International Journal of Control

ISSN: 0020-7179 (Print) 1366-5820 (Online) Journal homepage: https://www.tandfonline.com/loi/tcon20

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To cite this article: Xingwen Liu & Xudong Zhao (2016) Stability analysis of discrete-time switched systems: a switched homogeneous Lyapunov function method, International Journal of Control, 89:2, 297-305, DOI: 10.1080/00207179.2015.1075254

To link to this article: https://doi.org/10.1080/00207179.2015.1075254

Published online: 19 Aug 2015.

INTERNATIONAL JOURNAL OF CONTROL, 2016
VOL. 89, NO. 2, 297–305
http://dx.doi.org/10.1080/00207179.2015.1075254

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Stability analysis of discrete-time switched systems: a switched homogeneous Lyapunov function method

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ABSTRACT
This paper addresses the stability issue of discrete-time switched systems with guaranteed dwell-time. The approach of switched homogeneous Lyapunov function of higher order is formally proposed. By means of this approach, a necessary and sufficient condition is established to check the exponential stability of the considered system. With the observation that switching signal is actually arbitrary if the dwell time is one sample time, a necessary and sufficient condition is also presented to verify the exponential stability of switched systems under arbitrary switching signals. Using the augmented argument, a necessary and sufficient exponential stability criterion is given for discrete-time switched systems with delays. A numerical example is provided to show the advantages of the theoretical results.

ARTICLE HISTORY
Received 19 December 2014
Accepted 18 July 2015

KEYWORDS
Delays; dwell-time; homogeneous Lyapunov function; stability; switched systems

1. Introduction

As an important class of hybrid dynamic systems, switched systems inherit the feature of both continuous state and discrete state dynamic systems. Loosely speaking, a switched system consists of a family of dynamical subsystems and a rule, called a switching signal, that determines the switching manner among the subsystems (Sun & Ge, 2011; Sun & Wang, 2013; Yang, Xiang, & Lee, 2012). Many dynamic systems can be modelled as switched systems (Goebel, Sanfelice, and Teel, 2009) which possess rich dynamics due to the multiple subsystems and various possible switching signals (Deaecto, Fioravanti, & Geromel, 2013; Zhang, Cui, Liu, & Zhao, 2011). This paper focuses on the stability issue of discrete-time switched systems with arbitrary switching signals and switching signals satisfying the dwell-time constraint.

Stability is one of the most important properties of switched systems (Allerhand & Shaked, 2013; Zhang, Abate, Hu, & Vitus, 2009). For discrete-time switched systems with arbitrary switching signals, the switched quadratic Lyapunov function approach was proposed in Daafouz, Riedinger, and Iung (2002), which, compared with the common quadratic Lyapunov function method, has the advantage of less conservativeness. A switched linear copositive Lyapunov function method was presented in Liu (2009) to analyse the stability of switched positive systems. The method in Daafouz et al. (2002) was extended in Geromel and Colaneri (2006) to investigate the stability property of switched systems with guaranteed dwell-time. However, all these conditions are only sufficient, not necessary for guaranteeing the asymptotic stability of the considered systems. The Lyapunov functions in Daafouz et al. (2002), Geromel and Colaneri (2006) are the most frequently used quadratic Lyapunov function which may be high conservative in the context of switched systems. For example, exponential stability of a switched linear system under arbitrary switching is equivalent to the existence of a common Lyapunov function, but generally does not imply that there exists a common quadratic one for its constituent systems (Dayawansa & Martin, 1999).

Unlike the quadratic Lyapunov function method, homogeneous Lyapunov function of higher order, together with sum of square technique (Chesi, Garulli, Tesi, & Vicino, 2009), can lead to less conservative stability conditions. Recently, copositive polynomial Lyapunov function (a special form of homogeneous Lyapunov function) was proposed for continuous-time switched systems with arbitrary switching signals (Zhao, Liu, Yin, & Li, 2014). Chesi et al. proposed a nonconservative linear matrix inequality (LMI) condition to check the exponential stability of continuous-time switched systems with guaranteed dwell-time (Chesi, Colaneri, Geromel, Middleton, & Shorten, 2012). However, a result parallel to discrete-time switched systems with

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guaranteed dwell-time has not been established, which is the concern of this paper. In fact, homogeneous Lyapunov function was seldom applied to discrete-time switched systems before. Therefore, this work will also enlighten us to use the method in future study of other dynamic properties of discrete-time switched systems.

As far as dwell-time is concerned, one of the differences between continuous- and discrete-time switched systems lies in: for the latter, if the dwell-time is one sample time, then the switching signals are actually arbitrary. Based on this observation, if we establish a nonconservative stability condition for a discrete-time switched system with dwell-time, then we can obtain a nonconservative stability condition for the same system under arbitrary switching signals. Moreover, a discrete-time system with bounded delays can always be transformed into a delay-free system of higher dimension by using the so-called augmented method. Hence, it is possible to provide a necessary and sufficient stability condition for a discrete-time system with delays. It should be pointed out that in spite of low conservativeness of homogeneous Lyapunov function, it cannot, at least for the time being, be applied to general delayed systems. Therefore, on the basis of this work, it is possible to apply homogeneous Lyapunov function to delayed systems in the future.

On this ground, we study in this paper how to apply homogeneous Lyapunov function to discrete-time switched systems, focusing on the exponential stability rather than the asymptotic stability as existing ones. The main contribution lies in the following aspects: first, a necessary and sufficient condition checking the exponential stability of discrete-time switched system with guaranteed dwell-time is presented, which is formulated in a set of LMI conditions. Second, a nonconservative exponential stability condition is proposed for switched systems under arbitrary switching signals. Third, as an application of the proposed results, some necessary and sufficient exponential stability conditions are deduced for switched systems with bounded delays. In addition, several other interesting results are also presented, including the claim that if a system is asymptotically stable for all switching signals with a given dwell-time, then it is exponentially stable for all switching signals with any dwell-time greater than or equal to the given one.

The rest of this paper is organised as follows. Preliminaries are presented in Section 2. Sections 3 consists of two parts: Subsection 3.1 proposes nonconservative exponential stability conditions for switched systems with guaranteed dwell-time or with arbitrary switching signals, respectively; Subsection 3.2 discusses the stability property of switched systems with bounded delays. Section 4 gives a numerical example to show the advantage of the proposed results over reported papers. Finally, Section 5 concludes this paper.

The following notations are rather standard : $A > 0(< 0)$ means that square matrix A is a symmetrical positive (negative) definite matrix, and $A^{\mathrm{T}}(A^{-1})$ is the transpose (inverse) of matrix A. $\mathbb{R}$ is the set of real numbers, $\mathbb{R}^n$ is the set of n-dimensional real vector, and $\mathbb{R}^{n\times n}$ stands for the set of real matrices of $n \times n$ dimension. diag$(a_1, \ldots, a_n)$ denotes a diagonal matrix with diagonal elements $a_1, \ldots, a_n$. $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 = \{0\} \bigcup \mathbb{N}$. For $m \in \mathbb{N}$, $\underline{m} = \{1, 2, \ldots, m\}$. Given $p \in \mathbb{N}$, $\|\boldsymbol{x}\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}}$, where $\boldsymbol{x} = [x_1, \ldots, x_n]^{\mathrm{T}} \in \mathbb{R}^n$. Particularly, $\|\boldsymbol{x}\|_1 = \sum_{i=1}^n |x_i|$. $\|\boldsymbol{x}\|_\infty = \max\{|x_1|, \ldots, |x_n|\}$, $\|A\|_1 = \sup_{\boldsymbol{x}\neq 0} \frac{\|A\boldsymbol{x}\|_1}{\|\boldsymbol{x}\|_1}$. $\mathbb{S}^n$ is the set of $n \times n$ real symmetric matrices. $A\otimes B$ denotes the Kronecker product of A and B, and $A^{\otimes q}$ the qth Kronecker power in A, namely $A^{\otimes q} = \underbrace{A \otimes A \otimes \ldots \otimes A}_{q \text{ times}}$. $\mathbf{0}_{n\times m}$ is zero matrix of $n \times m$ dimension, and I_n the unit matrix of $n \times n$ dimension. Throughout this paper, the dimensions of matrices and vectors will not be explicitly mentioned if clear from context.

2. Preliminaries

Consider the following switched linear system

$$\boldsymbol{x}(k+1) = A_{\sigma(k)}\boldsymbol{x}(k), \quad k \in \mathbb{N}_0, \tag{2.1}$$

where $\boldsymbol{x}(k) \in \mathbb{R}^n$ is the state variable, the map $\sigma : \mathbb{N}_0 \to \underline{m}$ is a switching signal with m being the number of subsystems, $A_l \in \mathbb{R}^{n\times n}, l \in \underline{m}$, are system matrices. A subsystem, say the lth one, is activated at instant k if $\sigma(k) = l$. Let $\boldsymbol{x}_0 = \boldsymbol{x}(0)$ be the initial condition.

The following definitions will be used repeatedly.

Definition 1: A continuous function α: $[0, a) \to [0, \infty)$ belongs to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$, where $a > 0$ or equals $+\infty$. It belongs to class $\mathcal{K}_\infty$ if it belongs to class $\mathcal{K}$ and $\alpha(r) \to \infty$ as $r \to \infty$. A continuous function β: $[0, a) \times [0, \infty) \to [0, \infty)$ belongs to class $\mathcal{KL}$ if, for any fixed s, the mapping $\beta(r, s)$ belongs to class $\mathcal{K}$ with respect to r and, for any fixed r, $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to \infty$.

Definition 2: Consider system (2.1) and let $\mathbb{D}$ be a given set of switching signals. (2.1) is exponentially stable over $\mathbb{D}$ if there exist two scalars $\alpha > 0$ and $\gamma > 1$ such that $\|\boldsymbol{x}(k)\| \leq \alpha\gamma^{-k}\|\boldsymbol{x}_0\|$, $\forall k \in \mathbb{N}_0, \forall\sigma \in \mathbb{D}$, and is asymptotically stable over $\mathbb{D}$ if there exists a class $\mathcal{KL}$ function β such that $\|\boldsymbol{x}(k)\| \leq \beta(\|\boldsymbol{x}_0\|, k)$, $\forall k \in \mathbb{N}_0, \forall\sigma \in \mathbb{D}$, where the norm $\|\cdot\|$ can be any vector norm.

Definition 3 (Liberzon, 2003)**:** Suppose that a switching signal σ with switching sequence $\{k_\ell\}_{\ell=0}^\infty$, where

$k_0 = 0$, $k_{\ell+1} > k_\ell$ and $k_\ell \in \mathbb{N}_0$. $\kappa \in \mathbb{N}$ is said to be the dwell time of σ if $k_{\ell+1} - k_\ell \geq \kappa$, $\forall \ell \in \mathbb{N}_0$. Denote $\mathbb{D}_\kappa = \{\sigma : k_{\ell+1} - k_\ell \geq \kappa, \ell \in \mathbb{N}_0\}$.

Define the following symbols for later use:

$$\begin{aligned}\vartheta(n,d) &= \frac{(n+d-1)!}{(n-1)!d!}, \omega(n,d)\\ &= \frac{1}{2}\vartheta(n,d)(\vartheta(n,d)+1) - \vartheta(n,2d),\\ \mathscr{L}_{n,d} &= \left\{L \in \mathbb{S}^{\vartheta(n,d)} : \left(\boldsymbol{x}^{\{d\}}\right)^{\mathrm{T}} L\boldsymbol{x}^{\{d\}} = 0\right\}.\end{aligned}$$

Note that $\mathscr{L}_{n,d}$ is a linear space which can be completely characterised by a vector $\boldsymbol{v} \in \mathbb{R}^{\omega(n,d)}$ (Chesi et al., 2009).

For a given positive integer d, $\boldsymbol{x}^{\{d\}} \in \mathbb{R}^{\vartheta(n,d)}$ is a base vector containing all homogeneous monomials of degree d in $\boldsymbol{x}$ (Chesi et al., 2009). There may exist many specific configurations of $\boldsymbol{x}^{\{d\}}$ for given d and $\boldsymbol{x}$.

Fixing d and $\boldsymbol{x}^{\{d\}}$, there exists a full column rank matrix $K_d \in \mathbb{R}^{n^d \times \vartheta(n,d)}$ such that $K_d\boldsymbol{x}^{\{d\}} = \boldsymbol{x}^{\otimes d}$. By Brewer (1978, T3.7), $(A\boldsymbol{x})^{\otimes d} = A^{\otimes d}\boldsymbol{x}^{\otimes d}$, so $K_d(A\boldsymbol{x})^{\{d\}} = (A\boldsymbol{x})^{\otimes d} = A^{\otimes d}K_d\boldsymbol{x}^{\{d\}}$ and therefore $(A\boldsymbol{x})^{\{d\}} = \mathcal{A}_d\boldsymbol{x}^{\{d\}}$ with $\mathcal{A}_d = (K_d^{\mathrm{T}}K_d)^{-1}K_d^{\mathrm{T}}A^{\otimes d}K_d$, since K_d is of full column rank and thus its left inverse is $(K_d^{\mathrm{T}}K_d)^{-1}K_d^{\mathrm{T}}$. Moreover, $(A^i\boldsymbol{x})^{\{d\}} = (A(A^{i-1}\boldsymbol{x}))^{\{d\}} = \mathcal{A}_d(A^{i-1}\boldsymbol{x})^{\{d\}} = \cdots = \mathcal{A}_d^i\boldsymbol{x}^{\{d\}}$ for any $i \in \mathbb{N}$, that is,

$$\left(A^i\boldsymbol{x}\right)^{\{d\}} = \mathcal{A}_d^i\boldsymbol{x}^{\{d\}}, \quad \forall i \in \mathbb{N}. \tag{2.2}$$

Now we provide the definition of switched homogeneous Lyapunov function (SHLF).

Definition 4: Let $V(k,\boldsymbol{x},d) = \left(\boldsymbol{x}^{\{d\}}\right)^{\mathrm{T}} P_{\sigma(k)}\boldsymbol{x}^{\{d\}}$ with $P_i > 0$, $\forall i \in \underline{m}$. If V satisfies $V(k+1,\boldsymbol{x}(k+1),d) - V(k,\boldsymbol{x}(k),d) < 0$, $\forall k \in \mathbb{N}_0$, then it is called an SHLF of degree d for system (2.1) (under arbitrary switching signals). If $\kappa \geq 2$ and

$$\begin{aligned}&V(k+1,\boldsymbol{x}(k+1),d) - V(k,\boldsymbol{x}(k),d) < 0,\\ &\forall k \in \{k_\ell,\ldots,k_{\ell+1}-2\}, \forall \ell \in \mathbb{N}_0,\\ &V(k_{\ell+1},\boldsymbol{x}(k_{\ell+1}),d) - V(k_\ell,\boldsymbol{x}(k_\ell),d) < 0, \forall \ell \in \mathbb{N}_0\end{aligned} \tag{2.3}$$

holds for any $\sigma \in \mathbb{D}_\kappa$, where k_ℓ is the switching instant of σ, then V is called an SHLF of degree d for system (2.1) with guaranteed dwell-time κ.

Applying Liu and Liu (2014, Theorem 2) with $A_{il}(k) = 0 (\forall i \in \underline{p})$ and $A_{0l}(k) = A_l \left(l \in \underline{m}\right)$, the next lemma holds.

Lemma 1: *Fix $\kappa \in \mathbb{N}$. System (2.1) is exponentially stable over $\mathbb{D}_\kappa$ if it is asymptotically stable over $\mathbb{D}_\kappa$.*

Lemma 2: *Fix $\kappa \in \mathbb{N}$. If there exists an SHLF of degree d for system (2.1), then (2.1) is exponentially stable over $\mathbb{D}_\kappa$.*

Proof: Suppose first that $\kappa = 1$. It follows from Daafouz et al. (2002, Theorem 1) that system (2.1) is asymptotically stable under arbitrary switching signals. By Lemma 1, it is exponentially stable under arbitrary switching signals.

For $\kappa \geq 2$, the first inequality in (2.3) implies that

$$\begin{aligned}&V(k,\boldsymbol{x}(k),d) < V(k_\ell,\boldsymbol{x}(k_\ell),d),\\ &\forall k \in \{k_\ell+1,\ldots,k_{\ell+1}-1\}, \forall \ell \in \mathbb{N}_0.\end{aligned} \tag{2.4}$$

Let $\sigma(k_{\ell+1}) = j$, $\sigma(k_\ell) = i$. By definition of V,

$$\begin{aligned}&V(k_{\ell+1},\boldsymbol{x}(k_{\ell+1}),d) - V(k_\ell,\boldsymbol{x}(k_\ell),d)\\ &\quad = \left(\left(A_i^{k_{\ell+1}-k_\ell}\boldsymbol{x}(k_\ell)\right)^{\{d\}}\right)^{\mathrm{T}} P_j \left(A_i^{k_{\ell+1}-k_\ell}\boldsymbol{x}(k_\ell)\right)^{\{d\}}\\ &\qquad - ((\boldsymbol{x}(k_\ell))^{\{d\}})^{\mathrm{T}}P_i(\boldsymbol{x}(k_\ell))^{\{d\}}.\end{aligned} \tag{2.5}$$

Particularly, take a switching signal $\sigma \in \mathbb{D}_\kappa$ with $k_{\ell+1} - k_\ell = \kappa$, $\ell \in \mathbb{N}_0$, then it follows from (2.5) and the second inequality in (2.3) that

$$\begin{aligned}&V(k_{\ell+1},\boldsymbol{x}(k_{\ell+1}),d) - V(k_\ell,\boldsymbol{x}(k_\ell),d)\\ &\quad = \left((\boldsymbol{x}(k_\ell))^{\{d\}}\right)^{\mathrm{T}}\left(\left(\mathcal{A}_{i,d}^\kappa\right)^{\mathrm{T}} P_j\mathcal{A}_{i,d}^\kappa - P_i\right)(\boldsymbol{x}(k_\ell))^{\{d\}} < 0,\end{aligned}$$

which means that $(\mathcal{A}_{i,d}^\kappa)^{\mathrm{T}}P_j\mathcal{A}_{i,d}^\kappa - P_i < 0$. Thus, there exists a scalar $0 < \iota_{ij} < 1$, such that $(\mathcal{A}_{i,d}^\kappa)^{\mathrm{T}}P_j\mathcal{A}_{i,d}^\kappa - P_i < -\iota_{ij}P_i$. Define $\iota = \min_{i,j\in\underline{m},i\neq j}\{\iota_{ij}\}$, then $(\mathcal{A}_{i,d}^\kappa)^{\mathrm{T}}P_j\mathcal{A}_{i,d}^\kappa - P_i < -\iota P_i$ holds, which implies that $V(k_{\ell+1},\boldsymbol{x}(k_{\ell+1}),d) < \alpha V(k_\ell,\boldsymbol{x}(k_\ell),d)$ with $\alpha = 1 - \iota$ satisfying $0 < \alpha < 1$. Consequently, by virtue of (2.4), it holds that

$$V(k_\ell,\boldsymbol{x}(k_\ell)) < \alpha^\ell V(0,\boldsymbol{x}_0), \quad \forall \ell \in \mathbb{N}. \tag{2.6}$$

Note that $\sigma \in \mathbb{D}_\kappa$ may have only finite switching instants. In this situation, (2.4) means that the system is asymptotically stable. If σ do have infinite switching instants, then (2.6), together with (2.4), indicates that system (2.1) is asymptotically stable over $\mathbb{D}_\kappa$. By Lemma 1, (2.1) is exponentially stable over $\mathbb{D}_\kappa$. □

3. Main results

This section first considers the stability issue of switched systems with guaranteed dwell-time in Subsection 3.1, and then extends the results in Subsection 3.1 to switched systems with delays in Subsection 3.2.

3.1 Switched homogeneous Lyapunov function for switched systems

The following lemma whose proof is given in Appendix 1 is a key to establish our main result.

Lemma 3: *System (2.1) is asymptotically stable over $\mathbb{D}_\kappa$ if and only if there exist full row rank matrices X_i, matrices P_i with $\|P_i\|_1 < 1$, and square matrices R_{ij} with $\|R_{ij}\|_1 < 1$, such that*

$$A_i X_i = X_i P_i, \quad A_i^\kappa X_j = X_i R_{ij}, \quad \forall i, j \in \underline{m}.$$

By Blanchini and Miani (2008), Lemma 3 indicates that system (2.1) is asymptotically stable over $\mathbb{D}_\kappa$ if and only if there exists a set of polytopes $\mathscr{X}_i$ with vertex representation matrix X_i being, such that $\psi_i(\boldsymbol{x}) = \inf\{\|\boldsymbol{p}\|_1 : \boldsymbol{x} = X_i \boldsymbol{p}\}$ can serve as a Lyapunov function candidate. Moreover, suppose that F_i (full column rank matrix) is the plane representation matrix of $\mathscr{X}_i$, then $\|F_i \boldsymbol{x}\|_\infty = \inf\{\|\boldsymbol{p}\|_1 : \boldsymbol{x} = X_i \boldsymbol{p}\}$. These observations result in the following lemma.

Lemma 4: *System (2.1) is asymptotically stable over $\mathbb{D}_\kappa$ if and only if there exist full column rank matrices F_i, $\forall i \in \underline{m}$, such that $\psi_{\sigma(k)}(\boldsymbol{x}(k)) = \|F_{\sigma(k)}\boldsymbol{x}(k)\|_\infty$ is a valid Lyapunov function for (2.1).*

We are in a position to propose the following main result.

Theorem 1: *Suppose that $\kappa \geq 2$. The following statements are equivalent.*

(1) *System (2.1) is exponentially stable over $\mathbb{D}_\kappa$;*

(2) *System (2.1) admits an SHLF of degree d for some $d \in \mathbb{N}$;*

(3) *There exist a positive integer d, matrices $0 < P_i \in \mathbb{S}^{\vartheta(n,d)}$, and $L(\boldsymbol{v}_i), L(\boldsymbol{v}_{ij}) \in \mathscr{L}_{n,d}$ with $\boldsymbol{v}_i, \boldsymbol{v}_{ij} \in \mathbb{R}^{\omega(n,d)}$, such that*

$$\mathcal{A}_{i,d}^{\mathrm{T}} P_i \mathcal{A}_{i,d} - P_i + L(\boldsymbol{v}_i) < 0, \quad \forall i \in \underline{m}, \tag{3.1}$$

$$\left(\mathcal{A}_{i,d}^\kappa\right)^{\mathrm{T}} P_j \mathcal{A}_{i,d}^\kappa - P_i + L\left(\boldsymbol{v}_{ij}\right) < 0, \quad \forall i, j \in \underline{m},\ i \neq j, \tag{3.2}$$

where $\mathcal{A}_{i,d} = \left(K_d^{\mathrm{T}} K_d\right)^{-1} K_d^{\mathrm{T}} A_i^{\otimes d} K_d$.

Proof: (2)⇒(1) holds by Lemma 2. Now show (3)⇒(2) and (1)⇒(3).

(3)⇒ (2). Chose the Lyapunov function candidate as follows:

$$V(k, \boldsymbol{x}(k), d) = \left((\boldsymbol{x}(k))^{\{d\}}\right)^{\mathrm{T}} P_{\sigma(k)} (\boldsymbol{x}(k))^{\{d\}}. \tag{3.3}$$

Suppose that the switching signal σ has switching instants $\{k_\ell\}_{\ell=0}^\infty$. Assume that $k \in \{k_\ell, \ldots, k_{\ell+1} - 2\}$ and that $\sigma(k_\ell) = i$. It follows from (3.1) that

$$\begin{aligned} &V(k+1, \boldsymbol{x}(k+1), d) \\ &= \left((\boldsymbol{x}(k+1))^{\{d\}}\right)^{\mathrm{T}} P_i (\boldsymbol{x}(k+1))^{\{d\}} \\ &= \left((\boldsymbol{x}(k))^{\{d\}}\right)^{\mathrm{T}} \mathcal{A}_{i,d}^{\mathrm{T}} P_i \mathcal{A}_{i,d} (\boldsymbol{x}(k))^{\{d\}} \\ &< \left((\boldsymbol{x}(k))^{\{d\}}\right)^{\mathrm{T}} (P_i - L(\boldsymbol{v}_i)) (\boldsymbol{x}(k))^{\{d\}}. \end{aligned}$$

Since $L(\boldsymbol{v}_i) \in \mathscr{L}_{n,d}$, we have that

$$\begin{aligned} V(k+1, \boldsymbol{x}(k+1), d) &< \left((\boldsymbol{x}(k))^{\{d\}}\right)^{\mathrm{T}} P_i (\boldsymbol{x}(k))^{\{d\}} \\ &= V(k, \boldsymbol{x}(k), d). \end{aligned}$$

Therefore,

$$V(\boldsymbol{x}(k+1)) - V(k, \boldsymbol{x}(k), d) < 0. \tag{3.4}$$

Let $\sigma(k_{\ell+1}) = j$. It clearly holds that $k_{\ell+1} - k_\ell = \kappa + \imath$ with $\imath \in \mathbb{N}_0$. Then,

$$\begin{aligned} &V(k_{\ell+1}, \boldsymbol{x}(k_{\ell+1}), d) \\ &= \left(\left(A_i^{\kappa+\imath}\boldsymbol{x}(k_\ell)\right)^{\{d\}}\right)^{\mathrm{T}} P_j \left(A_i^{\kappa+\imath}\boldsymbol{x}(k_\ell)\right)^{\{d\}}. \end{aligned}$$

By (2.2), $\left(A_i^{\kappa+\imath}\boldsymbol{x}(k_\ell)\right)^{\{d\}} = \mathcal{A}_{i,d}^{\kappa+\imath}(\boldsymbol{x}(k_\ell))^{\{d\}}$, which combining (3.2) yields that

$$\begin{aligned} &V(k_{\ell+1}, \boldsymbol{x}(k_{\ell+1}), d) \\ &= \left(\mathcal{A}_{i,d}^\imath (\boldsymbol{x}(k_\ell))^{\{d\}}\right)^{\mathrm{T}} \left(\mathcal{A}_{i,d}^\kappa\right)^{\mathrm{T}} P_j \mathcal{A}_{i,d}^\kappa \mathcal{A}_{i,d}^\imath (\boldsymbol{x}(k_\ell))^{\{d\}} \\ &< \left(\mathcal{A}_{i,d}^\imath (\boldsymbol{x}(k_\ell))^{\{d\}}\right)^{\mathrm{T}} \left(P_i - L\left(\boldsymbol{v}_{ij}\right)\right) \mathcal{A}_{i,d}^\imath (\boldsymbol{x}(k_\ell))^{\{d\}}. \end{aligned}$$

It is not difficult to see that

$$\begin{aligned} &\left(\mathcal{A}_{i,d}^\imath (\boldsymbol{x}(k_\ell))^{\{d\}}\right)^{\mathrm{T}} L\left(\boldsymbol{v}_{ij}\right) \mathcal{A}_{i,d}^\imath (\boldsymbol{x}(k_\ell))^{\{d\}} \\ &\quad = \left(\left(A_i^\imath \boldsymbol{x}(k_\ell)\right)^{\{d\}}\right)^{\mathrm{T}} L\left(\boldsymbol{v}_{ij}\right) \left(A_i^\imath \boldsymbol{x}(k_\ell)\right)^{\{d\}} = 0, \\ &\left(\mathcal{A}_{i,d}^\jmath (\boldsymbol{x}(k_\ell))^{\{d\}}\right)^{\mathrm{T}} L(\boldsymbol{v}_i) \mathcal{A}_{i,d}^\jmath (\boldsymbol{x}(k_\ell))^{\{d\}} = 0, \quad \forall \jmath \in \mathbb{N}_0. \end{aligned} \tag{3.5}$$

Repeatedly using (3.5) and (3.1), one has that

$$\begin{aligned} & V\left(k_{\ell+1}, \boldsymbol{x}\left(k_{\ell+1}\right), d\right) \\ & \quad<\left(\mathcal{A}_{i, d}^{\imath}\left(\boldsymbol{x}\left(k_{\ell}\right)\right)^{\{d\}}\right)^{\mathrm{T}} P_i \mathcal{A}_{i, d}^{\imath}\left(\boldsymbol{x}\left(k_{\ell}\right)\right)^{\{d\}} \\ & \quad \leq\left(\mathcal{A}_{i, d}^{\imath-1}\left(\boldsymbol{x}\left(k_{\ell}\right)\right)^{\{d\}}\right)^{\mathrm{T}}\left(P_i-L\left(\boldsymbol{v}_i\right)\right) \mathcal{A}_{i, d}^{\imath-1}\left(\boldsymbol{x}\left(k_{\ell}\right)\right)^{\{d\}} \\ & \quad=\left(\mathcal{A}_{i, d}^{\imath-1}\left(\boldsymbol{x}\left(k_{\ell}\right)\right)^{\{d\}}\right)^{\mathrm{T}} P_i \mathcal{A}_{i, d}^{\imath-1}\left(\boldsymbol{x}\left(k_{\ell}\right)\right)^{\{d\}} \\ & \quad \ldots \\ & \quad \leq\left(\left(\boldsymbol{x}\left(k_{\ell}\right)\right)^{\{d\}}\right)^{\mathrm{T}} P_i\left(\boldsymbol{x}\left(k_{\ell}\right)\right)^{\{d\}} \\ & \quad=V\left(k_{\ell}, \boldsymbol{x}\left(k_{\ell}\right), d\right) . \end{aligned} \tag{3.6}$$

By (3.4) and (3.6), we claim that $V(k, \boldsymbol{x}(k), d)$ in (3.3) is the required SHLF in (2).

(1)⇒(3). By Lemma 4, there exists a set of full column rank matrices M_i such that $V_i = \|M_i\boldsymbol{x}\|_\infty, i \in \underline{m}$, are valid Lyapunov functions. By Blanchini and Miani (1999), $\|M_i\boldsymbol{x}\|_{2p}$ converges uniformly as p approaches ∞. Therefore, V_i may be chosen as $V_i = \|M_i\boldsymbol{x}\|_{2p}$ for some $p \in \mathbb{N}$. Moreover, it is easy to verify that V_i^α is also a valid Lyapunov function for any $\alpha \in \mathbb{N}$, and therefore we can take $V_i = \|M_i\boldsymbol{x}\|_{2p}^{2p}$. Then, following a process similar to that of Chesi et al. (2012, Theorem 6), the conclusion follows. □

In Theorem 1, the quantities of $\vartheta(n, d)$ and $\omega(n, d)$ are listed in (Chesi et al., 2009, Table 1.1), which shows that the quantities of $\vartheta(n, d)$ and $\omega(n, d)$ rapidly increase when n or d increases. Therefore, the necessary and sufficient condition is at the price of possible heavy computational effort. Similar comments can be made for later results.

By Lemma 1 and Theorem 1, the following corollary clearly holds since $\mathbb{D}_{\kappa_1} \subset \mathbb{D}_\kappa$ if $\kappa_1 > \kappa$.

Corollary 1: *If system (2.1) is asymptotically stable over $\mathbb{D}_\kappa$, then it is exponentially stable over $\mathbb{D}_{\kappa_1}$ for any $\kappa_1 > \kappa$.*

The following corollary reveals the monotonicity of Theorem 1 with respect to the parameter d whose proof is an analogy of that of Chesi et al. (2012, Theorem 5) and hence is omitted.

Corollary 2: *Suppose that there exist a positive integer d, matrices $0 < P_i \in \mathbb{S}^{\vartheta(n,d)}$ and $L(\boldsymbol{v}_i), L(\boldsymbol{v}_{ij}) \in \mathscr{L}_{n,d}$ with $\boldsymbol{v}_i, \boldsymbol{v}_{ij} \in \mathbb{R}^{\omega(n,d)}$ satisfying (3.1) and (3.2). Then for any $q \in \mathbb{N}$, there exist matrices $0 < P_i \in \mathbb{S}^{\vartheta(n,qd)}$ and $L(\boldsymbol{v}_i), L(\boldsymbol{v}_{ij}) \in \mathscr{L}_{n,qd}$ with $\boldsymbol{v}_i, \boldsymbol{v}_{ij} \in \mathbb{R}^{\omega(n,qd)}$, such that (3.1) and (3.2) are satisfied.*

Now let $\kappa = 1$. Following a similar proof of Theorem 1, we have the following important result which provides a necessary and sufficient condition to check the stability property of system (2.1) under arbitrary switching signal.

Corollary 3: *The fact that system (2.1) is exponentially stable for arbitrary switching signal is equivalent to any one of the following statements:*

(1) *System (2.1) admits an SHLF of degree d for some $d \in \mathbb{N}$.*
(2) *There exist a positive integer d, matrices $0 < P_i \in \mathbb{S}^{\vartheta(n,d)}$, and $L(\boldsymbol{v}_{ij}) \in \mathscr{L}_{n,d}$ with $\boldsymbol{v}_{ij} \in \mathbb{R}^{\omega(n,d)}$, such that*

$$\mathcal{A}_{i,d}^{\mathrm{T}} P_j \mathcal{A}_{i,d} - P_i + L(\boldsymbol{v}_{ij}) < 0, \quad \forall i, j \in \underline{m}. \tag{3.7}$$

Remark 1: Geromel and Colaneri (2006, Theorem 1) is a special case of Theorem 1 in this paper with $d = 1$, and Daafouz et al. (2002, Theorem 2) is a special case of Corollary 1 in this paper with $d = 1$. Recently, by meas of the concept of contractive set, Dehghan and Ong proposed a necessary and sufficient asymptotic stability condition for (2.1) with a certain constraint (Dehghan & Ong, 2012); however, it is difficult to apply the method to systems with delays.

3.2 Some extensions

This subsection extends the results in the previous subsection to switched systems with delays. Consider first the following switched system with constant delays:

$$\boldsymbol{x}(k+1) = A_{\sigma(k)}\boldsymbol{x}(k) + B_{\sigma(k)}\boldsymbol{x}\left(k - \tau_{\sigma(k)}\right), k \in \mathbb{N}_0, \tag{3.8}$$

where $\sigma : \mathbb{N}_0 \to \underline{m}$, $\tau_i \in \mathbb{N}$. Define $\tau = \max_{i \in \underline{m}}\{\tau_i\}$, $\boldsymbol{y}(k) = [\boldsymbol{x}^{\mathrm{T}}(k), \ldots, \boldsymbol{x}^{\mathrm{T}}(k-\tau)]^{\mathrm{T}}$, and $\jmath = n(\tau+1)$, then system (3.8) is equivalent to $\boldsymbol{y}(k+1) = \mathbf{A}_{\sigma(k)}\boldsymbol{y}(k)$ with

$$\mathbf{A}_i = \begin{bmatrix} A_i & \mathbf{0}_{n\times n(\tau_i-1)} & B_i & \mathbf{0}_{n\times n(\tau-\tau_i)} \\ & I_{n\tau} & & \mathbf{0}_{n\tau\times n} \end{bmatrix}.$$

Let $\mathcal{A}_{i,d} = \left(K_d^{\mathrm{T}}K_d\right)^{-1}K_d^{\mathrm{T}}\mathbf{A}_i^{\otimes d}K_d$. Applying Theorem 1 and Corollary 3, we have the following.

Theorem 2: *System (3.8) is exponentially stable over $\mathbb{D}_\kappa$ for $\kappa \geq 2$ if and only if there exist a positive integer d, matrices $0 < P_i \in \mathbb{S}^{\vartheta(\jmath,d)}$, and $L(\boldsymbol{v}_i), L(\boldsymbol{v}_{ij}) \in \mathscr{L}_{\jmath,d}$ with $\boldsymbol{v}_i, \boldsymbol{v}_{ij} \in \mathbb{R}^{\omega(\jmath,d)}$ satisfying (3.1) and (3.2). Particularly, system (3.8) is exponentially stable for arbitrary switching signal if and only if there exists a positive integer d, matrices $0 < P_i \in \mathbb{S}^{\vartheta(\jmath,d)}$, and $L(\boldsymbol{v}_{ij}) \in \mathscr{L}_{\jmath,d}$ with $\boldsymbol{v}_{ij} \in \mathbb{R}^{\omega(\jmath,d)}$ satisfying (3.7).*

Finally, consider

$$\boldsymbol{x}(k+1) = A_{\sigma(k)}\boldsymbol{x}(k) + B_{\sigma(k)}\boldsymbol{x}(k - \tau_{\sigma(k)}(k)), k \in \mathbb{N}_0, \tag{3.9}$$

where $\sigma : \mathbb{N}_0 \to \underline{m}$, $\tau_i(k)$ satisfies

$$0 \le \tau_{i1} \le \tau_i(k) \le \tau_{i2}, \quad \forall i \in \underline{m} \tag{3.10}$$

$\tau = \max_{i \in \underline{m}} \{\tau_{i2}\}$. Let $\boldsymbol{y}(k) = \left[\boldsymbol{x}^{\mathrm{T}}(k), \ldots, \boldsymbol{x}^{\mathrm{T}}(k-\tau)\right]^{\mathrm{T}}$ and $\tau_i = \tau_{i2} - \tau_{i1} + 1$. For each $i \in \underline{m}$, define

$$A_{i1} = \begin{bmatrix} A_i + B_i & \mathbf{0}_{n \times n\tau} \\ I_{n\tau} & \mathbf{0}_{n\tau \times n} \end{bmatrix},$$
$$A_{ij} = \begin{bmatrix} A_i & \mathbf{0}_{n \times n(j-2)} & B_i & \mathbf{0}_{n \times n(\tau+1-j)} \\ & I_{n\tau} & & \mathbf{0}_{n\tau \times n} \end{bmatrix},$$
$$j \in \{2, \ldots, \tau_i\}$$

for $\tau_{i1} = 0$ and

$$A_{ij} = \begin{bmatrix} A_i & \mathbf{0}_{n \times n(\tau_{i1}+j-2)} & B_i & \mathbf{0}_{n \times n(\tau-\tau_{i1}+1-j)} \\ & I_{n\tau} & & \mathbf{0}_{n\tau \times n} \end{bmatrix},$$
$$j \in \{1, \ldots, \tau_i\}$$

otherwise. Define $\mathcal{A}_{ij,d} = \left(K_d^{\mathrm{T}} K_d\right)^{-1} K_d^{\mathrm{T}} A_{ij}^{\otimes d} K_d$. Clearly, $A_{ij} \in \mathbb{R}^{\jmath \times \jmath}$ with $\jmath = n(\tau+1)$. Let $j(k) = \tau_{\sigma(k)}(k) - \tau_{\sigma(k)1} + 1$, and it is not difficult to verify that (3.9) can be recast into the following system:

$$\boldsymbol{y}(k+1) = A_{\sigma(k)j(k)} \boldsymbol{y}(k),\ k \in \mathbb{N}_0.$$

The follow theorem immediately follows from Theorem 1.

Theorem 3: *System (3.9) is exponentially stable for all delays satisfying (3.10) under arbitrary switching signal if and only if there exist a positive integer d, matrices $P_{ij_i} \in \mathbb{S}^{\vartheta(\jmath,d)}$, and $L\left(\boldsymbol{v}_{ij_i,lj_l}\right) \in \mathscr{L}_{\jmath,d}$ with $\boldsymbol{v}_{ij_i,lj_l} \in \mathbb{R}^{\omega(\jmath,d)}$, such that*

$$P_{ij_i} > 0, \quad \forall i \in \underline{m},\ j_i \in \{1, \ldots, \tau_i\},$$
$$\mathcal{A}_{ij_i,d}^{\mathrm{T}} P_{lj_l} \mathcal{A}_{ij_i,d} - P_{ij_i} + L\left(\boldsymbol{v}_{ij_i,lj_l}\right) < 0,$$
$$\forall i, l \in \underline{m},\ \forall j_i \in \{1, \ldots, \tau_i\},\ j_l \in \{1, \ldots, \tau_l\}.$$

Remark 2: Taking $m = 1$, system (3.9) is of the following form:

$$\boldsymbol{x}(k+1) = A\boldsymbol{x}(k) + B\boldsymbol{x}(k - \tau(k)), \quad k \in \mathbb{N}_0, \tag{3.11}$$

where $\tau(k)$ satisfies $0 \le \tau_1 \le \tau(k) \le \tau_2$. Clearly, by virtue of Theorem 3, the necessary and sufficient condition guaranteeing (3.11) is exponentially stable under arbitrary switching signal can be derived immediately.

Table 1. Stability regions computed by different methods.

a	Corresponding stability interval of b: (Daafouz et al., 2002, Theorem 1)	Corresponding stability interval of b: Corollary 3 with $d = 2$
0.602	$[-0.45, -0.04]$	$[-0.53, -0.04]$
0.604	$[-0.45, -0.05]$	$[-0.52, -0.05]$
0.64	$[-0.4, -0.14]$	$[-0.45, -0.14]$
0.676	$\emptyset$	$[-0.31, -0.24]$
0.68	$\emptyset$	$[-0.3, -0.25]$

4. Example

This section presents an example to illustrate previous theoretical results.

Example 1: Consider the following system:

$$\boldsymbol{x}(k+1) = A_{\sigma(t)} \boldsymbol{x}(k), \quad k \in \mathbb{N}_0, \tag{4.1}$$

where $\boldsymbol{x}(k) = [x_1(k), x_2(k)]^{\mathrm{T}} \in \mathbb{R}^2$, $\sigma : \mathbb{N}_0 \to \{1, 2\}$, and

$$A_1 = \begin{bmatrix} a & 0.8 \\ 0.2 & -0.9 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -0.8 & -0.1 \\ -0.7 & b \end{bmatrix}$$

with a and b being parameters. Clearly, the chosen a, b should make both A_1 and A_2 Schur matrices, since the main results presented previously require implicitly both A_1 and A_2 to be Schur matrices.

First consider system (4.1) under arbitrary switching signal. Applying Daafouz et al. (2002, Theorem 1) and Corollary 3 in this paper, we obtain the comparison result in Table 1. Note that Daafouz et al. (2002, Theorem 1) is in fact a special case of SHLF of degree one. It can be seen from Table 1 that, compared with Daafouz et al. (2002, Theorem 1), Corollary 3 in this paper can lead to larger stability regions.

Then, consider (4.1) with guaranteed dwell-time. Applying Geromel and Colaneri (2006, Theorem 1) (a special case of Theorem 1 in this paper with $d = 1$) and Theorem 1 in this paper yields Table 2 , which shows that Theorem 1, compared with Geromel and Colaneri (2006, Theorem 1), can result in smaller lower bound of dwell-time. Note that both Daafouz et al. (2002, Theorem 1) and Geromel and Colaneri (2006, Theorem 1) can only

Table 2. Dwell times computed by different methods.

Parameters: a	Parameters: b	Dwell time: (Geromel & Colaneri, 2006, Theorem 1)	Dwell time: Theorem 1 ($d = 2$)	Dwell time: Theorem 1 ($d = 3$)
0.68	-0.3	2	1	1
0.818	-0.6	3	3	2
0.8	-0.6	3	2	2
0.8	-0.64	4	3	2

check the asymptotic stability, however, Theorem 1 and Corollary 3 in this paper imply the exponential stability of system (4.1).

5. Conclusions

We have proposed an SHLF method for discrete-time switched systems, by which a series of nonconservative exponential stability conditions has been presented for switched systems with guaranteed dwell-time or with arbitrary switching signals. These results can be used to compute the lower bound of dwell-time for switched systems, or to compute the upper bound of delays for delayed (switched) systems. It should be pointed out that the computational burden is heavy when the number of subsystems or the difference between upper and lower bounds of delay is large, or when the dimension of system is high. So, future work will focus on how to reduce the computational effort.

Acknowledgements

We would like to thank the anonymous reviewers for their helpful suggestions. Furthermore, the first author would like to thank Dr. Chunming Wang, Professor at Department of Mathematics, University of Southern California, for his instructive advice.

Disclosure statement

No potential conflict of interest was reported by the authors.

Funding

This work was partially supported by National Nature Science Foundation [61273007], Sichuan Youth Science and Technology Fund [2011JQ0011], the Key Project of Chinese Ministry of Education [212203], SWUN Construction Projects for Graduate Degree Programs [2015XWD-S0805], and Innovative Research Team of the Education Department of Sichuan Province [15TD0050].

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Appendix 1. Proof of Lemma 3

Basically, the proof of Lemma 3 is similar to that of Blanchini and Colaneri (2010, Theorem II.1). The following definition and lemmas are required.

Definition 5 (Blanchini & Miani, 2008)**:** A set $\mathscr{S} \in \mathbb{R}^n$ is called C-set if it is convex and compact and includes the origin as an interior point. A C-set $\mathscr{S}$ is 0-symmetric if $\boldsymbol{x} \in \mathscr{S}$ implies that $-\boldsymbol{x} \in \mathscr{S}$. For a given C-set $\mathscr{S}$, its Minkowskii function is $\psi_{\mathscr{S}}(\boldsymbol{x}) = \inf\{\lambda \geq 0 : \boldsymbol{x} \in \lambda\mathscr{S}\}$, which in fact is a norm. If a C-set $\mathscr{S}$ is 0-symmetric, then the unit ball of norm $\psi_{\mathscr{S}}$ is $\mathscr{S}$ itself, that is, $\mathscr{S} = \{\boldsymbol{x} \in \mathbb{R}^n : \psi_{\mathscr{S}}(\boldsymbol{x}) \leq 1\} \triangleq \mathcal{N}[\psi, 1]$. A bounded polyhedral set is called a polytope. A norm $\psi_{\mathscr{S}}(\boldsymbol{x})$ is polyhedral if its unit ball $\mathcal{N}[\psi, 1]$ is a 0-symmetric polytope which can be represented either in its plane representation $\mathscr{S} = \{\boldsymbol{x} \in \mathbb{R}^n : \|F\boldsymbol{x}\|_\infty \leq 1\}$ with F a full column rank matrix or in its vertices representation $\mathscr{S} = \{\boldsymbol{x} = X\boldsymbol{p} : \|\boldsymbol{p}\|_1 \leq 1\}$ with X full row rank and $\boldsymbol{p}$ a vector of appropriate dimension. The Minkowskii function of $\mathscr{S}$ can be denoted by means of F and X in the form $\psi_{\mathscr{S}}(\boldsymbol{x}) = \|F\boldsymbol{x}\|_\infty = \inf\{\|\boldsymbol{p}\|_1 : \boldsymbol{x} = X\boldsymbol{p}\}$.

Lemma 5: *For two given polyhedral C-sets $\mathscr{S}_1$ and $\mathscr{S}_2$ with vertex representation matrices X_1 and X_2, $\mathscr{S}_1 \subseteq \mathscr{S}_2$ if and only if there exists a matrix P with $\|P\|_1 \leq 1$, such that $X_1 = X_2P$.*

Indeed, let $X_1 = [\boldsymbol{x}_1^{(1)} \dots \boldsymbol{x}_s^{(1)}]$ and $X_2 = [\boldsymbol{x}_1^{(2)} \dots \boldsymbol{x}_q^{(2)}]$, where $\boldsymbol{x}_j^{(1)} \in \mathbb{R}^n (j \in \underline{s})$ and $\boldsymbol{x}_j^{(2)} \in \mathbb{R}^n (j \in \underline{q})$. Since $\mathscr{S}_1 \subseteq \mathscr{S}_2$ is equivalent to the fact that there exists a vector $\boldsymbol{p}_j = [p_{j1} \dots p_{jq}]^{\mathrm{T}}$ for each vector $\boldsymbol{x}_j^{(1)}$, such that $\boldsymbol{x}_j^{(1)} = \sum_{i=1}^{q} p_{ji}\boldsymbol{x}_i^{(2)}$ and $\sum_{i=1}^{s} p_{ji} \leq 1$, which is actually another expression of $X_1 = X_2P$ with $P = [\boldsymbol{p}_1 \dots \boldsymbol{p}_s]$ and $\|P\|_1 \leq 1$.

Lemma 6: *System (2.1) is asymptotically stable over $\mathbb{D}_\kappa$ if and only if there exists a scalar $\epsilon > 0$*

$$\boldsymbol{x}(k+1) = \left(A_{\sigma(k)} + aI\right)\boldsymbol{x}(k) \tag{A.1}$$

is asymptotically stable over $\mathbb{D}_\kappa$ for any a satisfying $-\epsilon \leq a \leq \epsilon$.

Proof: It suffices to show the necessity. By Lemma 1, (2.1) is asymptotically stable over $\mathbb{D}_\kappa$ means it is also exponentially stable over $\mathbb{D}_\kappa$. Therefore, there exist two scalars $\alpha > 0$ and $\gamma > 1$ such that the solution $\boldsymbol{x}(k)$ to (2.1) satisfying

$$\|\boldsymbol{x}(k; \boldsymbol{x}_0, \sigma)\| = \|\boldsymbol{x}_0\| \left\| \boldsymbol{x}\left(k; \frac{\boldsymbol{x}_0}{\|\boldsymbol{x}_0\|}, \sigma\right) \right\| \leq \alpha\gamma^{-k} \|\boldsymbol{x}_0\|, \quad \forall k \in \mathbb{N},$$

which means that there is some $T \in \mathbb{N}$ with the property that

$$\sup_{\|\boldsymbol{x}_0\|=1} \|\boldsymbol{x}(k; \boldsymbol{x}_0, \sigma)\| \leq \frac{1}{2}, \quad \forall k \geq T.$$

The solution to system (A.1) is $\boldsymbol{x}_a(k; \boldsymbol{x}_0, \sigma) = \mathcal{A}(k, \sigma, a)\boldsymbol{x}_0$ with $\mathcal{A}(k, \sigma, a) = (A_{\sigma(k-1)} + aI)\dots(A_{\sigma(0)} + aI)$. Clearly, for given k and σ, $\mathcal{A}(k, \sigma, a)$ is continuous in a. Therefore, there exists a scalar $\epsilon > 0$ sufficiently small, such that

$$\sup_{\|\boldsymbol{x}_0\|=1} \|\boldsymbol{x}_a(T; \boldsymbol{x}_0, \sigma)\| \leq \frac{3}{4}, \quad \forall a \in [-\epsilon, \epsilon]. \tag{A.2}$$

Furthermore, there necessarily exists a scalar $\alpha_1 \geq 1$ satisfying

$$\sup_{\|\boldsymbol{x}_0\|=1, 0<k<T} \|\boldsymbol{x}_a(k; \boldsymbol{x}_0, \sigma)\| \leq \alpha_1, \quad \forall a \in [-\epsilon, \epsilon]. \tag{A.3}$$

By (A.2), (A.3), and linearity of (A.1),

$$\begin{aligned} \|\boldsymbol{x}_a(k; \boldsymbol{x}_0, \sigma)\| &\leq \alpha_1 \frac{3^{i-1}}{4^{i-1}} \|\boldsymbol{x}_0\|, \quad (i-1)T < k < iT, \\ \|\boldsymbol{x}_a(iT; \boldsymbol{x}_0, \sigma)\| &\leq \frac{3^i}{4^i} \|\boldsymbol{x}_0\| \end{aligned} \tag{A.4}$$

holds for $i = 1$. It follows from (A.2) and (A.3) that

$$\begin{aligned} \|\boldsymbol{x}_a(k; \boldsymbol{x}_0, \sigma)\| &= \|\boldsymbol{x}_a(iT)\| \left\| \boldsymbol{x}_a\left(k - iT; \frac{\boldsymbol{x}(iT)}{\|\boldsymbol{x}(iT)\|}, \sigma\right) \right\| \\ &\leq \frac{3^i}{4^i}\alpha_1 \|\boldsymbol{x}_0\|, \quad iT < k < (i+1)T, \\ \|\boldsymbol{x}_a((i+1)T; \boldsymbol{x}_0, \sigma)\| &= \|\boldsymbol{x}_a(iT)\| \left\| \boldsymbol{x}_a\left(T; \frac{\boldsymbol{x}(iT)}{\|\boldsymbol{x}(iT)\|}, \sigma\right) \right\| \\ &\leq \frac{3^{i+1}}{4^{i+1}} \|\boldsymbol{x}_0\|, \end{aligned}$$

which implies that (A.4) holds for any $i \in \mathbb{N}$. Clearly, (A.4) indicates that $\|\boldsymbol{x}_a(k; \boldsymbol{x}_0, \sigma)\| \leq \alpha_1 \frac{3^{i-1}}{4^{i-1}} \|\boldsymbol{x}_0\|$, $(i-1)T < k \leq iT, \forall i \in \mathbb{N}$. □

Similar to Blanchini and Colaneri (2010, Lemma III.2), the following lemma naturally holds.

Lemma 7: *System* (A.1) *is asymptotically stable for any* $\sigma \in \mathbb{D}_\kappa$ *if and only if the following system*

$$\boldsymbol{x}_{j+1} = \Phi\left(\imath, \sigma\left(k_j\right)\right)\boldsymbol{x}_j \tag{A.5}$$

is asymptotically stable for any $\sigma \in \mathbb{D}_\kappa$, *where* $\boldsymbol{x}_j = \boldsymbol{x}(k_j)$ *and* $\Phi(\imath, i) \in \{(A_i + aI)^{\kappa+\imath} : i \in \underline{m}, \imath \in \mathbb{N}_0\}$.

Lemma 8 (Blanchini & Colaneri, 2010, Lemma III.3)**:** *System* (A.5) *is asymptotically stable if and only if there exists a polyhedral norm* $\psi_{\mathscr{S}}(\boldsymbol{x})$ *which is a Lyapunov function, namely such that* $\psi(\boldsymbol{x}_{j+1}) \le \lambda\psi(\boldsymbol{x}_j)$ *holds for some positive scalar* $\lambda < 1$.

Remark 3: Let $\sigma(k) = i$, $\forall k \in \{k_0, \ldots, k_1 - 1\}$. Assume that there exists a polyhedral norm ψ with unit ball $\mathcal{N}[\psi, 1]$ as in Lemma 8. Just as in Blanchini and Colaneri (2010, Lemma III.4), this assumption is equivalent to the following fact: For all $\boldsymbol{x}_0 \in \mathscr{X} = \mathcal{N}[\psi, 1]$, the state of (A.1) is in $\mathscr{P}_i$, the largest invariant subset of $\mathscr{X}$ for $\boldsymbol{x}(k+1) = (A_i + aI)\boldsymbol{x}(k)$. Moreover, for $a = 0$ and $\boldsymbol{x}_0 \in \mathscr{X}$, $\boldsymbol{x}(k)$ reaches a polytope $\mathscr{X}_i \subseteq \lambda\mathscr{X}$ with $0 < \lambda < 1$, where $\mathscr{X}_i$ is contractive for system $\boldsymbol{x}(k+1) = A_i\boldsymbol{x}(k)$.

Proof of Lemma 3: The sufficiency part of this lemma is almost the same as that of Blanchini and Colaneri (2010, Theorem II.1), so we only provide the necessity part.

Suppose that system (2.1) is asymptotically stable over $\mathbb{D}_\kappa$, which, by Lemma 7, is equivalent to saying that so is (A.5). According to Lemma 8, there exists a polyhedral Lyapunov function $\psi(\boldsymbol{x})$ with unit ball $\mathcal{N}[\psi, 1]$. The condition $\boldsymbol{x}(0) \in \mathscr{X} = \mathcal{N}[\psi, 1]$ means that $\boldsymbol{x}(\kappa) \in \mathscr{X}_i \subseteq \lambda\mathscr{X}$ with $0 < \lambda < 1$ and $\mathscr{X}_i$ being invariant set for the ith subsystem, which, by Lemma 5, implies that there exists a matrix $P_i(\kappa)$ with $\|P_i(\kappa)\|_1 \le 1$, such that

$$A_i^\kappa X = X_i P_i(\kappa), \quad \forall i \in \underline{m}, \tag{A.6}$$

where X and X_i are the vertex describing matrices of the 0-symmetric polytopes $\mathscr{X}$ and $\mathscr{X}_i$. On the other hand, by Lemma 5, the condition $\mathscr{X}_i \subseteq \lambda\mathscr{X}$ implies the existence of matrices $\tilde{P}_i$, such that

$$X_i = X\tilde{P}_i, \quad \|\tilde{P}_i\|_1 \le 1, \ \forall i \in \underline{m}. \tag{A.7}$$

By Blanchini and Miani (2003, Lemmas 2.8, 2.9), $\mathscr{X}_i$ is invariant for the ith dynamics is equivalent to the existence of $\|P_i\|_1 < 1$ satisfying $A_iX_i = X_iP_i$. By means of (A.6) and (A.7), $A_i^\kappa X_j = A_i^\kappa X\tilde{P}_j = X_iP_i(\kappa)\tilde{P}_j = X_iR_{ij}$, where $R_{ij} = P_i(\kappa)\tilde{P}_j$ and $\|R_{ij}\|_1 = \|P_i(\kappa)\tilde{P}_j\|_1 \le \|P_i(\kappa)\|_1\|\tilde{P}_j\|_1 \le 1$. $\square$