objects and taking away two of them, or an interpretation of minus as taking away. When the situation becomes three minus one-half, it is incorrect to interpret this as starting with three objects and taking away half of them. Another example stems from sharing situations in division. In whole numbers, division situations can be interpreted as sharing problems. For example, $3 \div 4$ can be read as sharing three objects among four people. If the problem were instead $1\frac{3}{4} \div \frac{1}{2}$, it would be incorrect to say that you are sharing $1\frac{3}{4}$ of something among half of a person. From these examples, it is easy to see how language can confound students' ability to conceptualize situations involving fraction operations.

Children's Thinking

The shift to move from working with whole numbers to working with rational numbers is a difficult transition for children to make. Children must move away from using a single whole number to represent a quantity to using "a pair of numbers as a single quantity" (Hiebert & Behr, 1988, p. 6).

Children come into a classroom with a wide range of informal knowledge related to rational numbers (Carraher, 1996; Mack, 1990, 1993). The rational number knowledge that children bring to the classroom is largely based on the part-whole meaning of rational numbers (Mack, 1993). However, children also start developing other concepts, such as ratios and proportions, long before they experience formal instruction on rational numbers (Carraher, 1996).

Researchers have suggested that this informal understanding provides a good foundation for students to develop a conceptual understanding of rational numbers

(Mack, 1990, 1993). Mack (1993) suggests that students' informal knowledge is limited in that "students' informal strategies treat rational number problems as whole number partitioning problems, students' informal conception of rational number influences their ability to reconceptualize the unit, and students' informal knowledge initially is disconnected from their knowledge of formal symbols and procedures associated with rational numbers" (p. 87). For example, when a student was presented with a situation of having one-eighth of a pizza and then receiving another one-eighth of a pizza, the student said that the answer is two-sixteenths because "you have one whole pizza with eight pieces and you get another whole pizza with eight pieces, so there's two pizzas with sixteen pieces in all" (Mack, 1995, p. 432). This illustrates how students' informal understanding of rational numbers is largely based on whole number strategies, which often result in incorrect solutions.

Though children's informal knowledge of rational numbers can initially hinder their ability to understand rational number concepts, a curriculum must present situations in which a formal understanding can be built from this informal knowledge (Mack, 1993). Mack (1993) found that using children's part-whole understanding of rational numbers can provide a foundation for building other concepts, such as unitizing. In order for this to be done successfully, Mack (1993) suggests that problems must be presented in a realistic, contextualized situation.

Children's correct understanding of rational numbers is not always successfully transferred across contexts. This understanding varies depending on the context in which fractions are presented (Brizuela, 2005; Empson, 1999; Mack, 1990, 1993). Being familiar with a given situation or model does not imply that children will be successful with the same type of situation or model that is presented in a different context (Mack, 1990, 1993). For example, Brizuela (2005) found that children who are able to understand the concept of one-half when presented with an equal sharing problem, such as sharing one pizza for two people, do not necessarily have an understanding of one-half when presented with a topic, such as age. Within the sharing situation, children would understand that each person received one of two equal shares of a pizza, but did not understand how they could be $4\frac{1}{2}$ years old. In addition, children who do have a conceptual understanding of one-half, may have difficulties interpreting the symbolic representation of one over two. Thus, children need to be presented with a variety of contexts to fully conceptualize fraction meanings in addition to transferring these conceptual understandings to symbolic representations.

A curriculum must present different types of contexts so that children can develop a more complete understanding of rational numbers. These situations can then lead to children developing a variety of rational number models. When working with models, studies have shown that children are more successful with some models, than others. Post, Cramer, Behr, Lesh, and Harel (1993) found that children are able to identify part-whole relationships better with discrete sets as opposed to area models. At the same time, children are able to develop informal reasoning strategies for ordering and equivalence better with circular area models as opposed to using discrete sets (Cramer & Henry, 2002).

In addition to providing a curriculum that incorporates different fraction models, a curriculum must be comprehensive in incorporating the different subconstructs. For example, when ratios are presented in elementary mathematics, children develop understandings of rational numbers better than children who have not been exposed to ratios (Streefland, 1991). When ratios are introduced early, especially within the context of equivalence situations, children will be able to derive strategies that can then be applied in proportion situations (Vanhille & Baroody, 2002). In addition, Streefland (1991) found that children, who are able to reason in a ratio sense, are able to conceptually understand, for example, why fractions cannot be combined across numerators and denominators in addition situations. By incorporating multiplicative structures early, such as ratios, this void found within traditional curricula may be filled, and children may be able to better understand such topics as fraction operations as a result (Vanhille & Baroody, 2002).

Teacher Knowledge

In order for teachers to present a conceptually based rational number curriculum to their students, they themselves must have a conceptually based understanding of the topic. Research has shown that the knowledge that elementary and middle school teachers bring to the classroom is procedurally based and largely misunderstood (Ball, 1990a, 1990b; Kajander, 2005; Ma, 1999; Tirosh, 2000; Tzur & Timmerman, 1997). This is particularly true with teachers' understanding of fraction operations.

Within the operations, the most often studied is that of teachers' conceptions of division situations. Similar to Fischbein, Deri, Nello, and Marino (1985), many researchers have found that preservice and practicing teachers' conceptions of division situations are also largely based on the partitive, or sharing, meaning of division (Ball, 1990a, 1990b; Simon, 1993; Tirosh, 2000).

Ball (1990a) studied 252 preservice elementary and secondary teachers' conceptions of fraction division. Within the first task, prospective teachers had to match $4\frac{1}{4} \div \frac{1}{2}$ with a contextualized situation. Without a limit on the number of choices that could be correct, only 30% of the prospective teachers could identify the correct word problem. Within those, many chose the incorrect model for $4\frac{1}{4} \div \frac{1}{2}$. A different task required prospective teachers to provide a representation for $1\frac{3}{4} \div \frac{1}{2}$. Only 35 elementary and secondary preservice teachers were interviewed on this question. Almost everyone could obtain the correct answer of $3\frac{1}{2}$ by using the traditional invert and multiply algorithm. However, only 4 preservice teachers, all secondary, conceptually understood the situation. Ball concluded that preservice teachers' understanding of fraction division is limited to the partitive meaning of division. In addition, to be ready for any question that may arise in their own classrooms, preservice teachers must have the experiences to know mathematics "in sufficient depth to be able to represent it appropriately and in multiple ways – with story problems, pictures, situations, and concrete models" (Ball, 1990a, p. 458).

Research has shown that teachers need similar experiences with fraction operations to what children need before they can conceptually understand the algorithms used in fraction operation situations (Tzur & Timmerman, 1997). Rather than being presented with an algorithm first and then asked to solve several problems using that algorithm, teachers first need the opportunity to use models and pictures to solve the problems. From there, teachers can develop their own algorithms for solving problems involving fraction operations.

Teacher Preparation

Mathematics content courses for preservice teachers that provide nothing more than a reiteration of the traditional algorithms, do not aid in preservice teachers' development of pedagogical content knowledge any more than what was provided in their K-12 education. These courses instead need to present the material in such a way that preservice teachers are given the opportunity to evaluate and critique their current level of understanding so that they can start filling the gaps within their own knowledge (Sowder et al., 1993). At the same time, the curriculum must be comprehensive enough so that gaps in knowledge are not inadvertently created or ignored.

Topics, within a rational number curriculum for preservice elementary teachers, should not present a limited view of the meanings of rational numbers. Attention needs to be given to all of the rational number subconstructs (Kieren, 1993; Sowder et al., 1993). Kieren (1993) notes that "building a curriculum with the subconstructs in mind allows the study of fractional or rational numbers to become a significant window on the whole domain of mathematics" (p. 59). Focusing on more than just the part-whole meaning of rational numbers is important because other subconstructs "are more appropriate for demonstrating certain concepts and operations" (Sowder et al., 1993, p. 246). The meanings of rational numbers should also not be treated in isolation. The curriculum should highlight the relationships between each of the subconstructs (Post et al., 1993) as rational numbers can take on more than one meaning within a single problem. Having an understanding of the different meanings of rational numbers, and how they relate, is needed to be able to develop numerous models, which can then be used flexibly between and within various problem situations (Post et al., 1993). For example, using and developing an understanding of a linear model of rational numbers within unitizing situations can then lead to conceptualizing and using the linear model within fraction addition situations.

Rational number concepts, which include order and equivalence, should be the main emphasis of a rational number unit (Sowder et al., 1993). Rational number concepts are important to focus upon first, because the understanding of the rational number as a quantity, for example, is needed before operations (Post et al., 1993).

Once rational number concepts are understood, the curriculum can then shift to focusing on operations. Preservice teachers need to be able to distinguish between operation situations and "learn how to make a correct choice of operation" (Sowder et al., 1993, p. 249). Thus, the curriculum must include problems in which a particular operation strategy is not readily apparent. Once an operation is chosen, preservice teachers should not revert to an algorithm to solve the problem because research shows that preservice and inservice teachers do not understand why the algorithms work (Ball, 1990a, 1990b; Ma, 1999; Tirosh, 2000). Rather, they should use their knowledge of rational number concepts to arrive at an answer. From here, the algorithms can be derived, and thus conceptually understood.

Operations also provide an opportunity for preservice teachers to develop estimation strategies (Sowder et al., 1993). These strategies can be used as a way of estimating an answer as well as checking the reasonableness of a solution. Estimation strategies cannot be accurately developed within a curriculum that focuses on rote procedures (Sowder, 1988).

Tirosh examined a methods course where preservice teachers analyzed and anticipated children's abilities when solving fraction division problems. Her findings showed that the preservice teachers were quick to conclude that the only way to divide was by the traditional algorithm of reciprocating the second fraction and multiplying. Even when presented with the correct algorithm of dividing the numerators and denominators, Tirosh found that preservice teachers accepted this method to always work, but still did not prefer it because of the times for when a complex fraction was the solution. For example when solving $\frac{3}{4} \div \frac{5}{2}$ by dividing the numerators and denominators, the answer would be $\frac{3/5}{2}$.

Tirosh notes that "a major goal in teacher education programs should be to promote development of prospective teachers' knowledge of common ways children think about the mathematics topics the teachers will learn" (Tirosh, 2000, p. 5). Before preservice teachers can be asked to understand how children think about problems with rational numbers in methods courses, they first must have a conceptual understanding of rational numbers themselves for two reasons. First, a conceptual understanding will enable them to determine whether or not a child's method is correct. Second, teachers must be able to justify to their students why the mathematics that they teach always works.

Models in Mathematics

As previously mentioned teachers must have an understanding of representing mathematics in multiple ways in anticipation of the questions which may arise from students in their own classroom (Ball, 1990a). Multiple representations are important as students who are able to represent rational numbers in multiple contexts as well as translate among various representations develop a deeper understanding of the content (Post et al., 1993). Representations, which include area, linear, and set models “are grounded in the way that contextual problems are solved by the students (Gravemeijer & Stephan, 2002, p. 148) and are grounded through the tools used to represent each model. The tools, or physical shape, are used as a way for students to represent a solution and solution process. As a result, there are various ways in which students may incorporate tools when solving problems.

Gravemeijer (2004) distinguishes between four levels of activity with tools which include activity in the task setting, referential activity, general activity, and more formal mathematical reasoning. According to Gravemeijer (2004), students progress through these levels to develop an overarching concept or model where various tools act as stepping stones. As students progress through these levels of working with tools, tools move from initially being used as a way for students to represent their thinking to eventually being used to model more formal mathematical thought (Cobb, 2000; Gravemeijer, 2004). While students’ work with tools evolves, overarching models emerge (Gravemeijer, 2004). In addition, by working with tools, students can start to develop mental images for various overarching models (Cramer & Henry, 2002).

The ways in which tools are incorporated into a solution and solution process evolve as students work with tools over time (Walkerdine, 1988). Walkerdine describes this as a chain of signification in which the ways in which students use tools become taken-as-shared. As students use with a tool evolves, an overarching model also emerges (Gravemeijer, 2004). The taken-as-shared ways of using tools eventually come to signify the use of another. Thus, the instructional sequence and hypothetical learning trajectory must be created in such a way that students have opportunities to work with various tools.

Hypothetical Learning Trajectory

A hypothetical learning trajectory (HLT) is a theoretical path of the ways students are going to develop an understanding of a topic. An HLT incorporates several components. It is agreed that these components include the learning goals for each class session, the instructional activities needed to achieve these learning goals, and the tools students will use to aid in their mathematical development (Gravemeijer, 2004; Simon, 1995). Tools, within this respect, refer to the ways in which students use objects such as circles, rectangles, and the number line.

Even though a hypothetical learning trajectory is established at the beginning of a unit or topic, it is important to note that the HLT could easily change throughout a unit. This results from the ways in which students learn a topic. More often than not, students will learn a topic differently from how the teacher believes they will learn. Thus, the teacher must be flexible, so that if student learning does not proceed in a way that was originally thought, the HLT can be altered to accommodate student understanding.

The HLT then provides a foundation for which instructional activities will be used (Simon, 1995). Simon (1995) notes that an HLT is important because just having a set of instructional activities, though thought-provoking to ensure conceptual learning, is still not sufficient without a structure underlying the order in which the activities will be presented.

A discussion of the HLT and instructional sequence used for this study is included in chapter 3. Also included is a discussion of the theories and research used to develop the HLT and instructional sequence.

Conclusion

The need for understanding how children learn rational numbers is important when developing mathematical activities for preservice teachers as preservice teachers need mathematical experiences similar to what children need. Since research with adults' development of rational number understanding is limited, research with children was used in conjunction with research with adults to develop the hypothetical learning trajectory and instructional sequence for this study. These are further discussed in the next chapter.

CHAPTER THREE: METHODOLOGY

This study incorporated qualitative methods documenting preservice teachers' development of rational number understanding. This study analyzed data from an already existing data source. Using a design-based research methodology, the data came from a classroom teaching experiment that was conducted in a semester-long mathematics content course focusing on elementary school mathematics.

This chapter starts with a discussion of design-based research. Following this is a discussion of the instructional design theory used to create the hypothetical learning trajectory (HLT) and instructional sequence for the course as well as the theoretical framework, which informed the data collection. The chapter concludes with a description of how the data were analyzed.

Design-Based Research

Design-based research, also known as design experiments (Brown, 1992; Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003), have been incorporated into mathematics education research settings throughout the past few decades (Steffe & Thompson, 2000). Design experiments are conducted by first "developing instructional designs to support particular forms of learning and systematically studying those forms of learning within the context defined by the means of supporting them" (Cobb, 2003, p. 1). These types of experiments are not intended to verify the use of specific instructional activities but rather to develop theories about how students' learning progresses both individually and as a collective group and how the instructional activities foster this learning (Cobb et al., 2003).

A design experiment "that focuses on the development of local instruction theories basically encompasses three phases: developing a preliminary design, conducting a teaching experiment, and carrying out a retrospective analysis" (Gravemeijer, 2004, p. 109). Within the preliminary design, an HLT, which includes the learning goals and instructional sequence, is created and used as a basis of instruction. Through the refining process of design experiments the HLT could be revised as a study progresses (Cobb et al., 2003). This is due to students' actual learning and development of a mathematical topic.

Design experiments are iterative in nature in that the results should be used to continually "improve the instructional design" (Cobb, 2003, p. 11). The resulting HLT and instructional activities developed from these revisions are then used to inform another design experiment, and the cycle continues (Gravemeijer, 2004). In addition to using the results to make constant revisions, the data collected should also document the social or whole-class normative ways of reasoning, as well as individual students' mathematical learning as it happens within the context of the whole-class (Cobb, 2003). Since all of these components are needed "to contribute to reform in mathematics education" (Cobb, 2003, p. 10), each were included within the research focus of this study.

Design experiments can be conducted within various educational settings. These include one-on-one design experiments, classroom experiments, preservice teacher development experiments, in-service teacher development studies, and school district restructuring experiments (Cobb et al., 2003). Though this study was conducted with preservice teachers, the educational research setting chosen for this study was a classroom teaching experiment.

Classroom Teaching Experiment

Classroom teaching experiments started being incorporated into mathematics education studies in the United States in the 1970's (Steffe & Thompson, 2000). Teaching experiments, unlike traditional studies with control groups, provide detail as to how students learn throughout a study. Rather than just comparing pre-test scores to post-test scores, teaching experiments document students' learning and development as a study progresses (Steffe & Thompson, 2000). The classroom teaching experiment was chosen for this study because the intent of the study was to analyze the development of preservice teachers' rational number understanding and not to study the affect their learning has in their teaching practices.

The process of conducting a classroom teaching experiment includes testing a conjecture, such as the conjecture that fraction language needs to be taught before partitioning situations. This conjecture is then tested, refined, and tested again. A new conjecture may be created when students follow a learning path that is different from the original HLT (Steffe & Thompson, 2000). The way the students learn directly impacts the HLT and instructional activities. Thus, both the HLT and instructional activities are continually refined and retested throughout a study's duration.

Classroom teaching experiments incorporate research teams. The role of research team members are to observe class sessions and assist the teacher in assessing student learning (Steffe & Thompson, 2000). The purpose of this is for both the teacher and observers to keep each other informed as they each could have very different views of what students have learned and understand.

The research team for this study consisted of 8 people. This included an associate professor who was the instructor of the course, one visiting assistant professor, one doctoral student at the dissertation stage, and five doctoral students at the predissertation stage all in mathematics education. The visiting assistant professor was the instructor for the sixth day of the rational number unit. Members of the research team videotaped each class session.

The research team was also instrumental to the development and refinement of the hypothetical learning trajectory. Prior research and educational theories were used to initially develop the hypothetical learning trajectory for this study. Realistic Mathematics Education (RME) was the instructional design theory used to develop the HLT and instructional sequence implemented in this study.

Realistic Mathematics Education

Realistic Mathematics Education was developed at the Freudenthal Institute in the Netherlands (Streefland, 1991) and introduced a mathematics classroom structure different from traditional mathematics teaching. The premise of RME is that instructional materials should be created around "learning paths along which students can reinvent conventional mathematics" (Gravemeijer, 2004, p. 107).

In order for students to reinvent mathematics for themselves, RME suggests three design heuristics that must be taken into consideration when planning a comprehensive mathematics unit. These heuristics include guided reinvention, didactical phenomenology, and emergent models. Together these heuristics "help the research team in designing a possible learning route together with a set of potentially useful instructional activities that fit this learning route" (Gravemeijer, 2004, p. 110). In other words, the HLT is based off of these heuristics.

The guided reinvention principle is made up of two components. The first component states that students should be directed in such a way that they are able to reinvent mathematics (Gravemeijer, 2004). Instead of learning facts and algorithms from a teacher, as with traditional mathematics classrooms, students are given the opportunity to make sense of the mathematics for themselves. This is accomplished through the second component, which is presenting activities in a context that is experientially real for students. This does not mean that students need to have a personal connection to the problem situation, but rather believe that the problem 'could' be real (Streefland, 1991). The instructional activities should be created such that students are able to reinvent mathematics within topics that have real-world applications.

The second heuristic, didactical phenomenology, incorporates prior research to develop an order for the big ideas created from the first heuristic (Gravemeijer, 2004). These ideas must be ordered in a mathematically logical manner such that students are not asked to reinvent a higher-order mathematical topic prematurely. For example, within the context of rational numbers students need to have some understanding of ratios before being asked to solve proportion problems.

The third heuristic, emergent models, focuses on the progression of students' usage of mathematical tools and symbols over time. The instructional activities must be designed to "support the evolution of ways of symbolizing as part of a process of fostering the development of mathematical meaning" (Gravemeijer, 2002, p. 141). Once students are presented with activities, it is the teacher's role to "help students model their own informal mathematical activity" (Gravemeijer, 2004, p. 117). These models are initially used as a way for students to represent their thinking and are later used to model more formal mathematical thought (Cobb, 2000; Gravemeijer, 2004). Within this study, tools were presented as pictures. Students did not have access to pre-made manipulatives representing various models. This was because of the deeper cognitive thought processes needed to work with pictures as opposed to pre-cut fraction manipulatives.

Since teachers need to be able to provide their own students with experiences related to all three models, the classroom activities developed as part of the instructional sequence were designed so that students would have experiences with several tools related to these models within rational number concepts before working with fraction operations. By the time the class reached the fraction operations portion of the unit, they had worked with area models (circular and rectangular regions), a linear model (number line), and set models (groups of circles and squares).

The three components of RME were used in conjunction with past research to develop the HLT and instructional sequence implemented in this study.

Prior Research

A previous iteration of this classroom teaching experiment was conducted and a hypothetical learning trajectory and instructional sequence for rational numbers was developed (Wheeldon, 2008). As a result of Wheeldon's study several suggestions were made for improving future iterations, which directly affected the design of this study.

This is consistent with the cyclic nature of design-based research in that the results should be used as a basis for future studies (Gravemeijer, 2004).

The research focus of Wheeldon's (2008) study only pertained to the social aspect of the classroom environment. In order to contribute to reform in mathematics education, Cobb (2003) notes that both the social and individual aspect of a classroom environment should be analyzed. Thus, Wheeldon suggested that the individual aspect should also be incorporated within the research focus of future iterations. A second suggestion was to create a single overarching context throughout the entire rational number unit. Pizza was found to be a recurring topic students' preferred to use when solving fraction problems. Thus, Wheeldon suggested that a future iteration should use contexts related to pizza as the umbrella for the rational number unit. Wheeldon found that a sequence of activities that do not follow a single 'storyline' appear to be isolated from one another, even though the concepts presented within the activities build upon one another.

Hypothetical Learning Trajectory

As discussed in the previous chapter, hypothetical learning trajectories are the projected path by which students are going to learn a specific topic (Simon, 1995). The hypothetical learning trajectory developed by Wheeldon (2008) was used as a foundation for the development of the hypothetical learning trajectory used for this study.

Wheeldon's trajectory incorporated five phases of learning. These included using fractions to name an amount, developing fraction properties such as the larger the denominator the smaller the piece, developing reasoning strategies to compare and order fractions, adding and subtracting fractions, and multiplying and dividing fractions.

Results from current and past research with children's rational number learning and understanding also informed the HLT. Research from children was used because of the limited amount of research that analyzes preservice teachers' rational number learning and understanding.

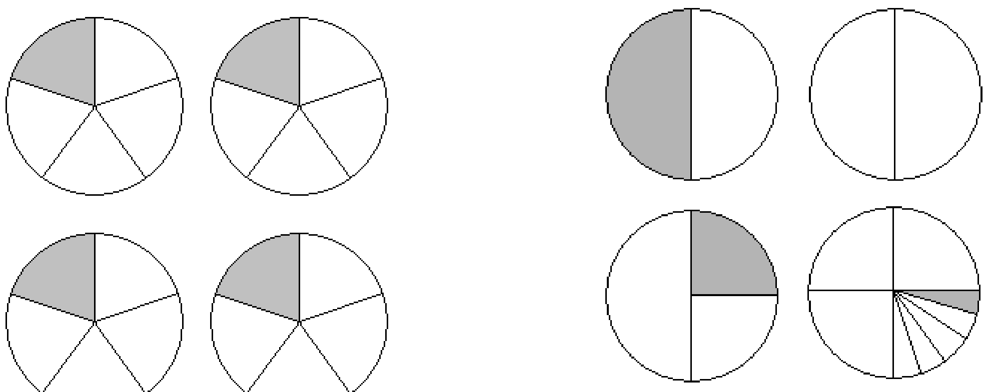
The HLT developed for this study incorporated five phases of rational number instruction (See Table 2).

Table 2: Initial Hypothetical Learning Trajectory

Phase	Overarching Topic
1	Define Fraction Based on Whole Compose and Decompose Fractions
2	Unitizing Multiplicative and Additive Relationship of Equivalent Fractions
3	Relational Thinking Comparing Fractions with Reasoning Ordering Fractions Fraction Density
4	Fraction Addition and Subtraction
5	Fraction Multiplication and Division

The goal of the first phase of instruction was to introduce basic concepts of rational numbers. This included defining a whole to name fractions less than, greater than, or equal to one. This was consistent with Streefland's (1991) findings that students need to have experiences with fractions greater than one from the beginning of a rational number unit. Another goal of this phase was to develop multiple ways to represent the same fraction using various partitioning strategies. For example, when sharing 4 pizzas equally among 5 people, students could show that $\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}=\frac{1}{2}+\frac{1}{4}+\frac{1}{20}$ by partitioning the pizza in different ways (See Figure 4). This will lead to a more complete understanding of the way that rational numbers can be composed and decomposed into varying amounts (Lamon, 1993).

Solution 1 Solution 2



$$\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}=\frac{4}{5} \qquad \frac{1}{2}+\frac{1}{4}+\frac{1}{20}=\frac{10}{20}+\frac{5}{20}+\frac{1}{20}=\frac{16}{20}$$

Figure 4: Sharing 4 Pizzas Equally Among 5 People

Once students had an understanding of representing fractions in multiple ways, they moved into the next phase, which is comprised of two ideas. The first idea, unitizing, incorporated a three-step process. Step one included decomposing fractions into a set of unit fractions, or fractions with a numerator of one. Once unit fractions were found, the next process included iterating the unit fraction, and finally developing a composite unit of one (Steffe, 1988). For example, the fraction $\frac{3}{5}$ can be decomposed into $\frac{1}{5}+\frac{1}{5}+\frac{1}{5}$. The unit fraction of $\frac{1}{5}$ can then be iterated or counted up to a composite unit of one or $\frac{5}{5}$. Upon doing this it is hoped that students will understand that $\frac{3}{5}$ is the same as having three one-fifths and similarly, $\frac{5}{5}$ is five one-fifths. As discussed in chapter two, an understanding of this reciprocal thinking is necessary to develop higher-level rational number concepts such as equivalence (Lamon, 1996).

Unitizing skills could then be used to aid in students' development of equivalence concepts, which was the second idea within this phase. It was intended that by the end of this phase students would understand that equivalent fractions are different names for the same amount in addition to using multiple strategies for determining if two rational numbers are equivalent. By incorporating unitizing strategies within equivalence situations, students will be able to use more than just a common denominator method. Other methods include using an additive strategy, such as $\frac{2}{3} = \frac{2+2}{3+3} = \frac{4}{6}$, and unitizing strategies, such as converting both $\frac{2}{3}$ and $\frac{4}{6}$ to $\frac{1}{1.5}$.

The next phase of the HLT progressed to comparing nonequivalent rational numbers using equivalence methods. Though the unitizing methods discussed in the previous paragraph are sufficient for comparing and ordering fractions, the goal of this phase was for students to develop the reasoning strategies of comparing fractions to a benchmark fraction, comparing using common numerators, comparing using common denominators, and comparing using missing pieces. This was because reasoning strategies, in some instances, are more efficient than an algorithm for finding a solution. For example, when comparing $\frac{43}{82}$ and $\frac{96}{95}$, it is easier to reason that $\frac{43}{82}$ is less than one and $\frac{96}{95}$ is greater than one, instead of using a common denominator algorithm or using a unitizing strategy to compare the two. Once students were able to identify equivalent fractions and compare and order fractions using reasoning strategies, they then used these strategies to find fractions between two given fractions, which was the second goal within this phase. It was within this idea that students were introduced to the fact that between any two given fractions, infinitely many fractions exist.

The final two phases moved away from rational number concepts and into operations. The goal of these two phases was for students to conceptually understand fraction operations. Students needed to apply their knowledge to develop non-traditional methods for addition, subtraction, multiplication, and division. For addition and subtraction, specifically, students were expected by the end of the phase to distinguish between these two types of situations and develop estimation strategies to check the reasonableness of a problem's solution.

Through various multiplication and division situations, the goal of this fifth phase was for students to develop an understanding of the traditional algorithms as well as the underlying concept of how the whole changes throughout the problem. For example, when multiplying $\frac{1}{2} \times \frac{3}{4}$, the multiplication situation becomes $\frac{1}{2}$ of $\frac{3}{4}$ where $\frac{3}{4}$ is represented out of a whole of 1. The $\frac{1}{2}$ is represented as $\frac{1}{2}$ of the $\frac{3}{4}$ which is where the whole changes from 1 to $\frac{3}{4}$. Then the final answer of $\frac{3}{8}$ is out of the original whole of 1, thus the whole changes twice. Within multiplication situations, the solution is in-terms of a unit of one.

In division situations the solution is in-terms of the divisor. For example, in the problem $\frac{3}{4} \div \frac{1}{2}$, both $\frac{3}{4}$ and $\frac{1}{2}$ are both represented out of a whole of 1. Finding how many groups of $\frac{1}{2}$ of a whole there are in $\frac{3}{4}$ of a whole, there is a whole group of $\frac{1}{2}$ with $\frac{1}{4}$ of a whole of 1 leftover. This leftover piece though for the final answer needs to be in terms of the divisor of $\frac{1}{2}$. Thus the answer is $1\frac{1}{2}$ not $1\frac{1}{4}$. These understandings of multiplication and division are not typically highlighted when traditional algorithms are taught.

Instructional Sequence

As suggested by Wheeldon (2008), the instructional sequence was set in a pizza parlor scenario. The instructional activities were designed so that each topic was presented in the context of a pizza situation (see Table 3).

Table 3: HLT Including Instructional Sequence for Rational Number Concepts and Operations

Day	Overarching Topic	Instructional Activities	Tools
1	Define Fractions Based On Whole	Opening Day	Pizza
2	Composing and Decomposing Fractions	Pizza Sharing	Pizza
3	Unitizing	Pizza Dough Machine Keeping Track	Number Line Discrete Sets
4	Multiplicative and Additive Relationships of Equivalent Fractions	Family Reunion Customers	Tree Diagram Ratio Table
5	Relational Thinking Comparing Fractions with Reasoning	Birthday Parties Comparing Fractions	
6	Ordering Fractions Fraction Density	Pizza Eating Contest Recipes	
7	Addition Subtraction	Pizza Parlor Situations 1 Pizza Parlor Addition Pizza Parlor Subtraction	Number line Discrete Sets Pizza
8	Multiplication	Pizza Parlor Situations 2 Pizza Parlor Multiplication	Number line Discrete Sets Pizza
9	Division	Pizza Parlor Situations 3 Pizza Parlor Division	Number line Discrete Sets Pizza
10	Comprehensive Exam over Rational Number Concepts and Operations		

On the first day of the unit, students were introduced to a pizza parlor scenario, which was carried throughout the unit. The task used on this day was intended to introduce students to defining fractions (See Appendix C). The task included discovering non-equivalent names for the same shaded regions. Naming the same shaded region in multiple ways was intended to reinforce the concept that fractions are named by their whole. Since defining the whole is a key component, correct mathematical language was also important.

Understanding how to define the whole was needed to successfully solve partitioning situations. Day two incorporated partitioning situations introduced through the context of fair sharing problems where a group of customers comes to the pizza parlor and shares a set number of pizzas equally with everyone in their party. The fair sharing situation also indirectly related the ratio, quotient, and part-whole meaning of rational numbers. The problem of sharing 4 pizzas with 5 people is a rate in the sense that there are 4 pizzas for every 5 people. The process of sharing incorporates division, or the quotient meaning. The final answer of everyone receiving $\frac{4}{5}$ of a pizza is the part-whole meaning. These problems also included situations where the solution was greater than one, because students need to have early experiences with mixed numbers (Streefland, 1991). Two diagrams, circular pizzas and rectangular dessert pizzas, were also included to start introducing students to various tools that can be used in rational number situations (Gravemeijer, 2004).

Unitizing was the focus of the third day of instruction. The process of decomposing a fraction into unit fractions and iterating the unit fraction was accomplished through a linear model represented by a roll of pizza dough (See Appendix M). By using the pizza dough representation, students were asked to find a given a set of fractions using paper folding techniques. The final step of unitizing was done through a task that included set, linear, and area models. The problems gave a picture and the fraction of the whole that the picture represented. The task then required the picture to be manipulated down to the unit fraction and then iterated to a second specified fraction (See figure 5)

The following is $\frac{2}{7}$ *of a pound of dough. Show* $1\frac{3}{14}$ *pounds of dough.*

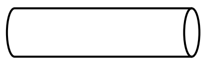


Figure 5: Example Unitizing Problem

Problems such as this were included where a composite unit of 1 had to be reunitized into a different set of composite unit fractions.

The next day of instruction focused on the additive and multiplicative relationships of equivalent rational numbers. This was done within ratio situations as Tarlow and Fosnot (2007) suggest. The first instructional task involved splitting up a family and pizza order such that no matter which table someone sits at they receive the same amount of pizza (See Appendix F). This task was developed based on Streefland's (1991) pancake sharing scenario, and introduced students to ratio tables. The second task was comprised of rate problems (See Appendix F). Within this activity the problems were presented in a part-whole ratio context. Students were then asked to use any strategy they wished to continue the rate for given additional amounts.

Day five moved students away from numeric methods and into reasoning strategies for comparing fractions. The first task presented word problems such that the solution could be found using a reasoning strategy. Students developed reasoning strategies such as comparing to benchmarks of one-half and one, using the relative size of a unit fraction, and the relative size of fractions that have the same number of pieces missing. The next task required that the reasoning strategies be applied to non-contextualized situations. Within these situations, two fractions were given and the task was to determine which fraction is greater. Each situation could be solved using multiple reasoning strategies. The reasoning strategies were then applied to ordering problems, which were presented on the sixth day. Using a pizza-eating contest, the task involved placing fractions in order from both least to greatest and greatest to least. The fractions chosen for the task were not fractions that were well-known, such as $\frac{96}{99}$. Since students did not have calculators, a least common denominator method was difficult, and subsequently it was easier to use reasoning to determine the answer. The next task used recipes to find either one, two, or three non-equivalent fractions in between two given fractions.

The last four days of instruction focused on the four operations. The activities, though easily solvable using traditional algorithms, required students to use pictures to solve the problems. Students were able to use any model to solve each situation.

Ethical Considerations

Before these instructional activities were used with this class of students, this study was approved through the university's Institutional Review Board (IRB) (See Appendix A). Every student in the course agreed to participate in the study, and indicated this by signing an informed consent letter (See Appendix B). To preserve anonymity, pseudonyms were used in the data analysis.

Participants and Setting

This study was conducted at a large metropolitan university in the southeastern part of the United States. There were 33 participants in this study. Participants were all female undergraduate students majoring in either elementary or exceptional education. They were all at least in their sophomore year of college.

This study was conducted during a spring semester. The course was a four credit hour semester-long undergraduate course focused on mathematics for teaching elementary school. The course met twice per week for one hour and fifty minutes each day. Students in the course were situated at tables of at least four and no more than six. The classroom was equipped with a document camera.

Research Questions

The intent of this study was to analyze the collective develop of preservice teachers' rational number understanding as well as the interaction between the social and individual environments, since learning is both an individual and social activity (Cobb & Yackel, 1996). The research questions were:

1. In what ways do classroom mathematical practices develop related to rational numbers?
2. In what ways do the social and individual environments impact one another?
3. In what ways does the instructional sequence facilitate the development of preservice elementary teachers' rational number understanding?

Data Collection

When conducting design experiments, a variety of data need to be collected including video recordings of whole class sessions and student interviews as well as student work (Cobb, Stephan, McClain, & Gravemeijer, 2001). Data were collected from 10 class sessions, which focused on a rational number unit. The rational number unit was part of a larger study which also included a unit focusing on place value and whole number operations in base 8 (Roy, 2008; Safi, in preparation). The rational number unit constituted the second unit in the course, thus students were already accustomed to being videotaped and observed in every class. The data collected included video recordings of whole-class discussions, audio recordings of small group discussions, student work, pre and post-test scores, and research team notes.

Three video cameras were used to capture varying aspects of the classroom. Facing the front of the room, one camera was situated at the back right of the classroom and focused on the whole class and individual students. The second camera was placed at the back of the classroom and focused on the work done at the board and the work presented on the document camera by both the instructor and students. The third camera focused on the instructor and individual students from the front left of the classroom.

Audio recordings documented small group interactions during class work activities. Three small groups were chosen to have audio recorders placed at their tables. Each of these groups consisted of at least one person who was interviewed individually, which is how the group was chosen to have the audio recorders placed at their table.

Five students were interviewed individually before and after the rational number unit. The students were selected because they were interviewed and observed during the whole number unit (Safi, in preparation). Each interview was videotaped and lasted approximately 40 minutes. During the interview students were asked to solve several rational number problems. Students' work from each interview was collected. These students also participated in a focus group session halfway through the rational number unit that focused on students' overall feelings of the classroom structure, their mathematical activities, and thoughts of the rational number unit thus far.

Other data collected from students included class work, homework, and exams. The data collected from the research team included field notes and reflective journals from class observations. The research team met after every class session to discuss if the learning goals for the day were met and to plan the next class session. Each of these team meetings were audio taped.

Data Analysis

The emergent perspective (Cobb & Yackel, 1996) was the interpretive framework through which the data were analyzed. The emergent perspective coordinates the social and psychological perspectives which are "two distinct theoretical viewpoints on mathematical activity" (Cobb et al., 2001, p. 118). Within each of these two perspectives there are three correlated aspects of students' mathematical activities (See Table 4).

Table 4: The Emergent Perspective

Social Perspective	Psychological Perspective
Classroom social norms	Beliefs about own role, others' roles, and the general nature of mathematical activity in school
Sociomathematical norms	Mathematical beliefs and values
Classroom mathematical practices	Mathematical conceptions and activities

Coordination of Social and Psychological Perspectives

The social perspective pertains to the normative ways students act, reason, and argue in the classroom (Cobb et al., 2001). The psychological perspective focuses on how individual students participate within the classroom community. Even though "the social perspective brings to the fore normative taken-as-shared ways of talking and reasoning, the psychological perspective brings to the fore the diversity in students' ways of participating in these taken-as-shared activities" (Cobb et al., 2001, p. 119).

The coordination of these perspectives implies that neither takes precedence over the other (Cobb & Yackel, 1996). Without individual students contributing to discussions, there would be no classroom community. Without having a classroom community established, there would be nothing within which students could participate. Therefore, both perspectives must be taken into consideration when analyzing students' classroom mathematical activities.

Aspects of Social and Psychological Perspectives

The ways in which students become accustomed to participating within a classroom community are a developmental process (Dixon, Andreasen, & Stephan, in press). Students do not come into the classroom on the first day knowing how to argue mathematically or question one another for example. Rather these communal processes "are considered to be jointly established by the teacher and students as members of the classroom community" (Cobb & Yackel, 1996, p. 178). The first step in establishing these communal processes was establishing social norms.

Social norms define both the teacher's and students' role in the classroom. Though social norms are established from the beginning of a course, they are continually negotiated and renegotiated throughout the course by both the teacher and students (Cobb, Wood, & Yackel, 1993; Dixon et al., in press). Social norms are not specific to mathematics and include explaining and justifying solution strategies, making sense of other students' strategies, questioning other students' solution strategies when misunderstandings occur, and agreeing/disagreeing with other students (Cobb, Yackel, & Wood, 1989). If a student's explanation or justification is not clear for example, then it is the responsibility of the teacher and other students to ask clarifying questions to the person whose solution process is vague.

Social norms cannot be established without contributions made from individuals in the classroom. When "making these contributions (social perspective), students reorganize their individual beliefs about their own role, others' roles, and the general nature of mathematical activity (psychological perspective)" (Cobb et al., 2001, p. 123). These reorganized beliefs then are what drive individuals to contribute to the renegotiation of established norms or to the negotiation of new ones.

Though social norms provide a foundation for the teacher's and students' roles in the classroom, norms also need to be established for students' mathematical activity. These are known as sociomathematical norms and include determining "what counts as a different mathematical solution, a sophisticated mathematical solution, an efficient mathematical solution, and an acceptable mathematical explanation" (Cobb & Yackel, 1996, p. 178). The establishment of the social norms in the classroom fosters the sociomathematical norms, as students are now expected not only to voice their solutions and solution processes, but also to analyze, critique, and correct one another's solutions in terms of their mathematical accuracy and clarity. Once these norms are established, the teacher must then guide students in such a way that they not only learn how to contribute to mathematical discussions but do so at an appropriate time and in an acceptable way (Cobb et al., 2001).

The students and teacher must decide for themselves what constitutes acceptable mathematical solutions and collectively add to these discussions when discrepancies arise. Developing this type of classroom community where a student's role includes perpetuating a portion of the mathematical discussions subsequently alters students' mathematical beliefs and values. The development of individual students' "mathematical beliefs and values [that] enable them to act as increasingly autonomous members of the classroom mathematical community" (Cobb et al., 2001, p. 124). When students contribute to the constant renegotiation of the sociomathematical norms they in turn continually alter their own perceptions to fit within the established sociomathematical norms of the classroom. Thus, there is a reflexive relationship between the sociomathematical norms and individual students' mathematical beliefs and values.

The establishment of the students' and teacher's role in the classroom, both as a collective group and as individuals, provides a foundation for students to develop mathematically both collectively and individually. The normative mathematical activity of the classroom community is known as classroom mathematical practices (Cobb, 1991). Classroom mathematical practices "focus on the taken-as-shared ways of reasoning, arguing, and symbolizing established while discussing particular mathematical ideas" (Cobb et al., 2001, p. 126). While the classroom community develops as a cohesive group of learners, individual students' mathematical activities or ways of reasoning, arguing, and symbolizing evolve within and throughout various mathematical ideas as well. However, the evolution of individual students' mathematical conceptions and activities may be diverse even though the mathematical practices are the same for everyone (Cobb & Yackel, 1996).

As with the social and sociomathematical norms, there is a reflexive relationship between the classroom mathematical practices and the mathematical activities of individual students. As the mathematical practices develop from individuals' contributions, these contributions are "enabled and constrained by the students' participation in the mathematical practices" (Cobb & Yackel, 1996, p. 180). Students then alter their own mathematical activities as a result.

As just described there is a reflexive relationship between individual students' learning and the social context within which they learn. The ways in which the classroom mathematical practices emerged as well as the ways in which the social and psychological was the focus of this study evolution of individual students' rational number learning as it occurred within the social context of the classroom will be the focus of the data analysis of this study. What follows is a discussion of the methods used to document collective activity and the methods used to coordinate both collective and individual mathematical activity.

Documenting Collective Activity

Documenting collective activity is important because "it offers an empirically grounded basis for design researchers to revise instructional environments and … is a mechanism for comparing the quality of students' learning opportunities across different enactments of the same intervention" (Rasmussen & Stephan, 2008, p. 196). There are three aspects of collective activity. These include "(a) a taken-as-shared purpose, (b) taken-as-shared ways of reasoning with tools and symbols, and (c) taken-as-shared forms of mathematical argumentation" (Cobb et al., 2001, p. 129). A taken-as-shared purpose includes "what the teacher and students are doing together mathematically" (Cobb et al., 2001). The taken-as-shared ways of reasoning with tools and symbols include the ways in which tools and symbols are used as well as the ways in which these are defined by the classroom community. The taken-as-shared forms of mathematical argumentation include the ways in which students provide explanations and justifications for their solutions and solution processes. Thus, when documenting collective activity it is important to analyze all three of these aspects. For this study the classroom mathematical discussions were analyzed to determine what became taken-as-shared by the classroom community.

The classroom discussions were analyzed using Toulmin's (2003) model for analyzing argumentation. Toulmin's model, which consists of four components, involves analyzing what is said and classifying this to ultimately determine what becomes taken-as-shared. Three of these components, the claim, the data, and the warrant, are considered to be the foundation of an argument. A claim is a mathematical statement or solution to a problem. For example, when solving a problem such as $\frac{3}{4} - \frac{1}{4}$, a claim would be that the answer is $\frac{1}{2}$. Data are used as a way to provide evidence for or to back up the claim (Rasmussen & Stephan, 2008). For the problem $\frac{3}{4}-\frac{1}{4}$ the data could be that $\frac{3}{4}-\frac{1}{4}=\frac{2}{4}$. If the data are challenged, then a warrant is needed. A warrant is a justification for why the data are valid. Within the example, a warrant to link the data and claim would be that $\frac{2}{4}=\frac{1}{2}$. When the warrant is challenged, then backing, Toulmin's fourth component, must be provided to justify why the warrant holds authority thereby validating the entire mathematical argument. For example, if questions still arise on how $\frac{2}{4}=\frac{1}{2}$, then the backing would be that 2 is half of 4 and 1 is half of 2, thus $\frac{2}{4}=\frac{1}{2}$. This is summarized in the following figure (See Figure 6).

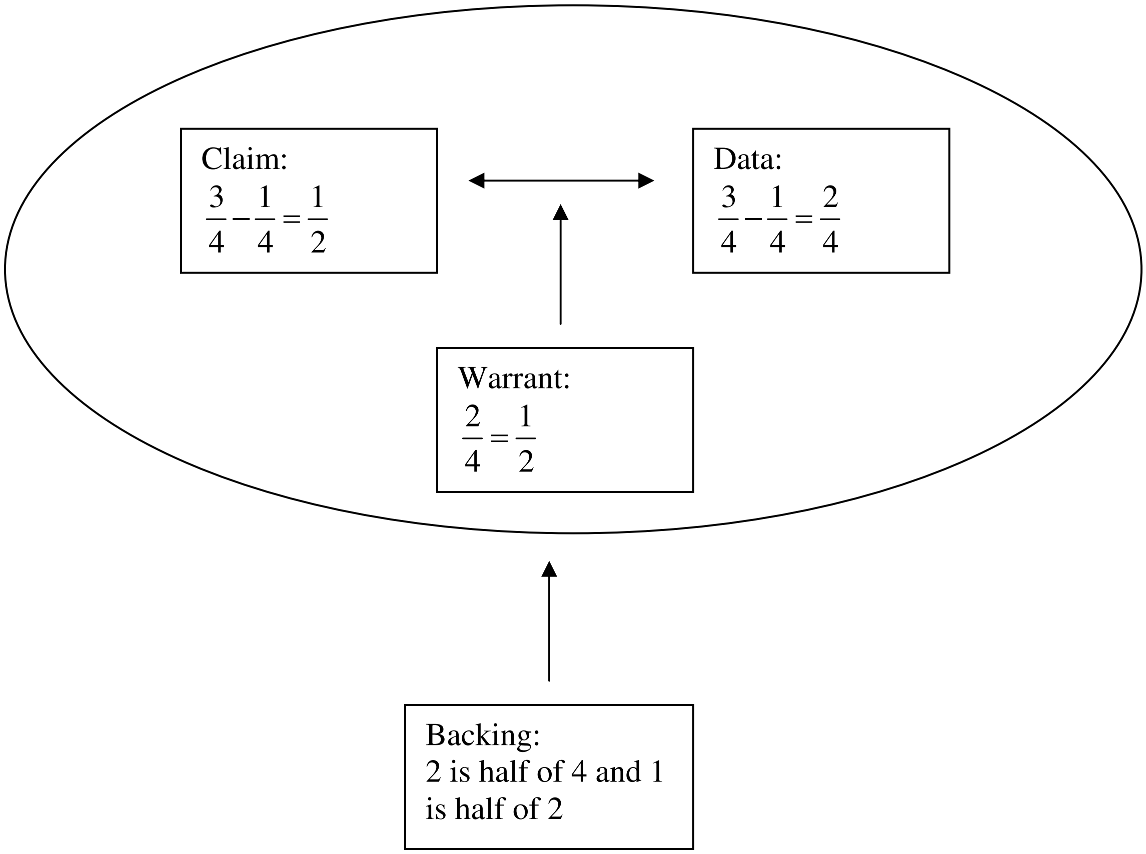


Figure 6: Toulmin's Argumentation Model

Toulmin's argumentation model has been used within other methodologies to document collective activity (Rasmussen & Stephan, 2008). Rasmussen and Stephan's methodology includes three phases for documenting collective activity and was the methodology used to organize and analyze the data in this study.

The first phase of this methodology involved transcribing the video recordings from each class session. Once the videos were transcribed they were analyzed in terms of Toulmin's argumentation model. For this study a team consisting of at least two researchers identified the claims, data, warrants, and backings within each of the transcripts (Rasmussen & Stephan, 2008). This then led to the creation of an argumentation log or scheme.

Within the second phase, the argumentation log was analyzed to determine what mathematical ideas were becoming taken-as-shared. Rasmussen and Stephan (2008) developed two criteria to determine when an idea is taken-as-shared. The first involves analyzing the dialogue and looking for where warrants and backings are no longer used. If claims and data are no longer challenged, then an idea is as-if-shared as no further justification is needed. The second criteria is looking for when the claim, data, warrants, or backing "shifts position (i.e. function) within subsequent arguments and is unchallenged (or, if contested, challenges are rejected)" (Rasmussen & Stephan, 2008, p. 209). If the warrant for an argument later becomes the data for a new mathematical idea, and is not challenged, then the idea is said to be as-if-shared. In the example described the warrant was that 2 is half of 4 and 1 is half of 2. If a future problem is presented, such as $\frac{2}{3}\times\frac{3}{4}$, and a student claims that the answer is $\frac{1}{2}$ because $\frac{2}{4}=\frac{1}{2}$, and this is not challenged, then the idea of $\frac{2}{4}=\frac{1}{2}$ is taken-as-shared.

Once the mathematical arguments were analyzed across the whole rational number unit, the taken-as-shared ideas were used to create a mathematical ideas chart for each class day. Rasmussen and Stephan discuss that the ideas chart should contain three categories including ideas that appear to be taken-as-shared, ideas to look for in future discussions to become shared ideas, and additional notes. The ideas chart from each day was then compared with one another to determine which ideas changed function throughout the rational number unit. This is analogous to Glaser and Strauss's (1967) constant comparative method, where ideas were analyzed and reanalyzed to determine patterns and eventually used to develop a theory.

For the final phase, the ideas chart was used to create a list of mathematical ideas that became mathematical practices. Using Rasmussen and Stephan's (2008) method, mathematical practices in this sense could result from more than one mathematical idea. Thus, the establishment of individual mathematical practices could overlap with one another as the same idea could fall within more than one practice. For example, two practices that could be developed are unitizing and equivalence. The idea of using unit fractions falls under both categories. In unitizing this would entail *creating* a unit fraction, and in equivalence this includes *using* unit fractions to determine if two fractions are equivalent. After the mathematical practices were determined, they were used to refine the hypothetical learning trajectory for future teaching experiments, as described in chapter six.

Coordinating Individual and Social Activity

The first step in coordinating the social and individual aspects of the emergent perspective was to determine the classroom mathematical practices (Cobb, 2000). Once the practices were determined, the next step included analyzing individuals' roles in the emergence of these practices as well as to determine the affect the emergence of the practices had on individual students' learning (Stephan, Cobb, & Gravemeijer, 2003).

Toulmin's argumentation model was used to analyze individuals' contributions to the emergence of the classroom mathematical practices. Since the transcripts had been analyzed using Toulmin's model, the claims, data, warrants, and backings were analyzed for who said them. The transcripts were assessed throughout the entire rational number unit to get a better understanding of an individual's role in establishing the practices.

To document student learning, the data collected from student work samples and classroom discussions were analyzed to determine the affect the social environment had on individuals' mathematical activity. Student learning was documented through two components. The first included looking for where students altered their mathematical activity due to what was presented to them by the classroom community be it through small group or whole class interactions. For example, if a student was shown a different way to solve a problem and the student then changed to using that method from that point on, this was noted. The second included looking for instances where the classroom community disagreed with a solution and/or solution strategy presented by an individual (Stephan et al., 2003). This would in turn cause students to disregard a mathematically incorrect strategy and seek alternative methods which are mathematically correct. At this point, students developed an understanding of not only correct mathematical methods but also developed an understanding of incorrect methods and why those methods were incorrect.

Individuals were analyzed for their contributions to the emergence of the practices as well as the affect the practices had on their learning. The ways in which individual students participated in the emergence of the practices was analyzed through the claims, data, warrants, and backings they contributed (Toulmin, 2003). To document the ways in which the social environment affected student learning, the transcripts were analyzed for the places where the classroom community either accepted or rejected individuals' mathematical arguments and where students altered their mathematical activity due to what was presented to them (Stephan et al., 2003).

Limitations

The intent of this study was to develop a theory about the ways in which preservice elementary teachers' develop an understanding of rational number concepts and operations. Due to the qualitative nature of classroom teaching experiments, with only studying one class of 33 students, the results are not necessarily generalizable to all classes of preservice elementary teachers. In addition, all of the students included within this study were all female, which may have added another limitation to the data.

All of the participants in this study were adult learners. They had prior experiences with learning rational numbers before enrolling in the course. Thus, new ideas presented to the class may not have necessarily been a new topic for students and the instructional sequence may not have been the sole contributor to students' development of rational number understanding.

The instructional sequence itself did not account for students' diversity. For example, language barriers, to be discussed further in the next chapter, required the hypothetical learning trajectory and instructional sequence to be altered. Thus, the diversity in the ways students already understood rational numbers was not taken into account in the planning of the unit.

Conclusion

This chapter presented an overview of the methodologies used for the design and implementation of this study. Using a design-based research methodology, the intent of this study was to document both the collective development of preservice teachers' rational number understanding and the ways in which the social and individual environments interacted with one another. This was done using a cyclical methodology to coordinate the affects the social aspect had on individual students' mathematical learning as well as how individual students contributed to the emergence of the classroom mathematical practices. The next chapter presents the results from this study.

CHAPTER FOUR: RESULTS

The purpose of this study was to document the ways in which preservice elementary teachers develop an understanding of rational number concepts and operations as a collective group as well as the ways in which the social and psychological environments interact with one another. Analysis of the social perspective included determining the social norms, sociomathematical norms, and classroom mathematical practices established as a result of this study. Analysis of the interaction between the social and psychological perspectives focused on the ways in which individuals contributed to the establishment of the practices as well as the impact the social environment had on individual students' knowledge reorganization.

This study was part of a larger study focusing on number and operations. Rational numbers constituted 9 days of class instruction followed by a unit test and was the second unit presented in the course. As described in chapter 3, a hypothetical learning trajectory (HLT) and instructional sequence were designed such that students worked on problems first either individually or in small groups followed by a whole-class discussion. The activities were designed to allow students to reinvent the mathematics for themselves.

A research team consisting of 2 mathematics education faculty and 6 doctoral students met after every class session to discuss students' development and determine whether or not the learning goals for the day were met. In the instances where the team determined that the learning goals were not met, activities were then either modified, added, or taken out to better aid in students' development. Though an initial HLT was created, as discussed in the previous chapter, this was continually refined throughout the duration of the study to meet the needs of the students. A separate refined HLT, determined from the results of this study, will be discussed in chapter five for use in future research.

This chapter presents the results from this study in terms of the social perspective and the ways in which the social and psychological perspectives interact. The social and sociomathematical norms established and sustained throughout the study are discussed first. This is followed by a discussion of the classroom mathematical practices that were established. The practices are discussed in terms of overarching mathematical topics. The ways that the norms and practices were established are illustrated through whole-class discussions. A select number of the practices are used to illustrate the ways in which the social and psychological environments impacted one another.

Social Norms

Social norms constitute the students' and instructor's role in the classroom and are jointly established by both the students and instructor. The social norms that were established within this study included a) explaining and justifying solutions and solution strategies, b) making sense of others' explanations and justifications, and c) questioning others when misunderstandings occur. Social norms were introduced on the first day of class and established and sustained throughout the rest of the semester. To introduce these norms, the first two days of class instruction focused on problem solving activities that were not specific to the content of the course. Though the problem solving activities could have easily been solved with algebra, students were asked not to use algebra when solving them and to develop alternative methods instead. For example, one of the problems the class was presented with was:

> *Fifteen people are at a party. If each person shakes hands with everyone else (JUST ONCE), how many handshakes are there in all?*
>
> *What if there were 20 people? 40 people?*

This problem could have easily been solved with the formula $\frac{n(n+1)}{2}$, but instead students were asked to draw pictures and use reasoning to arrive at an answer. These activities were used to introduce norms such as explaining and justifying solutions and solution strategies and making sense of others' explanations and justifications, and were introduced during whole-class discussions of the problems.

Explain and Justify

Explaining a solution meant that students had to be able to describe how they solved a problem to arrive at an answer. When justifying, students had to be able to describe mathematically why their explanation was valid. Though introduced in problem solving activities, the social norm of explaining and justifying solutions and solution strategies was not established until the first unit of the course focusing on whole-number concepts and operations (Roy, 2008). Roy described the evolution of this norm being established in various phases. Initially students were only prompted to explain their thinking. Following this conversation the instructor stated that students would frequently be asked to share their thinking on how they arrived at a solution. This norm was then negotiated during the second day of the unit. When students were discussing a problem, they only wanted to discuss the answer they got. The instructor then prompted the class to talk about their solution processes first before discussing the solution.

Within the second day of instruction Roy notes that this was the first time when the norm was negotiated to include what it means to explain and justify. By the end of the second day of instruction there was a shift from the instructor prompting students to provide an explanation to students initiating that themselves in which they automatically gave an explanation when discussing the answer they got within a problem. By the fifth day of the whole-number unit, the expectation to explain and justify was taken-as-shared by the class (Roy, 2008).

When the rational number unit started, the expectation to explain and justify was already established as just described. However, the research team had hypothesized that this expectation would have to be re-established due to the content area shifting from whole numbers to rational numbers. Also, past research has shown that the expectation to explain and justify had to be re-established when the content area shifted (Dixon et al., in press; Wheeldon, 2008).

On the first day of the rational number unit, students were introduced to the pizza parlor scenario. Just as they had been placed in the context of a candy shop for whole numbers, they were placed in the context of a pizza parlor for rational numbers. Students were presented with the following problem:

Name a fraction that represents the shaded amount.

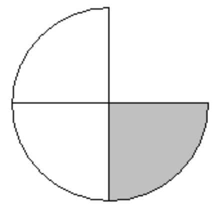


Figure 7: Restaurant Table 1

This was the first problem students were given in rational numbers. The restaurant tables shown in each problem represented the amount of leftovers the table had with the shaded area representing the pizza that had mushrooms on them. The following is the first rational number discussion in the unit. Jane presented two solutions to the problem and did so in the form of a question in which she automatically provided an explanation and justification for both answers.

> Jane: The question I have
> Instructor: Can you hear? Okay.
> Jane: which I think she tried to ask was is the empty space counting as pieces eaten or is it just not there? Because I did my answer to the fact that what I have is all that I'm counting and not as pizza eaten. But counting empty space. So my answer is 1/3 whereas there's is 1/4.

Within this discussion, though the answer came in the form of a question, Jane provided an explanation and justification for two different answers. The answer Jane got was 1/3, which she explained as being out of all that she has, whereas her other group members got 1/4 including the empty space that was missing from the pizza. Rather than just asking for which answer is correct, 1/3 or 1/4, Jane included the explanations and justifications for both responses without being prompted to do so.

When the class moved on to the second problem (see figure 8), students were again not prompted by the instructor to provide explanations and justifications. In the following problem, Edith was trying to explain and justify her answer of 2/8 and how that relates to 1/4. Though she struggled to do this, she knew that she needed to provide that explanation and justification, as indicated by the bolded dialog.

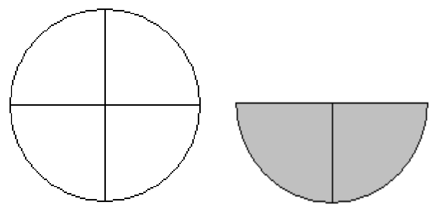


Figure 8: Restaurant Table 2

Edith: Well I had two over eight and I thought about you know simplifying it to 1/4 because you could look at it as like. **I'm trying to think of how to explain it in words.**
Instructor: **I need more than this.**
Edith: **I know.**

During the discussion, Edith's response of "I know" meant that she understood the need to explain and justify her answer of 2/8 or 1/4, though she struggled to do so.

By the fourth day of rational numbers, the idea of what needs to be explained and justified was initiated by the students. Students were given the activity Keeping Track (See Appendix E). Before students started working on this activity the instructor asked if there were any questions regarding the activity. Cordelia asked if they were supposed to explain and justify even though the directions in the activity did not specify that this had to be done. Within students' responses to Cordelia's question, everyone felt that they needed to either explain and justify or at least be able to explain and justify though they were not directly instructed to do so.

Instructor: What I didn't do before I started you working on these problems is ask you if there were general questions or concerns or issues. Yes Cordelia.

Cordelia: **I didn't catch what we were supposed to explain on this part because it doesn't say to.**

Instructor: Okay. For Keeping Track, while it doesn't say to explain and justify, that's a good point. What do you guys think?

Jackie: What was the question?

Instructor: **Do we need to explain and justify? It didn't directly say to.** Jocelyn.

Jocelyn: **I'm sure we're supposed to.**

Instructor: Why are you sure we're supposed to? You're sure we're supposed to.

Mary: **If we're not turning it in though for homework, shouldn't we just be able to explain this if you ask us to come up there?**

Instructor: Caroline.

Caroline: **I would prefer to practice and explain and justify.** So that you can look at it and know how we're doing it.

Instructor: **So I think it's clear to me that you all realize there's this expectation that you need to be able to.** That's consistent with each of your answers. And what you need to do for yourself at your own personal level really is what you need to do. If that practice is helpful, Caroline you want that practice of doing it, then you should do it. Right? But if we do need you to because we're going to collect it and look at it, we'll probably say that we need you to, but you might find it helpful for yourself anyway.

From each student's response, the class understood that they needed to explain and justify. The instructor then negotiated this norm to include when students would have to explain and justify for themselves and/or for the instructor and research team. Even in the activities, such as this, in which the directions were not explicit with telling students to explain and justify, students felt that they were expected to do so anyway.

Though the content area shifted from whole numbers to rational numbers, the expectation to explain and justify did not need to be re-established. Starting from the first day of rational numbers, students were providing explanations and justifications within each of their answers without being prompted by the instructor to do so.

Making Sense of Others

As with the social norm of explaining and justifying, the norm of making sense of others' explanations and justifications was also already established before the rational number unit started. Making sense of others' explanations and justifications was the second social norm taken-as-shared by the class (Roy, 2008). Within the whole-number unit, this norm was introduced by the instructor in the form of a question asking if everyone follows what a student did in her explanation and justification. This norm was then negotiated into two parts. The first involved the expectation to help others when misunderstandings occur. As described by Roy, when students in the class said they were confused, the instructor further negotiated this norm by stating that when a student is confused other students need to help the student understand. The second part of this norm included being able to restate what another student said. When making sense of what other students are doing, the class also had to be able to explain someone else's thought process in addition to understanding her solution method.

Within the first day of rational numbers, this norm was student initiated. The following conversation occurred within the discussion of the solution to the Restaurant Table 1 problem (see figure 8).

> Instructor: Alex do you want to respond?
> Alex: I'm sorry
> …
> Alex: I didn't quite understand what she was saying.

Alex commented that she did not understand another student's solution, and her comment was not initiated by the instructor. The instructor did not have to ask if someone did not understand, rather Alex stated this on her own.

When the class moved on to the Restaurant Table 2 problem (see figure 8), the following discussion involved students making sense of the answers of 1/4 and 1/3. Though Laney got an answer of 1/3, she was able to provide an explanation and justification for the answer of 1/4. After providing that explanation and justification, Laney then went on to state, "If that makes sense."

Instructor:	How many of you got 1/4? Okay. Who wants to talk about it? Laney.
Laney:	**I didn't really get 1/4 but I can see how she got 1/4.** It all goes back to the confusion of taking it from the total amount to taking from what's left over.
Instructor:	Okay.
Laney:	Because if it's just what's left over, it's 1/3 because the other 1/4 isn't there. But if you're taking it from the whole it's 1/4 because there's 4 groups of 2. **If that makes sense.**
Instructor:	Does that make sense?
Class:	Yeah.

Within this discussion two things were happening. First, Laney explained the answer of 1/4 even though she stated that this was not the answer she got. At this point she was making sense of another student's answer. Second, when Laney was providing an explanation and justification for both answers, she asked if her explanation made sense. Within this discussion, students themselves initiated both aspects of this norm without being asked to do so. This illustrates that when the content area shifted to rational numbers that students already expected to not only make sense of others' explanations and justifications, but also to be able to explain and justify others' solutions as well as verify if what they said made sense.

On the second day of rational numbers, Kassie was asked to come to the board to discuss her solution to the problem of sharing 4 pizzas among 5 people. As she came to the board Kassie noted that she could not justify what she had done, however she knew that if she could not provide a justification that someone would help her.

Instructor: Thank you Kassie.
Kassie: I can't really justify it. But I'll try my best.
Instructor: So what's going to happen if you can't justify it?
Kassie: People are going to help me.
Instructor: There you go.

Even though Kassie was willing to share her thinking she knew that others in the class would help her if she could not justify her thinking. During the whole number unit, the instructor told the class that when someone could not provide a justification for her solution, their responsibility as a class would be to help that student with a justification (Roy, 2008). Evident from this discussion, the instructor did not have to reiterate that the class would need to help Kassie. Rather Kassie took this to be understood evident from her response that "people are going to help me."

Though making sense of others' explanations and justifications did not have to be re-established, the instructor continued to sustain this norm throughout the rational number unit. The following conversation, from the fourth day of instruction, illustrates the ways in which the instructor sustained this norm by asking what someone did and why they did that. The class was discussing the following problem:

During a reunion, a family ordered 24 pizzas for 32 people. How could the workers split the pizzas if there were only two tables and one table was a table for 4?

Edith: So here's the pizza. So 12 pizzas go to so this is 12 pizzas for 16 people. So if you broke it down into 4 different groups, each group would get three pizzas. I was going to do the people since it would be easier to divide in 4 but.
Instructor: **What did she say?**
Edith: It would have been easier to go that way too. I just realized that. I'm not thinking right now.

Instructor: **So Winnie, what did she do?**
Winnie: She took the 12 pizzas and divided it by 4 and got 3 pizzas for 4 people.
Caroline: So each group is 4 people.
Instructor: **Do we know why she did that? Cordelia, you know why she did that?**
Cordelia: Yeah. Because the question asked if at one table there was 4 people, you need to find out how many pizzas 4 people will eat and then the rest all goes to the other table because there's only 2 tables.
Instructor: **Is that why you did that?**
Edith: Yeah. I mean how I drew it on my paper, I actually just did this and then had 4 people and so I did another group of 4. I did it that way all the way through.

Throughout this discussion, the instructor asked two different students to restate Edith's explanation and justification by asking them what she did and why she did that. Once Winnie and Cordelia answered those prompts, the instructor then moved the conversation back to Edith. This was done to verify that Edith not only explained herself in such a way that others could make sense of her solution, but also to verify that Winnie and Cordelia's understanding of what Edith had done was correct.

Similar to the norm of explaining and justifying, the norm of making sense of others' explanations and justifications also did not need to be re-established when the rational number unit started. Though the instructor sustained this norm throughout the rational number unit by asking students how and why someone did something, students were the ones who initiated the conversations of explaining others' thinking and asking if what they were doing made sense. The class also understood that if a student could not explain or justify a solution then they were also going to help that student with generating an explanation and/or justification.

Questioning

The social norm of questioning others when misunderstandings occur was the first social norm not already established before the rational number unit started. Roy (2008) notes that though this is one of the four social norms as described by Cobb and Yackel (1996), a conversation in which students questioned the solution processes of others did not surface in the whole number unit. Thus, the topic of rational numbers was the first time in which the norm of questioning students was introduced and established.

During the first day of instruction on rational numbers, the norm of questioning others was introduced. When the class was discussing how they arrived at their answers for the Restaurant Table 2 situation (see figure 8), Claire brought up a new idea of solving the problem with "undividing." When Jocelyn responded that she did not understand what Claire meant by undividing, the instructor then prompted Jocelyn to ask Claire a clarifying question.

Claire:	To get the 1/3 I looked at it as sections. I kind of looked at the whole piece is a half. The top part would be 1 the bottom would be 2, and then the shaded part would be 3 to get the 1/3. I divided them in further into sections.
Instructor:	You divided them further into sections?
Claire:	Well I guess I didn't divide further I kind of undivided them.
Instructor:	**So didn't you mean you can undivide to get one third? What does she mean she undivided? Jocelyn what did she mean when she said she undivided them?**
Jocelyn:	I have no idea.
Instructor:	**Ask her a question.**
Jocelyn:	What do you mean?

When Claire was explaining how she got the answer of 1/3, she introduced the idea of undividing to get the answer. When asked by the instructor what Claire meant, Jocelyn replied that she had no idea what Claire meant meaning that she could not make sense of

Claire's method of solving the problem. The instructor then asked Jocelyn to ask a question since she did not understand what Claire was saying. Claire then went on to justify her method of undividing further.

On the second day of instruction, the instructor again prompted the class to ask a question. However, this time the prompt was provided as a response to someone in the class getting an answer that was different from what was presented.

Share 4 Pizzas Among 5 People

Kassie:	Basically all I did was split each pizza into, is it 5, yeah 5 pieces and pretend they're even. So yeah and then I was like okay I need, I have 5 people so that's why I split it into 5 people, because I figured it would come out even. And I figured by doing this each person will get 4 pieces. Because there will be 4 here gone, 4 here, 4, and 4 for 4 people and then 1 left over in each. So I guess I got 4/20, because together it was 20 pieces and four for each person. Questions?
Instructor:	So raise your hand if you got exactly the same thing. **Okay raise your hand if you got something different. You've got a question.**
Mary:	Me?
Instructor:	Yeah.
Mary:	Well it says determine the fraction of pizza of a pizza each person will get. So I did 1/5 because when I divided each pizza into 5 pieces, each person would get 1 piece.
Kassie:	Oh I looked at it as a whole.

Within this conversation, the idea of students having questions came to the forefront when students indicated that they got a different answer from someone else. The idea of asking a question when an answer was different was introduced by the instructor.

On the third day of instruction, there was a shift from this norm being initiated by the instructor to being initiated by a student. Claudia was at the board explaining how she solved a problem. When she was finished, she left a pause in the conversation for students to ask her questions.

> Instructor: Are there questions?
> Claudia: **Do you guys have questions? I was waiting.**
> Instructor: You were waiting because you were expecting them to do it. Good. Good. How many of you did it just like Claudia? Interesting. Okay. Thank you so much.

By the third day of the unit the norm of questioning others became taken-as-shared by the class. This occurred when there was a shift from the instructor initiating students to ask questions to students taking the initiative in maintaining this themselves. By Claudia waiting after giving an explanation and justification, she indicated that she did so with the expectation that others in the class would ask her questions if they misunderstood something she had said.

Within the norm of questioning others, there were two instances where students needed to question. The first was when a student did not understand another student's thinking. The second was when a student arrived at a different solution from someone else. Throughout the remainder of the rational number unit both the students and the instructor continued to sustain this norm by asking if there were questions.

The three social norms that became taken-as-shared were a) explaining and justifying, b) making sense of others, and c) questioning others. Cobb and Yackel (1996) discuss a fourth norm of indicating agreement/disagreement. Indicating agreement/disagreement was part of conversations as evident in the following discussion:

> Instructor: **Raise your hand if you agree.** Okay several of you don't agree then. So ask a question. Lydia said if the answer was 1/2, it would need to say of 1 pizza is mushroom. Anyone have questions about that one? Because not everyone agreed. So how about table three? Have all of

you had a chance let's just stop and let you think a minute about table 3 and table 4 and then we're going to talk about those answers. **Do we agree?**

Caroline: Yeah.

Instructor: Okay. **Are we okay with that?**

Though there were instances in which students agreed/disagreed with another student, this was provided in conjunction with the expectation to question others. Also, there was never a shift from the instructor initiating this conversation to the students initiating this conversation. Thus, there was not enough in the classroom discussions to determine if agreeing/disagreeing was taken-as-shared.

Social norms, once developed, provide a foundation for new norms to be established. Once students could explain and justify their thinking, they then had to understand others' explanations and justifications. In the instances where students did not understand what another student was doing, they then had to ask that student a question to clarify her explanation and/or justification. As previously discussed, social norms only pertain to the students' and instructor's role in the classroom. Sociomathematical norms are specific to mathematics and pertain to students' mathematical activity.

Sociomathematical Norms

Sociomathematical norms include determining what constitutes an acceptable, different, sophisticated, and efficient solution and solution process (Cobb & Yackel, 1996). The sociomathematical norms established as part of this study included determining what constituted a) an acceptable and b) a different solution.

Acceptable Solution

During the whole-number unit of the course, the class established what constituted an acceptable solution (Roy, 2008). According to Roy, within whole numbers acceptable solutions were those that included both an explanation and justification. At the beginning of the whole-number unit, the instructor started by discussing the importance of providing an explanation when talking about a solution to a problem. This was later negotiated to include providing a justification as well. By the end of the whole-number unit, acceptable solutions were those that included both an explanation and a justification.

Once the rational number unit started, this norm had to be re-established. As previously discussed, the social norm of explaining and justifying did not need to be re-established. Thus, students knew they needed to explain and justify in the context of rational numbers; however, the idea of what it means to explain and justify in mathematically meaningful ways had to be re-established.

During the first day of the rational number unit, students were quick to provide explanations that were reiterations of the procedures they had learned as children. Mary used a "prior knowledge" argument in her explanation of how she went from 2/6 to 1/3.

> Mary: I got 2/6 but from my prior knowledge I know that I can divide that to make it a smaller fraction. So that would be 1/3.
> Instructor: You divided it?
> Mary: I knew you were going to do this to me. Oh you can break down two. I don't know how to explain that.

Mary's use of "prior knowledge" referred to the fraction knowledge she learned before taking this class. Mary's initial response to justify an answer of 2/6 or 1/3 immediately reverted to using the procedure of dividing to simplify the fraction. Apparent from this conversation, Mary knew that her answer would not be acceptable, however she still could not provide a more conceptual explanation. A few minutes later Mary attempted again to provide an explanation and justification.

Mary:	Well I got 2/6 because I counted all those pieces separately. But you could do 1/3 if you just
Instructor:	You said 2/6 or 1/3.
Mary:	Huh?
Instructor:	You said this.
Mary:	Well yeah because I knew that from prior knowledge.
Instructor:	So now here we are with this prior knowledge business. Right? **The prior knowledge is only okay if you can explain and justify it in mathematically meaningful ways.** That came to bite us with long division. Right? Because we can do long division **we can do it but we need to explain and justify it very very cautiously**.

At this point in the conversation, the instructor makes it clear that a prior knowledge argument does not suffice for constituting as an acceptable explanation or justification. Students then had to use other arguments to develop an acceptable explanation and justification.

Continuing to work with the Restaurant Table 2 problem (see figure 8), Caitlyn shared her explanation and justification with the class in which she asked if she had explained and justified in an acceptable manner.

Edith:	But in the first one how, I'm trying to think. No. Wait. No. How they did it with the 2 over 6 or the 1 over 3? Basically taking the sections of 2 slices. So the 2 slices that were mushroom was really 1/4 because it was 1/2 of the 2 pizzas. I'm going to have to come up (to the board). Basically these 2 slices together is 1/2 of this pizza and then there's 4 halves because there's this 1/2, this 1/2, this 1/2, and this 1/2. So there's 1, 2, 3, 4. And then that's 1 of the 4 halves.
Instructor:	Caitlyn
Caitlyn:	I need to, if we're going to need to explain and justify what it

means, 1/4 would just by saying. There are 2 pieces in each of the 4 sections and then 1 section is shaded in. **Is that explained and justified?**

Instructor: What do you guys think? Did you all hear her?

Students: No.

Caitlyn: If there are 2 pieces in each of the 4 sections, and mushroom

Instructor: So 2 pieces in each of the 4 sections.

Caitlyn: and mushroom represents 1 section of 2**. Is that explaining and justifying that it's 1/4?**

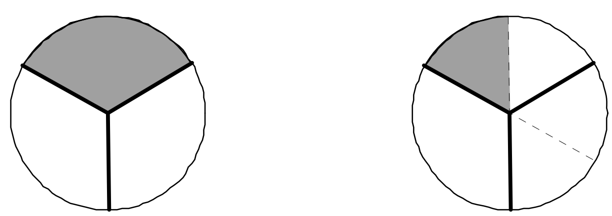
Students: Yeah

Instructor: Agree? Okay.

After the instructor said that it was not acceptable to use prior knowledge, students then started asking what constituted an acceptable explanation and justification. When Caitlyn presented an explanation and justification, she then asked the class to determine if that was acceptable or not.

The negotiation of this norm continued on the third day of instruction when students were discussing their answers to the problem presented in figure 9.

A student was given the following problem:
Share 2 pizzas equally with 4 people. The student did the following:



The shaded amount in the diagram represents the amount of pizza one person got.

Figure 9: A Student Example of Sharing 2 Pizzas with 4 People

The class was frequently given "A student did this…" situations in which the class had to determine what the fictitious student did and if what he or she did would be correct or not. This technique was used for the class to practice making sense of others' explanations and justifications.

Each group of students was asked to come up with an explanation and a justification that could explicate what the student did to solve the problem. The class discussed how to do an explanation and justification with a picture.

Caitlyn: And if you were to number it 1, 2, 3, 1, 2, 3 you could show that you only have 6 and 3 is half of 6. That's a way for you to show it.
Instructor: And what did we assume that makes my picture legal?
Alex: That they're all equal slices.
Instructor: What can't we do though?
Alex: Assume that they're equal slices.
Instructor: We can't look at it and say because it looks like it. Because we all know how I draw and we all know how your students are likely to draw. Looking at your tables you're much better than both of those situations, but we can't say because the picture looks like it. That's not an acceptable explanation or justification. It helps you solve the problem, but you need to explain and justify in ways that aren't because it looks like the picture.

When developing an explanation and justification with a picture, the instructor noted that saying that a picture looks like the answer is not enough to constitute an acceptable answer. Though using pictures was encouraged throughout the semester, students had to develop other means of explaining and justifying in conjunction with a picture.

Students continued to struggle to explain and justify rational numbers in mathematically meaningful ways. On the fifth day of instruction, students still reverted to procedures to explain how to solve a problem. In the following discussion, Jane discussed how she determined how much a piece is worth when sharing a fourth among 3 people.

Jane: On mine I explained it, I said it's not just 1/3 of a pie, it's

	1/3 of 1/4. So when I drew the picture out, I figured out I would have to explain how I got to 12 on that one.
Instructor:	Yes you would. Go ahead.
Jane:	But
Instructor:	Go ahead. Not what you did, but how would you explain how you got to 12?
Jane:	I just multiplied 3 times 4 equals 12. So then both of them would be the same. Instead of trying to find
Instructor:	**And since that's not acceptable, what would you do?** This is, you took this piece. How do you know? Just multiplying 3 times 4, you're pulling things out of the sky here.

As soon as Jane started explaining in terms of just multiplying to arrive at the answer the instructor immediately replied that that is not acceptable and asked for a different way to explain the problem.

The sixth day of rational numbers was when the idea of what constitutes an acceptable explanation and justification shifted from being initiated by the instructor to being initiated by the students. Within the following problem, students were developing ways in which to compare two fractions:

At the party, the trapezoid table was decorated with 5/6 of a spool of a ribbon. The rectangle table was decorated with 9/10 of a spool of ribbon. On which table was more ribbon used?

The second mathematics education faculty member on the research team taught the sixth day of class denoted by Instructor 2.

Katherine:	Well I start by drawing the 5/6 and the 9/10. And I saw that, just I know and by looking at it that this one's more. But I thought that wasn't enough explanation. So I changed it so that they have the same denominator, and this one became 25 over 30 and this one became 27 over 30. And then, like that. And then I explained it that this one used more ribbon because 27 over 30 is, 27 is more than 25 and that's it.
Instructor 2:	Questions for
Katherine:	And then it's still the same equal value even though the fractions changed.

Instructor 2: Questions for Katherine?
Suzy: **Was there a justification? Because I would like to do that, but I'm not going to do that if I really if I don't know how to explain it.**
Katherine: I put the thing that to compare them, I should use the same denominator for both. So I multiplied them by to make 30 because 30 was the first number that they would go into.
Suzy: So just by multiplying you found a common denominator. **Is that acceptable?**
Katherine: The way I wrote it, in my justification. I put I found the least common denominator which is 30. This is the number that both 6 and 10 can be multiplied to make or into. Then I put 6 times 5 equals 30, so 6 and then I went into it.
Instructor 2: So I heard somebody, **I think Caroline, say it didn't seem like it's sufficient.**
Caroline: Uh-huh.

Within the discussion of comparing 5/6 and 9/10 several students solved the problem similar to Katherine in which they just found a common denominator to compare the two. Both Suzy and Caroline questioned the validity of just saying you multiply by a number to get 30. This was the first instance where students initiated the conversation of an answer not being acceptable and it was the first time they did so in terms of only using a procedure to solve a problem. The class went on to develop an explanation and justification that was acceptable for this particular problem by replacing language for referring to procedures for finding a common denominator and instead looking at how many pieces each fraction was missing. Common denominators were an acceptable method; however, students' ways of explaining and justifying them were not, thus they could not be used.

Towards the end of the rational number unit, students initiated the conversations regarding which explanations and justifications were acceptable. Within this shift, students were discussing the use of pictures in explanations and justifications. Caitlyn was at the board and had just presented her explanation and justification for multiplying 2/3 x 3/4 with the use of a picture. The following conversation occurred immediately after Caitlyn finished discussing her solution:

Instructor:	What do you guys think [*about using pictures in explanations*]?
Caitlyn:	Are you asking if you can use a picture to explain?
Katherine:	Yeah.
Caitlyn:	Sure I think so.
Katherine:	Because other times
Instructor:	Is a picture enough?
Caroline:	I think in that case it's very clear.
Caitlyn:	You can't
Instructor:	Caitlyn.
Caitlyn:	**You can't just say because the picture looks like it. You have to provide what you did and why you did it on the picture.**
Instructor:	I agree. Olympia.
Olympia:	I mean we look at your picture we can see what you do, what you did. But I guess if you want to be safe you can just write, you can in writing write what you did. And how you, know added what you added
Caitlyn:	You just explain what you did
Olympia:	Yeah.
Instructor:	And why.
Caitlyn:	And why.
Instructor:	And know that she didn't just write it without speaking. She talked us through it so that it made sense to us why she was doing what she was doing.

This discussion occurred after Caitlyn, who was at the board, discussed with a picture how she got an answer to a fraction multiplication problem. As discussed by the class, the picture was used in conjunction with Caitlyn's explanation and justification, however Caitlyn's explanation did not include the argument that, "the picture looked like it," which made her explanation acceptable.

There were two aspects of acceptable solutions that had to be negotiated by the students and instructor. The first was the idea that reiterating known procedures did not suffice to being an acceptable explanation and justification. The second type of acceptable solution involved the use of a picture. The class had to negotiate that pictures could be used in solution strategies, however it was not acceptable to use the argument that, "the picture looks like it" when explaining and justifying.

Different Solution

The sociomathematical norm of determining what constitutes a different solution was established before the rational number unit started (Roy, 2008). Within whole numbers, this norm was negotiated such that there was a shift from identifying different solutions to explaining the differences between these solutions. By the end of the whole-number unit, students were embracing the fact that problems can be solved in multiple ways.

The rational number unit was designed so that students would arrive at different answers, even if the problem only had one correct solution, so that different solutions could be discussed. In some activities, the directions were left ambiguous so that students would understand that multiple solutions could be obtained. Before the first whole-class discussion of the first rational number activity, the instructor brought different solutions to the forefront of the conversation.

> Instructor: So you've got some different answers it seems… Raise your hand if you're at a table that has different answers from each other at one table. Look at that. Four out of the seven tables have different answers at the same table.

Before discussing the first problem, students were expecting that different solutions were going to be discussed because of the instructor highlighting the fact that students responded to the question in different ways.

When the class moved on to discussing the third problem in the activity (see figure 10), the idea of what constitutes a different solution was presented by the students. Within this discussion students were determining if 1 3/4 is equal to 7/8. Some students thought these two fractions were equal because the picture for each solution looks the same.

Name a fraction to represent the shaded amount.

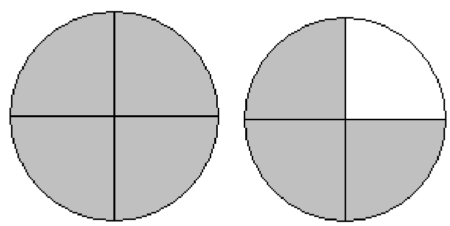


Figure 10: Restaurant Table 3

Kassie:	Okay what I was saying when I said that they're the same is that I looked at it as each individual part. And you do that with the 7/8 too. You look at it as each individual part. Whereas if you had broken it down into the halves, do you see what I'm saying?
Caroline:	You would have had a half left.
Kassie:	You would have had a different
Instructor:	But what happens now if I have this? You said it's either 1 3/4 or 7/8, so they're equal. What do I do to the child that walks up to that? Alex.
Alex:	**It's just a question of how you group your problem.** I grouped mine into 8 individual groups, so I have 7 of the 8 that are shaded.
Instructor:	Okay.
Alex:	Kassie did hers in fourths.
Instructor:	Okay.

Alex: So what hers is, is one group of 4, two groups of 4. Her one group of 4 is an entire mushroom pizza and the 3/4 is the second group of 4 that she worked with. **So she was looking at it, but just grouped it differently.**

The idea presented by several students, before this conversation started, was that 1 3/4 is equal to 7/8. Their line of reasoning stemmed from the fact that the picture in both solutions is going to be the same. Several students at the same time disagreed that the two are equal. Out of the students who disagreed, Alex, noted that the difference was in the way the picture was grouped. The instructor did not have to prompt the class to determine how the two solutions were different. Rather, the class was able to develop this themselves.

At the end of the first day of instruction, the instructor asked the following question:

Instructor: Anyone have another way of describing that they want to share? Questions? Different answer?

The instructor frequently asked this question throughout the other 8 days of instruction. From the beginning of the rational number unit, the class negotiated that different solutions come in two forms. One is a different answer. The other is a different way to represent the same answer.

Midway through the rational number unit, students started to present different solutions without the instructor asking for someone who got something different.

Instructor: Claire.
Claire: Should I show how I did it? Because I got something different.
Instructor: Yeah. Are there questions for Claudia? Or well put yours up there, and maybe you can both stay up there so we can respond.

This conversation was the first time students indicated that they had gotten something different from what was presented. Up until this point, the instructor always asked, "who got something different?", and then asked out of those people, "who would like to share?" Claire not only said she got something different, but also offered to come to the board to show how she got a different solution without being asked to do so. In activities after this, the instructor and students still continued to both initiate conversations regarding different solutions and solution processes.

Throughout the rational number unit, the instructor did not have to re-establish what constitutes a different solution. Though some students struggled with this idea on the first day of rational numbers, the conversations on what makes the solutions different were generated by the students. However, the instructor did keep the idea of different solutions in the forefront of conversations and there was a shift from the instructor initiating this to the students.

Sophisticated Solution

The sociomathematical norm of what it means to have a more sophisticated solution and/or solution strategy was addressed for the first time the second to last day of the rational number unit. The discussion of a sophisticated solution was initiated by Claudia when the class was discussing finding a common denominator for sixths and eighths in the problem 5/6 + 5/8.

Instructor: Claudia.
Claudia: I mean it just goes back to the fact of trying to find the more sophisticated way of solving things.
Instructor: So what does she mean by this trying to find the more sophisticated way? Which is more sophisticated? Finding 24 or finding 48?

Class:	24.
Instructor:	24? So is four, so then the question is using 48 acceptable?
Class:	Yes.
Instructor:	But not completely sophisticated.
Class:	Right.

Though the discussion was student generated, this was the only instance in the entire study, including during the instructional unit on whole numbers, where a discussion like this took place. Thus, there were not enough classroom episodes to determine that students understood what it means to have a more sophisticated solution, though they correctly identified the sophisticated solution within this discussion.

The idea of what constitutes an efficient solution was never mentioned with rational numbers. Efficient solutions were discussed within whole numbers but not mentioned enough to determine if they were taken-as-shared by the class before the rational number unit started (Roy, 2008).

The norms that were established and/or sustained within the rational number unit are summarized in the table below (See Table 5).

Table 5: Social and Sociomathematical Norms Established in Rational Numbers

Social Norms	**Sociomathematical Norms**
Explain and Justify	Acceptable Solution • Without using prior knowledge • Without using what a picture looks like
Making Sense of Others Explanations and Justifications Helping Others Asking if a Solution Makes Sense Explaining Someone Else's Thought Process	Different Solution • Different answer • Different process to obtain the same answer
Questioning Others	

The norms of explaining and justifying, making sense of others, and determining what constitutes a different solution were established before the rational number unit started. Though these norms were established they continued to be sustained and negotiated by the instructor and students throughout the rational number unit.

The only norm that had to be re-established during the rational number unit was what constitutes an acceptable solution. When the class moved on to rational numbers, students were quick to revert to the procedures they learned as children. In addition, the acceptable use of pictures in explanations and justifications had to be negotiated. The idea of what constitutes an acceptable explanation and justification had to be re-established.

The social norm of questioning others was the only norm completely established within the rational number unit. When the class was focused on whole numbers, questioning others was not a focus as most problems only had one solution. Even though different strategies became the focus during instruction on whole numbers, students never questioned what each other was doing (Roy, 2008). Within rational numbers, students frequently questioned the solutions and solution strategies of others. Though this norm had to be established in rational numbers, it was found to be taken-as-shared by the third day of class in which students were expecting others to ask questions for what they had done.

The social and sociomathematical norms that were established constituted the students' and instructor's roles in the classroom. These norms provided students with the foundation they needed for participating within the mathematical practices.

Classroom Mathematical Practices

Classroom mathematical practices are the taken-as-shared ways of reasoning mathematically by the class. The classroom mathematical practices that were established as part of this study were determined using Rasmussen and Stephan's (2008) three-phase approach for documenting collective activity. The first phase involved transcribing the videos from each class session and then analyzing the transcripts using Toulmin's argumentation scheme to develop an argumentation log (See Appendix O). The argumentation log was then analyzed to determine which ideas were becoming taken-as-shared. Finally, the last phase involved using the mathematical ideas chart to determine the classroom mathematical practices.

The interaction between the social and individual environments were determined using the constant comparative method (Glaser & Strauss, 1967) and looking for patterns among the ways in which individual students contributed to the establishment of the practices through Toulmin's (2003) argumentation model. The constant comparative method was also used to determine the ways in which individuals' reorganized their mathematical understanding as a result of the social environment.

The classroom mathematical practices are discussed in terms of overarching mathematical topics. A select number of the practices are used to illustrate the ways in which individual students contributed to the establishment of the practices as well as to illustrate the ways in which the social community impacted individual students' knowledge reorganization.

The rational number unit was placed in the context of a pizza parlor. Each activity that the class was presented with pertained to some situation that occurred within the pizza parlor, and the class was introduced to this starting on the first day of the rational number unit before being given the first activity.

Define Fractions

On the first day of instruction, the class was given an activity to define fractions based on a whole (See Appendix C). Within this activity, the class started developing different ways of defining fractions. These included defining a fraction in terms of the idea that parts need to be equivalent, that fractions represent parts of wholes, and that fractions represent a comparative index.

Equal-Sized Parts

Within all of the earliest conversations, the first idea discussed was that of needing equal-sized parts when naming fractions. This idea was brought to the forefront of the conversation starting with the second problem during the first rational number activity.

Name a fraction to represent the shaded amount.

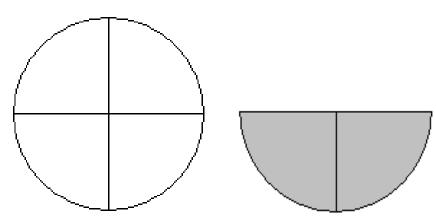


Figure 11: Restaurant Table 2

With the given restaurant tables, students were asked to name a fraction for the shaded amount. One student in the class, Claire, got 1/3 for her solution. When asked how she got that answer, the backing for her argument was the idea that fractions are in equal parts. Within the dialog, claims, data, warrants, and backings are labeled to start highlighting arguments and to note when they shift in function in later discussions.

Claire:	To get the 1/3 I looked at it as sections. I looked at the whole piece is a half. The top part would be 1, the bottom would be 2, and then the shaded part would be 3 to get the 1/3. I divided them further into sections. *(Data)*
Instructor:	You divided them further into sections?
Claire:	Well I guess I didn't divide further I undivided them. *(Data)*
Instructor:	So didn't you mean you can undivide to get 1/3? What does she mean she undivided? Jocelyn what did she mean when she said she undivided them?
Jocelyn:	I have no idea.
Instructor:	Ask her a question.
Jocelyn:	What do you mean?
Claire:	I meant if the whole piece, it's into 4 pieces. And to make that 2 pieces I erase one of the lines. To make it the 2 pieces instead of the 4. *(Warrant)*
Instructor:	Like the line?
Claire:	Yeah. And then I did, so then it's 1 2 3.
Instructor:	1 2 3 4
Claire:	**Well I erased that one too. I looked at them as equal parts.** *(Backing)*

Claire used equal parts as a way to validate her answer of 1/3, which she got by "undividing" or erasing partitioning lines.

Later in the activity, when the class moved on to the fourth problem (see figure 12), the idea of equal parts was brought into the conversation again. This time, equal parts shifted in function from backing to warrant, as Edith used the argument as a warrant for Katherine's solution.

Name a fraction to represent the shaded amount.

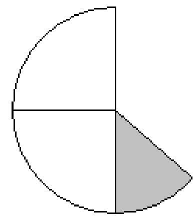


Figure 12: Restaurant Table 4

Katherine:	Well I got 1/5 and I said I broke the other 2 slices up in half. Right. And so I put 1/5 of the leftovers is mushroom. Or it could have been 1/8 if you would have done it out of a whole pizza. *(Data)*
Instructor:	Okay. You've explained what you did. And you've justified the pizza versus the leftovers, but you didn't justify why you did that.
Katherine:	Because the picture looks like there's 1/2 of a slice, so I didn't want to say 1/2 of a slice of 2 1/2 slices because that's too confusing.
Instructor:	1/2 of 2, 1/2 of a slice of 2 1/2. I see.
Katherine:	Right.
Instructor:	Do you guys see what she said?
Class:	Yeah.
Katherine:	That's kind of confusing in wording. So I wanted it to be 1/5 so that it could be. You could look at it out of the big picture instead of like
Instructor:	What word is she grasping for there?
Edith:	**She wants all the pieces to be equal.** *(Warrant)*

Within Katherine's answer of 1/5, the idea of equal parts became the warrant for why an answer of 1/5 is valid. At this point in the conversation, there was no need for a backing for what was meant by equal parts, and no one in the class questioned Edith.

The idea of fractions being in equal parts did not just pertain to fractions less than one. The idea of equal parts was also used to make sense of fractions greater than one as seen in the Restaurant Table 3 problem (see figure 13).

Name a fraction to represent the shaded amount.

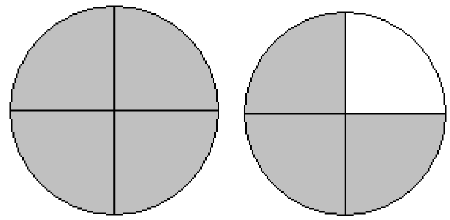


Figure 13: Restaurant Table 3

Students came up with the solution of 1 3/4, which they then converted to 7/4. When justifying how this was done, Claudia used the idea of equal parts to convert from one solution to another.

> Claudia: Because what you're doing is you're adding the one, which is equal to 4/4. When you're multiplying it you're multiplying the four times one because you're making the one pie into the four parts to kind of make it standardized? If you guys get what I mean. *(Data)*
>
> Instructor: What? I don't. What do you mean standardized?
>
> Claudia: **So you're trying to get with the one you're just seeing it as one whole but you want to break it up into the same amount of slices so that each slice is the same exact size.** *(Warrant)*

Claudia used the idea of needing a same-sized piece when talking about fractions in order to justify how to convert 1 3/4 into 7/4. By breaking the whole of one into four equal sized pieces, the pieces are then the same size as the pieces in the 3/4. Claire's argument for "standardizing" the one had to be warranted by the idea that the pieces need to be equal sizes.

The discussion of equal parts did not start to surface until the second problem in the activity. When the first problem was discussed, no one questioned students' solutions being in equal parts. Rather, this appeared to already be understood. The discussion of equal parts only surfaced to validate the new ideas, such as "undividing" or "standardizing," and never got discussed in terms of solutions needing to be in equal parts. Even with the first problem in the activity, students were arriving at answers that were based on equal parts, though this was never stated directly.

Equal parts appeared to already be understood as evident from students' solutions starting with the first question. Equal parts were discussed when students brought new ideas to the class, such as "undividing" or "standardizing," and to help students develop a better method for explaining and justifying their solution, as was evident with Katherine. Questions were never raised about fractions needing to be in equal parts. Though the idea of equal parts shifted from a backing to a warrant within these discussions, it appeared that the idea that fractions are comprised of equal parts was already established before students started the rational number unit. This is similar to the findings of previous research with preservice teachers (Wheeldon, 2008).

Fractions Represent Comparisons

The ideas that fractions are parts of wholes and/or a comparison of the number of pieces you have to the number of pieces in the whole, were discussed when solutions greater than one were obtained. During the discussion of the Restaurant Table 3 problem (see figure 13), once Claudia justified converting 1 3/4 to 7/4, the class then had to make sense of 7/4 as a fraction. In response to this, Caitlyn replied that 7/4 is not a fraction because you can't have 7 parts out of 4.

Caitlyn: Well seven, okay. **To me 7/4 is not a fraction.** Well it is a fraction**. But when I think fraction I think of a part of something. And when you have more on the top than on the bottom that's not a part of something. That's more.** So you have to convert it into a whole and then

your 3/4 left over. *(Data)*
…

Instructor: So you're saying this is not okay.
Caitlyn: Right. **Well it doesn't make sense because a fraction is supposed to be part of something. And you have more**
Caroline: More than part of something.
Caitlyn: We have more than part of something. We have an overflow I guess. **If you are trying to put 4 you can't make 7. You can't color in 7 pieces out of 4.** *(Warrant)*

Evident from Caitlyn's response, it appeared that students were coming into rational numbers with the understanding that fractions had to be parts of wholes. In the cases where there is an "overflow" you have to change the fraction into a mixed number in order to make sense of it. No one in the class questioned Caitlyn's statement that fractions are parts of wholes. Just as Caitlyn had difficulties, others in the class had difficulties making sense of 7/4 without going to 1 3/4. This was because a fraction of 7/4 is impossible in a part-whole situation (Streefland, 1991).

In order to make sense of 7/4 without going to 1 3/4 and without using a part-whole argument, students started developing other ways to define fractions. As evident in the following conversation, Barbara and Cordelia both contributed to this discussion by referring to 7/4 in terms of what the 7 and the 4 represent as opposed to using a part-whole relationship.

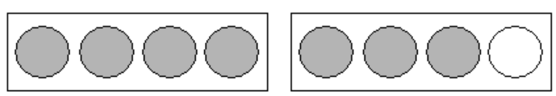
Barbara: But the 1 3/4 makes it understandable because you're turning 7/4 into 1 whole and then three-quarters. *(Claim, Data)*
Instructor: But since you can turn it into this [7/4], this [7/4] must be understandable also. How can we make sense of it? What does it mean?
Barbara: **It means that there's 7 pieces with one whole being 4 pieces.** *(Warrant)*
Instructor: Okay so there's 7 pieces with 1 whole being 4 pieces. So you're making meaning going this way instead of necessarily this way. Which is fine. What other meanings

could we make of that? Cordelia.

Cordelia: **Well if you look at it with a 4 in the bottom number, 4 represents how many there are in a whole.** So if you take 4 away from 7, that gives you 1 whole with still 3 leftover out of 4 wholes. Out of 4 pieces that make a whole. So that gives you 1 whole and then 3 parts of a whole. *(Backing)*

This was the first conversation in which fractions were referred to as representing the number of pieces you have compared to how many pieces are in a whole. Though both Barbara and Cordelia had to make 7/4 understandable by still going to the 1 3/4, they introduced the idea that 7/4 can represent two individual quantities of 7 pieces to 4 pieces total.

A few moments later, the conversation of making sense of 7/4 continued because the class still struggled with understanding 7/4 without going to 1 3/4. In an effort to make sense of 7/4, Cordelia drew the following representation at the board:



When discussing her picture, Cordelia still could not make sense of 7/4 without referring to 1 3/4. Claudia then added to the conversation by using the idea that fractions are a comparison of the pieces you have to the number of pieces in the whole as data and warrant for Cordelia's diagram.

Instructor: Claudia

Claudia: I think the question goes back to, what is a whole? A whole is the box as she says, because the whole is broken up into four parts. And so technically we have two wholes. **The four just says how many parts are in each whole. So there's four and then the seven is how many parts of the whole do we have.** We have seven parts of actually two wholes. *(Data)*

Instructor: You said you have seven parts of a whole but actually two wholes, and then I get confused about if I didn't have the picture I wouldn't know what you were talking about.

Claudia: Okay. **Because there's seven parts and then we know that each whole has four parts to it. So we actually have seven parts but the four just shows us how many parts are in each whole.** *(Warrant)*

Claudia's discussion was the first point in the conversation at which the fraction 7/4 was not related to 1 3/4. Using the idea that Barbara and Cordelia introduced, Claudia used the argument that 7/4 represents the number of parts you have to the number of parts in the whole to warrant her argument, which referred to the fraction as a comparison of individual quantities.

Evident from the previous discussions, the idea that fractions only represent parts of wholes was questioned because students could not make sense of 7/4, in terms of a part-whole relationship. This is similar to other research that found that students are not able to understand fractions greater than one with a part-whole definition (Charalambous & Pitta-Pantazi, 2007). Thus, this idea had to be altered to include that fractions also represent a comparison relationship, like a ratio, in which you compare the number of pieces you have to the number of pieces it takes to make a whole.

<u>*Denominator Represents Equal Parts in a Whole*</u>

The comparative nature of rational numbers led students to find different ways to define the denominator in a fraction. The first, as just described, referred to the denominator as the number of parts the whole is divided into. This definition was slightly altered when students moved on to making sense of situations in which pieces were combined together. Combining pieces was not discussed until the second day of instruction when the class was working on a fair sharing activity. One of the fair sharing problems students were presented with is presented in figure 14:

Share 5 Pizzas Among 3 People

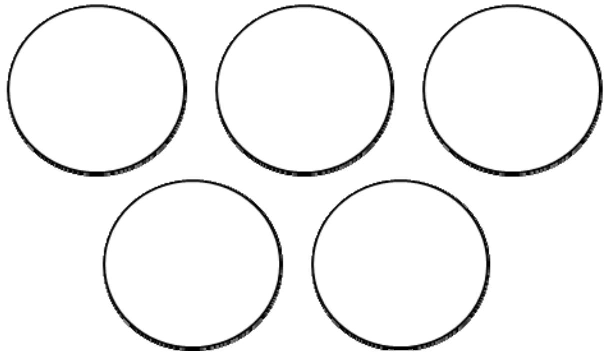


Figure 14: Share 5 Pizzas Among 3 People

Some students arrived at the solution of 5/3 by partitioning each pizza into three equal pieces. Once students did this, they first gave each person one piece from the first three pizzas giving away 3/3 of a pizza to each person, they then gave them an additional 2/3 from the last two pizzas. Within the conversation of combining pieces to arrive at an answer of 5/3, Cordelia discusses fractions in terms of what the "bottom number" of a fraction represents in order to justify how pieces are combined.

Instructor:	What did you do when you got 5/3? What did you do when you got 5/3? Veronica what did you do when you got 5/3?
Veronica:	I added 3/3 to 2/3. *(Data)*
Instructor:	How?
Veronica:	You combine them.
Instructor:	You combined them. How did you combine them? What do you mean you combined them?
Veronica:	Added them together. You have the same denominator so you can add them. *(Data, Warrant)*
Instructor:	Do you know what kids do? When we teach them that kind of thing.
Caroline:	Add both numbers.

Instructor: This is what kids do when we teach them those rules. What can we do to keep kids from, teaching those rules? Drilling leave the denominators the same and add the numerators clearly isn't it, because we've tried that. What do we do? Laura what do you think?

Laura: I don't know.

Instructor: Cordelia.

Cordelia: **I think they need to understand that the bottom number doesn't change because the bottom number is only representing how many equal parts the whole is divided into.** *(Backing)*

When combining pieces together, as Cordelia stated, the denominator does not change because it represents how many equal parts the whole is cut into. Before, Cordelia referred to the denominator as being the number of parts the whole is divided into. This time, she was more specific and referred to the denominator as being the number of equal parts the whole is divided into. As evident from this conversation, the definition of what the denominator represents directly came from the idea that fractions also represent a comparison in which the "bottom" number refers to the number of equal pieces in the whole.

The idea that fractions represent the number of pieces you have compared to the number of pieces in the whole was used as backing for Veronica's method of combining pieces together. This was the second time this idea shifted in function in a conversation. The first time this idea shifted, it went from backing to warrant. Within this conversation the argument shifted from warrant to backing and was never questioned. Thus, the idea that the denominator represents the number of pieces in the whole was taken-as-shared.

The idea that the denominator represents the number of pieces in a whole was the first denominator idea that was introduced and established. Later on in the unit, students introduced a new idea that the denominator represents all the pieces together in the whole.

<u>Denominator is All the Pieces Together</u>

Defining the denominator as being all the pieces together is the opposite of defining the denominator as being broken up into a set number of pieces. This reciprocal thinking is similar to understanding the difference between a composite whole of one and the number of pieces it takes to make that whole (Lamon, 1996).

The idea that the denominator can also represent all the pieces together was not discussed until later on in the rational number unit, after the unitizing activities had been presented. The unitizing idea, to be discussed further later, of developing a composite unit of one was taken-as-shared by this point. The conversation regarding the denominator was needed when Cathy questioned how half of the denominator could be written in the numerator. This was the first conversation in which the denominator was referred to as being all of the pieces in the whole together.

Cathy:	I just don't see how mathematically you would put 1/2 of the denominator on top of the denominator I guess.
Claire:	It equals 1/2.
Claudia:	Because you're looking at 1/2. *(Data)*
Caroline:	We're just comparing to 1/2 of the pieces *(Data)*
Edith:	We're just seeing which comes closest to 1/2 of the denominator. *(Warrant)*
Claudia:	**Because the denominator is all the pieces together.** So you want to know what 1/2 of the pieces are. *(Backing)*

The unitizing idea of having a composite unit of one was used to further define denominators. As mentioned, the idea of a composite unit was taken-as-shared several class days before this conversation. That composite unit was expanded as a second way of defining the denominator as evident from Claudia's argument.

From this point on, the class interchangeably used both ideas that the denominator represents the number of pieces in the whole as well as being all of the pieces together. This is consistent with unitizing in that one whole is equivalent to three individual pieces, for example, and at the same time it takes three pieces to make a whole (Lamon, 1996).

As illustrated, when the class developed ways to define fractions they also developed ways to define the denominator. The idea of what the denominator represents did not surface in the classroom conversations until the problem situation involved a fraction greater than one. By incorporating fractions greater than one, students had to alter their definition of fractions to include that fractions are also comparative in nature.

Fractions are Parts of Wholes

Even though the class defined fractions as a comparison, they only did so in the context of fractions greater than one. Thus, the idea that fractions represent parts of wholes still continued to be used throughout the duration of the rational number unit, specifically in the cases where the fraction was less than one and also when fractions were contrasted with ratios.

The parts of wholes discussions only surfaced in the conversations when the topic at hand questioned the part-whole relationship. As previously discussed the first time this occurred in the rational number unit was when a fraction greater than one was presented. The second time this occurred was when an activity was presented in the context of ratios.

The rational number unit included work with ratios in the context of equivalence situations. While working with the equivalence activities, several students noted that the problems were in terms of ratios and not fractions. As a result of the activity being placed in the context of ratios, students questioned the methods used to solve the problems because they contradicted the way the same situation with fractions would be solved, specifically in the context of operations.

While developing equivalence techniques with ratios, one method which can be used to find another equivalent rational number is by continually adding equivalent amounts together. For example, $\frac{1}{5} = \frac{1}{5} + \frac{1}{5} = \frac{2}{10}$. The activity was presented so that students could develop 1/5 and 6/30 as equivalent fractions by continually adding groups of 1/5 together.

Jane questioned why adding ratios allowed you to add both the numerators and denominators together. In the context of fractions, the denominator remains the same. The idea that fractions represent parts of wholes was used as backing for why the fives could be added together, in this problem and not in others.

Jane:	This just goes back because we kept saying. I understand everything that you guys are doing but we kept saying okay you have to have the same number pieces on so how are you going to explain that? That it works in this one but not in all the others?
Instructor:	It's a good question. I'm struggling with that question also. Cordelia.
Cordelia:	Because you were adding 1/5 to 6/30 and we just added a whole bunch of one-fifths together. *(Warrant)*
Instructor:	Okay but what she's saying is when we did this before we made sure we had thirtieths and thirtieths.
Cordelia:	But that was fractions and this is ratios. They're different things. **Fractions are parts of wholes**. **Ratios are not parts of wholes.** *(Backing)*

When Jane questioned why the denominators were allowed to be added together, Cordelia referred to the idea that the problem situation they were working on was in ratios not fractions. Since fractions are in parts of wholes, the whole remains the same. In the context of ratios, the rational number is a comparison of two distinct quantities which allow you to add each quantity together.

During the rational number unit, several conversations regarding how to define fractions surfaced in various discussions. When a solution pertained to fractions greater than one or involved denominators, such as when adding, students defined fractions as comparative indexes. In other instances, such as when ratios became the focus or the fractions were less than one, then fractions were defined as being part of a whole.

Three ideas became taken-as-shared by the class by the end of the second day of instruction. The first was that of fractions being comprised of equal parts, which appeared to be taken-as-shared before the rational number unit started. The second was that of fractions representing parts of wholes. This idea again appeared to be taken-as-shared by the class before the rational number unit started. The third idea, which was established during the rational number unit, was that of fractions representing a comparative index of the number of pieces you have to the number of pieces in the whole. This idea was only discussed in the context of describing fractions greater than one and when the situation incorporated ratios.

A fourth way of defining fractions is not within the fraction itself, but labeling a fraction in terms of its whole. This idea was also focused on starting the first day of instruction and highlighted throughout the rational number unit.

Defining the Whole

Defining the whole is important for students to understand numerous rational number topics (Simon, 1993). The first activity in the rational number unit was designed so that students could start developing ways in which to define the whole in their solutions of representing shaded regions (See Appendix C). The intent of the activity was to have students successfully name fractions, not only in terms of their relationship to the whole but also in terms of what the whole represented. For example, students were given the problem presented in figure 15 and instructed to name a fraction that represents the shaded amount.

Name a fraction to represent the shaded amount.

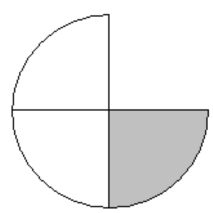


Figure 15: Restaurant Table 1

Within this activity, all of the pictures shown represented the amount of pizza that specific tables in the pizza parlor had leftover. The shaded amount represented the part of the pizza that had mushrooms on it. Being asked to name a fraction to represent the mushrooms or the shaded amount, instead of just naming that amount as 1/3, the intent of the activity was to have students label that region as 1/3 of the leftovers. This was to introduce students to the idea of providing enough information when labeling fractions so that an exact amount could be determined as opposed to an arbitrary amount.

Within the Restaurant Table 1 situation, the class initially used the argument that the directions for the activity were not clear enough to determine a single correct answer. By the end of the conversation regarding this problem, the class agreed that both 1/3 and 1/4 sufficed as answers to represent the mushroom pizza. However, 1/3 or 1/4 alone was not enough information, and Claire introduced the idea of including an "of what."

Instructor:	So here I am, I've got 1/3 and 1/4. How can they both be right? How can I just leave it? You say they both can be right. I need more information. What else would need to be here? Yeah.
Claire:	**You could write you need to fill 1/3 of what.** So 1/3 of the leftovers is mushroom or 1/4 of the whole pizza was leftover. *(Warrant)*

Claire's idea of including an "of what" was the first instance where the class started to develop the idea of defining the whole. The "of what" idea allowed students to arrive at multiple answers to the same problem and also have multiple wholes within a problem. This is similar to what students encounter within fraction multiplication situations as well, which will be discussed further later.

With an answer of 1/3, the whole is the leftover pizza. With an answer of 1/4, the whole is the entire pizza, if the missing piece were filled in. The directions were intentionally left arbitrary so that the class would have this conversation, including what the whole is when discussing fractions gives a more precise answer to a problem.

When the class moved on to a situation where the amount that was left was more than one whole pizza, the idea of defining the whole shifted in function in the conversation. Within the problem presented in figure 16, the class found two answers to be 1 3/4 and 7/8:

Name a fraction that represents the shaded amount.

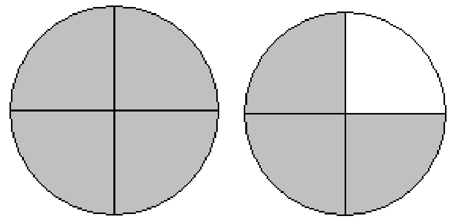
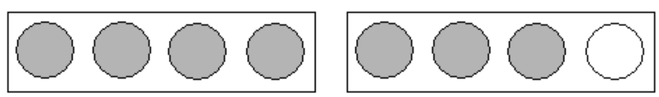


Figure 16: Restaurant Table 3

Several students in the class determined that these two answers are equivalent because nothing changed in the original picture to get those answers. Pieces did not have to be drawn in or taken out to arrive at either answer. The class continued to attempt to make sense of each answer and if they were in fact equivalent or not. Within this conversation the class also arrived at the answer of 7/4 by converting 1 3/4 procedurally. When the conversation shifted to justifying how to make sense of 7/4 without going to 1 3/4, Cordelia drew the following on the board:



As part of the conversation of making sense of 7/4 as a fraction, Claudia mentioned that the whole needed to be determined in order to make sense of the fraction.

Claudia:	**I think the question goes back to, what is a whole?** *(Backing)* A whole is the box as she [*Cordelia*] says, because the whole is broken up into 4 parts. And then so technically then we have 2 wholes. The 4 just says how many parts are in each whole. So there's 4, and then the 7 is how many parts of the whole we have. We have seven parts of actually 2 wholes. *(Data)*
Instructor:	Well because you said you have 7 parts of a whole but actually 2 wholes, and then I get confused about if I didn't have the picture I wouldn't know what you were talking about.
Claudia:	Okay. Because there's 7 parts and then we know that each

whole has 4 parts to it. *(Warrant)* So we actually have 7 parts but the 4 just shows us how many parts are in each whole. *(Data)*

The idea of defining the whole shifted in function in the conversation. The idea was used as backing for making sense of how 7/4 is understandable. Claudia then used this idea to start defining fractions, as previously discussed, in which the bottom number represents the number of parts in the whole. By the end of this conversation the class determined that 1 3/4 is 1 3/4 pizzas, 7/8 was 7/8 of the remaining slices, and that the two were not equivalent numbers though their pictures were the same.

On the second day of instruction, the class was still developing how to label an answer in terms of the whole. The idea of needing to define the whole shifted to a warrant in the following classroom episode. The discussion took place when the class was answering how much of a pizza each person would get if they shared 4 pizzas among 5 people. One student in the class, Kassie, determined the answer to be 4/20 from cutting each pizza into 5 equal pieces and giving everyone one piece of each pizza. Everyone received 4 pieces out of the 20 pieces total. In order to help students make sense of this answer, the instructor introduced a scenario of each piece being worth 6 points, then asked the class how to make sense of 4/20 in terms of how many total points it would be. Claudia then concluded that an answer of 4/20 does not include enough information by itself, which Katherine then replied that the 4/20 was out of 4 pizzas total.

Claudia:	**4/20 doesn't give her enough information because**
Instructor:	Okay are you listening?
Claudia:	all we know is how much 1 slice is worth, which is 6 points. So if we just say 4/20 of all of it, well we can't say how much you know we won't have enough information to figure out how many points of it is total. *(Warrant)*
Kassie:	But you do because you have the 4 out of the 20 slices.

Instructor: Do you want to

Edith: Well because you were saying how 4/20 could be reduced to 1/5. So if you're looking at it that way, at least that's the way I was thinking. The 4/20 is representative of just 1 serving of it. It's not really the entire, yeah they got 4 slices out of 20 but if you wanted to reduce it it's only 1/5. Which isn't, that's only of 1 serving of it. *(Claim)*

Instructor: I think we need to answer this question. 4/20. Okay Katherine.

Katherine: Well I don't think what she does is wrong because right when she wrote it I saw how she got that answer. But I did it of 4/5 because I did it out of the 5 slices of 1 pizza. *(Data)*

Instructor: So yours is 4/5 of, say it again, 1 pizza?

Katherine: Of 1 pizza *(Warrant)*

Instructor: Okay. And then she has what?

Katherine: **4/20, which is 20 of the 4 pizzas total.** 20 of all the slices put together. *(Warrant)*

Instructor: Now would we know how many points?

Claudia: Yeah.

Katherine: The same amount. 24.

Instructor: **So it sounds to me like if we knew it was 4/20 of 4 pizzas, it would be correct.**

Katherine: Right.

Claudia started the conversation that a fraction written by itself is not going to be enough information in determining exactly how much that fraction is worth. Katherine then used Claudia's idea of needing more information to determine that 4/20 represented 4/20 of the four pizzas.

Both Claudia's and Katherine's arguments were warrants in the conversation. This shifted from the previous day in which this same argument was a backing for the conversation. Though the argument shifted in function at this point, this idea was not taken-as-shared because the instructor had to present the class with a secondary scenario of making each slice worth six points for the class to determine that 4/20 was not enough information.

After this discussion, students included labeling the whole in their solutions automatically. When discussing the problem of sharing 3 dessert pizzas among 4 people, Mindy provided her answer in terms of the whole.

> Mindy: So I divided the first two into halves. So each person gets an equal half and then the last one I just divided into fourths. And then, to find out how much they got total, I found a common denominator which they both go into four, but I used 8 so we'll use four. Four goes into 4 one, and one times one is one. And two goes into four two and two times one is two. And then I just added that. **And I got 3/4 of a dessert pizza.** Any questions?

When Mindy gave her answer of 3/4, she included the whole automatically in her solution.

This idea also continued to be used throughout later activities. Even on the fourth day of the unit, students were providing a label in their answers without being prompted to do so as illustrated by Katherine's comment.

> Katherine: I got 3/4. Each person gets 3/4 of a pizza.

By the fourth day of the unit, students were able to define the whole when their solutions were less than one and did so without being prompted by the instructor to include that in their answer. For fractions greater than one, students continued to have difficulties defining the whole through the operations portion of the unit.

Defining the whole became taken-as-shared before the class reached the operations portion of the rational number unit. Even though this idea was taken-as-shared, students continued to struggle with how to do this correctly when the fraction was greater than one. One of the addition problems presented to the class during the operations portion of the unit was 5/6 + 5/8. The class determined that the solution was

70/48, however students such as Caitlyn commented they still had difficulties explaining the answer in terms of the whole.

Caitlyn:	**If we're using pizzas would that be 70/48 of a pizza? Because I still don't understand that whole pizza thing.**
Instructor:	So we had 5/8 of a pizza leftover and 5/6 of the same size pizza leftover. How much pizza do we have altogether? We have 5/8 of a pizza and 5/6 of a pizza. Edith.
Edith:	**Wouldn't it be 70/48 of one pizza because one pizza is 48?**
Instructor:	Are you listening Caitlyn? Say it again.
Edith:	**Wouldn't it be 70 of 48 or 70/48 of one pizza because one pizza is 48 slices?**
Caitlyn:	Okay. Oh I see why now.

As evident from Caitlyn's discussion on the eighth day of the unit, students still struggled with defining the whole for fractions greater than one. Edith used the idea of the whole being represented by the denominator to help Caitlyn in determining that the answer should be of a pizza or of one pizza. It may be that students continued to struggle with defining the whole for fractions greater than one because they did not have a complete understanding of a rational number as a quotient (Charalambous & Pitta-Pantazi, 2007).

Throughout the rational number unit, students frequently had difficulties with the differences between the language of *a* pizza, of *one* pizza, of *each* pizza, and of *the* pizza. Some of the difficulties students had with these discussions were language customs that students grew up with as the class consisted of students originally from different areas of the country. The instructional sequence was altered the third day of class to include an activity targeting these language issues (See Appendix N). The activity was designed so that students had to first determine if the picture represented a fraction of the pizza, of a pizza, of one pizza, and/or of each pizza. For the second problem, students were given each of the four statements and asked to determine what the picture would look like in each scenario. The activity only included fractions less than one. Less time was spent on developing ideas with defining the whole when the fraction was greater than one, which may have been why students still continued to struggle with them.

By the end of the second day of class, defining the whole in terms of the answer became taken-as-shared as students did this automatically without being prompted to do so by the instructor. In addition, by the end of the second day warrants and backings were no longer needed in the conversations. At this point, it was not as though students could do this seamlessly, as they continued to struggle with defining the whole especially when the fraction was greater than one. Rather, students understood that they needed to define the whole when discussing rational numbers. This was evident also in the shift from the instructor highlighting that students should define the whole to students questioning or including the whole when this was not done automatically.

Define a Whole of One

A second aspect of defining the whole that started being established on the second day of class was defining a whole of 1. This idea was only discussed during problem situations in which the answer was greater than 1. Given the problem of sharing 5 pizzas among 3 people, the class was discussing arriving at an answer of 10/6 or 1 4/6. Kassie mentions that 6 over 6 or one could be used in this situation.

Kassie:	I was going to say what she just said. If you did the 6 over 6 you would be adding it a different way. So you're final answer would be 10 over 6. *(Claim, Data)*
Instructor:	And this way isn't correct and the other is correct?
Kassie:	No. **Well, I think they're both correct. You don't have to have the 6 over 6.**
Instructor:	Why not?
Kassie:	**Because the one is the whole.** So if you know that you just add the other parts. And you still have the one.

(Warrant)

Instructor: Nancy.

Nancy: I kind of did it with the 6/6. I think it keeps it consistent with the rest of the

Instructor: Keeps what consistent?

Nancy: Because she was adding the 3/6 and the 1/6. And there's just the one there and I think if it's 6/6 even though it does represent one, it keeps I'd say it's easier for them to see it keeps the numbers consistent so. I don't know how to explain it. *(Warrant)*

Katherine: Because it's 6 and sixths *(Backing)*

Kassie introduced the idea that one whole would be the same as 6/6 for this particular problem. Nancy then provided a warrant that 6/6 would keep the numbers consistent with the 3/6 and 1/6. Then Katherine provided the backing of the 6/6 being consistent because it is 6 and sixths. Defining a whole of one became important for the situation where the answer was greater than one. As just described this could be represented as 1 or in the form x/x where $x \neq 0$.

During the discussion of the same problem other students arrived at an answer of 5/3 or 1 2/3. Through this discussion, Barbara and Mary further developed a whole of 1 by justifying how 3/3 is equal to 1 in terms of how fractions are defined.

Barbara: And then 3 of those is 3/3, which is 1. And then you have 2/3 leftover. There's your 1 2/3. *(Data)*

Instructor: How do you know 3 of those is 1?

Barbara: **Because you have a pizza and it's split into 3 pieces. So if you have 3 pieces in that 1 pizza.** *(Warrant)*

Instructor: Okay. And how do you know that this is 2/3?

Barbara: Because there's 2 pieces of a pizza that has 3 pieces. *(Warrant)*

Instructor: What do you guys think about that? Mary, you had your hand up.

Mary: **I was going to say that if you have 3 pieces of the 3 pieces that means you have the whole thing, which is 1.** *(Warrant)*

Unlike the previous discussion in which students just mentioned that 6/6 is equal to 1, Barbara and Mary took that idea a step further in defining how the fraction 3/3 is equal to 1. They did this by specifically referring to how fractions are represented by the number of pieces you have to the number of pieces total. This was also the first step in the class developing a composite unit of one.

Two ideas were prevalent in defining the whole. The first, which was established, was that of labeling an answer in terms of the whole. This was established on the second day of instruction within the fair sharing activities. The second idea was that of defining a whole of 1. This idea was established, but not until later in the unit, and is discussed further in the unitizing section.

Partitioning

Partitioning involves the process of breaking an object or quantity into a set number of pieces. Partitioning was addressed in the rational number unit in the form of fair sharing situations (See Appendix D). This activity was presented on the second day of instruction in which students were given situations, such that which is presented in figure 17.

Share 4 Medium Pizzas Equally Among 5 People

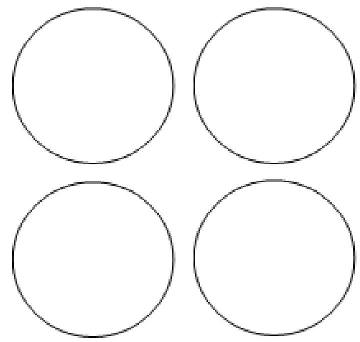


Figure 17: Share 4 Medium Pizzas Among 5 People

The diagram for each fair share situation was presented with the problem. Students were asked to not only draw in the partitions in each diagram but also to determine how much of a pizza each person would receive. With the situation being a fair-sharing situation, the idea of equal pieces was discussed early in the conversations as illustrated by Kassie's explanation for how she solved the problem.

> Kassie: Basically all I did was split each pizza into 5 pieces and pretend they're even. *(Data)*

Throughout the discussion of various partitioning situations, some students presented the idea that that their partitions were equal, such as what Kassie did, and others took it to be understood. Fractions being composed of equal parts was established on the first day of the rational number unit, thus not everyone included that in their explanations. For example, Mary did not say that her pieces were equal when describing how she partitioned each pizza into 5 equal parts.

> Mary: It says determine the fraction of a pizza each person will get. So I did 1/5 because when I divided each pizza into 5 pieces, each person would get 1 piece. *(Data)*

Equal pieces so far had been used under the premise that every piece is congruent. When the instructor presented the situation illustrated in figure 18, students questioned whether you would still get the correct answer because the slices were no longer congruent to one another.

A student did this to share 4 pizzas among 5 people
The shaded regions how much one person receives.

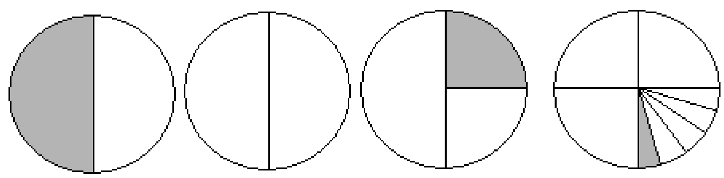


Figure 18: Share 4 Pizzas Among 5 People

When asked to determine if what the student did would be correct or not, Jocelyn replied that she did not think the student's partitioning method would be correct until another student in the class said that it was. To determine that partitioning in this way was valid, Jocelyn reverted to the procedure for adding fractions once she determined the amount each shaded piece was worth.

Jocelyn: **I never even thought it would work until Jackie said something.** And if you added up 1/4 of the pizza plus 1/2 of the pizza. If you add up all the ones. *(Data)*
Nancy: Can you go write it down?
Instructor: Yeah. Nancy would like you to write it.
Jocelyn: Thanks Nancy.
Nancy: Well no we all wrote it down then it's easier to understand.
Jocelyn: I added up all the ones, like somebody said earlier. **And it ends up working out.** You add up 1/4 here plus the 1/2 plus 1/20 because this is divided up into 5 so 5, 10, 15, 20 [*pointed to each fourth in the last circle*], and that comes to be 1/20. And you find a common denominator and that's 20. And 20 goes into 20 and 1 times 1 is 1 [*showed the procedure on the board separate from the picture*]. 2 goes into 20 is 10 times, 10 times 1 is 10. 4 goes into 20, 5

times, 5 times 1 is 5. You add them up, which gives you 16/20, which reduces down to 4/5. It's back to the same thing. Does that work? *(Claim and Data)*

Jocelyn's initial justification, which included using a procedure to add fractions was not sufficient, thus another student then went on to provide a justification for what Jocelyn did.

Granted Jocelyn reverted to a procedure for adding fractions to determine that the solution strategy does work, Jocelyn admitted that she never thought the strategy would work until someone else in the class said that it would. Though the solution strategy was not taken-as-shared at this point in the conversation, the social environment, which included the instructor, caused Jocelyn to reorganize her thinking that partitioning amounts does not always have to be done with congruent partitions. This illustrates one of the ways that the social environment impacts student learning.

The instructor then took the same problem and partitioned the pizzas in yet another way of splitting each pizza up into 40. When asked why that is allowable, Cordelia responded by commenting that it does not matter how you partition as long as everyone receives the same amount.

Instructor:	Now I have this friend who likes to draw. And you said that it had to be twentieths. My friend said nope. I cut each of these pizzas, full pizzas into 40 slices.
Olympia:	How would they do that?
Instructor:	Is that okay? They're big pizzas. Why is that okay? Cordelia.
Cordelia:	**It doesn't matter how many slices you cut each pizza up into as long as you give each person the same amount. If you give each person the same as many** *(Warrant)*

By presenting the class with a situation in which the partitions themselves were not congruent, some students had to reorganize their thinking as far as what is meant by an equal share of something. Cordelia then added that it does not matter how many partitions you make as long as each person receives an equal share or the same amount.

Within subsequent discussions on other partitioning problems, students no longer questioned partitions where the pieces were not all congruent to one another. For example, when the class discussed sharing 5 pizzas equally among 3 people, Winnie partitioned her pizzas in the following manner:

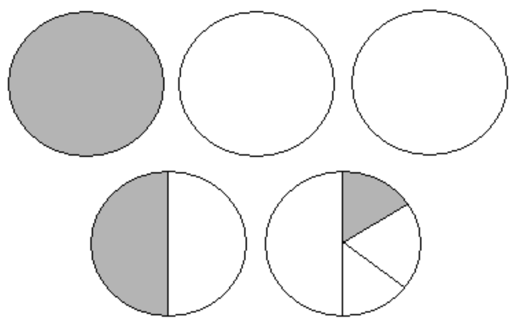


Figure 19: Share 5 Pizzas Equally Among 3 People

Winnie came to the board to present this picture and explained what she did.

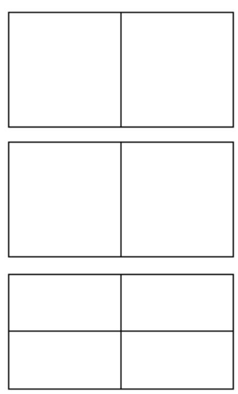
> Winnie: Okay. So this is how I divided them up. And I'll show you the math part that I did to get the answer. Since these are each a whole I had 1. And then for the halves, I have 1/2. And then for the little 1/3 part it's really 1/6. Which I first messed up, but I corrected myself. So 1/6, which then. That's what I got. 1 4/6. Did anyone else get that? *(Data)*

By this point, the class no longer questioned partitions that were not congruent to one another. Thus, the idea that partitions did not have to be congruent in order for everyone to still receive an equal amount had become taken-as-shared. Within the conversations, warrants and backings had to still be provided for how the answer was determined, as students continued to revert to procedures, but the act of partitioning itself no longer required warrants and backings to justify it further.

Unitizing

Unitizing incorporates four aspects of reasoning with rational numbers. The first is developing a unit fraction. The second includes iterating that unit fraction. The third is developing a composite unit of one. Lamon (1996) includes a fourth aspect of unitizing in terms of the whole, though this is the most difficult for students to understand. Unitizing ideas started being developed on the second day of the rational number unit.

The first unitizing idea that was established was developing a unit fraction from another fraction. The following conversation occurred within the problem of sharing 3 dessert pizzas equally among 4 people. Mary and Mindy were at the board explaining how to determine how much of a pizza everyone receives after partitioning the first two pizzas in half and the third pizza in fourths. Mary used a unit fraction of 1/4 to relate 2/4 to 1/2.



Mindy: Those are halves and those are fourths, so you have to make them equal. So in order to make them equal, like she was saying you can't, you're not dividing them into 6 so you can't add the two denominators so you have to find a common denominator to make them equal. *(Data)*

Instructor: Mary.

Mary: If you divided the two boxes that are divided in half into fourths so they are the same size portion as the bottom one. *(Data)* **To make 1/2 you have 2 one-fourths. So 2/4**

would equal 2 one-fourths, which is 1/2. *(Warrant)*

Mary's comment first pertained to making 1/2 from 2 one-fourths. Then she used this to relate 2/4 to 2 one-fourths as a warrant for making the slices equal in all the pizzas. Mary unitized by determining that 2/4 is equivalent to 2 one-fourth pieces.

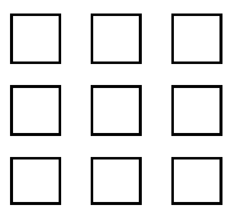
A similar argument was used by Barbara when the class was discussing their solutions to sharing 5 pizzas among 3 people. The class was converting 5/3 into 1 2/3 and Barbara used a unit fraction of 1/3 as data to justify going from 5/3 to 1 2/3. Barbara then used the 1/3 to start developing a composite unit of one.

Instructor:	So here I have this. You're telling me the procedure 3 goes into 5 once with 2 leftover, and I need more than that. Barbara.
Barbara:	**Could you break the 5/3 down into 1/3 five times?** *(Data)*
Instructor:	Why can you break the 5/3 down into 1/3 five times?
Barbara:	**Because it's kind of what we did for addition when you pulled. Like 121, made 100 and 20 and 1.** *(Warrant)*
Instructor:	Okay.
Barbara:	And then three of those is 3/3, which is 1. And then you have 2/3 leftover. There's your 1 2/3. *(Data)*
Instructor:	How do you know 3 of those is 1?
Barbara:	**Because you have a pizza and it's split into 3 pieces. So if you have 3 pieces in that 1 pizza.** *(Warrant)*
Instructor:	Okay. And how do you know that this is 2/3?
Barbara:	Because there's 2 pieces of a pizza that has 3 pieces. *(Warrant)*
Instructor:	What do you guys think about that? Mary, you had your hand up.
Mary:	**I was going to say that if you have 3 pieces of the 3 pieces that means you have the whole thing, which is 1.** (*Warrant)*

Barbara used unitizing concepts from whole numbers to warrant her data of breaking 5/3 into 5 one-thirds. Barbara and Mary then both used this idea to start developing a new idea of a composite unit of 1. This was warranted by Mary's statement that "3 pieces of the 3 pieces means you have the whole thing, which is 1." In other words, one whole in this problem, is equivalent to 3 one-thirds. The idea of finding a unit fraction from another composite fraction was becoming taken-as-shared at the same time that the idea of a composite unit of one was introduced.

Finding a unit fraction from another composite fraction did not become taken-as-shared until the next class session. During the third day of instruction the class was presented with the unitizing activity Keeping Track (See Appendix E). This activity was designed so that students would need to find a unit fraction, iterate that unit fraction, and develop the composite unit of one in order to solve the problem. The following was the first problem the class discussed:

> *Pete was taking inventory so that he could place an order with the local grocery store. Looking at his dwindling pepperoni stock, he saw that he only had 9 bags of sliced pepperoni left, which is 3/4 of a container of pepperoni. Show how many bags of pepperoni fill 1 5/6 containers.*



In the class discussion, Kristy came to the board to discuss how her group solved the problem. In this discussion, Kristy used unit fractions and a composite unit of one to assist her in her answer. Both arguments are represented as data in her discussion and are not questioned. She also introduced iterating a unit fraction as part of her solution process.

Kristy: Okay. So this is what they gave us and this is 3/4 of the entire thing. So we, my group, knew that there were 3 rows of 3, so there are 3 groupings. So in order to get the 4/4, we knew we needed another equal group. So we added 3 more because that would be 4. *(Data)*

Instructor: Mary has a question for you.

Mary: Okay. No I don't have a question, but you forgot to say, we got the 3 rows of 3 because there are 9 containers in the storage thing. *(Data)*

Kristy: Okay yeah. Sorry.

Mary: I was trying to help you.

Kristy: So this gave us 12 in all for the one. So now we knew we needed 5/6 of the 12. So we needed 6 equal parts. Right? Okay. So we decided to split them into groups of 2. So this would be one group, this would be two groups, 3, 4, 5, 6. So that gave us 6 equal groups of 2. So we needed 5 of the 6. So we took 1, 2, 3, 4, 5 and we put them over here, which would be 10 more. So it would be 12 and 10. So 22 in all. *(Claim, Data)*

Kristy's idea of needing six equal parts because of the 5/6 required her to find 1/6 of the whole first. Though this was not said directly, the idea of using 1/6 was implied in her conversation. Once Kristy found the unit fraction of 1/6, she then introduced iterating a unit fraction as implied from her counting to 6. Kristy stopped at six implying that she had stopped at the whole of one, thus developing a composite unit of one in the process. Only data were presented and no one in the class questioned Kristy's process of finding a unit fraction or her composite unit of one. Thus, those two ideas became taken-as-shared.

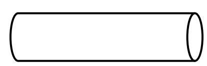
Also prevalent in this discussion was the idea that a composite unit of one can be composed of differing amounts depending on the situation (Lamon, 1996). Initially, the composite unit of one was developed from determining 4/4 of a group that was 3/4 of a whole. Once the 4/4 was found, this then had to get reunitized into a composite of 6 one-sixth sized groups of two.

As Kristy developed a composite unit of one, she introduced iterating a unit fraction. Kristy's notion of iterating the unit fraction of 1/6 was not questioned however this was the first conversation in which that idea was introduced. Thus it was not apparent from this discussion if iterating fractions was in fact taken-as-shared at this point as well.

<u>*Iterating a Unit Fraction*</u>

Kristy's idea of iterating a unit fraction was introduced in the form of counting out groups of two until six equal groups of two were obtained. The next time iterating a unit fraction was discussed was when the class moved on to discuss the fourth problem in the activity.

The following is $\frac{2}{7}$ *of a pound of dough. Show* $1\frac{3}{14}$ *pounds of dough.*



When developing a composite unit of one in this situation, a unit fraction of 1/7 was needed. Claudia determined the 1/7 by partitioning the 2/7 into two equal pieces.

Claudia: So we start off with this little rolly thing that is 2/7. So I thought of doing it this way of 1/7 and 1/7 to show that it's 2/7. And then I drew I kept drawing another one until I got to 1, because I broke it up so that I'd go get the 1 first and then I would get the 3/14. First I wanted to get the 1. So I knew I needed 7 of these half pieces of the roll in order to get 1. Because then it would be 7 pieces of the 7 parts that the whole is broken up into. Does that make sense? *(Data)*

Class: Uh-huh.

Claudia first cut the 2/7 piece in half to develop 2 one-sevenths. The 1/7 then was used to determine the composite unit of 1 being equivalent to 7 of the 1/7 pieces. As Claudia continued with her discussion, she iterated the 1/7 as evidenced by her counting out 7.

Claudia: So I kept going and I realized that when I got to that point, I needed half of this one and then this right here would give me a whole. 1, 2, 3, 4, 5, 6, 7, that's 7, so this is 1 because

> it's also 7 over 7 because it makes one whole, and I kept going. So I needed now 3/14 and 7 is half of 14 so I knew that I needed to split this one in half to make it 1/14 and then 1/14 over here. Can you guys see that or is it to small? Okay. And the reason I knew that was is if I would think of it as whole, if I split it and I knew that 7 was half of 14 and then if I split the 1/7 into 2 then that would give me 2 and another 7 into 2, that would give me 2, so eventually it would equal, the whole would be into 14 parts. *(Warrant)* If I broke each 7 into 2. So I have now I have two 1/14 and I needed 3, so I needed one other piece over here that was 1/14 to give me the 3/14. So that was, that's my 3 that's my 1 and 3/14 all of this right here. And this right here is 3/14 so altogether it's 1 3/14. *(Data)*

Claudia's explanation included iterating a unit fraction of 1/7 until she got to the whole of one. Claudia explained iterating a unit fraction as part of her data, and again no one in the class questioned her being able to do that. Though it never shifted and questions or challenges were never raised, iterating a unit fraction appeared to be taken-as-shared. As evident from these conversations, it appeared that iterating a unit fraction was taken-as-shared before the rational number unit started. Thus, it may be that taken-as-shared ideas also occur when only data are needed and are never questioned or challenged.

<u>*Unitizing in Terms of the Whole*</u>

On the fourth day of class, the idea of unitizing in terms of the whole was discussed. Unitizing in terms of the whole is similar to finding a unit fraction. For example, given the fraction 3/4, the unit fraction would be 1/4. In order to unitize in terms of the whole, 3/4 would be represented as 0.75/1. When unitizing in terms of the whole, the denominator would become one instead of the numerator.

The activity the class was presented with was an equivalence activity placed in the context of a ratio of pizzas to people. The class discussed changing 24 pizzas for 32 people to 3/4 of a pizza per person. During this discussion the class went back to make sense of a fair sharing division problem presented in the whole number unit of sharing 120_8 stickers among 10_8 friends in which Olympia then asked if the pizza problem could be solved by using 1 pizza instead of 1 person.

Olympia:	So if you do the same thing to 32 over 24, then what?
Instructor:	Then what
Olympia:	Because it doesn't go into it nicely.
Instructor:	It doesn't so
Olympia:	Then what do you do?
Instructor:	So what would we have
Olympia:	Beside it
Instructor:	if we wanted 1 pizza here?

Olympia introduced the idea that instead of looking for one person, the problem could be solved instead by looking for one pizza. When determining how many people would share one pizza, the class found the answer to be either 1 1/3 or 1 1/4.

Kassie:	Well actually I know what the answer is but I don't know how to say it.
Instructor:	What is the answer?
Kassie:	The answer's 1 wait hold on. I had it.
Jackie:	1 1/3 people. *(Claim)*
Kassie:	Yeah it's 1 1/4 *(Claim)*
Instructor:	How did you get that Jackie?
Jackie:	Well 24 goes into 32 once, subtract 24 from 32. *(Data)*
Instructor:	So that's one group of 24.
Jackie:	Right, plus 8. And 8 goes into 32, I guess it's 4, 4, 4. 1 1/4 because 8 goes into 32 four times. *(Data)*
Instructor:	What do you guys think? 1 1/3 or 1 1/4?
Students:	1 1/4 (Claim)
Claudia:	I can see why she got 1/3. Sorry.
Instructor:	So which one do you think it is Claudia?
Claudia:	No it's 1/4 but she was thinking of 8 into 24 the first time to get the 1/3. *(Data)*
Jackie:	Yeah.
Instructor:	Cordelia.
Cordelia:	I think it's 1/3 because if you were to do that you'd have one and then 8/24 and 8/24 would be simplified into 1/3. *(Data)*

...

Cordelia: 8 goes into 8 one time, 8 goes into 24 three times. *(Data)*

Evident from this conversation, the class struggled with determining if the answer was 1 1/3 or 1 1/4 because of the having 8/24 versus 8/32. At this point no one could determine which answer was correct. A few moments later, Claudia who initially thought the answer was 1 1/4, determined that the answer would in fact be 1 1/3 because the 24 was the number of parts that the problem was broken into.

> Claudia: Well I was going to go back to 1 1/3 and 1 1/4. It is 1 1/3 because 24 would be what we are breaking our whole into, since it's 24 parts. So that's why it's 1 1/3 because then it'd be 8/24, which is 1/3. Because 24 is our parts. *(Warrant)*

Unitizing in terms of the whole introduced students to explore the ways in which the remainder is represented in division situations. By having this conversation, students started developing division ideas before division was presented in the rational number unit.

Once the answer of 1 1/3 was determined, Edith commented that unitizing in terms of the whole or unitizing in terms of the number of pieces are both valid ways to solve the problem. When discussing both answers of 3/4 of a pizza per person and one whole pizza to 1 1/3 people, the second answer was disregarded because of having a fraction of a person. Though students viewed the strategy as valid, the solution was disregarded because having 1 1/3 people did not make sense.

> Edith: I was just going to say that I think both ways work just depending on what question you're trying to answer. If your answering how many pizzas for 32 people then you could say 24 pizzas for 32 people, but if you're saying how many people for 24 pizzas, you'd say 32 people for 24 pizzas. *(Claim, Data)*

...

Edith: I was just going to say you could show how I drew earlier, the 3 pizzas divide them up into fourths and then you could just show that each person would get a fourth of each pizza. So it would be 3/4. *(Data)*

Instructor: And you'd get down to the one person. And does this make sense this 3/4 of a pizza to one person.

Class: Yeah.

Instructor: Does this make sense this 1 1/3 persons?

Class: No.

No one rejected the idea of finding the number of people to one pizza, except for the fact that the answer did not make sense. The answer of 1 1/3 people to one pizza did not make sense because of the problem representing a set model in which getting 1/3 of a person is impossible.

Unitizing in this manner did present the class with a situation regarding how to find the remainder in a division problem. The class had to go back to make sense of what the whole would be in the problem to determine if the answer is 1 1/3 or 1 1/4. Similar to Lamon's (1996) findings, this discussion illustrated that several students struggled when unitizing in terms of the whole, however it provided the class with a way to start conceptualizing the remainder in division situations.

The idea of unitizing in terms of the whole was briefly mentioned in subsequent class days. When the class moved on to comparing situations, they again were presented with a comparison of people to pizzas problem:

> *A birthday party took up two tables. One table had 9 pizzas for 18 people. The other table had 2 pizzas for 4 people. Each table shared the pizzas equally. If you were invited to this party and came hungry, which table would you want to sit at?*

Olympia solved this problem by finding the number of people who would share one pizza and was not questioned.

Olympia: The way I drew it visually, it wouldn't matter what table I sit at I mean because you just took 2 people to 1 pizza. *(Data)*

The idea of unitizing in terms of the whole was taken-as-shared as Olympia only needed to present data for her solution and this was not questioned. This idea appeared to be taken-as-shared within the previous situation of having 1 pizza to 1 1/3 people as students commented that unitizing in this way would work, but because the solution was unrealistic it was not clear that this idea was taken-as-shared because of the class disregarding that solution. Within this problem, Olympia's solution of 1 pizza to 2 people made sense in this context, and her solution was not questioned. Others in the class unitized in terms of the people and found that each person would get 1/2 of a pizza. Thus, the class understood that unitizing can be done with either amount.

Three ideas within unitizing became taken-as-shared. These included, a) finding a unit fraction from a composite fraction, b) developing a composite unit of one, and c) unitizing in terms of the whole. Iterating a unit fraction was also taken-as-shared but appeared to be taken-as-shared before the rational number unit started.

Equivalence

Students who can flexibly think about equivalence situations are able to erase and insert partition lines as well as understand that equivalent rational numbers are different names for the same amount (Kamii & Clark, 1995). Within this study, activities were designed to focus on equivalence (See Appendix F); however, students started developing equivalence ideas on the first day of the rational number unit.

The first equivalence idea that was discussed was that of erasing partition lines to make a bigger piece equivalent to the original. Claire introduced this idea as 'undividing' and discussed this when explaining her answer of 1/3 to represent the shaded amount in the problem represented in figure 23. Claire's conversation also illustrates one of the ways individuals contribute to the social community by introducing ideas to the class.

Name a fraction that represents the shaded amount.

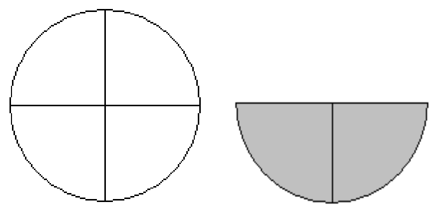


Figure 20: Restaurant Table 2

Claire:	To get the 1/3 I looked at it as sections. I looked at the whole piece is a half. The top part would be 1, the bottom would be 2, and then the shaded part would be 3 to get the 1/3. I divided them in further into sections. *(Data)*
Instructor:	You divided them further into sections?
Claire:	**Well I guess I didn't divide further I undivided them.** *(Data)*
Instructor:	So didn't you mean you can undivide to get one third? What does she mean she undivided? Jennifer what did she mean when she said she undivided them?
Jocelyn:	I have no idea.
Instructor:	Ask her a question.
Jocelyn:	What do you mean?
Claire:	**I meant if the whole piece, it's into 4 pieces. And to make that 2 pieces I erase one of the lines. To make it the 2 pieces instead of the 4.** *(Warrant)*
Instructor:	Like the line?
Claire:	Yeah. And then I did, so then it's 1, 2, 3 [*refers to each half shown*].
Instructor:	1, 2, 3, 4 [*points to each half in the problem including the one that is not shown*].
Claire:	Well I erased that [*the half not shown*] one too. I looked at them as equal parts. *(Backing)*

Claire's method of undividing meant that she could erase partition lines to have two pieces instead of four in the whole pizza and represent the shaded region as 1/3 instead of 2/6.

When new ideas were presented to the class, the class's responsibility included determining if the idea was mathematically correct or not. Claire's method of undividing was correct, though not everyone initially understood her method. Once this idea was introduced, other students in the class then continued to use undividing as part of their solution processes and explanations in finding answers to subsequent problems.

During the discussion of the next problem (see figure 21), Claire's idea of undividing became a warrant for Kassie's justification on why 1 3/4 is equal to 7/8.

Name a fraction that represents the shaded amount.

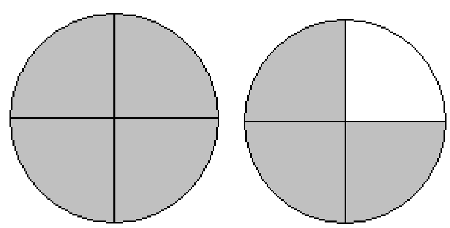


Figure 21: Restaurant Table 3

Instructor:	So what you're telling me is 1 3/4 is equal to 7/8?
Kassie:	Yeah.
Student:	No.
Instructor:	Okay someone just said no. Quickly. And then went like this [*put hand over mouth*].
Kassie:	You want me to explain why? Okay. Each individual piece is accounted for in that, we didn't regroup it any other way. *(Data)*
Instructor:	What do you mean we didn't regroup it any other way?
Kassie:	Like in the other problem we split it into 2.
Instructor:	**Like we undivided.**
Kassie:	**Yeah undivided.** *(Warrant)*

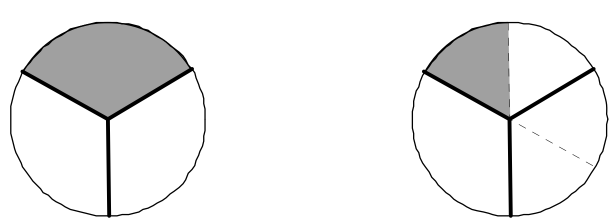
Though Kassie's idea that 1 3/4 is equal to 7/8 was incorrect, the idea of undividing to get an answer became synonymous with regrouping. Also evident in Kassie's explanation, she felt that since the picture did not require any undividing or regrouping, that the two answers were automatically equivalent implying that undividing or regrouping gives an answer that is not equivalent. Though students, like Kassie, seemed to understand that undividing meant regrouping or erasing lines, it was not clear at this point that they understood when doing that you arrive at an equivalent answer.

The idea of making pieces bigger or undividing continued to be discussed throughout the next two class days of instruction. During the sharing activity of sharing 5 pizzas among 3 people, Caroline justified the equivalent answers of 1 2/3 and 1 4/6 by referring to making bigger slices.

> Caroline: You look for equal portions. If you look at that and you see that you can put 2 slices together there, 2 slices together there, and 2 slices together there. **To come up with 3 bigger slices**, you'd have 2 of those 3 slices. *(Warrant)*

On the second day of instruction the idea of making pieces bigger was still used as a warrant for justifying equivalent fractions and done so in terms of putting slices together to make a bigger piece.

The third day of instruction is when undividing shifted in function in the conversation. The class was presented with a problem in which a student shared 2 pizzas among 4 people in the following way:



As part of the solution method to this problem, students determined that the 1/3 piece would ultimately be cut to make 2/6. The instructor then followed up by asking the class how to work backwards from 2/6 to 1/3.

> Instructor: Now if you did it that way, if you said okay I started with 2/6, how can you make 2/6 into 1/3? Caroline.
> Caroline: Split each of the three slices into two, in half. *(Data)*
> Instructor: Okay that's going from 1/3 to 2/6. What if I was giving you 2/6 and I wanted to know if it was 1/3?
> Edith: Just group pieces of two. Two slices together to have three different groups of two slices. *(Data)*
> Olympia: **You undivide.** *(Data)*
> Instructor: Undivide right. Alex is that what you were going to say?
> Alex: I didn't understand what she said.
> Edith: Group, if you originally had six slices for one pizza, to show that two of the slices is equal to 1/3, you could just group two slices together, group another two slices together, and group another two slices together to represent three different groups of two slices. *(Data)*

Undividing in this conversation was still referred to as being a way to regroup pieces. With both Edith's and Olympia's comments, undividing or regrouping was used as data and there was no longer a need to provide warrants or backings for the conversation. In addition, undividing to find an equivalent amount was not questioned. Thus, grouping pieces into bigger sections to make an equivalent amount was taken-as-shared.

These conversations also illustrated the ways in which individual students contributed to the social community. The first way was that of introducing ideas to the class as Claire had with her method of undividing. The second way individuals contributed to the social environment was by sustaining ideas once they were introduced. After Claire introduced undividing and the class determined that this method was mathematically correct, other individuals then became responsible for sustaining this idea, such that it became taken-as-shared on the third day of class.

Equivalent Fractions are Different Names for the Same Amount

Another idea with equivalence that also started being discussed on the first day of the rational number unit was the idea that equivalent fractions are different names for the same amount. This idea was first discussed when Cordelia reverted to a procedure to explain how 2/6 is equal to 1/3.

Cordelia:	Because I was taught that if you divide the same number on the top by the same number on the bottom that it gives you a smaller fraction of the same amount. No. *(Data)*
Instructor:	A smaller fraction of the same amount?
Cordelia:	**It represents the same thing.** *(Data)*

Cordelia corrected herself about what she meant by a smaller fraction of the same amount. The idea that equivalent fractions represent the same amount did not come about in the conversation until the focus was on breaking a fraction into smaller pieces or dividing "the same number on the top by the same number on the bottom." At this point, though the conversation initially was focused on equivalence in terms of breaking pieces apart, the class could not do this without using a procedure.

The idea that equivalent rational numbers are different representations of the same amount was not discussed again until the fourth day of instruction when a problem was presented that required students to break a group into smaller amounts to arrive at an equivalent solution (See Appendix F). This was also the first time students started to conceptualize the procedure of doing the same thing to the numerator and denominator.

The equivalence activities were placed in a ratio context (Tarlow & Fosnot, 2007) and required students to find an equivalent ratio by breaking an initial ratio into smaller groups. The equivalence activities focused around a family reunion coming into the pizza parlor. The following problem was the first equivalence situation presented:

During a reunion, a family ordered 24 pizzas for 32 people. There was not enough room at one table for the family so they split up into 2 tables. How could the workers split up the family and pizzas so that everyone receives a fair share of pizza?

No one questioned several students' solutions of 12 pizzas for 16 people. This solution then was used by students to answer the second question.

How could the workers split the pizzas if there were only two tables and one table was a table for 4?

Though students discussed breaking down the groups within the first question, the solution only required students to break each group in half. The second question required students to do more than just break each group in half.

Edith discussed her solution to the second problem using her solution from the first question. Following this, Beth and Caroline provided further justification for Edith's work by referring to the fact that all the same numbers are used thus the portion remains the same in each instance.

Edith:	Well I did it the same way she did and I simplified it or reduced or broke it down to 3 pizzas for 4 people because I was looking at when I broke it up among two tables. I looked at the 12 and the 16's in order to get a smaller number. So I realized that 3 pizzas for 4 people, that's what you would do if you broke it down even more. *(Data)*
Instructor:	Do you guys understand what she is saying?
Caroline:	No
Edith:	No. I confused myself.
Instructor:	What?
Beth:	Well it's the same thing as saying 4 times 3 is 12. **I mean it's all the same numbers. It's just broken down into small groups and I made it bigger.** *(Warrant)*
Instructor:	It's the same thing as saying.
Edith:	Well the smaller number just represents how much pizza they get. *(Warrant)*
Beth:	Yeah.
Caroline:	**It's still the same portion.** *(Backing)*

Two ideas were discussed within this conversation. First, Edith and Beth discussed the answer in terms of breaking bigger groupings down into smaller groups. The second idea involved creating smaller groups that still represent the initial amount. Caroline's response of the smaller groups representing the same portion refers back to the idea that equivalent fractions are different names for the same amount. The portion in this problem was found to be 3/4 of a pizza per person or 3 pizzas for 4 people, which is equivalent to a grouping of 12 pizzas for 16 people. Following this conversation, the class then went on to show that the portions in each solution were in fact the same.

At the beginning of the rational number unit the idea that equivalent rational numbers represent the same amount was discussed in terms of dividing the numerator and denominator by the same number, or undividing to make bigger pieces. Similarly, this idea was again discussed on the fourth day of the unit again in the context of dividing the numerator and denominator by the same amount, but this time in terms of making smaller groups. Conceptually, the mathematics behind each situation was the same however the physical process of breaking groups down was different. Within an area model this required partition lines to be erased, whereas in the set model this required groups to be partitioned even more. For example, when showing how $\frac{2}{4} = \frac{1}{2}$ with an area model, the process involves erasing one of the partition lines to create a bigger group (see figure 22).

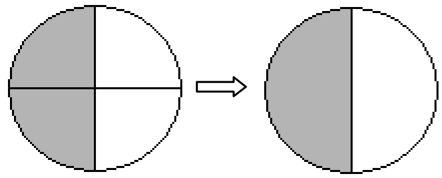


Figure 22: Area Model of 2/4 = 1/2

In a set situation, going from 2/4 to 1/2 involves breaking groups down rather than combining groups together. The two groups out of four become one group out of two (see figure 23).

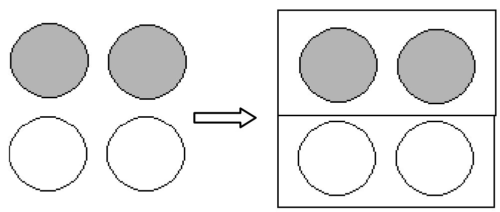


Figure 23: Set Model Showing 2/4 = 1/2

Once the three equivalence ideas of erasing partitioning lines, breaking groups into smaller groups, and that equivalent fractions are different names for the same amount were established, the class then moved onto making sense of the procedure for finding equivalent rational numbers. In response to a homework problem, one student asked specifically how to justify common denominators in the context of an addition situation. As an example, the instructor presented the problem 2/3 + 1/6. Equivalence was discussed as part of the conversation of breaking each fraction up into the same number of pieces. In this conversation, Jane initially reverts to a procedure to find a common denominator. Following this, Edith describes the procedure in terms of breaking pieces into smaller amounts.

Instructor:	What's our justification that our 2/3 is equal to 4/6? Jane.
Jane:	The way I always think of it is, how many times does 3 go into 6? *(Data)*
Instructor:	But that doesn't really follow the conversation we just had so readily. So now let me push you. We just sort of walked ourselves through. What did we do to make our 2/3 become 4/6?
Jane:	**We divided it in half.** *(Data)*
Instructor:	Divided what in half.
Jane:	**Each slice in half.** *(Data)*
Instructor:	Okay. So we put each slice in half or in two equal pieces.
Jane:	**So multiply the top and bottom by two.**
Instructor:	Right there I start to get a little confused. Edith.
Edith:	**Well because when you divide one of the original three slices in half, you get twice as many slices from it. So that's why you multiply everything by 2.** *(Warrant)*
Instructor:	That's why you. That's important. So what Jane's saying is a procedure. You multiply the top and the bottom by 2. Why? And several of you have why written on and justify written on your papers at this point. Why? What happened when we divided our thirds into two equal pieces? What happened to all the thirds?
Edith:	**You got twice as many.**

Edith discussed how cutting a piece in half will produce twice as many pieces. Instead of just discussing that you multiply the top and the bottom by two, which is what the class had been doing up until this conversation, students started making sense of the procedure that cutting up into smaller pieces will increase the number of pieces you have.

This idea was again revisited when the class moved on to the addition and subtraction activities in class. Within these situations, similar to the homework problem involving 2/3 + 1/6, the class continued to have difficulties finding equivalent fractions in terms of the procedure without being guided by the instructor. The class was given the following subtraction situation:

> *The pizza dough machine broke after it had made 2 1/2 pounds of dough. The cook used 2/3 of a pound. How many pounds of dough were left?*

Before the following discussion, Mary discussed her solution process of taking 2/3 of a pound of dough away from one whole, then adding 1/3 from what remained of the first pound, 1, and 1/2 to arrive at the answer of 1 5/6 pounds. Once the common denominator of six was determined, the class could still not discuss going from 1/3 to 2/6 without referring to multiplying the bottom number by something and then going back to multiply the top number by the same thing. Once this occurred, the instructor then used a picture of a circle to help students make more sense of what was happening with the procedure.

Instructor:	How did I make these thirds into sixths? What did I do?
Students:	**Cut/Divide them into half.** *(Data)*
Instructor:	Cut them into half? Each into half. And now I need to cut each into half?
Class:	No.
Instructor:	But that's what you said. After I cut my thirds each into two equal pieces, then I've got to go back
Caroline:	Right.
Instructor:	And cut something. The one.
Jackie:	**Well by cutting your thirds into two equal pieces you're making you're taking 3 pieces and making it 6 pieces total.** *(Warrant)*
Instructor:	Okay.
Caroline:	That's it.
Instructor:	So then I got to go back.
Erin:	You still have 2 pieces that's not shaded so that's 2 out of the 6.
Instructor:	But if I go back, didn't you, so what do I do here to get from 1/3 to 2/6?
Erin:	**Cut the thirds into halves.** *(Data)*
Instructor:	Cut the thirds into halves.
Erin:	And then you count your unshaded parts and you still have two out of the six pieces not eaten.

Erin and the rest of the class discussed cutting the thirds in half to get six pieces. This was not accomplished until after the instructor asked what happened in the problem in terms of the picture. Before this conversation, the class could still only discuss finding equivalent fractions in terms of the procedure of multiplying the numerator and denominator by the same number. By having the class look at what was physically happening when you multiply the numerator and denominator by the same number, they could see that the process of doing both occurs in one step. When going from 1/3 to 2/6, you cut each third in half which then doubles the number of pieces represented by the numerator.

Once equivalent fractions were found Nancy and Jackie then went on to continue the idea of what equivalent fractions represent. Nancy used the idea of proportions to justify going from 1/2 to 3/6. Then Jackie used the idea that equivalent fractions are different names for the same amount to further justify this same idea.

Nancy:	**Well I looked at it as looking at proportions to the whole.** You know she chose a common denominator, which makes it go into those, part of her explanation. To justify how you get it, say for the 3/6 how that's equal to 1/2. *(Data)*
Instructor:	Okay. So you're looking right here now [*pointing to the circle cut into 6 pieces with 3 pieces shaded*].
Nancy:	Right. How we make those equal. 3/6 out of 6 pieces is 1/2 of a whole. A half is 1 piece of 2 pieces, so it's half of a whole. So 3 pieces of 6 pieces is half of a whole. So it's the same proportion. It's just by choosing a common denominator, they're still the same, but it's I understand what she said it's easier to add.
Instructor:	I see and I think we're okay with that we need the same sized pieces, that we need a common denominator. How she got to the common denominator, I multiplied 2 goes into 6 three times, so I multiply the numerator by three. I need more with that. You need more with that. Jackie.
Jackie:	**Well to keep from changing like what each fraction is worth, if you multiply the denominator by a number. Since 3 over 3 and 6 over 6 equals one, if you multiplied the denominator by 3, then you have to multiply the numerator by 3 because it's in essence multiplying it by one to keep from changing the amount of each fraction.** *(Data)*
Instructor:	So tell me about this one.
Jackie:	So for that one since to go from 3 to 6 or 3 goes into 6 twice, that she'd in essence be multiplying the 3 by 2. 3 times 2 to

get the 6. So to keep from changing the amount that that fraction equals, she has to multiply the numerator by 2 as well. So it's like she's multiplying 1/3, 1 over 3 by 2 over 2. *(Warrant)*

Two arguments were prevalent within the above conversation. The first was that of equivalent fractions representing the same amount. With fractions that represent the same amount, Jackie discussed this in terms of multiplying the fraction by a quantity of one, thus not changing the amount that the fraction is worth.

The second argument was that equivalent fractions are proportional to one another. This was not the first time the idea of equivalent fractions being proportional was discussed. A similar conversation occurred during the equivalence activities when the class discussed the additive relationship of equivalent rational numbers with the following problem.

Pete noticed that for every 30 customers, 6 were under the age of 11. At this same rate, how many children would Pete see if he has 40 customers?

In the case when the numbers are large and a picture would take too long to draw, Kassie commented that the final answer is going to be proportional to the original fraction. In essence the picture will always represent the same thing no matter how many boxes are drawn.

Beth:	I have a question. I completely understand like I'm a visual learner, I understand the boxes. But I don't have the patience to like
Caroline:	Draw that many
Beth:	draw it. So how do you explain it like that because I did it with algebra but I don't know how to explain what she did with the boxes.
Instructor:	Okay.
Beth:	Without physically drawing them all out.
Instructor:	Kassie.
Kassie:	Basically I think it looks, it's the same thing. **You're still**

> **keeping it proportional to the 1/5.** You're still adding, I don't know how to explain that, **but it's still proportional.** *(Data)*

Within the above discussion, which occurred on the fifth day of the unit, the task involved finding part-whole ratios equivalent to 1/5 by adding groups of 1/5. The idea that equivalent rational numbers are proportional was presented as data and was not questioned, thus taken-as-shared.

The equivalence ideas that became taken-as-shared included a) erasing partitioning lines to make pieces bigger, b) cutting pieces into smaller pieces, and c) equivalent rational numbers are different names for the same amount. Throughout equivalence, other ideas were discussed, such as making sense of the procedure, however the class never got to a point where they could do this without the instructor's guidance.

Comparing and Ordering

Several ideas were established as part of the taken-as-shared ways of reasoning with comparing and ordering fractions. The reasoning strategies students developed included comparing using a) a benchmark fraction, b) common numerators, c) missing pieces, and d) common denominators.

In the comparing and ordering activities (See Appendix G and H), students initially developed these reasoning strategies through contextualized situations then applied them to non-contextualized problems. Though the strategies started developing simultaneously they are discussed individually.

<u>Compare to a Benchmark</u>

The first practice established within these activities was comparing fractions to a benchmark. The following situation was presented to the class:

> *The 22 people at the party sat in the party section of the restaurant, which holds 42 people. At the same time, 16 people were in the non-party section, which holds 36 people. Which section was more crowded?*

The two fractions students had to compare were 22/42 and 16/36. The class presented two different solutions to this problem. One solution that was presented, by Claudia, was that the party section was more crowded because it was more than half full. The other solution, presented by Claire, was that both sections were the same because each section had 20 empty seats. Though the intent of the question was for students to start developing the idea of comparing to a benchmark fraction, the question was also written for the common misconception of comparing by only using missing pieces to also come into the conversation. Though this conversation illustrates the ways in which comparing to a benchmark started being developed, this also illustrates the ways in which the social environment impacted individuals' knowledge reorganization by rejected the incorrect idea of comparing by only looking at the number of pieces missing.

Claudia:	For this one, I did it kind of just simple. So the first one was 22 people to 42 in the party section. Right? That was the party maximum of people that could be there. And then in the non-party section there were 16 people and there was 36 maximum allowed in the non-party. **So what I did was thought of it as finding what half of the party max was, and I found that half of the party max would be 22, I mean 21 people out of the 42. And then the second one the half of that would be 18 of the 36 max. And this one is less than half and this one is more than half, so this one would be closer to capacity.** *(Claim, Data)*
Instructor:	Claire.

Claire:	Should I show how I did it? Because I got something different.
Instructor:	Yeah. Are there questions for Claudia? Or well put yours up there, and maybe you can both stay up there so we can respond.
Student:	Which one is more capacity?
Claudia:	This one [*22/42*] is closer to capacity because it's more than half. This is half. *(Data)*
Claire:	Okay what I did I took
Instructor:	Okay take a look
Claire:	**I drew 2 pictures. This was for [*22/42*] I drew out the 42 people and then I shaded in the 22. And then for this one I did the 36 people I shaded in 16. And I saw that both of them had the same amount of seats leftover, so I figured they were both equal to capacity because. I don't know if that makes sense.** *(Data)*
Instructor:	What do you guys think? Caroline.
Caroline:	Well that's originally what I started out to do and then I realized that the portions aren't the same. We we're talking about the pizzas being out of certain slices.
Claire:	But, yeah I know what you mean and I started thinking it but it says which section is closer to capacity? And I guess
Caroline:	Oh right.
Claire:	I don't know. I guess
Caroline:	Huh.
Claire:	one could be, but I don't know.
Instructor:	Mindy.
Mindy:	I thought the same thing she did because when I subtracted, I know in fractions they both have 20 seats leftover. So wouldn't they be equal? *(Data)*
Claire:	See, I don't know because I'm saying I'm thinking about them like
Caroline:	Closer to capacity.
Claire:	But still one has more people than the other so I don't know how that plays in.
Instructor:	We did that on purpose. So that we could have this discussion. So let's say we had 1000 seats and 980 of them were taken. Which would feel, how about if we ask which one is more crowded? Which would feel more crowded with these then? Barbara.
Barbara:	I don't think so. They'd both be the same because when you have more people, you have more seats, so you have more space. So it wouldn't feel more crowded, any more crowded than the other one because you have less people

and you have less space. *(Warrant)*

Instructor: Edith.

Edith: Well because I didn't do it exactly like Claire did but I had the same question because I had the same number leftover. And now that I look at it I think the 16 out of 36 would probably feel more crowded just because the space is smaller. Maybe that's just how I think of *(Warrant)*

Instructor: Jane.

Jane: I think we'd have to go back to the ratios and try to match up ratios to see which one.

Instructor: Okay. So if we went back to ratios, which one is more crowded? How do we go back to ratios?

Jane: Well both of them, 6 goes into both of them. I think. Wait. Yeah 6.

Instructor: Okay

Jane: No. Wait. I'm just looking at the bottom numbers. I'm all about the bottom numbers if you can't tell. Hold on. Because I did it with Suzy and we just subtracted and got 20 but now I can see that *(Data)*

Instructor: Well could we use the ratio of 1 person to 2 party max to help us?

Jane: Yeah.

Instructor: Okay, even though it's not going to work perfectly. If this was for every, using Claudia's, for every 1 person we have a party max of 2.

Jane: 1 person gets 2 seats.

Instructor: Well, which of these would one of these be on one side of the ratio and one be on the other side of the ratio? Claudia what do you think?

Claudia: I'm sorry what was your question?

Instructor: What was my question? I didn't word it carefully so help me out? What am I getting at? Because Jane, I'm using your idea here.

Jane: I'm going to need help from others.

Instructor: Okay. Caroline.

Caroline: She's saying that there's a 1 to 2 for seats for people. And in one of those problems you were saying one of them is more than half one of them is less than half for.

Instructor: So one of those for every.

Jane: Yeah. I like the way Claudia solved it. She was close to it as far as ratios go and shows

Instructor: So which one of those has a less crowded feel? Cathy.

Cathy: I don't know if this is exactly what you're talking about but I was just rereading the question and if you're looking at the capacity for each separate room I don't understand why it wouldn't be the same because they both have 20 empty seats. And from their capacity it's the same although it's different numbers. For the wording of the question. *(Data)*

Instructor: I guess. You're right. And we need to change the question. But what if we were coming in and saying, I'm reporting to you how the day went. You own the pizza shop but I I'm the manager. And so I'm trying to let you know it was crowded today. Or maybe not so much. Which one of these would make us feel like it was more crowded today? Which one for the say one pizza shop has 42 maximum capacity. Another pizza shop has a 36 person maximum capacity. Then could we talk about which one had a better day?

Mary: The party max would seem fuller because it's more than half. *(Claim, Data)*

Instructor: Would that make the question better?

Cathy: Yeah.

Claire: But more people went to the other one. So as a I mean as a business owner, that's better for me.

Instructor: More people went to the other one, but if I have a bigger shop, I'll probably order more ingredients and have more stuff and more people working. So in one shop, using Claudia's reasoning, in one shop I had more than half of the place full. And in the other shop I had less than half of the place full. And that's what we were going for with this problem, but I can certainly see where you would think that that it that talking about the empty seats would take care of it. Thank you.

Even though the wording of the question confused students as far as why one fraction was greater versus the two fractions being equivalent, the class started establishing the idea of comparing the fractions to half. This was done in two different ways. Claudia presented the idea as using a fraction of 1/2 in terms of half of the capacity, whereas Jane presented the idea of using 1/2 in terms of a ratio of 1 person to 2 seats.

Claire introduced the idea of subtracting the two numbers to look at the number of missing pieces or empty seats. Other students in the class also used the same strategy as Claire when initially solving the problem. As typical of conversations when an idea is presented that is mathematically incorrect, some of the class agrees and others immediately disagree. Conversations regarding someone's new idea, in this case Claire's, focus around the validity of the strategy in question. The strategy of subtracting the two numbers was found to be incorrect when determining an overall quantity of the fraction as a whole, though students like Mindy and Claire initially felt that having the same number of pieces leftover meant that the two fractions were equivalent.

This illustrates the second type of situation in which students reorganized their thinking as a result of the classroom community. When an individual presented a new idea that they thought was correct but in actuality mathematically incorrect, these ideas were eventually rejected by the class. When an incorrect idea was rejected by the class, individuals had to reorganize their knowledge such that this strategy would no longer be used in subsequent problems.

This particular problem was written intentionally for the class to derive two solutions to the problem. The first was that the party section was more crowded because the section was more than half full. The second was that the two sections were equal because they both had 20 empty seats. The same conversation continued on the next class day when the class again discussed both strategies. After that discussion, Claire's idea of saying that the two fractions were equal because they were missing the same number of pieces was disregarded because of the total number of people affecting each fraction as well. Thus, Claire and other students who agreed with her needed to reorganize their thinking by rejected the idea that comparing fractions can be accomplished by just looking at how many pieces in each fraction are missing.

During the next activity, students moved onto non-contextualized problems in which they had to compare two fractions and determine which one is bigger. The fractions in these problems all could be solved using multiple methods. The first non-contextualized problem presented was to compare 4/9 and 4/5. When the class discussed solutions to this problem, Caroline used the strategy of comparing to half. In Caroline's explanation and justification, the half strategy shifted in function from data to warrant, and was also not questioned by the class.

Caroline:	Just that 4/5 is larger than 1/2 but 4/9 is smaller than 1/2. *(Data)*
Instructor 2:	And how do you know that?
Caroline:	Because 1/2 of 5 is 2.5 and 4 is bigger than 2.5. And 1/2 of 9 is 4.5 and 4 is smaller than 4.5. *(Warrant)*

During the previous day's discussion, when Claudia introduced the idea of comparing fractions to half, her data included both the strategy itself as well as finding half of each fraction. When Caroline used this strategy, finding half of each fraction became the warrant for her solution. This was the first time the strategy shifted in function. Though other students solved the problem differently, no one questioned the half strategy. Thus, comparing to half became taken-as-shared.

<u>*Common Numerators*</u>

The next strategy that was established was comparing fractions using common numerators. The class was given the following contextualized problem:

> *After everyone was done eating at the party, 1/6 of a large mushroom pizza was left and 1/8 of a large sausage pizza was left. Which pizza had more leftover?*

To develop the idea of using common numerators to compare fractions, the problem was intentionally written with fractions that already had a common numerator. As with the comparing to half strategy, students also developed two strategies for this problem. One was the common numerator. The other was common denominators.

Jane: Okay. I knew that the 1/6 had more leftover because it's being split amongst fewer people. I already know how to do common denominators, but I knew that would be really hard to explain and so I tried to draw a picture. You can think of it as like with kids. You give them a cake and say alright you have to split it amongst your friends. They're going to want to split it between fewer because they want more cake. So you can see in the picture like the 1/8 piece is smaller than that one. But if you don't draw it very proportionally then you wouldn't see that. So *(Claim, Data)*

Instructor 2: So the picture can get in the way if we don't draw the picture accurately.

Jane: Correct. Yes.

Instructor 2: Okay.

Jane: So when the common denominators the bottom numbers have to be the same, like over here. So I knew that 6 and 8 both go into 24. So whatever it takes to get to 24, which is 4. I multiply by the top. And whatever it takes to get to 24, which is 3. I multiply by the top. And since this one is obviously larger, 4 instead of 3 things. So 1/6 is bigger. Does that make sense? *(Data)*

Instructor 2: So what question do you think I have?

Suzy: Justify.

Instructor 2: Justify. Justify what?

Suzy: What she just said. But can't she just leave the picture and say assuming all pieces are equal. 1/6.

Instructor 2: Why?

Suzy: Why? What did she say about if they are equally distributed? So equal pieces

Instructor 2: So, Suzy right?

Suzy: Correct.

Instructor 2: Suzy said can't we just go back to the picture and if all the pieces are equal size in both, if I split both pizzas into equal size pieces that 1/6 is bigger.

Suzy: By viewing it.

Instructor 2: By viewing it. Can we use the picture like that?

Jane: It would just be difficult if you had 61 slices and the other one had 62 slices. That would be really close.

Instructor 2: If I don't draw my pizzas, my 2 squares exactly the same size. And maybe they look like they're close to the

	same region the same space. So Jane went to this common denominator. What's my question about that? Or what should your question be about that? Claudia.
Claudia:	How did she get the common denominator?
Instructor 2:	How did she get 24? How did you know 24?
Jane:	Because I had to pick a number that both 6 and 8 could multiply by and equal to. So, 6 times 2 is 12 but 8 times 2 is 16 so that doesn't equal, so I kept going until I got to 24 which they both can be multiplied by, or to get the same answer
Instructor 2:	Okay. Are we okay with that? What if I didn't want to go to the common denominator? Jane, you started to talk something about the kids and cake. Did you guys hear what she said? What did she say?
Claudia:	They don't like to share.
Instructor 2:	They don't like to share. Yeah my kids don't like to share. And what about sharing do they want?
Caroline:	Bigger piece.
Instructor 2:	Bigger piece.
Caroline:	Portion.
Instructor 2:	So which of these gives them a bigger piece?
Caroline:	1/6. (Claim)
Instructor 2:	Why Caroline?
Caroline:	Because the portions that are equal are larger than the ones, when you divide them up. *(Data)*
Instructor 2:	Why are these bigger pieces than these?
Jackie:	You're splitting up the same amount between 6 people versus 8 people. *(Warrant)*
Instructor 2:	Okay. So when I split them among fewer people
Jackie:	Your piece ends up bigger. *(Backing)*
Instructor 2:	My pieces end up bigger. So would I have to go to Jane's twenty-fourths?
Class:	No.
Instructor 2:	How could I justify 1/6 being bigger than 1/8 without going to the twenty-fourths? It's okay to go to twenty-fourths. Could I do it without? Anybody do it without? You guys? Somebody want to explain?
Claudia:	Edith you
Edith:	Well I kind of liked how she started out with the cake example about sharing because that does give a good idea for kids. If you go to a party, you want the most cake. So if there's 8 kids getting cake or there's 6 kids getting a cake, which group would you want to be in? That kind of comparison. Would you want to be with 7 other people or 5 other people getting cake? So they

	can picture it that way instead of drawing the actual pieces. *(Data)*
Instructor 2:	Okay. So because my pieces are bigger. What has to be true in that case to be able to say because my pieces are bigger?
Claudia:	Because you're using 1 piece because you're using 1 of the 6 and 1 of the 8.
Instructor 2:	Okay. Both pieces had 1 piece. Okay. Anybody do it differently? How many found common denominators to compare numerators? How many of you just looked at the size of the pieces?

Though this problem was set in a context of having pizza leftover, Jane turned the problem into a fair sharing situation in order to make sense of the solution. Though Jane immediately moved on to using common denominators, after not being able to justify how to get a common denominator, the instructor asked the class to think of another way to solve the problem. The class then moved back into using a fair sharing situation, in which Jackie brought forth the idea of when you split something up among fewer people, each share is bigger. Claudia then noted that in this situation the strategy is valid because of the fact that you are taking one of each.

This strategy then continued to be developed when the class moved on to the non-contextualized situations. The common numerator strategy was presented when the class was discussing their solutions when comparing 4/5 and 4/9.

Instructor 2:	Okay so which one's bigger? Who says 4/5? Who says 4/9? So why is 4/5 bigger? Jane.
Jane:	Because 4/5 and 4/9 you can look and see that each one has 4 pieces colored I guess. But then 5 and 9, that's how many pieces are in it. So 5 has bigger pieces therefore the pieces are going to be bigger. So 4 out of 5 slices are taken. Large slices. That's going to be more than just 4 of a lot smaller pieces. *(Data)*
Instructor 2:	Okay. Questions for Jane? Who thought of it that way? They both have 4 pieces, but the pieces are bigger in the 4/5. So the fraction's bigger.

The common numerator strategy shifted and was not questioned within this conversation. Jane contributed again to this strategy, but this time only needed to present the data for what she did. In the previous conversation on this strategy, the class as a whole contributed warrants and backings to the discussion, however these were not needed in this discussion. Also relevant in this discussion is again the idea of the size of a piece. Though this problem was placed out of context, Jane referred to the problem as if it were situated in a context pertaining to pieces or slices of something.

When the class was developing the common numerator strategy, the idea of the size of the piece was introduced. The class started developing the idea that the number of pieces that something is divided into is inversely related to the size of each piece. This idea then continued to be developed when the class was establishing comparing using missing pieces.

<u>*Missing Pieces*</u>

The most difficult of the comparing strategies was the strategy of missing pieces. Within this strategy, the number of pieces and size of the piece is needed to determine which fraction has the least amount leftover thus had the most to start with. To introduce students to this idea, the following example presented two fractions with only one piece missing.

> *At the party, the trapezoid table was decorated with 5/6 of a spool of a ribbon. The rectangle table was decorated with 9/10 of a spool of ribbon. On which table was more ribbon used?*

As the class discussed their method for solving the problem, Edith used the ideas from the previous problem of looking for the size of the piece to determine which fraction, 5/6 or 9/10, is bigger.

Edith:	But what we first thought of was kind of in comparison to the last question how we are looking for the largest piece. In this case we are kind of looking for the smallest piece leftover. Because it says which one has more ribbon used. So if you look at the fractions, 5/6 compared to 9/10. If you know, again if you said okay well 5/6 if you divide among 6 people you know 6 strips of ribbon you know 1 for each person and then 10 strips of ribbon you know 1 for each person or whatever. Then you could look at it and say okay well the 10 strips are going to be smaller than the 6 strips. *(Data)*
Instructor 2:	Are you following this?
Edith:	Is this making any sense? Okay. So if you have 9 out of 10, you're going to use more because the strips are smaller. Because you'll have more of the original ribbon used. *(Data)*
Instructor 2:	Does that make sense?
Claudia:	Because the piece is left
Edith:	Yeah because the piece that's left is the smallest piece left. *(Warrant)*

Edith provided data and warrants for using the strategy of looking for the smallest piece that is left, which then tells her which one has more leftover. The idea that the number of pieces is inversely related to the size of the piece had to be used in order to make sense of the missing piece strategy. No one questioned Edith’s comment that “the strips are smaller” when cutting something into 10 versus 6. This idea no longer needed warrants and backings, and was not questioned, thus became taken-as-shared.

The strategy of missing pieces however continued to be developed throughout the non-contextualized situations. When comparing 9/11 and 13/15, the missing pieces strategy can be used because both fractions have two pieces missing. When Caroline was explaining this strategy to the class, the strategy was questioned in terms of how it should be explained to a child.

Caroline:	Since they both have 2 pieces missing, the pieces on the bottom one are smaller. That means the rest of what was used, or whatever we're talking about is more. *(Data)*
Alex:	I'm sorry, could you repeat that one more time?
Caroline:	I hope so. When they're divided up they're each, each of the problems are missing 2 pieces.
Alex:	Gotcha.
Caroline:	But the pieces in the bottom are all teenier pieces. So there's more leftover that isn't missing. If you think of a pie, there's 2 little pieces leftover, so there's more leftover. *(Data)*
Instructor 2:	So Caroline does that capture what you. Two pieces missing. This one's smaller pieces so it's a bigger fraction. So because they're both 2 away, and it's the same number of pieces, then I can look at. Does that make sense? People that were confused before. Less confusion? I don't want to confuse you. That's not my goal. So what did Caroline do? Somebody who hasn't talked much today. Who's confused?
Jackie:	I don't see how that would make sense to a kid that you're teaching that to. They both have 2 pieces missing but the bottom one has smaller pieces and the bigger fraction, so therefore that one's bigger. How is that going to make sense?
Instructor 2:	If I draw this [*9/11*]. And I draw this [*13/15*]. In both cases I have 2 pieces leftover.
Jackie:	Right.
Instructor 2:	I think that kids would see that.
Jackie:	Aren't we not allowed to draw it though?
Instructor 2:	You can draw it, but you can't say the picture looks bigger, so it's bigger.
Caroline:	Right.
Instructor 2:	Because drawing thirteenths or drawing elevenths and drawing fifteenths my picture probably won't be very accurate. Like this. If I just looked at this picture and said this one looks bigger. Not so good. But if I can look at the picture and say but look they're both missing 2 pieces. That can come from a picture. Maybe we have that picture in our head because I told you not to write them down.

Jackie:	But how do we keep going? You say you're missing 2 pieces but then
Instructor 2:	So how do we keep going? Somebody who hasn't talked much, who's getting it. How big are these pieces compared to these pieces? If they're the same size whole, same size pizza, same size cake, same size brownie, whatever. How big are these pieces compared to these pieces? Bigger or smaller?
Class:	Bigger.
Instructor 2:	Why?
Katherine:	Because it's cut into 11.
Instructor 2:	This one's cut into 11. This one's cut into 15. So these pieces are bigger. In both cases I have 2 pieces left. Which of these is bigger? These two pieces, or these two pieces?
Jackie:	The top two. *(Claim)*
Instructor 2:	These two are bigger. So what's this? Bigger or smaller than that?
Students:	Smaller. *(Claim)*
Instructor 2:	If this was a pizza, who ate more pizza?
Students:	The bottom one. *(Claim)*
Instructor 2:	This had more here than here. So if there's more left who ate more to start with?
Students:	The bottom one.
Instructor 2:	This one. So it's hard reasoning. It's backwards from what we think. When there's more left, it was smaller to start with. Because it's both 2 pieces, we can compare it that way. Does that make a little bit more sense now? Questions on that? It's hard reasoning. The missing pieces.

Jackie questioned the strategy in terms of how to explain it to a child. As was evident from the continuing conversation, students still had difficulties explaining the strategy of comparing by using a missing pieces method. The method of comparing using missing pieces was not taken-as-shared until the ordering activity when the class was given the following problem:

> *Pete held a pizza-eating contest. The following table shows how much of a large pizza each contestant ate. Rank the five contestants in order from first to fifth place.*

Colin	*7/8 of his pizza*
Amanda	*7/13 of her pizza*
Brandon	*9/20 of his pizza*
Stephanie	*23/24 of her pizza*
Jessica	*3/20 of her pizza*

Claire used the missing pieces strategy to compare 7/8 and 23/24. During this conversation no one questioned the strategy and only data was needed for Claire's explanation of 23/24 being bigger.

Claire: Well I first saw that 23 out of 24, I saw that 24 it's the smallest but that pizza's kind of the smallest amount. And then I got confused because I was thinking well maybe 7/8. But then I was like if they're smaller and 23 out of 24, one is missing, so that one's smaller than 7/8. So like the amount that they ate. Or the amount leftover was smaller so I knew that that was bigger than 7/8. *(Data)*

Instructor 2: So what did she say?

Caitlyn: There's more pieces in the first one. So therefore if the pizzas are the same size, those pieces are going to be smaller. And then since they're each missing one piece, the piece is the smallest one leftover, has the most. *(Data)*

Instructor 2: Okay. Are we okay with that? They're both missing one piece. This one's smaller pieces so it's more eaten. So this one's bigger than that one.

When Claire and Caitlyn finished explaining the missing pieces strategy, the strategy was no longer questioned by anyone else in the class. In addition, warrants and backings were no longer required to further justify the strategy. Comparing by using missing pieces became taken-as-shared.

<u>*Common Denominators*</u>

The fourth comparing method of using common denominators appeared to be taken-as-shared before the strategy was introduced with the comparing activities. Students knew that they could use a common denominator to compare fractions as evident from several students using that idea within some of the problems just described.

For example, Jane used common denominators when comparing 1/6 and 1/8.

Jane:	Okay. I knew that the 1/6 had more leftover because it's being split amongst fewer people. I already know how to do common denominators, but I knew that would be really hard to explain and so I tried to draw a picture. You can think of it as like with kids. You give them a cake and say alright you have to split it amongst your friends. They're going to want to split it between fewer because they want more cake. So you can see in the picture like the 1/8 piece is smaller than that one. But if you don't draw it very proportionally then you wouldn't see that. So *(Claim, Data)*
Instructor 2:	So the picture can get in the way if we don't draw the picture accurately.
Jane:	Correct. Yes.
Instructor 2:	Okay.
Jane:	So when the common denominators the bottom numbers have to be the same, like over here. So I knew that 6 and 8 both go into 24. So whatever it takes to get to 24, which is 4. I multiply by the top. And whatever it takes to get to 24, which is 3. I multiply by the top. And since this one is obviously larger, 4 instead of 3 things. So 1/6 is bigger. Does that make sense? *(Data)*
Instructor 2:	So what question do you think I have?
Suzy:	Justify.
Instructor 2:	Justify. Justify what?
Suzy:	What she just said. But can't she just leave the picture and say assuming all pieces are equal. 1/6.
Instructor 2:	Why?
Suzy:	Why? What did she say about if they are equally distributed? So equal pieces
Instructor 2:	So, Suzy right?
Suzy:	Correct.
Instructor 2:	Suzy said can't we just go back to the picture and if all the pieces are equal size in both, if I split both pizzas into equal size pieces that 1/6 is bigger.
Suzy:	By viewing it.
Instructor 2:	By viewing it. Can we use the picture like that?
Jane:	It would just be difficult if you had 61 slices and the other one had 62 slices. That would be really close.
Instructor 2:	If I don't draw my pizzas, my 2 squares exactly the same size. And maybe they look like they're close to the

	same region the same space. So Jane went to this common denominator. What's my question about that? Or what should your question be about that be? Claudia.
Claudia:	How did she get the common denominator?
Instructor 2:	How did she get 24? How did you know 24?
Jane:	Because I had to pick a number that both 6 and 8 could multiply by and equal to. So, 6 times 2 is 12 but 8 times 2 is 16 so that doesn't equal, so I kept going until I got to 24 which they both can be multiplied by, or to get the same answer.

Jane's comment of using common denominators was never questioned. Since she could not justify how to find a common denominator, *the method* for finding common denominators was questioned.

Jane was not the only one who reverted to a procedure for finding common denominators. When the class moved to the non-contextualized situation of comparing 1/3 and 3/5, Claire solved the problem by finding a common denominator, but also could not justify it.

Instructor 2:	Which one's bigger?
Claire:	3/5. (Claim)
Instructor 2:	3/5. Why? Claire.
Claire:	I found a common denominator just because I'm familiar with 3 and 5, but I can't really justify *(Data)*
Instructor 2:	Okay. So you found a common denominator in your head.
Claire:	Yeah.

Just as Jane was not able to explain common denominators outside of the procedure, Claire too admitted not being able to justify how to get a common denominator. Again, no one questioned using common denominators to compare though students could not justify how to find a common denominator.

The class quickly moved away from using common denominators in the cases where the denominators were not already the same. This method did get used in the cases where the fractions had a common denominator to begin with. These problems were not presented until the ordering activities. The first question with the pizza eating contest included comparing the fractions 9/20 and 3/20. By the time the class moved on to the ordering activity, students were able to compare using common denominators when the fractions had a common denominator to begin with. Caroline justified this as both fractions being out of the same portion.

Instructor 2:	So how do you know this one's bigger than that one?
Caroline:	Because they're both out of the same portion. I didn't even look at that. *(Data)*
Instructor 2:	They're both out of 20. And we forget that strategy sometimes. They're both out of 20 and 9 is bigger than 3.

The next ordering problem included comparing the fractions 10/71 and 15/71. Caitlyn used common denominators as her warrant for determining that 15/71 is greater.

Caitlyn:	I started guessing and I said 10 over 71 would be smallest because 10 is way less than half of 71. *(Data)*
Instructor:	Okay. Is that enough for us to know that it's the smallest?
Caitlyn:	No. Then I went and started comparing more and realized that my guess was accurate. I compared the other fractions to 1/2. *(Warrant)*
Instructor:	Okay, so let's, keep going with that reasoning and then we'll compare the rest.
Caitlyn:	So then I thought that maybe 15 over 71 was the next smallest. And I compared 15 to half of 71 and 10 is smaller than 15 so 10 over 71 must be smaller. *(Data)*
Instructor:	What do we need to hear in that explanation?
Claire:	Common denominator.
Instructor:	What?
Claire:	Common denominator. *(Warrant)*
Instructor:	Okay it's that common denominator also.
Caitlyn:	Right.
Instructor:	Okay, so you said it's smaller than 1/2, bigger than 10 over

71 and the strategy used is common denominator, more pieces. Is that? Okay. Is this representing your thinking?

Caitlyn: Uh-huh.

When the class was presented with the ordering activity, some of the fractions were presented with a common denominator already. As previously mentioned, it appeared that comparing with common denominators was taken-as-shared before the comparing and ordering activities were presented. No one ever questioned using common denominators to determine which fraction is greater. Instead the component that was questioned was justifying how to find a common denominator. This issue was never raised in the instances where the denominator was common already.

The strategies that became taken-as-shared during the comparing activities were the common numerator and comparing to a benchmark presented as comparing to half. The missing pieces strategy was taken-as-shared during the ordering activity. Comparing using common denominators appeared to be taken-as-shared before the comparing activities were presented.

The idea that the number of pieces into which a fraction is broken is inversely related to the size of the piece was taken-as-shared within the comparing activity as well. Though this was not a comparing strategy, this idea was indirectly developed from the common numerator and missing pieces strategy where students had to compare the size of the pieces.

Between Fractions

After the comparing and ordering activities and before moving on to the operations, the class was given an activity to find fractions in between two given fractions (See Appendix I). The intent of that activity was to have students develop the density property of rational numbers, which states that there are infinitely many fractions between any two fractions. The classroom conversations never included students stating that there were infinitely many answers. However, students were providing solutions with decimals. For example when asked to find three non-equivalent fractions between 1/6 and 1/3, Claudia responded that one fraction would be 1 1/4 over 6 or 1.25/6. No one in the class questioned Claudia for having a fraction in a fraction or a decimal in a fraction. Immediately following Claudia presenting her solution, the instructor asked the class to change the fractions so that they were not complex fractions. Having a fraction in a fraction or a decimal in a fraction was only briefly discussed, but was never questioned. It may be that the idea of having infinitely many fractions in between two given fractions was already taken-as-shared before the rational number unit started.

The between activity itself did nothing more than reiterate the comparing strategies that students had already taken-as-shared. Students did not develop any new methods from those they had just learned in the comparing and ordering activities. The between activity was the last concept activity presented before the class moved on to the operations.

Operations

The last two and a half days of the rational number unit focused on adding, subtracting, multiplying, and dividing fractions. As with the previous activities, the operations were presented in contextualized situations first. In addition, students were not told which operation the word problem represented and were left to determine this for themselves (Sowder et al., 1993). Addition and subtraction were presented in conjunction with one another (See Appendix J) and multiplication and division were separated into individual activities (See Appendix K and L).

Addition and Subtraction

Addition concepts were discussed starting on the second day of the unit. The sharing activity that the class was presented with not only asked them to share a given number of pizzas with a given number of people but to also determine how much of a pizza each person would receive. When determining how much of a pizza each person would receive, students had to add quantities together. Initial conversations regarding adding fractions focused around common denominators and that the denominator stays the same and the numerators are combined. These conversations also focused around the procedure for adding fractions.

Mindy:	So I divided the first two into halves. So each person gets an equal half and then the last one I just divided into fourths. **And then to find out how much they got total, I found a common denominator, which they both go into four, but I used 8 so we'll use four. Four goes into 4 one, and one times one is one. And two goes into four two and two times one is two. And then I just added that.** And I got three fourths of a dessert pizza. Any questions? *(Data)*
Instructor:	Questions? What's my question? You guys don't want to put Mindy on the spot so you don't want to ask the question I'm going to ask. What was this something goes into something so multiplies something?
Mindy:	**You need the lowest number. When adding you need the same denominator.** *(Warrant)*
Instructor:	Why?
Mindy:	**Because when you add, you don't add the denominators, you just add the numerators together.** *(Backing)*

Initial conversations regarding fraction addition occurred on a very surface, procedurally-based level. The backing for Mindy's argument was nothing more than keeping the denominator the same and adding the numerators.

When the focus shifted to a new problem, that same class day, a similar discussion occurred. When sharing 5 pizzas among 3 people, Cordelia expanded Mindy's initial backing to include what the numerator and denominator represent to justify why the denominator stays the same when adding.

Instructor:	What did you do when you got 5/3? Veronica what did you do when you got 5/3?
Veronica:	I added 3/3 to 2/3. *(Data)*
Instructor:	How?
Veronica:	You combine them.
Instructor:	You combined them. How did you combine them? What do you mean you combined them?
Veronica:	Added them together. You have the same denominator so you can add them. *(Data, Warrant)*
Instructor:	Do you know what kids do? When we teach them that kind of thing.
Caroline:	Add both numbers.
Instructor:	This is what kids do when we teach them those rules. What can we do to keep kids from, teaching those rules? Drilling leave the denominators the same and add the numerators clearly isn't it, because we've tried that. What do we do? Laura what do you think?
Laura:	I don't know.
Instructor:	Cordelia.
Cordelia:	**I think they need to understand that the bottom number doesn't change because the bottom number is only representing how many equal parts the whole is divided into.** *(Backing)*

Cordelia provided a more conceptual explanation that to add fractions the denominators do not get combined because they only represent how many parts the whole is divided into. After this discussion, the class moved on to the unitizing activity, thus adding was not discussed again until the addition activity was presented later on in the rational number unit.

Six class days differentiated between the first discussion of adding fractions and the next time it was discussed. When the addition discussions started, students again reverted to procedures. When given the following subtraction problem, Mary discussed the addition she did in terms of keeping the denominators the same, and did not go into any further mathematical explanation as to why that needs to be done.

The pizza dough machine broke after it had made 2 1/2 pounds of dough. The cook used 2/3 of a pound. How many pounds of dough were left?

Mary:	Alright this is one pound plus one pound plus a half which is the 2 1/2 that we originally have. It says the cooks used 2/3 of a pound and it wants to know how much is leftover. I cut this one into thirds, and I took 2/3 of it away. So I did 1 minus 2/3. And one can equal 3/3. So 3/3 minus 2/3 equals 1/3. And then I had the 1/3, the whole, and the half left so I had 1 plus 1/2 plus 1/3. **Then I want to get the denominators the same so I can add them.** *(Data)*
Instructor:	You know what you said that and like 18 of you looked at me. Why was that?
Mary:	I can't say that now?
Instructor:	Can she? Why'd you guys look at me?
Mary:	**So I want to get the denominators the same so I can add them. Because if the denominators aren't the same then it's really going to be hard to add.** *(Warrant)* So I figured the 2 and 3 could both go into 6, so I changed this to sixes on the bottom or as the denominator. And because I multiplied 2 times 3 to get 6 I had to multiply the top by 3. And because I multiplied the 3 times the denominator by 2, I had to multiply the top by 2, so the numerator was 2. And then for this, to get the denominator the same, I just changed this to 6/6. And I added which gave me 11 over 6. And you want it simplified it could be 6 goes into 11 one time with 5 leftover. *(Data)* So I get 1 5/6. *(Claim)*

Mary did not revert to a procedure for subtracting fractions. Procedures did not surface in the conversation until addition was involved. Even though six class days prior to this students started to slowly make more conceptually-based arguments for adding fractions, such as “the bottom number doesn't change because the bottom number is only representing how many equal parts the whole is divided into,” Mary, as with several other students in the class, quickly reverted back to procedures and using them on a superficial level.

When the conversation continued, Mary used what the denominator and numerator represent to further justify why the numerators are combined and the denominators are not.

Mary:	Since 6 is our whole, the whole doesn't change. Only the amount of pieces we take out of the whole does. *(Data)*
Instructor:	Laney.
Laney:	Because if you, I've heard it said that say you had a pan that could only fit 6 pieces, you can't shove another piece in there.
Instructor:	Couldn't I cut some of those pieces up? Once it's sixths is it always sixths? Because she had a pan that had 2 pieces in it, or that one that has 3 pieces in it, and she changed the number of pieces in that pan.
Mary:	You're not changing the whole, you're just changing the number of pieces in the whole. *(Data)* You're not putting another piece in, you're just cutting what was there into another piece. *(Warrant)*

Mary used a similar argument to what Cordelia had used in the sharing activity of referring to the numerator and denominator as representing the number of pieces you have and the number of pieces in the whole. As part of Mary’s argument, since the denominator represents the whole this is not going to change when combining pieces together. The idea that the denominator stays the same because the whole does not change when combining fractions together became taken-as-shared. This idea shifted from backing in the sharing activity to data in Mary's argument and no one questioned that the whole does not change.

Mary's warrant regarding cutting what is already there into another piece was the beginning of the conversations focusing on finding a common denominator. Up until this point, whenever a common denominator was discussed, it was done so in terms of students already knowing what the common denominator would be. Within this problem, needing a common denominator for two and three, students automatically knew that the denominator would be six. In response to the instructor's comment on the sixths remaining sixths, the class then moved on to developing ways to find a common denominator, since most of them seemed to already know how to do this procedurally.

A "trick" way was developed to find a common denominator for any two given fractions. In the problem of 1/2 + 1/3 from Mary's subtraction problem, Claire decided to cut the 1/2 into thirds and the 1/3 in half to create six pieces in each.

Claire:	**What I was going to say was one way that I did it to find the 6 was I took the 1/3 and I divided it in half because you see the other one says two and then you divide the third into half.** Does that make sense? And so that was just a trick to see that it was six. *(Data)*
Instructor:	So what did you do to the half?
Claire:	**Into thirds I meant. So then they both equal the same amount and that's how I knew that it was 6.** *(Data)*
Instructor:	Will that always work?
Claire:	I think so.
Instructor:	And why? So say I gave you this problem. [*Put 5/6 + 5/8 on board*]
Jackie:	It could get ugly.
Instructor:	It could get ugly.
Claire:	But I
Instructor:	Like this ugly.
Claire:	But even if it gets ugly it's a way to find the number that

they both can equal.
Instructor: Why will that always work?
Claire: **Because you're multiplying the two numbers and so they'll both equal.** *(Warrant)*

Claire's method of cutting the pieces in one fraction into the number of pieces given in the denominator of the other fraction was found to give the same number of pieces as multiplying both denominators together. Initially the instructor presented the problem 5/6 + 5/8 as a way to show the class that the numbers may not always be ideal to work with. Claire's method was further justified when the class then used the same process for finding a common denominator for 6 and 8.

Caroline In the 5/8 one you would cut each 1/8. Right? Into 6 equal pieces. And then on the other one you would cut each piece into 8 equal pieces. *(Data)*
Instructor: And how do I know that'll work?
Caroline Because you're cutting into the same number of equal pieces in each whole. *(Warrant)*
Instructor: How do we know that's going to happen? What if I had 97ths and 43rds?
Caroline Because 6 times 8 is 48 and 8 times 6 is 48. *(Backing)*
Instructor: Susan what is it?
Susan: That's right. She was just saying that it doesn't matter um in the problem where the numbers are. *(Warrant)* 8 times 6 is equal to 6 times 8.
Instructor: What property do we call that?
Olympia: Commutative. *(Claim)*
Instructor: Commutative property. That's an important property. We're finding a lot of reasons to use this commutative property.

Claire's method was found to work because of the commutative property of multiplication. Evident from the previous conversation, the class found that Claire's method of slicing pieces was equivalent to multiplying the two numbers together. This then led to the idea of using the commutative property of multiplication to find a common denominator for two fractions. In the same conversation, as the class finished solving 5/6

+ 5/8, Cordelia used Claire's method as data to explain the problem, and was not questioned.

Instructor: I want to know how. Now what? So where we talked about breaking these eighths into sixths and the sixths into eighths, who can provide the explanation and justification now? Cordelia.

Cordelia: I think so. You want to find how many pieces you can divide your whole into so that way they both have the same amount of pieces in that whole. So if I use

Instructor: We're okay so far?

Cordelia: So if I use 48, because 6 times 8 is 48, which is just a fast common denominator. **I would divide my 8 pieces up into 6 pieces in each piece. Each eighth is divided into 6 pieces.** So my 5 out of my 8 pieces that are shaded, because each eighth was divided into 6 pieces, I now have 30 out of 48 pieces shaded. *(Data)*

Instructor: Are we okay with that?

Class: Uh-huh.

Cordelia: And for 5/6 I'm doing the same thing. **I'm taking my 5 pieces that are shaded out of my 6 and dividing each sixth into eight pieces.** So now I have forty of my pieces shaded out of my 48 pieces. *(Data)* So now I have 70/48. *(Claim)*

Cordelia used Claire's method of dividing each piece into the number of pieces listed in the denominator of the other fraction and was not questioned.

The idea of using 48 as a common denominator was questioned in a previous conversation regarding this problem because of it not being the lowest common denominator.

Mary: But there's a lower common denominator than 8 times 6. 24 is, so that wouldn't work because it wouldn't be the least common denominator if you multiply 8 times 6. I mean it's still a common denominator. Don't we teach later on least common denominator?

Though the idea of using common denominator that is not necessarily the least common denominator was questioned because of how common denominators are typically taught, the class agreed that a common denominator of 48 was acceptable to use in this problem.

The idea of slicing pieces up to create a common denominator became the second taken-as-shared idea established in the context of adding fractions. The first idea that was taken-as-shared was that the denominator does not change when combining fractions because the whole stays the same.

With the addition and subtraction activities being presented together, the two practices of keeping the whole the same and slicing pieces up to find a common denominator both became taken-as-shared when addition was discussed. This was in part because the subtraction problem was converted into an addition situation. Both practices referred to the ways to make sense of the algorithm for adding and subtracting fractions. When adding and subtracting fractions, the whole remains the same which is why the denominators do not get combined. In addition, in order to combine or take away fractions, the pieces must be the same size, thus the denominators must be the same. When the class moved onto multiplication and division, the practices that were established did not pertain to the algorithm as they did with addition and subtraction. Rather, the practices pertained to underlying concepts when multiplying and dividing.

Multiplication and Division

Multiplication and division were presented as separate activities (See Appendix K and L). Within the multiplication problems, the first questions that arose were not that of how to solve the problem, but how to write the number sentence. The following was the first problem presented:

A cook made four pizzas that had 3/5 of a package of mushrooms on each. How much of a package of mushrooms were used?

Initial conversations focused around the number sentence to represent the situation. Though most students solved this problem as 3/5 + 3/5 + 3/5 + 3/5, the intent of having students write a number sentence was to have them determine if this should be represented as 4 x 3/5 and/or 3/5 x 4. Kassie initially presented the answer of 3/5 x 4 and then changed this answer when the instructor referred back to multiplication with whole numbers.

Instructor:	Okay. So what's a number sentence that can be used to solve this? That that to describe this problem.
Jackie:	3/5 plus 3/5 plus 3/5 plus 3/5.
Instructor:	When we repeat the same in addition over and over again, what's another way of writing this problem?
Kassie:	3/5 times 4.
Instructor:	Many of you will say 3/5 times 4, but we have a convention and we're not following this convention in this instance. We were trying to find 3/5 plus 3/5 plus 3/5 plus 3/5. Think back to our multiplication with whole numbers. When we have a problem like this and it's groups, what number represents what?
Kassie:	Oh it would be 4 times 3/5 because
Instructor:	Why Kassie?
Kassie:	It's four groups of 3/5.
Instructor:	Right. And that matters in this class. Right? The 4 here in this problem [*3 x* 4] tells us how much in each group. What does the 3 tell us?
Students:	How many groups. *(Data)*
Instructor:	How many groups. In this problem, what does the 4 tell us?
Caroline	How many groups. *(Data)*
Instructor:	How many groups. And what does the 3 tell us 3/5 I mean?
Caroline	How much in each group. *(Data)*

As with multiplication with whole numbers, the instructor noted that multiplication with fractions follows the same convention of how many groups times the number in each group. The groups of notation then became the focus of the discussion of the second problem:

> *Sue ate some pizza. 2/3 of a pizza is left over. Jim ate 3/4 of the left over pizza. How much of a whole pizza did Jim eat?*

With this problem, the number of groups was no longer a whole number. As a result, students such as Caroline struggled to understand if the problem represented 3/4 x 2/3 or 2/3 x 3/4.

Instructor:	What's the multiplication problem that's represented by this problem this story? Order matters.
Kassie:	3/4 times 2/3.
Instructor:	Kassie says 3/4 times. Kassie says ¾
Kassie:	Times 2/3.
Instructor:	How many of you agree with Kassie? Not so many. Jessica why do you agree with Kassie?
Jessica:	Because you're finding 3/4 of 2/3.
Instructor:	So you're saying because we're finding 3/4 of 2/3 that tells us the order of our factors?
Caroline:	No.
Instructor:	Caroline doesn't agree.
Caroline:	Well it doesn't, it's the same thing. It looks the same to a child. Okay hold on.
Instructor:	Do you agree with this order or not?
Caroline:	That order? 3/4 times 2/3, no.
Instructor:	No. Why?
Caroline:	Because first comes the number of groups.
Instructor:	Okay.
Caroline:	Is the 2/3. And then comes the number of objects in that group. Because the 3/4 is how much of that group. So the 3/4 should come next.

Though Caroline correctly identified the placement of the numbers as the first number representing the number of groups and the second being the number of objects in the group, she felt that the number sentence should be 2/3 x 3/4 instead of the correct answer of 3/4 x 2/3. This may have been due to the fact that the 2/3 is what you start with and then 3/4 is what you do to the 2/3.

As this conversation continued, Olympia used Caroline's idea of starting with the 2/3 to explain why the 2/3 is actually placed second in the number sentence.

Olympia:	The number of groups go first that's why.
Instructor:	The number of groups go first. That's correct.
Olympia:	It's times the objects
Instructor:	So
Olympia:	because objects is what the original problem
Instructor:	Right. So how is 2/3 the objects?
Caroline:	Yeah.
Olympia:	**Because it's of the pizza you start with.**
Instructor:	Because it's of the pizza
Olympia:	Yeah. That's what we have.
Instructor:	Okay. 2/3 is the pizza we have. That describes the amount of pizza we have. 3/4 describes what we had of what we had. The number of groups or the part of group of the pizza that we had. 2/3 describes the objects. 3/4 refers to how much of that object we have.

From Olympia's explanation of the pizza that you start with, 2/3 was the object being acted upon, thus represented the objects in the problem or the second factor in the number sentence. Given the next problem, this meaning then shifted to simply becoming taking a fraction of a fraction.

There was 4/5 of a pound of pizza dough in the freezer from the previous day. The cook thawed out 3/8 of that dough. How much of a pound of dough did the cook thaw?

Several students thought this problem represented a subtraction situation of 4/5 – 3/8. The convention of taking a fraction of a fraction was used to explain the difference between this situation and a subtraction situation.

Kassie:	I'm confused on why it was multiplication.
Instructor:	Okay. What did you think it was? Subtraction?
Kassie:	Yeah.
Instructor:	Okay. Cordelia why is it multiplication and not subtraction?
Cordelia:	Because we're finding out how many groups we have out of how much of a group we had.

Instructor: How is that different than this? What is this [*4/5 – 3/8*] telling us? Cordelia.
Cordelia: That we're taking one whole. We're taking part of a whole from another part of a whole and that's not what we're finding. We're finding how many groups we have of what we have.
Olympia: How do we know?
Alex: You're not making any sense.
Instructor: Caitlyn help.
Caitlyn: **We're finding a fraction of a fraction. I don't know if that helps.**
Caroline: **A portion of fraction.**
Caitlyn: **A portion of a fraction.**
Caroline Whereas there, we're taking away from that fraction.
Caitlyn: Yeah.
Kassie: But it's saying
Caitlyn: It's hard.
Caroline: It's hard yeah.
Instructor: Olympia.
Olympia: I'm in the same boat. I don't understand why.
Instructor: Okay. Kassie.
Kassie: The question, maybe I read it wrong, but it said the cook thawed out 3/8 of the dough. How much did he the cook thaw? So
Instructor: So if it had, "There was 4/5"
Kassie: Oh I get it.
Instructor: Alright tell us. Help us out.
Kassie: Okay. It's basically what she said. They're looking at the 4/5
Instructor: Of a pound
Kassie: Yeah.
Instructor: of dough. Okay.
Kassie: I looked at it as he thawed out just that amount. How much didn't he thaw? So that would then be subtraction right?
Instructor: So what helped you? So if it had said there was 4/5 of a pound of dough. The cooked thawed 3/8 of a pound of dough. How much dough is left?
Kassie: Yeah.
Instructor: But really he thawed 3/8 of that dough.
Kassie: Yeah.

The question that was raised during the discussion of this problem was the difference between multiplication and subtraction situations. When Caitlyn and Caroline used the idea that multiplication involves taking a fraction or portion of a fraction, no one questioned this. The "groups of" convention evolved from taking the number of groups times the number of objects in the group to taking a fraction of a fraction. Since this was not questioned, the groups of convention was taken-as-shared.

When the class moved on to multiplication situations that were not placed in a context, the groups of context was automatically placed on each situation. When given the problem 1 1/5 x 1 2/3, and asked what that means, Caroline immediately interpreted this as 1 1/5 groups of 1 2/3.

Caroline: What is 1 1/5 of a group that is 1 2/3?

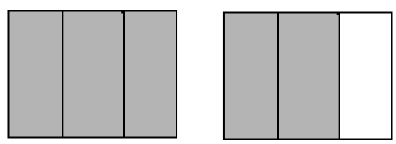
Even though the class understood that this problem represented 1 1/5 groups of 1 2/3, several students, like Jane, still did not understand how to start solving the problem without reverting to a procedure for doing so.

Jane:	I understand how to multiply fractions and I have no idea how you get started with that to get the answer. But if you do it this way, I know that 1/5 of 1 2/3, one times one is one. And then 1/5 of 2/3, like I did it in my head and if you multiply across it's 2/15.
Instructor:	So what you're saying is you would solve this by saying 1 times 1 plus 1/5 times 2/3?
Jane:	Yeah. Which is 2 pieces because there's the two, I guess. I don't know how you would explain it. It just worked in my head and gave me an answer anyway.
Instructor:	What do you guys think?
Olympia:	I don't understand what she said.
Instructor:	Olympia doesn't understand. Wait.
Jane:	1/5 of 2/3 is 2. I don't. So like basically you can just disregard the one. But it's 2/3.
Instructor:	So really this is equal to that? [*1 1/5 x 1 2/3 = 1/5 x 2/3*].
Jane:	Yeah.

Jane felt that she understood the procedure since she could use the procedure to get the correct answer of 2. Evident from Jane's explanation, she felt that the 1 in each fraction could be disregarded and had no affect on the answer.

Immediately following this, the instructor then put the problem in a context of a breakfast pizza serving being 1 2/3 pizzas, and eating 1 1/5 servings. Claudia then came to the board to explain the problem outside of the procedure and started developing the idea of a "new whole" in multiplication.

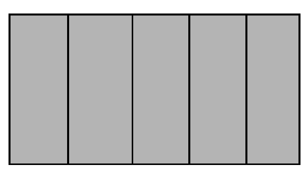
Claudia: So if we start off with two of these and divide them into thirds and then we find 1 2/3. Right? So then it would be this one, this one, this one, that's one. And then 2/3 would be this much.



So let's like draw because now 1 2/3 is our new whole because we're trying to find 1 1/5 of it. So if we just draw it altogether and make that our new whole. That's right, right? Yeah.

Instructor: How did she know that was right? What did she just check? You guys following her so far?

Claudia: So I'm just combining these into here to make that our whole.

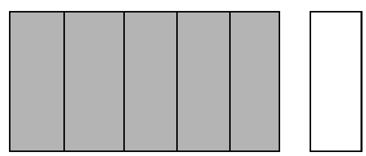


Everybody follows?

Class: Yeah.

Claudia: Okay. So now this is our whole and now this is our whole and it's divided already into five. So then this is one and then one more would be 1/5, right? Because this would be 1/5 since this is our whole and then this is one piece of that.

Suzy: I thought it had thirds, is that not a third?
Claudia: Yeah this is a third of one thing. But then
Suzy: Each of those thirds right?
Claudia: But then of this new whole which is 1 2/3, this is 1/5 of that
Suzy: Alright.
Claudia: because it's divided into five. Does anybody not follow that?
Alex: I'm not following it. Sorry.
Suzy: Say it again. It took me a second, but that made sense.
Claudia: This is what 1 2/3 are right? Right here what I shaded.
Suzy: Because we're finding 1 1/5 times
Claudia: And 1 2/3 is going to be our new whole though because we're trying to find 1 1/5 of that. So I just drew that as a new whole. Everybody okay up to there?
Class: Yeah.
Student: So you have that and then 1/5?
Claudia: Yeah so then one more of these would be 1 1/5 because this is 1/5, right of this 1 2/3? And then one more would be that and it would be two.



Claudia's idea of a "new whole" showed that in fraction multiplication the whole changes throughout the problem. The initial whole was defined and used to represent the first fraction, in this case the 1 2/3. The whole group of 1 2/3 then became the new whole so that 1 1/5 of the 1 2/3 could be taken. For the answer, the whole changed back to the original or initial whole of one.

When given the situation of 1 1/5 x 1 2/3, several students had difficulties solving the problem because of having two fractions greater than one within the problem. Though the class was able to interpret this problem as 1 1/5 groups of 1 2/3, they could still not solve the problem until it was placed in a contextualized situation. Once this was done, new concepts such as a “new whole” started being developed.

After Claudia’s explanation, the class then was able to make more sense of multiplication situations by using Claudia’s diagram to discuss the fact that when you multiply, you can break groups apart and take individual groups separately, or distribute to find the answer.

Laney:	You've taken the one and 5/5 and you've taken
Instructor:	1 1/5. Yeah.
Laney:	Yeah. One and 1/5, and you've taken it apart so you
Instructor:	Apart like this?
Laney:	Yeah.
Instructor:	Okay.
Laney:	And you're still multiplying 1/5 by 1 2/3. You're still multiplying the whole 1 1/5 by 1 2/3 you're just doing it separately. As in steps.
Instructor:	Okay.
Laney:	Instead of doing it all at once you're doing, you're first step would be to multiply one times 1 2/3. So that's half of that when you split it up. And then your second step would be to multiply 1/5 of that 1 and 2/5.
Instructor:	1 2/3.
Laney:	1 2/3.
Instructor:	Okay. There's a hand up.
Edith:	I was going to say aren't you just distributing the 1 2/3 to the, like if you had written
Instructor:	What are we distributing?
Edith:	The 1 2/3 to 1 1/5, like you did the one plus 1/5. You're just distributing the 1 2/3 to that. That's the way I see it.

After Claudia’s explanation of her diagram, Laney and Edith went on to discuss that 1 1/5 x 1 2/3 can be solved as finding 1 group of 1 2/3 and 1/5 of a group of 1 2/3 or distributing the 1 1/5 to the 1 2/3.

The class was then presented with the problem of 2 1/3 x 3/5. When asked what this problem means, the distributive property was used, though not mentioned specifically.

Students: 2 1/3 of 3/5, 2 groups of 3/5 and 1/3 of 3/5

Several students answered this at once, and both the groups of interpretation and distributing interpretation were mentioned and not questioned. When the class went on to solve this problem, it was solved as (1 x 3/5) + (1 x 3/5) + (1/3 x 3/5). The idea of distributing to solve multiplication situations became taken-as-shared. This was only in the multiplication situations in which the number of groups was greater than one.

Two ideas became taken-as-shared with fraction multiplication. The first was that multiplication can be represented as "a groups of" situation or taking a fraction of a fraction. The second was the idea of distributing when the number of groups is greater than one. Though the class could solve each multiplication problem with a procedure, they never got to a point where they demonstrated conceptual understanding of the procedure for multiplying fractions.

Fraction Division

A similar process happened for fraction division. The class never got to a point where they demonstrated conceptual understanding of the procedure for dividing fractions, but rather discussed concepts underlying fraction division situations. The only concept discussed was that of how to represent the remainder.

As mentioned in the unitizing section, representing the remainder was first discussed during the equivalence problem of having 32 people for 24 pizzas, when the class unitized to put this in terms of people per pizza. This discussion did not occur again until the division activities (See Appendix L).

The following is the first division problem that the class was presented with.

> *1 7/8 pounds of pizza dough were made. How many 1/4 pound servings can be made? What part of another serving will be left?*

When arriving at the answer, the two answers that the class came up with were 7 1/2 and 7 1/8. A student had drawn the problem on the board and the class discussed what to do with the leftover piece. Some of the class referred to this piece as 1/2 because it represented 1/2 of a serving, whereas other students, like Jackie, represented the remainder in terms of the whole and arrived at an answer of 7 1/8. As evident from the following discussion, though Jackie initially responded with an answer of 7 1/8, she stated that the 1/8 is actually 1/2 of a fourth. By the end of the conversation, she determined that the correct answer was 7 1/2.

Instructor:	What's the answer?
Mary:	7 1/2.
Jackie:	7 1/8
Instructor:	7 1/8 or is it 7 1/2? As I heard from over here.
Jackie:	It's 7 1/8 because what's leftover is half of a fourth.
Instructor:	Okay so
Jackie:	That one section is 1/8 of the whole.
Susan:	Is that the same difference or no?
Jackie:	Yeah it's its
Instructor:	Only one's correct. Kassie.
Kassie:	Isn't it 1/2 because now you've broken the portions up into fourths and it's half of one? *(Data)*
Instructor:	Jackie did you hear what Kassie said?
Jackie:	Yeah sort of what she said. If you look at is as a serving is 1/4, you just you just have to explain that the 1/2 is not 1/2 of a whole it's 1/2 of a serving. *(Warrant)*

The discussion occurred after another student explained her solution at the board by drawing circles to represent the division situation of 1 7/8 divided by 1/4 illustrated in figure 24.

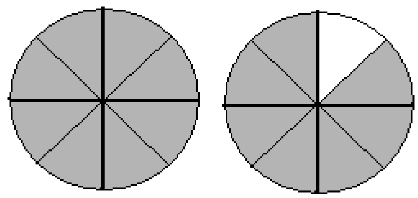


Figure 24: 1 7/8 Divided by 1/4

The remaining piece was 1/8 of a whole circle, but 1/2 of a serving. This discussion then led the class to discuss what 7 1/8 and 7 1/2 would each mean in terms of the problem.

Instructor:	Why do you think so many people give the wrong answer of 1/8? Yeah.
Edith:	Because they look at the whole pizza instead of just the serving. *(Backing)*
Instructor:	So when they do that, what does the 7 refer to?
Jackie:	7 servings.
Instructor:	Servings. And what does the 1/8 refer to?
Students:	Pounds
Instructor:	Pounds. The whole pie or pizza, pounds, right. Or pounds of dough. And so now we have two different wholes, which isn't legal. This is our how many servings? 7 1/2 servings.

The answer of 7 1/8 was incorrect because the 7 referred to servings whereas the 1/8 referred to pounds. In the 7 1/2, both the 7 and 1/2 refer to the number of servings. The discussion of representing the remainder led the class to discuss how mixed numbers are represented. Within mixed numbers, both components need to be representative of the same amount, or servings in this case.

A similar discussion occurred within the discussion of the next problem:

There was 5/6 of a gallon of lemonade leftover from a party. Pitchers that hold 2/3 of a gallon were filled. Using all of the lemonade, how many pitchers can be filled?

When discussing their solutions to this problem, again two answers were discussed. The correct answer of 1 1/4 gallons and the incorrect answer of 1 1/6 or 1 pitcher and 1/6 of a gallon.

Edith: You have the one pitcher and 1/4 of a pitcher.
Instructor: Jane you have a question?
Jane: Yeah. How does 1/3 plus 1/3 equal one pitcher?
Students: One pitcher is 2/3.
Jane: Okay. I never saw that.
Instructor: Good. Okay. Is that correct? How many of you said the answer was this? Be careful. Why is this [*1 1/6*] not correct? Those of you that said this [*1 1/6*], tell us why you think it's not correct. Jackie.
Jackie: Because we're not finding the amount in gallons we're finding the amount in pitchers. *(Backing)*

Again, backing had to be provided to explain the difference between the two answers and why the answer of 1 1/6 is incorrect. The answer of 1 1/4 was questioned by Katherine.

Katherine: I don't get it.
Instructor: Okay. Is that your question?
Jane: No. I remember.
Instructor: Okay wait. Let's take care of Katherine's question. I don't get it. What don't you get?
Katherine: The answer.
Instructor: Okay. Do you get the one?
Katherine: Um
Instructor: Okay well let's start out by writing the number sentence and then come back to you. Who has the number sentence? Alex.
Alex: 5/6 divided by 2/3.
Instructor: Because it's a sentence it's going to have equal or inequality or something there, but in this case an equal sign. So Alex do you want to respond to Katherine? How do you, what?
Alex: I don't quite know what you don't understand. What portion of it?
Katherine: I only got up to where we separate it into groups. And

then I don't understand the rest.

Instructor: Okay she got to

Katherine: I got the 1/3, 1/3, 1/3.

Instructor: Why did you do that?

Katherine: Because we were looking for 2/3. So then there's 2/3 there. Yeah that's 2/3 to find out that.

Alex: So that equals one pitcher according to the question itself. So that's how we got the one pitcher because the question had told us 2/3 equals one pitcher. And then when you're looking at that you still have the 1/3 left,

Katherine: Right.

Alex: but you're looking at the 1/3 of the pitcher. So you have the 1/3, 1/3, 1/3, 1/3, so there's 4 thirds there in the pitcher and then the 1/3 is leftover.

Susan: Actually the quarter the 1/3

Instructor: The 1/3 is leftover?

Alex: Or the 1/4. The one, so you're looking at the 1/3, is actually 1/4 of

Instructor: It's not really 1/3. This whole thing is 1/3. This little piece

Katherine: Right. Right.

Instructor: is a sixth. Right. So go ahead Alex. The 1/6 is

Katherine: A fourth of a pitcher. I get it.

Defining the whole became important in the context of division. Though the 1/6 was 1/6 of a whole of one, it represented 1/4 of a pitcher at the same time.

As evident from Katherine's discussion, the idea of representing the remainder in division was still being questioned. This was the last division situation the class was presented. It was not apparent from the discussions of both problems if representing the remainder became taken-as-shared. Backings were still needed in both problems to justify why an answer was incorrect, and this idea was still being questioned.

Conclusion

This chapter presented the taken-as-shared ideas that were established throughout the nine days of instruction. These are summarized in the table below (See Table 6).

Table 6: Taken-as-Shared Ideas Established

Classroom Mathematical Practices	Taken-as-Shared Ideas
Naming Shaded Regions to Define Fractions	• Fractions are parts of wholes • Fractions are comprised of equal parts • Fractions are a comparison of the number of pieces you have to the number of pieces in the whole
Naming Shaded Regions to Define the Whole	• Fractions need to be labeled in terms of a specified whole
Partitioning Circles and Rectangles to Create Fair Shares	• Partitions do not have to be equivalent as long as everyone receives the same amount
Using Linear, Set, and Area Models to Unitize	• Finding a unit fraction • Iterating a unit fraction • Develop a composite unit of one • Unitizing in terms of the whole
Partitioning and Undividing Circles to Find Equivalent Fractions	• Different names for the same amount • Make pieces bigger • Break groups into smaller
Comparing with Reasoning	• Compare to a benchmark • Common numerators • Common denominators • Compare using missing pieces
Using Comparing Reasoning Strategies to Determine Fraction Density	• The number of pieces is inversely related to the size of the piece
Partitioning Areas and Combining Pieces to Add Fractions	• Whole stays the same • Commutative property to find a common denominator
Partitioning Partitions to Multiply Fractions	• Groups of convention • Distributing when groups are greater than one

A select number of the practices were used to illustrate the ways in which the social and psychological environments impacted one another. Upon completion of the data analysis, there were found to be two ways in which individuals contributed to the practices. The first was that of introducing a new idea in the form of data, warrants, or backings. The second was that of contributing additional evidence again in the form of data, warrants, and backings to sustain an idea.

There were two ways in which the classroom community impacted individual students' knowledge reorganization. The first was that when a mathematically correct idea was presented that students did not think was correct, those students had to alter their thinking to accept the idea as correct (Stephan et. al., 2003). Second, when a mathematically incorrect idea was presented that students thought was correct, students had to alter their thinking to no longer accept that idea as correct when the classroom community rejected that idea. The ways in which these two environments impacted one another are illustrated in figure 25.

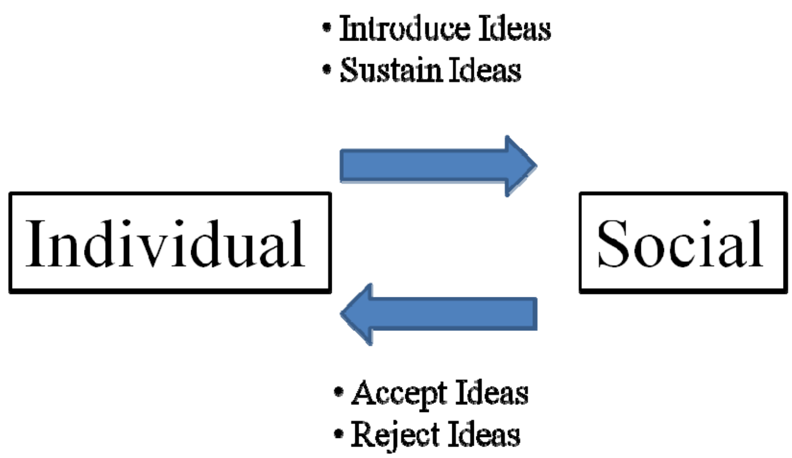


Figure 25: Cycle of Classroom Learning

The next chapter presents a summary of the findings as well as implications from the results. The refined hypothetical learning trajectory and ways in which the data analysis contributed to the suggested revisions are discussed.

CHAPTER FIVE: CONCLUSION

This study was part of a semester-long classroom teaching experiment focusing on the ways in which preservice elementary teachers develop an understanding of numbers and operations. The results from this study focused on the rational number unit in the course, which was the second unit presented to the class. This chapter provides a discussion of the results as well as the implications of the results for future iterations of this study.

The results from this study were presented in terms of the social environment as well as the ways in which the social and individual environments interact with one another (Cobb et al., 2001; Cobb, Yackel, & Wood, 1992). The learning of the class was determined from the ideas that were taken-as-shared throughout the rational number unit. The taken-as-shared ideas were determined using Rasmussen and Stephan's (2008) three-phase approach for documenting collective activity. Individual students' contributions to the practices was determined from the data, warrants, and backings they provided to the classroom conversations (Toulmin, 2003). In addition the ways in which the social community impacted student learning was determined using the constant comparative method (Glaser & Strauss, 1967).

An initial hypothetical learning trajectory was developed out of a combination of previous research with children's and adults' learning and understanding of rational numbers. The initial hypothetical learning trajectory incorporated five phases. The first three phases included work with rational number concepts of partitioning, unitizing, finding equivalent rational numbers, comparing and ordering, and finding fractions between other fractions. The last two phases focused on the four operations with fractions.

The rational number unit was designed so that students would not only work with various concepts and operations, but would also work with the five subconstructs of rational numbers. These included the part-whole, quotient, ratio, operator, and measure meanings of rational numbers (Kieren, 1976). These meanings were incorporated throughout the activities. For example, the partitioning situations were presented as a quotient situation of sharing a number of pizzas with a set number of people. In addition, the rational number unit was placed into the context of pizza as suggested by previous research with preservice elementary teachers (Wheeldon, 2008).

The activities presented in the rational number unit were designed so that students would work with contextualized situations first, work to solve the problems in their small groups, then discuss the problems in a whole class setting (Gravemeijer, 2004; Streefland, 1991). While students participated in a whole-class discussion, several social and sociomathematical norms were established. The social norms included a) explaining and justifying solutions and solution processes, b) making sense of others' explanations and justifications, c) helping others, d) asking if a solution makes sense, e) explaining someone else's thought process, and f) questioning others when misunderstandings occur. The sociomathematical norms established included determining what constitutes a) an acceptable solution and b) a different solution in the form of either a different answer or a different process to arrive at the same answer.

As a result of the whole-class discussions, several classroom mathematical practices were established. These included a) defining fractions, b) defining the whole, c) partitioning, d) unitizing, e) finding equivalent rational numbers, f) comparing, and g) ideas related to addition, subtraction, and multiplication. Several established practices fell under an overarching topic. For example, unitizing included a) finding a unit fraction, b) iterating a unit fraction, c) developing a composite unit of one, and d) unitizing in terms of the whole. The practices are summarized in the table below.

Table 7: Established Practices

Classroom Mathematical Practices	Taken-as-Shared Ideas
Naming Shaded Regions to Define Fractions	• Fractions are parts of wholes • Fractions are comprised of equal parts • Fractions are a comparison of the number of pieces you have to the number of pieces in the whole
Naming Shaded Regions to Define the Whole	• Fractions need to be labeled in terms of a specified whole
Partitioning Circles and Rectangles to Create Fair Shares	• Partitions do not have to be equivalent as long as everyone receives the same amount
Using Linear, Set, and Area Models to Unitize	• Finding a unit fraction • Iterating a unit fraction • Develop a composite unit of one • Unitizing in terms of the whole
Partitioning and Undividing Circles to Find Equivalent Fractions	• Different names for the same amount • Make pieces bigger • Break groups into smaller
Comparing with Reasoning	• Compare to a benchmark • Common numerators • Common denominators • Compare using missing pieces
Using Comparing Reasoning Strategies to Determine Fraction Density	• The number of pieces is inversely related to the size of the piece
Partitioning Areas and Combining Pieces to Add Fractions	• Whole stays the same • Commutative property to find a common denominator
Partitioning Partitions to	• Groups of convention

Multiply Fractions	• Distributing when groups are greater than one

Several ideas related to the practices appeared to be taken-as-shared before the rational number unit started. These included the idea that a) fractions are parts-of-wholes, b) fractions are comprised of equal parts, c) a unit fraction can be iterated, d) there are infinitely many fractions between any two fractions, and e) common denominators can be used to compare fractions. As described in chapter four, these ideas did not always shift in function and were never questioned. Thus, taken-as-shared ideas may not always shift in arguments. Rasmussen and Stephan (2008) outlined two criteria for documenting collective activity. Their criteria included that when ideas shift in function and are not questioned, the idea is taken-as-shared. Evident from this data analysis, ideas also appeared to be taken-as-shared when only data are needed and that data are never questioned. Thus, this may be a third criterion in documenting collective activity.

Defining the whole and partitioning were presented first because of their importance in providing students with a foundation for learning subsequent topics (Carpenter et al., 1993; Moskal & Magone, 2002). Moskal and Magone found that it is difficult to accurately assess students' understanding of future tasks if they are unable to define the whole. Thus, the Opening Day activity where students had to develop ways of defining the whole was presented first. The partitioning situations were presented immediately after that as several researchers have suggested that partitioning is the natural link between whole numbers and fractions (Carpenter et al., 1993; Kieren, 1976).

While the class developed how to define the whole, more misconceptions with how to do this correctly surfaced in the partitioning activity in which several students represented their solutions out of all the pizzas instead of representing the answer in terms of a pizza. This may have been because more than one answer was acceptable for the Opening Day activity, whereas the sharing activity specified that answers had to be put in terms of a pizza. Though the intent of the Opening Day activity was to have students start defining the whole, other ideas started surfacing as well such as undividing. With students prematurely developing equivalence concepts, it is suggested that future studies should start with partitioning first and then use an activity like Opening Day if students still have difficulties defining the whole after completing a partitioning activity.

Unitizing was presented after the partitioning for two reasons: one, students need to be able to partition and define the whole before they can successfully unitize (Moskal & Magone, 2002). Two, unitizing ideas can be used in later activities where students need to determine if two fractions are equivalent for example (Lamon, 2002). The unitizing activity, Keeping Track, was designed so that students only needed to work with finding a unit fraction, iterating a unit fraction, and developing a composite unit of one (Lamon, 1996). Unitizing in terms of the whole was not a focus of the activity, nor was it a focus of any activity because students did not need to understand how to unitize in terms of the whole to successfully complete other activities. Though it was not a focus, unitizing in terms of the whole was discussed during the equivalence activities and eventually became taken-as-shared.

The equivalence activities were presented through ratios as suggested by Tarlow and Fosnot (2007). Equivalent fractions are proportional to one another and follow the same between and within relationships as ratios (Noelting, 1980a, 1980b). Before the class was presented with the equivalence activities, the ideas that equivalent fractions represent the same amount and that partition lines can be erased to find an equivalent fraction were already established.

The first equivalence activity was designed so that students would work to break groups down into smaller groups representing equivalent amounts. As part of this activity, an unintended idea which was introduced was unitizing in terms of the whole. Unitizing in terms of the whole was introduced as the class discussed breaking groups down into equivalent groups of smaller amounts. Though unitizing in terms of the whole was problematic because of the context of the problems resulting in a fraction of person, students understood that they could unitize in terms of the whole. As discussed in chapter four, unitizing in terms of the whole allowed the class to discuss the remainder in division situations before being presented with a division situation. Thus, unitizing in terms of the whole should be included in a rational number unit though it is the most difficult unitizing concept for students to understand (Lamon, 1996). This should be presented in a situation where having a fraction of something does not present an issue such as 1 1/3 of a person. For example, in a situation of having 3 cups of water to 2 cups of concentrate, unitizing in terms of the whole would result in 3/2 cups of water to 1 cup of concentrate, where 3/2 cups of water is a realistic solution.

The comparing and ordering activities were developed so that students would discover reasoning strategies to compare fractions as opposed to using a numeric procedure (Post et al., 1993; Post et al., 1985). The ideas related to comparing and ordering fractions included primary ideas of comparing to a benchmark, common numerators, common denominators, and missing pieces, which were the same strategies children were found to develop in previous research studies (Behr et al., 1984). The secondary idea that the number of pieces something is cut into affects the size of the piece was also established. The comparing and ordering activities also introduced fractions as quantities (Post et al., 1985).

The ideas related to the operations pertained to the algorithms for addition and subtraction, and underlying concepts for multiplication. Due to the nature of the multiplication and division activities the algorithms for multiplication and division were never discussed. Rather, the multiplication and division activities were designed so that students would develop concepts related to the meanings of multiplication and division and of the solutions. In addition, students were not given enough experiences related to division for ideas in division to become taken-as-shared. With addition and subtraction, the ideas discussed pertained specifically to the algorithm. These included the ideas that the whole does not change, which is why the denominator stays the same, and using the commutative property to find a common denominator. Within multiplication the ideas that became taken-as-shared were not related to the algorithm but to multiplication in general. The first was the groups of meaning of multiplication of taking a part of a group. The second was that when multiplying with fractions greater than one, the distributive property can be used to take a whole group or groups and combine that with a part of that same group. The idea related to division that was introduced was representing the remainder. Though this was discussed in the equivalence activity in terms of unitizing to the whole and discussed again in the context of division situations, the class was not given enough remainder situations for the idea to become taken-as-shared. Thus, more division problems with a remainder need to be included so that ideas can be fully developed. In addition, estimation was included in the activities; however, because of estimation not being brought to the forefront of conversations it was not clear if students accurately developed estimation strategies or not because estimation was only briefly discussed.

Two of the activities presented in the rational number unit only reiterated the ideas that the class had already established and did not add anything new to students' learning. These included the number line activity and the between activity. The number line was introduced as a way for students to start developing unitizing strategies. Recipes was an activity related to finding fractions between two given fractions.

The number line activity was designed so that students would use paper folding techniques to find a unit fraction, iterate a unit fraction, and develop a composite unit of one. This activity was presented in between the partitioning and Keeping Track activity. Though the intent of the activity was to have students start developing unitizing techniques, the activity only reiterated partitioning strategies.

The between activity, Recipes, did nothing more than reiterate the various comparing strategies. The intent of the activity was for students to develop the idea that between any two fractions there are infinitely many fractions. With students initially responding with decimal solutions and these solutions not being questioned, it appeared that the idea that there are infinitely many fractions between any two given fractions was already taken-as-shared before the rational number unit started.

As individuals participated in the classroom environment, there were found to be two ways that individuals contributed to the practices as well as two ways in which the social environment affected student knowledge reorganization. Individuals participated in the establishment of the practices by introducing and sustaining ideas. Individuals reorganized their knowledge as a result of ideas being either accepted or rejected by the classroom community. This reflexive relationship is important for both the establishment of the practices as well as individual student's knowledge development (Cobb & Yackel, 1996).

The linear and set tools were introduced to students at the beginning of the semester when the class was focused on whole numbers (Roy, 2008). In whole numbers, the linear tool was introduced as a way for students to represent their informal counting strategies. The set tool was introduced as pieces of candies, which then were packaged 10_8 to a roll and 100_8 to a box. By the start of the rational number unit, the class was familiar with working with the linear and set models.

The rational number unit was introduced with an area tool. The area tool was initially represented as circles and this was expanded to include rectangles starting the second day of the unit. Area tools continued to be used throughout the rational number unit and were the predominant tools used. This was in part because several rational number concepts, such as partitioning, were more conducive to an area model.

Linear and set tools were also used throughout the rational number unit, but not as frequently. When these tools were incorporated into an activity, students tended to convert the situation into an area situation. In the instances where the tool was not changed into an area tool, students struggled with finding the solution and/or disregarded the solution. For example, when unitizing 32 pizzas for 24 people down to one pizza, the solution of 1 1/3 people was disregarded. The only exception to this was in the unitizing activity Keeping Track. Keeping Track incorporated all three models and was the only activity where the linear and set models did not hinder students' solutions or solution processes. In addition, it was the only activity where students did not convert a linear or set model into an area representation.

As discussed in the results, several ideas were introduced and established while the class worked with area tools. The data analysis illustrated that though students could work with both circles and rectangles in an area situation, some practices were easier for students to develop while working with circles. For example, circles were easier for students to work with in multiplication situations than working with rectangles.

Though past research has suggested that children need experiences with various types of models (Cramer & Henry, 2002; Post et al., 1993), the instructional unit implemented in this study was mainly conducive to work with an area model. Thus, the instructional activities may need to be designed better in future iterations so that all three models can be developed fully.

Instructional Sequence Revisions

The hypothetical trajectory used in this study was developed as a result of previous research with children's and adults' learning and understanding of rational number concepts and operations (Lamon, 2005; Mack, 2001; Streefland, 1993; Wheeldon, 2008). From the results of previous research the HLT incorporated several phases of learning in which the activities were designed so that students would work with all five subconstructs of rational numbers (Kieren, 1976). The activities were also designed so that students would work with contextualized situations first before being asked to solve problems out of context (Gravemeijer, 2004; Streefland, 1991).

Wheeldon (2008) suggested that an overarching scenario, such as pizza, should be used throughout a rational number unit. The pizza parlor scenario was used because of Wheeldon's findings that preservice teachers tended to use pizza to represent fractions even when the original problem had nothing to do with pizza. What follows is a discussion of the suggested revisions to the instructional sequence.

Revision: Tools

The intent of the rational number unit was to incorporate the area, linear, and set models throughout as teachers need to provide their own students with experiences related to each model. Though an overarching topic was used for this study, the model presented in the context of some problems caused confusion within these situations. This was particularly true for problems represented in a set context. The rational number unit was not designed for students to develop strategies related to unitizing in terms of the whole, for example. However, when this idea was introduced in the equivalence activity students unitized down to 1 1/3 people, which caused students to disregard the answer because of it not being realistic. If the problem would have been placed in an area context, where having a fraction of a quantity was realistic, then the class may have accepted the solution as representing a correct strategy of unitizing in terms of the whole.

As students progressed through the rational number unit, they almost exclusively chose to use an area model to solve the problem, even when the problem was presented in a different context. The only time this did not occur was during the unitizing activity.

Thus, better activities need to be incorporated in future iterations so that students can develop ideas while incorporating all three models and not just area.

Revision: Ratio Context for Concepts

Rational number ideas that the class were familiar with before starting the rational number unit, such as common denominators, tended to initially confound students' understanding of topics as they frequently reverted to the procedures they had learned in the past to solve problems. The only time procedures did not present an issue with students' developing an understanding of a topic was when the class moved on to the equivalence activities which were placed in a ratio context. Initially students did not agree with writing a ratio in fraction notation because of ratios representing two different quantities. Thus, when the class moved on to ratios, discussions occurred regarding how ratios and fractions are different. Though the class worked with developing equivalent fractions, they did not revert to a procedure to find the answer. Rather, students relied on working with what was given to them in the problem to break groups apart into smaller groups and then develop the procedure of dividing the groups by the same number. This may have been because the problem was presented in the context of a ratio situation.

With the class viewing ratios and fractions as two completely different types of rational numbers, it may be that the concepts portion of a rational number unit should be taught exclusively through ratios. Ratios follow all of the same properties as fractions and they may be different enough for adults where preservice teachers are not going to be as quick to use a known procedure to solve a problem similar to the ways in which preservice teachers learn whole numbers in the context in base 8 (Andreasen, 2006; Roy, 2008). As previously discussed, ratios in which a quantity represented a discrete set caused students to disregard answers, thus ratios will need to comprise of continuous quantities such that solutions are realistic.

Revision 3: Deletions

Two activities are suggested to be deleted within future iterations. The number line and between activities were found not to contribute anything new to the established practices.

The number line activity was incorporated into the rational number unit as a way to introduce unitizing ideas of finding a unit fraction, iterating a unit fraction, and developing a composite unit of one. Rather, the number line task did nothing more than reiterate the partitioning strategies that became taken-as-shared within the previous activity. Thus, students did not introduce or establish any ideas that were not already taken-as-shared.

The between activity is also suggested to be deleted. Similar to the number line task, students did not introduce or establish any ideas different from those that had already been introduced or established. In addition, students relied on concepts and imagery within an area model to determine a correct answer, and did not use tools in a way different from what they had before.

Revised HLT

The proposed HLT incorporates four phases of learning illustrated in table 8.

Table 8: Proposed HLT for Future Iterations

Phase	Idea	Activity
One	Partitioning	Sharing Opening Day if needed
	Unitizing	Unitizing with Ratios
	Equivalence	Family Reunion Customers and Community
Two	Comparing	Compare with Ratios
	Ordering	Ordering with Ratios
Three	Addition Subtraction	Pizza Parlor Situations 1
Four	Multiplication	Pizza Parlor Situations 2
	Division	Pizza Parlor Situations 3

Within the first phase, basic concepts of rational numbers are introduced. This phase incorporates the ideas of defining the whole for a rational number as well as developing ideas related to composing and decomposing where the rational number is viewed as a portion of an amount which comprise of equal sized pieces. The second phase incorporates the idea that rational numbers are quantities. This phase introduces the idea that rational numbers are relative in size to one another. The last two phases incorporate the four operations. It is suggested that these phases include concepts within each operation as well as methods for reinventing the algorithms and estimation strategies.

By starting a rational number unit with partitioning situations in which a whole is already specified, students can start to develop ways to partition amounts and define the whole at the same time. Some of the difficulty with the partitioning situations in this study included students finding how much of a pizza each person got. When determining how much of a pizza everyone receives students tended to use the traditional algorithm for combining pieces together. With this activity being presented second in this study, the ideas needed to add fractions, such as equivalence, had not yet been taken-as-shared. Thus, this activity had students solving problems prematurely, which may have been why they reverted to an algorithm for solving them. Though it is suggested that the sharing activity should be presented first within subsequent iterations, the activity should be altered so that students are only partitioning and not being asked to determine how much everyone receives. For example, students should instead be asked a situation such as:

> *Share 4 pizzas among 5 people in two different ways. How do you know that each situation results in everyone receiving an equal amount?*

By presenting a situation like this, students do not need to determine how much of a pizza everyone gets, thus they will not be as quick to revert to the addition algorithm. Rather, the problem can be solved by making the pieces equivalent by cutting them or "undividing" them to show that in each instance the same amount is received though the partitions are not congruent to one another. In addition, students themselves can develop different ways of partitioning so that the idea does not have to be generated by the instructor.

The unitizing concepts of finding a unit fraction and developing a composite unit of one were two of the unitizing ideas established during the class. The idea of iterating unit fractions appeared to be taken-as-shared before the rational number unit started. With the three unitizing ideas, these ideas can be placed in a ratio context. Two continuous quantities will have to be used, such as Noelting's (1980a, 1980b) orange juice problems of having a given number of cups of orange juice to a given number of cups of water. Within this study, with one of the quantities being discrete, arriving at a fraction answer did not make sense. In this study the problem of unitizing 32 pizzas to 24 people resulted in 1 pizza to 1 1/3 people, thus students disregarded the solution, though it was correct, because of the context of not being able to take a fraction of a person. In addition, unitizing in terms of the whole should be focused on as this topic introduces students to representing the remainder in division situations.

Comparing and ordering strategies can also be developed from ratio situations. For example:

Which mixture will have a bigger lemon taste?

3 cups of lemon concentrate to 4 cups of water

OR

4 cups of lemon concentrate to 5 cups of water

This type of problem is similar to Noelting's (1980a, 1980b) orange juice problems. Students can then use unitizing and equivalence strategies of finding unit rates and can still develop all of the same reasoning strategies of comparing using a) benchmark fractions, b) common numerators, c) common denominators, and d) missing pieces.

The operations will still have to be taught in a fraction context. Within this study, students had no problem moving from ratios back into fractions when going from the equivalence to the comparing activity. Thus, students may not have any issues going from learning concepts in terms of ratios and then working with the operations in terms of fractions. Better tasks do need to be created for learning the operations so that students are able to fully discuss estimation strategies as well as establish why the algorithms work. Within this study, algorithms were not discussed in multiplication and division because the problem situations presented to the class were not conducive to students developing the algorithms.

Implications for Future Research

This study included analyzing both the social classroom environment and the ways in which individual students participated in that environment and how their learning was affected as a result. The hypothetical learning trajectory and classroom activities were designed so that students would progressively develop concepts which would ultimately lead to their understanding of the operations.

A proposed HLT was presented for use in future research studies focusing on the ways in which preservice elementary teachers develop an understanding of rational number concepts and operations. The proposed HLT includes more work with ratios, in which the concepts are taught through ratios. As indicated from the results in this study, ratios were deemed "different" from fractions. While working with ratios, the preservice teachers in this study did not revert to using procedures as they had when working with fractions. Though ratios follow all of the same properties as fractions, they may be different enough from fractions where preservice teachers will not revert to a procedure to solve a problem and instead develop concepts which then lead to a conceptual understanding of the procedures. Future research studies will need to address this as the results from this study did not directly indicate that preservice teachers will be able to apply concepts learned in ratio situations to fraction concepts.

In future studies, better designed tasks need to be presented within the four operations. With the questions pertaining more towards underlying concepts, specifically within multiplication and division, students were not able to develop the multiplication and division algorithms. In addition, more focus should pertain to estimation strategies within all four operations. This discussion was never brought to the forefront of the conversations of students' solution strategies.

For future studies, the tasks may also need to be redesigned so that the area, linear, and set models can all be developed. The majority of the practices in this study were all established while the class was working with an area model. When the tasks incorporated the linear and set models, students either turned the situation into a problem where an area model could be used or students relied on area model concepts to solve the problem.

Conclusion

The instructional sequence supported students' learning of several rational number topics. Throughout the instructional sequence the class developed ideas related to topics such as defining fractions, defining the whole, partitioning, finding equivalent fractions, unitizing, comparing and ordering, adding, subtracting, and multiplying. In addition, the instructional sequence supported students' learning of ideas that were not originally intended, such as unitizing in terms of the whole.

The results also provided insight to the knowledge that preservice teachers bring to teacher education programs. Previous research with preservice teachers have also analyzed the knowledge they bring to teacher education programs but have done so only in terms of fraction division (Ball, 1990a, 1990b; Tirosh, 2000). The results indicated that preservice teachers come to teacher education programs with several understandings related to rational number concepts as well.

This study has several implications for the ways in which preservice teachers may be taught rational number concepts and operations. By understanding the knowledge that preservice teachers bring to teacher education as well as the ways in which they develop an understanding of rational number concepts and operations, the results provide insight into the types of experiences preservice teachers need in mathematics content courses.

APPENDIX A: IRB APPROVAL

THE UNIVERSITY OF CENTRAL FLORIDA
INSTITUTIONAL REVIEW BOARD (IRB)

IRB Committee Approval Form

PRINCIPAL INVESTIGATOR(S): Juli Dixon, Ph.D. #06-4028

PROJECT TITLE: Prospective Teachers' Development of Understanding of Mathematical Concepts during a Classroom Teaching Experiment

[X] New project submission
[ ] Resubmission of lapsed project #_____
[ ] Continuing review of lapsed project #
[ ] Continuing review of #
[ ] Study expires
[ ] Initial submission was approved by expedited review
[ ] Initial submission was approved by full board review but continuing review can be expedited
[ ] Suspension of enrollment email sent to PI, entered on spreadsheet, administration notified _____

Chair

[X] Expedited Approval

Dated: 12/14/06
Cite how qualifies for expedited review:
minimal risk and #6, #7

[ ] Exempt

Dated: __________
Cite how qualifies for exempt status:
minimal risk and __________

[X] Expiration Date: 12/13/07

IRB Reviewers:

Signed: Tracy L Dietz
Dr. Tracy Dietz, Chair

Signed: ____________________
Dr. Craig Van Slyke, Vice-Chair

Signed: ____________________
Dr. Sophia Dziegielewski, Vice-Chair

Complete reverse side of expedited or exempt form

[ ] Waiver of documentation of consent approved
[ ] Waiver of consent approved
[ ] Waiver of HIPAA Authorization approved

NOTES FROM IRB CHAIR (IF APPLICABLE): ____________________

APPENDIX B: STUDENT CONSENT FORM

January 8, 2007 (Student Consent)

Dear Student:

I am conducting a study, the purpose of which is to investigate the ways in which prospective elementary school teachers understand number and operation and geometry and measurement concepts. I am asking you to participate in this study because you have been identified as a student in one of the elementary mathematics content courses at UCF. Researchers will observe and videotape class sessions of the Instructional Mathematics for Elementary School (MAE 2801). Selected groups may also be audiotaped during class discussions.

Selected students will be asked to participate in several interviews lasting no longer than 15 minutes each. You will not have to answer any question you do not wish to answer. The interviews will be conducted at your convenience on campus after we have received a copy of this signed consent from you. With your permission, we would like to audiotape and videotape these interviews.

Only the research team will have access to the audio and video tapes, which may be professionally transcribed, removing any identifiers during transcription. The tapes will then be kept in a locked file cabinet. After we have received a copy of this signed consent from you, you will be asked to complete a questionnaire about your mathematics content knowledge at the beginning and the end of the course. Your name will not appear on the questionnaires, but a unique code will be used for identification purposes. Only the researchers will have access to the identification codes which will be destroyed after the end of course questionnaire. Copies of your course assignments may be used as data for this study. Additionally, video tape segments may be used in presentations and/or publications related to this study. Your name will be kept confidential and will not be revealed in the final manuscript(s) or any related presentations.

There are no anticipated risks, compensation or other direct benefits to you as a participant in this study. You are free to withdraw your consent to participate and may discontinue your participation in the study at any time without consequence.

If you have any questions about this research project, please contact Dr. Juli K. Dixon at (407) 823-4140 or jkdixon@mail.ucf.edu. Questions or concerns about research participants' rights may be directed to the IRB Coordinator, Institutional Review Board (IRB), University of Central Florida (UCF), 12201 Research Parkway, Suite 501, Orlando, Florida 32826-3246. The phone number is (407) 823-2901 and the fax number is (407) 823-3299. The hours of operation are 8:00 am until 5:00 pm, Monday through Friday except on University of Central Florida official holidays.

Please sign and return one copy of this letter. A second copy is provided for your records. By signing this letter, you give me permission to videotape you and report your responses anonymously in the final manuscript(s) You also give me permission to use videotape segments as a part of related publications and/or presentations.

Sincerely,

Juli Dixon, Ph.D.

_______________ I have read the procedure described above for this research study.

_______________ I agree to participate in the research.

_______________ I do not agree to participate in this research.

_______________ I confirm that I am 18 years or older.

____ /

Participant Date

APPENDIX C: OPENING DAY

Opening Day

Pete wanted to thank his customers personally for coming to his grand opening, so he went around and boxed up their leftovers. For some tables, Pete noticed that a portion of the pizzas had mushrooms on them.

Below are some of the pizzas Pete boxed up. The shaded portion represents mushroom pizza. Assuming that no one at the table ate mushroom pizza, name a fraction to represent the mushroom pizza. Explain and Justify.

Table 1

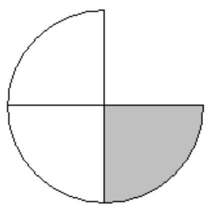


Table 2

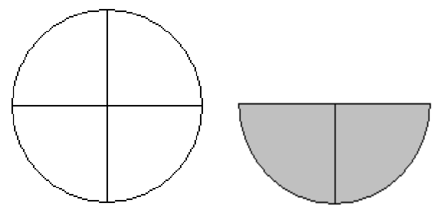


Table 3

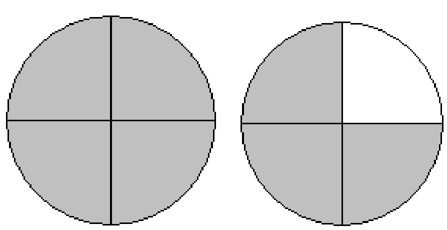
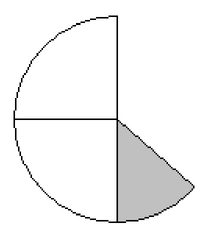


Table 4

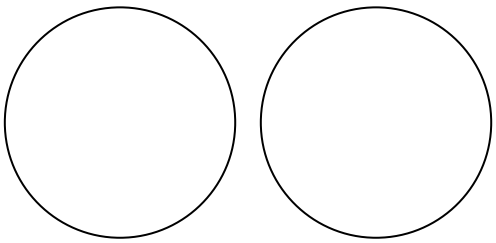


APPENDIX D: SHARING

Pizza Sharing

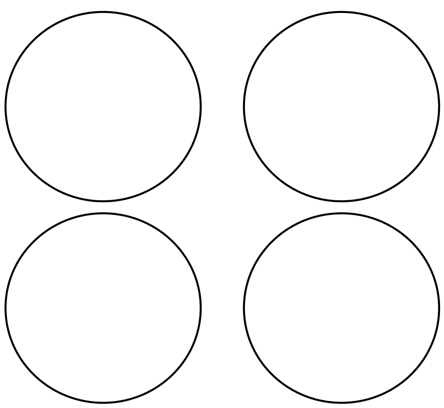
During the first week of business, Pete made some notes on how many pan pizzas and dessert pizzas that groups of customers ordered. Sharing equally, determine what fraction of a pizza each person received.

1. 2 large pizzas among 4 people



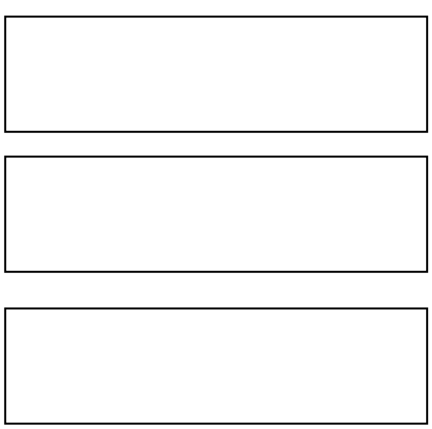
Explain and justify your reasoning.

2. 4 medium pizzas among 5 people



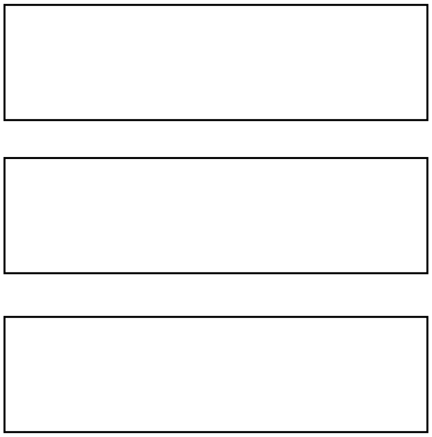
Explain and justify your reasoning.

3. 3 medium dessert pizzas among 4 people



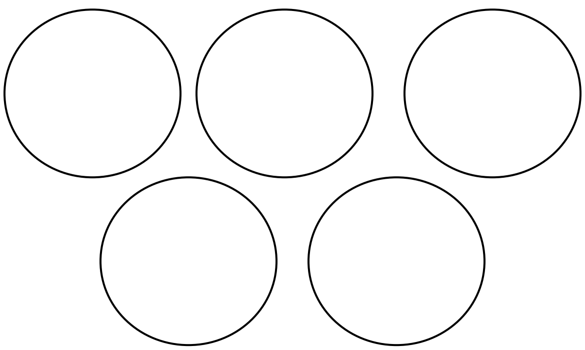
Explain and justify your reasoning.

4. 3 medium dessert pizzas among 7 people



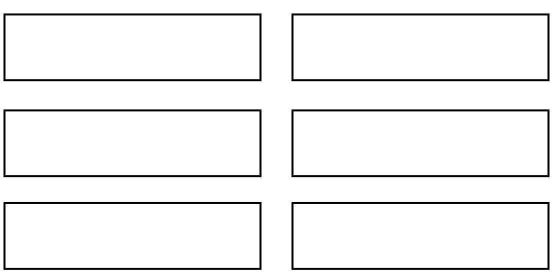
Explain and justify your reasoning.

5. 5 small pizzas among 3 people



Explain and justify your reasoning.

6. 6 small dessert pizzas among 5 people



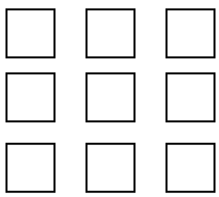
Explain and justify your reasoning.

APPENDIX E: KEEPING TRACK

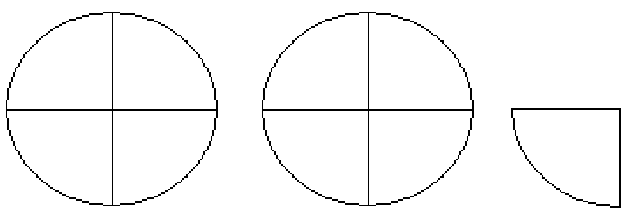
Keeping Track

Use the given picture to solve each of the following problems.

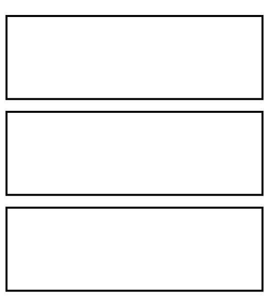
1. Pete was taking inventory so that he could place an order with the local grocery store. Looking at his dwindling pepperoni stock, he saw that he only had 9 bags of sliced pepperoni left, which is $\frac{3}{4}$ of a container of pepperoni. Show how many bags of pepperoni fill $1\frac{5}{6}$ containers.



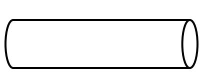
2. The following is $\frac{3}{4}$ of the leftover sausage pizza. Show $\frac{7}{12}$ of the leftovers.



3. The following is $1\frac{1}{2}$ dessert pizzas. Show $\frac{5}{6}$ of a dessert pizza.



4. The following is $\frac{2}{7}$ of a pound of dough. Show $1\frac{3}{14}$ pounds of dough.



APPENDIX F: EQUIVALENCE ACTIVITIES

Family Reunion

1. During a reunion, a family ordered 24 pizzas for 32 people. There was not enough room at one table for the family so they split up into 2 tables. How could the workers split up the family and pizzas so that everyone receives a fair share of pizza? Explain and Justify.

For each of the following, decide how many people should sit at a table and/or how many pizzas should go on a table so that everyone receives a fair share of pizza. Explain and justify.

2. How could the workers split the pizzas if there were only two tables and one table was a table for 4?

3. How could the workers split the family into three tables if one table had 6 pizzas placed on it?

4. Four tables: One table has 3 pizzas placed on it. One table seats 16 people. What could all four tables look like?

Customers and Community

1. Pete noticed that for every 12 customers he has, 4 are under the age of 11. At this same rate, how many customers would Pete see if he sees 20 children? How many children will he have if sees 36 customers? Explain and justify.

2. Pete noticed that for every 30 customers, 6 were under the age of 11. At this same rate, how many children would Pete see if he has 40 customers? How many customers can Pete expect if he sees 14 children? Explain and justify.

3. Pete had a fundraiser and donated $2 out of every $10 in sales. How much will Pete donate if he has sales of $20, $100, or $35? How much in sales did Pete have if he donated $63? Explain and justify.

APPENDIX G: COMPARING ACTIVITIES

Birthday Party

For each problem, use reasoning strategies (not an algorithm) to determine the answer. Explain and justify your solutions.

1. A birthday party took up two tables. One table had 9 pizzas for 18 people. The other table had 2 pizzas for 4 people. Each table shared the pizzas equally. If you were invited to this party and came hungry, which table would you want to sit at?

2. The 22 people at the party sat in the party section of the restaurant, which holds 42 people. At the same time, 16 people were in the non-party section, which holds 36 people. Which section was more crowded?

3. After everyone was done eating at the party, 1/6 of a large mushroom pizza was left and 1/8 of a large sausage pizza was left. Which pizza had more leftover?

4. At the party, the trapezoid table was decorated with 5/6 of a spool of a ribbon. The rectangle table was decorated with 9/10 of a spool of ribbon. On which table was more ribbon used?

Comparing Fractions

For each pair of fractions, use reasoning strategies (not an algorithm), to determine which fraction is greater.

Compare	Solution and Strategy
1. $\frac{4}{5}$ and $\frac{4}{9}$	
2. $\frac{1}{3}$ and $\frac{3}{5}$	
3. $\frac{4}{7}$ and $\frac{3}{8}$	
4. $\frac{7}{8}$ and $\frac{5}{4}$	
5. $\frac{3}{8}$ and $\frac{5}{8}$	
6. $\frac{3}{7}$ and $\frac{5}{8}$	
7. $\frac{9}{11}$ and $\frac{13}{15}$	
8. $\frac{5}{8}$ and $\frac{4}{6}$	

APPENDIX H: PIZZA EATING CONTEST

Pizza Eating Contest

For each problem, explain and justify your solution.

1. Pete held a pizza-eating contest. The following table shows how much of a large pizza each contestant ate. Rank the five contestants in order from first to fifth place.

Colin	7/8 of his pizza
Amanda	7/13 of her pizza
Brandon	9/20 of his pizza
Stephanie	23/24 of her pizza
Jessica	3/20 of her pizza

2. Put the following fractions in order from least to greatest.

$\frac{96}{95}$ $\frac{43}{46}$ $\frac{10}{71}$ $\frac{96}{99}$ $\frac{43}{82}$ $\frac{15}{71}$

APPENDIX I: RECIPES

Recipes

For each problem, explain and justify your solution.

1. For the dessert pizza, if 2/5 of a cup of sugar or less is used in the dough recipe, the dough isn't sweet enough. If at least 4/5 of a cup of sugar is used, the dough is too sweet. Find one amount of sugar that can be used in the recipe?

2. If 10 1/2 cups of flour or less are used in the dough recipe, the dough won't stick together. If 10 2/3 cups of flour or more are used, the dough will be too dry. Find one amount of flour that can be used so that the dough is the right consistency?

3. Name 2 non-equivalent fractions between 3/8 and 5/8.

4. Name 3 non-equivalent fractions between 1/6 and 1/3.

5. Name 3 non-equivalent fractions between 8/9 and 1. Choose fractions that have the same numerator.

APPENDIX J: ADDITION AND SUBTRACTION ACTIVITIES

Pizza Parlor Situations 1

For each of the following problems, draw a picture to solve each problem, write a number sentence, and explain and justify your reasoning.

1. Martha came into the pizza parlor and ate 3/4 of a medium cheese pizza. Then she ate 5/8 of a medium pepperoni pizza. How much pizza did Martha eat altogether?

2. The pizza dough machine broke after it had made 2 1/2 pounds of dough. The cook used 2/3 of a pound. How many pounds of dough were left?

3. A birthday party brought in a balloon bouquet. In the bouquet, 1/4 of the balloons are blue and 1/6 of the balloons are yellow. How much of the bouquet are blue and yellow balloons?

4. In a container of after-dinner mints, 1/4 of the mints are pink and 2/3 of the mints are green. How much more of the container of mints are green than pink?

Addition

Write a word problem for each. Then, estimate each answer. Check how close your estimate was by solving each problem with a model. Explain and justify your reasoning.

1. 1/4 + 5/6

2. 1/3 + 3/8

3. 3/10 + 1 2/5

4. 5/8 + 5/6

5. 1 1/2 + 2/5

<u>Subtraction</u>

Write a word problem for each. Then, estimate each answer. Check how close your estimate was by solving each problem with a model. Explain and justify your reasoning.

1. 5/8 – 1/2

2. 5/6 – 2/9

3. 3/4 –2/3

4. 2 1/3 –1/2

5. 1 1/3 –5/6

APPENDIX K: MULTIPLICATION ACTIVITIES

Pizza Parlor Situations 2

For each of the following problems, write a number sentence, draw a picture to solve each, and explain and justify your reasoning.

1. A cook made four pizzas that had 3/5 of a package of mushrooms on each. How much of a package of mushrooms were used?

2. Sue ate some pizza. 2/3 of a pizza is left over. Jim ate 3/4 of the left over pizza. How much of a whole pizza did Jim eat?

3. A party dessert pizza measures 2/3 of a yard by 3/4 of a yard. How much of a square yard is the party dessert pizza?

4. There was 4/5 of a pound of pizza dough in the freezer from the previous day. The cook thawed out 3/8 of that dough. How much of a pound of dough did the cook thaw?

Multiplication

Write a word problem for each. Then, estimate each answer. Check how close your estimate was by solving each problem with a model. Explain and justify your reasoning.

1. 3/5 X 2

2. 1 1/3 X 3/4

3. 1 1/5 X 1 2/3

4. 3/5 X 5/6

5. 5/6 X 3/8

APPENDIX L: DIVISION ACTIVITIES

Pizza Parlor Situations 3

For each of the following problems, write a number sentence, draw a picture to solve, and explain and justify your reasoning.

1. 1 7/8 pounds of pizza dough were made. How many 1/4 pound servings can be made? What part of another serving will be left?

2. Four friends shared 3/4 of a strawberry dessert pizza. Sharing equally, how much of a dessert pizza will each person get?

3. There was 5/6 of a gallon of lemonade leftover from a party. Pitchers that hold 2/3 of a gallon were filled. Using all of the lemonade, how many pitchers can be filled?

Division

Write a word problem for each. Then, estimate each answer. Check how close your estimate was by solving each problem with a model. Explain and justify your reasoning.

1. 3 ÷ 5/8

2. 3/4 ÷ 1/8

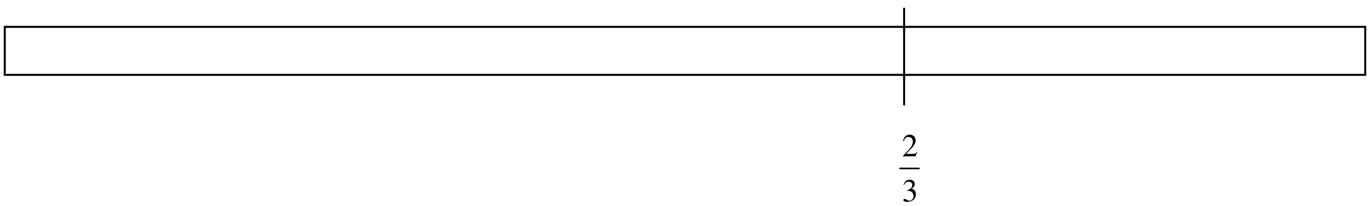
3. 1 1/2 ÷ 2/3

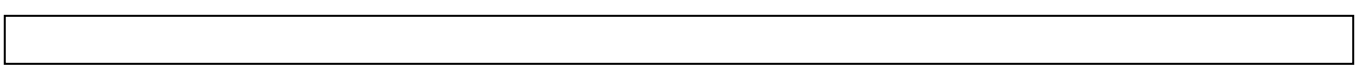
4. 2/3 ÷ 4

APPENDIX M: NUMBER LINE ACTIVITY

Pizza Dough Machine

Pete lined up his sheet of paper with the dough the machine produced and drew the following. Help Pete figure out how to fold the cutting guides so that the dough can be cut the whole way across for each type of pizza that Pete offers.





APPENDIX N: LANGUAGE ACTIVITY

Fraction Language

Draw two pizzas (circles) on the board. Underneath the circles write:

1/2 of a pizza
1/2 of the pizza
1/2 of each pizza
1/2 of one pizza

Have a whole class discussion on what the picture will look like in each situation.

To extend the question, cut each pizza into fourths and shade in 1/4 of each pizza. Ask if the shaded portion represents:

1/4 of the pizza
1/4 of one pizza
1/4 of a pizza
1/4 of each pizza

APPENDIX O: SAMPLE ARGUMENTATION LOG

Claim	Data	Question	Warrant	Backing
Picture	We start off with this little rolly thing that is 2/7. I thought of doing it this way of 1/7 and 1/7 to show that it's 2/7. And then I kept drawing another one until I got to 1, because I broke it up so that I'd go get the one first and then I would get the 3/14. First I wanted to get the 1. I knew I needed seven of these half pieces of the roll in order to get 1. Then it would be 7 pieces of the 7 parts that the whole is broken up into. I kept going and I realized that when I got to that point, I needed half of this one and then this right here would give me a whole. 1,2,3,4,5,6,7, that's 7, so this is one because it's also 7 over 7 because it makes one whole, and I kept going. I needed now 3/14 and 7 is half of 14 so I knew that I needed to split this one in half to make it 1/14 and then 1/14 over here. . . . I have two 1/14 and I needed 3, so I needed	How'd you know how long to make the other one?	I knew that I knew that was is if I would think of it as whole, if I split it and I knew that 7 was half of 14 and then if I split the 1/7 into 2 then that would give me 2 and another seven into two, that would give me two, so eventually it would equal, the whole would be into 14 parts if I broke each seven up into 2. I put it right below it.	

	one other piece over here that was 1/14 to give me the 3/14. That's my 1 all of this right here. And this right here is 3/14 so altogether it's 1 3/14. I came out with the same answer to same way, but I did one long roll to make the 7,7,7. . . I cut it into 14 pieces and took 3.		So it's the same length.	
	They gave us 2/7, so then I just drew another one and labeled that 2/7 and drew another one and labeled that 2/7. Then I drew half of one because half would be 1/7. That way I'd have 2,4,6,7 sevenths. Then I knew that 7 was half of 14. I knew that that would be two 1/14 in the 1/7. Then I knew that I would need one more half of 1/7 so then I drew a bigger cylinder and labeled it 3/14.			

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