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Developing Mathematical Practices In A Social Context:an Instructional Sequence To Support Prospective Elementary Teachers'

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DEVELOPING MATHEMATICAL PRACTICES IN A SOCIAL CONTEXT: AN INSTRUCTIONAL SEQUENCE TO SUPPORT PROSPECTIVE ELEMENTARY TEACHERS' LEARNING OF FRACTIONS

by

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ABSTRACT

This teaching experiment used design-based research (DBR) to document the norms and practices that were established with respect to fractions in a mathematics content course for prospective elementary teachers. The teaching experiment resulted in an instructional theory for teaching fractions to prospective elementary teachers. The focus was on the social perspective, using an emergent framework which coordinates social and individual perspectives of development. Social norms, sociomathematical norms, and classroom mathematical practices were considered.

A hypothetical learning trajectory (HLT) including learning goals, instructional tasks, tools and imagery, and possible discourse, was conjectured and implemented in the mathematics class. Video tapes of the class sessions were analyzed for established norms and practices. Resulting social norms were that students would: (a) explain and justify solutions, (b) listen to and try to make sense of other students' thinking, and (c) ask questions or ask for clarification when something is not understood. Three sociomathematical norms were established. These were expectations that students would: (a) know what makes an explanation acceptable, (b) know what counts as a different solution, and (c) use meaningful solution strategies instead of known algorithms.

Two classroom mathematical practices with respect to fractions were established. The first was partitioning and unitizing fractional amounts. This included (a) modeling fractions with equal parts, (b) defining the whole, (c) using the relationship of the number of pieces and the size of the pieces, and (d) describing the remainder in a division problem. The second practice was quantifying fractions and using relationships among

these quantities. This included: (a) naming and modeling fractions, (b) modeling equivalent values, and (c) using relationships to describe fractions.

Finally, recommendations for revising the HLT for a future teaching experiment were made. This will contribute toward the continuing development of an instructional theory for teaching fraction concepts and operations to prospective elementary teachers.

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LIST OF ACRONYMS/ABBREVIATIONS

DBR	Design-Based Research
HLT	Hypothetical Learning Trajectory
NAEP	National Assessment of Educational Progress
NCTM	National Council of Teachers of Mathematics
PCK	Pedagogical Content Knowledge
PSSM	Principles and Standards for School Mathematics
PUFM	Profound Understanding of Fundamental Mathematics
RME	Realistic Mathematics Education
RNP	Rational Number Project
SMK	Subject Matter Knowledge

CHAPTER ONE: INTRODUCTION

The purpose of this study was to describe how a specific instructional sequence supported the collective learning of fraction concepts and operations with a group of prospective elementary teachers enrolled in an elementary mathematics content course. The teaching experiment resulted in an instructional theory for teaching fractions to prospective elementary teachers. The idea for the study originated in conversations and class discussions about the mathematical content knowledge of teachers. From those discussions, a research team was formed and began to plan the investigation. This introduction sets the stage for teaching in reform-based classrooms, presents a case for improving teacher preparation, and explains why the topic of fractions was the content chosen to be studied. A statement of the problem and a discussion highlighting the significance of the study are also provided.

Mathematics Education Reform

It is important to consider the vision of mathematics education reform for this study because the instructor of the class implemented reform-based methods as a model for students who would become elementary school teachers. A brief introduction to mathematics education reform is appropriate here and serves to frame the context of the class in which the study was conducted. In 1989 the National Council of Teachers of Mathematics (NCTM) released their first Standards document (NCTM, 1989) describing a vision of mathematics education. In 2000, NCTM released the *Principles and Standards for School Mathematics* (PSSM), updating the vision from the original Standards. This vision is summarized in the following excerpt from the PSSM document:

> Students confidently engage in complex mathematical tasks chosen carefully by teachers. They draw on knowledge from a wide variety of mathematical topics, sometimes approaching the same problem from different mathematical perspectives or representing the mathematics in different ways until they find methods that enable them to make progress. Teachers help students make, refine, and explore conjectures on the basis of evidence and use a variety of reasoning and proof techniques to confirm or disprove those conjectures. Students are flexible and resourceful problem solvers. Alone or in groups with access to technology, they work productively and reflectively, with the skilled guidance of their teachers. Orally and in writing, students communicate their ideas and results effectively. They value mathematics and engage actively in learning it (p. 3).

The Standards are based on a socioconstructivist view of mathematics in which mathematical knowledge is thought to be the product of a community, and teachers facilitate the development of mathematical thought. Thus, the traditional approach to instruction where teachers deliver concepts, facts, and skills is not viable (Nelson, 1997). Though the ideas of reform-based mathematics are complex, this description offers a broad image of what should take place in a reform-based classroom. Much has been written about reform-based methods in mathematics classrooms, and detailing all of the literature here would not greatly contribute to the purpose of this particular research report. Instead, reform-based instruction will be painted with broad strokes to give the reader a general idea of the classroom climate.

In a classroom implementing a vision aligned with reform, the role of the teacher is that of a facilitator rather than one who instructs by telling. The role of the student also differs from that in a traditional instructional setting. In a reform-based classroom, students explore and make conjectures that they verify or disprove. This is in contrast to a traditional classroom setting where students may simply receive information passed on to them by the instructor. Students are expected to interact with one another in addition to having a dialogue with the teacher. Mathematical activities are conducted in an environment where the teacher considers presentation, mental activity, pupil reflection, and socialization of the learning. Teachers should consider how students process ideas and instruction and capitalize on students' prior knowledge and background understandings (Brooks & Brooks, 1993; Lederman & Niess, 1996; Smith, 1999; Stiff, 2001).

Activities such as hypothesizing, trying things out, executing mathematical procedures, communicating and defending conclusions, and reflecting on the methods selected and the results are part of engaging in mathematics from a constructivist perspective (Davis, Maher, & Noddings, 1990). There is an emphasis on creating a mathematical community in constructivist classrooms. This means that there is significant interaction in the classroom. Students talk to one another in addition to talking exclusively to the teacher.

Ball (1991a) describes two contrasting approaches to teaching which can further operationalize the difference between reform-based instruction and traditional instruction. One teacher, who presumably teaches to the vision of the NCTM Standards, helps students develop the mathematical skills and understanding they need to judge their own ideas and the ideas of fellow students. This is achieved, in part, by asking students to explain their thinking and establishing a classroom discourse in which validation becomes the responsibility of the learner rather than the teacher. The students become participants in the mathematics. Ball describes a contrasting approach by a teacher who has a more procedural method. That is, she shows students how to perform procedures, then assigns and monitors practice. If students have difficulty, she will use remediation techniques to help them. Her goal is to have students become independent in performing procedures to arrive at correct answers, which she validates. Little attention is given to discourse and reasoning. Note the differences between the two classrooms. In the former, students are participants in the mathematics. In the latter, the teacher is the authority that validates the mathematical thinking and correct answers. The former invites students to engage in discourse and the latter is teacher directed.

Goldin (1990) describes the difference more succinctly when he discusses rote learning and meaningful learning. He writes that in rote learning there is a heavy reliance on stating rules and giving examples in a procedural manner. Meaningful learning highlights student thinking and investigation to discover patterns and test conjectures. His ideas provide a simple way to remember the basic difference between the two instructional approaches.

With this brief introduction to mathematics education reform ideas, one should realize the importance of teaching prospective elementary teachers with methods that will deepen their understanding of mathematics and provide a model of good instruction for them. In fact, Sowder, Armstrong, et al. (1998) warn that teachers must have appropriate preparation if curricular changes are to succeed. Thus, the next section addresses preparing teachers to teach mathematics in ways that are aligned with reform ideas.

Teacher Preparation

Teachers need to understand the mathematics they teach in order to provide instruction that matches the vision described by the NCTM Standards (Carpenter & Lehrer, 1999). This is supported by Ball (1990a) who states that teachers need to understand mathematics in order to respond to student questions and interpret and assess students' ideas and thinking. Additionally, an understanding of mathematics will allow teachers to be able to use a variety of representations in their teaching. Cooney (1994) asserts that elementary teachers' knowledge of mathematics is weak and inhibits their ability to use reform methods in their teaching. He advocates helping teachers develop an understanding for how students learn and think about mathematics so they will be able to analyze their students' thinking.

The importance of teachers gaining deep conceptual understandings of the mathematics they teach has also been substantiated by research studies. Sowder, Philipp, Armstrong, and Schappelle (1998) found that teachers may underestimate the difficulty of the elementary mathematics curriculum. However, once they begin to learn mathematics conceptually, they seem to gain a new appreciation for the elementary curriculum. Also, teachers are not likely to be able to teach conceptually if they lack conceptual understanding of the content (Stoddart, Connell, Stofflett, & Peck, 1993; Sowder, Philipp, et al.). Teaching conceptually is aligned with the vision of the PSSM. Thus, the goal to build deeper conceptual understandings will help teachers teach in ways envisioned in the PSSM.

An idea that guided instruction in this teaching experiment was that if prospective teachers are expected to teach in ways that align with mathematics education reform, they need to experience instruction in such ways. Cooney (1994) has cited research findings that support this assumption. He suggests that what teachers take to their classrooms is a function of their learning experiences, not education. That is, if teachers are taught with traditional methods, they will more likely teach in ways that reflect those methods as opposed to what they may have learned in courses about reform methods. Thus, giving prospective teachers an experiential base for teaching mathematics with understanding is important. Their conception of teaching mathematics is likely formed from their experiences accumulated as students (Stoddart et al., 1993). If teachers are expected to create mathematical communities, then they must participate in such communities themselves (Cooney). Finally, Even and Lappan (1994) stress the importance of teachers knowing mathematics. They extend that idea and suggest that teachers also need to have learning experiences that build deep understandings of what mathematics is. In addition to understanding what mathematics is, teachers must know what it means to do mathematics.

The fact that students consistently perform poorly with respect to fractions is commonplace in the literature (Kloosterman, 2004; Lacampagne, Post, Harel, & Behr, 1988; Steffe & Olive, 1991). It has been suggested that this is because teachers do not understand fundamental concepts of fractions and teach in a very rote and shallow manner (Lester, 1984; Tirosh, Fischbein, Graeber, & Wilson, 1998). Ball (1990a) found prospective teachers' understanding of mathematics to be compartmentalized and dependent on rules. The literature is replete with studies of practicing and prospective teachers who are deficient in the mathematical understandings necessary to teach fractions (Ball, 1990a; Borko et al., 1992; Cramer & Lesh, 1988; Lacampagne et al., 1988; Lehrer & Franke, 1992; Ma, 1999; Simon, 1993; Tirosh, 2000; Tirosh & Graeber, 1989). Further, teachers may not realize that they lack sufficient understanding to teach fractions in a meaningful way (Sowder, Philipp, et al., 1998). These ideas motivated the researchers in this teaching experiment to investigate how prospective elementary school teachers develop understandings of fraction concepts and operations.

The Case for Fractions

The topic of fractions was chosen for this investigation for several reasons. First, students in the United States have a long history of not demonstrating understanding of fractions on standardized tests (Armstrong & Bezuk, 1995; Kloosterman, 2004; Kloosterman et al., 2004; Sowder, Wearne, Martin & Strutchens, 2004). In addition, Post (1989) refers to the ubiquitous nature of rational numbers in mathematics. He claims this makes them one of the most important conceptual domains to be studied. Behr and Post (1992) claim that students may have difficulties in algebra because they lack complete understanding of fractions. Saxe, Gearhart, and Nasir (2001) state that the domain of fractions is an important part of the mathematics curriculum for upper elementary grades. Important as they are, one must acknowledge that fractions are cognitively complicated and difficult to teach (Smith, 2002).

Leinhardt and Smith (1985) suggest that fractions may be difficult to teach because there are different conceptual meanings, two numbers actually represent only one quantity, and there are many names for the same amount. Lamon (1999) also identifies new types of units, a new notational system, new concepts of operations, and interference with whole numbers as reasons students may have difficulty with fractions.

Students may also have difficulty with fractions because of their teachers' inadequate understanding of fractions (Lester, 1984; Saxe et al., 2001). That is, teachers who do not fully understand fractions cannot teach their students so they develop a deep understanding of fractions (Ball, 1990a; Cramer & Lesh, 1988; Lacampagne et al., 1988; Ma, 1999; Tirosh, 2000; Tirosh & Graeber, 1989). These difficulties among students and teachers prompted the research team in this project to develop an instructional sequence that would deepen the understanding of prospective elementary teachers. The instructional theory with respect to fractions that resulted from this teaching experiment may contribute to alleviating the concerns.

Statement of the Problem

Much has been written about what teachers may be lacking in order to teach mathematics, and specifically fractions (Ball, 1990a, 1991a; Borko et al., 1992; Cramer & Lesh, 1988; Lacampagne et al., 1988; Leinhardt & Smith; 1985; Ma, 1999; Post, Harel, Behr, & Lesh, 1998; Simon, 1993; Tirosh 2000; Tirosh & Graeber, 1989). However, Ball (1990a) notes that the understandings of prospective teachers are rarely explored. She continues to say that there is a need to find out what prospective teachers know and how they learn it. This is the focus of this research.

The primary goal of this study was to analyze how classroom experiences supported learning in order to develop an instructional theory for teaching fraction concepts and operations to prospective elementary teachers. The goal to have the teacher candidates build deeper conceptual understanding with respect to fractions was fundamental in the teaching. The question that guided the work of this research was: How do instructional experiences in an elementary school mathematics content course for prospective elementary teachers support learning fraction concepts and operations from a social perspective?

In order to answer this question, a teaching experiment was conducted. The research team developed and implemented a hypothetical learning trajectory (HLT) in a classroom of prospective elementary teachers enrolled in a mathematics content course. Following the ideas of Cobb, Stephan, McClain, and Gravemeijer (2001), the developers of the conjectured learning trajectory envisioned mathematical practices and how they may develop. This HLT, which will be described in detail in Chapter Three, included learning goals related to the following big ideas: (a) using fractions to name amounts; (b) understanding differences between whole number relationships and fraction relationships, (c) replacing rote procedures with reasoning to build meaning; (d) reasoning with addition and subtraction; and (e) reasoning with multiplication and division. The teaching episodes were analyzed to determine the actual learning trajectory of the classroom community. The observed practices that were established contributed to an instructional theory by providing feedback for the revision of the enacted HLT.

The instructor for the elementary mathematics content course had recently changed her approach to teaching the course, and this was her third semester of implementing HLT-based instruction with elementary education majors. She was motivated by her beliefs that prospective teachers need to develop a deeper understanding of mathematics and they should be taught in ways they should be expected to teach. The research team attempted to create a mathematical community where process and reasoning took priority over procedures and correct answers. Specific effort was made to establish classroom norms that would facilitate learning in such an environment. The tasks and activities selected by the research team were deliberately chosen to further the mathematical development of the students in the class and evoke the desired discourse.

This interaction among the students and between the teacher and students provided much of the data for the research.

Significance of the Study

Sowder, Philipp, et al. (1998) observed that as teachers' conceptual knowledge increased, so did their tendency to teach more conceptually. This provides justification for trying to increase conceptual knowledge among the prospective teachers in this study. Further justification is found in the studies that have demonstrated teachers' lack of adequate knowledge for teaching. For example, some researchers have documented that teachers may not be able to explain procedures even though they can calculate a correct answer (Borko et al., 1992; Tirosh et al., 1998). Teachers need to know how students may reason about a particular task and how to respond to their reasoning (Ball, 1990a, 1991a; 1991b; Ball & Bass, 2000); yet some researchers have shown that teachers may not be able to understand the level of students' understanding. They also may not know how to analyze student errors (Lehrer & Franke, 1992; Sowder, Philipp, et al., 1998; Tirosh, 2000). Additionally, teachers have difficulty generating verbal representations of number sentences involving division with fractions (Ball 1990a, 1990b; Ma, 1999; Simon, 1993). Another finding of research with teachers is that they tend to have the same misconceptions as children have with respect to some fraction concepts (Lacampagne et al., 1988; Post et al., 1988; Tirosh et al., 1998). For all of these reasons, Borko et al. (1992) recommend including more courses on conceptual development for mathematics topics in teacher education programs. This study sought to investigate how to increase prospective elementary teachers' conceptual understanding of fractions.

The design of this study falls into the teaching experiment category of design-based research (DBR). Teaching experiments can offer important contributions to the field. This type of experiment allows researchers to gain firsthand experience with students' mathematical learning and reasoning (Steffe & Thompson, 2000). With this firsthand experience, researchers can generate a body of knowledge about teaching and learning. This knowledge may then be adapted by a teacher to his or her own class (Clements & Sarama, 2004). Gravemeijer, Bowers, and Stephan (2003) express this idea another way by writing that DBR gives teachers a "global learning route to be tailored to the specific situations by classroom teachers" (p. 56). This means that results of the experiment can be used and adapted by any teacher who wishes to engage in similar teaching.

Several characteristics DBR resonated with the purposes and goals of this study. Originating in the Netherlands, where design and research have long been integrated, this model of curriculum development is guided by theory, yet produces a theory. Gravemeijer (1994) describes this approach to curriculum development as being embedded in a framework of "educational development," and credits Freudenthal with creating this model. With this educational development approach, the end goal is to change educational practice—not merely to develop curriculum. Design experiments are highly interventionist and test innovations in instruction (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003). This intent to change practice is one aspect of this methodology that attracted the researchers in this study to DBR.

Another aspect of the methodology that was attractive to the researchers was its foundation in Freudenthal's Realistic Mathematics Education (RME). In RME students should engage in and follow a similar process to that of when the mathematics they are studying was invented. Students are presented with situations which they mathematize. That is, they organize the situation mathematically so it makes sense to them. Then they analyze their mathematical activity (Gravemeijer, 1994, 2004). Mathematizing should not be confused with having concrete experiences. To illustrate mathematizing as a process, consider a situation in which students are asked to find out how many students are in their grade at their school. If they have already developed a sense of number, they may realize they should combine the numbers of students in all the classrooms of their same grade. They may mathematize the process by using cubes or counters to represent the students and combine all of them. They may also realize that making groups of ten makes the counting easier. What is important here is that the students decide on the process. This is in contrast to a classroom where a teacher may provide base-ten blocks to the students and tell them how to model the numbers in order to add them. In the latter example, the mathematics is simply concretized. That is, the teacher is imposing a concrete model for the mathematics that was created by someone other than the students. In RME, instructional activities make mathematizing the main learning principle. This enables students to reinvent mathematics. This idea supports the goals for the instruction that took place in the research classroom in this study.

Simon (2000) writes that we do not understand the developmental process for teacher knowledge because we do not observe teachers as they develop this expertise. He calls for identifying the key aspects of teacher knowledge and creating useful frameworks to describe how that knowledge is developed. In this research, prospective elementary teachers were observed as they built an understanding of fraction concepts and operations in order to create a framework of the type called for by Simon. This framework is presented as a revised HLT in Chapter Five. So, this study contributes to the body of knowledge about how teachers develop their expertise with respect to fractions. In addition, the results of this teaching experiment will be examined and the revised HLT may be implemented in a future teaching experiment. This cycle will continue, and with each iteration, improvements can be made to the resulting instructional theory.

For this teaching experiment, the resulting instructional theory documents one attempt to develop the critical deep understandings of fractions that teachers need in order to teach fractions to their students in meaningful ways. It should be considered as the beginning of a continuing investigation focused on developing deep understanding of fractions with prospective elementary teachers.

Conclusion

Design-based research methodology with the purpose of developing an instructional theory for teaching fractions to prospective elementary teachers was used in this research. The topic of fractions was the focus of this study because of its complexity and importance in the elementary mathematics curriculum (Lamon, 1999; Saxe et al., 2001; Smith, 2002).

Chapter Two summarizes the literature related to this teaching experiment. First, the mathematical foundations for fractions are discussed. This information about interpretations for fractions and unifying elements is relevant to the study because it informed the development of the HLT. This is followed by a summary of student-focused research including student achievement with respect to fractions, and instructional strategies. The final section documents results of teacher-focused research, including types of knowledge, the effect of teacher knowledge on teaching practices, and teachers' knowledge of fractions.

Chapter Three describes the DBR methodology used in this study. The specific procedures used in this study are detailed. This includes a description of the setting, an overview of DBR, details of planning the HLT, and the implemented HLT. The specific processes used for data collection and analysis are also discussed. A discussion of the limitations and assumptions associated with this teaching experiment closes the methodology section.

Chapter Four discusses the results of the teaching experiment. This includes the social and sociomathematical norms that were negotiated. Two classroom mathematical practices were established in the teaching experiment as well. These norms and practices are discussed in Chapter Four.

Chapter Five provides an overview of the teaching experiment results. The norms and mathematical practices are reviewed. A revised HLT is proposed for a future teaching experiment. Finally, implications of the study and suggestions for future research close the chapter.

CHAPTER TWO: REVIEW OF LITERATURE

Two themes related to fractions emerged from the literature. These themes reflect the mathematical foundations for fractions and the knowledge of fractions that teachers and students have. Each of these themes will be discussed in sections that follow, beginning with mathematical foundations. Fractions are founded deeply in mathematical theory and the rational number set. Since this includes several concepts well beyond the scope of elementary school mathematics, it is not important to review all the mathematical theory related to fractions. However, it is important to have knowledge of different interpretations of fractions and the unifying elements related to learning fractions in elementary school. This information informed the hypothetical learning trajectory (HLT) developed for this research and has implications for curriculum and instruction in general.

Student-focused research related to fractions is discussed after the mathematical foundations section. These research findings substantiate the need to improve student achievement. Following that, teacher-focused research related to fractions is reported. This builds the case for improving student achievement through better instruction provided by more knowledgeable teachers.

Mathematical Foundations of Fractions

Fractions are part of what is known as a multiplicative conceptual field which consists of "all situations that can be analyzed as simple and multiple proportion problems and for which one usually needs to multiply or divide" (Vergnaud, 1988, p. 141). Vergnaud includes concepts such as linear and *n*-linear functions, vector spaces, dimensional analysis, fraction, ratio, rate, rational number, and multiplication and division in the multiplicative conceptual field. Clearly, this perspective on the multiplicative conceptual field includes more advanced mathematics than was considered in this study which was directed toward fractions in elementary school mathematics. For prospective elementary teachers, it is sufficient to recognize that fractions are a subset of a larger mathematical structure. Ohlsson (1988) suggests this by noting that in order to understand the meaning of "fraction" attention must be given to the mathematical theory and real world applications of fractions. This discussion provides a broad look at the theory in order to situate fractions within their mathematical structure. First, various interpretations of fractions relevant to elementary school mathematics are discussed. Following that, several unifying elements, or big ideas, are discussed.

Interpretations

Several researchers have proposed varying interpretations for rational numbers. Table 1 provides a summary of four different sets of interpretations that have been proposed. This development clarifies how the particular interpretations used in this study were determined. A brief description of each follows the table.

Table 1: Summary of Interpretations for Fractions

Kieren (1976)	Kieren (1980)	Behr, Lesh, Post & Silver (1983)	Ohlsson (1988)
• Fractions • Decimal fractions • Equivalence classes of fractions • Ratio numbers • Multiplicative operators • Elements of an infinite ordered quotient field • Measures or points on a number line	• Part-whole relationships • Ratios • Measures • Quotients • Operators	• Fractional measure • Ratio • Rate • Quotient • Linear coordinate • Decimal • Operator	• Comparison • Partitioning • Composite operations • More specific types of comparison

Kieren (1976) argued that an individual should have experience with multiple interpretations of fractions in order to understand them. Initially, Kieren identified the following interpretations for rational numbers: (a) fractions, (b) decimal fractions, (c) equivalence classes of fractions, (d) ratio numbers, (e) multiplicative operators, (f) elements of an infinite ordered quotient field, and (g) measures or points on a number line. Later, he referred to the following five constructs: (a) part-whole relationships (b) ratios, (c) quotients, (d) measures, and (e) operators in his model for personal rational number knowledge (Kieren, 1980). Kieren's work was the basis for subsequent rational number research (Sowder, Armstrong, et al., 1998; Sowder, Philipp, et al., 1998). For example, Behr, Lesh, Post, and Silver (1983) redefined Kieren's interpretations of rational numbers, which they called subconstructs. Their list included the following subconstructs: (a) fractional measure, (b) ratio, (c) rate, (d) quotient, (e) linear coordinate, (f) decimal, and (g) operator. These subconstructs provided part of the theoretical foundations for their Rational Number Project (RNP), which may be the most comprehensive study of instruction and learning fractions in classrooms. The RNP research addressed the teaching and learning of multiplicative structures including the role of manipulative materials and the knowledge of middle grades teachers (Leavitt, 2002). More details from the RNP are included later in this chapter.

Ohlsson (1988) concluded that there are four basic interpretations for fractions: (a) comparison, (b) partitioning, (c) composite operations, and (d) more specific types of comparison. The first of Ohlsson's interpretations, comparison, allows for the quantities to be compared in relation to each other. An example might be two dogs for every cat. In this case, both numerator and denominator are interpreted as quantities. In the second case, the numerator is interpreted as a quantity and the denominator is considered as a parameter. This interpretation corresponds to Ohlsson's idea of partitioning. Interpreting both numerator and denominator as parameters results in the idea of composite operations. A statement such as, "the hamburger shrank to half of its size when it was cooked" would be an example of this situation. Finally, further constraining these interpretations results in specific types of comparisons such as proportions, ratios, division, and rates.

The purpose of including the preceding information is to make the point that mathematicians do not agree on one way to organize rational numbers. Although there does not seem to be a single agreed upon set of subconstructs or interpretations of fractions, there are commonalities. Ohlsson (1988) credits Kieren and Behr, Lesh, Post, and Silver for agreeing on some central concepts. These are: (a) quotient, (b) ratio, (c) operator, and (d) a version of the part-whole interpretation. In this study, the part-whole, measure, quotient, and operator interpretations of fractions were considered. These were selected because of their inclusion in elementary school mathematics. A more detailed discussion of these interpretations follows.

Part-Whole Interpretation

The part-whole interpretation refers to a fraction representing one or more parts of a unit that has been divided into a number of equal-sized pieces (Lamon, 1999). That is, a unit is partitioned into equivalent pieces, and the fraction represents the number of pieces being considered. Sometimes a distinction is made between parts of a whole and parts of a set. This distinction is not really necessary because the part-whole interpretation applies to a continuous quantity as well as a set of discrete objects (Sowder, Armstrong et al., 1998; Sowder, Philipp, et al., 1998). Thus, whether the situation involves part of a whole or part of a set, it is still considered to be a part-whole interpretation. However, the distinction becomes important in instruction because it is desirable to match models of a situation to the situation itself. Post, Behr, and Lesh (1982) state that the structure of concrete materials used in instruction should reflect the concept being taught. Thus, if a situation is about a part of a set, discrete objects would be appropriate to model it. If a situation includes part of a whole, such as a cake, an area model would be appropriate to model it.

Behr et al. (1983) suggest that partitioning and the part-whole subconstruct are basic to learning other subconstructs of rational number. This may explain why the part-whole interpretation of fractions has traditionally served to introduce students to instruction on fractions. Lamon (1999) further notes the importance of this interpretation by stating that it provides the language and symbolism for rational numbers in general. In addition, Mack (1993) recognizes that students acquire informal knowledge about fractions before they come to school. She also notes that studies have indicated that students' informal strategies to solve rational number problems are generally founded in the part-whole interpretation of fractions.

Powell and Hunting (2003) acknowledge that part-whole relationships are the foundation for young children's developing multiplicative structures. They advocate spending time developing the foundations for fraction concepts in the early grades. One suggestion they offer is for teachers to introduce fraction language in problem situations. Several contexts to develop the concept of part-whole relationships are available to young children. Gaining a firm understanding of this concept will further support learning and concept building in later grades.

In their discussion of part-whole concepts related to fractions, Steffe and Olive (1991), note that it is part-whole operations that separate students with prefractional concepts from those with part-whole fraction concepts. Once children have achieved an understanding of part-whole fraction concepts they can determine the whole when given part of it. In contrast, children operating with prefractional concepts would not be able to determine fractional parts for wholes partitioned into a number of parts different from the unit. For example, a student at the prefractional level would not be able to determine $\frac{1}{3}$ of a set of 6 objects. This student would have difficulty because his or her mental construction for $\frac{1}{3}$ is 1 out of 3 parts, not some other number of parts out of 6 parts.

While the part-whole interpretation of fractions may be considered to be a critical foundation for fraction concepts, it should not be the only situation students associate with fractions. Kerslake (1986) warns that learning only the part-whole model can result in serious limitations on children's understanding of fractions. Sowder (1992) also noted that elementary textbooks rely heavily on the part-whole interpretation of fractions. The decision in this investigation to explicitly include several interpretations for fractions in addition to the part-whole interpretation is supported by these findings. The next interpretation discussed is the measure interpretation.

Measure Interpretation

Behr et al. (1983) note that the measure interpretation is a reconceptualization of the part-whole interpretation. Like the part-whole interpretation, the measure interpretation considers how much of a quantity there is in relation to a particular unit of the quantity. Ohlsson (1988) also discusses the measure interpretation with respect to having a fixed reference quantity and a fixed partitioning parameter that results in a fixed part. For instance, a given unit such as a foot is divided into smaller units of inches by partitioning. So for measurement, the reference quantity (foot) and partitioning parameter (12 inches) are fixed. The difference between measures and fractions is that the partitioning parameter and unit are arbitrary with fractions. In this case, the numerator refers to the number of parts that make up the resulting part that is of interest. The denominator tells how many parts the reference quantity is divided into. The value of the fraction made up of the numerator and denominator refers to the amount of the fixed part. Ohlsson makes the point that fractions as measures build on the partitioning application that is present in the part-whole interpretation.

Kieren (1980) acknowledged the similarity of the measure construct to the part-whole construct as well. However, he also noted a difference in stating that the focus is not on the part-whole relationships, but on the arbitrary unit instead. He also states that in the measure interpretation a number is assigned to a region in order to tell how much there is. Kieren (1995) considered rational numbers as measures or points on the number line as well.

Sowder, Philipp, et al. (1998) further describe the measure interpretation as “the number assigned to some measurable quantity” (p. 9). It is something that occurs when a chosen unit of measure does not fit into something to be measured a whole number of times. Thus, the whole needs to be partitioned into parts and a fraction is used to express how much of the something there is. Thus, the measure interpretation tells how much.

A final point to be made about the measure interpretation is that partitioning plays a role in interpreting fractions as a measure (Lamon, 1999). It is not necessary to measure by comparing to a fixed number of equal parts. Instead, the number of equal parts in a unit can vary and the name given to the fractional amount depends on the number of partitions made. Performing the successive partitioning tasks is difficult for young children. So, Lamon suggests introducing this interpretation to fifth and sixth graders, after they have had experience with other interpretations. The quotient interpretation is often introduced to children early to provide a foundation. It is discussed next.

Quotient Interpretation

The quotient interpretation is often presented as fair or equal sharing problems. Thus, partitioning plays an important part as it did in the part-whole and measurement interpretations (Sowder, Philipp, et al., 1998). In fact, Post et al. (1982) identify partitioning as the major cognitive structure involved in the quotient interpretation of fractions. When two quantities are divided, the result is a quotient. The quotient interpretation for $\frac{a}{b}$ is that *a* is divided by *b* (Behr et al., 1983).

This interpretation of fractions provides a foundation for studying rational numbers as a quotient field (Lamon, 1999). Developing ideas of a quotient field is well beyond the scope of this study and would serve little purpose here. For this discussion, it is only necessary to note that the field properties apply and allow definition of equivalence and other properties and operations when rational numbers are considered elements of a quotient field. In addition, proving theorems about the structure of the system is also possible (Kieren, 1976). Seeing fractions as elements of a quotient field is one level of sophistication of the quotient interpretation. Another level of sophistication is establishing equivalent values (Post et al., 1982; Behr et al., 1983). That is, when $\frac{6}{3}$ and $\frac{1}{2}$ are interpreted as an indicated division the results are equivalent values of 2 and 0.5, respectively.

Operator Interpretation

The final interpretation to be considered here is that of operator. Lamon (1999) succinctly states that, “the operator notion of rational numbers is about shrinking and enlarging, contracting and expanding, enlarging and reducing, or multiplying and dividing” (p. 94). The operator interpretation can be thought of algebraically as a function that transforms geometric figures or sets of objects. (Behr et al., 1983; Post et al., 1982). That is, when a fraction operates on a continuous object, it stretches or shrinks the object. For example, if a length of 1 is operated on by $\frac{p}{q}$, then the length is stretched to *p* times its length, and shrunk by a factor of *q*. So, for a length of 6 and an operator of $\frac{2}{3}$, the result would be $2 \times 6 \div 3$, or 4. Similarly, when a fraction operates on a discrete set, it is a multiplier or divider. For example, a set containing *n* elements operated on by $\frac{p}{q}$ results in $pn \div q$. Thus, if a set contained 12 objects and was operated on by $\frac{2}{3}$, the result would be $24 \div 3$, or 8.

This concludes the discussion of different interpretations for fractions. The next section focuses on commonalities for all the interpretations and discusses underlying processes and concepts that are important in working with fractions.

Unifying Elements

Although the previous section presented four different interpretations for fractions, there are several unifying elements that connect the interpretations. These serve as big ideas around which to organize elementary mathematics with respect to fractions. The first of these unifying elements, or big ideas, is the notion of multiplicative, or relative, thinking. Deriving meaning for a quantity by comparing it to another quantity requires multiplicative thinking (Lamon, 1999). Lamon explains that this type of thinking is foundational to understanding several important ideas related to fractions. These include: (a) the relationship between the size of pieces and the number of pieces, (b) the need to compare fractions relative to the same unit, (c) the meaning of a fractional number, (d) the relationship between equivalent fractions, and (e) the relationship between equivalent fraction representations.

A notion of quantity is the second element that will be discussed. This idea is fundamental to later fraction work. One aspect of this notion of quantity is that a fraction is a single number, not two independent numbers (Sowder and Schappelle, 1994). In addition to a notion of quantity, Carpenter, Fennema, & Romberg (1993) identify unitizing and partitioning as two more unifying elements. The idea of equivalencing as described by Kieren (1988) is another unifying element that will be discussed. The final unifying element is that of common measures to add and subtract (Mack 1995). Beginning with multiplicative thinking, each of these big ideas related to fractions will be discussed in the following sections.

Multiplicative Thinking

Several ideas contribute to an understanding of multiplicative thinking. Lamon (1999) describes multiplicative thinking by discussing absolute versus relative thinking. Smith (2002) discusses fractions as relational numbers, noting that they represent relationships between two discrete or continuous quantities. Finally, Behr and Post (1992) address the significance of multiplicative thinking when they point out that the rational number set is the first set of numbers students encounter in their study of mathematics that is not based on a counting algorithm of some type. These ideas will be considered in this section on multiplicative thinking.

One way Lamon (1999) addresses the differences between multiplicative thinking and additive thinking is to differentiate between an absolute quantity and a relative quantity. An absolute quantity is independent and not related to another quantity. This notion of an absolute quantity employs additive thinking with which children are familiar before being introduced to fractions. When children are asked to understand change compared to something else, they need to engage in relative thinking, and deal with relative quantities. Thus, relative thinking involves comparing a quantity to something else. This is also known as multiplicative thinking or reasoning.

In order to illustrate the difference between additive and multiplicative thinking, Lamon (1999) describes a scenario similar to the following. There are two boxes of candy. The first box has 4 pieces of candy in it, and the second box has 10 pieces. Children would use additive thinking to answer questions like: “How many pieces of candy are in the first box?” and “How many more pieces of candy are in the second box than in the first box?” In contrast, multiplicative thinking would be required to answer these questions: “What part of a dozen do the pieces in each box represent?” and “The number of pieces in the second box is how many times as great as the number in the first box?” Notice that the answers to the multiplicative questions describe how much of something versus how many. The multiplicative quantity described is relative to something rather than being a countable amount.

Another way to think about the difference between multiplicative thinking and additive thinking is expressed by Behr and Post (1992). They state that rational numbers are not based on counting algorithms. This means that some form of simple counting will not solve every problem. Until the introduction of rational numbers, counting could be used to solve problems. One reason counting does not work for rational numbers is because there is no next rational number. They are continuous and one can always find a rational number between any two given rational numbers. Smith (2002) notes that even though fractions are the first set of numbers students encounter that express relationships, students have life experiences in relative thinking. For example, they share things with others. These experiences should help students transition from additive thinking to multiplicative thinking.

Multiplicative thinking is one aspect of the broader idea of multiplicative structures to which fractions belong. Lamon (1993, 1994a, 1994b) describes the complexity of multiplicative structures as requiring conceptual coordination of multiple compositions. This means having to compose units. For example, finding $\frac{3}{4}$ of 16 things entails beginning with the 16 things as one-units. Then make units of units so there are 4 four-units. Finally, make 1 three-unit from 3 of the four-units. This process, shown in Figure 1, represents a three-tiered composition of units.

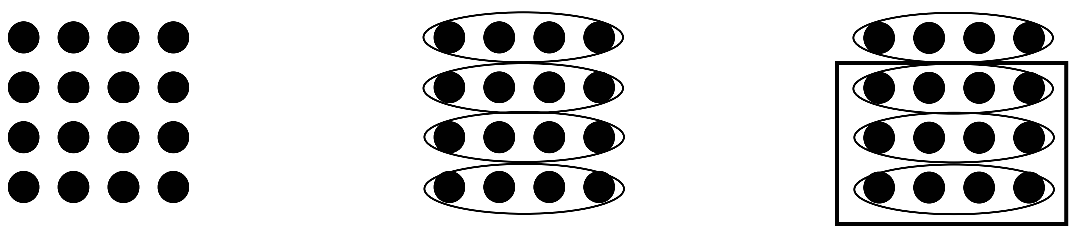


Figure 1: Multiple compositions of units

Furthermore, most multiplicative structures combine two quantities with different labels to create a third quantity with an altogether different label (Lamon, 1999). For example, three packages of 4 cookies per package yields 12 cookies. There are three distinct labels for the three quantities—packages, cookies per package, and cookies.

According to Lamon (1993) multiplicative structures are cognitively complex. This discussion serves to point out that multiplicative thinking is based on relationships between quantities rather than on counting. It may involve composing and decomposing units as well as partitioning. For these reasons, students who are accustomed to additive reasoning may have trouble shifting to multiplicative reasoning (Behr & Post, 1992; Lamon, 1999). It is therefore important to be aware of this need to shift from additive to multiplicative thinking. This discussion of multiplicative thinking provided background information for the other unifying elements. A notion of quantity is the next element considered.

Notion of Quantity

Post, Behr, and Lesh (1986) describe what they call a notion of quantity related to rational numbers. Their research suggests that students may lack a quantitative notion of rational numbers. Behr and Post (1992) discuss the often cited problem on the National Assessment of Educational Progress (NAEP), $\frac{12}{13}+\frac{7}{8}$. They conjectured about what students do not understand. Apparently, students do not realize that both of the addends are close to 1, so the sum will be close to 2. They may not realize fractions have size. If they do realize fractions have size, they may not be able to determine that size. Students seem to simply apply rote procedures for adding (incorrectly) from memory. They don't seem to have a good sense of what would be a reasonable answer to the problem. An analysis of students' responses suggests that students do not differentiate between operations with whole numbers and operations with fractions. Finally, there is evidence to suggest that they do not perceive the fraction as a number with a single value.

The problems noted in the preceding paragraph are related to a quantitative understanding of rational numbers. This includes realizing that rational numbers are numbers. Further, rational numbers may be expressed in many ways. Another aspect of this quantitative understanding is realizing that rational numbers can be ordered, but the procedures to do so are different and more complex than ordering whole numbers. Dealing with these aspects of rational numbers requires realizing that it is the relationship between the numerator and denominator—not their absolute magnitudes independently—

that defines the meaning of a fraction. In order to compare $\frac{1}{3}$ and $\frac{5}{8}$, each fraction must be seen as a single quantity and not two separate numbers. Whole number ideas often interfere with this. In this case, a student may state that $\frac{5}{8}$ is greater than $\frac{1}{3}$ because 5 and 8 are greater than 1 and 3.

Post et al. (1986) include the ability to determine whether or not an answer makes sense as part of a quantitative notion of rational number. This component, of course, is not unique to rational numbers. A characteristic that is new to rational numbers, and different from whole numbers, is understanding that rational numbers have relative and absolute sizes. That is, they can be compared with respect to their relationship to the whole that defines them. This relative magnitude depends on the size of the whole. This also speaks to the importance of the unit, which is discussed in the next section. When comparing relative magnitudes, it may be found that one half of a small pie is actually less than one third of a larger pie. While comparisons of relative magnitude can be made with parts of different-sized wholes, it is imperative that comparisons of absolute magnitude be made with respect to a common unit. That is, the amounts being compared must be based on the same-sized whole.

Ordering fractions is one way students use their rational number sense and a notion of quantity. When ordering numbers, the density property of rational numbers may present a challenge for some students. The density property stated simply says that there is an infinite number of fractions between any two fractions and it is possible to get as close to any point as desired with a fraction (Lamon, 1999). Unlike whole numbers, there is no “next” number and another number can be found between any two given numbers. This density property may be a source of confusion for some students. The characteristics of fractions related to density are important to consider when comparing and ordering fractions. For example, students may think there are no fractions between $\frac{1}{3}$ and $\frac{1}{4}$ because the denominators are consecutive numbers. One strategy to find values between two fractions is to partition the interval so there is a common denominator for the given fractions. So, for $\frac{1}{3}$ and $\frac{1}{4}$, the interval could be divided into 12 or 24 equal parts. This results in fractions that are equally spaced in the interval.

Finally, comparing fractions can build a quantitative understanding of fractions. A student who understands equality and transitive properties can apply them to comparing fractions. Post et al. (1986) explain that when comparing $\frac{3}{4}$ and $\frac{7}{8}$, a student who knows that $\frac{3}{4}$ is equivalent to $\frac{6}{8}$ would be able to conclude that $\frac{3}{4}$ is less than $\frac{7}{8}$ since $\frac{6}{8}$ is less than $\frac{7}{8}$, and $\frac{3}{4}$ is equivalent to $\frac{6}{8}$. A quantitative notion of rational number as described by Post et al. is important to making fractions meaningful to students. Understanding that fractions are numbers that express a relationship and whole number ideas and procedures may not be valid with fractions are at the core of a quantitative understanding of rational number. Also central to the idea is that fractions name a quantity and can be compared and ordered. The concept of the unit is an essential part of that quantity. In fact, "The unit is the context that gives meaning to the represented quantity" (Hiebert & Behr, 1988, p. 3). A discussion of the unit follows.

Concept of Unit

Lamon (1999) discusses new types of units that children encounter when they begin to study fractions. A unit may be a single object, a group of several objects, or part of an object, to name a few possibilities. These new types of units may be the source of common misconceptions for children. Thus, developing a thorough understanding of the whole, or unit, when working with fractions is critical.

This understanding includes the fact that the unit may be composed of more than one object, or several objects packaged as one. This is complex because when such a unit is partitioned, a new kind of number refers to the fractional part. For example, when a package of three cookies is divided into three equal parts, the result is one cookie in each part. Thus, one cookie represents $\frac{1}{3}$ of the original unit. However, children may view this one cookie as the whole. If there were two cookies in the unit, $\frac{1}{3}$ would be yet another quantity. Furthermore, a single cookie may represent $\frac{1}{2}$ of a package of two cookies, but $\frac{1}{4}$ of a package of four cookies. Additionally, each fraction can be represented by equivalent fractions, meaning there are many other names for the same quantity.

One important application of understanding the concept of the unit occurs in interpreting the remainder in a quotitive division problem. Lamon (1999) gives an example of a pie shop. At this pie shop, a slice of pie is $\frac{1}{3}$ of a pie. If there are $4\frac{1}{2}$ pies, how many slices of pie are there? In this scenario (and other quotitive division situations) the remainder is compared to the divisor. Thus, the divisor of $\frac{1}{3}$ becomes the unit. So the answer of $13\frac{1}{2}$ means there are $13\frac{1}{2}$ slices of pie. The half slice is actually $\frac{1}{6}$ of a whole pie. Lamon emphasizes that children need to learn that the unit is different for different problems. Thus, they should always identify the unit before solving the problem.

Lamon (2002) refers to unitizing as “the process of mentally constructing different-sized chunks in terms of which to think about a given commodity” (p. 80). For example, there are several ways to unitize 24 cans of soda. They could be “chunked” as

24 individual cans, 1 case of 24 cans, 2 packs of 12 cans, or 4 packs of 6 cans. This ability to reconceptualize quantities in different ways adds flexibility and usefulness to one's knowledge. When students are able to unitize, they don't need to memorize rules for finding equivalent fractions. Unitizing also emphasizes that a fraction is a number. It names a relative amount—the same amount regardless of the size of the chunks.

The context of a problem should give students a way to determine the unit (Lamon, 1999). For example, a situation involving 2 pizzas with 8 slices each could have different units. The unit would be a pizza if the question asked how much of *a pizza* was left. The unit would be 2 pizzas if the question asked how much of *the pizza* was left. Units are not always explicitly defined, however. Sometimes students may be asked to determine the unit given only part of the whole. For example, given 3 triangles that represent $\frac{3}{4}$, they would need to determine that 4 triangles represent the whole, or the unit. Determining the unit is useful in later work with fractions.

Lamon (2002) reports that fourth-grade students who were taught unitizing with part-whole fractions were able to perform fraction operations without being taught any operation rules. In fact, Lamon states that, "Students who develop strong reasoning processes based on unitizing surpass students who have had many years of rule-based instruction, both in their conceptual knowledge and in their ability to perform fraction computation" (p. 85). Unitizing is related to partitioning, the next unifying element to be discussed.

Partitioning

From the discussion of the various interpretations, one should surmise that partitioning plays an important role in understanding fractions. In fact, Lamon (1999) considers partitioning to be critical for rational number understanding. She points out that fractions are formed by partitioning. Thus, the process is fundamental to building rational number concepts and operations. For example, a result of partitioning activities should be that children realize that there is a relationship between the number of partitions made and the size of the parts.

According to Pothier and Sawada (1983), the ability to partition, or to divide an object or set into equal parts, develops gradually. They developed a theory about the emergence of partitioning capabilities. This theory was developed through a series of clinical interactions with 43 children as they performed tasks that were designed to reveal their partitioning capabilities. The resulting theory made use of three mathematical constructs to help the researchers analyze the partitioning behaviors. They were: (a) odd/even, (b) prime/composite, and (c) factor/multiple. Five levels of partitioning were found. They were: (a) sharing, (b) algorithmic halving, (c) evenness, (d) oddness, and (e) composition.

At the sharing level, children can usually partition rectangular and circular regions to show halves and fourths. However, they may make errors such as making unequal pieces, making an incorrect number of shares, or not using the entire region. These children allocate pieces without regard to size and what a fair share should be. At the second level, algorithmic halving, children are able to partition rectangular and circular regions into numbers of parts that are powers of 2. This is accomplished by halving the pieces in successive partitions. As with the sharing level, this procedure does not always result in equal-sized pieces. Children who consider the size of the pieces and evaluate the sizes with respect to being equal are said to be functioning at the third level called evenness. At this level, children can partition the regions into even numbers that are not powers of 2. The fourth level, oddness, is marked by children recognizing that their process of halving is not efficient for fractions with odd denominators. Instead, these children use a counting algorithm. They count the pieces as they are produced one at a time. Children will frequently need to adjust the partitioning lines in order to make equal shares. The final level was implied by the study, in that the researchers did not actually observe any of the children functioning at this level. They inferred that the fifth level, composition, would be achieved by older children. Children at this level would also make use of a counting algorithm. However, they would recognize that the counting algorithm used at the fourth level is inefficient for larger numbers such as 9 and 15. To partition a region into nine shares, they may first make thirds. They would then further divide the thirds into thirds resulting in nine equal shares. Children who use this multiplicative algorithm are able to construct any unit fraction.

Pothier and Sawada (1990) believe that using prepartitioned models does not facilitate students' representing fractions nonsymbolically. That is, it is important for students to do the partitioning themselves. When students work with prepartitioned shapes they do not focus on the geometric properties of the whole or the parts. Therefore, students should engage in tasks in which they actually make the partitions. Suitable tasks will vary, depending on the unit being partitioned. Pothier and Sawada identify five distinct types of units. They are: (a) discrete objects, such as $\frac{2}{3}$ of 12 bottle caps; (b) discrete sets of objects with the elements divisible, such as 3 people sharing 6 cookies; (c) discrete sets with subsets separable, such as 5 people sharing 8 packages of gum; (d) continuous quantity with subsets separable, such as 4 people sharing a prepartitioned candy bar; and (e) continuous quantity, such as 8 people sharing a pizza.

Finally, Pothier and Sawada (1990) suggest that experiences with partitioning help students to construct meaning with respect to fraction concepts. They also suggest that this facilitates solving fraction problems and helps children to verify symbolic computations with fractions. In the next section, partitioning will be related to equivalence and ordering.

Equivalence and Ordering

Equivalence and ordering are related to a notion of quantity, and were even discussed to some extent in that section. They are treated separately here to highlight the benefit of students engaging in activities that require equivalence and ordering. Lamon (1999) recommends providing many informal experiences with fractions before students perform formal operations with fractions. These experiences will help develop fraction sense. For example, children might use flexible thinking to order fractions before they have been introduced to an algorithm. Details of research with respect to children's ordering strategies are discussed in the section student-focused research in this chapter. A summary of strategies described by Lamon will suffice for this the discussion in this section.

First, when fractions have common denominators, the parts are the same size. In this case, the fraction with the greater numerator is the greater fraction. Lamon (1999) called this the Same-Size Parts strategy. The next strategy considers the number of parts.

When the numerators are the same, the same number of parts is being considered. In this case, the fraction with the greater denominator is less than the other one. That is, the parts are smaller, so the total size is smaller. This is called the Same Number of Parts strategy. The third strategy is Compare to a Reference Point. This strategy is useful when neither the numerator nor denominator is the same in either fraction. In this strategy, the fractions are compared to a third reference point such as $\frac{1}{2}$ or 1. These strategies developed through flexible thinking should be encouraged. Lamon (1999) points out that students who rely on the area model to order fractions may become overly dependent on drawing models. In addition, inaccurate drawings may lead them to wrong conclusions.

Equivalence with respect to fractions can be thought of as different names for the same amount or quantity (Mack, 1995). There is not a unique rational number for every fraction (Lamon, 1999). Fractions can be expressed in many ways and Vance (1992) acknowledges that realizing this is an important rational number concept. Kieren (1992) also identifies equivalencies as an important concept because it provides the foundation for operations, especially for addition and subtraction. Mack explains that equivalence ideas explain why computational algorithms work.

When an area or length that is divided into equal-sized pieces is further divided into smaller equal-sized pieces equivalent fractions can be named. Consider a pie cut into 4 pieces. One slice would be $\frac{1}{4}$ of the pie. Now, consider the same pie cut into 8 pieces as if each of the 4 pieces was cut in half. One piece would then be $\frac{1}{8}$ of the pie. Two of the $\frac{1}{8}$-size pieces would be the same amount as one of the $\frac{1}{4}$-size pieces. This illustrates that $\frac{1}{4}$ and $\frac{2}{8}$ are equivalent. Partitioning experiences such as these help children develop equivalence ideas in ways that are meaningful to them. Dividing models into an equal number of same-sized pieces also builds the concept of common measures for addition and subtraction. This is discussed in the next section.

Common Measures to Add or Subtract

Adding and subtracting fractions is a relatively easy task when there are common measures. Mack (2004) acknowledges that when children focus on the idea of operating on like-sized units they can avoid common misconceptions. One such misconception is adding the denominators when adding fractions. In working with students, Mack emphasized the idea of working with like-sized units. The students seemed to grasp the idea that like-sized units were needed, but were not sure how to determine the size. To guide them through the problems, Mack helped her students realize that renaming fractions can be helpful. She also helped them to connect the problems to previously solved problems. Making connections between strategies with manipulatives and those with just number sentences seemed to help the students.

In the case of renaming fractions, students need to use the ideas of equivalence and partitioning presented in previous sections of this chapter. Thus, it could be said that adding and subtracting common measures is a special application of equivalence and partitioning concepts. When students connect symbolic problems to manipulatives, they are also connecting to ideas of equivalence. The manipulatives can be used to model equivalent fractions as well. This is one way students may develop an efficient algorithm for adding and subtracting fractions.

Although it may seem that finding common measures to add or subtract fractions is unique to rational numbers, Mack (1995) points out similarities to adding and subtracting whole numbers. She writes that the place value structure assures that like-sized units will be added or subtracted. The difference for fractions is that the denominator, not place value, determines the like-sized units. In whole number operations, lining up place values in columns to add or subtract serves to combine like-sized parts. That is, the ones are added to ones, tens are added to tens, etc. When adding or subtracting fractions, the denominator determines the size of the pieces and thus needs to be the same for the fractions to be added or subtracted.

In this section, big ideas, or unifying elements for fractions were discussed. These big ideas were: (a) multiplicative thinking, (b) a notion of quantity, (c) concept of unit, (d) partitioning, (e) equivalence and ordering, and (f) common measures to add or subtract. Attention will now turn to research studies involving students and teachers. First, student-focused research will be discussed. This includes sections on student performance and instructional strategies.

Student-Focused Research

Student-focused research was important to this teaching experiment because it informed the development of the initial HLT. This section reports students' performance with respect to fractions, reasons students have difficulty with fractions, and instructional strategies that may improve student performance. The researchers considered all this information in their planning for the teaching experiment.

Student Performance

Students' performance in mathematics has been monitored by the National Assessment of Educational Progress (NAEP) for more than 30 years (Kloosterman, 2004). Information about students' performance on specific mathematics topics is often reported. In the case of fractions, the results cited are generally negative. Although the more recent data available from the 2003 NAEP show some improvement, the status of student achievement with respect to fractions is still cause for concern. Table 2 shows a comparison of the 2000 and 2003 NAEP results for items administered to fourth graders (Kastberg & Norton, 2007). This table does not include all the items administered to fourth graders. Some items were included on both the fourth- and eighth-grade tests. They will be discussed later. Of the items reported in Table 2, the only significant change from 2000 to 2003 was an increase on identifying the correct fraction.

Table 2: Fourth-Grade NAEP Results Reported in Percent Correct

Item Description	2000	2003
Reason using fraction concepts.	21	20
List equivalent fractions.	54	53
Identify the correct fraction.	65	70
Justify that $\frac{1}{4}$ of 20 is 5.	11	11
Solve a problem with fractions.	18	18
Name and shade fractions equivalent to $\frac{1}{2}$.	13	19

Eighth-grade results with respect to fractions are shown in Table 3. The increase from 2000 to 2003 on arranging fractions in order from least to greatest was significant. Another significant change, the percentage of students who were able to write a word problem using fractions, was a decrease (Kastberg & Norton, 2007).

Table 3: Eighth-Grade NAEP Results Reported in Percent Correct

Item Description	2000	2003
Arrange fractions in order from least to greatest.	43	46
Read a weight from a scale with increments of $\frac{1}{2}$ ounce.	82	82
Write a word problem using fractions.	23	12
Complete missing values on a rational number line.	68	70

Finally, Table 4 reports results of items that were common to the fourth-grade and the eighth-grade tests. Significant changes from the 2000 administration to 2003 were generally positive. The exception was determining the length of an item not placed at the end of the ruler. The percentages of both fourth graders and eighth graders responding correctly decreased. Another significant decline for eighth graders was in identifying a model of $\frac{3}{4}$. Significant changes in the positive direction were made by fourth graders in shading $\frac{1}{3}$ of a rectangle, locating $\frac{3}{4}$ on a number line, and identifying a model of $\frac{3}{4}$. For eighth graders significant gains were made in shading $\frac{1}{3}$ of a rectangle and recognizing equivalent fractions (Kastberg & Norton, 2007).

Table 4: NAEP Results Reported in Percent Correct for Common Items

Item Description	**Fourth Grade**		**Eighth Grade**	
	2000	2003	2000	2003
Divide a string that measures $\frac{3}{4}$ yard into eighths.	26	27	54	55
Shade $\frac{1}{3}$ of a rectangle.	25	28	68	73
Determine the length of an object that is placed on the ruler beginning at the 8-inch mark.	25	20	64	60
Locate $\frac{3}{4}$ on a number line divided into eighths and showing the location of $\frac{1}{2}$.	33	37	63	64
Recognize equivalent fractions.	47	49	74	71
Identify $\frac{3}{4}$ from a picture showing 3 of 4 equal parts of a rectangle shaded.	81	83	92	94

These data allowed Kastberg and Norton (2007) to conclude that almost all students had a concept of fractions as equal-sized pieces in a whole. Developmentally, students first consider fractions as numbers between two whole numbers. They can read and recognize these numbers. Then they use fractions for counting and forming a part-whole concept. They further concluded that many fourth-grade students can use part-whole reasoning. Even more eighth-grade students use part-whole reasoning. Finally, the scores did increase between fourth and eighth grades, indicating growth of fraction knowledge.

Armstrong and Bezuk (1995) reflected on past performance of students on NAEP items and noticed that students could solve multiplication problems that simply require computing when the problems are set up for them and look familiar. However, they could not use multiplication with fractions to solve word problems. Armstrong and Bezuk also made the point that students who are taught rule-based procedures will be able to successfully apply the rules if they remember them. However, they will not recognize when to apply the rules in problems that are not structured for them.

The Rational Number Project (RNP) conducted informative studies with students to assess their knowledge of fractions. The thought processes of fourth-grade students when comparing and ordering fractions was the focus of one teaching experiment conducted as part of the RNP (Behr, Wachsmuth, Post, & Lesh, 1984). According to the researchers, the ability to compare and order fractions is a measure of the quantitative notion of rational number previously discussed in this chapter. A summary of students' strategies for ordering fractions follows. In this experiment, each of 12 fourth-grade students was interviewed 11 times during the teaching experiment. The experiment took place over 18 weeks with six students at sites in St. Paul, Minnesota and DeKalb, Illinois. The students worked individually and as part of small groups during the 13 lessons which comprised the instructional component of the experiment. The lessons included instruction on five topics: (a) naming fractions, (b) equivalent fractions, (c) comparing fractions, (d) adding fractions with common denominators, and (e) multiplying fractions. These lessons were the extent of the formal instruction on rational numbers the students received during the experiment.

Students were given ordering items in which they were asked to identify which of two or three fractions was less or the least. The students were also given equivalence items that asked them to decide if two fractions were equivalent or to name a missing numerator or denominator to find a fraction equivalent to a given fraction. For both ordering and equivalence items, students were also asked to explain their reasoning. The items included fractions from three classes: (a) same numerator, (b) same denominator, and (c) different numerators and denominators. The analysis of the data suggested that children use distinct strategies to compare and order fractions (Behr et al., 1984). Table 5 lists the strategies used, and a brief summary of the strategies follows.

Table 5: Strategies to Order Fractions

Common Numerators	**Common Denominators**	**Different Numerators and Denominators**
• Numerator and denominator • Denominator only • Reference point • Manipulative • Whole number dominance	• Numerator and denominator • Reference point • Manipulative • Whole number consistent • Incorrect numerator and denominator	• Application of ratios • Reference point • Manipulative • Addition • Incomplete proportion • Whole number dominance

Five strategies were observed when ordering fractions with common numerators. Behr, Wachsmuth, Post, and Lesh (1984) name the strategies (a) numerator and denominator, (b) denominator only, (c) reference point, (d) manipulative, and (e) whole number dominance. Children who used the numerator and denominator strategy recognized that the fraction with the greater (or greatest) denominator had the smaller (or smallest) parts. They were then able to determine that when there are the same number of parts being considered (the numerators) the fraction with the larger (or largest) parts would be the greater (or greatest) fraction and vice versa. Children who used the denominator only strategy used similar reasoning, but referred only to the denominators in their explanations. They still recognized that the denominator was related to the size of the parts in the whole. Behr et al. suggest that using these strategies indicates that the child has an understanding of the inverse relationship of the number of parts in a whole and the size of each part.

Children using the reference point strategy compared the given fractions to a third number. Common numbers were $\frac{1}{2}$ and 1 (Behr et al., 1984). Explanations about this strategy referred to the amount needed to complete a whole (or another reference point) or the amount or number of pieces greater than a whole (or another reference point). For example, a child giving an explanation about completing a whole may have reasoned that $\frac{8}{9}$ is greater than $\frac{4}{5}$ because each fraction is lacking one piece from being a whole. The child may have compared the missing pieces and determined that the one piece for the $\frac{8}{9}$ ($\frac{1}{9}$) is smaller than the one piece for the $\frac{4}{5}$ ($\frac{1}{5}$). Using this reasoning, the child may then have concluded that $\frac{8}{9}$ is greater than $\frac{4}{5}$. An example of using the strategy of the number of pieces greater than a reference point might be used when comparing the fractions $\frac{5}{8}$ and $\frac{7}{12}$. A student who recognizes that each of these fractions is one piece more than $\frac{1}{2}$ will compare the pieces. In this case the pieces are $\frac{1}{8}$ and $\frac{1}{12}$. Since $\frac{1}{8}$ is greater, $\frac{5}{8}$ is the greater fraction.

Children using the manipulative strategy incorporated manipulatives or drawings into their explanations. For example, a child might draw models of $\frac{1}{3}$ and $\frac{1}{5}$ and explain that $\frac{1}{3}$ is larger because there are more pieces in the model divided into fifths. This child may recognize the relationship between the number of parts and the size of the parts, but still needs a visual cue to express it.

The whole number dominance strategy is the only strategy that was invalid. Children who used this strategy focused on the values of the denominators but erroneously applied whole number ordering rules to them. Thus, they may have concluded that $\frac{1}{3}$ is less than $\frac{1}{4}$ because 3 is less than 4. Use of this strategy suggests that children's schemas for ordering whole numbers are strong and interfere with their ordering of fractions. Data suggest that, with instruction, this dominance of whole numbers diminishes (Behr et al., 1984).

Behr et al. (1984) reported that students also used five distinct strategies when comparing fractions with the same denominator. The strategies were (a) numerator and denominator, (b) reference point, (c) manipulative, (d) whole number consistent, and (e) incorrect numerator and denominator. Again, these include four valid strategies and one invalid strategy. The reference point and manipulative strategies were the same as for fractions with the same numerators and will not be detailed in this discussion.

Children who used the numerator and denominator strategy indicated that the size of the parts was the same, but one had more parts. Behr et al. (1984) suggest that children who use this strategy use a mental image of manipulatives based on their experiences with manipulative aids. This kind of thinking represents thought that is not dependent on embodiment as the manipulative strategy is. Children who used the whole number consistent strategy ordered the fractions by comparing the sizes of the numerators only.

The final strategy, incorrect numerator and denominator, is the invalid one. Children who used this strategy inverted the relationship between the numerator and denominator. That is, they incorrectly thought that when the numerator was less it meant the parts were larger. These children did not understand the compensating relationship between the size of the parts and the number of parts into which a whole is equally partitioned (Behr et al., 1984).

The final type of fractions students were asked about had different numerators and denominators. For this type of fraction, students were asked to not only compare, but to also find a fraction equivalent to a given fraction. Behr et al. (1984) reported six strategies used by students for this type of fractions. They were (a) application of ratios, (b) reference point, (c) manipulative, (d) addition, (e) incomplete proportion, and (f) whole number dominance. The reference point and manipulative strategies were the same as for the previous two types of fractions and will not be discussed with respect to fractions with different numerators and denominators.

When children used the application of ratios strategy, they used ratios to determine if the fractions were equivalent. For example, an explanation for writing 12 to complete $\frac{3}{4} = \frac{9}{?}$ would have addressed the fact that 3 “goes into” 9 three times and 4 “goes into” 12 three times. The application of ratios, reference point, and manipulative strategies are the only valid strategies of the six used for fractions with different numerators and denominators. Thus, the remaining strategies are invalid.

Children who used the addition strategy added a number to the numerator and the denominator of a fraction to compare the fractions. That is, when asked to find a fraction equivalent to $\frac{3}{4}$ with a denominator of 8, the child added 4 to the numerator and the denominator to arrive at the answer of $\frac{7}{8}$. Children who used the incomplete proportion strategy gave an explanation that used one of the ratios in the proportion, but did not apply it correctly. An example of this strategy might be completing $\frac{6}{4} = \frac{?}{8}$ with a 3 and explaining it as 3 “goes into” 6 twice and 2 times 4 is 8. The final strategy is whole number dominance. Children who used this strategy compared numerators and denominators separately using whole number ordering rules. That is, they might have stated that $\frac{3}{5}$ is less than $\frac{6}{10}$ because 3 is less than 6 and 5 is less than 10 (Behr et al., 1984).

In summary, this study of how children think with respect to order and equivalence of fractions resulted in noting some commonalities across the three classes of fractions. These are: (a) thinking that considers the numerator and denominator of each fraction, (b) thinking that relies on manipulatives, (c) thinking that compares the given fractions to a third fraction, and (d) thinking that is influenced by previous knowledge of whole numbers. This thinking facilitates one's conclusions only when the denominators of the fractions considered are the same (Behr et al., 1984).

Instructional Strategies

This section considers how student performance with fractions may be improved. A variety of researchers have proposed reasons that students have trouble with fractions. First, teachers may not take advantage of students' informal experiences they have outside of school. Moss and Case (1999) refer to this as being adult-centered. That is, teachers tend to teach from their perspective rather than that of their children. This means children's informal knowledge about fractions may be ignored.

Fraction knowledge is rooted in children's informal knowledge and, with instruction, key concepts develop. Mack (1993) acknowledges that students have considerable informal knowledge about fractions when they come to school. Leinhardt (1988) refers to intuitive knowledge and states that it is not a result of direct instruction. Instead, this knowledge is based in real life and is circumstantial. While this knowledge can be built on to develop fraction concepts (Leinhardt, 1988; Mack, 1990; Streefland,

1993) it can also interfere with children's development of important ideas. An example of such interference is that, given two unit fractions such as $\frac{1}{6}$ and $\frac{1}{8}$, children may choose $\frac{1}{8}$ as the greater fraction because they apply the rule for whole numbers to fractions. The argument is that these informal conceptions of rational numbers have more characteristics of whole number knowledge than of rational numbers. Post et al. (1988) attribute this interference to insecure rational number concepts, and warn that teachers need to carefully consider this in their instruction. They further suggest that children need to learn how to determine when whole number schemas are appropriate and when they are not. It is not reasonable for children to rely on teachers to tell them if their schemas are appropriate.

Another cause of student difficulty is rooted in students' use of concrete materials. Bezuk (1988) states that manipulatives are not used with the maximum benefit, or are removed too soon in the instructional process. Moss and Case (1999) suggest that representations may help students develop meanings for fraction concepts. Appropriate representations are critical to that end. Teachers should be able to select representations that help to clarify fraction concepts and not confuse students.

Post and Cramer (1987) write about the importance of initial fraction concepts being introduced concretely. This is to allow children to operate meaningfully on fractions when they are later represented abstractly. Using manipulatives helps students to develop strong mental images. Post and Cramer observed that children with such mental images used strategies that had not been taught to order fractions. They also suggested that physical models may help children overcome the influence of whole-number schemas when working. Further, Cramer and Henry (2002) noted that using concrete models over extended periods of time helps children to develop mental images for fractions. This helps them think conceptually about fractions. Cramer and Henry claim that when students use concrete models, they develop mental images for fractions. These images, in turn, help students to understand about fraction size. This understanding then leads to students being able to perform fraction operations in meaningful ways.

Though it is generally accepted that using concrete models is helpful, specific recommendations as to their use vary. Bezuk (1988) recommends using a variety of manipulatives and real-world objects in fraction instruction. She further recommends that after being introduced to several manipulatives, students should be able to choose the manipulative they prefer to use. Additionally, Bezuk cautions that activities should be done with manipulatives, real-world objects, and pictures of real-world objects before introducing diagrams which are more abstract. Fennema (1972) found that children will choose the model that makes the idea most meaningful to them if they are given a choice. So, the student is the best indicator of which type of representation should be used. While concrete models are useful in introducing new concepts, students for whom the symbols have meaning may choose to work with symbolic models.

Empson (1995) does not advocate giving children manipulatives specific to fractions (as with fraction circles, precut into equal pieces). Her thinking is that working with neatly precut equal pieces may cause the students not to realize the importance of understanding that the pieces must be the same size. Empson recommends that children’s thinking, not the manipulatives themselves, should dictate the use of manipulatives. For example, using a context of equal sharing would help children realize that the pieces to be shared should be of equal size. If children were simply handed prepartitioned manipulatives such as fraction circles, they may not realize the importance of equal pieces.

Thompson (1994) offers a slightly different perspective on using manipulatives. While he is in favor of using manipulatives in mathematics instruction, he warns that just using concrete materials is not enough to assure students gain understanding. He further warns that it is easy to misuse concrete materials. This happens frequently when teachers use manipulatives to model a procedure students may be asked to do symbolically. According to Thompson, the focus should be on what students should understand instead of on what students will do. Accepting this, Thompson supports using manipulatives for two reasons. First, they give the students and teacher an opportunity to engage in conversation. This conversation should be about how to think about the materials and the meanings attached to them. The second reason to use concrete materials is to give students something on which to act. This will enable them to reflect on their actions and think about the mathematics. Baroody (1989) supports this position when he writes that students need to reflect on the use of manipulatives in order to assimilate them into their existing knowledge. He suggests this is more likely to happen if students use what they know rather than being shown how to use the manipulatives. In fact, Baroody suggests that the manipulative itself may be less important than the experience itself. This experience must be meaningful to the students.

There are several ideas to develop fraction understanding with an emphasis on moving to abstract representations. Sowder and Schappelle (1994) state that it is important for students to move back and forth between symbols and concrete experiences while their fraction knowledge is developing. They also write that before being introduced to symbolic representations for fractions, students should have experiences that build on their informal knowledge of fractions. Bezuk (1988) also advocates delaying the introduction of symbol names until the concept is fully understood. She recommends using word names, and is supported by Hendrickson's (1983) position that children should not write or interpret symbols until they can be meaningful and facilitate thought. In fact, he states that thinking is more important than memorizing symbols and steps of symbol manipulation. Similarly, Hamrick (1980) suggests that children should master the objectives orally before being introduced to symbols. She states that mathematical symbols may lack meaning if children don't have adequate verbal experiences. This seems to support Streefland's (1993) statement that "symbolic fractions can be understood only when the fractions stand for something" (p. 298). Empson (1995) offers a guiding principle for introducing symbols to children. She recommends making sure that the symbols are related to concepts and situations that are meaningful to them. Additionally, symbols will be a natural extension of ideas established in conversations between teachers and children (Powell & Hunting, 2003).

Hiebert (1989) writes that "written symbols offer a convenient and powerful way to represent mathematical situations and to manipulate mathematical ideas" (p. 39). He acknowledges that many students are capable of solving problems outside of school, but are not able to apply that knowledge to school tasks. The fact that informal mathematics outside of school is performed largely without written symbols tends to limit its usefulness. Written symbols allow students to move beyond the limitations of oral and mental arithmetic. Using written symbols requires students to establish meanings for them. Students need to understand that symbols represent problem situations and manipulating the symbols parallels actions taken on the things being represented by the symbols.

Using symbols wisely involves three things: (a) interpreting the symbols appropriately, (b) manipulating the symbols using well-understood strategies, and (c) judging the reasonableness of the answer (Hiebert, 1989). Hiebert suggests promoting these characteristics through instruction that helps students connect their new knowledge about the symbols with their understandings from outside of school. He further suggests three points in the problem solving process where connections should occur. First, students should develop meaning for symbols. This entails connecting quantities to the symbols that represent them as well as connecting operations and relationships to the symbols that represent them. Hiebert cautions that students should learn the meaning for the symbol before they learn the computational procedure associated with it. That is, developing meaning for the symbol is not the same as knowing the rule or procedure for finding the answer. Second, students should develop meaning for rules. Hiebert suggests that students need to associate manipulation of the symbols with the manipulation of the quantities. This is because most of the rules are reflections of actions on the actual quantities. Recognizing the parallels between the actions on the symbols and the quantities is how students develop meanings for the rules and algorithms. Unless students make these connections, they will forget the rules or apply them to problems inappropriately. The final point of connection is that students should check the reasonableness of their solutions. Hiebert points out that most students show few connections here and don’t realize their answers should make sense.

Lamon (1999) calls for instruction on fractions to "provide children the opportunity to build a broad base of meaning for fraction symbols, to become flexible in moving back and forth among meanings, to establish connections among them, and to understand how the meanings influence the operations one is allowed to perform" (p. 4). This can be achieved by spending time in the beginning helping students to develop meanings for the different representations. Students should work with physical models, pictures, realistic contexts, and verbal descriptions (Kilpatrick, Swafford, & Findell, 2001).

Another reason students have difficulty is that they are not allowed to spend enough time exploring equivalence and ordering. Bezuk (1988) asserts that there is not enough time spent on developing the concept of fractions and activities related to order and equivalence before operations are introduced. This becomes problematic when students are asked to perform operations with fractions before they really have a good understanding of what they are. Such practice can lead to students simply following rote procedures without meaning. When this occurs they do not know whether their results are reasonable or not. Thus, students should have ample opportunity to develop a sense of fractions and work with fractions before operating on them.

Research also shows that children may attribute whole number properties to fractions (Lamon, 1999; Post et al., 1986). One example of interference from whole number concepts can often be seen when children compare fractions. They fail to realize that greater denominators result in smaller pieces, and therefore greater numbers do not mean greater values. That is, a student comparing $\frac{1}{5}$ and $\frac{1}{2}$ would think $\frac{1}{5}$ is greater because 5 is greater than 2. Children also often confuse whole number results in multiplication and division with those in fractions. They think the product of two numbers is always greater than both of the factors and the quotient of two numbers is always less than the dividend. Additionally, children may try to apply repeated addition and repeated subtraction models to make sense of multiplication and division respectively. While these models may have served them well in working with whole numbers, they are of little use in the domain of fractions.

The preceding discussion highlights the importance of building meaning and moving carefully from concrete to symbolic representations. The effect is to deemphasize procedures in order to develop meaning. Teachers play an important role in this process. Therefore, research related to teachers is reported in the next section. This includes types of knowledge, the effect of knowledge on teaching practices, and teachers' knowledge of fractions.

Teacher-Focused Research

This section on teacher-focused research was important to the teaching experiment for two reasons. First, it documents that teacher knowledge is critical in their teaching. Second, as with student-focused research, the research reported informed the development of the initial HLT. Types of knowledge, the effect of knowledge on teaching practices, and teachers' specific knowledge of fractions are discussed in this section.

Types of Knowledge

Teacher knowledge with respect to fractions is important to this research. Because other kinds of knowledge are also important, this section begins by describing types of knowledge. Recall that this teaching experiment originated from discussions about teacher education. The work of Ma (1999), Shulman (1986), and Ball (1990a, 1990b, 1991a) were considered. Ma (1999) believes that a teacher must have a profound understanding of fundamental mathematics (PUFM) in order to teach students to understanding. She describes PUFM as deep, vast and thorough knowledge, and being able to connect a topic with more conceptually powerful ideas. Knowing the relationships among topics in mathematics, understanding the conceptual structure of the subject, and being able to carry on discourse in the teaching of mathematics are all part of PUFM. It was in discussing PUFM that the idea for this teaching experiment originated. Thus, one might say that PUFM is at the very core of this investigation, even if it is not the object directly studied. In initial discussions of PUFM, the research team also explored other types of knowledge often associated with teachers.

One of these, introduced by Shulman (1986), is the idea of pedagogical content knowledge (PCK). This is one of the three kinds of content knowledge Shulman identified. The other two types were subject matter content knowledge and curricular content knowledge. While he acknowledged that content knowledge void of pedagogy may be useless, he believed that knowledge of theories and teaching methods came in second to a teacher’s knowledge of the subject matter. Shulman suggested a blending of the two types of knowledge to form what he called PCK. This kind of knowledge goes beyond the subject matter in that it considers how to teach the subject matter. In mathematics, this might include relationships among topics and different ways to represent a concept. Knowing what misconceptions students may have and how to correct them are examples of PCK.

Ball, Bass, Hill, and Schilling (2005) propose four types of knowledge that teachers need in order to teach mathematics. These are: (a) common content knowledge, (b) specialized content knowledge, (c) knowledge of content and students, and (d) knowledge of content and teaching and curriculum. Further, Ball, Bass, and Hill (2004) identify teaching tasks that require mathematical knowledge. These include analyzing students' errors, explaining an algorithm and its validity, and using representations. Ball (1991a) states that one type of knowledge necessary for teaching involves being able to talk about mathematics. It is not enough to talk only about the steps in an algorithm. Teachers also need to be able to discuss their teaching decisions and mathematical relationships and procedures with meaning. The decisions to which Ball refers involve deciding which student comments to pursue, creating tasks that encourage exploration, and conducting productive class discussions.

This brief introduction to types of knowledge serves to highlight the kind of background the researchers in this teaching experiment were striving to construct with the prospective teachers. Content knowledge specific to fractions was the focus of this study. However increasing PUFM and PCK was also a goal of the instruction. These types of knowledge were all considered when planning for the instruction in the teaching experiment. The following section discusses why this knowledge is important to effective teaching.

Effect of Knowledge on Teaching Practices

Several researchers have reported findings about a relationship between teachers' knowledge and their teaching practices. This information is included here to validate the belief that deeper conceptual understanding improves teaching. This was the reason the research team wanted to increase conceptual understanding of the students in the teaching experiment class. Sowder, Philipp, et al. (1998) conducted a two-year study that investigated middle-grades teachers' knowledge of mathematics and how it related to their instructional practices. The study focused on multiplicative structures. They selected eight teachers who felt they needed to improve their mathematical backgrounds to change their teaching methods. Of the eight teachers selected to participate in the study, only four completed the full two years. Another teacher participated fully for the first year, and on a limited basis after he moved away in the second year. The participants completed an assessment of their content knowledge. Results of this instrument were used to plan seminars that would highlight careful reasoning based on conceptual understanding. Data were collected through interviews and classroom observations. Sowder, Phillip, et al. gained several insights from the study. Some of these are summarized in the paragraphs that follow.

One insight offered by Sowder, Phillip, et al. (1998) is that teachers often underestimate the difficulty of the elementary mathematics curriculum. This seems to be the viewpoint of teachers who accept mathematics as a set of procedures to follow. Once these teachers began to consider the curriculum conceptually, they realized it was more difficult than they originally thought. They also realized that they did not have opportunities to learn the mathematics they were expected to teach.

Another observation was that once the teachers began to gain understanding of the mathematics they needed to teach, they were able to teach more conceptually (Sowder, Philipp, et al., 1998). The researchers commented that one participant did not seem to be aware that he taught in a procedural manner. They noted that once he understood the mathematics more conceptually, his teaching changed as well. They also commented that another participant who had been teaching conceptually surprised them on one observation by teaching from the textbook. Upon reflecting on the situation, they realized the teacher was teaching a topic that had not been presented in the seminars. Noting that conceptual understanding is about specific content, this seemed reasonable to the researchers since the participant had not had an opportunity to think about what conceptual teaching would be in that particular content topic. The researchers also noticed instances of the teachers reverting back to procedural teaching when they began to feel uncomfortable with the discourse in the classroom. Thus, the comfort level of a teacher may actually interfere with conversations and important relationships may not be discussed.

Sowder, Phillip, et al. (1998) also identified two key issues regarding teaching mathematics conceptually. First, teachers may have difficulty teaching conceptually because they lack full understanding of the mathematics. Second, the teachers may not fully understand their students' understanding of the mathematics. At the end of the study, the researchers concluded that when teachers have conceptual understanding of the mathematics, they can help their students better understand mathematics conceptually. It should also be mentioned that these researchers did note that developing knowledge is fragile. At times, the participants seemed to understand something, but could be easily confused about the same concept later. It was difficult for the participants to realize that their understanding would continue to grow and develop, and thus, would never be complete.

Lehrer and Franke (1992) reported case studies of a second-grade teacher and a fifth-grade teacher. The subjects were selected because of noted differences in their teaching practices. The second-grade teacher interacted with students in her class and often posed problems to them. She attempted to understand students' thinking and listened to their solutions. The fifth-grade teacher seemed to follow the textbook and would provide additional examples if students expressed lack of understanding. The researchers used personal construct theory in order to explore associations between the teachers' personal constructions and their classroom teaching.

With this process, each teacher was presented a subset of 3 problems from a set of 12 problems. The teachers were asked a series of questions about each triad. The triads were formed to assess teachers' knowledge of identifying and representing fractions, ordering fractions, equivalent fractions, and representation of fractions. First, they were asked to determine how the problems in each triad were the same or different and which two of the problems were more alike and why. Then the teachers were asked if the problems were alike or different with respect to how their students would solve them. The final phase of questioning probed pedagogical knowledge. At this time, teachers were asked how the problems were alike or different based on how they teach and the actions they use in working with students.

Four different categories of constructions resulted. These were: (a) knowledge of fractions, (b) general pedagogical knowledge, (c) pedagogical knowledge specifically related to fractions, and (d) cognitional knowledge. The results indicated differences between the two teachers. First, the second-grade teacher provided 33 constructs, with some in each category. About 30% of her constructs were classified as cognitional knowledge, or related to how students think. The fifth-grade teacher provided 18 constructs, distributed across three of the categories. She had no constructs in cognitional knowledge category. The researchers interpreted the results as indicators of knowledge application. Thus, it was seen as an indication whether the teacher knew when and how to apply her knowledge. Lehrer and Franke (1992) concluded that the fifth-grade teacher's knowledge was less refined than the second-grade teacher's knowledge. Ultimately, the fifth-grade teacher's practices reflected a lack of connection between pedagogy, pedagogical content, and content. This supports the idea that a teacher's knowledge of the mathematics being taught impacts the way in which it is taught and subsequently what the students learn.

Tirosh (2000) states that a major goal of teacher education programs should be to analyze the knowledge of prospective teachers with respect to common responses school students might give on mathematical tasks. She investigated prospective teachers' knowledge with respect to common difficulties children experience when dividing fractions. She was interested in finding out if prospective teachers were aware of the common difficulties and to what they attributed the difficulties. The prospective teachers completed a questionnaire that included two items about division of fractions. The first item asked them to calculate answers to four division expressions, list common mistakes seventh-grade students might make after studying fractions, and describe possible sources for the mistakes. The other question presented three problems and asked the prospective teachers to write an expression to solve the problem (but not calculate the expression), write common incorrect responses, and describe possible sources of the errors.

Analysis of this questionnaire given before instruction indicated that some prospective teachers were aware of children’s tendencies to apply properties of operations with natural numbers to operations with fractions. These prospective teachers used this knowledge in their predictions about what students’ responses may be to the given tasks. They also used this knowledge to describe the possible sources of student errors. If teachers were not aware of the tendency to apply natural number properties to fraction operations, they attributed the students’ errors to algorithmic difficulties or reading comprehension problems. Tirosh (2000) suggested that these differences would impact how the prospective teachers will later react to students’ errors in their classes.

The prospective teachers then participated in several activities designed by Tirosh (2000) based on the results of the questionnaire and research about children’s and teachers’ conceptions of rational numbers. Tirosh reported that, by the end of the course, most of the participants were familiar with common sources of incorrect responses. She was also careful to point out that simply knowing students’ misconceptions is not adequate. Teachers also need to understand why the students make the mistakes. That is, knowledge of students’ ways of thinking is also important. Thus, Tirosh concluded that teacher education programs should include instruction about how students think, and common misconceptions.

Borko et al. (1992) considered beliefs of the novice teachers and their effect on their teaching. In the case of the novice teacher in this report, she believed her mathematics background would enable her to teach mathematics to elementary students. Borko et al. recommended that prospective teachers be given an opportunity in their university courses to strengthen their subject matter knowledge. However, they concluded that simply taking more mathematics courses will not assure that prospective teachers will gain the subject matter knowledge they need. They recommended that there should be courses that focus on conceptual development of topics that are important in elementary mathematics. They also recommended that university courses should address PCK. Such courses should develop the language and concepts to connect representations and applications to algorithms and procedures. A final recommendation was to allow students in teacher preparation courses to talk about their reasoning with others who are more proficient and can act as models for them.

In another study that involved novice teachers, Leinhardt and Smith (1985) explored the subject matter knowledge (SMK) of expert teachers compared to that of novice teachers. They defined SMK as including “concepts, algorithmic operations, the connections among different algorithmic procedures, the subset of the number system being drawn upon, the understanding of classes of student errors, and curriculum presentation” (p. 247). A total of eight fourth-grade teachers participated in the study. Four were novice teachers and four were considered expert teachers. They attained the expert status based on the unusual growth of mathematics scores of their students over the previous five years. The novices were student teachers in their final year of teacher preparation.

The researchers conducted interviews and classroom observations. The participants were also given a set of cards containing 40 mathematics problems related to fractions. These problems were randomly selected from the computation sections of fourth-grade textbooks. The teachers were asked to sort the cards and give a rationale for the way they sorted the cards. The sort indicated natural breaks between the novice and expert teachers. There was also a distinction evident between experts with high and low knowledge of mathematics. The high knowledge experts sorted the problems into approximately 10 categories and ordered the topics by difficulty to teach or to do. The novice teachers made categories for every one or two problems. They did not see much difference in problem difficulty. Nor did they make connections among the problems in the set. Leinhardt and Smith (1985) concluded that the more experienced teachers' knowledge was more refined and had a hierarchical structure. That is, the expert teachers were able to provide categories that were more connected and deeper than those of the novice teachers.

This section can be summarized by saying that teachers with deeper conceptual knowledge are better able to teach the concepts to the students. Teachers identified as having PCK and SMK were better prepared to address students' errors. They were able to organize the concepts in elaborate and connected ways. Next, studies about teachers' and prospective teachers' knowledge with respect to fractions is presented.

Knowledge of Fractions

Several studies have been conducted to examine teachers' knowledge with respect to fractions. Many of these studies suggest that teachers lack an adequate knowledge of fractions. From the preceding discussion, it can be seen that teachers who are expected to build understanding of mathematical concepts with students need to have a deep understanding themselves. They need that understanding in order to teach their students and to be able to respond to their questions and react to their thinking. (Ball & Bass, 2000; Ball et al., 2004). The research reported in this section suggests that teachers do not have such understanding and may lack information to teach conceptually.

Tirosh et al. (1998) investigated prospective elementary teachers' knowledge of rational numbers. The 147 subjects were in their first year of teachers' college. There were 21 subjects who had chosen mathematics as an area of concentration. All the subjects completed a questionnaire to assess their understanding of rational numbers. Some subjects also participated in three phases of interviews to help the researchers better understand the knowledge of the prospective teachers. The first phase of interviews involved 25 prospective teachers who demonstrated difficulties in solving multiplication and division problems. The second phase involved 18 prospective teachers who demonstrated serious deficiencies in their formal understanding of rational numbers. In the third phase, 32 of the subjects who did not have an area of concentration in mathematics were interviewed. A summary of the study results with respect to fractions follows.

Addition and subtraction of fractions did not seem to be difficult for the prospective teachers. The errors made reflected the same misconceptions that students have. For example, a common error in addition was to add the numerators and denominators. The multiplication and division problems had different results. The prospective teachers specializing in mathematics performed better than the others, but performance for neither group was considered satisfactory. A common error in dividing fractions was to invert the dividend rather than the divisor. The researchers also noted that the prospective teachers were generally not able to explain the steps of the standard algorithms of the operations with fractions. Also of note is the fact that the prospective teachers seemed surprised that there should be a need to explain the steps of the algorithm.

The prospective teachers' formal knowledge of fractions was tested with items that addressed properties and representations. With respect to the density property of rational numbers, the researchers reported that only 24% of the prospective teachers knew that there were infinitely many numbers between $\frac{1}{5}$ and $\frac{1}{4}$. In fact, 43% thought there were no numbers between $\frac{1}{5}$ and $\frac{1}{4}$. Additionally, 30% claimed that $\frac{1}{4}$ is the number that comes right after $\frac{1}{5}$. With respect to the effect of the operations, 60% of the prospective teachers thought that in a multiplication problem, the product is always greater than or equal to each factor. Similarly, 51% thought that in a division problem, the quotient is always less than the dividend. Thus, a majority of the prospective teachers shared the same beliefs with children that "multiplication always makes bigger" and "division always makes smaller." Interestingly, this was in conflict with the algorithmic calculation of the division problems. All the subjects who thought division always makes smaller were able to perform the algorithm and obtain correct answers to at least some of the division problems.

The prospective teachers were also asked to construct as many representations as possible for several fractions and one example of adding fractions. The vast majority of them used common ways to illustrate fractions. Their graphic representations were mostly circular or rectangular regions in which the unit was evident and the parts seemed to be reasonably equal. They relied heavily on the part-whole model for fractions. Even though they were asked to construct as many representations as possible, most of them did only one. When they were prompted to give additional examples, they tended to simply change the shape of the area model shown. The researchers also noted that very few constructed set models and no number line models were used. This was limiting when they were asked to construct a model for an improper fraction. When asked to model an addition example, the prospective teachers had much more difficulty. Only about one third of them successfully modeled $\frac{1}{2}+\frac{1}{3}$, and all of the satisfactory models were area models. The most common error was that the prospective teachers overlooked the need to use the same unit (common denominators) for the addends.

The overall findings suggested the prospective teachers' knowledge with respect to fractions was "insufficient, rigid and segmented" (Tirosh et al., 1998, p. 11). They were generally unable to explain the steps of the algorithms they used. They also lacked the ability to construct appropriate representations for fractions and addition with fractions. Although the prospective teachers with a concentration in mathematics performed better that the others, neither group's performance was judged to be acceptable.

In another study, Cramer and Lesh (1988) tested 48 elementary education majors to assess their conceptual understanding of rational number ideas. The overall mean on the 45-item test was 63%. The content of the assessment reflected the content that had been taught to fourth-grade students as part of the RNP. The preservice teachers were junior and senior level students, and all but one student had met the university requirements for mathematics courses in their degree programs. Several had taken courses beyond the requirements. In fact, 29% of the students had taken classes from college algebra to calculus, and 4% had a minor in mathematics.

Cramer and Lesh (1988) reported that 20% of the education majors answered fewer than 42% of the items correctly. The categories of test items with the percent correct are shown in Table 6.

Table 6: Reported Results of Preservice Elementary Teachers on a Fraction Test

Category	**Percent Correct**
Fraction equivalence	45
Fraction division	51
Concept of unit	59
Ordering	61
Qualitative questions	65
Part-whole concepts	69
Division story problems	78

Note the difference between the placement of fraction division and division story problems. This suggests that the prospective teachers used informal understanding of division to solve the story problem, but not the division sentence. In fact, none of the students wrote a division number sentence for the story problem. To summarize, the prospective teachers who took this assessment of fraction conceptual knowledge were not as successful as the researchers had assumed they would be. More alarmingly, 20% of the participants did not seem to have the mathematics background they needed in order to teach fractions with understanding.

Lacampagne et al. (1988) reported similar results with inservice teachers. They administered an instrument to assess rational number multiplication and division understandings to 218 fourth- through sixth-grade teachers. The teachers were asked to answer 58 short-answer or multiple-choice questions. There were also six problems on the test that asked them to solve, then explain how they would describe their solutions to children. Results indicated that teachers did not have a clear understanding of partitioning and the part-whole concept. The teachers also had difficulty solving division problems with fractions. However, they were often able to solve such problems if they drew a picture when the suggestion was made. They also had difficulty composing a word problem to represent a division situation given a number sentence with division of fractions. These results suggested that the teachers may have had the same misconceptions as students with respect to multiplication and division. Namely, the misconceptions are that multiplication results are greater than the factors and division results are less than the dividend (Lacampagne et al., 1988).

Stoddart et al. (1993) reported that elementary teacher candidates entering their teacher education programs were lacking in knowledge of the mathematics they would teach. Results of a paper and pencil test with 83 students enrolled in a mathematics methods course showed 54% had neither procedural nor conceptual knowledge about rational numbers. Another 42% of the students demonstrated procedural knowledge, but not conceptual knowledge. Only 4% demonstrated both procedural and conceptual knowledge with respect to rational numbers. The majority of the prospective teachers were able to answer the addition and subtraction computational problems with fractions correctly. However, only about half could correctly solve multiplication, division, and equivalence problems. Performance on word problems was about the same. In the conceptual arena, only about 10% of the subjects could provide adequate explanations of their solutions.

In reflecting on a seminar that focused on division of fractions Sowder, Philipp, et al. (1998) conjectured that the participants had indeed received prior instruction about the meaning of division with fractions, but did not recall the experience. This is perhaps because it may have been a brief lecture. The participants in the study were expected to understand the content. The researchers conjectured this may have been the first time the participants had been in an environment where they were expected to understand the mathematics. Armstrong and Bezuk (1995) support this by writing that teachers who are interested in changing the ways they teach must first approach the topics in ways that are very different from their previous experiences in mathematics learning. They need to reflect on how to conceptualize the operations. They also need to consider how to test, model, and apply the concepts related to the operations. Armstrong and Bezuk also report that in one of the seminars, a researcher asked the participants what made composing a word problem for division of fractions difficult. The teachers thought the task was difficult because: (a) it was hard to tell if the problem was a multiplication or division problem, (b) the partitive and measurement interpretations of division were confusing, (c) using fractions makes the word problems harder, and (d) they lacked practice in making up word problems to go with an expression.

In a study investigating prospective teachers' understanding of division with fractions, Ball (1990b) found their knowledge to be generally fragmented. The study involved 19 prospective teachers. Of these, 10 were secondary education majors, and 9 were elementary education majors. The prospective teachers were interviewed just prior to taking their first education course. They were asked how they would solve a division with fractions problem. The problem was $1\frac{3}{4} \div \frac{1}{2}$. The prospective teachers were then asked to think of a real-world situation or story problem for which the problem could be used to solve. After describing a situation, the participants were asked how it fits with the given mathematical expression. In the event that the answer to the situation was different from the computed answer, and the participant noticed the difference, he or she was asked why it was different. If a participant could not find a situation or other representation, he or she was asked what made it difficult. Responses included that division is just hard and there is not real world application for division with fractions.

All of the elementary education students were able to calculate the correct answer to the given division problem. However, none was able to offer an appropriate representation for the given problem. Three gave an inappropriate representation, and six were unable to generate any representation. One error was to give a representation for dividing by 2 instead of $\frac{1}{2}$. Another error was to give a representation for $1\frac{3}{4} \times 2$. Ball (1990b) suggested that this participant focused on the “invert and multiply” rule for division with fractions. Of the six who were not able to generate a representation, one realized the situation generated was dividing in half—not by one half. The others seemed to think the expression could not be represented with a real-world context. Ball suggested meaning in mathematics seemed to be lacking for these students. She concluded that prospective teachers do not have adequate subject matter preparation for teaching, and more attention should be given to subject matter preparation of prospective teachers.

Ball (1990a) reported similar findings from another study. In that study, 252 prospective teachers’ responses on questionnaires and interviews were analyzed. Again, Ball found their mathematical understandings to be “rule-bound and thin” (p. 449). Of the 252 study participants, 217 were elementary education majors. They were given a questionnaire item which asked them to select a good story problem to illustrate the meaning of $4\frac{1}{4} \div \frac{1}{2}$. The item was presented in multiple-choice format and respondents were told to mark all that apply even though only one option was mathematically appropriate. The interview task was very similar to the one described in the previous study. In this study, 30% of the prospective elementary teachers selected the appropriate representation for the division problem. Ball noted that making a choice from several options is easier than generating a representation. She also noted that many of the respondents who selected the correct response also chose other incorrect responses. Also of note, is the fact that 10% of the prospective elementary teachers selected the "I don't know" option. Thus, in the elementary group, 30% selected a correct response, 60% selected inappropriate representations, and 10% were unable to generate a representation.

On the interview task, Ball (1990a) reported that nearly all the students were able to calculate the answer to the division problem $1\frac{3}{4} \div \frac{1}{2}$. None of the students in the elementary group was able to generate an appropriate representation. Ten students generated an inappropriate representation, and 15 were not able to generate any representation. As with the previous study, the most common error was describing a situation which required dividing by 2 instead of $\frac{1}{2}$. This interview task also revealed that the participants were preoccupied with the fractions rather than focusing on the division. That is, the fractions seemed to make the task more difficult to generate a real-world example. The participants also tended to divide in half rather than by one half. This was also reported in the previous study. Ball suggested this erroneous thinking is a result of confounding everyday language with mathematical language. The participants who were not able to generate a representation made the same mistakes as in the study previously summarized in this section. Ball again concluded that prospective teachers' understanding of division was narrow.

McDiarmid and Wilson (1991) investigated prospective teachers pursuing alternative certifications with respect to their understanding of division of fractions. They interviewed eight mathematics majors who were becoming certified as secondary teachers, and eight non-mathematics majors who would teach math as elementary teachers. When presented with a division problem with fractions, they approached it procedurally, and had difficulty generating a verbal representation for the problem. Of the eight prospective secondary teachers, six generated a situation which required dividing into halves, or dividing by 2. None of the prospective elementary teachers was able to generate an appropriate situation. McDiarmid and Wilson concluded that the teachers seeking alternative certifications had mastered the algorithms, which may be a first step toward conceptual understandings. They also noted that some of the teachers did learn more mathematics during their first year of teaching. Nevertheless, not all teachers will learn more mathematics on their own. Teachers must have a deeper understanding of the mathematics they teach in order to answer the questions students have and be able to explain why.

Ma (1999) gave a similar division task to Chinese teachers and U. S. teachers and compared their responses. Not only did all of the Chinese teachers calculate the correct answer, they knew alternative ways to calculate the answer. In addition, they knew the meaning of the alternative ways to calculate. They were also able to solve the problem in simpler ways. In contrast, only 43% of the U. S. teachers were able to calculate the answer. The Chinese teachers did better than the U. S. teachers on creating a word problem to represent the division problem as well. Of the 72 Chinese teachers, 65 generated conceptually correct word problems. In fact, 12 Chinese teachers created more than one appropriate representation. Only six Chinese teachers could not think of a word problem, and one teacher gave an incorrect problem. Almost all of the U. S. teachers failed to generate a representation of the given division problem. Of the 23 U. S. teachers, only 1 provided a conceptually correct problem, and that problem was pedagogically flawed. Sixteen of the U. S. teachers created stories with misconceptions, and six could not make up any kind of problem.

The misconceptions identified by Ma (1999) were similar to the findings reported by Ball (1990a, 1990b). They included: (a) confounding division by $\frac{1}{2}$ with division by 2; (b) confounding division by $\frac{1}{2}$ with multiplication by $\frac{1}{2}$; and (c) confusing division by $\frac{1}{2}$, division by 2, and multiplying by 2. The pedagogical flaw was that the word problem for the division problem given resulted in an answer of $3\frac{1}{2}$ children. The teacher knew that half a child was problematic. She interpreted it as the answer would be that four children could have the snack, but one of them would only get a half portion compared to the others. The conclusion was that the U. S. teachers lacked connections and links. This is cause for concern, as Ma concluded that teachers need to have a comprehensive understanding of a topic in order to create representations for it.

Borko et al. (1992) followed novice teachers through their final year of teacher preparation and first year of teaching in order to examine the process of becoming a middle school mathematics teacher. They found that even though one such novice teacher had beliefs aligned with the mathematics education reform ideas, she lacked the conceptual understanding to explain division of fractions beyond the rote procedure. This novice teacher had completed several mathematics courses because she had started college as a mathematics major. In fact, she was granted credit for two of the education courses in mathematics by meeting the requirements on a test. This information would seem to suggest that she would have adequate mathematical understanding for teaching mathematics. Yet, when confronted with a question about why the procedure to divide fractions is to invert and multiply, she was not able to give an adequate explanation. In fact, her initial attempt resulted in drawing a model which turned out to be an illustration of multiplying fractions.

Tirosh and Graeber (1989) conducted a study of preservice teachers to examine their beliefs that “multiplication always makes bigger” and “division always makes smaller.” The participant group consisted of 136 preservice teachers who were enrolled in mathematics methods courses, having typically completed at least the first of two required mathematics content courses. All the participants completed written instruments and 71 of them were interviewed as well. Although the scope of the study included multiplication and division with decimals as well as with fractions, the results summarized here are focused on the domain of fractions. Results showed that 85% of the preservice teachers were able to respond correctly that the product of a multiplication problem is not always greater than either factor. However, 40% of the preservice teachers used whole number examples, such as multiplying by 1 or 0, to justify their responses. Also, 25% of the preservice teachers justified their answers by stating a rule about multiplication with fractions or decimals. It is not clear from the data reported what part of these mentioned fractions specifically. With respect to division, only 45% responded correctly that the quotient does not need to be less than the dividend. Again, the preservice teachers used whole number examples to justify their responses. In fact, 15% fell into this category. An additional 21% cited fractions or decimals as counter-examples. In responding to a false statement that the divisor in a division problem must be a whole number, 80% answered correctly, with 33% referring to fractions or decimals being possible divisors. The interviews revealed that some interviewees rejected division by a fraction because "you invert and multiply." The final item on the written instrument that can provide insight to preservice teachers' beliefs about fractions asked the participants to tell if the quotient for $70 \div \frac{1}{2}$ is less than 70. For this item, 60% answered correctly that the quotient is not less than 70. Justifications included 29% who computed the answer or referred to the invert and multiply procedure, indicating a reliance on procedural knowledge.

Tirosh and Graber (1989) pointed out that the two beliefs that "multiplication always makes bigger" and "division always makes smaller" are logically equivalent. However, nearly half of the participants accepted the belief about division while refuting the belief about multiplication. The researchers concluded that the conceptions of the preservice teachers, like their justifications, tended to be limited to the whole number domain.

Simon (1993) administered an open-response written instrument to prospective elementary teachers in order to investigate their knowledge of division. The 33 participants had completed the mathematics content portion of their teacher preparation program and were awaiting student teaching. One of the items asked the participants to write a word problem that could be solved by using $\frac{3}{4} \div \frac{1}{4}$. Results indicated that 70% of the prospective teachers were not able create an appropriate word problem. In fact, 36% wrote a problem that could be solved by another expression. The most common type of word problems could be solved by $\frac{3}{4} \times \frac{1}{4}$. Simon suggested that the difficulty with this task may have been students' lack of understanding of quotitive division. If participants had used a quotitive division example, it may have made more sense. They would have been thinking about making groups of $\frac{1}{4}$ instead of $\frac{1}{4}$ of a group. Simon noted that the participants who created partitive examples ($\frac{1}{4}$ of a group) were less likely to succeed. He also conjectured that they likely did not realize that the partitive context limited their abilities to make sense of the fraction expression.

Post et al. (1988) reported more results of testing fourth- through sixth-grade teachers' knowledge of rational number concepts. The test items were developed to reflect what they called the "conceptual underpinnings" of rational number topics for fourth- through sixth-grades. Admittedly, they added the operations with fractions section in order to document that teachers were competent in the types of problems found in the curriculum. However, they did not find this to be true. They reported that 10% to 25% of the teachers answered basic level items incorrectly. Further, some fundamental items were answered correctly by only 54% of the teachers. Also disturbing, was the fact that in any item category, there was a significant percentage of the teachers who missed one half to two thirds of the items. Generally, 20% to 30% of the teachers scored below 50% on the overall instrument.

The results of the work of Post et al. (1988) indicated that many teachers simply do not know enough mathematics. Additionally, many of the teachers that are able to solve the problems correctly are not able to explain their solutions in ways appropriate for instruction. It is possible that teachers are being asked to teach concepts that they were not exposed to as students. The fact remains that these teachers need to be able to present and explain mathematical concepts. They need to know what the right questions are and know when to ask them (Post et al.).

In summarizing this section on teacher-focused research, two striking observations can be made. First, there is cause to be concerned about teachers' knowledge with respect to fractions. Of particular concern are basic concepts about the structure of rational numbers (the density property, for example). Understanding the meaning of operations with fractions and being able to compose word problems for computation problems is also an area of difficulty for teachers. In addition to composing word problems, teachers do not seem to know the effect of multiplication and division with fractions. Second, the literature contains studies predominantly of testing or interviewing methods. Design-based research seemed to be lacking in the literature. This investigation contributes to the literature by proposing an instructional theory for teaching fractions to prospective teachers.

Conclusion

This review of literature has described the mathematical foundations for fractions. Different interpretations for fractions discussed included: (a) part-whole, (b) measure, (c) quotient, and (d) operator. Unifying elements, or big ideas for fractions were also discussed. These included: (a) multiplicative thinking, (b) a notion of quantity, (c) concept of unit, (d) partitioning, (e) equivalence and ordering, and (f) common measures to add and subtract. Though these were treated as separate entities, they are very much related and overlap one another. Research about fractions related to children and teachers was also presented. Children have historically performed poorly on standardized tests with fractions. Suggestions related to improving instructional experiences for students were given. Finally, research indicates that teachers may lack the conceptual understandings necessary to teach fractions to their students. Specific deficits in teacher knowledge were reported.

Based on this information, the research team for the teaching experiment developed an HLT to investigate how prospective teachers develop their understanding of fraction concepts and operations in a teaching experiment. Details of the HLT as well as the research design are discussed in Chapter Three.

CHAPTER THREE: METHODOLOGY

The purpose of this chapter is to explain how this study was designed to answer the research question posed in Chapter One: How do instructional experiences in an elementary school mathematics content course for prospective elementary teachers support learning fraction concepts and operations from a social perspective? First, the setting of the experiment is described. This is followed by a discussion of the design-based research (DBR) process used. Sections on planning for the hypothetical learning trajectory (HLT) and the implemented HLT describe the instructional tasks. Finally, methods for data collection and data analysis follow, with a discussion of limitations and assumptions closing the chapter.

Setting

This study was conducted at a university in the southeastern United States. Student enrollment at the university was approximately 43,000. About 5,000 students were enrolled in the College of Education. The research took place in a course called Elementary School Mathematics offered in the College of Education. The purpose of the course is to provide instruction in mathematics appropriate for elementary teachers. It is a prerequisite for a subsequent course that focuses on methods of teaching elementary school mathematics. College Algebra, Finite Mathematics, or documentation of an equivalent level of mathematics meets the prerequisite requirements for the course. Topics in the course for the research semester included problem solving, place value, whole number operations, fraction concepts and operations, geometry, and measurement.

Data collection occurred in the summer semester. During this 6-week semester, the course met three evenings a week for 3 hours and 10 minutes. There was usually a 10-minute break in each class period, leaving a total of 9 hours of instruction per week. Table 7 lists the topics addressed in each class session.

Table 7: Course Topics

Class Session	**Topic**
1	Course Introduction; Problem Solving (establishing class norms)
2	Problem Solving; Whole Number Place Value and Operations
3	Whole Number Place Value and Operations
4	Whole Number Place Value and Operations
5	Whole Number Place Value and Operations
6	Whole Number Place Value and Operations
7	Test
8	Fraction Concepts and Operations
9	Fraction Concepts and Operations
10	Fraction Concepts and Operations
11	Fraction Concepts and Operations
12	Fraction Concepts and Operations
13	Test
14	Geometry and Measurement
15	Geometry and Measurement
16	Geometry and Measurement
17	Geometry and Measurement
18	Final Exam

As the table indicates, five of the class sessions were devoted to instruction on fractions. Other sessions of interest to this study took place prior to the fraction instructional sequence. For example, the first class sessions focused on problem solving in order to begin setting classroom norms. In these sessions, attention was given to negotiating classroom norms with respect to participation and mathematical explanations and justifications. The table also indicates that the instructional sequence for fractions began approximately halfway through the semester. At this point, some norms had already been established. These included social norms for expecting students to explain and justify their solutions and solution processes and to attempt to make sense of other students' solutions, asking questions if something is not understood. Two sociomathematical norms were partially established prior to the instruction on fractions. These included determining the criteria for what counts as a different or unique solution and what makes a good explanation (Andreasen, 2006).

Students enrolled in the course were mostly sophomores majoring in elementary education or exceptional education. In the semester of this study, there were 14 elementary education majors, 2 exceptional education majors, 1 mathematics education major, and 1 undeclared major. For the purposes of this study no distinction was made among students with varying majors. Students signed consent forms allowing video and audio taping of the class sessions and any individual interviews that may have been conducted. No individual interviews were conducted for this particular study, but another study that occurred in the same class required interviews.

The classroom was furnished with tables, allowing students to sit in small groups. The instructor asked that no more than five people sit in any one group. As a result, there were six groups with two to five students each. Although seats were not assigned, students generally sat in the same seats every class period. Appropriate manipulatives for the planned activities were distributed at the beginning of each class. This allowed students to use them as desired, but the choice was left to them. There was a whiteboard at the front of the room where students often went to explain their solution strategies. A document camera also provided a means for students to present their explanations to the class. The instructor used the whiteboard and document camera as well.

The research team consisted of the instructor, who was a mathematics teacher educator on the faculty of the university and three mathematics education doctoral students. There were other contributors to the research team who were not able to attend every class session. These included two additional mathematics education doctoral students and a second mathematics teacher educator. The second mathematics teacher educator had conducted this type of DBR and advised the team on methodology as well as the supporting learning activities. The doctoral students had planned the instruction with the instructor for the previous two semesters as well as this semester. Their role in the classroom was to be researchers-observers. They were sometimes called upon by the instructor to reflect on their observations during class sessions and assess the need for any change in direction for the discussion or activities. The doctoral students were free to interject comments and questions into the class discussion, but primarily observed, recorded field notes, and operated video equipment. As members of the research team they also contributed to reflection sessions held after classes to discuss and document the instruction.

The instructor had been teaching the course for five years at the time the study was conducted. She had recently changed her methods of instruction from predominantly lecture-based to an inquiry approach. The semester in which the study occurred was the third iteration of this format for the instructor. The inquiry approach employed was based on the description by Richards (1991). In writing about the inquiry approach, Richards described it as one in which students have opportunities to construct their own mathematics and learn the language of mathematical literacy. He argued that discourse is central to teaching mathematics with an inquiry approach. Thus, the discourse in the teaching experiment allowed for students to interact with other students as well as the instructor. The inquiry format of the class also provided opportunities for talking among students and between students and the instructor. Yackel (1995) stated that talking serves two purposes in the classroom, depending on the point of view. For students, the purpose is to explain their own thinking and to challenge others' thinking. Additionally, teachers gain information about students' progress from the talking in the classroom. For these reasons the course instructor encouraged talking about mathematical ideas.

The inquiry method of the course instruction placed emphasis on developing deep conceptual understanding of mathematics. Students engaged in an instructional sequence for fractions to support that development of understanding. They had access to manipulatives and were encouraged to create any kind of other tool to help them make sense of a given situation. The instructor encouraged students to interact with one another and provide justifications to a greater extent than they may have experienced with more traditional instructional methods. Teacher lecture and presentation time were minimized in the class, while student interaction was increased.

Throughout the course, an emphasis was placed on explaining students' thinking and the underlying mathematical reasons for the familiar procedures. A typical class session included the posing of a problem, followed by small group work. Modeling with manipulatives or drawings was an integral part of the small group work. Members of the research team circulated among the groups and observed the solution strategies, making note of interesting strategies to be presented to the class. Once groups had completed their work, the teacher led a discussion in which different solution methods were highlighted. The solutions were chosen for their potential to develop students' mathematical understandings. This process was often repeated several times in a class session.

Homework and optional practice work were also given to the students. At the end of the fraction instruction, a test was administered. Students had the option to take the test in a small group, mirroring the interaction that took place in the class. Most of the students chose this option, meaning that they worked with their group to arrive at a group answer. In choosing to take the test as a group, the students agreed to accept the recorded work of the group as their own. In addition to this test, items about fractions also appeared on the final exam for the course. Unlike on the fraction test, students answered these items individually.

The following section on DBR begins with background information. A discussion of teaching experiments, the specific type of DBR used in this study, follows. Then a discussion of the phases of DBR, with specific attention to teaching experiments closes the discussion.

Design-Based Research

This experiment used DBR methodology which describes an approach to theory development that may be referred to by several names, but all share common features, process, and results. Whether it is called developmental research (Cobb, 2000; Cobb et al., 2001; Cobb & Yackel, 1996; Gravemeijer, 1994, 1998;), design research, (Bannan-Ritland, 2003; Collins, Joseph, Bielaczyc, 2004), design studies, (Bannan-Ritland) or design experiments (Bannan-Ritland; Brown, 1992; Cobb et al., 2003; Collins et al.; Gorard, Roberts, & Taylor, 2004), the approach seeks to generate instructional theory through teaching experiments.

Gravemeijer (1998) describes developmental research as “a mixture of curriculum development and educational research in which the development of instructional activities is used as a means to elaborate and test an instructional theory” (p. 277). This captures the essence of DBR. Gravemeijer (1994) describes the process of developmental research by writing that “the developer will envision how the teaching-learning processes will proceed, and afterwards he will try to find evidence in a teaching experiment that shows whether the expectations were right or wrong” (p. 449). Thus, developmental research is a cyclic process of development and research. The result is a domain-specific instructional theory (Gravemeijer, 1998; Cobb et al., 2003).

Background

Design-based research was adapted from the design and engineering fields (Kelly, 2003). For example, in the field of product design, a new design is proposed and tested. The proposal itself is a suggestion that the product will be improved. The testing and refinement of the product is an iterative activity (Zaritsky, Kelly, Flowers, Rogers, & O’Neill, 2003). That is, designing begins with brainstorming followed by prototype creation. This is followed by testing in which problems are identified and possible solutions are proposed. The process is repeated until a viable design has been determined. Collins et al. (2004) describe the process as being similar to design sciences such as aeronautics, artificial intelligence, and acoustics. They explain that design sciences aim to see how designed artifacts behave under certain conditions. In the case of education, the effect of different learning environments on dependent and independent variables in learning and teaching is studied. Collins et al. use the term "progressive refinement" to describe the iterative nature of design experiments. This means that the design is constantly revised as a result of testing it until the desired goals are achieved.

Design methodology is useful because there is a need to explain mathematical development in the context of classrooms (Steffe & Thompson, 2000). This is in contrast to traditional experimental designs where a sample is selected, a treatment is applied, and differences are analyzed. In DBR the students become the focus of the analysis. The DBR methodology allows for researchers to see how students made meaning of the content and how they learned specific concepts. That is, DBR allows for instructional theory to be developed based on how the learners construct their own knowledge.

Cobb et al. (2003) identify several types of design experiments of varying scope. However, they also identify five features common to all types of design experiments in educational research. These are: (a) the purpose is to develop a theory about the process of learning as well as the means to support the learning, (b) they are highly interventionist in nature, (c) they are both prospective and reflective—they are implemented with the intent of being refined, (d) they are iterative cycles of invention and revision, and (e) there is a domain-specific theory that accounts for the activity of the design results.

Additionally, Cobb et al. (2001) name three criteria for a design research approach. They state that it should: (a) enable documentation of collective mathematical development, (b) enable documentation of individuals' development of mathematical reasoning, and (c) result in analyses to provide feedback to improve the instructional designs. These criteria, in conjunction with the common features identified by Cobb et al. (2003), apply to teaching experiments, a specific type of DBR.

Teaching Experiments

The teaching experiment is at the core of classroom DBR (Gravemeijer, 2004). There were two driving forces that caused the teaching experiment to emerge in about 1970 (Steffe & Thompson, 2000). First, the models for research at the time had been developed outside of mathematics education. Researchers were looking for a way to account for the progress of students as they participated in interactive mathematical communication. The existing models did not serve this purpose. Second, there was a gap between teaching and research. Classical experimental design did not support investigating students' processes to make sense of mathematics. There was also a belief that teaching experiments could contribute to making mathematics education an academic field (Steffe & Thompson).

A teaching experiment involves a repeated process of developing instructional sequences, testing the sequences in classroom instruction, and analyzing the learning. Revisions to the sequence are made based on the analysis and the process begins again (Gravemeijer et al., 2003). Thus, in order to develop an instructional theory, instructional activities are developed then tested and redesigned in an iterative process (Gravemeijer, 1998).

Although there are different types of teaching experiments, Steffe and Thompson (2000) identify elements common to all teaching experiments. These include: (a) a sequence of teaching episodes, (b) a teacher, (c) one or more students, (d) an observer to witness the teaching episodes, and (e) a method to record what happens in the teaching episodes. Teaching experiments allow for the testing and generating of hypotheses. These hypotheses are tested during teaching episodes. Because it is never certain what may transpire during a teaching episode, it may be necessary to revise a hypothesis during an episode. The teacher's goal is to promote the greatest progress in all the students. If the direction of the instruction needs to change in order to have this happen, then the hypothesis is changed. An observer is present to witness the teaching episodes (Steffe & Thompson). The teacher cannot truly be a self-observer while teaching, so the observer serves to assist the teacher in interpreting students' reasoning. The witness may be able to offer another interpretation for the events that took place. Another opinion is sometimes helpful. The observer also meets with the teacher after the teaching episode to plan the next teaching episode.

Researchers are present in the classroom as the teaching experiment takes place. This enables them to participate in an ongoing analysis of the classroom activity and make adjustments to the activities as needed. This results in modification of learning goals and instructional activities on an almost daily basis. In the end, the instructional sequence is reconstructed with activities that the research team believes will accomplish the learning goals (Gravemeijer, 2004). Thus, all the researchers, including the classroom teacher, participate in theory development. In practice, this means that as the teaching takes place the researchers look for indications whether the students' thinking matches the thinking that was anticipated. These observations become the basis for modifying conjectures about students' thinking. So, instructional activities are tested, revised, and designed on a daily basis (Gravemeijer). In addition to the daily analysis, there is retrospective analysis.

This retrospective analysis shapes the redesign of the instructional sequence (Gravemeijer et al., 2003; Steffe & Thompson, 2000). Thus a method, such as video taping, to record what happens in the teaching episode is critical. In this stage of the research, the video tapes are analyzed with respect to students' learning. The original interpretations may be either confirmed or modified in some way. There may also be an opportunity for new interpretations. The retrospective analysis aids the researchers in developing models of the students' learning throughout the teaching experiment. In this teaching experiment, the retrospective analysis provided a means to determine what norms and practices emerged in the class. The next section details the phases of DBR.

Phases of Design-Based Research

Recall that DBR is an iterative process whereby an instructional theory is constructed in the context of the classroom. It involves a repeated process of developing instructional sequences, testing the sequences in classroom instruction, and analyzing the learning. Revisions to the sequence are made based on the analysis and the process begins again (Gravemeijer et al., 2003). Design experiments allow researchers to develop instructional designs and study the learning that takes place. Further, they have a theoretical and a practical goal in that they result in a domain-specific instructional theory as well as instructional design (Cobb, 2003).

Gravemeijer (2004) identified three phases of this type of design research. They are: (a) developing a preliminary design, (b) conducting a teaching experiment, and (c) completing a retrospective analysis. In the preliminary design phase, a conjectured local instruction theory is formed (Gravemeijer). Simon (1995) calls this the HLT. It is the pathway the instructor expects the learners to take on the way to accomplishing the intended learning goals. It is a conjectured path and is subject to change based on how students develop understanding. Cobb (2003) describes an HLT as “an envisioned sequence of mathematical practices together with conjectures about the means of supporting and organizing the emergence of each practice from prior practices” (p. 11). According to Cobb, the HLT addresses not only the tasks to support students’ learning, but considers the tools students may use, the classroom discourse, and activity structure.

Once the HLT is determined, the instructional phase of the teaching experiment begins. In this phase, the teacher engages in a series of teaching episodes planned to meet the learning goals (Gravemeijer, 2004). In the teaching episodes, the research team is constantly analyzing the understanding of the students. The path changes if the conjectured learning trajectory does not result in the expected outcomes. Therefore, adjustments are made to the activities as the HLT is implemented (Cobb, 2000).

The retrospective analysis phase of the teaching experiment allows researchers to construct an instructional sequence that is comprised of effective activities (Gravemeijer, 2004). Such analysis should be conducted with respect to several criteria (Cobb, 2003; Cobb et al., 2001; Gravemeijer, 2004). The first of these is that the results should provide feedback so the instructional design can be improved. Second, the methodology should allow for the collective mathematical learning of the class to be documented. Finally, the analysis should allow for documenting the development of individual students as members of the community. The following section describes the process of planning the HLT.

Planning for the Hypothetical Learning Trajectory

The HLT was developed and adjusted based on information from several sources. First, the initial sequence was based on research findings about fractions. A detailed discussion of these findings appears in Chapter Two. This section details other factors that had an impact on the initial HLT. These include the means of support, principles of Realistic Mathematics Education (RME), and exploratory teaching. It is important to consider the means of support when planning a HLT because some of them are not typically considered when planning classroom experiences. Instructional tasks and tools are generally considered, but the classroom discourse and activity structure are not. These means of support are central to the HLT (Cobb, 2003). The research team was guided by principles of RME in developing the HLT in that they made an effort to embed the mathematics in realistic contexts so students would be able to reinvent the mathematics for themselves. The team also needed to consider what tools and models may emerge from the activities. Finally, the exploratory teaching was a source of input for the planning of the HLT. Some tasks from previous semesters were omitted, and new ones were added.

Means of Support

Cobb (2003) identifies four means of support in teaching experiments. They are: (a) the instructional tasks, (b) the tools students use, (c) the nature of the classroom discourse, and (d) the classroom activity structure. He discusses means of support independently, yet makes the point that they are interrelated. In fact, he suggests that they be viewed as a single classroom activity system.

Instructional tasks include the activities in which students engage in order to develop their reasoning. These activities should be designed to further the students' conceptual developments as well as to be problematic for students. Problematic situations provide learning opportunities for students. Such situations may present a variety of opportunities to students. These would include: "(a) resolving obstacles or contradictions that arise when they attempt to make sense of a situation in terms of their current concepts and procedures, (b) accounting for a surprising outcome, (c) verbalizing their mathematical thinking, (d) explaining or justifying a solution, (e) resolving conflicting points of view, or (f) developing a framework that accommodates alternative solution methods and formulating an explanation to clarify another child's solution attempt" (Wood, Cobb, & Yackel, 1995, p. 413). In planning the instructional tasks in this teaching experiment, the research team focused on how the task might further the students' conceptual understanding and attempted to create tasks that would be problematic and evoke discussion to push conceptual development even further.

Cobb (2003) identified tools as a second means of support. These tools serve to support students in the reorganization of their reasoning. Stephan (2003) lists physical materials, tables, pictures, computer graphs and icons, and both conventional and nonstandard symbols as tools. Gravemeijer (2004) suggests that students should invent the tools they use, but this is not feasible. Instead, teachers should carefully introduce new tools as a solution to a problem. That is, students should be given a problem and an opportunity to think about a solution. After this opportunity to think and discuss, they should be asked to evaluate if the new tool would be an acceptable solution to the problem. In this way, tools emerge somewhat from students' activity. Gravemeijer also recommends that the use of tools should be grounded in imagery. Supporting this notion is Thompson (1996), claiming that "mathematical reasoning at all levels is firmly grounded in imagery." Thompson does not limit imagery to mental pictures. He considers images to include sensory and affective fragments of individual experiences. For example, if a child's parent had divided a piece of cake into two equal shares, that experience may be recalled and used as imagery. The experience, not an isolated mental picture, is recalled. Since imagery can be from many sources, it is very individualistic. Students who have shared common experiences in instruction may recall similar images. Further, images contribute to understandings and these understandings, in turn, contribute to strengthening the imagery. So, tools and imagery were considered when planning for the HLT in order to create common experiences for students to recall. This was to facilitate the students' reasoning with tools and imagery as they reorganized their conceptions of fractions.

The classroom discourse is based on certain standards, or norms (Cobb, 2003). These norms set the expectations for the discourse and are found in two categories. Social and sociomathematical norms are central to the participation structure in the classroom (Cobb et al., 2001; Cobb & Yackel, 1996; Gravemeijer, Cobb, Bowers, & Whitenack, 2000; Stephan & Cobb, 2003). Examples of norms include expectations that students: (a) explain and justify their reasoning, (b) listen to others' explanations, and (c) indicate when they do not understand something and ask questions to clarify it. This teaching experiment fostered the establishment of social and sociomathematical norms. The specific norms are discussed in detail in Chapter Four.

Activity structure refers to how the classroom is organized. For example, instructional activities may be completed by small groups of students or presented to the whole class (Cobb, Yackel, & Wood, 1995). The activity structure of the class certainly impacts the discourse that occurs. The activity structure in this teaching experiment included small group work and whole class discussion as described in Chapter One. The instructor purposefully selected various interpretations and solutions to be shared with the class. This contributed to the development of mathematical practices which are detailed in Chapter Four.

These means of support were considered in the planning of the instructional sequence. The HLT discussed later in this chapter included the tasks, tools and imagery, and possible discourse topics. As stated earlier, the means of support are all interrelated and work together to create a classroom activity structure (Cobb, 2003). Another aspect of that structure was RME. This is discussed in the following section.

Realistic Mathematics Education

Gravemeijer (1999) writes that there is a reflexive relationship between developmental research and RME. He explains that RME theory guides design and research while developmental research contributes to refining the RME theory. (More details of RME theory follow this paragraph). Further, he states that the goal of RME is to present mathematics education to students in a way that facilitates their reinvention of mathematics. This goal is compatible with the instructional goal of the teaching experiment class, which was to deepen prospective teachers' understanding of mathematics. The reinvention of mathematics associated with RME would require deeper understanding. This section describes three principles of RME.

The RME theory emphasizes the construction of mathematics as opposed to the reproduction of mathematics (Streefland, 1991). The theory is founded on the idea that mathematics is a human activity and three principles provide the basis for the process called mathematization (Gravemeijer et al., 2003). These principles are: (a) guided reinvention, (b) didactic phenomenology, and (c) mediating, or emergent, models (Gravemeijer, 1998; 2004).

Guided Reinvention

The guided reinvention principle considers how specific mathematical practices evolved through history. The designer then conjectures whether the students will follow similar paths of development, including potential roadblocks and breakthroughs (Gravemeijer et al.,2000). The goal for students is to develop solution methods. This differs from a goal of simply teaching strategies to students (Gravemeijer, 2004). Students' informal strategies are a starting point for them to mathematize a contextual problem. In doing this, they make the problem accessible for a mathematical approach. This brings their mathematical activity to a higher level (Gravemeijer, 2004). It should be noted that the emphasis is not on mastering small pieces of the concept in a hierarchical manner. Instead, the focus is on gradual development and increasing sophistication to the subject matter (Gravemeijer, 1998). Following the guided reinvention principle, the teaching experiment conducted for this study encouraged students to mathematize situations to develop the mathematics they needed to solve problems. The goal was for students to organize their ideas into increasingly sophisticated ways of reasoning about fractions.

Didactic Phenomenology

The second principle, didactic phenomenology, suggests that students' mathematical development is promoted through engaging in problems that make the mathematics accessible. The problems should be experientially real for the students. That is, students need to be engaged in activities that are meaningful for them. They don't need to actually have had the real experience, but they do need to be able to imagine themselves in the situation or role presented by the problem. Thus, the problems are grounded in real-world situations that allow students to develop increasingly sophisticated solutions (Gravemeijer et al., 2000). It is important to distinguish between problems that afford students the opportunity to construct the mathematics and problems that attempt to teach abstractions by concretizing them. With the former, students must organize the mathematics for themselves and manipulate their means of organizing. In the latter, students are given a concrete embodiment of an abstract concept and do not engage in organizing the mathematics for themselves (Gravemeijer, 2004). Gravemeijer (1998) also calls for presenting problems to students before they have learned the standard procedure or concept. This gives the teacher an opportunity to observe the students' informal strategies for solving the problem. Since the second principle of didactic phenomenology calls for making problems experientially real for students, the learning tasks in this teaching experiment were presented to students within a contextual setting. When problems were not presented in a context, students were encouraged to create one.

Emergent Models

The emergent model principle calls for students to model their mathematical activity (Gravemeijer, 2004). When students model their own mathematical activity, the model will develop into a model for more formal mathematics. In this case a model refers to “a task setting or to a verbal description as well as to ways of symbolizing and notating” (Gravemeijer et al., 2000, p. 240). In RME students’ informal activities provide a foundation on which to build up to conventional ways of symbolizing. In more traditional instruction, the instructor may show students how to use manipulatives such as fraction circles, to model fractions and operations with fractions. This is what Gravemeijer et al. (2003) refer to as concretizing the formal abstract mathematics. That is, the models embody the mathematics to be taught as perceived by a person that already understands the mathematics. On the other hand, in RME, the models are generated by students as a means of organizing their mathematical activity. This leads to a series of symbolizations that becomes a chain of signification for a concept.

Use of models in mathematics classrooms is a complex process. One goal of the designer is to facilitate in students’ transition from representation serving as a model of their informal activity to being a model. This transition can be explained by four activity levels. These levels are: (a) activity in the task setting, (b) referential activity, (c) general activity, and (d) more formal mathematical reasoning (Gravemeijer, 2004; Gravemeijer et al., 2000). In this process, the model arises out of a context. This contextual reference gives the model meaning. Generalization occurs after several similar problems are solved. Eventually, the mathematical activity is no longer dependent on the model (Gravemeijer, 1998, 2004; Gravemeijer et al.). A more detailed description of the activity levels follows.

As stated previously, the first level involves a contextual problem. In RME, the situations are experientially real for the students (Gravemeijer et al., 2003; McClain & Cobb, 1998). This means that the students should be able engage in meaningful activity. This does not necessitate the experiences being ones that students have had or could have had. Instead, students need to be able to imagine themselves in the scenario. At this level involving a contextual problem, understandings depend on the specific setting, often outside of school. This activity in the task setting enables students to create situation-specific imagery (Gravemeijer, 2004; Gravemeijer et al., 2000). At the referential level the models are grounded in students' understanding of experientially real settings. These are instructional settings, mostly in school (Gravemeijer et al.). The model has meaning for students because it signifies the activity in the contextual setting (Gravemeijer). The next level, general activity, sees students' reasoning shift away from dependency on situation-specific imagery. Students begin to reason about the mathematical relations involved, and the meaning of the model is tied to this framework of relations (Gravemeijer). Finally, the students no longer need the model (Gravemeijer). At this level of reasoning with conventional symbolizations, the model "serves more as a means of mathematical reasoning than as a way of symbolizing activity grounded in specific settings" (Gravemeijer et al., p. 243). Students experience a gradual change as they move from situational, to referential, to general, and finally to formal mathematical activity as their mathematical reasoning develops.

An example may clarify this process. An activity in the task setting may be cutting a pizza into equal-sized pieces so a given number of people can have equal shares. Here the model is tied to a real setting. At the referential level, models are grounded in experientially real settings. Thus, students may recall the pizza experience if they are asked to cut a cake so that a certain number of people will get an equal share. Students engaging in activity at the general level begin to reflect on their referential activity. Thus, general activity may be expressing the relationship that as the number of pieces a whole is divided into increases, the size of each slice decreases. This would be a reasonable generalization from the partitioning experiences previously mentioned. Finally, more formal mathematical reasoning occurs when students can express relationships such as $\frac{1}{2}$ is greater than $\frac{1}{4}$. This is an application of the generalization that the greater the number of pieces a whole is divided into, the smaller the pieces are. In this example, students shifted from reasoning tied to an experience of equally sharing a pizza to reasoning about the amounts named by fractions. This is why models are an important consideration for an HLT. The final influence for the initial HLT was the exploratory teaching that took place prior to the teaching experiment.

Exploratory Teaching

Prior to the teaching experiment, some members of the research team were involved in what would be considered exploratory teaching, following the suggestion of Steffe and Thompson (2000). They encourage any researcher who has not conducted a teaching experiment to first engage in exploratory teaching. The purpose of exploratory teaching is to become acquainted with students' thinking and reasoning with respect to a domain of mathematics—fractions in this case. This phase of research also serves to uncover any issues that result from the teacher educator changing his or her style of teaching. This exploratory teaching took place during the two semesters immediately preceding the teaching experiment. Members of the research team observed the classes with the same instructor as in this experiment and met to plan and debrief between classes.

The exploratory teaching guided planning for the HLT in more than one way. First, it allowed the researchers an opportunity to observe and make changes perceived as needed before the actual data collection phase. Two doctoral students worked with the instructor to develop the instructional sequence for the first semester. Based on observations and students' work, minor changes were made to this sequence for the second semester of instruction. A summary of the instructional sequences for all three semesters is shown in Table 8.

Both instructional sequences of the exploratory teaching semesters began with some scenarios that involved fractions and questions for students to answer. Some changes were made to the wording of these scenarios for clarity. The next task in each semester was about equal sharing situations. In the first semester, the students completed this in class. It was completed as homework in the second semester and discussed in class. This was done in order to use class time for other activities. The next task was to use manipulatives to compare fractions. In the second semester, an effort was made to include different types of models (area, set, and linear). This was followed by an activity in which students located fractions on a number line in the first semester. In the second semester, the number line activity was moved to be after the comparing without manipulatives task, and was preceded by a task in which students found fractions between two given fractions. From that point on the tasks in the first two semesters were the same. Some tasks were not included in the teaching experiment semester. These were the initial fraction situations activity, comparing fractions with manipulatives, problems on equivalent fractions, and representing fractions with two-color counters. Several tasks were added to the instructional sequence for the teaching experiment. These are denoted with an asterisk (*) in the summary table.

Table 8: Summary of Instructional Sequences for Fraction Concepts and Operations

Exploratory Teaching—Semester 1	**Exploratory Teaching—Semester 2**	**Teaching Experiment—Semester 3**
Fraction Situations	Fraction Situations	On the Bus*
Sharing Problems	Equal sharing as take home	Familiar Fraction Situations and Models*
Compare fractions with manipulatives	Compare fractions with manipulatives	How Much?*
Number Line Challenge	Compare fractions without manipulatives	Equal Sharing
Compare fractions without manipulatives	Between	The Candy Bar*
Homework	Homework	Number Line Challenge
Problems on equivalent fractions	Number Line Challenge	Between
Represent equivalent fractions with two-color counters	Problems on equivalent fractions	Compare
Addition	Represent equivalent fractions with two-color counters	Fraction Kit
Subtraction	Addition	Introduction to Addition and Subtraction
Practice Problems	Subtraction	Addition and Subtraction Practice
Multiplication and Division Situations	Practice Problems	Practice Problems
Multiplication with manipulatives	Multiplication and Division Situations	Multiplication and Division Situations
Multiplication practice	Multiplication with manipulatives	Practice Problems
Reading and problems from book	Multiplication Practice	Multiplication
Division Sharing and Measurement Situations	Division Sharing and Measurement Situations	Practice Problems
Division problems with manipulatives	Division problems with manipulatives	Division
Division Practice	Division Practice	Practice Problems
		Estimate Products and Quotients*

* Denotes new tasks for the teaching experiment

The exploratory teaching semesters also guided planning in that a prestudy instrument (similar to the instrument shown in Appendix A) was administered during the first semester so it could be revised if necessary. The purpose of the instrument was to gain information about students' knowledge of fractions before the instruction began. This information would be used in planning the HLT for the teaching experiment. A slight modification was made to questions that asked students to compare fractions. In the prestudy instrument, students were asked to model the fractions with a manipulative then sketch their models as part of their explanations. In the instrument used for the pre-instruction questionnaire for the teaching experiment class, students were simply asked to name the greater of two fractions and justify their answers. This change was made because during the two semesters of exploratory teaching, the research team changed their view of how students should interact with tools such as manipulatives.

The post-instruction questionnaire for the exploratory teaching was the pre-instruction questionnaire, with a few additional questions to gather data on students' opinions with respect to fractions and the teaching in the class. These questions probed students' comfort level with mathematics and fractions. Students were also asked what they liked and disliked about the class. For the teaching experiment class, a question was added that asked students how important they thought it was to explain how they arrived at an answer in math class. The questions about what they liked and disliked about the class were not included, because the information collected in the exploratory teaching semester was not judged to be useful by the research team. The post-instruction instrument for the teaching experiment is shown in Appendix B. Additional information was collected from students' responses to final exam questions (Appendix C). The next section details the implemented HLT for the teaching experiment semester.

The Implemented Hypothetical Learning Trajectory

The HLT implemented resulted from observations in exploratory teaching as well as a review of research on children's and teachers' thinking and knowledge of fractions. Means of support and RME were also considered. This particular HLT evolved over the fraction instruction part of the course. That is, although the team had a general notion of the broad goals, the specific activities often were not finalized until a day or two before they were presented to the class. The pace at which the class gained understanding of concepts drove the progression of the activities. Often, more activities were planned for a class session than time allowed to complete. This resulted in delaying some activities until the next class session. While time constraints had an impact on the delivery of the instructional sequence, there was continuity from teaching episode to teaching episode.

Five learning goals, or big ideas, were at the center of this HLT. These were determined to encompass the unifying elements discussed in Chapter Two. Furthermore, the concepts and tasks were chosen in order to address specific deficits of teachers and students reported in Chapter Two. However, this does not mean that every concept or task is included just to address a deficit. There was consideration given to building a complete understanding of fractions. Therefore, concepts and tasks that the research team thought would accomplish that were grouped into the five learning goals. They were: (a) using fractions to name amounts, (b) understanding differences between whole number relationships and fraction relationships, (c) replacing rote procedures with reasoning to build meaning, (d) reasoning with addition and subtraction, and (e) reasoning with multiplication and division. Each of these learning goals was associated with several concepts and comprised one phase of the HLT. The concepts are more specific ideas about the intended learning for the students. A more detailed description of the HLT follows, and Table 9 summarizes the implemented HLT.

Stage One: Using Fractions to Name Amounts

The first stage of the HLT focused on fundamental concepts important to understanding fractions. Stage One was more complex than the learning goal associated with it may seem to indicate. Using fractions to name amounts included relational thinking, partitioning, modeling fractions, different interpretations and models of fractions, the concept of unit, and the relationship of the number of pieces in the whole to the size of the pieces. Based on the results of the pre-instruction questionnaire, most of the students in the research class could name fractional parts of areas before instruction began, but they may not have been explicitly aware of these fundamental ideas. For this reason, the first stage of the HLT was devoted to basic ideas needed in order to build the desired deep conceptual understanding.

Table 9: Implemented HLT

Stage	Learning Goal	Concepts	Supporting Tasks	Tools and Imagery	Possible Discourse
One	Using fractions to name amounts	Relational thinking	On the Bus	School bus	Fullness is relative
		Different interpretations and models of fractions	Familiar Fraction Situations and Models	Drawings	Different interpretations for fractions
		Significance of and defining the whole	Unit	Squares—area model; Pattern blocks	A fraction is named relative to the unit.
		Modeling a fraction amount given the value of a model	How Much?	Set and area models	Fraction representations are relative to the whole
		Partitioning; relationship of number of pieces to size of pieces; and fractions have equal parts	Equal Sharing	Area models representing cookies or candy bars	Fractions need equal-sized pieces; as the number of pieces increases, the pieces get smaller; there is more than one way to divide a whole into fair shares
Two	Understanding differences between whole number relationships and fraction relationships	Equivalent values	The Candy Bar	Candy bar divided into 16 sections	Name an amount with several fraction names; Efficient ways to make fair shares
		Relationships among fractions; compare and order; and density of fractions	Number Line Challenge	Number line	Linear representation of fractions; sequencing fractions; equivalent fractions; relationships of fractions
		Name fractions between; density; and relationship of number of pieces to size of pieces	Between	Number line	Making smaller pieces to have equal pieces in different fractions
Three	Replacing rote procedures with reasoning to build meaning	Relative magnitudes of fractions and relationship of number of pieces to size of pieces	Compare	Models—area, set, number line	Relationship of size and number of pieces; benchmark values
		Relationships and create a tool for operations	Fraction Kit	Area model and colors	Modeling fractions; add and subtract fractions

Stage	Learning Goal	Concepts	Supporting Tasks	Tools and Imagery	Possible Discourse
Four	Reasoning with addition and subtraction	Solve addition and subtraction of unlike fractions given in context	Introduction to Addition and Subtraction	Context of problem; drawings	Need a way to express the answer (common denominators); using different models for area, set, and linear situations
		Model addition and subtraction with fractions; understand why same-size wholes are necessary to add and subtract fractions; and develop meaning and make sense of addition and subtraction with fractions	Addition and Subtraction	Creating contexts	Improper fractions; regrouping in subtraction;
		More modeling of addition and subtraction (Extra practice)	Practice Problems		
Five	Reasoning with multiplication and division	Develop meaning and make sense of multiplication and division with fractions	Multiplication and Division Situations	Context of problem; Models—set and area	Need to define the whole in order to describe the remainder; measurement model for division; sharing model for division
		Understand the effect of multiplying with fractions	Estimate Products		The effect of multiplication with fractions
		Model products with fractions	Multiplication	Creating contexts; drawings	
		More modeling of products (Extra practice)	Multiplication Practice Problems		
		Model division with fractions	Division Situations	Context of problem; models	Define the unit to name the remainder
		More modeling of division with fractions (Extra practice)	Division Practice Problems	Creating contexts; modeling	
		Reinforce and assess understanding of the effect of multiplication and division	Estimate Products and Quotients		The effect of multiplication and division with fractions

The instructional sequence began with On the Bus (Appendix D), a task designed to highlight the relative nature of fractions. Students were presented with two scenarios about how crowded a school bus was. The instructor invited the class to ask questions about the scenarios with the intent that they would realize that how crowded a bus is depends, not only on the number of people on the bus, but on the size of the bus as well. This was designed to point out the difference between absolute and relative amounts and to serve as an introduction to relative thinking (Lamon, 1999). Once students realized that fractions provide a useful way to express relational comparisons such as how crowded a bus is, they were challenged to think of similar situations in which fractions are a good way to express relational values. This task, Familiar Fraction Situations and Models, was intended to lead into a discussion of different interpretations for fractions. The instructor visited with small groups to find examples of the part-whole, measure, and quotient interpretations of fractions. Although a whole-class discussion was planned to highlight the differences among the interpretations, it did not occur and any thoughts about different interpretations were not explicitly addressed. The common attribute of the fraction situations created by the students was that they answered the question "How much?" rather than "How many?" This was addressed in the subsequent discussion. The research team conjectured that students would use drawings to communicate their fraction situations in this task, thus introducing tools and imagery to this sequence.

The purpose of the next task, Unit (Appendix E), was for the class to realize the importance of defining the unit. A square with three of four equal parts shaded was displayed. The students were asked to write the amount they thought was represented. The research team conjectured that they would write $\frac{3}{4}$, since the square was shown in the conventional representation for $\frac{3}{4}$. After polling the class, the instructor asked if anyone wanted to change their answer, conjecturing that most would stay with $\frac{3}{4}$. At that time, the instructor showed two squares the same size as the original one and told the class they represented the unit. Students were asked again if they wanted to change their original answers about the shaded part of the square. Following this discussion, similar problems with pattern blocks were presented to the class. The pattern blocks introduced a tool students could use in their reasoning about fractions. For example, students were shown a yellow hexagon and told that it represented the whole. They needed to tell what part of the whole another block, such as a red trapezoid would be. They had pattern blocks available to them when completing this task. The intent of the discussion related to these problems was to have students realize the importance of the unit when working with fractions. This concept is basic to naming fractions, and was discussed again with comparing, adding, subtracting, and dividing fractions. The square and pattern blocks provided imagery for the students to use in later tasks.

With their realization about the importance of the whole, students moved on to constructing an amount given a value as they worked on a task called "How Much?" (Appendix F). For example, students were given four squares that together represented $\frac{1}{2}$. They were asked to construct another value, such as $1\frac{3}{4}$. The researchers conjectured that this task would begin to challenge some of the students if they lacked conceptual understanding. The instructor and observers visited with the different groups and made note of their strategies and solutions. A whole-class discussion highlighted the strategies and explanations. The researchers conjectured that procedural knowledge of fractions would possibly interfere with this task. That is, students may try to find common denominators as a starting point instead of reasoning about fraction concepts. A discussion followed, with the intent of helping students accept that procedures are not necessary to solve these problems.

In the final task in Stage One, Equal Sharing (Appendix G), students were asked to partition sets among different numbers of people. This task was designed to prompt students to think about three ideas. The first of these was that fractions are based on equal-sized parts. The second was that the more parts a whole is divided into, the smaller the parts are. The final idea was that there is more than one way to divide the whole into fair shares. The researchers conjectured that some of the students in this class may use the largest pieces possible in forming the equal shares. In fact, they thought if students did not use the largest pieces possible, they may lack conceptual understanding. The subsequent discussion addressed the different strategies in order to accomplish the fair sharing. Many students made drawings to help in their thinking. These tools would provide imagery to be called upon later. The task also called for students to explain their reasoning. This was intended to provide students with an opportunity to write about their thinking. The research team hoped this would help them distinguish between conceptual explanations and simply writing what procedures were performed. This was the last task presented in Stage One of the HLT.

Stage Two: Differences between Whole Numbers and Fractions

The focus for Stage One of the HLT was to develop basic concepts that are fundamental to fractions. In Stage Two, students were able to extend their understandings and explicitly investigate ways in which reasoning with fractions needs to be different from whole numbers. This includes realizing: (a) that there are infinitely many fractions between any two given fractions, (b) comparing denominators requires an inverse logic compared to whole numbers, and (c) many names can represent the same amount. These concepts are grouped together because many children demonstrate misconceptions based on over-generalizing whole number concepts to the fraction domain. Research has shown teachers may share the same misconceptions.

The first task in Stage Two was the Candy Bar (Appendix H). It followed the equal sharing task for two reasons. First, it was related to fair shares. Second, it was designed to explicitly name an amount with several fraction names. In the last task of Stage One, students may have begun to explore different names for the same amount. The Candy Bar task reinforced these ideas, and required students to think about using different fraction names for the same amount. The task was to explain how four different people could have had the same amount of a candy bar even though their shares were different shapes and had different fraction names. Again, the research team conjectured that students would possibly rely on procedural approaches to verifying that the amounts were the same. Therefore, the instructor clarified that the explanations were to be conceptually-based and not rely on rote procedures. The task was an opportunity for students to explore efficient ways to make fair shares. That is, it is more efficient to divide a candy bar into four large pieces rather than 16 smaller pieces then deal the 16 pieces out.

Having been introduced to partitioning and equivalent fractions, students completed the Number Line Challenge (Appendix I) next. This task required students to reason about the location of fractions on the number line. They needed to know how to use equivalent fractions and distance from a given point in order to locate missing fraction values on the number line. Students were given a number line with 0, $\frac{1}{5}$, $\frac{2}{3}$, and 1 marked. They were instructed to find other values on the number line. This task was designed to promote thinking about relationships among fractions. For example, $\frac{1}{5}$ is the same distance from 0 on the number line as $\frac{4}{5}$ is from 1. The completed number line would show the relative positions of several fractions. In the follow-up discussion, students were asked why some fractions were seemingly omitted. The purpose of this query was to prompt students to discuss equivalent values on the number line. In the number line activity, students began to think about the structure of fractions and relative magnitudes of fractions.

In the next task, Between (Appendix J), students were given pairs of fractions and asked to find from one to three fractions between the two values. It followed the number line task because the research team thought students would possibly recall images of the number line to help organize their thinking. The task began with a relatively simple pair of fractions and required increasingly sophisticated reasoning to complete it. The students were told that they needed to explain their solutions conceptually because the research team did not want students to simply find common denominators. Instead, the team thought students would rely on drawings for tools and imagery to help them complete the task. This may have introduced them to reasoning about common denominators as a concept. That is, if common denominators were used, it would be in terms of making smaller pieces until each fraction had the same-sized pieces. This task concluded Stage Two of the HLT.

Stage Three: Using Reasoning Instead of Algorithms

In Stage Three of the HLT, students continued reason about fractions rather than performing rote procedures. The intent of this stage was to have students apply what they knew about fractions to solve problems instead of simply using known algorithms without meaning. The results in Chapter Four discuss how this actually became a negotiated norm. It is addressed here because it was a stage in the initial HLT. The focus was on relationships between fractions, beginning with comparing fractions (Appendix K). This task required students to compare two fractions to determine which was greater. The fraction pairs were chosen to facilitate reasoning strategies similar to those used by children (Behr et al., 1984; Lamon, 1999) Students were told that drawings or manipulatives may not be an adequate justification of answers for this task. When the numbers of pieces are close to the same size, drawings cannot be accurate enough to judge size. Manipulative pieces may also be too close in size to definitively judge. Thus, students were pushed to use other tools and imagery to complete the task and provide explanations of their reasoning.

The comparing fractions task completed the actual concepts portion of the instructional sequence. The next task served as a bridge between the concepts and operations portions of the fraction unit. The task was to make a fraction kit (Appendix L). The fraction kit is a tool with which students could further explore the relationships discussed to this point. Due to time constraints in the class, the fraction kit was given to students to complete at home independently. There was little follow-up discussion in class about it, but it was intended to be used in addition and subtraction, which would begin the next stage of the HLT.

Stage Four: Addition and Subtraction

Stage Four of the HLT began the operations portion of the fraction unit, specifically addition and subtraction. The first task (Appendix M) in this stage was an introduction to addition and subtraction. Students were given an addition problem in context. They were asked to solve it in small groups without using known algorithms. Afterwards, they shared their strategies. The research team conjectured that students would possibly struggle with how to express the answer; and this perturbation would provide conceptual underpinnings for why common denominators are used with the addition and subtraction algorithms. Two more problems were presented in the same manner. The three problems were designed to illustrate an area model, linear model, and set model of fractions. These different situations were presented in order to expose students to a variety of models.

After working with addition and subtraction in context, students were asked to work problems without a context (Appendix N). The instructor encouraged them to create scenarios for the problems. The purpose of this was twofold. First, putting the problem in a context would give students something by which to judge the reasonableness of their answers. Secondly, it would give them practice making up problems for their students and help to reinforce the meaning of the operations for them. A class discussion addressed the strategies and reasoning they used. One of the problems resulted in an answer greater than 1. This was included to introduce the concept of improper fractions to the students. In a similar fashion, they encountered subtraction problems with fractions that required regrouping. The research team conjectured that solving the problems conceptually would lead to generalizations about the reason for the familiar algorithms. These addition and subtraction tasks were followed by the operations of multiplication and division.

Stage Five: Multiplication and Division

In this final stage, multiplication and division were explored. The instructional sequence in this stage began with multiplication and division situations (Appendix O), presented to students in context. Again, students were asked to model the situation rather than use known algorithms. The research team thought they might struggle with representing multiplication, and students might confuse the multiplication and division operations. The division situations were designed to prompt a discussion about the importance of defining and knowing what the unit is. The subsequent discussion highlighted these ideas.

The next task in Stage Five was to practice estimating products. In this task, students previewed the multiplication practice that would be assigned next. They were instructed to look at each problem and estimate the product as being less than or equal to 1, between 1 and 2, or greater than 2. The intent was to reinforce the effect of multiplication with fractions, namely that multiplication does not always make bigger. Additionally, it suggested that estimating is a good strategy to use when performing any computation. If students estimate results, they should be more likely to know when they have made errors.

The next task was on multiplication with problems presented without context (Appendix P). Students were again encouraged to create an appropriate situation for the problems. They were to model the problem and be prepared to explain their reasoning to the class. Additional practice problems (Appendix Q) were also given to students who wanted more practice. This consisted of two contextual problems and two problems without context. Discussion on all of these problems highlighted the strategies students used and the obstacles they may have encountered.

Two division situations comprised the next task (Appendix R). By design, one situation was a quotitive, or measurement, model of division and the other was partitive, or sharing, model. With the former, the total number and the number in each group are known. In the latter, the total number and number of groups are known (Behr & Post, 1992). The first problem was the measurement situation. There were two questions, and the unit was different for each of the questions. This was done to prompt students to think about the concept of the unit as previously discussed. These problems were followed by division practice (Appendix S). Students again were encouraged to create scenarios to go with the division problems. Like other problems, they were to work them without using known algorithms. They also were to provide a model and be able to explain their thinking. Division practice problems (Appendix T) were given to students who wanted more practice. This problem set contained two problems in context and two without contexts.

The instructional sequence concluded with an estimation task (Appendix U). Students were shown multiplication and division expressions and they selected the best estimate for the answer from three choices. This task was intended to review multiplication and division reasoning. A discussion highlighted students' strategies and reasoning. More details and analysis of the tasks in the HLT are given in Chapter Four. A description of data collection methods is next.

Data Collection

Data collected included video tapes of the classes, questionnaires completed by students, and notes of observers in the classroom. Research team meetings held after each class session were also documented. Student work was collected as well. Each data source is further described in this section.

Each class session was video taped. Two cameras were used to capture the activity in the classroom. During whole class discussions one camera focused on the presenter (either the instructor or students) at the front of the classroom. This camera recorded what was written on the board or displayed on the document camera. The other camera focused on the interactions among the students in other parts of the room. When students were engaged in small group work, the cameras focused on specific groups. The goal was to capture as much of the class activity as possible on video tape. This enabled the subsequent analysis of the class session.

Recall that class members responded to a questionnaire (Appendix A) that was designed to assess conceptual understanding of fractions. The questionnaire contained questions about whole number concepts and operations as well as fractions. The whole number section did not pertain to this study, but was administered as a data source for another study being conducted in the same class. Questionnaires were completed at the beginning of the semester and after the relevant teaching had taken place. The post-instruction questionnaire was not identical to the first one. Questions similar to some of the initial questions appeared on the fractions test and the final exam. Additionally, some questions were omitted because students demonstrated a thorough understanding of the concepts on the initial questionnaire. Some questions were designed to collect demographic data and were not mathematically oriented. These were administered only once. Student responses to the mathematical questions at the beginning of the semester gave the research team some insight into students' current thinking and reasoning with respect to fractions and were used to guide them in developing the initial instructional sequence. The observers on the research team took field notes to further document the class activity. Field notes included the time frame for the activity, the mathematical activity taking place, and notes about the discourse. The intent was to document as much of the classroom discourse as possible in order to supplement the video taped record of the sessions.

The research team met after each class session to discuss the instruction and the students' mathematical development. These meetings resulted in changing the proposed instructional sequence, so it was important to document the thinking of the researchers in this process. The researchers kept notes of these meetings, and the meetings were audio taped. Additionally, the researchers kept journals as a record of their thoughts and reflections on the experiment. Artifacts of student work were collected to look for evidence of their understanding. This work was copied so the students could keep their original work. In-class work, homework, and tests were included. Some students submitted their class notes as well. The next section details how the data were analyzed for this teaching experiment.

Data Analysis

The data analysis was completed through the lens of an interpretive framework that coordinates the psychological and social aspects of learning. This emergent perspective is described in the next section. Analysis of the classroom activity contributed significantly to the results of this teaching experiment and is discussed after the interpretive framework. The specific methodology for this analysis is also discussed.

Interpretive Framework

Cobb and Yackel (1996) developed a framework that considers a psychological perspective and a social perspective to analyze individual and collective activity. These two perspectives together make up the emergent viewpoint on which this teaching experiment was founded. This emergent view was chosen for this study for several reasons. First, it coordinates a social perspective and a psychological perspective. That is, instead of considering the two perspectives as opposing viewpoints, they are seen in a reflexive relationship (Cobb & Bauersfeld, 1995). This means that neither individual activity nor collective activity of the class can be explained without considering the other.

Even though the analysis in this study considered only the social perspective, the coordination of social and individual perspectives was important. The research team acknowledged that this experiment was the beginning of a series of similar teaching experiments. Since this was the first experiment, the scope was limited in order to focus more on logistic challenges of the research and learning a methodology new to a majority of the team members. The HLT revisions suggested as a result of this experiment will be the starting point for another teaching experiment in which the individual and social perspectives may be analyzed.

The emergent perspective also provides a construct that allows for analysis of collective mathematical learning and resulting mathematical practices. This collective aspect was considered in this analysis. Cobb (2003) identifies a need to explicitly address collective mathematical learning and further states that the notion of mathematical practices was developed to answer that need. The emergent perspective framework meets that need.

Finally, the framework is compatible with the style of teaching used in the teaching experiment. Since the classroom activity structure emphasized explaining and justifying solutions and strategies, it was important to view it through a social lens. The social perspective constructs of the framework provided such a lens. Both social and psychological components of the framework are shown in Table 10.

Table 10: Interpretive Framework for Analyzing Individual and Collective Activity

Social Perspective	**Psychological Perspective**
Classroom social norms	Beliefs about our own role, others' roles, and the general nature of mathematical activity
Sociomathematical norms	Specifically mathematical beliefs and values
Classroom mathematical practices	Mathematical conceptions and activity

The components of the Social Perspective column were the focus of this study. Classroom social norms, sociomathematical norms, and classroom practices involve examining how students reason and argue in a classroom community (Cobb et al., 2001). More details of these components follow.

Classroom Social Norms

Social norms serve to describe the participation structure in the classroom (Stephan & Cobb, 2003). As students participate in the class, social norms are negotiated. Participation in the teaching experiment class occurred on two levels. First there was small group work. Students worked in small groups and shared their ideas and reasoning to complete relatively short tasks. There was an expectation that this time would be used, not only to make sense of the problems, but also to be sure students were able to explain their solutions and reasoning. The students frequently used tools (such as manipulatives or drawings) to model an expression or situation. They often recorded their solutions and explanations so they could present it to others in the class. Members of the small groups were frequently called upon to present their strategies to the whole class. Thus, discussion in the small groups also served to facilitate students' articulation of their own strategies and reasoning.

The second level of participation was at the whole-class level. There was an expectation that the students in the class would explain and justify their answers. When something was not clear to another student, that student needed to ask for further clarification. If a student did not agree with an answer or line of reasoning presented, that student needed to say so. Students were also encouraged to share alternate strategies for finding solutions. Additionally, they may have asked for help or suggestions when a strategy was not working out as expected. These expectations became social norms in the teaching experiment class. The specific classroom social norms that were negotiated for this class are detailed in Chapter Four.

Sociomathematical Norms

Sociomathematical norms focus on whole class discussions that deal specifically with the mathematics. Cobb (2000) cites examples that address mathematical solutions, particularly what counts as a different, sophisticated, or efficient solution. Another example of a sociomathematical norm would be what counts as an acceptable explanation and justification. This was important to this teaching experiment because many of the students may have been experiencing the process of explaining their thinking to others for the first time. Therefore, time was spent on the difference between justification and simply stating a procedure. Students came to accept that justifications did not need to be provided with words only. They may include drawings or models to help explain their reasoning. Like classroom social norms, the sociomathematical norms established in this teaching experiment are discussed in Chapter Four.

Classroom Mathematical Practices

Cobb et al. (2001) describe mathematical practices as “taken-as-shared ways of reasoning, arguing, and symbolizing established while discussing particular mathematical ideas” (p. 126). Mathematical practices do not require any justification. Students actively contribute to mathematical practices as they reorganize their knowledge (Cobb & Yackel, 1996). Thus, practices emerge from the activity in the class. Documenting classroom mathematical practices provided the researchers with information to answer the research question. An analysis of these practices reveals how the activities supported the learning of fraction concepts and operations. The following section details the specifics of the analysis.

Analyzing the Classroom Activity

This section addresses how the teaching episodes were analyzed to determine the classroom mathematical practices. It includes a brief discussion of argumentation followed by a methodology used by Rasmussen and Stephan (in press). Finally, the specific procedures used for this study are discussed including modifications to Rasmussen and Stephan's methodology.

Argumentation

Krummheuer (1995) refers to argumentation as primarily a social phenomenon when cooperating individuals present rationales for their actions in order to convince others to adjust their intentions. Further, argumentation arises when several participants engage in the interaction. Toulmin (2003) identifies four components of an argument in his discussion of argumentation. Three of these components make up what Toulmin refers to as the core of an argument. They are the data, the claim, and the warrant. The fourth component is the backing. The mathematical claim is what will need support. The claim is also called the conclusion by some (Krummheuer; Whitenack & Knipping, 2002). The data provide the initial support for the claim. A speaker may provide evidence for a claim, whether or not challenged. This evidence is referred to as data in Toulmin's argumentation scheme. If further clarification is still needed after the data are presented, a warrant is given. A warrant offers additional information about how the data support the claim. A backing may be given to explain why the warrant should be accepted. Figure 2 illustrates Toulmin's model with a mathematical example.

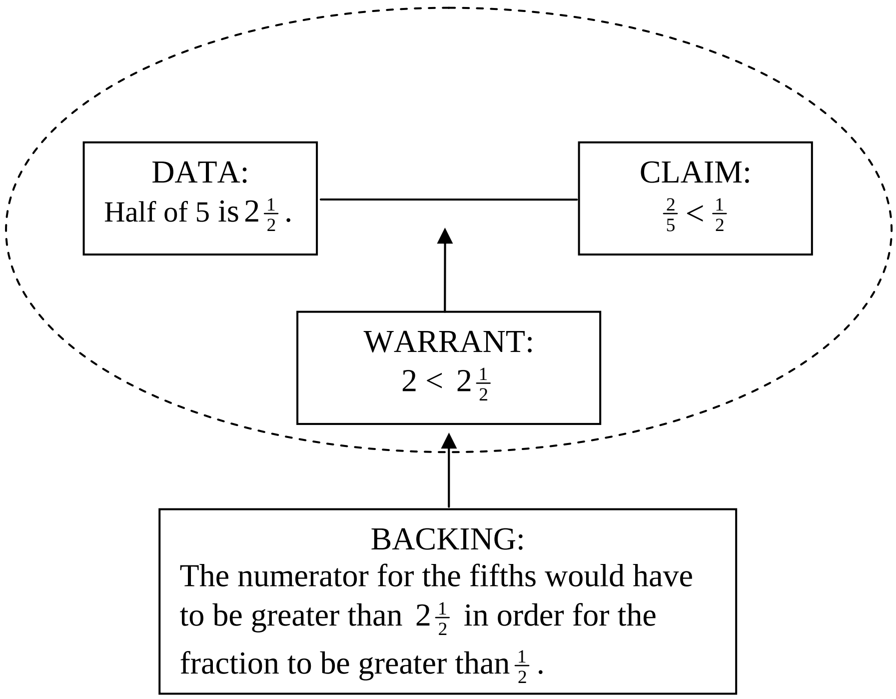


Figure 2: Toulmin's Model of Argumentation

In Figure 2, the Data, Claim, and Warrant make up what Toulmin (2003) calls the core of the argument. These three parts are necessary to have an argument. The claim is an assertion or conclusion. In this case, a student claims that $\frac{2}{5}$ is less than $\frac{1}{2}$. If that claim goes unchallenged, there is no argument. However, if that claim is challenged, it is necessary to provide data. In this case, "half of 5 is $2\frac{1}{2}$" serves as the data. This statement leads to the conclusion that $\frac{2}{5}$ is less than $\frac{1}{2}$. If the challenger does not understand how the statement of data leads to the conclusion, a warrant is provided. Here, the warrant states that 2 is less than $2\frac{1}{2}$. This serves to explain why the data support the conclusion. Finally, if the challenger disagrees that the warrant actually supports the data, a backing is necessary. The backing serves to justify the validity of the warrant. In this example, the backing explains that the numerator in $\frac{2}{5}$, 2, would need to be greater than $2\frac{1}{2}$ (half of 5) in order for the fraction to be greater than $\frac{1}{2}$. Since the numerator, 2, is less than $2\frac{1}{2}$, the fraction $\frac{2}{5}$ is less than $\frac{1}{2}$. With this, the validity of the core of the argument is also established. The next section will describe how Toulmin's model may be used to analyze the classroom activity.

General Methodology

Analyzing classroom discourse with respect to the argumentation that takes place is one way to determine what has become taken-as-shared knowledge. This leads to identifying classroom mathematical practices (Stephan & Cobb, 2003). This analysis took place within the format set forth by Rasmussen and Stephan (in press). Their approach for documenting collective activity consists of three phases. The first phase is to create transcripts of every whole-class discussion. This is followed by watching the video recordings and noting when a claim is made. Then Toulmin's model is used to create an argumentation scheme for each claim. This results in an argumentation log.

In the second phase, the argumentation logs are reviewed looking for evidence of ideas being taken-as-shared. A mathematical idea is considered taken-as-shared if the backings or warrants are no longer necessary. Another condition that indicates an idea is taken-as-shared is when any of the claim, data, warrant, or backing changes function. That is, if students use a claim that was previously justified in a different function, the idea is said to be taken-as-shared. Using the example in Figure 2, suppose later a student justifies a claim that $\frac{4}{5}$ is greater than $\frac{1}{2}$ by stating that the numerator for the fifths would have to be greater than $2\frac{1}{2}$ in order for the fraction to be greater than $\frac{1}{2}$, and is not challenged. This is an example of a former backing changing to function as data, and an indication that the idea of comparing a fraction to $\frac{1}{2}$ by using the value of half the denominator has become taken-as-shared.

This analysis of the activity is included in a mathematical ideas chart. This chart identifies ideas that are taken-as-shared. In addition, notes are included on ideas to keep an eye on. It is important to keep an eye on some ideas because they may become taken-as-shared and this helps to document the process. Finally, the chart also captures additional comments. Thus, the mathematical ideas chart serves as a record of how ideas that are taken-as-shared become so. In the third phase, the ideas that are taken-as-shared are organized according to the mathematical activity that was taking place when the ideas became established. The same process was followed in the teaching experiment with minor modifications. The following section details the specific processes used to analyze the data from this teaching experiment.

Specific Methodology

As stated previously, the methodology used followed that of Rasmussen and Stephan (in press). However, there were some modifications. Recall that the first phase is to create transcripts of whole-class discussions. This was done from the video tapes that recorded the class activity. Two tapes were used to record each class session, and a spreadsheet was created for each tape. The spreadsheet format was chosen so the researchers could readily locate the corresponding video to watch the episode if desired. For this phase, the spreadsheet contained columns for time, speaker, and what was spoken. The time column was in 5-second intervals that coordinated with the time of the video segment. After the transcripts were completed, attention turned to identifying claims.

Before work to identify claims began, members of the research team met to discuss Toulmin's argumentation model. The purpose of these meetings was to be sure everyone applied the same criteria in their analyses. Once claims were identified, the related data, warrants, and backings were identified. Several people analyzed each segment of the transcript. Then they met to discuss their analyses. If there was disagreement, they discussed it until a consensus was reached. This was done to prevent a single judgment from being accepted without review. This process, in which opinions were challenged, strengthened the analysis.

For tracking purposes, each claim was given a code number by which it could be easily located in the transcripts to be reviewed in context. The code contained the date of the class session and a designation for the video tape, the activity number, and sequential number indicating claims for that activity. For example, a claim coded 2A1.2b meant that it could be located in the part of the transcript for the first video tape on June 2. Further, the claim occurred during activity 1, specifically in the discussion of number 2. If there were multiple claims for a particular item, a letter denoting sequence followed the number. The "b" in this code indicates that the claim was the second claim for item number 2. This coding scheme allowed the researchers to quickly situate the argument within the instructional sequence.

Rasmussen and Stephan (in press) use the argumentation scheme to create an argumentation log. Rather than create a separate log, this information was entered into the spreadsheet in a column labeled “Argumentation.” This allowed the arguments to be reviewed in the context of the entire transcript rather than as an isolated segment. Then the ideas from the argumentation column were recorded in a column labeled “Ideas” in the spreadsheet. This column served as the idea log that Rasmussen and Stephan describe. The next step was to categorize these ideas according to the activity taking place in the class. This was noted in another column titled “Group.” The purpose of this was to categorize the ideas into similar groups and determine practices that emerged. The final step was to determine the practices, which are discussed in Chapter Four. This concludes the discussion of how the data were collected and analyzed. The next section addresses limitations and assumptions associated with this teaching experiment.

Limitations and Assumptions

Simon (2000) discusses several limitations of teaching experiments. First, as with any qualitative study, the knowledge and skills of the researcher are a factor. This was the first such study for this researcher which may limit the overall results. However, the team was not working without guidance, as there was an experienced researcher on the team. She was available during all aspects of the investigation from planning to analysis.

Simon (2000) warns that teaching experiments are labor intensive and costly, generating a great deal of data. The research team was small and there was no outside funding for the project. This may limit the study in that the video taping and transcribing was done by amateurs. In a related limitation, Simon states that a period of several years is appropriate for teaching experiments. This study was conducted over a six-week period in a summer session. However, others will follow up in subsequent teaching experiments.

It is important to note that the data analysis method was somewhat subjective, in that the argumentation schemes identified were not always clear. At least two researchers analyzed every segment of the transcript and conferred about their judgments regarding the argumentation. This meant that the conclusions were not made by an isolated person. At times, there were disagreements that were discussed and resolved by the research team members.

The class setting presented some limitations. First, the students were adult learners who had some knowledge of the content being taught. Therefore, they were not truly discovering the concepts and ideas for the first time, and it may be presumptuous to think that the established mathematical practices are due solely to the instructional tasks. Second, students were motivated by their grades in the class. This may have contributed to their willingness to discuss and explain their solutions. It is not certain if the same motivation to explain answers would have been seen had concern about grades not been present. The instructor may have required explaining when the class did not really need it. Thus, in analyzing the argumentation, some data and warrants may have been present when they actually were not necessary. This was taken into account and noted in the analysis.

Conclusion

This chapter reviewed the methodology used for this teaching experiment that took place in a mathematics content course for prospective elementary teachers. First, the setting of the teaching experiment was described. Then, the general methodology of DBR was discussed. Planning for the HLT was discussed. Then, the implemented HLT was described. Finally, procedures for data collection and analysis were shared. A discussion of limitations of the study closed the chapter. The next chapter discusses the results of the analysis.

CHAPTER FOUR: FINDINGS

This chapter begins with a review of the stages in the enacted instructional sequence for fractions. Following the summary of the instructional sequence, social and sociomathematical norms that developed over the course of class are discussed. As some norms were established before the fraction segment of the class began (Andreasen, 2006), this section will provide evidence for continuation of the norms and new norms that were established. Finally, the observed mathematical practices are discussed.

Instructional Sequence

The classroom teaching experiment began on the first night of the semester. However, the fraction instructional sequence did not begin until the eighth class session. Problem solving was the specific topic for the first one and a half class sessions. The research team thought that the discourse resulting from presenting rich problems for students to solve in groups would aid in the establishment of social norms for the class (Andreasen, 2006). After the problem solving class sessions, there was a unit on number and operations with an emphasis on whole-number place value. This unit lasted for five class sessions, and was followed by an exam, then the instructional sequence on fractions. The complete instructional sequence for fractions was presented in Chapter Three. The discussion here is included as a reminder of the activities and their learning goals in order to help the reader realize the intent of the activities and better position the class discussion in a context for analyzing norms and practices.

Five broad learning goals with respect to fractions were identified and activities were structured in a way that the research team thought these ideas would become mathematical practices for the students. These goals were: (a) using fractions to name amounts, (b) understanding differences between whole number relationships and fraction relationships, (c) replacing rote procedures with reasoning to build meaning; (d) reasoning with addition and subtraction, and (e) reasoning with multiplication and division. The first of these learning goals focused on using fractions to name amounts represented in various ways including sets, areas, and locations on a number line. Several types of models were presented so that students would be exposed to a variety of representations and interpretations. Several concepts considered to be fundamental to building understanding with fractions were central to the activities in this stage. These included: (a) fractions are based on relational, or multiplicative, thinking; (b) it is important to define the whole and its size is important; (c) partitioning into equal parts is foundational to fractions; and, (d) as the number of parts increases, the size of the parts decreases.

The second goal of understanding differences between whole number relationships and fraction relationships included: (a) equivalent values, (b) relationships among fractions, (c) comparing and ordering fractions, and (d) naming fractions between two given values. The emphasis was on having students realize the differences between working with whole number relationships and fraction relationships. Among the differences expected to be pointed out were: (a) there is more than one name for a given amount, (b) another fractional value can be found between any two given values, and (c) when comparing fractions, greater denominators do not result in greater values. These ideas have been cited as the reason children have difficulty with fractions (Lamon, 1999) so it seemed prudent to include activities directed toward clarifying any misconceptions the students may have.

The goal of replacing rote procedures with meaningful processes was not limited to the activities in a specific stage of the hypothetical learning trajectory (HLT), even though it appears as the third stage of the HLT. Using reasoning strategies instead of rote procedures was encouraged throughout the course, and was identified as a sociomathematical norm. The intent of the goal for the third stage was to be sure students tried to understand the processes instead of simply applying rote procedures they had already learned. The goal was to have them understand the meaning behind the procedures. This was especially the focus of, but not limited to the following topics: (a) equivalent (b) fractions, (c) relationships among fractions (comparing and ordering), and (d) performing operations with fractions with and without a context.

Stages Four and Five addressed operations with fractions. At first, problems were presented in a context to help students find the solutions through reasoning instead of known algorithms. Later, bare computation problems were presented and students were encouraged to generate contexts to help them find the answers. The multiplication and division stage of the HLT included estimating. This was intended to encourage students to check the reasonableness of their results as well as to help them understand the effect of multiplication and division with fractions.

Norms

As stated in Chapter Three, norms are central to the participation structure in the classroom (Cobb et al.,2001; Cobb & Yackel, 1996; Gravemeijer et al.,2000; Stephan & Cobb, 2003). With that in mind, the research team planned the instruction for the semester to begin with problem solving to facilitate negotiating norms. This singular focus on problem solving began on the first night of class and continued midway into the second session. While the problems could have been solved algebraically, the class was asked to reason out the answers without using algebra. The intent of this was to provide an opportunity negotiate social and sociomathematical norms in the classroom. Following the problem solving class sessions, the class engaged in a unit on whole number place value and operations. The development of norms was documented for this period by Andreasen (2006). She discussed two social norms becoming established. These included the expectations that students would: explain and justify solutions and solution processes, and attempt to understand other students' solutions. The social norms established before the fraction unit began were similar to the norms observed during the unit on fractions, which were that the students would: (a) explain and justify solutions, (b) listen to and try to make sense of other students' thinking, and (c) ask questions or ask for clarification when something is not understood. Andreasen included the third norm about asking questions as part of the norm to make sense of other students' thinking. It is included as a third and separate norm for fractions because questions served to clarify and challenge processes and statements. This seemed to be different than only making sense of others' thinking. Andreasen also discussed two sociomathematical norms that were emerging before the fraction unit began. These were that students would recognize: what counts as a different and unique solution to the same problem, and what makes a good explanation. As with the social norms, similar sociomathematical norms along with one additional norm were found in the fraction unit. These included criteria that students would: (a) know what makes an explanation acceptable, (b) know what counts as a different solution, and (c) use meaningful solution strategies instead of known algorithms. The norms for the fraction unit are summarized in Table 11.

Table 11: Classroom Norms for the Fraction Instructional Sequence

Social Norms	Sociomathematical Norms
• Explain and justify solutions • Listen to and try to make sense of other students' thinking • Ask questions or ask for clarification when something is not understood	• Know what makes an explanation acceptable • Know what counts as a different solution • Use meaningful solution strategies instead of known algorithms

The following sections review Andreasen's (2006) observations with respect to norms and document the process of observing and establishing norms after the whole-number unit.

Social Norms

Recall that social norms describe the participation structure of the classroom (Stephan & Cobb, 2003). Participation in this class occurred on two levels. Students were expected to interact within a small group structure as well as the whole class. The tasks were generally given to students to work out in small groups, and then some students were asked to share their solutions with the entire class. This sharing often meant coming to the front of the class to show a solution on the board, overhead projector, or document camera. This was part of the expectation that students would explain and justify their solutions.

Explain and Justify Solutions

Andreasen (2006) documented the establishment of the norm to explain and justify solutions and solution processes before the fraction unit began. She described how at first, the instructor asked questions to prompt students to explain and justify. By the third night another student, instead of the instructor, asked a student for an explanation. Andreasen reports that the norm to explain and justify solutions became taken-as-shared on the third night of the instructional sequence for whole-number place value and operations. She described an episode in which a student gave the answer to a problem and freely explained how she arrived at that answer.

Although Andreasen (2006) reported a social norm to explain and justify solutions being established before the fraction unit, the instructor began the study of fractions with a reminder of that expectation. She believed this was necessary because of the many questions about explanations students asked during the exam for the whole-number unit. In introducing students to the fraction unit, she said:

> We're starting fractions tonight and just as you thought very hard in eight world [referring to base eight problems in the previous unit on place value and operations] … fraction world is going to involve some thinking. So, once again, when you make statements be prepared to explain them…Hopefully you'll continue making statements and being involved in the discussion as you were in the other unit in base eight and base ten because you had some wonderful discussions going on there.

This set the expectation that students should continue to explain how they arrived at their solutions. After this brief reminder, the class began to investigate the first fraction task. During the same class session, the instructor showed a square divided into four equal parts with three of the parts shaded as shown in Figure 3.

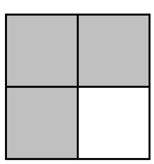


Figure 3: Representation of $\frac{3}{4}$

The students were asked to tell what part of the square was shaded. The following exchange took place during the discussion of this problem. It illustrates that this student needed to be prompted by the instructor to give an explanation.

Instructor: What's a fraction that will tell us how much is shaded? Amy.

Amy: Three fourths.

Instructor: How did you know?

Amy: Because three of the four total pieces are shaded.

Instructor: And from our discussion earlier what was important about the four pieces?

Amy: That they're all equal size.

Amy explained her answer, but only in response to the instructor's questions. It seemed that the instructor would need to prompt students for explanations. This is similar to what Andreasen (2006) observed in the beginning of the course. An examination of the class transcript reveals similar prompting by the instructor in the first half of the class on that first night of the fraction unit. Examples of such prompts were, "Tell me why," "How did you get it?" and "Why did you do that?" In the second half of that class session, explanations were given without prompts. For example, Sarah offered the following explanation when the instructor asked her to share how she found $\frac{3}{4}$ of a box of chocolates by looking at representation of $\frac{2}{5}$ of a box as shown in Figure 4.

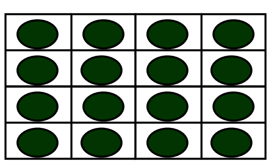


Figure 4: Diagram for $\frac{2}{5}$ of a box of chocolates

> I started off with $\frac{2}{5}$ and I divided it in half so I knew that each part was going to be a fifth. I went ahead and added 3 more of my $\frac{1}{5}$ so I would have the whole. And then I looked at it and I was trying to decide how I could divide it up into quarters because that's what was looking for—$\frac{3}{4}$ of a box. I looked at the rows going across and there's 4 rows going across. And I knew if I took 3 of those rows and fill the top 3 rows with chocolate that would be three quarters.

Not only did Sarah tell each step she took to arrive at her answer, she explained why the steps were valid. This suggests that the norm to explain and justify solutions had been established. However, there is evidence that the instructor periodically asked students questions to prompt them to explain their solutions. By the fourth class session on fractions, these prompts had diminished, and students were offering explanations on their own. An example is Amy's explanation of her solution of $\frac{17}{24}$ to $\frac{1}{3}+\frac{3}{8}$. Figure 5 shows the drawing she made to support her explanation.

Figure 5: Amy's drawing for $\frac{1}{3}+\frac{3}{8}$

> Amy: OK, I knew that I had to find a way to find $\frac{1}{3}$ and $\frac{3}{8}$ of the same picture of the same amount because like on number one [referring to the previous problem of $\frac{1}{4}+\frac{3}{8}$] you had fourths and eighths so you know when you make your little circles you can make halves and then figure out the fourths and eighths as the same size pieces. But I couldn't do that with my thirds and eighths. I had to figure out a way to represent thirds and eighths in the same way. So my eighths are going across and my threes are going down [referring to a 3 x 8 array of small circles]. So to represent my $\frac{1}{3}$, I look at my picture and figure out what will be $\frac{1}{3}$ of all my circles and that would be 1 out of the 3 rows. And for the $\frac{3}{8}$, it will be three rows.
>
> Instructor: Three columns.
>
> Amy: Three columns, sorry, out of the eight. And then I counted all the circles—all my shaded circles—and ended up with seventeen. And then I counted up how many circles I had all together in one picture, and that was 24.

This type of explanation became typical of the students after this class session. They gave answers and thoroughly explained how they arrived at the answer. Thus the norm to explain and justify solutions seemed to be established in the fraction unit in the fourth class session. Students not only were expected to explain solution methods, but they were also expected to listen to others' explanations and make sense of them. This is the next social norm discussed.

Listen to and Try to Make Sense of Other Students' Thinking

Andreasen (2006) reports that a norm to understand other students' solutions began to be established during the problem solving sessions early in the semester. She notes that the instructor followed up students' comments by asking the class if there were any questions. This set the expectation that students should be listening to explanations of others and making sense of them. Asking for questions was not the only strategy the instructor employed to help the norm become established. Andreasen also notes that the research team decided the instructor should ask students to explain what another student had said. This began on the second night of class. By the end of the second class period, Andreasen reports that the norm to attempt to understand others' solutions had become taken-as-shared.

When the fraction unit began, the instructor still used the methods discussed in the previous paragraph. After a student finished with an explanation, she would ask if there were questions. On the first night of the fraction unit, asking for questions was in addition to prompting the students for more complete explanations. This is mentioned here to make the point that explanations early in the fraction unit were often more in response to the instructor's questions, rather than being independently generated by the students. Therefore, making sense of other students' explanations may have been more difficult. Nevertheless, students often spoke directly to another student who was explaining something. This indicates that the norm of listening to and trying to make sense of one another's solutions was established in the fraction unit.

In the example that follows Katrina was explaining how she determined the number of chocolates in a box of chocolates given that sixteen pieces were $\frac{2}{5}$ of the box. This episode took place in the second part of the first class session on fractions, indicating that the norm may have been maintained from the previous unit.

> Katrina: First, I want to find out how much the whole was. How much $\frac{5}{5}$ were. So I know that $\frac{2}{5}$ is like the one we already have. So, if I divide that in half I find out what $\frac{1}{5}$ is. So I need 5 of those to make the complete whole. That's where I get my picture over there [pointing to her drawing of the whole box of chocolates]. …Since I want $\frac{3}{4}$, I know that out of 100 that's 75 because 25 plus 25 plus 25 is 75 and that's like three parts if I break 100 down into four parts.
>
> Doug: Where do you get the hundred? I only see 8, and 8, and 8, and 8, and 8.
>
> Katrina: 40. Because I wanted to make $\frac{3}{4}$ so I took a hundred to use percentage. I figured it was easier for me to do it in percentages.
>
> Doug: I don't see that.
>
> Instructor: Doug needs an explanation.
>
> Doug: I can't figure out where you got a hundred.
>
> Katrina: Because I wanted to get a percentage. I picked it out of my head I don't know. I saw three fourths and I thought percent. So if I want three fourths of that picture I want to know how many that is out of 100. Because I have ten rows. So if I can convert it into 100 that's now a number that I can look at that and find a way to find out how much I need. With $\frac{3}{4}$ I really can't visually see how I'm going to divide that up. Does that make sense?
>
> Doug: It would have to be a hundred. Comparing to this [pointing to the drawing of the whole containing 40 pieces] it would be 40.

Katrina: I have ten rows across if I want $\frac{3}{4}$.

Instructor: Ten columns.

Katrina: Yes I have ten columns so if I want $\frac{3}{4}$ I can't look at this [referring to the drawing of the whole box candy] and find like $\frac{3}{4}$ out of that. I just don't look at it and see that picture set up in a way that I can find $\frac{3}{4}$ of them. So I want to know what percentage is $\frac{3}{4}$ so I pick 100 because I have ten rows. I explained it better I think on the second one I came up with. So I take a hundred so that would be 75 out of a hundred which is $\frac{3}{4}$ so I can get percentages. So that's 75%. So with that 75% if I have 10 rows I know that that's seven and a half rows to get the 75%. So, I color in seven rows and then half of the eighth row to get the $7\frac{1}{2}$. When I did it the second way I did it as each row represents ten percent. So then I get my 70 percent and to get that five percent I did half of it—the eighth row.

Doug: Oh I see.

Doug did not grasp why Katrina was using 100 as a reference point. He interrupted her explanation to tell her he did not understand her solution. She explained it again in another way until he understood. Doug saw the sets of eight given in the problem, but Katrina was explaining that there were 100. She was able to eventually explain her method so that it made sense to Doug.

Another indication that the norm had been established occurred when several students entered into the discussion about the solution to the skyscraper problem on the final night of instruction on fractions. The problem was: "Pete is building a model of a city for a school project. He needs to cut lengths of a board that measure $\frac{1}{4}$ foot each to make the skyscrapers. How many whole skyscrapers can he cut from a board that is $1\frac{7}{8}$ feet long?" Prior to the following excerpt April had claimed that the answer to the problem was $7\frac{1}{2}$ skyscrapers. Matt thought the answer should be $7\frac{1}{8}$ skyscrapers. Lilly supported Matt's answer by saying, "It would go seven times and then $\frac{1}{8}$ left." Doug joined the discussion and the following exchange took place.

> Doug: The question is what part of a skyscraper… You're dividing by $\frac{1}{4}$ that's what we're looking at—the $\frac{1}{4}$ and the eighth that's left over—eighth of this one section is a half of one quarter. So it has to be $7\frac{1}{2}$.
>
> Instructor: Lilly.
>
> Lilly: But half of a board is not $\frac{1}{2}$ of $\frac{1}{2}$.
>
> Doug: …You are dividing by a quarter, and so 4 equal parts of that and so an eighth is really only half of a quarter.
>
> Instructor: Matt.
>
> Matt: Yeah, it's $7\frac{1}{2}$ because what Doug said. Because it was half of $\frac{1}{4}$ and still you're dividing by $\frac{1}{4}$ and if you keep going by $\frac{1}{4}$, half the distance you could keep going in even numbers then you get to $1\frac{6}{8}$ then you can only go half that amount of times to get to $1\frac{7}{8}$. So, it's $7\frac{1}{2}$ because you can only go $\frac{1}{8}$ more so it's half of that.
>
> Instructor: Lilly is still shaking her head…
>
> David: The best way I can think about it is to think right now this represents feet and this represents a skyscraper. This is how many feet divided by a skyscraper gives you seven skyscrapers and a half. You can't say 7 skyscrapers and a half feet so you use the same language. It's $7\frac{1}{2}$ skyscrapers. It wouldn't be $7\frac{1}{8}$ skyscrapers.
>
> Instructor: What does the $\frac{1}{8}$ refer to?
>
> David: $\frac{1}{8}$ refers to feet now. You can't say 7 skyscrapers, $\frac{1}{8}$ feet. …

The discussion continued, but the excerpt here is enough to see that students were listening to others' explanations and trying to make sense of them. One way they did this was by challenging the explanation when it didn't fit with their own expectations. This episode was finally resolved when everyone in the class agreed the answer was $7\frac{1}{2}$ skyscrapers. Episodes like this occurred somewhat regularly. However, the instructor still prompted the class for questions following explanations. That is not to suggest that the norm was not established. It is more likely a habit of the instructor. The episodes discussed here occurred with no prompting from the instructor, giving reason to believe that it would have occurred at other times as well.

Andreasen (2006) discussed the instructor asking students to explain what another student had said as a technique to help establish the norm. This practice continued throughout the fraction unit. In the beginning, it served as a way to communicate that listening to each others' explanations was valued while establishing the norm. However, later in the fraction unit, after the norm had been established, the instructor still asked students to reiterate what someone had just said. In the following excerpt from the second night of the fraction unit, David was explaining how he found three fractions between $\frac{1}{6}$ and $\frac{1}{3}$.

> David: With $\frac{3}{9}$ the ninths are bigger pieces than eighteenths and when you cut them in eighteenths they're smaller than ninths. So tenths would have to be bigger than eighteenths and smaller than ninths. So if you had three of them, they would be bigger than eighteenths and smaller than ninths.
>
> Instructor: Kim, what did he say? What did he mean?
>
> Kim: He meant that if you've got a whole candy bar and cut the candy bar into nine pieces and take away three, then cut the same size candy bar into eighteen pieces and take away three, then like any number between the nine and eighteen if you cut it into 10 pieces the pieces are still going to be between. Like the size of those three pieces together that you took will be between the three pieces from the nine you took and the eighteen you took because the pieces are smaller than the eighteenths no than the ninths.

Kim paraphrased David's explanation and added imagery of a candy bar. David's original explanation simply addressed the pieces. Kim placed pieces in the context of cutting up a candy bar. This imagery may have helped her make sense of the explanation. Students were asked about other students' explanations during the fraction unit on several occasions. Although the technique may have originated in an attempt to convey the importance of making sense of the explanations of other students, by continuing to use it, the instructor could monitor the level of understanding of the students.

Ask Questions or Ask for Clarification When Something is Not Understood

This social norm is related to the previous one in that one way to understand the solutions of others is to follow the explanation by listening carefully and asking questions to clarify points that are not understood. In fact, Andreasen (2006) included asking questions as a part of the norm to attempt to make sense of others' explanations. While one purpose of asking questions may be to help make sense of explanations, asking questions is treated as a separate norm here. Asking questions is indeed an indication students are attempting to make sense of others' explanations. Although it may be a fine distinction, asking questions for the purposes of this norm also indicates students are trying to understand the mathematical processes and build their own meaning. Thus, they are accepting responsibility for their own learning.

Asking for clarification was encouraged by the instructor who frequently asked for questions at the end of explanations by the students. Often, questions were asked, leading to further explanation and clarification. Examples of this are prevalent in the class discourse. In the excerpt below, the instructor noticed a student shaking her head, and interrupted the explanation so the student could ask her question. In the following exchange from the latter part of the second class period on fractions, David was explaining how he found three fractions between $\frac{8}{9}$ and 1. Lilly, Kathy, and Sarah needed clarification for parts of his explanation.

> David: We've got to find three fractions between $\frac{8}{9}$ and 1. What I did was I changed the $\frac{8}{9}$ into $\frac{800}{900}$ just because I knew it had to be big numbers…and 1 will equal 800 over 800. I'm trying to keep the same numerator…You just have to find any number between here [pointing to 800 and 900 in denominators] using 800 and it will be between them. It will be 800 and 850 and 800 over 849 [writing $\frac{800}{850}$ and $\frac{800}{849}$ on the board].
>
> Instructor: Lilly is shaking her head.
>
> Lilly: Aren't you trying to get to 900 over here [pointing to the 1 on the number line] though? Where that 800 to 900 is, aren't you trying to get to 900 over 900 for it to be 1?
>
> David: I did that the first time but then I realized that we're trying to get the same numerator so I changed it to $\frac{800}{800}$ so we'd still have the same numerator and we'd work from there. It still equals 1. …

David’s explanation continued to describe how to divide the pieces into smaller pieces to show equivalent fractions. It is lengthy and not included here because it is not relevant to the establishment of the norm to listen to and make sense of others’ explanations. After he explained about making smaller pieces Kathy entered into the conversation.

> Kathy: I understand that part, but did you just randomly pick 800 over 800?
>
> David: I picked 800 over 800 because we're trying to keep the same numerator.
>
> Kathy: OK…
>
> Sarah: Could you have used 80 over 90 and done at 80 over 80 as your 1?
>
> David: Yes.

As Lilly was listening to David's explanation, she apparently expected him to name 1 as $\frac{900}{900}$. When he used $\frac{800}{800}$, she was puzzled. She asked him for clarification. This indicates she did not understand his choice of $\frac{800}{800}$. Kathy later asked about the same thing in a different manner. To her, David's choice of $\frac{800}{800}$ seemed like a random choice. She needed clarification as to why he chose that particular fraction. Finally, Sarah seems to have generalized his method and wonders if another number may also have been appropriate. She may well have understood the explanation, but asked a question to check her own understanding. Each of these questions demonstrates that the norm to ask questions or ask for clarification when something is not understood may have been established early in the fraction unit.

In the next example, Kim was explaining her answer to this problem: "Betty had $2\frac{1}{2}$ yards of ribbon. She gave $\frac{2}{3}$ of a yard of her ribbon to Wilma. How much ribbon did Betty have left?" The discussion took place about midway through the fraction unit, on the third night. It illustrates that students asked questions of other students when they did not understand something. Figures 6 and 7 will help the reader make sense of the following explanation.

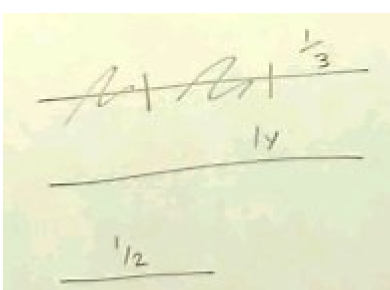


Figure 6: Linear model for subtracting fractions

Kim: What I did is drew out my 2 yards as a line, and then my half yard…So I said here's one yard. Here's one yard. Here's half of a yard [drawing Figure 6]. And then I knew I had to get rid of $\frac{2}{3}$ of one of the full yards so I split one of the yards into thirds…So here goes this yard and this yard and so I'm left with $\frac{1}{3}$ and I'm left with 1 yard and I'm left with half. So then I said obviously this is less than one yard and that's less than one yard [pointing to the remaining $\frac{1}{3}$ and $\frac{1}{2}$ so I want to see if I can combine them to get closer to one yard. So I took a yard and divided it into thirds because I had a third. So this would be $\frac{1}{3}$ and this would be $\frac{2}{3}$, and this would be my third third. And I was like I don't know how to add half to that so I had to split these into halves, which would give me $\frac{1}{6}$, $\frac{1}{6}$.

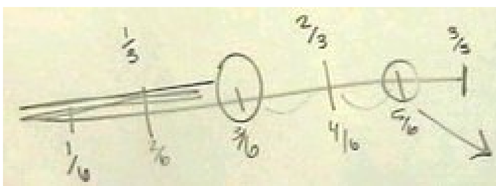


Figure 7: Number line showing subtraction

Instructor: You're writing $\frac{1}{6}$, $\frac{2}{6}$, $\frac{3}{6}$, $\frac{4}{6}$, $\frac{5}{6}$.

Kim: So each area is a sixth but then like I'm keeping track of it here. I know that $\frac{3}{6}$ is half so my half is right here [circling the mark at $\frac{3}{6}$]. So this part right here takes up my half. To add $\frac{1}{3}$ to that I can't go to this $\frac{2}{3}$ because that's not a full third, but I know that from $\frac{1}{3}$ to $\frac{2}{3}$ is that so I can go from here over 1 and over 1 and end up here and know that I have another $\frac{5}{6}$ here. So $\frac{1}{3}$ plus $\frac{1}{2}$ is $\frac{5}{6}$.

David: So how would you explain why you divided each piece of ribbon twice?

Kim: Because I wanted to find a half and you can't find a half from the third. Or I couldn't in this case. So I knew that the midpoint between this; there's going to be a midpoint between this so I separated it in half.

David didn’t seem to understand why Kim divided each piece of ribbon twice. He immediately asked her a question and she provided further clarification. This is just one example of a student asking questions for clarification. It was a common occurrence.

Another example of students talking to one another to clarify points in other students' explanations follows. The excerpt begins with Kim's summary of her explanation of how she determined what $\frac{3}{4}$ of a box of chocolates would be based on seeing a representation of $\frac{2}{5}$ of a box. This occurred early in the fraction unit, before the episode with Katrina and Doug discussed as an example of the norm to make sense of others' explanations.

> Kim: So if you count up by twos there's 20 sets of twos. So then in my other box or to show $\frac{3}{4}$ of a box you need to have 15 sets of the twenty.
>
> April: I'm a little confused. If I'm just looking at that, it looks like you need to go from $\frac{8}{20}$ to $\frac{15}{20}$. Why wouldn't she just like add 7 boxes?
>
> Kim: You would. It's 7 boxes of 2 pieces, not 7 boxes of 1 piece. So like you have [interrupted]
>
> Instructor: Do you want to come up and make some drawings or point to things so it shows what we're talking about?
>
> Kim: OK, in my schematic right here [pointing to diagram shown in Figure 8] one box of two pieces is $\frac{1}{20}$ and I got that because I saw this part that's drawn as $\frac{8}{20}$ of a box. But there's 16 individual pieces there. So 16 individual pieces separated into 8 parts would make it 2 pieces per part. So then I said OK that's 8 of my parts. And then, just to complete my box I drew the other twelve so here's 4, 8, 12. So then $\frac{3}{4}$ of the box would be [pauses] you can either think of it as that part right there [pointing to the first two columns of black ovals in Figure 8] which is 8 plus another 4 plus another 7 because 8 and 7 is fifteen. So that's where I got my 15 out of 20. Or I just counted 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15.

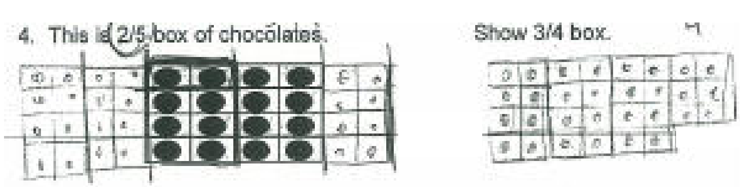


Figure 8: Drawing to show $\frac{3}{4}$, given $\frac{2}{5}$

April: So you didn't really have to draw the other chocolates on the other one. At that point as soon as you figured out that 2 squares was $\frac{1}{20}$ you could have just really just added them.

Kim: But I understood that, but when I was trying to explain to them where I was getting them, they didn't understand why.

In this episode, April admits to being a little confused, and followed her statement with a specific question. Kim provided a drawing to enhance her explanation and then April understood. She indicated her understanding by repeating an alternate strategy, verifying what she was thinking. The questions from the students in the episodes discussed here would not have been asked if others had not been explaining their thinking and solution methods. Thus, these explanations from students were at the core of the social norms. If students were not expected to explain their solutions, there would be no need to expect them to make sense others' explanations. And, without the expectation to make sense of others' thinking, questions from the students may have been asked less frequently. These norms helped to define the general participation structure in the class. The sociomathematical norms discussed in the next section helped to describe the expectations with respect to mathematics in the class.

Sociomathematical Norms

Sociomathematical norms focus on whole class discussions that deal specifically with the mathematics. Andreasen (2006) identified two such norms in developmental stages. The first was what counts as a different or unique solution. The second was what makes a good explanation. She concluded that student work did indicate that they had some understanding of what counted as a different solution and what made a good explanation. However, there was not sufficient evidence to conclusively state these norms had been fully established during the whole-number portion of the course. Recall that the instructor began the fraction unit by reminding students they would again be expected to explain their thinking. She did this because during the whole-number place value and operations test, students were unsure of their explanations. They seemed to need approval that what they wrote would be judged as adequate. In the fraction unit, three sociomathematical norms were observed. Two were similar to those identified in developmental stages by Andreasen. The third norm was to use meaningful solution strategies instead of rote algorithms. A discussion of these sociomathematical norms follows.

What Makes an Explanation Acceptable

Since students were expected to explain their solutions, it follows that knowing the criteria for what makes a good explanation would need to be negotiated in this class. Students often expressed concern about their explanations being sufficient. Frequently, this seemed to be motivated by their desire to make a good grade more than wanting to develop a deeper understanding of the mathematics they were learning. Nevertheless, qualities of a good explanation were negotiated in the class. During the first class session on fractions, the instructor frequently asked students to extend their explanations by asking why. This set the expectation that explanations should include why the procedure that resulted in the answer is valid. Thus, simply stating the procedure used is not an adequate explanation.

During the first session on fractions, Kim explained how she found $\frac{3}{4}$ of the box of chocolates when she knew that 16 pieces made up $\frac{2}{5}$ of the box. To solve the problem, Kim essentially found a common denominator for $\frac{2}{5}$ and $\frac{3}{4}$. She expressed each fraction as twentieths. The following is an excerpt from the discussion that contains the part of her explanation that described the process of finding the common denominators.

> Instructor: How did you know you were looking for $\frac{15}{20}$?
>
> Kim: Because if $\frac{8}{20}$ represents $\frac{2}{5}$ of a box then $\frac{15}{20}$ represents $\frac{3}{4}$
>
> Instructor: How did you know?
>
> Kim: I made a common denominator.
>
> Instructor: What's a common denominator?
>
> Kim: It's what both $\frac{2}{5}$ and $\frac{3}{4}$ can go into. Like if I have this number as 5 over 5 would equal one whole. Then I want to make 5 over 5 equal to 20 over 20. And then on this side I would have 4 over 4 equals 20 over 20. And then since my number isn't 5 over 5 on $\frac{2}{5}$, I would have to multiply the 2 by whatever I multiplied. I multiplied the 5 by 4 to give me 20 so then I need to multiply this 2 by 5 to give me 8.…
>
> Instructor: I feel like you've explained a procedure of finding common denominators. What I don't have a clear understanding is why we find common denominators and why that procedure is what we use.

The instructor’s comment about wanting to understand why common denominators are helpful indicated that the explanation was lacking some conceptual information. The discussion continued with the instructor asking questions to try to get to the conceptual origin of the procedure. Finally another student, Sarah, gave a succinct explanation that seemed to satisfy the criteria for being conceptual and explaining why.

> I started off with $\frac{2}{5}$ and I divided it in half so I knew that each part was going to be a fifth. I went ahead and added 3 more of my $\frac{1}{5}$ so I would have the whole. And then I looked at it and I was trying to decide how I could divide it up into quarters because that's what was looking for—$\frac{3}{4}$ of a box. I looked at the rows going across and there's 4 rows going across.

> And I knew if I took 3 of those rows and fill the top 3 rows with chocolate that would be three quarters.

Though not explicitly stated, the previous two exchanges communicated that an acceptable explanation should tell why a mathematical procedure works. Kim gave an explanation of how to find common denominators, but could not tell why finding common denominators was a good strategy. When Sarah gave her explanation, it was conceptually based. She explained each step she took and justified it in terms of the amounts she was working with or wanted to find.

Later in the same class period Katrina approached the subject of what makes an explanation acceptable. The problem that had been presented was how to divide 4 cookies equally among 5 people. She knew that only the numerators are added when adding fractions. The specific case she was working with was $\frac{1}{5}$ added 4 times. She was questioning whether or not she could explain her work satisfactorily.

> Katrina: I'm just trying to figure out if my explanation would be sufficient. When I say I'm adding 4 parts of $\frac{1}{5}$, and I'm adding that four times. So, I'm adding $\frac{1}{5}$ to that one part and then another fifth so that's $\frac{2}{5}$. But I'm trying to figure out how to explain it where I only add the top and not the bottom.
>
> Lilly: Because when you're adding you don't have to have to have a least common multiple on the bottom. You just add the top numbers. If you have the same number on the bottom all the way across you add all the numbers on the top.
>
> Katrina: But why?
>
> Carrie: Because if you're looking at the picture you're always going to have only five pieces. And the top is like one section of the pieces.
>
> Instructor: Keep going.
>
> Carrie: You're not getting more pieces.

Instructor: Anyone want to take up where Carrie left off?

Carrie and Lilly offered their suggestions to make Katrina's explanation complete. In doing this they are adding to Katrina's explanation to make it acceptable. Their comments helped to define what makes a good explanation. Lilly's comments were more procedural in nature. After she stated the procedure, Katrina asked why. This seemed to be what was concerning her when she asked her initial question. She needed to be able to explain why the procedure is mathematically valid. Including why had become a criterion of an acceptable explanation.

Another criterion for an acceptable explanation emerged early on the first night of the fraction unit. The class was discussing the fraction represented by pattern blocks. The whole was a figure made from a red trapezoid and a blue rhombus. Two red trapezoids made up the part, resulting in a fraction of $\frac{6}{5}$.

> Instructor: Now this group in the back said we found 6, but we can call that $1\frac{1}{5}$. After they used the $\frac{6}{5}$ I think with the manipulatives then they knew well, that's the same thing as $1\frac{1}{5}$. Tell me why. Katrina.
>
> Katrina: Well, I did 5 goes into 6 one time. So that's one whole. Six minus 5 is 1 so you have one more piece and you put that on top and you have a total of 5 pieces in the unit that would be on the bottom. …
>
> Instructor: So she said you put the 5 into 6 and it goes in 1 time and there's 1 left over. Does anyone have any questions about why she can even do that?
>
> Instructor: Would you be able to explain that on a written assignment so that someone else could understand your thinking? Say someone wasn't here today—Matt. We need to explain it to Matt, Katrina's process and explain why she can do that.

The criterion that was discussed here was that an explanation should be thorough enough so that someone could read it and understand the thinking behind it. Matt happened to be absent that class, and he was used as an example of a reason to provide complete explanations. Now, two criteria for acceptable explanations have been shared. An explanation must tell why the mathematical process is valid and it must be thorough enough so that someone who was not in class would be able to understand the thinking.

There was evidence that these criteria were considered in students' explanations later in the class. On one occasion Matt commented to a fellow student explaining something that he thought they needed to explain it better than the student had. He added that the other student should explain why. Again, the message was to tell why. Lilly was explaining her solution to a problem in which she needed to add $\frac{1}{3}$ and $\frac{1}{2}$.

> Lilly: I couldn't add these two numbers. So I broke it down into something where I could figure out how many were in a half and how many were in each third. So I broke down both pies into six.
>
> Matt: I think we have to explain it better than that. Like why.
>
> Joe: The only problem I had was explaining how to get from that point where you're at.
>
> Lilly: I broke down. I know both of these can be turned into $\frac{1}{6}$, so I just turned it into 6 pieces.
>
> Instructor: They see that you did that. And they see how you did it. They want to know why you did it.

Matt told Lilly she needed to explain why. This seems to indicate that the criterion for explaining why as part of an acceptable explanation had become taken-as-shared. It is quite possible that when the "why" was not included in an explanation, the student simply had trouble recognizing the "why." There is evidence that the students knew that an explanation needed to include why. However, they seemed insecure about their explanations and were concerned mainly that their explanations would be adequate for the test and a good grade. It is difficult to judge the extent to which the norm for knowing what is an acceptable explanation was fully established. Students seemed to recognize the criteria for acceptable, but had some difficulty always meeting the criteria of explaining why and being thorough in their explanations. That difficulty may be attributable to their mathematical knowledge. In addition to providing good explanations, students were expected to know what made a solution different. That is the next norm discussed.

What Counts as a Different or Unique Solution

The instructor set the expectation to discuss multiple solutions by frequently asking the class if anyone did the problem a different way. Students usually were able to offer different solutions. At times, they volunteered their different solutions without the instructor's prompt. Early in the fraction unit, a student shared an explanation of her solution to a problem. The instructor asked, “How many of you solved it that way?” Then she added, “I've never seen it that way you explained it and I understood it. That's interesting.” Her reaction to the explanation served to let students know that different solution strategies were valued in the class.

In discussing fractions that are between $\frac{4}{7}$ and $\frac{5}{7}$, one student gave $\frac{45}{70}$ as an answer. In the following discussion that took place on the third night of the fraction unit, criteria for a different solution included a different answer as well as a different way to solve the problem. In this excerpt it can be seen that not only did the instructor ask for other answers; she asked how the students could have arrived at that answer.

> Instructor: Are these the only answers we could have come up with? Did anybody get anything else?
>
> Amy: I got $\frac{17}{28}$ and $\frac{18}{28}$.
>
> Instructor: How could she have gotten 17 out of 28 and 18 of 28?
>
> Instructor: Sarah.
>
> Sarah: Instead of dividing each piece into 10, she divided each piece into 4. Then the same concept—you 're going to shade 4 big pieces and that's going to give you 16 small pieces. And then the 5 is going to give you 20 small pieces.

Sarah's explanation contrasted the process for using 28 pieces to the process described by the student before her who used 70 pieces. This criterion seemed to be implicitly known by the students. When the instructor asked if anyone did a problem in a different way, students responded appropriately with different processes. Specifying what makes a solution different occurred frequently as part of explanations in the class.

In another example that occurred on the third night of the fraction unit, Katrina referred back to the previous solution then explained what she did differently in comparing $\frac{4}{7}$ and $\frac{3}{8}$.

> I did the half that we did before [referring to having used $\frac{1}{2}$ as a benchmark to compare $\frac{1}{3}$ and $\frac{3}{5}$]. I didn't draw a picture I just looked at the denominator. I know that half of 8 is 4, and then half of 7 is $3\frac{1}{2}$. I looked back at the denominator and I know that 3 is less than 4 because you count 1, 2, 3, and $3\frac{1}{2}$. And 4 is greater than $3\frac{1}{2}$, so $\frac{4}{7}$ is bigger because 4 is larger than half of 7.

Other instances of students recognizing what is different about their strategies follow. On the fourth night of the fraction unit, Kathy said, "I solved it like Doug, but with pies." She recognized that the basic difference between her solution and Doug's was in the model used. The basic procedure was the same. On the final night of the fraction unit, David also explained the difference between his solution process and the previous one shared with the class by explaining how he estimated the quotient of $6\frac{1}{3}$ divided by $\frac{3}{4}$. He said, "The way I did it was I didn't think about it being 1 and 7 at all. I did $6\frac{1}{3}$ as 6 and $\frac{3}{4}$ as 1, and 6 divided by 1, that was 6. And a little more than 6 was 6 and not all the way to 18." These examples indicate students knew what makes a solution different, as they included that information in their explanations.

There were times when the instructor indicated what would be different by her questioning. On the fourth night, several students solved the first subtraction problem presented ($2\frac{1}{2} - \frac{2}{3}$) by subtracting $\frac{2}{3}$ from 2, then adding $1\frac{1}{3}$ and $\frac{1}{2}$ to find the difference for the original problem. Two students had given explanations using different models. The instructor then said, "Did anyone do this without using addition?" By doing this, she was providing a suggestion about what kind of solution would be different from the ones already presented.

Students seemed to be comfortable with knowing what made a solution different almost from the beginning of the fraction unit. The preceding examples indicate that using different models, number of pieces, rounding strategies, or operations made a solution different. That is, a solution that used different tools or strategies was considered different. When the instructor asked if anyone did a problem differently, they offered their solutions, including what made theirs different from ones previously discussed.

Use Reasoning Strategies Instead of Algorithms to Solve Problems

Students in the teaching experiment class brought some knowledge of procedures with them. These were sometimes incomplete and not meaningful. In order to deepen the students' understanding, the research team wanted the students to use reasoning instead of relying on meaningless procedures to complete tasks. This led to the final sociomathematical norm, which was to use reasoning strategies instead of algorithms to solve problems. This was related to the norm to explain and justify solution processes in that simply reiterating a procedure that followed an algorithm was not an acceptable solution. It is considered as a separate norm because this expectation was pervasive throughout the unit on fractions, beginning with the first task. In this task, students described how full two school buses were. In doing so, they compared the buses using relational thinking. In other words, they described how full the buses were by telling how much of the capacity was filled compared to the total capacity. One bus had 60 seats and 24 of them were taken. The other bus had 30 seats and 15 of them were taken. An excerpt from the discussion follows.

> Katrina: Well, if you look at the 15 and 15 you know that 15 is half of 30 so that bus is half full. But if you look at the 24 and 36, half of 60 is 30 so you know that 24 is less than 30 so that's less than half. So the 15 would be half full so that would be more than the other bus which is less than half full.
>
> Instructor: … So what you said is 15 over 30 is half full because 15 is half of 30 and you said 24 over 60 is less than half full because why?
>
> Katrina: Because half of 60 is 30 so 24 is less than 30.

> Instructor: …So, you just compared these two—a bus representing 15 seats full out of 30, and a bus representing 24 seats out of 60—and compared them for fullness without following procedures and rules for finding, for comparing fractions…As you might guess, that will be what we do in here now. We're going to start thinking about fractions and make sense of them without relying back on our old rules and procedures we can't make sense of.

Katrina used reasoning to explain why the bus with fewer passengers was actually more full than the bus with more passengers. She did this by comparing the part of each bus that was full to $\frac{1}{2}$. The instructor reinforced her thinking by stating the expectation that the students would be working with fractions, but not relying on known rules and procedures. If Katrina had used an algorithm, her explanation may have simply been that she found a common denominator of 60, and multiplied $\frac{15}{30}$ by $\frac{2}{2}$ to rename the fraction as $\frac{30}{60}$. Then she would have been able to easily compare $\frac{24}{60}$ to $\frac{30}{60}$. Another example of using reasoning instead of an algorithm occurred later on the first night of the fraction unit.

In the following excerpt, Doug explained how his group determined that there would be 30 chocolates in $\frac{3}{4}$ of a box of chocolates. They were given the fact that $\frac{2}{5}$ of the box of chocolates contained 16 chocolates. The group reasoned that if 16 pieces make up $\frac{2}{5}$ of the box, then $\frac{1}{5}$ of the box would be 8 pieces. With that information, they were able to determine a whole box would have 40 pieces.

> We saw that in the original thing up there, the picture of the chocolates, the $\frac{2}{5}$, well $\frac{1}{5}$ would equal 8 chocolates and so we filled up the box so we added 3 more fifths and made a big box which equals 40 chocolates, which would be $\frac{5}{5}$. That'd be a whole box. And so, this box over here [referring to another box drawn representing 40 chocolates] is the same as this box that has 40 chocolates in it, and we want to show $\frac{3}{4}$ of that. So, we saw that to divide that into 4 equal groups of chocolates each one would

> have to have 10 chocolates in each group. And so, we're looking at 3 of them—3 times 10 would be 30. So we just colored in 30 chocolates.

This explanation demonstrates reasoning because the group extended the drawing to show the whole box, then found $\frac{3}{4}$ of that amount. The mathematics used for this reasoning was that $\frac{5}{5}$ make a whole. They were able to find the whole because they used the fact that $\frac{1}{5}$ is half of the $\frac{2}{5}$ that was given. An algorithmic approach to solving the problem may have been to use a proportion to find that the whole box contained 40 chocolates, then find $\frac{3}{4}$ of 40. This would have simply involved the operations of multiplication and division, and perhaps not had the meaning that the group's explanation had.

On the second night of the fraction unit, the class participated in a number line activity (Appendix I). They were given a number line that had the locations of 0, $\frac{1}{5}$, $\frac{2}{3}$, and 1 marked. They were challenged to find the points for one half, thirds, fourths, fifths, sixths, eighths, and tenths. The activity had a constraint that they could not use rulers. Students were able to use reasoning to find all the points by halving the distance between given points and new points they located. Here is David's explanation of how he used the halving method to find the location of $\frac{1}{3}$.

> You have $\frac{2}{3}$. Half of those $\frac{2}{3}$ would be one of those thirds. … We already know this is $\frac{1}{3}$ [pointing to one section of a rectangle divided into 3 parts]. There's 1, 2, 3 thirds. This is $\frac{2}{3}$, and you can see that that's half. One third is half. So, we line it up again. To find out that $\frac{1}{6}$ is half of $\frac{1}{3}$, so you put the 0 on the $\frac{1}{3}$.

David used his knowledge that $\frac{1}{3}$ is half of $\frac{2}{3}$ in order to locate the $\frac{1}{3}$ on the number line. He folded the number line with the $\frac{2}{3}$ and 0 aligned. This gave him the location of $\frac{1}{3}$ on the number line. Then he used his knowledge that $\frac{1}{6}$ is half of $\frac{1}{3}$ and repeated the process of folding to locate $\frac{1}{6}$ on the number line. This reasoning process was perhaps easier than if students had tried to use a ruler to mark the points on the number line. In using the halving process, they built the number line based on relationships of the given fractions and the fractions they needed to find. This was the intent of the activity and the reason for the constraint of not using rulers.

Later in the same class period, students were presented with the problem to find fractions between two given fractions. The algorithmic way to solve the problem would have been to find a common denominator for the two fractions, then name two more fractions with the same denominator and numerators with values between the two given fractions expressed in higher terms. Doug modeled the algorithm, but explained it conceptually in terms of breaking the fractions down into smaller pieces. He was able to reason through the process of finding common denominators and justify his method without ever referencing the algorithm or using the term “common denominator.” His explanation for how he found a fraction between $\frac{1}{6}$ and $\frac{1}{3}$ follows. Figure 9 is included to help the reader understand his reasoning. Doug drew this figure on the board to support his explanation.

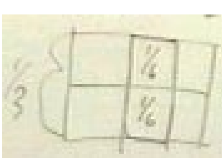


Figure 9: Area model to show $\frac{2}{6}$ is equivalent to $\frac{1}{3}$

Doug: The only way I could figure this one out was to make $\frac{1}{6}$ and $\frac{1}{3}$ the same. And I know that $\frac{1}{6}$ is $\frac{1}{2}$ of $\frac{1}{3}$ so this is $\frac{2}{3}$, I'm sorry $\frac{2}{6}$. And this is $\frac{1}{6}$ and $\frac{2}{6}$ so I know I have to make two fractions in between and I've got have to have smaller pieces because there's no difference between 1 and 2. …I have to make smaller pieces so this is $\frac{4}{12}$ and that's 2 [twelfths] and there's a difference between 2 and 4 [twelfths] and that's $\frac{3}{12}$.

Instructor: Can you hold on for just a minute? What questions do we want Doug to explain?

Joe: How he got the $\frac{2}{12}$ and $\frac{4}{12}$

Doug: How do you get the $\frac{4}{12}$? If I take this [drawing the rectangle shown in Figure 9 in thirds] and this is thirds and make this one [pointing to $\frac{1}{6}$] equal to this one [pointing to $\frac{1}{3}$] is this and this is $\frac{1}{6}$ and $\frac{1}{6}$ and $\frac{2}{6}$ equals $\frac{1}{3}$.…So, this whole is made into 6 pieces so each of these is 1 of 6 pieces. So, you add $\frac{1}{6}$ of the whole and another $\frac{1}{6}$ of the whole and it comes to the same as $\frac{1}{3}$ which equals this [pointing to the middle column of Figure 9]. So, for me to understand this is a smaller number, I have to make them equal to each other. So, $\frac{1}{3}$ is the same as $\frac{2}{6}$. Now I can compare these two numbers and try to find something in between there. And so to find a number between there they have to make them a smaller number so I have to divide them up again—1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 [dividing each of the 6 sections into 2]. Now there are 12 parts and $\frac{1}{3}$ equals $\frac{4}{12}$ and $\frac{1}{6}$ now equals $\frac{2}{12}$ now because there's a difference between the 4 and 2 [twelfths] halfway between them is $\frac{5}{12}$, I'm sorry $\frac{3}{12}$. So that's one number in between these two numbers.

In the preceding explanation, Doug used a drawing to show that $\frac{1}{3}$ is equivalent to $\frac{2}{6}$, and later $\frac{4}{12}$. He did not use the procedure to find a common denominator. He explained the process conceptually in terms of making smaller pieces so each fraction can be represented with the same number of pieces in the whole. This idea of making smaller pieces so each whole has the same number of pieces was used again when addition was introduced.

About midway through the fourth night of the fraction unit, Amy presented a solution to adding $\frac{3}{10}$ and $1\frac{2}{5}$. Amy succinctly explained what she did in the following statement.

> It was $\frac{3}{10}$ plus $1\frac{2}{5}$. Right away when I saw the 10 and the 5 I knew if I drew a pie with 5 pieces—if I halved each of my pieces I'd end up with 10 and that way I could easily work with the same size pieces and I can compare my fractions better.

Here Amy used the strategy to make same-sized pieces to add the fractions. This is modeling common denominators conceptually. The relationship between the denominators of 5 and 10 may have made the problem easier to think about. That is, in order to make same-sized pieces, Amy only needed to divide the fifths in half. The other "pie" could remain as it was. Another problem later on the same night was more complex.

Doug explained how he subtracted $\frac{1}{2}$ from $1\frac{1}{3}$. These denominators were not related in the same way as the 10 and 5 were in the previous problem. However, Doug still explained his process with reasoning supported by drawings rather than applying an algorithm. Note that he first subtracted the $\frac{1}{2}$ from 1 instead of $1\frac{1}{3}$. His explanation follows, and his work is shown in Figures 10 and 11.

> So, all I did was I'm not going to change this area here [pointing to the diagram showing $\frac{1}{2}$].

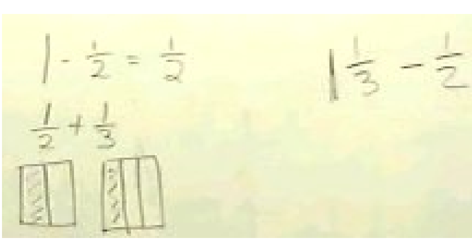


Figure 10: Doug's work

> I can divide it equally any way I want and still be the same thing. So here I'm going to divide this [pointing to the first figure drawn in Figure 10] into three parts and this one [pointing to the second figure drawn in Figure 10] I'm going to divide in half. Nothing has changed.

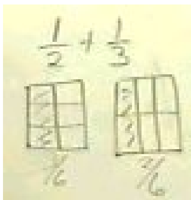


Figure 11: Doug's diagram to show addition

> So now I have 1, 2, 3—3 of 6 parts and this [pointing to the figures in Figure 11] is 2 of 6 parts and now I can combine them which is $\frac{5}{6}$, so $\frac{5}{6}$ is what's left.

Doug's reasoning diverged from the traditional algorithm in that he first subtracted $\frac{1}{2}$ from 1 instead of the $1\frac{1}{3}$. The model he drew may have helped him see that once he did that he needed a way to combine $\frac{1}{2}$ and $\frac{1}{3}$. His reasoning included modeling each of these fractions as sixths—$\frac{3}{6}$ and $\frac{2}{6}$. Once he had completed this stage of the modeling it was easy to see that combining them results in $\frac{5}{6}$. Had Doug used the algorithm, he would have had to rename the $1\frac{1}{3}$ as $\frac{4}{3}$, then find a common denominator. His method was based on modeling and reasoning.

The preceding excerpts from the fraction unit illustrate how students used reasoning strategies instead of known algorithms. They used a benchmark of $\frac{1}{2}$ to make informal comparisons early in the unit. They used their knowledge of fraction relationships to determine the size of a whole and then find a portion of that whole. On the number line activity, they again used the concept of $\frac{1}{2}$ to find different fraction locations on the number line. Finally, the students explored the concept of common denominators by dividing the whole into pieces of the same size to find fractions between given values and to perform the operations of addition and subtraction. These types of reasoning were pervasive throughout the fraction unit. Students were required to explain their solutions and algorithms were not considered to be acceptable explanations. In addition to norms, classroom mathematical practices were established during the fraction instructional sequence. The next section discusses the two practices that were established.

Classroom Mathematical Practices

Recall that classroom mathematical practices emerge from the activity in the classroom. They are taken-as-shared ways of reasoning. The methodology for documenting collective activity detailed by Rasmussen and Stephan (in press) provided the means for analyzing the classroom participation in order to identify the classroom mathematical practices. Rasmussen and Stephan describe a three-phase approach which was discussed in Chapter Three. The specific process followed in this research is reviewed here as a reminder. In the first phase of documenting collective activity, transcripts were created from the classroom video tapes. After the transcripts of the video taped class sessions were created, the researchers noted when claims were made while watching the video recordings. The final step in this first phase was to apply Toulmin's (2003) model to the argumentation and make notes about the argumentation in the spreadsheet. This served as the argumentation log. Other members of the research team also identified claims and analyzed the argumentation schemes. This information was shared and discussed at team meetings and smaller meetings with some of the team members. The analyses were compared and discussed. When there was disagreement, the team members discussed the analysis and tried to reach agreement. Once the argumentation analysis was complete, the second phase of documenting the collective activity began.

In the second phase, the argumentation log created from the spreadsheet served as the data to be analyzed. In looking across all the class sessions focused on fractions, taken-as-shared ideas were extracted. These were determined using Rasmussen and Stephan's (in press) criteria for taken-as-shared ideas. When the backings and/or warrants are no longer included in students' explanations, the idea is said to be taken-as-shared. Additionally, if there is a challenge that is rejected by other members of the class an idea can be considered taken-as-shared. Another criterion for an idea attaining taken-as-shared status is that the claim, data, warrant, or backing may change function without being challenged. For example, when a claim from a previous argument serves as an unchallenged justification in a subsequent argument, the idea is considered to be taken-as-shared. So, suppose a student makes a claim that $\frac{1}{2}$ of 6 is 3, and another student sketches 6 circles and divides the set into 2 equal subsets as data for the claim. If a student later justifies that $\frac{5}{6}$ is greater than $\frac{1}{2}$ by stating that $\frac{1}{2}$ of 6 is 3, the original claim has shifted position in the argument. What was first stated as a claim was used later as data, and thus is considered as taken-as-shared.

A mathematical ideas chart was then constructed after the taken-as-shared ideas were identified based on information in the spreadsheet. This chart summarized the progression of the ideas becoming shared. Information for ideas was entered into the spreadsheet for each identified idea. The spreadsheet captured what ideas were taken-as-shared or if the research team should "keep an eye on" an idea. The ideas to keep an eye on were emerging as taken-as-shared ideas, but were not quite functioning that way yet. An example of such ideas can be taken from the class discussions in the teaching experiment.

During one episode the instructor showed a group of pattern blocks relative to a hexagon block and asked students to name a fraction to describe the group. When they were able to do that easily, the instructor redefined the whole as two hexagons. Again, students could easily name an appropriate fraction. A note was made in the ideas column that defining the whole seemed to be taken-as-shared, but more observation was needed. Actually, defining the whole became taken-as-shared later, as will be shown in the discussion in this chapter. Comments were also entered into the spreadsheet. This ideas list became the data for the next phase of the analysis.

In the third and final phase, the mathematical ideas list was used to identify ideas that became taken-as-shared. Several ideas were identified and were then organized according to the mathematical activity taking place. This level of general mathematical activity is defined by Rasmussen and Stephan (in press) as a classroom mathematical practice. This definition guided the determination of mathematical practices in this research. The resulting practices included partitioning and unitizing fractional amounts, and quantifying fractions and using relationships among these quantities. This procedure describes how the discourse was analyzed to find what mathematical practices were established.

Partitioning and Unitizing Fractional Amounts

Lamon (1999) places partitioning at the center of rational number understanding. Stated simply, partitioning is dividing a whole into parts (Pothier & Sawada, 1983). More information about partitioning and its importance to learning fractions is presented in the Partitioning section in Chapter Two. Likewise, information about unitizing is presented in the Concept of Unit section in Chapter Two. Unitizing is also fundamental to fraction understanding. However, as a quick review of important points, the fact that fractions are formed by partitioning makes it fundamental to building rational number concepts and operations (Lamon, 1999). Additionally, unitizing supports concepts such as equivalent fractions, proportional reasoning, fractions as numbers, and complex fractions (Lamon, 2002).

This broad practice of partitioning and unitizing has been segmented into four smaller ideas for this discussion. These ideas became taken-as-shared during the teaching experiment and were grouped together under the partitioning and unitizing practice. They are related to one another and only discussed separately for simplicity. They are: (a) modeling fractions with equal parts, (b) defining the whole, (c) using the relationship of the number of pieces and the size of the pieces, and (d) describing the remainder in a division problem. A discussion of each of these ideas related to the practice of partitioning and unitizing follows.

Modeling Fractions with Equal Parts

Early in the fraction unit the notion that equal parts are necessary when working with fractions was discussed. During the first session of fraction instruction, the instructor pressed students to consider this idea. The following discussion took place as part of a conversation about one fourth of a pizza. A student had just described how to represent one fourth of the pizza.

> Instructor: Why would that show one fourth? [Referring to a circle with one of the equal parts shaded]
>
> Lilly: Because there would be four parts and one of them would be shaded so it would be one out of four.

The instructor followed that explanation with another question about how fifths could be shown. She was pushing the students to recognize that the parts must be equal in size.

> Instructor: What if I were to break this (Figure 12) down into five parts? I could say these two are one part [pointing to sections labeled “3”] , these two are one part [pointing to the two lower sections labeled “2”], this is one part [pointing to the upper section labeled “2”], this is one part [pointing to the right section labeled “1”], and this is one part [pointing to the 2 left sections labeled “2”]. Could I do that and call it one fifth?

Figure 12: Model to show equal parts

> Sarah: All your parts have to be equal when you're dealing with fractions.
>
> Instructor: What do you mean?
>
> Sarah: Like there [pointing to the fourth containing sections labeled “1” and “2”] two of your parts are only equal to half of one other part.
>
> Instructor: So this is equal to half of this?
>
> Sarah: Yeah, and so you're not going to get an accurate perception of how much your answer is if they're not all equal.

This excerpt was the first mention of equal parts in the fraction unit. Although students did not offer the requirement of equal parts without a prompt, they did reject the idea that one part of four could be divided into two and the figure would show fifths when the instructor presented it. Later in the same class session, Doug referred to equal groups in an explanation. He said, “Well, I'm just thinking it doesn't matter where you cut it as long as each piece is cut into four equal pieces.” He was referring to a situation in which a candy bar with pre-partitioned sections was divided evenly among four people in different ways. His explanation acknowledged that the idea of partitioning into equal parts is necessary when dealing with fractions. There were no challenges to any of the examples discussed. Although students may not have always thought to specify that the parts must be equal, they seemed to share the idea. That is, the idea that the parts must be equal when dealing with fractions was likely taken-as-shared by the students when the instruction on fractions began. Having recognized this, students drew representations of fractions that did not have equal parts, but they would mention that the parts are supposed to be equal.

Defining the Whole

Students in the teaching experiment class defined the whole for different reasons. One reason they defined the whole was to determine what part was modeled. For example seeing 3 out of 4 equal-sized pieces suggests the fraction $\frac{3}{4}$. However, the actual value depends on the size of the whole. Another reason students defined the whole was to model a fraction when another fraction was given. For example, students may have been shown a model for $\frac{3}{4}$ and were asked to find $\frac{2}{3}$. They completed tasks similar to this by first finding the whole on which $\frac{3}{4}$ was based. Then they could find $\frac{2}{3}$ of the whole. Thus, students were unitizing, or chunking the quantity in different ways. The development of these ideas is discussed in this section.

Early in the first session on fractions, the instructor asked students to write a fraction to tell how much of the following figure was shaded.

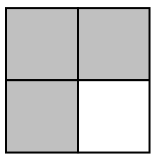


Figure 13: Figure to demonstrate defining the whole

Amy confidently answered that $\frac{3}{4}$ of the figure was shaded. She justified her response by saying that three of the four total pieces were shaded. She also recognized that the pieces were equal in size. The instructor acknowledged the students' comfort level with such a problem and asked them what would happen if the whole being considered was made up of the following figures.

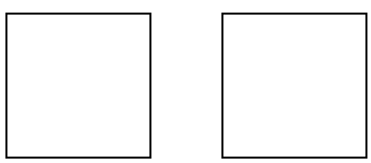


Figure 14: Squares to redefine the whole

Several students knew that the shaded portion would represent $\frac{3}{8}$ if the size of the whole doubled. Doug acknowledged that the whole would have eight parts in his explanation of why the figure represented $\frac{3}{8}$: "This one unit [pointing to the original figure as shown in Figure 13] would equal two of those [pointing to one of the squares in Figure 14], and so you already have them divided into four equal parts. So this would be in four equal parts. That would be in four equal parts; seven, eight. Eight altogether, eight equal parts." Kathy confirmed Doug's thinking by stating that, "If you took away the lines and you counted the different amount of quadrants or whatever you want to call that you'd still have eight. You'd still have only three shaded in. So that's three out of eight." These conclusions seemed guided by the instructor's suggestion to change the whole. That is, when directly shown the whole, it seemed easy for the students to name the shaded part. As the tasks became more complex, this was not always the case.

In the next task, still on the first night, the instructor used pattern blocks to elicit fractional names. First, she showed a red trapezoid on a yellow hexagon as shown in Figure 15 and asked what part of a whole the red represented. Joe quickly answered $\frac{1}{2}$, which was correct.

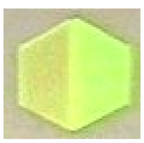


Figure 15: Red trapezoid on yellow hexagon

Figure 16: Whole defined as two yellow hexagons

The instructor then showed two hexagons (Figure 16) and stated that they now represented the whole. The following exchange between the instructor and Joe indicates that naming fractional parts of a different whole did not seem to be difficult for students, but naming the whole seemed more challenging.

Instructor: This is my whole [showing two yellow hexagons as in Figure 16].

Joe: It would be $\frac{1}{4}$.

Instructor: Why did you change your answer?

Joe: Because the whole changed.

Instructor: So, when you look at it here [referring to Figure 15] it was $\frac{1}{2}$ of the yellow hexagon, and here [referring to Figure 16] it is $\frac{1}{4}$ of what?

Joe: uh

Instructor: Two yellow hexagons.

Joe was quick to identify the red trapezoid as $\frac{1}{4}$ of the new whole, but had trouble naming the whole. This was a bit puzzling because he was looking at the image shown in Figure 16. Perhaps Joe was thinking there was a different name for the shape formed by the two hexagons.

After the work on naming fractions with respect to the whole, the class engaged in an activity in which they needed to define the whole, given a part of it. In the following discussion, Joe was explaining how he determined what $1\frac{1}{8}$ would be given $\frac{3}{4}$ of a set. The $\frac{3}{4}$ of the set was represented by six large dots as shown in Figure 17. In order to find $1\frac{1}{8}$, Joe first found the whole.

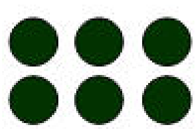


Figure 17: Representation for $\frac{3}{4}$ of a set

Joe: Since the first two dots equal $\frac{1}{4}$, the second two dots equal $\frac{2}{4}$ and the third dots equal $\frac{3}{4}$ the fourth two dots equal $\frac{4}{4}$. That's 4 over 4—a whole, which is one. That whole thing is 1. That actual dot [pointing to a single dot] is $\frac{1}{8}$.

Instructor: How do you know that's $\frac{1}{8}$?

Joe: Because the whole [interrupted]

Instructor: Katrina.

Katrina: The way I looked at it was the one dot was like you were starting a whole again but we only had 1. We only wanted 1 of 8. So, it would be the 1 dot.

Joe explained that each column of dots represented $\frac{1}{4}$ of the whole. He referred to the fourth pair of dots that are not shown in the drawing. He meant that if there were 2 more dots, for a total of 8, the whole would be represented. He also explained that a single dot would be $\frac{1}{8}$. Katrina helped him finish the explanation. She had also found the whole and knew that 1 more dot would be $\frac{1}{8}$ of the whole. The next problem was similar.

Students discussed how to find $\frac{5}{7}$ of a set, given that 9 triangles represented $\frac{3}{7}$ of the set as shown in Figure 18.

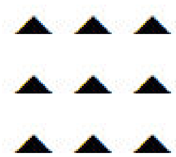


Figure 18: Representation for $\frac{3}{7}$ of a set

Carrie: I added four more rows of 3 triangles.

Instructor: Four more rows?

Carrie: Yeah, and that made it a whole which is $\frac{7}{7}$. And, then to do $\frac{5}{7}$, I drew 5 rows of the 3 triangles.

Like Joe, Carrie's strategy was to first determine what the whole set would be. In this case, it was 7 rows of 3 triangles. Once she knew that, she could identify 5 rows of triangles as $\frac{5}{7}$. At this point defining the whole seemed to be taken-as-shared. Students completed several problems similar to the previous examples. In each case, they defined the whole and then determined the amount for which they were asked. However, after completing these tasks, students did an equal sharing problem. In this situation presented in context, they did not seem to remember to define the whole.

The equal sharing problem occurred late on the first night of the fraction unit. Students had completed several tasks focused on defining the whole. This task required students to tell what part of a cookie each person would get if 4 cookies were shared equally among 5 people. The following discussion eventually resulted in the conclusion that it is important to define the whole when answering a question. The class was discussing why some people arrived at an answer of $\frac{4}{20}$, or $\frac{1}{5}$, and others thought the answer was $\frac{4}{5}$. First, Doug explained how his group arrived at and answer of $\frac{4}{20}$. Then Kim justified her group's answer of $\frac{1}{5}$.

> Instructor: OK so who got $\frac{4}{20}$ again? Do you want to explain to us how you did this?
>
> Doug: What I did, since there are 5 people I took each cookie and divided it into 5 parts. So this would make 5 pieces, and this would be 5 pieces and 5 pieces and 5 pieces [pointing to each cookie in Figure 19]. Add all the pieces together would be 20 pieces. If each person took one piece out of each one that would be 4 over 20…

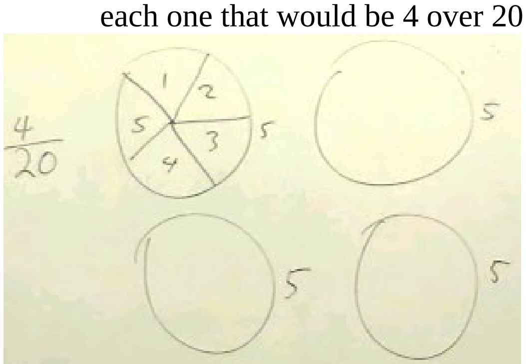


Figure 19: Sharing 4 cookies among 5 people

> Kim: Since each cookie is broken into five pieces each person would get one of those 5 pieces, which is $\frac{1}{5}$ of each cookie. That's how we're saying that you got $\frac{1}{5}$ or how you could explain getting $\frac{1}{5}$.

Instructor: So they got $\frac{1}{5}$ of each cookie. What does $\frac{4}{20}$ mean?

Kim: $\frac{4}{20}$ means they split each of the four cookies into 5 pieces so it's a total of 20 pieces and each person got 4 of the 20 pieces…

Carrie In the $\frac{4}{20}$ you're comparing all the pieces as separate pieces, not as cookies. Like it would be as if you drew like 20 triangles, the $\frac{4}{5}$ is you're comparing them as cookies. It would be the $\frac{4}{5}$ is representing 4 pieces out of each cookie…

Katrina: We've got 5 people and every cookie divided into 5 pieces. Each person gets 1 piece of that first cookie. So they're going to get $\frac{1}{5}$ of each cookie and then the same thing with the other 3 cookies…

April: What I think he did was he looked at the 4 cookies as the unit as the whole and he's giving you the portion of the whole unit of 4 cookies which is correct if you're looking at it that way. It's $\frac{1}{5}$ of the whole. …

Instructor: So you've shown me that our answer could be $\frac{1}{5}$ and our answer could be $\frac{4}{5}$.

Kathy: It depends on what you say after the $\frac{1}{5}$ or the $\frac{4}{5}$.

Instructor: What do you mean by that?

Kathy: Like the $\frac{4}{5}$ is $\frac{4}{5}$ of 1 cookie. And the $\frac{1}{5}$ is of the total cookies.

In this episode, there was confusion about the correct answer because some students used 1 cookie as the unit. Other students answered in terms of the set of all the cookies. Doug began the discussion by stating that he divided each cookie into 5 parts since there were 5 people. When he added all the fifths and expressed the answer as $\frac{4}{20}$, he did not realize he had merely said each of the 5 people got $\frac{1}{5}$ of the group of cookies. Carrie introduced the idea that each person got $\frac{4}{5}$ of 1 cookie. Katrina agreed with Carrie and explained that each person would get $\frac{1}{5}$ of each cookie, and since there were 4 cookies, that would be $\frac{4}{5}$ of a cookie. April was able to identify the reason for the discrepancy and explained that Doug defined the whole as all of the cookies rather than as 1 cookie. Kathy reinforced that idea when she said it depends on what you say after the fraction.

This is one way to think about defining the whole. Although no one used the exact phrase "define the whole," this episode was all about defining the whole. The exchange served to point out that it is important to know what the unit, or whole, is when solving problems with fractions. This was not the end of discussion related to defining the whole. Other opportunities to examine the unit, or whole, occurred with division problems, namely describing what is left over in a division problem. It is discussed later as a separate idea related to this practice. The division problems were presented near the end of the fraction unit, and other ideas related to partitioning and unitizing emerged prior to that. The next idea discussed here is using the relationship between the number of pieces and the size of the pieces.

Using the Relationship between the Number of Pieces and the Size of the Pieces

The relationship between the number of pieces and the size of the pieces was often referred to in justifications and explanations yet did not immediately become taken-as-shared. The first time the idea was mentioned occurred in a discussion late in the second class session on fractions. The discussion was about children's misconception that a greater number of pieces means a greater amount than fewer pieces when the amounts are actually equal. The example in this case had to do with a candy bar that was divided into equal amounts several ways (see Appendix H). The candy bar was segmented into 16 equal-sized pieces, and each of 4 people divided it into fourths in a different way.

Quentin divided it into 4 pieces, with each piece representing $\frac{1}{4}$ of the whole candy bar. Randy and Stephanie each divided it into 8 pieces in different ways, with each piece representing $\frac{2}{8}$. Finally, Tina divided it into 16 pieces, with each share representing $\frac{4}{16}$. The instructor asked a question about the misconception that the greater number of pieces would be more. Lilly, who responded, may have been thinking about the relationship between the number of pieces and the size of the pieces, but did not explicitly state it as such. Lilly did mention that the pieces would be smaller, however.

> Instructor: What would you say to the student who thought that $\frac{4}{16}$ was larger than the others?
>
> Lilly: It's only 4 out of 16 pieces that they're getting. That 16 pieces means their candy bar is going to be cut into smaller pieces. So it's going to seem like they're getting more but they're getting the same amount.

In this excerpt, Lilly set the stage further discussions about the relationship between the number of pieces and the size of the pieces. When she said it seems like they will get more, she was referring to the fact that they would get 4 pieces instead of 1 or 2 larger pieces in the other shares.

Later in the same class session, a student explicitly stated the relationship. Carrie referred to the relationship when she explained that $\frac{1}{4}$ and $\frac{1}{5}$ are two fractions between $\frac{1}{3}$ and $\frac{1}{6}$. She used imagery of pieces of pie to explain her thinking. Her group had discussed that as more pieces are cut, the pieces get smaller.

> Carrie: We didn't know how to find it. Like we couldn't divide anything to get these numbers. We decided to make pies and we realized the pieces were getting smaller. As the denominators increased the pieces got smaller.

Carrie gave a succinct explanation of her thinking. She noted that the pieces of pie she used for imagery got smaller as more pieces were cut. Carrie’s statement was not challenged by anyone else and it seemed that perhaps the relationship of the number of pieces and the size of the pieces could be taken-as-shared. However, this idea was presented several more times; and, it was further explained.

One such example occurred later in the same discussion about naming fractions between $\frac{1}{3}$ and $\frac{1}{6}$. The instructor had drawn models on the board during Lilly’s explanation. Kim suggested that she draw them again in order from least to greatest. The drawing is shown in Figure 20, and the discussion follows.

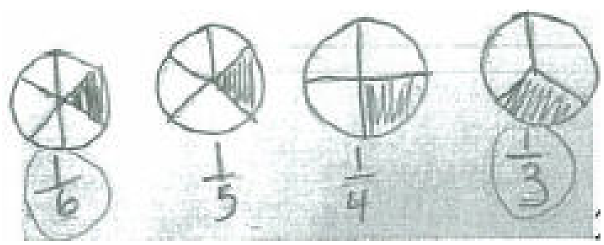


Figure 20: Drawings to order fractions

> Instructor: How do these pictures [referring to Figure 20] help you see that it's going from smallest to greatest?
>
> Lilly: Because there's less lines.
>
> Instructor: Why is that important?
>
> Lilly: There's a less amount of equal parts to split up.
>
> Instructor: What does that have do with each equal part?
>
> Lilly: It makes it bigger. When you go lower numbers, as the denominator gets smaller the pieces get bigger.
>
> Instructor: Why do the pieces get bigger?
>
> Lilly: There is less pieces that you have to cut in the pie.

David: When you have like one pie and six people you're going to have smaller pieces.

Instructor: So it's better to invite fewer people to your party? Each had one piece. How do we know that $\frac{1}{5}$ and $\frac{1}{4}$ are between $\frac{1}{6}$ and $\frac{1}{3}$?

Lilly: That [referring to the circle showing $\frac{1}{6}$] has more slices and the pieces are smaller. And that one [referring to the circle showing $\frac{1}{3}$] has the least amount of slices and the slices are bigger.

Instructor: Don't they each have the same number one, one, one? They each have one slice.

Lilly: But that's not the size that the slices are going to be when it's split into equal parts. As the number on the bottom gets smaller, the pieces get bigger.

The instructor was probing the student in the preceding example and pushed the discussion to support the development of the idea that the greater number of equal pieces a whole is divided into, the smaller the pieces are. Because the instructor wanted to make this point, she challenged students to explain this in several different ways. This may be why the idea did not appear to be taken-as-shared. Lilly rejected the instructor's suggestion that the shares were each just one piece and therefore equal. The relationship of the number of pieces and the size of the pieces was discussed several more times in subsequent problems, and seemed to be becoming taken-as-shared.

Another mention of the relationship between the pieces and the size of the pieces was during a discussion of three fractions between $\frac{8}{9}$ and 1. This took place later on the same night as the previous example. In this case, Matt is reiterating an explanation given by David in which he chose to use fractions with a numerator of 800.

It's the same example as before [referring to an earlier problem worked by using the same numerators] because when the numerator is the same and

> the denominator is bigger or smaller your pieces are going to be bigger or smaller because you're taking the same amount. But it's from a smaller; because you're cutting it off more times. You're cutting it off less times from the same amount. You're going to have bigger pieces in the 800 and smaller in the 900. That's all you have to tell.

Matt gave a matter-of-fact justification that drew upon the imagery of pieces of a pie or pizza. He did not actually draw a model to help him explain it. This may indicate that the concept had become more familiar to him and he did not need the visual stimulus to understand it. It should be noted that, with the preceding example, the justification perhaps was not necessary for the class, but the norm to explain answers had been established. Matt may have been simply justifying his answer due to that expectation.

The next reference to the relationship between the number of pieces and the size of the pieces occurred in the same discussion of fractions between $\frac{8}{9}$ and 1. April was trying to help Kathy understand the relationship. She offered the following help.

> You're cutting it into more pieces. Therefore, each of the pieces is getting smaller and smaller. You're using the same number of pieces, but the pieces are getting smaller and smaller. If you cut a candy bar into sixths would you rather have one piece or would you rather have one piece if you cut it into thirds? I'd rather have it if you cut it into thirds."

April used imagery of a candy bar to make her point. By asking which piece the class would rather have given a choice of 1 of 6, or 1 of 3, she added a context that was easy for students to relate to. The imagery of equal shares of some type of food played an important role in the development of the partitioning and unitizing practice.

Later examples were not challenged, but included some kind of imagery as part of the explanation. One such instance occurred when students were comparing $\frac{1}{3}$ and $\frac{1}{2}$ on the third night of the instructional sequence. David said, "I'd rather share [pizza] with one other guy than two other guys. I know if I share it with one other guy I have a bigger slice of pizza. That's why $\frac{1}{2}$ is greater than $\frac{1}{3}$." David also explained later that, "You can say that $\frac{1}{7}$ is bigger than $\frac{1}{8}$ because if you cut a pie into seven pieces you have big pieces. If you cut it into eighths, the pieces are a little smaller. So, if you have four of the big pieces and three of the smaller pieces the four is bigger." In both of the justifications, David relied on imagery of pizza or pie to convey the relationship. In the latter example, David first used the relationship of the size of pieces to the number of pieces to justify another answer. First he stated the relationship, noting that $\frac{1}{7}$ is bigger than $\frac{1}{8}$, then he went on to use that information to explain why $\frac{4}{7}$ is greater than $\frac{3}{8}$. His reasoning was simply that having more of the larger pieces makes a larger portion than having fewer smaller pieces.

The final evidence that, the relationship between the size of the pieces and the number of pieces could be considered taken-as-shared was on the third night of the fraction unit when Kathy referred to the "theory of the bigger the denominator the smaller the pieces." She was justifying why she said $\frac{2}{5}$ is greater than $\frac{2}{7}$. Note that Kathy succinctly stated the relationship in her reference to the theory.

> I'd say $\frac{2}{5}$ is bigger because you go back to the theory of the bigger the denominator the smaller the pieces. So, you know if you have to divide up the pizza by 5 people and only 2 of them take a slice you have 3 slices left over. …It all goes back to the point of the bigger the denominator the smaller the pieces. So you know that $\frac{1}{5}$ is larger than $\frac{1}{7}$ on the theory of what Lilly was saying earlier.

It seems that she may have continued her explanation beyond what was actually needed. In this explanation, she restates the same idea several times. This may again be attributed to the norm to explain answers. Nevertheless, the students accepted her explanation. In fact she did not reference cutting anything. Her explanation was given in terms of a denominator and pieces. At this point the relationship of the number of pieces and the size of the pieces was taken-as-shared. Students used this relationship to explain and justify their solutions to other problems.

Describing what is left over in a division problem

The final aspect of the partitioning and unitizing practice is related to defining the whole. It is treated separately because it is a specialized application of defining the whole that was introduced late in the fraction unit. In order to answer the division problems presented in class correctly, students needed to describe what was left over in a division problem. This meant they needed to know what the unit was and describe the remainder in terms of that unit. On the fourth night of the fraction instructional sequence, the class was presented with word problems that could be solved by dividing by a fraction. The first problem was: “It takes $\frac{3}{4}$ foot of wood to make a picture frame. How many $\frac{3}{4}$-foot lengths can Pat cut from an 8-foot board? What part of another picture frame would she have left?” The second question was included to prompt students to think about the answer with respect to the unit, a picture frame in this case. In the following exchange, Doug explained how he arrived at his answer of 10 frames with enough for $\frac{2}{3}$ of another frame left over. In order to explain his solution, Doug drew a representation of the 8-foot board (Figure 21) and showed the $\frac{3}{4}$-foot segments on it. First he took $\frac{3}{4}$ foot out of each foot segment. Then he combined the leftover fourths to make the ninth and tenth groups (See Figure 22). He had two pieces that were one fourth of a foot left over. His modeling may have helped him to see that the leftover $\frac{2}{4}$ foot was only enough for $\frac{2}{3}$ of another picture frame.

Figure 21: Drawing of 8-foot board

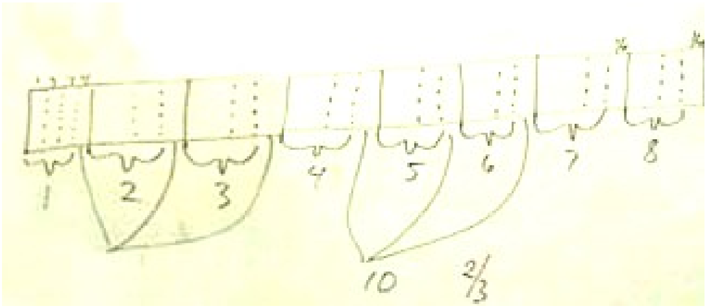


Figure 22: Drawing showing divisions for picture frames

> Doug: The best way I could understand it was I made one large 8 foot long plank, piece of wood, and divided it equally into eight feet.…[Figure 21]. It took $\frac{3}{4}$ of a foot of wood to make a picture frame, so, I knew this would be a foot [pointing to one section of the drawing, shown in Figure 22], so $\frac{3}{4}$ would be half of a half [implying $\frac{1}{2}+\frac{1}{4}$, as shown in Figure 23]. So this would be $\frac{3}{4}$ [pointing to the bracket portion in Figure 23].

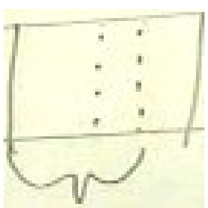


Figure 23: Detail indicating $\frac{3}{4}$ of a foot

> Instructor: Do you see what he's done? Because I only see that he's dividing it into three sections.

> Doug: I can make this into four pieces one, two, three, four. But at the same time I just did half and another half. So I saw here's one frame, two frames, three frames, four frames five, six, seven, eight. I know it takes three of the quarters to make one. Here's one, here's two, here's three, so this is number nine. Here's one, here's two, here's three. So that's number ten. Then I have one quarter left, another quarter left. It takes three of them so I have ten with enough for $\frac{2}{3}$ of a frame left over.

After Doug’s explanation, the class engaged in more discussion of the problem because other students in the class had arrived at an answer of $10\frac{2}{4}$. Matt shared his thinking and why he believed the answer was $10\frac{2}{4}$.

Matt: I added $\frac{3}{4}$ as many times.

Instructor: So you added $\frac{3}{4}$.

Matt: To $\frac{3}{4}$ to $\frac{3}{4}$ until I got as close as I could to 8 total.

Instructor: How many did you do?

Matt: I got to 10 of them. On the tenth picture, it was 30 over 4.

Instructor: Ten times was 30 over 4. What did you do with that?

Matt: And that converts to $7\frac{2}{4}$, and that's where I got to and that was on the tenth picture. So, I said it could make 10 picture frames. So, what I have left over was $\frac{2}{4}$ of wood.

Instructor: Because $7\frac{2}{4}$ plus $\frac{2}{4}$ is 8. So, how in the world can we do this? They're clearly not the same [referring to $10\frac{2}{3}$ and 8].

Carrie: Because it asks what part of another picture frame would you have left, and you only need $\frac{3}{4}$ to make a picture frame—not $\frac{4}{4}$.

Instructor: Matt.

Matt: Yeah, that’s good.

Instructor: What's good about it?

Matt: Well, I'm just assuming, you know, a whole. The next number would be a whole $\frac{4}{4}$ because we're used to that making a whole thing. So instead we need $\frac{3}{4}$.

Instructor: We still have $\frac{2}{4}$ left. You still have that left. Right, but $\frac{2}{4}$ isn't half of $\frac{3}{4}$. It’s $\frac{2}{3}$ of $\frac{3}{4}$. April.

> April: What you have there, that $\frac{2}{4}$ is $\frac{2}{4}$ of feet of wood not of a frame. So it's a half of foot left.

Matt eventually recognized that the remainder was $\frac{2}{4}$ of one foot. He admitted that he had assumed there were $\frac{4}{4}$ in the whole. He realized that in this situation the unit was $\frac{3}{4}$. April explained the remainder is actually only enough for $\frac{2}{3}$ of a picture frame. The next problem presented another opportunity to express the remainder in terms of the whole.

The problem was: “Pete is building a model of a city for a school project. He needs to cut lengths of a board that measure $\frac{1}{4}$ foot each to make the skyscrapers. How many whole skyscrapers can he cut from a board that is $1\frac{7}{8}$ feet long? What part of a skyscraper will he have left over?” April gave the following explanation.

> April: So what I did was I drew a board to represent the big whole long thing Figure 24] and I said this is 1foot, this is 2 feet. The problem is I only have $1\frac{7}{8}$ so I'm actually $\frac{1}{8}$ short so what I said from the very beginning is since I'm $\frac{1}{8}$ short I might as well cut everything up into eighths. And that way I can easily cut off what I don't actually have. So here's $\frac{7}{8}$ I don't actually have that [pointing to the last $\frac{1}{8}$] but here's the rest of it.

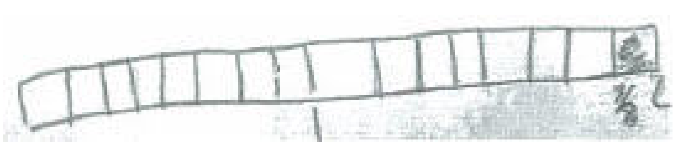


Figure 24: Drawing of board measuring $1\frac{7}{8}$ foot

> So I went ahead and marked it one eighth, two eighths, three eighths, four eighths, five eighths, six eighths, seven eighths. And I did the same thing down there [Figure 25]. And I said well by looking at this I can tell that $\frac{2}{8}$ and $\frac{4}{8}$ are $\frac{6}{8}$. This [pointing to two of the $\frac{1}{8}$ parts] is equivalent to $\frac{1}{4}$ so that means one skyscraper. Here's another quarter so that's another skyscraper here's a third one here's my fourth one. These are eighths as well so I said OK here's another one that's 5. Here's another one. That's 6,

here's another one. Here’s another one. That's 7. But when I get here I can't do that because I don't have this [pointing to the final section that is shaded]. It's gone so I only have half of what I needed. So it's only $\frac{1}{2}$ of a skyscraper. So I have 7 whole skyscrapers plus $\frac{1}{2}$.

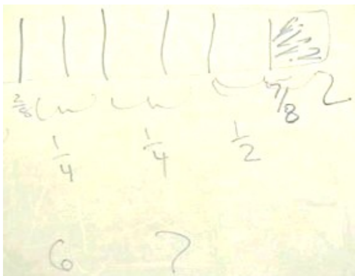


Figure 25: Drawing of board marked to show “skyscrapers”

There were no questions about April’s explanation when the instructor asked for them. Seeing an opportunity to reinforce the notion of how to interpret the remainder in a fraction problem, she prompted the students to think about the remainder and to confirm that April’s answer was correct. A portion of that discussion follows.

Instructor: Then if we were to write a number sentence to go with this what would it look like? Is it multiplication, division, addition, subtraction? Sarah?

Sarah: $1\frac{7}{8}$ divided by $\frac{1}{4}$.

Instructor: But what would $1\frac{7}{8}$ divided by $\frac{1}{4}$ equal?

Sarah: Seven and a half

Instructor: So even if we didn't have skyscrapers you think that half would be the answer? It would be $7\frac{1}{2}$ and not $7\frac{1}{8}$? …

Sarah: OK what's left is $\frac{1}{8}$ left over. But you're dealing with, you're dividing it by $\frac{1}{4}$ and $\frac{1}{8}$ is $\frac{1}{2}$ of $\frac{1}{4}$ and so $\frac{1}{4}$ goes into $1\frac{7}{8}$ seven and a half times.

The discussion continued, and Matt voiced his disagreement with the answer of $7\frac{1}{2}$. Recall that Matt disagreed with the answer in the previous example. Matt thought the answer was $7\frac{1}{8}$ and challenged the answer of $7\frac{1}{2}$. His answer was based on $\frac{1}{8}$ foot of the board being left over. He did not seem to realize that the left over portion was only enough for half of another skyscraper. The continued discussion follows.

> Matt: It wouldn't equal $7\frac{1}{2}$ because we're not doing it by a whole number…
>
> Instructor: So, what do you think the answer would be?
>
> Matt: I think it would be seven and $\frac{1}{8}$ if you just did it straight.…
>
> Sarah: I haven't checked it so I don't know for sure. With how I look at it it's not going to be that way because you're not dividing it. You have your 8 pieces and you have 7 of them. You find out how many times $\frac{1}{4}$ goes into it. It goes into it 7 whole times and then a half more.
>
> Instructor: Lilly.
>
> Lilly: On the drawing, if you look at it, it shows there's that dark shaded area—that one [pointing to the shaded section]. It would go 7 times and then $\frac{1}{8}$ left.…
>
> Instructor: So some people think it's $7\frac{1}{2}$ and some people think it's $7\frac{1}{8}$. …

Students attempted to validate their answers in different ways. Matt seemed to imply that the answer to a bare computation problem, without the context, could be different from the answer to the problem in context. He made a point to say he thought the answer would be "$7\frac{1}{8}$ if you just did it straight." Sarah seemingly wanted to rely on the algorithm. She said, "I haven't checked it so I don't know for sure." Then Lilly explained her reasoning by using a drawing. The discussion continued, and Doug justified his answer by defining the unit as $\frac{1}{4}$. Matt was subsequently convinced that the answer was $7\frac{1}{2}$. However Lilly still was not sure as the discussion continued.

> Doug: You're still looking at $\frac{1}{4}$. You're dividing by $\frac{1}{4}$. That's the unit you're comparing everything to $\frac{1}{4}$. And so when you have $\frac{1}{8}$ left it's half of a fourth. You're dividing by $\frac{1}{4}$ that's what we're looking at—the $\frac{1}{4}$ and the eighth that's left over—eighth of this one section is a half of one quarter. So it has to be $7\frac{1}{2}$.
>
> Instructor: Lilly.
>
> Lilly: But half of a board is not $\frac{1}{2}$ of $\frac{1}{2}$.
>
> Doug: You have to look at the quarter. The quarter is what you are looking at. That's what you are comparing everything to. You are dividing by a quarter and so 4 equal parts of that and so an eighth is really only half of a quarter so what we're looking at instead of an eighth is going to be a half of that quarter, so it will still be $7\frac{1}{2}$.
>
> Instructor: Matt.
>
> Matt: Yeah, it's $7\frac{1}{2}$ because what Doug said. Because it was half of $\frac{1}{4}$ and still you're dividing by $\frac{1}{4}$ and if you keep going by $\frac{1}{4}$, half the distance, you could keep going in even numbers then you get to $1\frac{6}{8}$ then you can only go half that amount of times to get to $1\frac{7}{8}$ so it's $7\frac{1}{2}$ because you can only go $\frac{1}{8}$ more so it's half of that.
>
> Instructor: Lilly is still shaking her head.…
>
> Kim: OK forget the skyscrapers. If I'm saying I want to divide $1\frac{7}{8}$ feet into $\frac{1}{4}$ foot, $\frac{1}{4}$ of a foot sections, then you're still talking about feet.
>
> David: Then you would get $7\frac{1}{2}$ sections of a foot. You're not talking about distance from here to here. Now you're asking how many slots you have.
>
> Instructor: Matt.

Matt: You could draw it out so if you kept adding you could see it's half of what you need.

Instructor: Of the part she's got shaded.

Matt: You could do it without shading. You could start with $\frac{1}{4}$ and convert it into eighths because the answer's $1\frac{7}{8}$. You've got $\frac{2}{8}$ and you want to fill the first part so you want to get like $\frac{8}{8}$, so you have to do $\frac{2}{8}$ plus $\frac{2}{8}$ is $\frac{4}{8}$ then $\frac{4}{8}$ plus $\frac{2}{8}$ is $\frac{6}{8}$. And then you add and you've got $\frac{8}{8}$ and you've done that four times. And then you have to get your $\frac{7}{8}$ of a foot and you're doing it by $\frac{2}{8}$. So you start by $\frac{2}{8}$ plus $\frac{2}{8}$, which is $\frac{4}{8}$. And then you would $\frac{4}{8}$ plus $\frac{2}{8}$ which is $\frac{6}{8}$ and you still be short $\frac{7}{8}$ you can't put $\frac{2}{8}$ into $\frac{7}{8}$. It doesn't go. So you have to take [pause] it's only going to go up once. So, you have to go up $\frac{1}{8}$ from $\frac{6}{8}$. That's $\frac{1}{4}$. I mean it's $\frac{1}{8}$. So it's half of the amount you have to go. You went in it 7 times evenly, and the last time you had to go half as you had been going in every other time. Because you go in $\frac{2}{8}$ and you know you're going $\frac{1}{8}$. And $\frac{1}{8}$ is half of $\frac{2}{8}$. You know it's half a time, so $7\frac{1}{2}$….

David: The reason why it's not $7\frac{1}{8}$ is because when you get to here [pointing to the end of the seventh segment in Figure 26], the 7, to get another one it's only halfway. The eighth would be this right here [pointing to the $\frac{1}{8}$ foot of the board left in Figure 26].

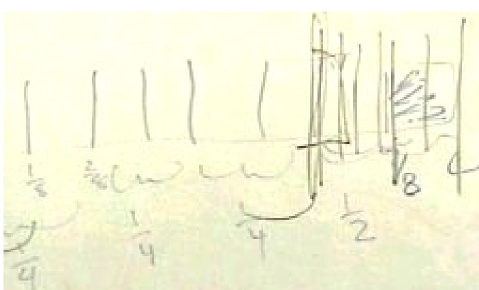


Figure 26: Drawing to show the remainder is half of the amount needed

Lilly nodded her head in agreement and said she was "getting it." With his description of number of slots, as opposed to the linear distance, David communicated that the unit was not feet. The confusion may have been that the students were more familiar with measurement problems in which the unit, or label, was a unit of measure such as foot. In this case, several students explained how to define the unit in terms of a picture frame.

Another discussion of a division problem with fractions occurred about midway through the last night of the instructional sequence. The problem, $1\frac{1}{2} \div \frac{2}{3}$, was presented without a context, which may have made it more difficult to describe the remainder. Nevertheless, one group created a scenario to help them make sense of the problem. First, Lilly proposed a scenario for the problem. Her scenario was actually a subtraction situation—taking away $\frac{2}{3}$ pie from $1\frac{1}{2}$. The following discussion begins with a division situation proposed by Sarah's group and continues with Lilly's explanation. Her drawing is shown in Figure 27.

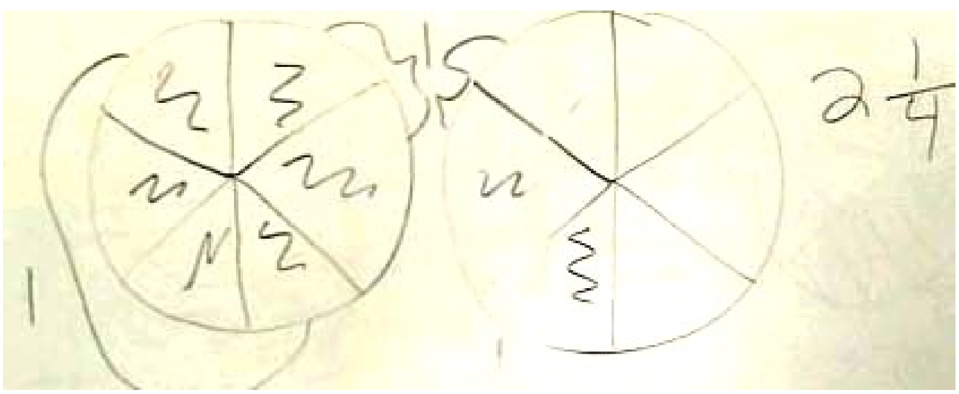


Figure 27: Dividing with a circular area model

> Sarah: We thought it was easier to do the problem if we come up with the situation first and we just kind of took from the skyscraper one. We said that $\frac{2}{3}$ of a foot make one skyscraper and you have one and a half feet of board, so how many skyscrapers can you get out of that?
>
> Instructor: [To Lilly] How is that different from the one you are doing?
>
> Lilly: I broke mine into equal pieces so I could see how much each one could equal—one total slice.

Instructor: Keep going and let's see if we can stick this skyscraper scenario on top of what you're doing.

Lilly: So then I broke down my thirds into six pieces. It's not coming out how I did on my paper. Oh I did it like this and I knew I had this [shaded $\frac{1}{3}$] and I broke down this one. And I knew I had all of this [shaded entire first pie]. And I know that $\frac{2}{3}$ is equal to $\frac{4}{6}$. So, I got this to equal one portion of the pie, and then I got these and these to equal another portion of the pie out of 6 six total pieces. And then I had this one left. So altogether I had two and a sixth of a pie.

Instructor: Questions for Lilly?

Lilly: Sarah.

Sarah: You got the right answer but I don't think the problem you came up with would have given you two and one sixth. Because you said that you had $1\frac{1}{2}$ pies and you're taking $\frac{2}{3}$ away and you have $\frac{5}{6}$ left.

Instructor: Unless you said how many groups of $\frac{2}{3}$ can you take away.

Lilly: And then I know that $\frac{2}{3}$ is equal to $\frac{4}{6}$ so then I took $\frac{4}{6}$.

Instructor: David

David: I think she did it fine.

Instructor: Kim.

Kim: I did something similar but I did mine like a number line. Anyway, I still ended up with 2 groups and a sixth left. But then I looked back to our other problem where I thought it was an eighth but it wasn't an eighth—it was a half, and I knew that that $\frac{1}{6}$ that was left was only $\frac{1}{4}$ of what I needed. So, my answer was $2\frac{1}{4}$.

Instructor: So. Kim did the same problem and got $2\frac{1}{4}$ as an answer. How many people got $2\frac{1}{6}$ as the answer? Several people got $2\frac{1}{6}$. How many people got $2\frac{1}{4}$? About half the class got $2\frac{1}{4}$. Did anyone get another answer? Now we need to decide is it $2\frac{1}{6}$? Is it $2\frac{1}{4}$?

Instructor: Amy.

Amy: I was imagining a number line. We're dividing by $\frac{2}{3}$ so what is $\frac{1}{6}$ of your $\frac{2}{3}$? $\frac{1}{4}$ — you're looking at $\frac{1}{6}$ of your $\frac{2}{3}$, not OK I have $\frac{1}{6}$ left. Because you're dividing by $\frac{2}{3}$.

Instructor: April.

April: I looked at it this way. How many even size pieces do I need to make one? In this case she drew an arrow around 1, 2, 3, 4. And on the second one she drew an arrow to 2 and 2 she needed 4 even size pieces to make the whole. You only have 1 left. You only have 1 of the 4 you need so you have $\frac{1}{4}$. That's the way I think of it— $\frac{1}{4}$ of what I need.

There were no further challenges to the answer of $2\frac{1}{4}$. However, it cannot be said that unitizing the remainder in a division problem had become taken-as-shared. Several students answered $2\frac{1}{6}$. One final division problem was discussed immediately after this problem. It was $3 \div \frac{5}{8}$, without a context.

Darren went to the board to present his group's solution. His drawing is shown in Figure 28. In his explanation, he refers to "squares," meaning the rectangles he had drawn. The following exchange indicates he had some trouble describing the meaning of the operation, but he did successfully describe the remainder.

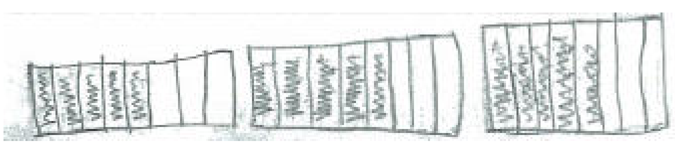


Figure 28: Dividing with a rectangular area model

Darren: What we did was to divide 3 squares up into eighths. OK, we need to find 3 groups of $\frac{5}{8}$ so we shade in 5 of the 8 squares.

Instructor: Wait, what are we trying to find? Say that again.

Darren: I don't know what I just said.

Instructor: Then say what you mean. What's the problem asking you?

Darren: We need to find $\frac{5}{8}$ split up 3 ways.

Instructor: Laura, what do you think?

Laura: We need to find how many times $\frac{5}{8}$ goes into 3.

Instructor: [To Darren] Does that make sense?

Darren: Yeah.

Instructor: OK.

Darren: So we shade in 5 squares on each box. So now we have 3 times already. But there's still enough to equal 5 of the blank ones. So we can color in here--1, 2, 3, 4, 5 [marking unshaded segments] to make 4. But we only have 1, 2, 3, 4 empty boxes, so we can't fill another 5. So, that's 4, and since the whole—the unit we're looking for —is 5, it's 4 out of 5. So, it's $4\frac{4}{5}$.

Darren’s group successfully described the portion of the rectangle left over as $\frac{4}{5}$ of a group of $\frac{5}{8}$. There were no challenges to his solution. It seemed that describing the leftover portion in a division problem had become taken-as-shared. This discussion occurred near the end of the final class session on fractions, and it was the last division problem to be discussed in the fraction instructional sequence. There was not a chance to see if this taken-as-shared status was maintained.

This concludes the discussion on the practice for partitioning and unitizing. Students came to share knowledge about using equal parts to model fractions. This was often assumed as models were not always drawn with equal parts. Students also shared knowledge about defining the whole, or unit, when naming fractions. This was true when fractions were presented with models and not within a context. Students also relied on the relationship between the number of pieces and the size of the pieces to explain their solution processes. They used imagery of pies, pizza, and candy bars to help them explain the relationship. Finally, describing what is left over in a division problem was presumably established. Unfortunately, there was not an opportunity to observe this practice being maintained. The next practice is quantifying fractions and using relationships among the quantities.

Quantifying Fractions and Using Relationships among These Quantities

A "notion of quantity" is discussed in Chapter Two. Part of this quantitative understanding of rational numbers is knowing that fractions have quantitative values and that those values are related in a variety of useful ways. As with the partitioning and unitizing practice, there are several ideas associated with this practice. The first idea deals with naming and modeling fractions. In a sense, this is the core definition of fractions. It involves drawing a model for a fraction name as well as assigning a value to a fraction model. This naming and modeling is the foundation on which later dealings with quantities are based. The second idea associated with this practice extends modeling of fractions in that it involves naming a fraction that is equivalent to a given fraction. Finding equivalent forms of fractions by breaking pieces into smaller pieces is discussed in this idea. The final idea associated with this practice involves using relationships among fractions to describe quantities. This includes describing one fraction in terms of another, such as "$\frac{1}{4}$ is half of $\frac{1}{2}$." Another way students used relationships to describe quantities was to use benchmark fractions to compare and order fractions. Each of these ideas related to this practice will be discussed in this section.

Naming and Modeling Fractions

Quantifying fractions involved creating a model for a given fraction and naming a fraction to describe an amount that is modeled. Naming the fraction that best matches the model, even if it is not in simplest form is inherent in the naming aspect of the practice. When students create a model for a fraction, they are demonstrating that they understand what the fraction symbol means. Likewise, when students name a fraction to describe a model, they are assigning a quantitative value to that model. This, of course, assumes the whole has been defined. Both situations require understanding that there are specific quantities associated with fractions, whether they are represented by models or symbols.

An opportunity to create models for given fractions occurred on the first night of the fraction unit. Students were given the picture in Figure 29 and told that it represented$1\frac{1}{2}$. With that information, students were instructed to create a model to represent 1.

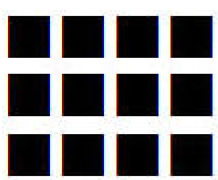


Figure 29: Model for $1\frac{1}{2}$

Students seemed to be able to complete this task with little difficulty. In the following excerpt, Lilly explained how her group determined the representation for 1.

> Lilly: I drew a line under the second row and I counted the top portion as one and the bottom portion as half.
>
> Instructor: How did you know to do that?

Lilly: Because the only way you can split three rows into one piece is to leave the one behind and that is the half. Because if you add that row and put another one under, it would be 2.

Instructor: Questions for Lilly?

Lilly may have simply told how she justified her answer of 8 squares representing 1, instead of really explaining her solution process. That is, she may have arrived at this answer simply by trial and error and then justified that her answer was correct. There is no way to be certain, but it does not seem obvious why Lilly drew a line under the first two rows. Due to this uncertainty, more evidence of being able to model fractions was sought.

The next problem was to find $1\frac{1}{8}$ given that 6 objects represented $\frac{3}{4}$. The given model for $\frac{3}{4}$ is shown in Figure 30.

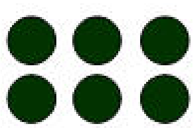


Figure 30: Model for $\frac{3}{4}$

In the following explanation, Joe shares his group's reasoning about solving the problem.

Joe: I split the three rows. I split it by twos and put the line between the columns.

Instructor: Why did you do that?

Joe: Because we came up to the conclusion that two [dots] equal $\frac{1}{4}$, and the second one equal $\frac{2}{4}$ and the third one is $\frac{3}{4}$ so what we did in order to get $1\frac{1}{8}$; we added another two dots to get $\frac{4}{4}$ and that equals the whole. And in order to get $\frac{1}{8}$ we added one more dot. $1\frac{1}{8}$,I mean.

Instructor: Why did you do that? …

> Joe: Since the first two dots equal $\frac{1}{4}$, the second two dots equal $\frac{2}{4}$ and the third dots equal $\frac{3}{4}$ the fourth two dots equal $\frac{4}{4}$. That's 4 over 4—a whole, which is one. That whole thing is 1. That actual dot [pointing to a single dot] is $\frac{1}{8}$.

Their strategy was to find a model for one whole, then add on to that to represent $1\frac{1}{8}$. This is the second example of students easily modeling fractions and this time, the explanation was clear. Students continued to model fractions in the class whenever they were presented with a situation. In fact, most of the explanations by students began with the student saying he or she modeled the fraction with a pizza, pie, or in some other way. Therefore no further discussion of modeling fractions is included here. It is sufficient to note that students consistently modeled fractions, and seemed to have that skill when they came into the class.

In addition to modeling a given fraction, it was important for students to be able to name a fraction for a given model. This would be needed to describe the result of modeling an action such as addition or subtraction. Such models are not always clear, perhaps making naming a fraction for a model more complex than creating a model for a given fraction. Clues from the situation need to be used at times in order to arrive at the fraction name that best matches the model. Such situations have already been discussed in the section on defining the whole in conjunction with the previous practice. What follows is evidence that students were able to name a fraction when given a model in a simple context. That is, the whole was defined, and the model was shown clearly as a part of that whole.

On the first night of the fraction unit, students named fractions modeled with pattern blocks. One such task defined the whole as three red trapezoid pattern blocks. The part was represented by 5 green triangles. The following exchange indicates students were able to name fractions represented by a model when the whole and part are clearly defined.

> Instructor: Three red trapezoids represent your whole, and the part is 5 green triangles. What fraction does that represent? Three red trapezoids are your whole and 5 green triangles are the parts. Laura.
>
> Laura: $\frac{5}{9}$
>
> Instructor: How did you get $\frac{5}{9}$?
>
> Laura: Lay down 5 green pieces [covering part of the trapezoids]. And there is one [space] left on the trapezoid [referring to uncovered parts of the trapezoids]…3 of the triangles lie on one trapezoid…plus the one that's already empty so that's 4. And you already had 5. Five plus 4 is 9. That's how much you have as a whole, and there's 5 pieces on it, it's $\frac{5}{9}$.
>
> Instructor: Questions for Laura?

There were no questions for Laura. The students seemed to understand her process. She explained her answer in terms of the total number of pieces in the whole, 9, and the number of pieces being considered, 5. Thus, naming a fraction for a model seemed to be taken-as-shared. However, the next problem presented some challenges for the students. The part and the whole did not have the same relationship as in the previous task (naming $\frac{5}{9}$). The fraction $\frac{6}{5}$ was modeled. This was the first in a series of problems students encountered in which the modeled fraction may have been simplified. That series is introduced with a discussion centered on naming $\frac{2}{8}$ as a fraction modeled. This had occurred earlier in the class and set the stage for writing what was modeled. The series is detailed here to trace the development of naming the fraction that is modeled, even when it is not in simplest form.

One aspect of assigning values to fractions that seemed to be difficult for students was writing the fraction that represented the model if it was not expressed in lowest terms. Early in the fraction unit, students showed a solution to a problem, but gave an answer that did not match the model. They seemed to do this because, as one student said, they had previously learned to "put it in lowest terms." In the following excerpt, a model drawn by the student showed 8 equal pieces with 2 of them shaded, representing $\frac{2}{8}$.

> Lilly: When you divide $\frac{2}{8}$ by 2 you get $\frac{1}{4}$ and you're supposed to put it in lowest terms.
>
> Instructor: Why are you supposed to?
>
> Lilly: I don't know. That's the way I was taught…
>
> Kathy: It's the simplest form of a fraction. And, when it's in the simplest form of a fraction it's easier to compare it to other fractions.
>
> Instructor: Well, to me the simplest form based on this picture is 2 out of 8. There's 2 slices out of the 8 slices. To me that's simpler because it's right there in front of you. …So for the purpose of our instruction for the time being I'd like you to write the symbol that represents your model.

Lilly's initial statement about dividing $\frac{2}{8}$ by 2 was incorrect and no one questioned her. Perhaps the other students were thinking the same thing Lilly was likely thinking. Instead of dividing by 2, perhaps she meant to say divide both the numerator and denominator by 2. In this case, students' prior knowledge seemed to interfere with the expectations for this class. In the discussion above, the instructor stated that she expected students to write what was modeled. This was the first mention of this expectation. It is evident from this exchange that students expected to simplify fractions, but did not really know why.

Later during that same class session, there was another opportunity to see the idea of naming and modeling fractions developing. Amy had just answered a question about a fraction she modeled. She said the model represented $\frac{6}{5}$. The model was made with a red trapezoid and a blue rhombus pattern block representing the whole. The students were to name what two red trapezoids would represent. The pattern blocks are shown in Figure 31. The set labeled "A" represents the whole and the set labeled "B" represents the part being considered. The original shapes have been covered with green triangles in both figures.

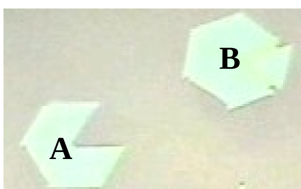


Figure 31: Model for $\frac{6}{5}$

> Doug: Six fifths is hard to understand. Would it be better to say one unit with $\frac{1}{5}$ left over?
>
> Amy: We weren't sure. We were dealing with improper fractions. …
>
> Doug: One unit and one piece left over
>
> Amy: Didn't she say to write the symbol that represents your model, and that [referring to $\frac{6}{5}$] represents the model.
>
> Instructor: I said what fraction describes this?
>
> Katrina: They're both the same. One's just easier to explain.

Amy pointed out that the instructor had previously asked students to write the symbol that represents the model. This indicates she realized that $\frac{6}{5}$ reflected the model more accurately than $1\frac{1}{5}$. Katrina recognized that the fractions represented the same amount, but she thought one was easier to explain. While the previous discussion indicates that writing what is modeled may be developing in some students, it was not yet taken-as-shared.

Three class sessions later, another discussion involved writing a fraction to match what was modeled. In justifying the answer to $\frac{5}{8}+\frac{5}{6}$, students were focusing on making meaning for the representation of $\frac{70}{48}$. The problem had been modeled with circles as shown in Figure 32.

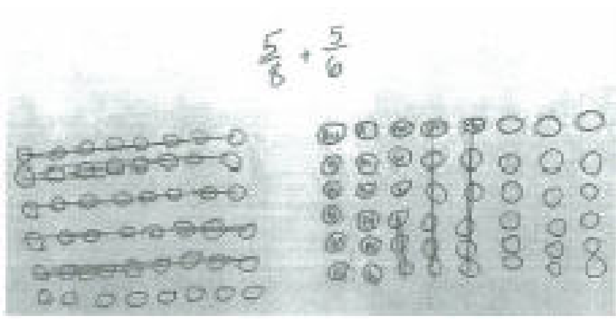


Figure 32: Model for $\frac{70}{48}$

> David: One unit has 48 and so you counted how many dots. If you use a slice of pizza you count how big are the slices. It's $\frac{1}{48}$ and you've got seventy $\frac{1}{48}$ slices or $\frac{1}{48}$ dots.
>
> Instructor: What does he mean by that?
>
> Lilly: Seventy out of 48 slices; I don't see how that one goes into the other. I don't see it.
>
> Instructor: Amy?
>
> Amy: I guess you have more than one whole. That's why your top is greater than the bottom. I guess your 48 is your whole, your 1, and the 70 is how many you have. How many you have plus what was left over.
>
> Instructor: What you have and what's left over—and why are they combined like they are the same?

Amy: You filled up your whole.

Instructor: If I filled up my whole I should see a 1.

Kathy: In that picture [pointing to the drawing shown in Figure 32] what you did was at the bottom. You took 8 away from the right hand picture and filled it up whole. And then, if you take their remaining circles on the right and put them over the ones on the left in a different color you would count 70 circles out of the 48.

When some students seemed confused that there were more pieces (70) in the answer than in the initial whole (48), David explained that the "forty eighths" are simply referring to the size of the pieces. Then Amy realized that $\frac{70}{48}$ was naming a fraction greater than 1. As the discussion continued, April stated that she saw it as 1 and 22 over 48. The following exchange tells how she thought of the situation.

April: If I was doing it this way the only thing I would do differently is when I got to the answer I would put 1 for my whole set and then put the 22 over 48.

Instructor: So you would say 1 and 22 over 48.

April: Yes, because I think of that first one as one whole. And then put what was left as the fraction.

Instructor: Matt.

Matt: Do you want us to simplify the fractions on the test? So like if we had 70 over 48 do you want us to put 35 over 24?

Instructor: April, did you simplify?

April: I didn't do it that way, but in looking at it I wouldn't because I was thinking of it as 22 parts of 48 so I would leave it.

Instructor: But did you go straight to here [pointing to $1\frac{22}{48}$] or did you go to here [pointing to $\frac{70}{48}$] first?

April: No, I wouldn't have seen it as $\frac{70}{48}$. I would have seen it as $1\frac{22}{48}$.

April saw the answer as a set of 48 with 22 left over. Because of this, she described the answer as $1\frac{22}{48}$. It's worth noting that Matt asked if the instructor wanted them to simplify answers on the test. Even though the discussion had been about writing what was modeled, his prior experience still made him think simplifying was necessary. The practice of writing what was modeled was not yet established.

By the time the following explanation was given on the fifth night of fraction instruction, the idea seemed to be taken-as-shared. Kathy explained how she determined the number of gallons of tea each of 4 people would have if there were $1\frac{2}{3}$ gallons of tea to be shared equally.

> I started off by dividing a pie into 3 equal thirds and shading 2 of those pieces to get $\frac{2}{3}$. Then to get equal pieces for 4 people you divide the pieces into fourths [drawing on board]. And then, I went through it and I know $\frac{2}{3}$ equals $\frac{8}{12}$. And I went through and gave away 4 pieces. And then I went through and gave 4 pieces away again. And then I started out $\frac{1}{12}$ plus $\frac{1}{12}$ equals $\frac{2}{12}$ for each person.

There was no response to the instructor's prompt for questions following Kathy's explanation. The answer of $\frac{2}{12}$ was not in simplest form, yet was accepted by the class. Kathy had explained her process by showing a model of twelfths, thus her answer of $\frac{2}{12}$ was expected. So, in terms of writing a fraction for a model, students had accepted that they should write a fraction that matches the model, even if the fraction is not in simplest form.

Naming and modeling fractions had become taken-as-shared knowledge. Students were able to name a fraction to match a model and create a model for a situation. Not only were students expected to model a given fraction, they sometimes needed to model a fraction in more than one way—equivalent fractions. Modeling equivalent fractions instead of using a known procedure to find common denominators is discussed next.

Modeling equivalent values

Equivalence, the fact that numbers can be named in a variety of ways is an important rational number concept (Vance, 1992). According to Kieren (1992), equivalence provides the foundation for operations. In this class, students modeled equivalent fractions especially to make meaning of common denominator algorithms. They modeled equivalent fractions by making smaller pieces so two fractions had the same number and size of pieces in their respective wholes.

Students were informally introduced to equivalent fractions early on the first night of the instructional sequence when they arrived at answers that were not in simplest form. However, the modeling of equivalent fractions first took place in the context of the solution to the following task. Students were shown $\frac{2}{5}$ of a box of chocolates represented by Figure 33. Given this model for $\frac{2}{5}$, they were asked to determine how many chocolates would be in $\frac{3}{4}$ of a box. Kim created equivalent fractions.

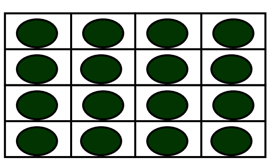


Figure 33: Diagram representing chocolates

> OK, so I said $\frac{2}{5}$ is equal to $\frac{8}{20}$ … Well, first I said $\frac{2}{5}$ and $\frac{3}{4}$; I want to make a denominator that both of those have in common …so I divided my box into 20 parts and made my $\frac{2}{5}$ into a number that's $\frac{2}{5}$ of the twenty parts, which would be $\frac{8}{20}$.

In her explanation, she tried to justify why she was able to say that $\frac{2}{5}$ is equal to $\frac{8}{20}$. Though she did not simply state the procedural way to find a common denominator, her explanation seemed to be focused on the common denominator algorithm. Her language included a “denominator that both have in common.” Explaining further, Kim said, “If $\frac{8}{20}$ represents $\frac{2}{5}$ of a box, then $\frac{15}{20}$ … represents $\frac{3}{4}$ [of a box].” Upon being challenged by the instructor, Kim added.

> I made a common denominator. … It's what both $\frac{2}{5}$ and $\frac{3}{4}$ can go into. Like if I have this number as 5 over 5 would equal one whole. Then I want to make 5 over 5 equal to 20 over 20. And then on this side I would have 4 over 4 equals 20 over 20. And then since my number isn't 5 over 5 on $\frac{2}{5}$, I would have to multiply the 2 by the whatever I multiplied; I multiply the 5 by 4 to give me 20, so then I need to multiply this 2 by 4 to give me 8.

Kim called upon her previous knowledge of common denominators to name an equivalent fraction. She simply explained the procedure for finding a common denominator. What conceptual understanding of common denominators Kim had did not come through in her explanation. Also, Kim did not make use of any models in her explanation. A later task would provide a more conceptual approach to naming equivalent fractions.

On the second night of the fraction unit, the Number Line Challenge (Appendix I) was presented to students. The primary intent of this task was to introduce relationships and ordering on the number line. However, it promoted discussions of equivalence as well because equivalent fractions were identified on the number line. The following discussion took place when the instructor queried why the activity did not ask students to find $\frac{2}{6}$ on the number line.

Kathy: It's the same as $\frac{1}{3}$.

Instructor: How do you know?

Kathy: If you look at that drawing [pointing to a drawing of a rectangle divided into sixths] and color $\frac{1}{6}$ and another $\frac{1}{6}$ you have $\frac{1}{3}$.

In this brief exchange, Kathy indicated that two sections on the number line with a length of $\frac{1}{6}$ are equivalent to $\frac{1}{3}$. This served as a model for equivalent fractions in that the number line was a representation of the fractions and Kathy used it to justify her answer. The discussion continued and the instructor introduced the term “equivalent fractions” in the next exchange. Students accepted the term without question. When the instructor asked for other equivalent fractions, a student explained that $\frac{2}{4}$, $\frac{3}{6}$, $\frac{4}{8}$, and $\frac{5}{10}$ are all equivalent to $\frac{1}{2}$. This student grounded the explanation in the experience of folding the number line. In fact, she used the knowledge that if the number line is folded into four parts, then two of those parts would be $\frac{1}{2}$. This exchange suggests that the class understood the idea of equivalence and was on its way to incorporating it into a practice. However, more discussions were to come.

The following discussion took place during a task that followed the Number Line Challenge on the second night of the instructional sequence for fractions. Students were asked to name three fractions between $\frac{1}{6}$ and $\frac{1}{3}$. They had already identified $\frac{3}{12}$, $\frac{1}{4}$, and $\frac{1}{5}$ as three fractions between $\frac{1}{6}$ and $\frac{1}{3}$. Katrina noticed that $\frac{3}{12}$ is equivalent to $\frac{1}{4}$.

Katrina: Isn't $\frac{3}{12}$ [equal to] $\frac{1}{4}$? …

Instructor: So then, so far we have answers of $\frac{3}{12}$, $\frac{1}{4}$, and $\frac{1}{5}$ as between $\frac{1}{3}$ and $\frac{1}{6}$. But we've noticed that $\frac{3}{12}$ is equivalent to $\frac{1}{4}$. How did we know that? …

David: Take 1 of those fourths [referring to one of the sections in the left figure shown in Figure 34] and put three in 3 of those little pieces [referring to three of the sections in the right figure shown in Figure 34] equals 1 of those big pieces.

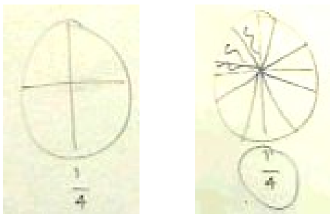


Figure 34: Area model showing fourths and twelfths

Instructor: And those little pieces are?

David: Twelfths

Instructor: So we've shown that $\frac{3}{12}$ is equivalent to $\frac{1}{4}$.

David explained his method of finding equivalent fractions without referring to the algorithm. Instead, he spoke about dividing the pieces up into smaller equal-sized pieces. This is the conceptual reason behind the algorithm. It seemed that the class was making progress toward modeling equivalent fractions becoming taken-as-shared. However, Lilly reverted back to a procedural explanation during the next class period. Here, she explained a procedural approach to finding equivalent fractions to help her find two fractions between $\frac{4}{7}$ and $\frac{5}{7}$:

Lilly: I got 41 over 70 and 42 over 70.

Instructor: How did you get that?

Lilly: I multiplied both sides—the $\frac{4}{7}$ and the $\frac{5}{7}$ by 10, getting an equivalent of $\frac{40}{70}$ and $\frac{50}{70}$. And then basically I just found a number between 40 and 50 and kept the denominator the same.

Instructor: What questions do we have for Lilly so she explains everything she needs to explain?

Amy: Why can you multiply like $\frac{4}{7}$ by 10 on the top and the bottom and it is still the equivalent of $\frac{4}{7}$?

Lilly: It's the same fraction but it has a 0 at the end of it.

This discussion continued, and the instructor pushed the students to consider the meaning of the multiplication to which Lilly referred. Note that Lilly said she multiplied "by 10." Amy more specifically stated that $\frac{4}{7}$ was multiplied "by 10 on the top and bottom." Eventually a student was able to describe the process as breaking each of the pieces into ten smaller equal-sized pieces. This clarified why the algorithm of multiplying by 1 written as a fraction with the same numerator and denominator works to rename a fraction to higher terms. Katrina drew on this discussion later when the instructor asked for an explanation of $\frac{5}{7}$ being equivalent to $\frac{50}{70}$. She said, "You would do basically the same thing. Now each slice has 10 little pieces inside of it, but we would take 5 whole slices. So we have 50 little pieces …out of 70 total." There were no questions when the instructor asked for them.

Later in discussing the same problem of two fractions between $\frac{4}{7}$ and $\frac{5}{7}$, a student added, "I got $\frac{17}{28}$ and $\frac{18}{28}$." Sarah called upon imagery of cutting pieces of a pie into smaller pieces to help explain the equivalent fractions.

Instructor: How could she have gotten 17 out of 28 and 18 of 28? Sarah.

Sarah: Instead of dividing each piece into 10, she divided each piece into 4. Then the same concept—you 're going to shade 4 big pieces and that's going to give you 16 small pieces. And then the 5 is going to give you 20 small pieces.

Instructor: Out of how many pieces altogether?

Sarah: 28.

The instructor, not other students, was questioning responses in these excerpts of the class discussion. This may indicate that other students had internalized the meaning of the process to find equivalent fractions. Later, imagery of dividing fraction models into smaller pieces was used as a strategy to solve addition and subtraction problems. This was another step toward the modeling equivalent fractions becoming a taken-as-shared.

The first discussion of making equal-sized pieces to add or subtract occurred in this subtraction scenario on the third night of the instructional sequence: “Betty had $2\frac{1}{2}$ yards of ribbon. She gave $\frac{2}{3}$ of a yard of her ribbon to Wilma. How much ribbon did Betty have left?” Lilly explained that first she subtracted the $\frac{2}{3}$ yard of ribbon from one of the whole 2 yards, leaving $\frac{1}{3}$ yard. That left her needing to combine one whole, $\frac{1}{3}$, and $\frac{1}{2}$ in order to express the amount of ribbon left. The following discussion took place.

> Lilly: So instead of subtracting, I added to get my answer. By adding the remainder of what was left. And then I added 1yard plus a half a yard plus $\frac{1}{3}$ [shown in Figure 35].

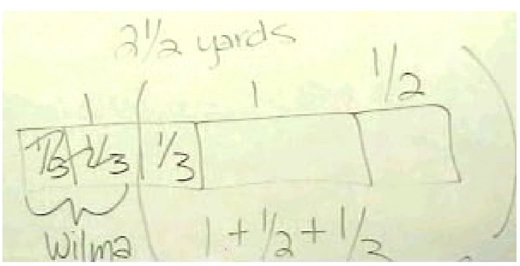


Figure 35: Model for subtracting

And then what I did was I turned this into pies to show $\frac{1}{2}$ and $\frac{1}{3}$ [shown in Figure 36].

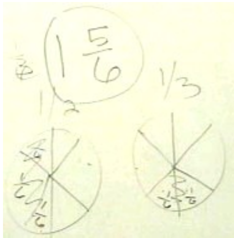


Figure 36: Area model for adding

> I split this one [pointing to the region on the lower left in Figure 36] in half. I split this one [pointing to the region on the lower right in Figure 36] into thirds. And then I knew that they were getting $\frac{1}{2}$ already so this is $\frac{1}{2}$ [pointing to the lower left circle]. And then I broke it down. I couldn't add the $\frac{1}{2}$ and $\frac{1}{3}$ so I broke this down into sixths. I cut them down both and I know that there were 1, 2, 3 sixths. And then I had 2 more sixths. And if I add them altogether I get $\frac{5}{6}$. So that's 1 yard and $\frac{5}{6}$.
>
> Instructor: Questions for Lilly? David.
>
> David: How did you get the sixth? Why did you divide it into 6?
>
> Lilly: Because I couldn't add these two numbers. So I broke it down into something where I could figure out how many were in a half and how many were in each third. So I broke down both pies into 6.
>
> Matt: I think we have to explain it better than that. Like why.
>
> Joe: The only problem I had was explaining how to get from that point where you're at.
>
> Lilly: I broke it down. I know both of these can be turned into $\frac{1}{6}$. So I just turned it into 6 pieces.
>
> Instructor: They see that you did that, and they see how you did it. They want to know why you did it. They see how they became 6. Heidi.
>
> Heidi: Can I help her? I looked at it as pizza. Everything is pizza, but out of the $\frac{1}{2}$ you just cut the $\frac{1}{2}$ into 3 slices on each side and $\frac{1}{3}$ cut into half. The reason why you did that is because you wanted to see how much ribbon you had together instead of half of a ribbon and $\frac{1}{3}$ of the other ribbon and one full ribbon. So in order to group it altogether you'd keep on splitting it up. I don't know why.

When Lilly gave a procedural explanation describing what she did, the instructor prompted her to explain why she did that. She was not able to explain why. Heidi tried to help and offered her explanation, but in the end admitted she did not know why she needed to have equal-sized pieces either.

In the next problem, students needed to combine $\frac{1}{6}$ and $\frac{1}{4}$. The following is an account of Kim's attempt to explain making equal-sized pieces.

> Kim: Well, to start out trying to decide how I would add $\frac{1}{6}$ and $\frac{1}{4}$; and I decided to go for a circle like a pizza, but not a pizza. So I started with two circles and I divided one into fourths and one into sixths…Barney ate one of these [pointing to a one fourth segment in Figure 37] and Andy ate one of these [pointing to a one sixth segment in Figure 37].

Figure 37: Model for addition

> And then I make those added together. So I need to separate both of those circles into a similar number, and I said 6 times 4 and 4 times 4 is the same thing. So I said I'm going to split the 4 two more times and the 6 one more time. OK forget 6 times 4 and 4 times 6. So I split this like this [drawing what is shown in Figure 38]. That's what I did first. And then I counted out how many pieces, and I got 12. And then I was like well how do I make this into twelve pieces? I knew I'd have to split each one of these [pointing to the figure on the right in Figure 38] into 2. I didn't mean fourths. I meant 6 times two and 4 times 3. Now they're even then I drew another one. And I added 2 for this one [pointing to the figure on the right in Figure 38], and then 3 for this one [pointing to the figure on the left in Figure 38] and I got $\frac{5}{12}$.

Figure 38: Model for equal-sized pieces

> Instructor: Questions for Kim?
>
> David: Why did you divide the right one in two?
>
> Kim: First, I say you have to divide the sixth like in half because I knew there were more of the sixths already so I just divided those in half and I saw that I came up with 12 and I knew that 4 times 3 was 12. I knew that I could get each fourth to be 3 pieces so two lines in that would make 3 pieces in each one. Also 12 pieces because each one had the same number pieces.

Kim may have started to use the common denominator algorithm in her explanation. She started to say she multiplied 6 times 4 (and later realized she meant 6 times 2). This may be how she would have solved the problem if she had not had the constraint of using a method other than finding common denominators. She stopped herself and continued to provide a more conceptually based explanation. That is, she used her drawing and explained how she reasoned through the process. Her explanation seemed to end abruptly. She implies that getting the same number of pieces was the goal, not adding the parts.

In the following discussion, there were no challenges to equivalent fraction language. This is evidence that the idea was taken-as-shared. In this explanation from the fourth night of the instructional sequence, Kathy is explaining how she subtracted $\frac{1}{2}$ from $1\frac{1}{3}$. She drew what is shown in Figure 39 to support her explanation.

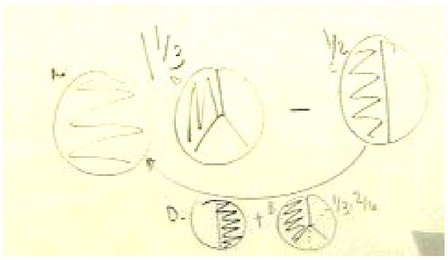


Figure 39: Model for subtraction

> Kathy: I drew $1\frac{1}{3}$ and I took away the half from my whole and this gave me a half left over which became my pie B. And then I saw I had $\frac{1}{3}$ left over so I had to add my $\frac{1}{3}$ … So these [referring to the half and third] have the equal pieces. I knew that by dividing the B pieces in half that $\frac{1}{3}$ and that equals $\frac{2}{6}$. Do you understand why I did that?
>
> Instructor: Does she need to explain that at this point?

There was no response to the instructor's question. Students accepted the explanation without further queries. Modeling equivalent fractions had become taken-as-shared knowledge. The quantifying practice thus far has focused on modeling. In the final part of the practice, relationships are the focus. In the next section, how students used relationships among fractions to describe other fractions is discussed.

Using Relationships to Describe Fractions

Students in the teaching experiment used relationships to describe fractions in two ways. One way was to compare fractions. That is, they may have described $\frac{1}{3}$ as being less, or smaller, than $\frac{1}{2}$. The other way they used relationships was to express an exact value of a fraction in terms of another fraction. For example, they may have described $\frac{1}{4}$ as half of $\frac{1}{2}$. Using relationships to describe fractions began early in the fraction unit. In fact, on the first night of the instructional sequence, there was an early discussion about amounts relative to a total capacity. This is an example of the first way students used relationships between quantities—to compare fractions.

On the Bus (Appendix D) was designed to introduce multiplicative reasoning to the students. The task presented stated that one bus had 30 seats and 15 were taken. Another bus has 60 seats, and 24 were taken. Students were to tell which bus was more full. During the activity, it was apparent that the prospective teachers brought some knowledge of fraction quantities to this class. In the following discussion, Katrina used the concept of one half in order to determine which bus was more full.

> Well, if you look at the 15 and 15, you know that 15 is half of 30 so that bus is half full. But if you look at the 24 and 36, half of 60 is 30 so you know that 24 is less than 30 so that's less than half. So the 15 would be half full so that would be more than the other bus which is less than half full.

In this statement, Katrina used her prior knowledge of the concept of one half to justify why the bus with 15 students was more full than the bus with 24 students. The instructor summarized her statement for the class and asked if there were any questions. The lack of response seemed to indicate her fellow students followed Katrina's reasoning and accepted her reference to one half without a formal definition of fractions. The concept of one half seemed to be a taken-as-shared idea in the class. There were no questions about what it meant to describe half of something, and it was accepted as a reasonable way to describe a quantity.

Later in the same class session, Kathy used similar reasoning, but actually applied it to a symbolic representation of fractions. She was explaining how she determined which of two containers was more full. A gallon container had 50 ounces of water in it;

and a quart container had 16 ounces of water. She gave the following explanation in response to a challenge from the instructor:

> Kathy: I did 16 over 32 and 50 over 128. And then I figured out that half of 32 is 16. Then I figured out that half of 128 is more than 50. Since the 16 over 32 is half full it has more than the 50 over 128…
>
> Instructor: But I would think the gallon would be more full because it has 50 ounces in it and the quart only has 16 ounces in it. …
>
> Kathy: Maybe it would be easier to use a different example other than a quart and gallon container. If you have a compact car like a small and a full size car.…If five of us piled into the compact car and even six of us piled into the minivan which one would have seats left?…Let's say you put five people into a five-passenger car, your compact car. And you only put six people into your minivan. Which one has more room? Even though it's a bigger car, and you're still putting more people in you still have more room in the minivan.
>
> Instructor: So you're saying it would be less full.
>
> Kathy: Correct. Just like that one the gallon is less full. Just because you're putting more substance into something doesn't mean that it's more full. It has to do with the quantity that you can put into it total.

No one challenged Kathy’s conclusion. Thus, it may be concluded that thinking about one half to compare fullness was taken-as-shared. Although the fraction “$\frac{1}{2}$” was never actually written, it was the benchmark to which these situations were compared. The class seemed to accept the idea of using $\frac{1}{2}$ as a benchmark to describe a quantity.

On the third night of the fraction unit students were given two fractions to compare. In order to do this, students often compared the two fractions to a third benchmark value such as 1 or $\frac{1}{2}$. Sarah explained why she believed $\frac{1}{3}$ was less than $\frac{3}{5}$.

> Sarah: I said that one half of three is one point five, and since one is less than one point five, we know that $\frac{1}{3}$ is going to be less than $\frac{1}{2}$.
>
> Instructor: Keep going.

> Sarah: And then half of five is two point five and three is more than two point five, so $\frac{3}{5}$ is going to be more than $\frac{1}{2}$ …
>
> Katrina: To show $\frac{1}{2}$ of three, I drew three circles and drew a line down the middle one and portrayed that. I did a [gesturing a bracket] to connect them to show that's $\frac{1}{2}$ of three and then on the other side was $\frac{1}{2}$ and then I just counted on the top. That first circle was 1 and the next one was a half. So it was $\frac{1}{2}$.

In this excerpt, Sarah compared the two fractions indirectly by comparing each to $\frac{1}{2}$. Since $\frac{1}{3}$ is less than $\frac{1}{2}$ and $\frac{3}{5}$ is greater than $\frac{1}{2}$, she was able to conclude that $\frac{1}{3}$ is less than $\frac{3}{5}$. This kind of thinking builds a sense of quantity with respect to fractions by describing fractions as relative values. The previous example was the first instance of such a strategy and it was observed several more times before becoming taken-as-shared.

The next problem presented to the class in the same session was to compare $\frac{4}{7}$ and $\frac{3}{8}$. Katrina offered her explanation after another student had already explained how he determined the answer by using the relationship of the size of pieces and the number of pieces. Katrina's explanation follows.

> Katrina: I didn't draw a picture I just looked at the denominator. I know that half of 8 is 4, and then half of 7 is $3\frac{1}{2}$. I looked back at the numerator and I know that 3 is less than 4 because you count 1, 2, 3 and three and a half. And 4 is greater than three and a half so $\frac{4}{7}$ is bigger because 4 is larger than half of 7. Half of 7 is three and a half. …
>
> April: I guess if I just saw that I would wonder why you're comparing three and a half to 4 and not to 3. I'm still not following that. Maybe I'm the only one.
>
> Instructor: So when you started with this, what were you looking for? You wanted to see if [interrupted].
>
> Katrina: I wanted to find half of 8 and half of 7.

Instructor: Because?

Katrina: Once I found half of them I would know if the numerator is bigger than half of it. It would be a larger number if the other one was smaller than half of the denominator. The numerator of the other one.

Katrina used the method of comparing each fraction to $\frac{1}{2}$. She determined half of each denominator to determine if the numerator was greater than or less than that amount. This resulted in her comparing each numerator to the value representing half of the denominator value. So, for $\frac{3}{8}$, Katrina compared the 3 to the 4, concluding that $\frac{3}{8}$ is less than $\frac{1}{2}$. For the $\frac{4}{7}$, she compared the 4 to $3\frac{1}{2}$ and determined that $\frac{4}{7}$ is greater than $\frac{1}{2}$. April questioned why Katrina was comparing to $3\frac{1}{2}$ and not to 4. Katrina extended her explanation to clarify her process. Katrina is using a relationship to $\frac{1}{2}$ to quantify other fractions.

Using $\frac{1}{2}$ was not the only benchmark strategy used for comparing fractions. Still on the third night of the fraction unit, David explained another problem in which he used 1 as a benchmark. He had drawn the diagram shown in Figure 40 to support his explanation. His strategy for comparing $\frac{6}{7}$ and $\frac{8}{9}$ follows.

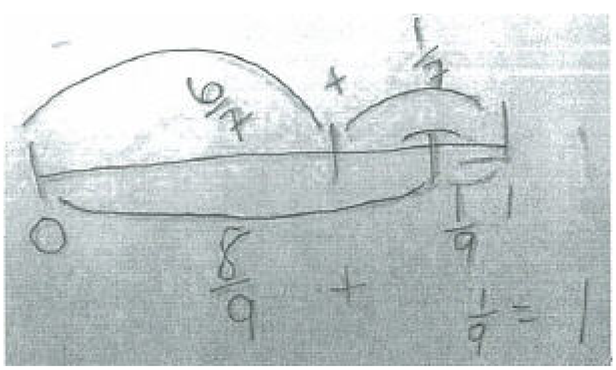


Figure 40: Using 1 as a benchmark

> David: Six sevenths was here [pointing to $\frac{6}{7}$ on the number line] plus $\frac{1}{7}$ equals one. And I knew that $\frac{8}{9}$ plus $\frac{1}{9}$ equals one. …I knew $\frac{1}{7}$ is greater than $\frac{1}{9}$; it's $\frac{8}{9}$ from here to here [pointing from 0 to the mark for $\frac{8}{9}$]. And if this is $\frac{1}{7}$ that means $\frac{6}{7}$ will be from here to here [pointing to from 0 to the mark for $\frac{6}{7}$]…And $\frac{8}{9}$ is longer than $\frac{6}{7}$.

David compared 2 fractions by comparing them to 1. That is, instead of comparing the fractions, he compared the distances on the number line to 1 from the location for each fraction. He found that $\frac{1}{9}$, the complement of $\frac{8}{9}$, was shorter than $\frac{1}{7}$. Therefore, $\frac{8}{9}$ was closer to 1, and greater than $\frac{6}{7}$. There were no questions for David when he finished his explanation. The strategy of using a benchmark such as 1 or $\frac{1}{2}$ seemed to be taken-as-shared for the students. Recognizing relationships to these values helped students quantify fractions and build on relationships among quantitative amounts.

Later in the same class period, David used $\frac{1}{2}$ as a benchmark in another way. In explaining his strategy for determining which of $\frac{5}{8}$ and $\frac{4}{6}$ is greater, he showed that each fraction to be compared could be thought of as one piece more than $\frac{1}{2}$ and then compared those pieces. David made an error in his explanation, but corrected it later when he was challenged on his assertion that $\frac{1}{2}$ plus $\frac{1}{4}$ is $\frac{5}{8}$. He clarified that he was thinking $\frac{1}{4}$ of $\frac{1}{2}$, which would be $\frac{1}{8}$. This part of the discussion is not addressed here because it is not relevant to the reasoning. Figures 41 and 42 show David’s work.

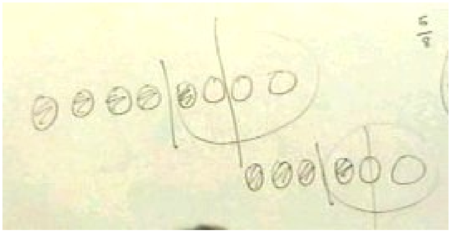


Figure 41: Model for $\frac{5}{8}$

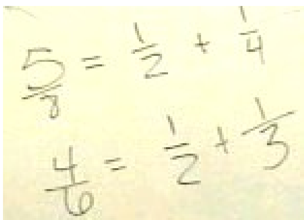


Figure 42: Symbolic notation for $\frac{5}{8}$ and $\frac{4}{6}$

> David: I divided it in half again. This is $\frac{5}{8}$. So I've got $\frac{5}{8}$ equals $\frac{1}{2}$ plus $\frac{1}{4}$, and $\frac{4}{6}$ equals $\frac{1}{2}$ plus $\frac{1}{3}$. So I looked at the $\frac{1}{4}$ and the $\frac{1}{3}$ and I knew that $\frac{1}{3}$ is greater because if you have a pie you divide it into less pieces have bigger pieces. So $\frac{1}{3}$ is more than $\frac{1}{4}$. So $\frac{4}{6}$ is greater than $\frac{5}{8}$ because $\frac{1}{2}$ plus $\frac{1}{4}$ is less than $\frac{1}{2}$ plus $\frac{1}{3}$.

David's strategy was to use $\frac{1}{2}$ as a benchmark again, but in a different way. In this case both the fractions to be compared were greater than $\frac{1}{2}$. In fact, David showed that each fraction was one piece more than $\frac{1}{2}$, then compared the size of the "extra" pieces. After he explained his strategy there was discussion related to his error mentioned previously, not his basic reasoning. The class accepted his reasoning, and once the error was clarified they were satisfied with the strategy. Thus, David successfully described two fractions as sums of $\frac{1}{2}$ and "an extra piece."

In addition to comparing fractions to benchmarks, students also used fraction relationships to quantify exact values for fractions. That is, a fractional amount was defined with respect to another fraction. The first instance of this occurred during the number line activity on the second night of the instructional sequence. Recall that in the Number Line Challenge (Appendix I) students were asked to find locations of various fractions on a number line, given locations for only 0, $\frac{1}{5}$, $\frac{2}{3}$, and 1. After having a chance to find the values, the instructor asked the class which ones were difficult to find. The following discussion came as a result of the students not naming $\frac{1}{6}$ as a fraction that was difficult to locate on the number line.

> Instructor: I'm curious then. How did you get $\frac{1}{6}$, if that wasn't one that caused issues? How did you get $\frac{1}{6}$?
>
> Sammi: Half the thirds.
>
> Instructor: What do you mean, "half the thirds"?
>
> Sammi: I can draw it.
>
> Instructor: OK.
>
> Sammi: So $\frac{1}{6}$ would be right there [referring to the shaded section in Figure 43]

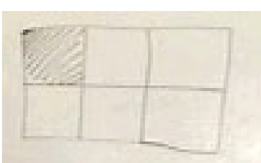


Figure 43: Area model for $\frac{1}{6}$

> Instructor: Thank you Sammi. You've given us a context to start with, but I've got a question about half of $\frac{1}{6}$ is $\frac{1}{3}$? [referring to an error Sammi made earlier in the discussion] …How do you know?
>
> Sammi: Oh yeah, I'm sorry. Because when you have thirds and you divide each third in a half, then you're going to have six pieces…
>
> Instructor: So how does that help with the number line? That's a rectangular region.

Sarah: …Once you've already found $\frac{1}{3}$ and you know that $\frac{1}{6}$ is half of $\frac{1}{3}$ you can find the space between 0 and $\frac{1}{3}$ and that's going to give you $\frac{1}{6}$.

In the preceding exchange, Sammi used the relationship between $\frac{1}{6}$ and $\frac{1}{3}$ in order to locate $\frac{1}{6}$ on the number line. She modeled $\frac{1}{6}$ by drawing a rectangular region divided into six pieces. Her explanation included that sixths result from dividing thirds in half because there would be six pieces instead of three. Sarah was able to apply the rectangular area model to the linear model of the number line. She located $\frac{1}{6}$ on the number line by dividing the distance between 0 and $\frac{1}{3}$ in half. This halving method was also applied to fifths to find tenths on the number line. Students used this halving method again on the fourth night of the instructional sequence.

Amy explained how she added $\frac{3}{10}+1\frac{2}{5}$ by partitioning the models into the same number of pieces: “It was $\frac{3}{10}+1\frac{2}{5}$. Right away when I saw the 10 and the 5 I knew if I drew a pie with 5 pieces; if I’d half each of my pieces, I'd end up with 10. And that way I could easily work with the same-sized pieces and I can compare my fraction better.” Although Amy’s explanation is about making the same-sized pieces in order to combine fractions, she used the relationship between fifths and tenths to determine how many pieces to make. Thus she divided the fifths into two pieces to make tenths. Halving was just one of the relationships students called upon to describe the value of a fraction.

Another strategy students used was to think of fractions in terms of their complements. In other words, the complement of $\frac{4}{7}$ is $\frac{3}{7}$, and there is a relationship between the two fractions. This idea was used on the third night of the instructional sequence when students were comparing $\frac{3}{7}$ and $\frac{5}{8}$. In reading the following excerpt it is important to know that the class had previously compared $\frac{4}{7}$ and $\frac{3}{8}$. David refers to that earlier problem in his explanation.

> Instructor: What about number nine, $\frac{3}{7}$ and $\frac{5}{8}$? What strategy did you use for that?
>
> David: I used the same strategy [referring to the previous problem which was to compare one piece out of the whole because the fractions were both one piece form the whole], but instead of taking one piece I took the piece from a whole and compared those. And I found out that same $\frac{4}{7}$ and $\frac{3}{8}$ that was already a problem.
>
> Instructor: Did you follow that? …
>
> Instructor: Can you start that over?
>
> David: All right, number nine is $\frac{3}{7}$ and $\frac{5}{8}$.
>
> Instructor: OK.
>
> David: And $\frac{3}{7}$ is $\frac{4}{7}$ away from a whole, and $\frac{5}{8}$ is $\frac{3}{8}$ away from the whole.
>
> Instructor: OK.
>
> David: And now compare which one is bigger or smaller. I know that $\frac{3}{8}$ is smaller than $\frac{4}{7}$ from an earlier problem. So, we know that $\frac{5}{8}$ is bigger because it's closer to one.

David recognized that the fractions to be compared, $\frac{3}{7}$ and $\frac{5}{8}$, were complements of two other fractions that were compared earlier. He used this relationship to 1 and compared how far away from a whole the complements of $\frac{4}{7}$ and $\frac{3}{8}$ were. He remembered that $\frac{3}{8}$ was less than $\frac{4}{7}$ from a previous problem. Therefore, $\frac{5}{8}$ is greater than $\frac{3}{7}$ because it is closer to 1.

This discussion illustrated how students developed a sense of quantity with respect to fractions. They were able to model fractional amounts, given a fraction in symbolic form. Likewise, when their work resulted in a model for a fractional amount, they were able to name a fraction to describe it and use symbols to write it. This aspect of the quantitative practice was fundamental to developing the practice fully. Students relied on models to solve other problems. By modeling fractions and writing names for models, students were assigning values to the fraction models or symbols. In doing so, they recognized that fractions represent quantities that can be compared, ordered, and used in operations. In order to complete tasks that required comparing and ordering, students modeled equivalent fractions by partitioning the equal pieces into smaller pieces. This provided imagery for renaming fractions with common denominators, which was also useful in addition and subtraction with fractions. Students also called upon their knowledge of quantitative relationships to describe fractions. Whether this meant simply comparing or renaming a fraction in terms of other fractions, this required a sense of quantity with respect to fractional numbers.

Conclusion

Norms and classroom mathematical practices were discussed in this chapter. In addition to norms developed earlier in the semester, new norms were established in the fraction unit. Social norms established were that students were expected to: (a) explain and justify solutions, (b) listen to and try to make sense of other students' thinking, and (c) ask questions or ask for clarification when something is not understood. Sociomathematical norms established called for students to: (a) know what makes an explanation acceptable,

(b) know what counts as a different solution, and (c) use meaningful solution strategies instead of known algorithms.

Two practices were established in the fraction instructional sequence. The first practice was partitioning and unitizing fractional amounts. Ideas that contributed to this practice include: (a) modeling fractions with equal parts, (b) defining the whole, (c) using the relationship of the number of pieces and the size of the pieces, and (d) describing the remainder in a division problem. The second practice was quantifying fractions and using relationships among these quantities. This practice includes: (a) naming and modeling fractions, (b) modeling equivalent values, and (c) using relationships to describe fractions.

CHAPTER FIVE: CONCLUSION

Recall that in design research, the results of one experiment feed back into another (Cobb, 2003; Cobb et al., 2001; Collins et al., 2004; Gravemeijer,1998). This feedback is considered in this chapter. Thus, the teaching experiment will be reviewed and recommendations for a future cycle of research will be made. First, an overview of the teaching experiment is presented. That is followed by commentary on the implemented hypothetical learning trajectory (HLT). These comments provide feedback for a revised HLT. This revised HLT, along with other suggestions for change, are proposed for a future cycle of research to build on an instructional theory for teaching fractions to prospective elementary teachers.

Overview

This teaching experiment used design-based research (DBR) to document the norms and practices that were established with respect to fractions in a mathematics content course for prospective elementary teachers. The DBR methodology is iterative in nature and the results of one cycle feed back into the next (Cobb, 2003; Cobb et al., 2001; Collins et al., 2004; Gravemeijer, 1998). This was the first iteration for this research on fractions. Two exploratory teaching semesters had occurred prior to this teaching experiment, but a full analysis of the classroom activity was not completed in those semesters.

The research team developed the HLT in accordance with ideas discussed in the review of literature. Classroom tasks addressed the big ideas of: (a) multiplicative thinking, (b) notion of quantity, (c) concept of unit, (d) partitioning, (e) equivalence and ordering, and (f) common measures to add or subtract (Behr & Post, 1992; Kieren, 1992; Lamon, 1993, 1999, 2002; Post, Behr et al., 1986; Pothier & Sawada, 1983, 1990; Vance, 1992). At times the sole intention of a task was to build background for one of these ideas. For example, the school bus task (Appendix D) was added to the HLT after the exploratory teaching to explicitly address the idea of multiplicative reasoning. Multiple interpretations of fractions were also considered in developing the HLT. The tasks included contexts for each of the following interpretations: (a) part-whole, (b) measure, (c) quotient, and (d) operator interpretations. (Behr et al., 1983; Kieren, 1976, 1980; Ohlsson, 1988). While not presented explicitly in the instruction, the different interpretations were represented in the tasks the students completed.

Research about students' performance and instructional strategies that may contribute to improved student performance influenced the instructional strategies as well as the tasks designed for this teaching experiment. An example of an instructional strategy that was influenced by the research involves using manipulatives. One suggestion in the literature was that students should be able to choose the manipulatives they use (Bezuk, 1988). Another suggestion came from Baroody (1989), who recommends that students should use what they know instead of being shown how to use manipulatives. For these reasons, manipulatives were available to students, but there was not specific instruction on how to use them to model situations. An example of a task that was influenced by the research was comparing fractions. The fractions to be compared in this task were chosen to elicit the same strategies that students in the research studies generated to compare and order fractions (Behr et al., 1984; Lamon, 1999).

Instructional strategies and tasks also followed principles of Realistic Mathematics Education (RME). This instructional theory calls for guided reinvention, didactic phenomenology, and emergent models (Gravemeijer, 1998; 2004). These principles are described in detail in Chapter Three. Their influence on the HLT was found in the tasks being grounded in a context when possible. Students in the teaching experiment were guided to discover the mathematics instead of being told how to do it. Based on the emergent models principle, the students created their own models instead of being shown how to use manipulatives to model the mathematics.

The resulting HLT included instructional tasks, tools, and possible discourse. These things, along with the classroom activity structure are identified by Cobb (2003) as means of support. The HLT addressed each of these means of support in that the research team created instructional tasks and conjectured about tools the students might use to complete the tasks. Possible discourse topics were included as part of the HLT. The classroom activity structure was implicit in the tasks. That is, the tasks themselves defined the structure. Generally, tasks were presented to the class and students worked in small groups to complete them. After small group work, there was generally a whole-class discussion about the tasks.

The activity structure was further defined by social and sociomathematical norms established in the class. Social norms called for students to: (a) explain and justify solutions, (b) listen to and try to make sense of other students' thinking, and (c) ask questions or ask for clarification when something is not understood. Sociomathematical norms established called for students to: (a) know what makes an explanation acceptable, (b) know what counts as a different solution, and (c) use meaningful solution strategies instead of known algorithms. Analysis of the discourse also revealed two mathematical practices.

Mathematical Practices

Several unifying elements were discussed in Chapter Two. These big ideas were discussed as a way to organize elementary mathematics with respect to fractions. They were: (a) multiplicative thinking, (b) notion of quantity, (c) concept of unit, (d) partitioning, (e) equivalence and ordering, and (f) common measures to add and subtract. These elements were considered when developing the HLT. During the class several ideas became taken-as-shared. These were: (a) modeling fractions with equal parts, (b) defining the whole, (c) using the relationship between the number of pieces and the size of the pieces, (d) using relationships to describe fractions, (e) modeling equivalent values, (f) naming and modeling fractions, and (g) describing the remainder in a division problem. These ideas formed the two practices that emerged during the fraction instruction. The ideas of modeling fractions with equal parts, defining the whole, using the relationship between the number of pieces and the size of the pieces, and describing the remainder in a division problem all address underlying concepts about forming fractions. These ideas which addressed the processes students need to build foundations of fraction understanding comprised the first practice of partitioning and unitizing fractional amounts. The remaining taken-as-shared ideas of modeling equivalent values, naming and modeling fractions, and using relationships to describe fractions dealt with the quantities expressed by fractions. That is, these ideas allowed students to work with fractions and recognize them as numbers that have specific values. The second practice of quantifying fractions and using relationships among those quantities included these taken-as-shared ideas.

Partitioning and Unitizing Fractional Amounts

The development of the partitioning and unitizing practice was documented in Chapter Four. This practice addresses underlying concepts about forming fractions. These include: (a) modeling fractions with equal parts, (b) defining the whole, (c) using the relationship between the number of pieces and the size of the pieces, and (d) describing what is left over in a division problem. The learning goal for Stage One of the HLT was using fractions to name amounts. This stage of the HLT was intended to provide background to students and focused on using fractions to name amounts. The ideas of partitioning and unitizing were central to this stage.

Lamon (1999) acknowledges the importance of partitioning and notes that fractions are formed by partitioning. Thus, partitioning is fundamental for building fraction concepts. For this reason, the instructional sequence included several tasks to encourage students to consider partitioning. The first opportunity to partition elicited a discussion about equal parts. Students almost immediately seemed to understand that the parts in fractions must be equal. The eventual outcome of partitioning tasks was the realization that as the number of pieces the whole is divided into increases, the pieces become smaller. While this idea emerged in Stage One, it did not become taken-as-shared until the third stage of the HLT. By that time students had used the idea to support their reasoning in several situations including justifying equivalent fractions, finding fractions between given values, and comparing two fractions. This relationship, resulting from partitioning became a powerful reasoning tool for the students.

Lamon (1999) acknowledges that when children begin to study fractions they may encounter new kinds of units. Understanding these new units is important. The concept of unit was addressed in the first stage of the instructional sequence. One task that explicitly addressed the size of the whole was used to introduce the concept. Students were shown a model and asked to name a fraction it represented relative to another model for the whole. As the model that defined the whole was changed, students realized that the fraction shown by the first model changed as well. In another task, students were shown models of specific values and asked to find another value. In order to complete this task, they needed to unitize and work from the whole. Finally, students came to realize the importance of knowing the unit when they solved problems from the equal sharing task. One problem from this task asked students to describe different ways 5 people could share 4 cookies. When there were two different answers, the discussion led students to understand that the situation and question help define the whole. This supports Lamon's assertion that the context of the situation should give students a way to determine the unit.

A special application of defining the whole was discussed as describing the remainder in a division problem. In a quotitive division situation the left over portion is compared to the divisor. Thus, it is important to know the unit for the divisor. That is, the answer will be different if the divisor is a slice of pizza rather than a whole pizza. The instructional sequence included several quotitive division problems to allow students to describe the left over portion. The idea became taken-as-shared on the final night of fraction instruction.

The fact that partitioning and unitizing appear in the literature as unifying elements of fractions validates that this practice is important for building deeper conceptual understanding with respect to fractions. Additional support for this practice can be found in the research reported about teachers' knowledge. Recall that teachers did not have a clear understanding of partitioning and the part-whole concept. (Lacampagne, Post et al., 1988). Further, once teachers began to understand the mathematics they needed to teach, their teaching became more conceptual (Sowder, Philipp, et al., 1998).

Quantifying Fractions and Using Relationships among These Quantities

The development of the quantifying practice was also documented in the previous chapter. This is the practice that deals with fractions as amounts and knowing that these amounts can be named, modeled, ordered, compared, and expressed in multiple ways. These amounts have relationships, and these relationships can be used to further describe the amounts or to work with the amounts to solve a problem. The ideas related to this practice are: (a) naming and modeling fractions, (b) modeling equivalent values and (c) using relationships to describe fractions. These encompass ideas related to three of the unifying elements identified in Chapter Two. These are a notion of quantity, equivalence and ordering, and common measures to add or subtract. The tasks in the instructional sequence that supported this practice built upon the foundational experiences of the first practice. That is, students had experiences with forming fractions and representing them. They had learned to name fractions, and then needed to assign values to those fractions.

In order to realize this goal, students needed to develop an understanding of the differences between whole number and fractional relationships. The second stage of the HLT focused on understanding these differences, namely realizing that greater denominators result in smaller parts, fractions have more than one name, and the density property for fractions (Lamon, 1999).

Post et al. (1986) identify several aspects of a quantitative notion of fractions when they suggest that students do not have a quantitative understanding of fractions. That is, they do not realize that fractions have size—that $\frac{7}{8}$ is close to 1. This prevents them from knowing what would be a reasonable answer to a computation problem. There is also evidence that students may not think of a fraction as a single number that can be expressed in many ways. Several tasks in the HLT were intended to build this quantitative notion of fractions. Even though this notion of quantity was the focus of Stage Two of the HLT, other stages contributed as well. In fact, the analysis showed that this practice was developing in every stage except the final one.

The first component of the practice involved modeling a fraction and naming a fraction for a model. Thus, students either named a fraction to describe an amount modeled or they created a model for a given fraction in order to make sense of a situation. This included naming a fraction in terms of what was modeled, even if it was not in simplest form. Prior experiences of the students caused them to want to simplify fractions when it wasn't necessary. Modeling and naming fractions in isolation seemed to be easy for the class. Naming fractions that made sense for a situation took longer to become taken-as-shared. It wasn't until the final night of fraction instruction that students accepted fraction names that were not expressed in lowest terms as representing a model.

This modeling and naming of fractions supported building students' notion of quantity for fractions and allowed them to work with multiple representations of fractions.

Students also worked with equivalence and ordering fractions in establishing the quantitative practice. Equivalence is an important concept to develop with respect to fractions (Lamon, 1999; Vance, 1992; Kieren, 1992). Mack (1995) describes equivalence simply as different names for the same amount. Generating equivalent fractions through modeling supported developing an understanding of common denominators. Students modeled equivalent fractions in two ways. The first way shared in class used a set model. The other, more common, way was to divide parts of area or linear models into smaller equal parts. Students modeled equivalent fractions in order to unitize with respect to different amounts, to find fractions between given fractions, and to perform addition and subtraction operations with fractions. Thus, modeling equivalent fractions was part of the quantifying practice. This reflects the unifying elements, or big idea of having common measures to add or subtract as well.

Students began to divide models into smaller sections in order to name common denominators early in the instructional sequence. By the time they reached the addition and subtraction portion of the instructional sequence, they were experienced in finding same-sized pieces for different fractions. This illustrates how equivalence provides the foundation for operations (Kieren, 1992).

Another aspect of equivalence and ordering concepts reflected in the quantifying practice was that students used fractions to describe other fractions. They did this by describing one fraction as smaller than or larger than another. This indicates they had developed a sense of quantity for fractions because they were comparing relative values.

Another way students described fractions was to name one fraction in terms of another. This was grounded in partitioning tasks. When students started with an amount and divided it into two equal parts they were able to describe the new parts in relation to the original. For example, if a student drew a set of four small circles and drew a line between the second and third circles, dividing the set in half, they concluded that half of 4 is 2. This proved to be especially valuable when they compared fractions, as $\frac{1}{2}$ was a common benchmark to which fractions were compared.

Research findings reported in Chapter Two validate this quantifying practice. These findings report that teachers have difficulty with fraction equivalence, the concept of unit, and ordering (Cramer & Lesh, 1988; Lacampagne et al., 1988). Additionally, Tirosh et al. (1998) noted that although addition and subtraction of fractions did not seem to be difficult for prospective teachers, a common error in addition was to add the denominators as well as numerators.

In the process of establishing this practice to quantify fractions and use relationships among the quantities, students began to express fractions in equivalent terms. They compared and ordered fractions. They named fractions between other fractions, recognizing the density property of fractions. Perhaps most importantly, they built the foundation for their work with operations.

The preceding discussion described how the partitioning and quantifying practices became established in the teaching experiment classroom. The unifying elements of a notion of quantity, the concept of unit, partitioning, equivalence and ordering, and common measures to add or subtract were represented in the practices. Multiplicative thinking, one of the unifying elements identified in Chapter Two, was not explicitly represented in the discussion about the established practices. That is not to suggest that this element was missing from the practices. Lamon (1999) describes multiplicative thinking as foundational to other important ideas related to fractions. These ideas include: (a) the relationship between the size of pieces and the number of pieces, (b) the need to compare fractions relative to the same unit, (c) the meaning of a fractional number, (d) the relationship between equivalent fractions, and (e) the relationship between equivalent fraction representations. Although multiplicative reasoning was not included specifically in the preceding discussion on practices, it is still reflected in the practices. For example, the act of unitizing with respect to fractions is based in multiplicative reasoning. Without the ability to engage in multiplicative thinking, the practices may not have been established. So, the learning trajectory that was implemented in the teaching experiment addressed each of the big ideas identified in the literature review. Reflecting on the practices and the instructional sequence has resulted in some proposals for the next iteration of the HLT. That revision is discussed in the next section.

Revisions for a Future Iteration

Just as the HLT implemented in this teaching experiment was based on evaluations of previous semesters, the HLT for a future iteration can be based on findings from this teaching experiment. There are two categories of revisions discussed in this section. First, revisions to the HLT are discussed. Following that, revisions to the classroom activity structure are discussed.

The Hypothetical Learning Trajectory

The recommendations that follow resulted from the analysis which was presented in Chapter Four. While much of the HLT remains unchanged, there are suggestions that may enhance the classroom mathematical practices. These suggestions are made largely because the analysis did not reflect the expected taken-as-shared ideas for the activities. For example, making the fraction kit is not included in the proposed HLT. The fraction kit was not widely used by the students and no reference to it was made in the excerpts of the class discourse selected to follow the development of the taken-as-shared ideas.

Another change that should be made involves planning for the use of tools. Due largely to inexperience, the research team for this teaching experiment was not successful in planning for students' use of tools. Thought should be given to an overarching model and how progressive tool use can help students work with that model. For example, in a teaching experiment about measurement, a ruler was the overarching model. Footsteps, connecting cubes, masking tape and other tools were other tools that students used in leading up to the ruler (Gravemeijer et al., 2003). The adult students in this experiment used drawings to help them reason about fractions. Though situations were included in the class that would suggest set and linear models, students seemed to prefer area models. This suggests that perhaps there should be more attention given to the different kinds of models for fractions in the next iteration of the HLT. Careful consideration should be given to tools in the context of the proposed HLT and the new activities that are suggested.

One of the principles of Realistic Mathematics Education (RME) is that students should begin with a realistic situation and mathematize it (Gravemeijer et al., 2003). While a consideration, this principle was not fully implemented in this teaching experiment. Although realistic situations were presented for the students to mathematize, they were often disjointed. This is not a direct result of the analysis presented in Chapter Four. Instead, it is based on observations of the research team. A better approach may be to present all the instructional tasks within the context of a single scenario. Since the imagery of pizza was so prevalent in the class discussions, this may provide such a scenario. Additionally, the progression of tools may be more easily implemented in a single scenario.

The stages of the HLT were grounded in the information about the mathematical structure of fractions reported in the literature review. The first stage was designed to address multiplicative thinking, partitioning, and the concept of unit. The second stage was intended to address the ways in which whole number processes differ from fraction processes. This included a notion of quantity, equivalence and ordering, and common measures to add or subtract. The purpose of the third stage was to compare fractions with reasoning strategies instead of known algorithms. In retrospect, this seems like an arbitrary distinction from Stage Two. In developing the HLT, there seemed to be a distinction because there are known procedures for comparing fractions. The tasks in Stage Two were not thought to likely be part of students' prior experiences, so replacing known procedures was not so much of a concern. After focusing on fundamental fraction concepts in the first three stages of the HLT, attention turned to operations in the final two stages—one for addition and subtraction, and one for multiplication and division.

The HLT implemented in this teaching experiment resulted in two classroom mathematical practices. Both the partitioning and quantifying practices supported students work with operations, but a practice specifically for operations did not emerge. This may have been a factor of time. Due to schedule constraints in the teaching experiment, the operations portion of the instruction was rushed and condensed. Research findings reported in Chapter 2 indicate teachers lack understanding of operations. Common errors are to add the denominators as well as numerators in addition and to invert the dividend rather than the divisor in division. Teachers in research studies were not able to explain the steps in standard algorithms for operations with fractions. They also seem to share a misconception that the dividend must be greater than the quotient (Tirosh et al., 1998). Several studies have indicated that teachers lack conceptual knowledge of division of fractions. They have difficulty solving division sentences as well as generating word problems to represent a division sentence (Ball, 1990a, 1990b; Cramer & Lesh, 1988; Lacampagne, Post et al., 1988; Ma, 1999; McDiarmid & Wilson, 1991; Simon, 1993; Sowder, Philipp, et al., 1998; Stoddart et al, 1993; Tirosh et al.). These findings warrant devoting more time in the instructional sequence to operations. In this study, many problems were presented within a context to aid the students in reasoning out solutions to the problems. When problems were presented as bare computation, without context, students were encouraged to create a situation to help them find the solution. This did not become an established practice, so attention should be given to generating word problems to help develop meaning for operations.

Another aspect of fraction operations that did not emerge as a practice was evaluating the reasonableness of answers. Estimation strategies were discussed briefly in an attempt to help students understand the effect of operations and to subsequently evaluate the reasonableness of their solutions. Again, lack of time was likely a contributing factor in this not becoming an established practice. The tasks and opportunities were presented in the implemented HLT, but time did not allow for extensive thought and reasoning with the operations.

Table 12 shows a proposed HLT for a future iteration of teaching. The analysis of discourse resulting from the enacted HLT for this teaching experiment guided the revisions. As previously noted, much of the HLT is unchanged. Note that the first two tasks, On the Bus and Familiar Fraction Situations and Models, were deleted from the enacted HLT. These tasks were not judged to be effective for their intended purposes, which were to develop multiplicative thinking and build understanding for different interpretations for fractions. However, some aspects of these tasks did contribute to the practices. They prompted discussion of equivalent fractions and using $\frac{1}{2}$ as a benchmark. Thus, when replacing the tasks attention should be given to these concepts. After the first two tasks, the HLT remains in tact for the remainder of Stage One.

Table 12: Proposed HLT for a Future Iteration

Stage	Classroom Mathematical Practice	Supporting Tasks	Tools and Imagery	Possible Discourse
One	Partitioning and Unitizing	New Task—to be determined		Multiplicative thinking; Answering “How much?” vs. “How many?”
		Unit	Squares—area model; Pattern blocks	A fraction is named relative to the unit.
		How Much?	Set and area models	Fraction representations are relative to the whole
		Equal Sharing	Area models representing cookies or candy bars	Fractions need equal-sized pieces; as the number of pieces increases, the pieces get smaller; there is more than one way to divide a whole into fair shares
Two	Quantifying and using relationships among fraction quantities	New Task—to be determined		
		Number Line Challenge	Number line	Linear representation of fractions; sequencing fractions; equivalent fractions; relationships of fractions (thirds to sixths)
		Between	Number line	Making smaller pieces to have equal pieces in different fractions
		Compare	Models—area, set, number line	Relationship of size and number of pieces; benchmark values
Three	Building meaning for operations and understanding their effects	Introduction to Addition and Subtraction	Context of problem; drawings	Need a way to express the answer (common denominators); using different models for area, set, and linear situations
		Addition and Subtraction	Creating contexts	Improper fractions; regrouping in subtraction;
		Multiplication and Division Situations	Context of problem; Models—set and area	Meaning of multiplication and division; Need to define the whole in order to describe the remainder; measurement model for division; sharing model for division
		Multiplication	Creating contexts; drawings	Meaning of multiplication
		Division Situations	Context of problem; models	Meaning of division; Define the unit to name the remainder

Stages Two and Three of the enacted HLT have been combined. The first task of Stage Two, The Candy Bar, was deleted. This task lacked depth and did not accomplish the goal of building understanding of equivalent fractions. This may be due to the fact that the fractions used were familiar to students and clearly equivalent. If the task could be restructured with unfamiliar fractions, perhaps it would have more impact on the students' learning. In the post-instruction research team meeting, the lack of imagery building for the candy bar was discussed. This should also be a consideration in revising the task. The fraction kit task, the last task in Stage Three of the enacted HLT was also deleted. This was intended to give students a tool to help them make sense of fractions. It was not used by many students in that way. They created their own tools for thinking mostly by drawing representations of the situations.

Finally, Stages Four and Five of the enacted HLT were combined to become Stage Three of the proposed HLT. This stage addresses building meaning for operations and understanding their effects. The instructional tasks in this proposed stage remain largely in tact from the enacted HLT. However, emphasis should be placed on generating suitable contexts for mathematical sentences. Practice problems for the operations were deleted because they did not contribute to significant in-class discussion. In future iterations, the instructor could certainly decide to offer extra practice to students. Another change is that estimation activities were deleted. Instead of including specific tasks that focus on estimating results of operations, this should be incorporated as an expectation for evaluating reasonableness of results. This may be established as a sociomathematical norm in future iterations.

Classroom Activity Structure

Activity structure refers to how the classroom is organized. For example, instructional activities may be completed by small groups of students or presented to the whole class (Cobb et al., 1995). In this teaching experiment, tasks were generally presented to the whole class. Students worked in small groups to complete the tasks. Whole-class discussion followed the small-group work, giving students an opportunity to share their solutions and listen to how others solved problems. Typically, the students worked in small groups on several similar problems at once. It seems that there may be greater opportunity for students to consider other students' reasoning when problems are solved and presented one at a time. However, there is likely a trade off as well. If students see others' solutions to similar problems, they may be influenced to use that solution method instead of finding one of their own. This is mentioned here as a precaution. An instructor should balance the amount of unguided work with the amount of explanation before exploration. At times, being able to see other students' strategies and reasoning may be beneficial. This should be considered in a future iteration of the fraction sequence.

Ideally, more time could be found to allow a more in depth treatment of the operations sequence, since lack of time seemed to be a problem for developing practices with respect to fraction operations. If more time cannot be made available, perhaps a different classroom activity structure could result in more efficient use of the time available. One suggestion is to assign different problems to small groups, allowing each group more time to work on fewer problems. Others could present their solutions to different problems. The problems would need to be chosen carefully to allow students similar opportunities. This suggestion may alleviate the concern about solving multiple problems before seeing others' solutions as well.

Sociomathematical norms help to document the participation structure in the classroom (Cobb et al.,2001; Cobb & Yackel, 1996; Gravemeijer et al.,2000; Stephan & Cobb, 2003). As a result of revisions to the instructional sequence, a new norm is being proposed. Instead of specific activities that focus on estimation, the instructor may try to negotiate another sociomathematical. This would set the expectation that students should quantify fractions and estimate results of operations. Doing this may also strengthen the operations portion of the instructional sequence.

Changing the tasks to all be set in a single realistic scenario would impact the activity structure. Although it is difficult to determine what that impact may be, a more integrated sequence may affect the discussion and the time needed to work on tasks due to increased transfer. The connections and transitions between the mathematical ideas may be more obvious. These effects on the activity structure should be considered in planning for the next iteration.

It can be gleaned from the preceding discussion that this teaching experiment demonstrated some success, but not in every aspect. Students were able to apply what they knew about fractions to solve problems presented to them. They learned to represent fractions, compare and order fractions, find equivalent fractions, and perform operations with fractions by using reasoning instead of algorithms. They also became comfortable with word problems involving fractions. In fact, one student actually said she preferred to have the context with the operations because it helped her to make sense of it. The class discussions demonstrated that they relied on basic concepts to explain their solutions.

These concepts emerged as taken-as-shared ideas among the students in the class. The hope was that students would make meaning for algorithms and procedures they may have already known. Explicit connections to the algorithms and known procedures were not made, so it is still undetermined whether they built that meaning. In a future iteration attention should be given to connecting the taken-as-shared ideas to the known procedures.

Implications

The purpose of this study was to describe how specific instructional activities support prospective elementary teachers' understanding of fraction concepts and operations from a social perspective. Social and sociomathematical norms were negotiated in the class. Classroom mathematical practices with respect to fractions were also established. The enacted HLT was evaluated, and a revised HLT was proposed to suggest how future researchers may direct their efforts. In addition, suggestions for changes to the classroom activity structure were made. Each of these will contribute to the research base of information about prospective teacher education with respect to fractions.

The norms reported in this study, along with one possible additional norm, can inform teachers as to norms they may wish to establish in their classrooms. Likewise, the classroom mathematical practices could inform teachers about goals they may wish to strive to achieve in their classrooms. In addition, the discussion about a practice for operations, could provide guidance to teachers. The enacted HLT in this study establishes a baseline for instructional tasks. This provides one way for supporting prospective elementary teachers to develop their understanding of fraction concepts and operations. Interested instructors or researchers could make the suggested revisions to the HLT and classroom activity structure and continue to develop effective methods for deepening prospective elementary teachers' understanding of fractions. However, it is not likely that the HLT will produce the same results in another classroom. In other words, outcomes of the HLT will not generalize (Gravemeijer et al., 2003). However, the ideas presented in Table 12, the instructional theory including the tasks, tools, imagery, and possible discourse may generalize to other classrooms. Suggestions for future research follow.

Future Research

As stated in the preceding paragraph, this study provides a baseline for an instructional sequence that may deepen prospective teachers' understanding of fraction concepts and operations. The analysis of the resulting norms and practices suggest several areas for follow up research. First, this study considered only the social perspective of the emergent model. The individual perspective should also be examined in a similar study.

A careful examination of how prior knowledge affects the establishment of mathematical practices would also contribute to the body of research. In this study, the learning goals included fraction content from the most basic concepts through operations. For example, the requirement for equal parts when describing fractions seemed to be taken-as-shared almost from the beginning. It seems that perhaps time was spent on instruction in this classroom that may not have been necessary. The students' prior knowledge and existing practices should play a greater role in the HLT. Also, students' prior knowledge seemed to interfere with establishing the new practices at times. For example, students wanted to simplify fractions because that is what they had been taught. This prior knowledge likely impacts the development of practices. It may be helpful to know the effect that it has.

Finally, research should be done to follow students into their subsequent mathematics courses and teaching. Generally, students taking this course follow it with an elementary mathematics methods course. It would be helpful to know if the understanding they developed in the content course had a lasting effect. That is, were students able to apply their knowledge of fractions in the methods course? In addition, did their work on projects or assignments in the methods course reflect the inquiry method of instruction in which they participated? Finally, the same questions could be asked of the students once they begin to teach mathematics in elementary classrooms.

Conclusion

The HLT was created with the overall goal of having students develop a deep understanding of fractions to increase their effectiveness as elementary mathematics teachers with respect to fractions. Students built conceptual understanding by replacing traditional rote algorithms with modeling and reasoning strategies. The tasks in the HLT supported the development of two mathematical practices that built meaning for students. These were partitioning and unitizing fractional amounts and quantifying fractions and using relationships among these quantities. The tasks in the HLT may have also supported developing a practice related to building meaning for operations and understanding the effect of operations if more time had been available.

The results of this teaching experiment may have implications for improving elementary teachers' preparation for teaching mathematics. The negotiated norms and established practices may inform instructors and help them plan better instruction for prospective teachers. In turn, elementary students may benefit. Continuing the line of research presented in this study may produce elementary school teachers with a greater understanding of fractions, thus allowing them to become more effective teachers.

APPENDIX A: PRE-INSTRUCTION QUESTIONNAIRE

PRE-INSTRUCTION QUESTIONNAIRE

START HERE

ID Number: ________________________

1. What do you feel are your strengths for teaching mathematics? ________________
   ____
   ____
   ____
   ____

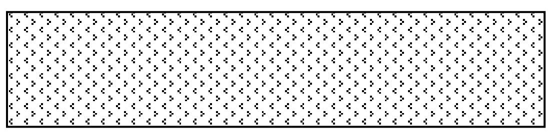
2. What do you feel are your challenges for teaching mathematics? _______________
   ____
   ____
   ____
   ____

3. What do you expect to learn from this course? ______________________________
   ____
   ____
   ____
   ____

4. Suppose your principal came to you and said you had to teach another grade this year. What would your reaction be? Would teaching mathematics at that level be a concern for you? Why or why not?______________________________________
   ____
   ____
   ____
   ____

5. A group of people selling homemade goods at a bake sale have agreed upon sharing the leftover items equally. Two and one half pans of brownies are shared among 8 people. How are they shared? Explain your reasoning in writing in as much detail as you feel necessary.

6. This is 2/3 of a chocolate bar. Draw the rest of the chocolate bar. Explain your reasoning in writing in as much detail as you feel necessary.



7. A teacher gave her class the challenge to find how many ways the number 572 could be thought about. The following are three children's answers. For each answer, mark whether is it correct or incorrect. If it is incorrect, please explain.

   a. 572 could be thought of as 57 tens and 2 ones.
      i. Is this answer correct or incorrect? ___ Correct ___ Incorrect
      ii. If this answer is incorrect, please explain the error.

   b. 572 could be thought of as 5 hundreds and 72 tens.
      i. Is this answer correct or incorrect? ___ Correct ___ Incorrect
      ii. If this answer is incorrect, please explain the error.

   c. 572 could be thought of as 5720 tenths.
      i. Is this answer correct or incorrect? ___ Correct ___ Incorrect
      ii. If this answer is incorrect, please explain the error.

8. Is Johnny's reasoning correct? Explain.
   Jessica: "I still have half my spelling words to learn and 3/4 of my vocabulary words to learn."
   Johnny: "Well, 3/4 is more than a half because 1/2 = 2/4. So, you have more vocabulary words than spelling words still to learn."

APPENDIX B: POST-INSTRUCTION QUESTIONNAIRE

POST-INSTRUCTION QUESTIONNAIRE

Name____________________________________ Date________________

Answer all questions in the space provided. Provide an explanation where asked and give complete answers.

For **Questions 1-3**, please circle the choice that best describes your opinion. You may use the Comment space to elaborate on your choice.

1. **After completing the unit on fractions in this class, how comfortable are you with fractions?**

   Very comfortable Somewhat comfortable Somewhat uncomfortable Very uncomfortable

   Comment:____

   ____

   ____

2. **How would you compare your learning about fractions in this class to what you think you would have learned in a class with traditional teacher-directed instruction?**

   I learned more I learned about the same I didn't learn as much

   Comment:____

   ____

   ____

3. **How important is it to explain how you arrived at an answer in a math class?**

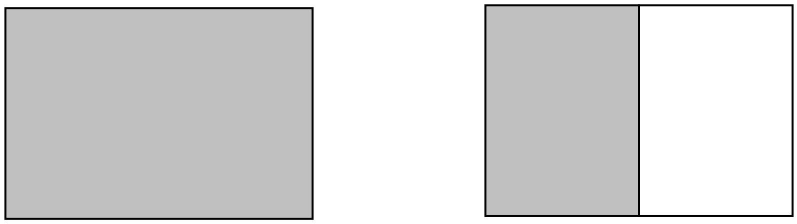
   Very important Somewhat important Not very important Not important at all

   Comment:____

   ____

   ____

4. A group of people selling homemade goods at a bake sale have agreed upon sharing the leftover items equally. One and one half pans of brownies are shared among 4 people. Explain how the people can share the brownies so that each person has the same amount. Tell how much of a pan of brownies each person gets.



5. Abby painted 1/3 of the wall yesterday. She painted 1/4 of the wall today.

What fractional part of the wall has she painted altogether? ___________________
Use the drawing to justify your answer.

APPENDIX C: QUESTIONS FOR FINAL

QUESTIONS FOR FINAL

Use reasoning strategies (not an algorithm) write the following fractions order from least to greatest. Explain how you determined the order.

97/100 47/90 47/97 38/85 87/90

Write a word problem that can be solved with the expression 2 3/4 ÷ 1/2. Then solve the problem you wrote.

APPENDIX D: ON THE BUS

On the Bus

Liz rides the bus to school. On Tuesday, she counted 24 students on the bus.

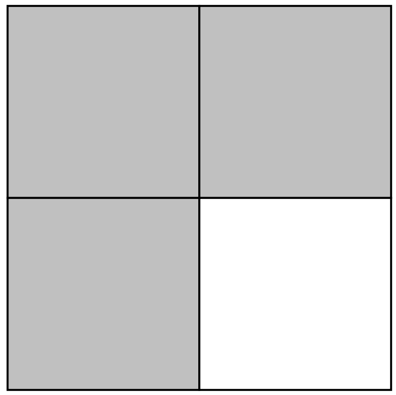
Riley rides the late bus home after school. There are 14 other students who also ride the late bus.

Liz rides the bus to school. She counted 24 students on the bus Tuesday. There were 36 empty seats.

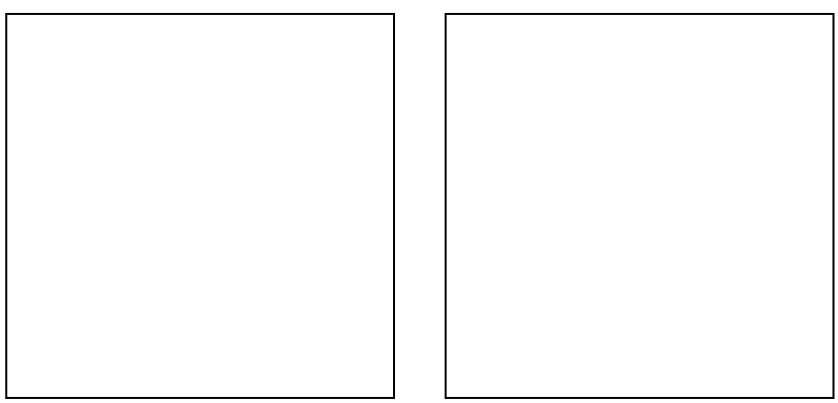
Riley rides the late bus home after school. There are 14 other students who also ride the late bus. Riley noticed there were 15 empty seats.

APPENDIX E: UNIT

Write a fraction to tell how much is shaded. Justify your answer.



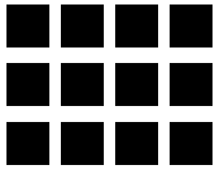
How much is shaded if this is the unit?



APPENDIX F: HOW MUCH?

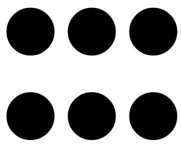
HOW MUCH?

1. This is 1 1/2.



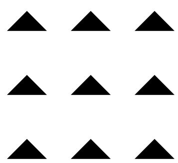
Show 1.

2. This is 3/4.



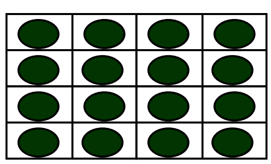
Show 1 1/8.

3. This is 3/7.



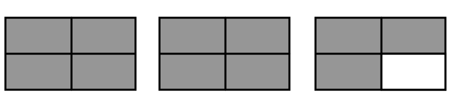
Show 5/7.

4. This is 2/5 box of chocolates.



Show 3/4 box.

5. This is 2 1/5.



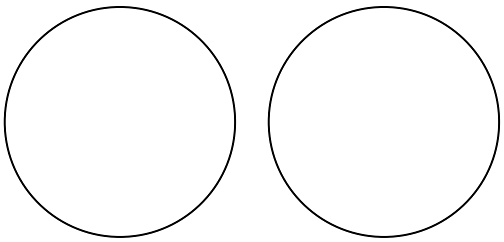
How much is ☐ ?

APPENDIX G: SHARING

Sharing

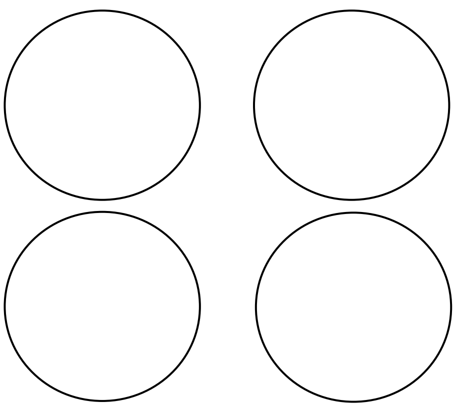
What fraction of a cookie or candy bar will each person get if they share equally?

1. 2 cookies among 4 people

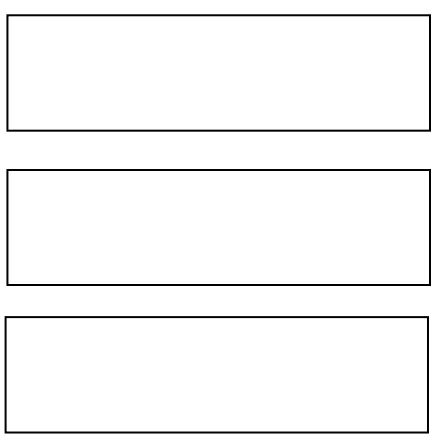


Explain your reasoning.

2. 4 cookies among 5 people



3. 3 candy bars among 4 people



Explain your reasoning.

4. 3 candy bars among 8 people

Explain your reasoning.

5. 3 candy bars among 7 people

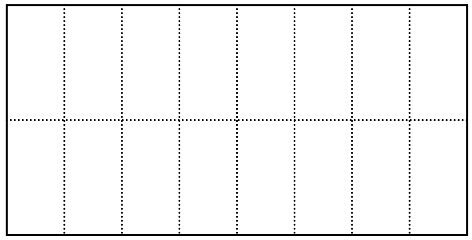
Explain your reasoning.

6. 4 candy bars among 6 people

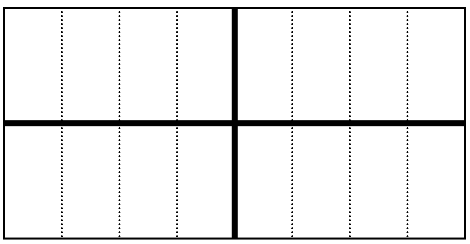
Explain your reasoning.

APPENDIX H: THE CANDY BAR

The Candy Bar



Quentin, Randy, Stephanie and Tina want to share 1 candy bar that is partitioned into the pieces shown. Each of them proposed one of the following ways to divide the candy bar so each one had an equal amount. Write a fraction to represent how each person divided the candy bar. Then justify why each one represents an equal amount.

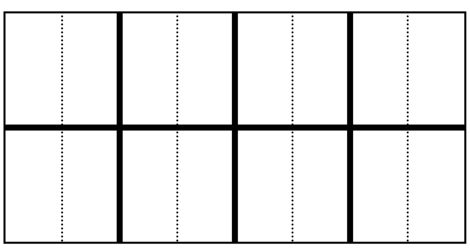


Quentin divided it into ___________.

Each person gets __________.

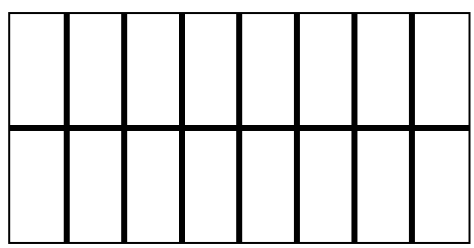
Randy divided it into ___________.

Each person gets __________.



Stephanie divided it into __________.

Each person gets __________.



Tina divided it into ___________.

Each person gets __________.

How do you know each person gets an equal amount?

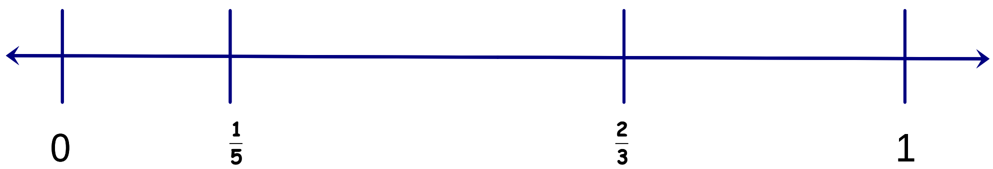
____

____

____

APPENDIX I: NUMBER LINE CHALLENGE

Number Line Challenge



Locate the following points on the number line:

$\frac{1}{2}$	$\frac{1}{4}$	$\frac{3}{4}$	
$\frac{1}{3}$	$\frac{1}{6}$	$\frac{5}{6}$	
$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	
$\frac{1}{8}$	$\frac{3}{8}$	$\frac{5}{8}$	$\frac{7}{8}$
$\frac{1}{10}$	$\frac{3}{10}$	$\frac{7}{10}$	$\frac{9}{10}$

APPENDIX J: BETWEEN

Between

Name each fraction as described and explain how you determined your answer. Use only whole numbers as numerators and denominators.

1. Name a fraction between 2/5 and 4/5.

2. Name a fraction between 1/5 and 2/5.

3. Name 2 fractions between 3/8 and 5/8.

4. Name 3 fractions between 1/6 and 1/3.

5. Name 3 fractions between 8/9 and 1. Choose fractions that have the same numerator.

APPENDIX K: COMPARE

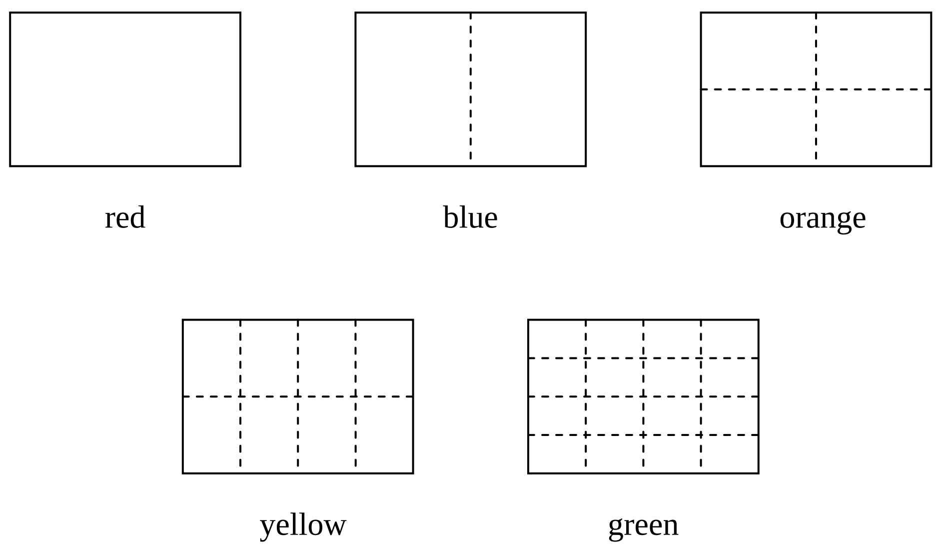
1. 4/5 and 4/9
2. 1/3 and 3/5
3. 4/7 and 3/8
4. 7/8 and 5/4
5. 3/8 and 5/8
6. 2/5 and 2/7
7. 3/4 and 9/10
8. 6/7 and 8/9
9. 3/7 and 5/8
10. 5/8 and 4/6

APPENDIX L: FRACTION KIT

Fraction Kit

Materials: 5 sheets of construction paper in the colors indicated below

Directions: Fold and tear the construction paper as indicated by the dashed lines.
Fold each piece at the midpoints of the long sides.



APPENDIX M: ADDITION AND SUBTRACTION SITUATIONS

Martha ate 1/4 of a medium cheese pizza. Then she ate 1/8 of a medium pepperoni pizza. What fractional part of a pizza did Martha eat?

Betty had 2 1/2 yards of ribbon. She gave 2/3 of a yard of her ribbon to Wilma. How much ribbon did Betty have left?

Lila brought some cupcakes to a picnic. Andy ate 1/6 of the cupcakes. Barney ate 1/4 of the cupcakes. What part of the cupcakes did they eat altogether?

APPENDIX N: ADDITION AND SUBTRACTION

Addition and Subtraction

Model each sum. Provide a drawing of your model and explain your reasoning.

1. 1/4 + 3/8

2. 1/3 + 3/8

3. 3/10 + 1 2/5

4. 5/8 + 5/6

5. 1/2 + 2/3

Model each difference. Provide a drawing of your model and explain your reasoning.

6. 5/8 – 1/2

7. 5/6 – 2/9

8. 3/4 –2/3

9. 1 1/3 –1/2

10. 1 1/3 –5/6

APPENDIX O: MULTIPLICATION AND DIVISION SITUATIONS

1. Four workers each painted 3/5 of a wall. How many walls did they paint in all?

2. It takes 3/4 foot of wood to make a picture frame. How many 3/4-foot lengths can Pat cut from an 8-foot board? What part of another picture frame would she have left?

3. Four friends buy a total of 3/4 pound of chocolate. How much will each person get if they share the chocolate equally?

4. Sue ate some pizza. 2/3 of a pizza is left over. Jim ate 3/4 of the left over pizza. How much of a whole pizza did Jim eat?

APPENDIX P: MULTIPLICATION

Model each problem. Provide a drawing of your model and explain your reasoning.

1. 2 X 3/5

2. 3/5 X 5/6

3. 1 3/4 X 2/3

4. 5/6 X 3/8

APPENDIX Q: MULTIPLICATION PRACTICE

Multiplication Practice

Model each problem. Provide a drawing of your model and explain your reasoning.

1. Jenny ate 1/2 of 3/8 of a pizza that was left over. How much of the whole pizza did Jenny eat?

2. Sam gave his friends 2/3 of the candy he made. If Sam made 4 pounds of candy, how many pounds did he give his friends?

3. 1/3 X 3/4

4. 3/5 X 2 1/2

APPENDIX R: DIVISION SITUATIONS

Pete is building a model of a city for a school project. He needs to cut lengths of a board that measure 1/4 foot each to make the skyscrapers.

How many whole sky scrapers can he cut from a board that is 1 7/8 feet long?

What part of a skyscraper will he have left over?

Sarah made 1 2/3 gallons of tea. What part of a gallon will each of 4 people have if they share it equally?

APPENDIX S: DIVISION

Model each problem. Provide a drawing of your model and explain your reasoning.

1. 3 ÷ 5/8

2. 3/4 ÷ 1/8

3. 2/3 ÷ 4

4. 1 1/2 ÷ 2/3

APPENDIX T: DIVISION PRACTICE

Make a drawing to show how you found each answer.

1. Jill put 1/8 of a pound of jelly beans into individual snack bags. How many bags can she make from 1 1/3 pounds? What part of a bag will she have left?

2. Mrs. Dunn's class had a pizza party. They had 1 2/3 pizza left over. Five teachers shared the left over pizza equally. How much of a pizza did each teacher get?

3. 4/5 ÷ 3/10

4. 2 ÷ 3/4

APPENDIX U: ESTIMATION

Estimation

Choose the estimate that best describes the product.

9/10 X 7 5/6

a. between 7 and 8
b. between 8 and 9
c. about 4

3/20 X 17 1/2

a. between 3 and 4
b. between 2 and 3
c. about 15

1 1/2 X 4 2/3

a. between 4 and 5
b. between 5 and 6
c. about 7

4 3/4 X 6 1/10

a. between 24 and 25
b. between 28 and 29
c. about 20

Choose the estimate that best describes the quotient.

11 3/8 ÷ 1/2

a. between 5 and 6
b. between 22 and 23
c. about 20

6 1/3 ÷ 3/4

a. between 6 and 7
b. between 18 and 20
c. about 8

25 1/4 ÷ 9/10

a. between 24 and 25
b. between 27 and 28
c. about 20

10 5/12 ÷ 1/8

a. between 83 and 84
b. between 10 and 12
c. about 88

APPENDIX V: IRB APPROVAL

THE UNIVERSITY OF CENTRAL FLORIDA

INSTITUTIONAL REVIEW BOARD (IRB)

IRB Committee Approval Form

PRINCIPAL INVESTIGATOR(S): Janet Andreasen **IRB #: 05-2548**
Debra Wheeldon

PROJECT TITLE: Preservice Elementary School Teachers' Understanding of Number and Operation through the Use of an Emerging Instructional Sequence

[X] New project submission [ ] Resubmission of lapsed project #_____
[ ] Continuing review of lapsed project #____ [ ] Continuing review of #_____
[ ] Study expired _______ [ ] Initial submission was approved by expedited review
[ ] Initial submission was approved by full board review but continuing review can be expedited
[ ] Suspension of enrollment email sent to PI, entered on spreadsheet, administration notified ______

Chair

[X] Expedited Approval
Dated: 6 APRIL 2005
Cite how qualifies for expedited review:
minimal risk and #7

IRB Co-Chairs:

Signed: [signature]
Dr. Sophia Dziegielewski

[ ] Exempt
Dated: ___________
Cite how qualifies for exempt status:
minimal risk and _______________

Signed: ______________________
Dr. Jacqueline Byers

[X] Expiration
Date: 5 April 2006

[ ] Waiver of documentation of consent approved
[ ] Waiver of consent approved

NOTES FROM IRB CHAIR (IF APPLICABLE): Researcher will verify via e-mail all participants are 18 and older.

Received verification from PI and supervisor. 4/8/05 BW

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