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Closed-loop stability analysis of a gantry crane with heavy chain and payload

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ABSTRACT
In this paper, we analyse a systematically designed and easily tunable backstepping-based boundary control concept for a gantry crane with heavy chain and payload. The corresponding closed-loop system is formulated as an abstract evolution equation in an appropriate Hilbert space. Non-restrictive conditions for the controller coefficients are derived, under which the solutions are described by a C_0-semi-group of contractions, and are asymptotically stable. Moreover, by applying Huang's theorem we can finally even show that under these conditions the controller renders the closed-loop system exponentially stable.

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1. Introduction

This paper deals with the rigorous stability analysis of a control concept presented by Thull, Wild, and Kugi (2006) applied to the infinite-dimensional model of a gantry crane with heavy chain and payload. The model consists of a cart of mass m_c, which moves horizontally along a rail, a heavy chain of length L with mass per length ρ, attached to the cart[1], and a payload of point mass m_p at its end. The chain is assumed to be inextensible and perfectly flexible. For the derivation of the equations of motion, it is further assumed that no friction occurs in the system. The force F acting on the cart serves as the control input. The situation is sketched in Figure 1 on the left-hand side.

Let $w(t, x)$ denote the horizontal chain position. Then, under the assumption that the chain slopes $\partial_x w(t, x)$ remain sufficiently small for all $t > 0$, the dynamics of the system are described by the following wave equation with dissipative higher order boundary conditions (see, e.g. Mifdal, 1997b; Petit & Rouchon, 2001; Thull et al., 2006)

$$\rho\partial_t^2 w(t, x) - \partial_x(P(x)\partial_x w(t, x)) = 0, \tag{1.1a}$$

$$m_p\partial_t^2 w(t, L) + P(L)\partial_x w(t, L) = 0, \tag{1.1b}$$

$$m_c\partial_t^2 w(t, 0) - P(0)\partial_x w(t, 0) = F(t). \tag{1.1c}$$

The function $P(x)$ represents the tension in the chain at height x, given by $P(x) = g[\rho(L - x) + m_p]$, where g denotes the gravitational acceleration. Note that $P \geq g\,m_p > 0$ holds uniformly on $[0, L]$. In the following, it is only required that $P \in H^2(0, L)$ and that $P(x) \geq P^0 > 0$ holds uniformly on $[0, L]$ for some constant P^0. Thus, the density of the chain does not need to be constant, as it was the case for instance in Thull et al. (2006). Moreover, the following notation $v := \partial_t w$ will be used in the sequel.

For the system (1.1), many different control laws can be found in the literature (see, e.g. Conrad & Mifdal, 1998; Coron & d'Andrea Novel, 1998; Mifdal, 1997b; Thull et al., 2006), with $v = w_t$. Basically, the common structure of these controllers looks like

$$\begin{aligned} F(t) = {} & \vartheta_1 v(t, 0) + \vartheta_2\partial_x v(t, 0) + \vartheta_3 w(t, 0) \\ & + \vartheta_4\partial_x w(t, 0), \end{aligned} \tag{1.2}$$

but they differ in the fact which parameters ϑ_j, $j = 1, \dots, 4$ are equal to zero, which conditions have to be fulfilled to render the closed-loop system asymptotically (or even exponentially) stable, and how the controller parameters can be systematically tuned.

Thus, for instance Conrad and Mifdal (1998) show that a (simple) passive controller with $\vartheta_2 = \vartheta_4 = 0$ in Equation (1.2) already ensures asymptotic stability of the closed-loop system. While this is a nice result from a theoretical point of view, this controller is not able to damp vibrations of the chain in case of stick-slip effects in the cart, which are always present in the real experiment. To make this clear, let us assume that the cart is in a sticking position, which also entails that $v(t, 0) = 0$ and $w(t, 0) = c$, with a constant c, then the cart will not move as long as the absolute value of the sum of the internal force in the

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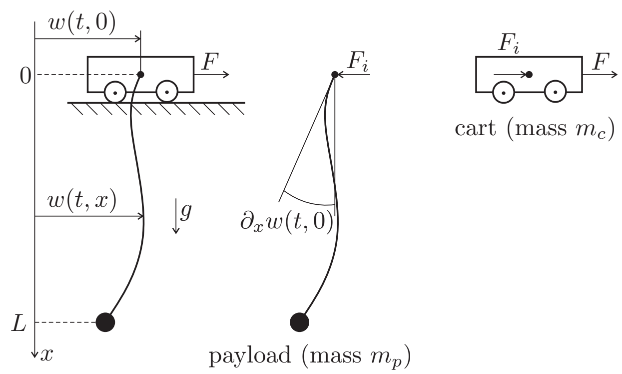

Figure 1. Gantry crane with heavy chain and payload: schematics (left) and representation of the internal force F_i (right).

pivot bearing carrying the chains $F_i = P(0)\partial_x w(t, 0)$ and the input force $F(t)$, see Equation (1.1c), is smaller than the sticking friction. As a consequence the chain keeps on vibrating, but the cart stands still and the control law (1.2) with $\vartheta_2 = \vartheta_4 = 0$ only produces a constant input force $F(t) = \vartheta_3 c$.

In Mifdal (1997b), the exponential stability of the closed-loop system is proven for the control law (1.2) with $\vartheta_4 = 0$ under certain further conditions by using an energy multiplier approach. The proof of stability is performed in a rigorous and elegant way, however, there is no systematic design of the control law and it is not clear how to specifically tune the controller parameters. In contrast, the control law (1.2) used in this paper overcomes these deficiencies, but then the controller parameters for tuning appear in all parameters $\vartheta_j, j = 1, \dots, 4$, which will be shown subsequently, and ϑ_4 has to be unequal to zero. For the authors, it is not obvious and does not seem to be straightforward to extend the stability proof of Mifdal (1997b) to the case $\vartheta_4 \neq 0$ under the same non-restrictive conditions on the controller parameters as presented in this paper.

For a controller with the same structure as in Equation (1.2) but, compared to the controller considered in this paper, with totally different conditions on the parameters $\vartheta_j, j = 1, \dots, 4$, the asymptotic and exponential stability of the closed-loop system is shown by Coron and d'Andrea Novel (1998). In this paper, the dynamics of the payload (1.1b) was neglected. Moreover, the controller was designed based on a backstepping approach; however, the authors mention in their paper that this boundary feedback law ensures asymptotic stability, but it is not clear if it produces exponential stability. Therefore, they modify the backstepping controller and based on this they are able to show exponential stability of the closed-loop system.

Inspired by Coron and d'Andrea Novel (1998), a controller was systematically designed and experimentally validated for the system (1.1) of Thull et al. (2006). In order to reveal the connection between the controller parameters for tuning and the parameters $\vartheta_j, j = 1, \dots, 4$ in Equation (1.2), the control design will be shortly revisited, see Thull et al. (2006) for more details. In contrast to the (simple) passive controller presented by Conrad and Mifdal (1998), the (damping) controller design by Thull et al. (2006) is based on the idea to specifically influence the energy flow between the cart and the chain, which is represented by the collocated variables cart velocity $\partial_t w(t, 0) = v(t, 0)$ and internal force in the pivot bearing carrying the chains $F_i = P(0)\partial_x w(t, 0)$, thus forming an energy port, see right-hand side of Figure 1. The sum of the potential energy of the chain and the kinetic energy of the chain and the payload according to Equation (1.1) reads as

$$\begin{aligned}\bar{H} = \frac{1}{2}\int_0^L P(x)\,(\partial_x w\,(t,x))^2 + \rho v^2\,(t,x)\,\mathrm{d}x \\ + \frac{1}{2} m_p v^2\,(t,L)\,. \end{aligned} \tag{1.3}$$

The change of $\bar{H}$ along a solution of Equation (1.1) yields $\frac{\mathrm{d}}{\mathrm{d}t}\bar{H} = -v\,(t,0)\,F_i$. Thus, if $v(t, 0)$ were the (virtual) control input, the control law $v(t, 0) := \chi_1 F_i$, with the controller parameter $\chi_1 > 0$, would render the closed-loop system passive. However, this would ensure a good damping of the chain vibrations, but the cart position $w(t, 0)$ remains unconsidered within this approach. Therefore, the energy functional $\bar{H}$ from Equation (1.3) is extended by the potential energy of a virtual linear[2] spring attached to the cart in the form[3]

$$\bar{V} = \chi_1 \bar{H} + \frac{\chi_2}{2} w^2\,(t,0)\,, \tag{1.4}$$

with the controller parameters $\chi_1, \chi_2 > 0$. The change of $\bar{V}$ along a solution of Equation (1.1)

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{V} = (\chi_2 w\,(t,0) - \chi_1 F_i)\,v\,(t,0) \tag{1.5}$$

immediately shows that the control law $v(t, 0) := -(\chi_2 w(t, 0) - \chi_1 F_i)$ for the virtual control input $v(t, 0)$ makes the closed-loop system passive again. The controller parameters χ_1 and χ_2 facilitate a simple tuning of the closed-loop behaviour, because a larger χ_1 (wrt χ_2) brings along a higher damping of the chain vibrations at the cost of larger deviations of the cart position $w(t, 0)$ from the equilibrium and for a larger χ_2 (wrt χ_1) the control of the cart position is becoming more important. Since $v(t, 0)$ is not the real control input, a simple backstepping approach, see, e.g. Krstić, Kanellakopoulos, and Kokotović (1995), is applied to Equation (1.1c). In a nutshell, the functional $\bar{V}$ from Equation (1.4) is extended in the form:

$$V = \bar{V} + \frac{1}{2}\left(v\left(t,0\right) + \chi_2 w\left(t,0\right) - \chi_1 F_i\right)^2 \quad (1.6)$$

and the control law is designed in such a way that

$$\frac{\mathrm{d}}{\mathrm{d}t}V = -v^2\left(t,0\right) - \chi_3\left(v\left(t,0\right) + \chi_2 w\left(t,0\right) - \chi_1 F_i\right)^2, \quad (1.7)$$

with the controller parameter $\chi_3 > 0$. The control law finally takes the form (see also (1.2)):

$$\begin{aligned} F\left(t\right) = &\underbrace{-\left(\chi_3 + \chi_2 + 1\right) m_c}_{\vartheta_1} v\left(t,0\right) \\ &+ \underbrace{\chi_1 P(0) m_c}_{\vartheta_2} \partial_x v\left(t,0\right) + \underbrace{\left(-\chi_3\chi_2 m_c\right)}_{\vartheta_3} w\left(t,0\right) \\ &+ \underbrace{\left(\chi_3\chi_1 m_c - 1\right) P(0)}_{\vartheta_4} \partial_x w\left(t,0\right). \end{aligned} \quad (1.8)$$

The third controller parameter χ_3 weights the deviation of the virtual control input $v(t, 0)$ from its desired time evolution $\chi_2 w(t, 0) - \chi_1 F_i$. Note that in addition to the feedback control law (1.8) the control concept of Thull et al. (2006) consists of a flatness-based feedforward controller as presented by Petit and Rouchon (2001). Due to the linearity of the system (1.1), the trajectory error dynamics are identical and thus also the stability proof remains the same.

In Thull et al. (2006), energy dissipation of the closed-loop system was shown for Equation (1.8). Thull, Wild, and Kugi (2005) attempted to show asymptotic stability by using LaSalle's invariance principle (see, Luo, Guo, & Morgül, 1999). This is common practice in the context of (hyperbolic) control systems, see, e.g. Miletić, Stürzer, and Arnold (2015), Miletić, Stürzer, Arnold, and Kugi (2016), Chentouf and Couchouron (1999), Conrad and Morgül (1998), Coron and d'Andrea Novel (1998), Kugi and Thull (2005) and Morgül (2001) for the control of an Euler–Bernoulli beam, and d'Andréa Novel, Boustany, and Conrad (1992), Morgül, Rao, and Conrad (1994) and d'Andréa Novel and Coron (2000) for the control of hanging cables. With the (energy) inner product chosen by Thull et al. (2005), Thull et al. (2006) the proof of the closed-loop stability did not work out, which was also correctly pointed out by Grabowski (2009). The backstepping approach presented by Thull et al. (2006) is quite intuitive from a control point of view, but it brings along the drawback that the control law depends on $\partial_x v(t, 0)$, see Equation (1.8), which makes it impossible to analyse the closed-loop system in the space $H^1 \times L^2$. Therefore, an appropriate Hilbert space $\mathcal{H}$ and a convenient inner product, see Equation (2.6), are introduced in this paper which allow the application of the Lumer–Phillips theorem and the rigorous proof of the closed-loop stability of Equation (1.1) with Equation (1.8). Due to the change of the abstract model setting, the strategy to prove asymptotic stability does not make use of LaSalle's invariance principle. Instead, techniques from spectral analysis are used, see Luo et al. (1999). In order to show exponential stability, Huang's theorem, see, e.g. Huang (1985), is employed.

In order to prove stability of the closed-loop system, in a first step (1.1) with Equation (1.8) is rewritten as an abstract evolution equation $\dot{y} = Ay$ in an appropriate Hilbert space $\mathcal{H}$, where the generator A is a linear operator, see, e.g. d'Andréa Novel and Coron (2000), Grabowski (2008), Grabowski (2009), Morgül et al. (1994) and Luo et al. (1999) for the formulation of similar systems describing hanging cables. We start by showing that A generates a C_0-semi-group of contractions, by using the Lumer–Phillips theorem. To this end an inner product, equivalent to the natural inner product in $\mathcal{H}$, is used. Then, we show that the inverse A^{-1} exists and is compact. This implies that the spectrum $\sigma(A)$ consists entirely of eigenvalues. Since A generates a C_0-semi-group of uniformly bounded operators, the Hille–Yosida theorem implies that $\sigma(A) \subseteq \{\zeta \in \mathbb{C} : \mathrm{Re}\,\zeta \leq 0\}$. We then show that $\mathrm{i}\mathbb{R}$ lies in $\rho(A)$, by demonstrating that for all $\lambda \in \mathbb{R}$ the eigenvalue equation $Ay = \mathrm{i}\lambda y$ only has the trivial solution. According to Luo et al. (1999, Theorem 3.26), this proves the asymptotic stability of the system. Finally, we show uniform boundedness of the resolvent $(\mathrm{i}\lambda - A)^{-1}$ for $\lambda \in \mathbb{R}$. Huang's Theorem (cf. Corollary 3.36 of Luo et al., 1999, see also Huang, 1985) then implies exponential stability of the closed-loop system.

The paper is organised as follows. In Section 2, we prove that A generates a C_0-semi-group of uniformly bounded operators. In Section 3, we show the asymptotic stability of this semi-group, and Section 4 is devoted to the proof of the exponential stability. Finally, Section 5 contains some conclusions.

2. Formulation as a Dissipative Evolution Equation

For the mathematical analysis of the system (1.1) with Equation (1.8) it is convenient to eliminate most numerical coefficients. To this end, we rescale length and time, i.e. we introduce new variables $\tilde{x} = \frac{P(L)\rho}{m_p}x$ and $\tilde{t} = \frac{P(L)}{m_p}\sqrt{\rho}t$. With $\tilde{w}(\tilde{t}, \tilde{x}) := w(t, x)$ and $\tilde{P}(\tilde{x}) = P(x)$ the system (1.1) is equivalent to

$$\partial_{\tilde{t}}^2 \tilde{w}(\tilde{t}, \tilde{x}) = \partial_{\tilde{x}}(\tilde{P}(\tilde{x})\partial_{\tilde{x}}\tilde{w}(\tilde{t}, \tilde{x}), \quad \tilde{x} \in (0, \tilde{L}), \tilde{t} > 0, \tag{1.1a}$$

$$\partial_{\tilde{t}}^2 \tilde{w}(\tilde{t}, \tilde{L}) = -\partial_{\tilde{x}}\tilde{w}(\tilde{t}, \tilde{L}), \tag{1.1b}$$

$$\partial_{\tilde{t}}^2 \tilde{w}(\tilde{t}, 0) = \tilde{\vartheta}_1 \tilde{v}(\tilde{t}, 0) + \tilde{\vartheta}_2 \partial_{\tilde{x}}\tilde{v}(\tilde{t}, 0) + \tilde{\vartheta}_3 \tilde{w}(\tilde{t}, 0) + \tilde{\vartheta}_4 \partial_{\tilde{x}}\tilde{w}(\tilde{t}, 0), \tag{1.1c}$$

in new coordinates. In Equation (1.1c'), m_c and all additional factors arising from the change of coordinates as well as the term $\tilde{P}(0)\partial_{\tilde{x}}\tilde{w}(\tilde{t}, 0)$ have been merged in the new coefficients $\tilde{\vartheta}_i$, $i = 1, \dots, 4$. In the following, we only consider the system (1.1'). However, for the sake of readability, we will omit the superscript tilde in the sequel and simply write x, t, ϑ_i, w, and P.

For the analysis of Equation (1.1'), we define the (complex) Hilbert space:

$$\mathcal{H} = \{z = (w, v, \xi, \psi) : w \in H^2(0, L),\ v \in H^1(0, L),\ \xi = v(L),\ \psi = v(0)\}, \tag{2.1}$$

which is a closed subspace of $H^2 \times H^1 \times \mathbb{C} \times \mathbb{C}$. Here, $H^n(0, L)$ denotes the Sobolev space of functions whose derivatives up to order n are square-integrable (see Adams, 1975, for details). The auxiliary scalar variables ξ, ψ are introduced here in order to include the dynamical boundary conditions (1.1b') and (1.1c') into the initial value problem. $\mathcal{H}$ is equipped with the natural inner product:

$$\langle z_1, z_2 \rangle = \langle w_1, w_2 \rangle_{H^2} + \langle v_1, v_2 \rangle_{H^1} + \xi_1 \bar{\xi}_2 + \psi_1 \bar{\psi}_2, \tag{2.2}$$

where $\bar{\xi}$ denotes the complex conjugate of ξ. Let the linear operator $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ be defined as

$$A : \begin{bmatrix} w \\ v \\ \xi \\ \psi \end{bmatrix} \mapsto \begin{bmatrix} v \\ (Pw')' \\ -w'(L) \\ \vartheta_1 v(0) + \vartheta_2 v'(0) + \vartheta_3 w(0) + \vartheta_4 w'(0) \end{bmatrix}, \tag{2.3}$$

where w' denotes the spatial derivative of w, i.e. $w' = \partial_x w$. The (dense) domain of A is defined as

$$D(A) := \big\{z = (w, v, \xi, \psi) : w \in H^3(0, L),\ v \in H^2(0, L),\ \xi = v(L),\ \psi = v(0),\ (Pw')'(L) = -w'(L),\ (Pw')'(0) = F[w, v]\big\}, \tag{2.4}$$

with

$$F[w, v] := \vartheta_1 v(0) + \vartheta_2 v'(0) + \vartheta_3 w(0) + \vartheta_4 w'(0)$$

due to Equation (1.1').

The boundary conditions stated in $D(A)$ arise naturally from the requirement that $\operatorname{ran} A \subset \mathcal{H}$. With these definitions, we can rewrite the system (1.1') as the following initial value problem in $\mathcal{H}$:

$$\begin{cases} \dot{z}(t) = Az(t), \\ z(0) = z_0 \in \mathcal{H}. \end{cases} \tag{2.5}$$

For some of the following proofs, the natural inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{H}$ is unpractical. Therefore, we define an equivalent inner product, which is more suitable for the considered problem:

$$\begin{aligned} \langle z_1, z_2 \rangle_{\mathcal{H}} := {} & \alpha_1 \int_0^L \left[\gamma (Pw_1')'(P\bar{w}_2')' + Pw_1'\bar{w}_2'\right] \mathrm{d}x \\ & + \alpha_1 \gamma P(L) w_1'(L) \bar{w}_2'(L) + \alpha_2 w_1(0) \bar{w}_2(0) \\ & + \alpha_1 \int_0^L \left(\gamma P v_1' \bar{v}_2' + v_1 \bar{v}_2\right) \mathrm{d}x + \alpha_1 P(L) \xi_1 \bar{\xi}_2 \\ & + \alpha_2 \gamma \psi_1 \bar{\psi}_2 \\ & + \frac{1}{2}\big(\psi_1 - 2\alpha_1 P(0) w_1'(0) + 2\alpha_2 w_1(0)\big) \\ & \times \big(\bar{\psi}_2 - 2\alpha_1 P(0) \bar{w}_2'(0) + 2\alpha_2 \bar{w}_2(0)\big), \end{aligned} \tag{2.6}$$

where α_1, α_2, and γ are positive constants to be specified later (in Lemma 2.5 and the corresponding proof). We have the following lemma:

Lemma 2.1: *The norm $\| \cdot \|_{\mathcal{H}}$ is equivalent to the natural norm $\| \cdot \|$ on $\mathcal{H}$.*

Proof: We have to prove the existence of constants c_1, $c_2 > 0$ such that $c_1\|z\| \le \|z\|_{\mathcal{H}} \le c_2\|z\|$ holds for all $z \in \mathcal{H}$. To verify the first inequality, it remains to show the existence of $\tilde{c}_1$ such that

$$\int_0^L \left[\gamma |(Pw')'|^2 + P|w'|^2\right] \mathrm{d}x \ge \tilde{c}_1 \int_0^L \left[|w''|^2 + |w'|^2\right] \mathrm{d}x \tag{2.7}$$

holds for all real-valued $w \in H^2(0, L)$. Using the properties of P mentioned above, Lemma A.1 (see Appendix 1) can be applied pointwise in x with $a = \sqrt{\gamma}\, P'(x)$, $b = \sqrt{\gamma}\, P(x)$, $\varepsilon = P(x)$, $x_1 = |w'(x)|$, and $x_2 = |w''(x)|$, which directly yields the desired inequality (2.7).

To verify the second inequality, it suffices to apply Cauchy's inequality $ab \le \frac{a^2}{2} + \frac{b^2}{2}$, $a, b \in \mathbb{R}$, to the terms obtained by expansion of the last term in $\|z\|_{\mathcal{H}}^2$. ■

The main statement of this section is the following theorem, which will be proved in several steps:

Theorem 2.1: *Let there be constants $a, b > 0$ satisfying $(a + b - 1)^2 < 4ab$, such that*

$$\vartheta_1 = \frac{\vartheta_3}{b} - a, \quad \vartheta_2 = \frac{\vartheta_4}{b}, \tag{2.8}$$

and ϑ_1, $\vartheta_3 < 0$ and ϑ_2, $\vartheta_4 > 0$. Then, the operator A is the infinitesimal generator of a C_0-semi-group of uniformly bounded operators $\{T(t)\}_{t\geq 0}$ on $\mathcal{H}$.

The conditions of Theorem 2.1 on the parameters ϑ_j, $j = 1, \ldots, 4$ from Equation (1.1c') are fulfilled if the controller parameters χ_1, χ_2, and χ_3 according to Equation (1.8) meet the following non-restrictive inequality constraints:

$$\chi_1 > 0, \quad \chi_2 > 0, \quad \text{and } \chi_3 > \frac{\left(m_p - P(L)\sqrt{\rho}\right)^2}{4m_p P(L)\sqrt{\rho}} \tag{2.9}$$

where

$$\begin{aligned} b &= \frac{\tilde{\vartheta}_4}{\tilde{\vartheta}_2} = \frac{m_p\alpha_3}{P(L)\sqrt{\rho}} > 0 \quad \text{and} \\ a &= \frac{\tilde{\vartheta}_3}{b} - \tilde{\vartheta}_1 = \frac{(\alpha_3+1)\, m_p}{P(L)\sqrt{\rho}} > 0. \end{aligned} \tag{2.10}$$

In Theorem 2.1, the admissible parameters (a, b) lie inside a parabola in the first quadrant, and this parabola is tangent to the positive $a-$ and $b-$axes.

Remark 2.1: Let us briefly compare the model (1.1') subject to Equation (2.8) with the closed-loop system from Mifdal (1997b) and even its extension in §0(a) of Mifdal (1997a). We recall that the latter two models correspond to a control law with $\vartheta_4 = 0$. Equations (1.1') and (2.8) can only be matched with the models analysed by Mifdal when using the special parameters $a = 0$ and $b = \frac{P(0)}{P(L)}\frac{M}{\vartheta_2 m}$ in Equation (2.8). Here, M and m denote parameters related to the masses of the payload and the cart in Mifdal (1997b). Since Theorem 2.1 requires $a > 0$, these two types of models and results are complementary to each other.

Remark 2.2: With respect to the inner product specified in Lemma 2.5, this semi-group is even a semi-group of contractions, see the proof of Theorem 2.1 .

We shall prove Theorem 2.1 by applying the Lumer–Phillips theorem (cf. Pazy, 1983). But before, we verify some basic properties of A.

Lemma 2.2: *The domain $D(A)$ defined in Equation (2.3) is dense in $\mathcal{H}$.*

Proof: Let $z_0 = (w_0, v_0, \xi_0, \psi_0) \in \mathcal{H}$. Since the inclusions $H^3(0, L) \subset H^2(0, L) \subset H^1(0, L)$ are dense, there exists a sequence $z_n = (w_n, v_n, \xi_n, \psi_n) \in H^3(0, L) \times H^2(0, L) \times \mathbb{C}^2 \cap \mathcal{H}$ such that $z_n \to z_0$ in $\mathcal{H}$. Now, in general, the second derivatives $\partial^2 w_n(0)$, $\partial^2 w_n(L)$ will not satisfy the boundary conditions necessary for $z_n \in D(A)$.

The fact that $H_0^1(0, L) \subset L^2(0, L)$ is dense ensures the existence of a sequence $\{u_n\} \subset H^1(0, L)$ satisfying $u_n(0) = a$ for all $n \in \mathbb{N}$ and any fixed $a \in \mathbb{C}$, with $\|u_n\|_{L^2} \to 0$. The sequence $\{y_n\}$ defined by

$$y_n := \int_0^x \int_0^\xi u_n(\zeta)\, \mathrm{d}\zeta\, \mathrm{d}\xi$$

satisfies $\partial^2 y_n(0) = a$ for all $n \in \mathbb{N}$, and $\|y_n\|_{H^2} \to 0$.

This shows that, for the sequence $\{w_n\}$, the values $\partial^2 w_n(0)$, $\partial^2 w_n(L)$ can be modified such that the modified sequence $\{\tilde{z}_n\} \subset D(A)$, but still $\tilde{z}_n \to z_0$ in $\mathcal{H}$. ■

Lemma 2.3: *Under the condition $\vartheta_3 \neq 0$, the operator A is injective and* ran $A = \mathcal{H}$, *i.e. A^{-1} exists and $D(A^{-1}) = \mathcal{H}$.*

Proof: We prove this lemma by showing that the equation $Az = (f, g, g(L), g(0))$ has a unique solution $z \in D(A)$ for every $(f, g, g(L), g(0)) \in \mathcal{H}$. This equation reads in detail:

$$\begin{bmatrix} v \\ (Pw')' \\ -w'(L) \\ \vartheta_1 v(0) + \vartheta_2 v'(0) + \vartheta_3 w(0) + \vartheta_4 w'(0) \end{bmatrix} = \begin{bmatrix} f \\ g \\ g(L) \\ g(0) \end{bmatrix}. \tag{2.11}$$

From the first line we immediately find $v = f \in H^2(0, L)$, which also fixes the values $v(0)$ and $v'(0)$. After integration of the second line we obtain

$$w'(x) = -\frac{P(L)}{P(x)} g(L) + \frac{1}{P(x)} \int_L^x g(y)\, \mathrm{d}y, \tag{2.12}$$

where we used $w'(L) = -g(L)$ from the third line. Since $1/P \in H^2(0, L)$ and $g \in H^1(0, L)$, we find $w' \in H^2(0, L)$. This equation also determines $w'(0)$. In combination with the already known values $v(0)$ and $v'(0)$, we obtain $w(0)$ from the fourth line in Equation (2.11), since $\vartheta_3 \neq 0$. Hence, $w(x)$ is uniquely determined as

$$\begin{aligned} w(x) = w(0) &- \int_0^x \frac{P(L)}{P(y)} g(L)\, \mathrm{d}y \\ &+ \int_0^x \frac{1}{P(y)} \int_L^y g(\zeta)\, \mathrm{d}\zeta\, \mathrm{d}y. \end{aligned} \tag{2.13}$$

All integrals exist, since $P(x) > 0$ holds uniformly. Finally, $w \in H^3(0, L)$ holds. Thus, the inverse A^{-1} exists and is defined on $\mathcal{H}$. ■

Lemma 2.4: *If $\vartheta_3 \neq 0$, the operator A^{-1} is compact.*

Proof: We show that for $(f, g, g(L), g(0)) \in \mathcal{H}$ the norm of $z = A^{-1}(f, g, g(L), g(0))$ in $\mathcal{J} :=$

$H^3(0, L) \times H^2(0, L) \times \mathbb{C}^2$ is uniformly bounded by $\|(f, g, g(L), g(0))\|_{\mathcal{H}}$.

Due to the continuous embedding $H^1(0, L) \hookrightarrow C[0, L]$ in one dimension (see, e.g. Adams, 1975), we have the estimates $|g(L)|$, $|g(0)| \le C\|g\|_{H^1}$. Here and in the sequel, C denotes positive, not necessarily equal constants. From the third line in Equation (2.11) we therefore get $|w'(L)| \le C\|g\|_{H^1}$. With this and Equation (2.12) we find the estimate

$$\|w'\|_{L^2} \le C\|g\|_{H^1}. \tag{2.14}$$

Next we will apply this result to the identity $Pw'' = g - P'w'$, which is obtained from the second line in Equation (2.11), and use $P' \in L^\infty(0, L)$ and $P(x) \ge P^0 > 0$ (with P^0 introduced right after Equation (1.1)). This yields

$$\|w''\|_{L^2} \le C\|g\|_{H^1}. \tag{2.15}$$

Similarly, from $(Pw')'' = g'$ we obtain the estimate:

$$\|w'''\|_{L^2} \le C\|g\|_{H^1}. \tag{2.16}$$

For v we immediately get $\|v\|_{H^2} = \|f\|_{H^2}$ using the first line in Equation (2.11). Due to the continuous embedding $H^k(0, L) \hookrightarrow C^{k-1}[0, L]$ in one dimension (cf. Adams, 1975), we find the following estimates:

$$|v(0)| \le C\|f\|_{H^2}, \tag{2.17}$$

$$|v'(0)| \le C\|f\|_{H^2}. \tag{2.18}$$

Using the above estimate for $w'(L)$ and Equation (2.12), we obtain

$$|w'(0)| \le C\|g\|_{H^1}. \tag{2.19}$$

Applying Equations (2.17)–(2.19) to the fourth line of Equation (2.11) and using $|g(0)| \le C\|g\|_{H^1}$ yields

$$|w(0)|^2 \le C(\|f\|_{H^2}^2 + \|g\|_{H^1}^2). \tag{2.20}$$

Altogether, we get

$$\|w\|_{H^3}^2 \le C(\|f\|_{H^2}^2 + \|g\|_{H^1}^2). \tag{2.21}$$

Thus, we have $\|w\|_{H^3}^2 + \|v\|_{H^2}^2 \le C(\|f\|_{H^2}^2 + \|g\|_{H^1}^2)$, which shows that A^{-1} maps bounded sets in $\mathcal{H}$ into bounded sets in $\mathcal{J}$. Since the embeddings $H^3(0, L) \subset\subset H^2(0, L) \subset\subset H^1(0, L)$ are compact, A^{-1} is a compact operator. ■

From the previous lemma we know that A^{-1} is a closed operator, therefore we have:

Corollary 2.1: *For $\vartheta_3 \ne 0$, the operator A is closed, and $0 \in \rho(A)$, the resolvent set of A.*

Now we turn to the application of the Lumer–Phillips theorem in order to prove Theorem 2.1. To this end we shall prove the dissipativity of A with respect to the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$.

Lemma 2.5: *Let the assumptions of Theorem 2.1 hold, and let $\mathcal{H}$ be equipped with the inner product (2.6), where we choose the coefficients*

$$\alpha_1 := \frac{\vartheta_2}{2P(0)}, \qquad \alpha_2 := -\frac{\vartheta_2\vartheta_3}{2\vartheta_4}, \tag{2.22}$$

and $\gamma > 0$ is sufficiently small. Then, the operator A is dissipative in $\mathcal{H}$.

The proof is deferred to Appendix 3. Now, Theorem 2.1 follows directly from the above results:

Proof of Theorem 2.1: First, we prove this result under the additional assumptions of Lemma 2.5 (on α_1, α_2, and γ). Then, A is dissipative in $\mathcal{H}$ equipped with $\|\cdot\|_{\mathcal{H}}$, and Corollary 2.1 implies $0 \in \rho(A)$. Since $\rho(A)$ is an open set, there exists some $\zeta \in \rho(A)$ with positive real part. So the requirements of the Lumer–Phillips theorem are fulfilled, and we obtain that A generates a C_0-semi-group of contractions on $\mathcal{H}$ with respect to $\|\cdot\|_{\mathcal{H}}$.

Next we drop the additional assumptions of Lemma 2.5, and consider an equivalent norm on $\mathcal{H}$. Then, the semi-group $\{T(t)\}_{t\ge 0}$ is not necessarily a contraction semi-group any more, but still a C_0-semi-group of uniformly bounded operators. ■

The following corollary follows as a consequence of Theorem 2.1, due to elementary properties of generators of C_0-semi-groups of operators (for more details, see Pazy, 1983):

Corollary 2.2: *Under the assumptions of Theorem 2.1, the initial value problem*

$$\begin{cases} \dot{z}(t) = Az(t) \\ z(0) = z_0 \end{cases} \tag{2.23}$$

has a unique mild solution $z(t) := T(t)z_0$ for all $z_0 \in \mathcal{H}$, where $\{T(t)\}_{t\ge 0}$ is the C_0-semi-group generated by A. If $z_0 \in D(A)$, then $z(t)$ is continuously differentiable on $[0, \infty)$ and $z(t) \in D(A)$ for all $t \ge 0$, and therefore is a classical solution. Furthermore, the norm $\|z(t)\|_{\mathcal{H}}$ remains bounded as $t \to \infty$.

3. Asymptotic stability

After having shown that the norm of every solution of the initial value problem (2.23) is uniformly bounded with respect to $t \geq 0$, we now prove that the norm even tends to zero as $t \to \infty$, i.e. the C_0-semi-group $\{T(t)\}_{t \geq 0}$ generated by A is asymptotically stable, by applying the following theorem (see Luo et al., 1999, Theorem 3.26):

Theorem 3.1: *Let $\{S(t)\}_{t \geq 0}$ be a uniformly bounded C_0-semi-group in a Banach space X with generator $\mathcal{A}$, and assume that the resolvent $R(\lambda, \mathcal{A})$ is compact for some $\lambda \in \rho(\mathcal{A})$. Then, $\{S(t)\}_{t \geq 0}$ is asymptotically stable if and only if $\mathrm{Re}\,\lambda < 0$ for all $\lambda \in \sigma(\mathcal{A})$.*

Remark 3.1: The compactness of the resolvent $R(\lambda, \mathcal{A})$ for one $\lambda \in \rho(\mathcal{A})$ already implies its compactness for all $\lambda \in \rho(\mathcal{A})$ (cf. Kato, 1966, Theorem III.6.29).

Theorem 3.2: *Let the assumptions of Theorem 2.1 hold. Then, the C_0-semi-group $\{T(t)\}_{t \geq 0}$ generated by A is asymptotically stable.*

Proof: According to Lemma 2.4 the operator A has compact resolvent, and the associated semi-group $\{T(t)\}_{t \geq 0}$ is uniformly bounded due to Theorem 2.1. As a consequence of the Hille–Yosida theorem (see Pazy, 1983, Corollary 1.3.6), this implies $\sigma(A) \subseteq \{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda \leq 0\}$. Hence, in order to apply Theorem 3.1, it remains to prove that $\mathrm{i}\mathbb{R} \subset \rho(A)$. Since the resolvent is compact, $\sigma(A)$ consists only of eigenvalues. Thus, it is sufficient to show that $A - \mathrm{i}\tau$ is injective for all $\tau \in \mathbb{R}$, that is to show that the system

$$\begin{bmatrix} v - \mathrm{i}\tau w \\ (Pw')' - \mathrm{i}\tau v \\ -w'(L) - \mathrm{i}\tau v(L) \\ \vartheta_1 v(0) + \vartheta_2 v'(0) + \vartheta_3 w(0) + \vartheta_4 w'(0) - \mathrm{i}\tau v(0) \end{bmatrix} = 0 \tag{3.1}$$

only has the trivial solution in $D(A)$. We can rewrite this system in terms of the following equivalent boundary value problem for $w \in H^3(0, L) \hookrightarrow C^2[0, L]$:

$$(Pw')' + \tau^2 w = 0, \qquad x \in (0, L), \tag{3.2a}$$

$$w'(0) = c_0 w(0), \tag{3.2b}$$

$$w'(L) = c_L w(L), \tag{3.2c}$$

where $c_0 := -\frac{\vartheta_3 + \tau^2 + \mathrm{i}\tau\vartheta_1}{\vartheta_4 + \mathrm{i}\tau\vartheta_2}$ and $c_L := \tau^2$. It is important to note that the conditions (2.8) on the ϑ_i imply that $c_0 \notin \mathbb{R}$ for all $\tau \in \mathbb{R}$. We now multiply Equation (3.2a) by the complex conjugate $\bar{w}$ and integrate by parts, which yields

$$-\int_0^L P|w'|^2\,\mathrm{d}x + \tau^2\|w\|_{L^2}^2 + P(L)w'(L)\bar{w}(L) = P(0)w'(0)\bar{w}(0).$$

Due to the boundary conditions (3.2b) and (3.2c) the left-hand side of above identity is real, but the right-hand side is either non-real or zero. Thus $w'(0)\bar{w}(0) = 0$, and Equation (3.2b) implies that $w(0) = w'(0) = 0$. Therefore, every solution of the boundary value problem (3.2) also satisfies the initial value problem:

$$\begin{aligned} (Pw')' + \tau^2 w &= 0, \qquad x \in (0, L), \\ w(0) &= 0, \\ w'(0) &= 0. \end{aligned}$$

Hence, $w \equiv 0$, and this shows that $A - \mathrm{i}\tau$ is injective for all $\tau \in \mathbb{R}$. ■

4. Exponential stability

Here we show an even stronger result, namely the exponential stability of the semi-group $\{T(t)\}_{t \geq 0}$, i.e. we prove that every solution of the initial value problem (2.23) tends to zero exponentially. We follow a strategy similar to the one applied by Morgül (2001).

Definition 4.1 (Exponential stability): A C_0-semi-group $\{S(t)\}_{t \geq 0}$ is said to be *exponentially stable* if there exist constants $M \geq 1$ and $\omega > 0$ such that $\|S(t)\| \leq M\exp(-\omega t)$ for all $t \geq 0$.

To investigate exponential stability of a C_0-semi-group, we use the following theorem (see Luo et al., 1999, Corollary 3.36):

Theorem 4.1 (Huang): *Let $\{S(t)\}_{t \geq 0}$ be a uniformly bounded C_0-semi-group in a Hilbert space, and let $\mathcal{A}$ be its generator. Then, $\{S(t)\}_{t \geq 0}$ is exponentially stable if and only if $\mathrm{i}\mathbb{R} \subset \rho(\mathcal{A})$ and*

$$\sup_{\tau \in \mathbb{R}} \|R(\mathrm{i}\tau, \mathcal{A})\| < \infty. \tag{4.1}$$

Theorem 4.2: *Assume that the conditions in Theorem 2.1 are satisfied. Then, the C_0-semi-group $\{T(t)\}_{t \geq 0}$ generated by A is exponentially stable.*

Proof: We know from Theorem 3.2 that $\{T(t)\}_{t \geq 0}$ is asymptotically stable, and that $\mathrm{i}\mathbb{R} \subset \rho(A)$. The map $\lambda \mapsto R(\lambda, A)$ is analytic on $\rho(A)$ (cf. Yosida, 1980), so, in particular, $\lambda \mapsto \|R(\lambda, A)\|$ is continuous on $\mathrm{i}\mathbb{R}$. In order to apply Theorem 4.1, it therefore remains to prove that $\|R(\mathrm{i}\tau, A)\|$ is uniformly bounded as $|\tau| \to \infty$. To this end, we need to find a τ-uniform estimate for the solution $z = (w, v, v(L), v(0))$ of the equation

$$(A - \mathrm{i}\tau)z = (f, g, g(L), g(0)) \in \mathcal{H} \tag{4.2}$$

in terms of the right-hand side. The corresponding homogeneous problem (3.1) only has the trivial solution (cf. the proof of Theorem 3.2). Hence, we show that the unique solution (w, v) of the BVP

$$v - \mathrm{i}\tau w = f, \qquad x \in (0, L), \tag{4.3a}$$

$$(Pw')' - \mathrm{i}\tau v = g, \qquad x \in (0, L), \tag{4.3b}$$

$$-w'(L) - \mathrm{i}\tau v(L) = g(L), \tag{4.3c}$$

$$\vartheta_1 v(0) + \vartheta_2 v'(0) + \vartheta_3 w(0) + \vartheta_4 w'(0) - \mathrm{i}\tau v(0) = g(0) \tag{4.3d}$$

satisfies the estimate

$$\|w\|_{H^2} + \|v\|_{H^1} \le C(\|f\|_{H^2} + \|g\|_{H^1}) \tag{4.4}$$

uniformly for all $f \in H^2(0, L), g \in H^1(0, L)$ and for all $|\tau|$ sufficiently large.

Since v and w are directly related via Equation (4.3a), we replace v in Equations (4.3b)–(4.3d) by $v = f + \mathrm{i}\tau w$ to obtain the following BVP for w:

$$(Pw')' + \tau^2 w = g + \mathrm{i}\tau f, \qquad x \in (0, L), \tag{4.5a}$$

$$-w'(L) + \tau^2 w(L) = (g + \mathrm{i}\tau f)(L), \tag{4.5b}$$

$$\underbrace{(\vartheta_4 + \mathrm{i}\tau\vartheta_2)}_{=:\gamma_1} w'(0) + \underbrace{(\vartheta_3 + \tau^2 + \mathrm{i}\tau\vartheta_1)}_{=:\gamma_2} w(0) = (g + \mathrm{i}\tau f)(0) - \vartheta_1 f(0) - \vartheta_2 f'(0). \tag{4.5c}$$

With this, we first show the desired estimate for w.

Step 1: Homogeneous boundary conditions

To begin with, we shall transform Equation (4.5) into a BVP with homogeneous boundary conditions. To this end, we use Equation (4.5a) to eliminate the terms $w(0)$ and $w(L)$. This yields, after differentiating Equation (4.5a), the following BVP for $\tilde{y} := Pw'$:

$$\tilde{y}'' + \frac{\tau^2}{P}\tilde{y} = g' + \mathrm{i}\tau f', \qquad x \in (0, L), \tag{4.6a}$$

$$\tilde{y}(L) + P(L)\tilde{y}'(L) = 0, \tag{4.6b}$$

$$\frac{\gamma_1}{P(0)}\tilde{y}(0) - \frac{\gamma_2}{\tau^2}\tilde{y}'(0) = \underbrace{-\frac{g(0)}{\tau^2}(\vartheta_3 + \mathrm{i}\tau\vartheta_1) - \frac{\mathrm{i}\vartheta_3}{\tau}f(0) - \vartheta_2 f'(0)}_{=:R_1}. \tag{4.6c}$$

In order to make the second boundary condition homogeneous, we determine a first order polynomial $h(x) = a_1 x + a_0$, such that $h(x)$ satisfies the boundary conditions (4.6b) and (4.6c). The coefficients can be determined uniquely:

$$a_1 = -\frac{\tau^2 P(0) R_1}{\gamma_1 \tau^2 (L + P(L)) + P(0)\gamma_2}, \qquad a_0 = -(L + P(L))a_1. \tag{4.7}$$

We note that, as already mentioned in the proof of Theorem 3.2, $\gamma_1/\gamma_2 = 1/c_0 \notin \mathbb{R}$, and so a_1 is always well defined. For $|\tau| > 1$ we find the estimate

$$|a_j| \le \frac{C}{\tau^2}(\|g\|_{H^1} + |\tau|\|f\|_{H^2}), \qquad \text{for } j = 0, 1, \tag{4.8}$$

by using the continuous embedding $H^k(0, L) \hookrightarrow C^{k-1}[0, L]$ in one dimension (cf. Adams, 1975) to estimate the terms occurring in R_1. Now, the function $y := \tilde{y} - h$ satisfies the following problem with homogeneous boundary conditions:

$$y'' + \frac{\tau^2}{P}y = H := g' + \mathrm{i}\tau f' - \frac{\tau^2}{P}h, \qquad x \in (0, L), \tag{4.9a}$$

$$y(L) + P(L)y'(L) = 0, \tag{4.9b}$$

$$\frac{\gamma_1}{P(0)}y(0) - \frac{\gamma_2}{\tau^2}y'(0) = 0. \tag{4.9c}$$

Step 2: Solution estimate

Now we determine the solution of Equation (4.9). Let $\{\varphi_1, \varphi_2\}$ be a basis of solutions of the homogeneous equation $y'' + \frac{\tau^2}{P}y = 0$. Then, the general solution of the inhomogeneous Equation (4.9a) can be obtained by variation of constants:

$$y(x) = c_1\varphi_1(x) + c_2\varphi_2(x) + \int_0^x H(t)\frac{\varphi_1(t)\varphi_2(x) - \varphi_2(t)\varphi_1(x)}{\varphi_1(t)\varphi_2'(t) - \varphi_1'(t)\varphi_2(t)}\,\mathrm{d}t \tag{4.10}$$

$$= c_1\varphi_1(x) + c_2\varphi_2(x) + \textstyle\int_0^x H(t)J(x, t)\,\mathrm{d}t, \tag{4.11}$$

where $J(x, t)$ is the Green's function introduced in Lemma B.1 in Appendix 2, and $c_j \in \mathbb{C}$ are arbitrary constants. The derivative $y'(x)$ satisfies

$$y'(x) = c_1\varphi_1'(x) + c_2\varphi_2'(x) + \int_0^x H(t)\partial_x J(x, t)\,\mathrm{d}t. \tag{4.12}$$

In order to determine the constants c_j we now specify the initial conditions of the solutions φ_1, φ_2:

$$\varphi_1(0) = 0, \quad \varphi_2(0) = 1,$$
$$\varphi_1'(0) = \tau, \quad \varphi_2'(0) = 0.$$

These conditions imply that the functions φ_j are real-valued. From the boundary conditions (4.9b) and (4.9c), we then find

$$c_1 = \frac{-\int_0^L H(t)J(L,t)\,\mathrm{d}t - P(L)\int_0^L H(t)\partial_x J(L,t)\,\mathrm{d}t}{\varphi_1(L) + P(L)\varphi_1'(L) + \frac{\gamma_2 P(0)}{\gamma_1 \tau}[\varphi_2(L) + P(L)\varphi_2'(L)]},$$
$$c_2 = \frac{\gamma_2 P(0)}{\gamma_1 \tau} c_1. \tag{4.13}$$

Again, since $\gamma_2/\gamma_1 \notin \mathbb{R}$ and φ_1, φ_2 are linearly independent, the coefficients c_1, c_2 are well defined. Next, we estimate these coefficients. First, we find that

$$\lim_{|\tau|\to\infty} \frac{\gamma_2 P(0)}{\gamma_1 \tau} = -\frac{\mathrm{i}P(0)}{\vartheta_2}.$$

Therefore, we can find some constant $C > 0$, independent of $|\tau| > 1$, such that the denominator N of c_1 can be estimated as follows:

$$|N|^2 := \left|\varphi_1(L) + P(L)\varphi_1'(L) + \tfrac{\gamma_2 P(0)}{\gamma_1 \tau}[\varphi_2(L) + P(L)\varphi_2'(L)]\right|^2$$
$$\geq C\left(|\varphi_1(L) + P(L)\varphi_1'(L)|^2 + |\varphi_2(L) + P(L)\varphi_2'(L)|^2\right).$$

From the initial conditions of φ_1, φ_2 and Lemma B.1 we find that the Wronskian satisfies $\varphi_1'(L)\varphi_2(L) - \varphi_1(L)\varphi_2'(L) = \tau$. Since $\|\varphi_j\|_{L^\infty}$ is uniformly bounded for all τ sufficiently large by Lemma B.2, this implies $|\varphi_1'(L)| + |\varphi_2'(L)| \geq C\tau$, for some constant $C > 0$ independent of τ. With this result, we obtain the estimate

$$|N| \geq C|\tau|, \tag{4.14}$$

for all $|\tau| > 1$, and C independent of τ.

Now it remains to estimate the integrals occurring in c_1 and those in Equations (4.11) and (4.12). To this end, we split these integrals according to $H = (H - g') + g'$. In order to estimate the integrals corresponding to the first term, we apply Theorem B.1 and use the estimates for h found in Equation (4.8). For the other integrals we apply Hölder's inequality, and obtain

$$\left\|\int_0^x g'(t)J(x,t)\,\mathrm{d}t\right\|_{L^\infty} \leq \|g'\|_{L^1}\|J\|_{L^\infty((0,L)^2)}$$
$$\leq \frac{C}{|\tau|}\|g\|_{H^1},$$

where we used Lemma B.2 to estimate $\|J\|_{L^\infty}$. The integrals with $\partial_x J$ instead of J can be estimated analogously. Altogether we obtain

$$\left\|\int_0^x H(t)J(x,t)\,\mathrm{d}t\right\|_{L^\infty} \leq \frac{C}{|\tau|}(\|g\|_{H^1} + \|f\|_{H^2}), \tag{4.15}$$

$$\left\|\int_0^x H(t)\partial_x J(x,t)\,\mathrm{d}t\right\|_{L^\infty} \leq C(\|g\|_{H^1} + \|f\|_{H^2}), \tag{4.16}$$

for all $|\tau| > 1$, with $C > 0$ independent of τ. Therefore, we conclude that the estimate $|c_j| \leq \frac{C}{|\tau|}(\|g\|_{H^1} + \|f\|_{H^2})$ holds uniformly in τ. Applying these results and the estimates for the basis-functions φ_1, φ_2 found in Lemma B.2 to Equations (4.11) and (4.12), we find that the following estimates hold uniformly for $|\tau| > 1$:

$$\|y\|_{L^2} \leq C\|y\|_{L^\infty} \leq \tfrac{C}{|\tau|}(\|g\|_{H^1} + \|f\|_{H^2}), \tag{4.17}$$

$$\|y\|_{H^1} \leq C(\|y'\|_{L^\infty} + \|y\|_{L^\infty}) \leq C(\|g\|_{H^1} + \|f\|_{H^2}). \tag{4.18}$$

Using Equation (4.8), we see that the same estimates hold for $\tilde{y}$. Furthermore, by using $\tilde{y} = Pw'$ and the Equation (4.5a) to express w in terms of w' and w'', we find

$$\|w\|_{H^1} \leq \tfrac{C}{|\tau|}(\|g\|_{H^1} + \|f\|_{H^2}), \tag{4.19}$$

$$\|w\|_{H^2} \leq C(\|g\|_{H^1} + \|f\|_{H^2}). \tag{4.20}$$

Finally, from Equation (4.3a) and by using Equation (4.19) we get the desired estimate

$$\|v\|_{H^1} \leq C(\|g\|_{H^1} + \|f\|_{H^2}),$$

which completes the proof. ■

5. Conclusions

In Thull et al. (2006), a backstepping-based controller was proposed for the infinite-dimensional model of a gantry crane with heavy chain and payload. This controller shows excellent results, which was also verified experimentally by Thull et al. (2006). In particular the control law was designed in a systematic way, it features to be robust with respect to unmodelled (stick-slip) friction effects, which are always present in real applications, and it can be easily tuned. Though energy dissipation of the closed-loop system could be shown by Thull et al. (2006), the proof of the closed-loop stability did not work out, which was also correctly pointed out by

Grabowski (2009). In this paper, a rigorous proof of the asymptotic and exponential stability of the closed-loop system is given. For this, it was necessary to formulate the dynamics of the closed-loop system as an abstract evolution equation in an appropriate Hilbert space which differs from the space $H^1 \times L^2$ which is usually used in the context of heavy chain systems. Moreover, under very mild conditions on the controller parameters, which were explicitly derived, it was proven that the solutions of the closed-loop system are described by an asymptotically stable C_0-semi-group of contractions. Finally, by employing Huang's theorem it was even possible to show that under the same conditions the backstepping-based boundary controller renders the closed-loop system exponentially stable.

Notes

1. Note that here only a single chain is considered, unlike the pair of parallel chains as used by Thull et al. (2006). This change corresponds to the substitution $\rho \to \rho/2$.
2. Note that in this paper the virtual spring force $f_s(\cdot)$ in Thull et al. (2006) is considered linear.
3. Here the equilibrium of the cart position is set to zero, but can, of course, take any other value in the operating range.

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Appendix 1. Useful inequalities

Lemma A.1: *Let a_0, b_0, $\varepsilon_0 > 0$ be given. Then there exist positive constants c, d such that*

$$(ax_1 + bx_2)^2 + \varepsilon x_1^2 \geq cx_1^2 + dx_2^2 \tag{A.1}$$

holds uniformly for all $x_1, x_2 \in \mathbb{R}$ and $|a| \leq a_0$, $b \geq b_0$ and $\varepsilon \geq \varepsilon_0$.

Proof: Inequality (A.1) can be rewritten in the equivalent form:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} a^2 + \varepsilon - c & ab \\ ab & b^2 - d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geq 0,$$

where the occurring matrix will be denoted as M. Since this inequality has to hold for all $x_1, x_2 \in \mathbb{R}$, it is equivalent to M being positive semi-definite. Applying the Sylvester criterion yields the following conditions:

$$b^2 - d \geq 0, \tag{A.2}$$

$$(\varepsilon - c)(b^2 - d) \geq a^2 d. \tag{A.3}$$

If $a = 0$, we can take $c = \varepsilon_0$ and $d = b^2$. Otherwise, we see from the conditions (A.2) and (A.3) that $d < b^2$, so that Equation (A.3) can be written as

$$c \leq \varepsilon - a^2 \frac{d}{b^2 - d}. \tag{A.4}$$

Because of the monotonicity of the right-hand side we find the estimate:

$$\varepsilon - a^2 \frac{d}{b^2 - d} \geq \varepsilon_0 - a_0^2 \frac{d}{b_0^2 - d}.$$

So, for Equation (A.4) to hold, it is sufficient that c, d satisfy the stricter inequality:

$$c \leq \varepsilon_0 - a_0^2 \frac{d}{b_0^2 - d}. \tag{A.5}$$

For d sufficiently small, the right-hand side becomes positive, and therefore a $c > 0$ satisfying Equation (A.5) exists. ■

Lemma A.2: *Let $\alpha, \beta, \delta \in \mathbb{R}$ and*

$$\begin{aligned} P_3(x_1, x_2, x_3) := {} & x_1^2 + x_2^2 + x_3^2 + 2\alpha x_1 x_2 \\ & + 2\beta x_2 x_3 + 2\delta x_1 x_3 \end{aligned}$$

be a polynomial. Then, the inequality $P_3(x_1, x_2, x_3) \geq 0$ holds for all $x_1, x_2, x_3 \in \mathbb{R}$ if and only if the coefficients satisfy the conditions:

$$\begin{gathered} \alpha^2 \leq 1, \quad \beta^2 \leq 1, \quad \delta^2 \leq 1, \\ \alpha^2 + \beta^2 + \delta^2 \leq 1 + 2\alpha\beta\delta. \end{gathered}$$

Proof: The polynomial can be written as

$$P_3(x_1, x_2, x_3) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}^T \begin{bmatrix} 1 & \alpha & \delta \\ \alpha & 1 & \beta \\ \delta & \beta & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

with M denoting the 3×3 matrix. Now the property $P_3(x_1, x_2, x_3) \geq 0$, $\forall x_1, x_2, x_3 \in \mathbb{R}$ is equivalent to M being positive semi-definite. Applying the Sylvester criterion to M yields the desired conditions. ■

Appendix 2. ODEs with a parameter: Uniform estimates

In this section, we discuss the behaviour of classical solutions $y \in C^2[0, L]$ to the equation:

$$y'' + \frac{\tau^2}{P(x)} y = 0, \qquad x \in (0, L), \tag{B.1}$$

where $\tau \in \mathbb{R}$ and $P \in C^1[0, L]$ is a real-valued function satisfying $P^0 \leq P(x) \leq P^1$ uniformly for $x \in [0, L]$ for some positive constants P^0, P^1. Since τ only occurs squared, we can assume that $\tau \geq 0$ holds in the following.

Lemma B.1 (Birkhoff & Rota, 1962)**:** *Let (φ_1, φ_2) be an arbitrary pair of linearly independent solutions of Equation*

(B.1). *Then the Green's function of the equation is given by*

$$J(x,t) := \frac{\varphi_1(x)\varphi_2(t) - \varphi_2(x)\varphi_1(t)}{\varphi_1'(t)\varphi_2(t) - \varphi_1(t)\varphi_2'(t)}. \tag{B.2}$$

Furthermore, the Wronskian $W(t) := \varphi_1'(t)\varphi_2(t) - \varphi_1(t)\varphi_2'(t)$ *is constant for* $t \in [0, L]$. *Hence, Equation* (B.2) *simplifies to* $J(x, t) = C[\varphi_1(x)\varphi_2(t) - \varphi_2(x)\varphi_1(t)]$.

With the prescribed initial data $\varphi(0)$ and $\varphi'(0)$, we shall denote the unique classical solution of Equation (B.1) by φ_τ. The behaviour of solutions of Equation (B.1) is stated in the following lemma. For the proof, see Proposition 2.1 in Arnold et al. (2011).

Lemma B.2: *There exists a constant* $C > 0$ *such that for any family of solutions* $\{\varphi_\tau\}_{\tau>1}$ *of Equation* (B.1) *the following estimates hold uniformly for* $\tau > 1$:

$$\|\varphi_\tau\|_{L^\infty} \le \frac{C}{\tau}\left(\tau|\varphi_\tau(0)| + |\varphi_\tau'(0)|\right),$$
$$\|\varphi_\tau'\|_{L^\infty} \le C\left(\tau|\varphi_\tau(0)| + |\varphi_\tau'(0)|\right).$$

Now we are able to prove the following theorem:

Theorem B.1: *Let* $\{J_\tau\}_{\tau>1}$ *be the family of Green's functions defined in Lemma B.1. Then, there exists a constant* $C > 0$ *such that the following estimates hold uniformly for all* $f \in H^1(0, L)$ *and* $\tau > 1$:

$$\left\| \int_0^x f(t) J_\tau(x,t)\,\mathrm{d}t \right\|_{L^\infty} \le \tfrac{C}{\tau^2}\|f\|_{H^1}, \tag{B.3}$$
$$\left\| \int_0^x f(t) \partial_x J_\tau(x,t)\,\mathrm{d}t \right\|_{L^\infty} \le \tfrac{C}{\tau}\|f\|_{H^1}. \tag{B.4}$$

Proof: We are going to show Equation (B.3), the proof of Equation (B.4) can be done analogously. The index τ is omitted for the sake of simplicity. First, we make the substitution $t = x - \xi$ in the left hand integral, and define the family of functions $\psi_x: \xi \mapsto J(x, x - \xi)$ with parameter x. These functions are solutions of the equation

$$\psi_x'' + \frac{\tau^2}{P(x-\xi)}\psi_x = 0, \tag{B.5}$$

with $'$ denoting here derivatives with respect to ξ. ψ_x takes the initial values $\psi_x(\xi = 0) = 0$ and $\psi_x'(\xi = 0) = 1$. Now, integrating by parts yields

$$\begin{aligned}
&\left| \int_0^x f(x-\xi)\psi_x(\xi)\,\mathrm{d}\xi \right| \\
&= \left| -\int_0^x \partial_\xi[(fP)(x-\xi)] \int_0^\xi \frac{\psi_x(\zeta)}{P(x-\zeta)}\,\mathrm{d}\zeta\,\mathrm{d}\xi \right. \\
&\quad \left. + f(0)P(0)\int_0^x \frac{\psi_x(\zeta)}{P(x-\zeta)}\,\mathrm{d}\zeta \right| \\
&\le 2\frac{\|\psi_x'\|_{L^\infty}}{\tau^2}\left(\int_0^x |\partial_\xi(fP)(x-\xi)|\,\mathrm{d}\xi + |f(0)P(0)| \right) \\
&\le C\frac{\|\psi_x'\|_{L^\infty}\|f\|_{H^1}}{\tau^2},
\end{aligned}$$

where we used Equation (B.5) in the second step. And in the last step we used the continuous embedding $H^1(0, L) \hookrightarrow C[0, L]$. From Lemma B.2 and the known initial conditions of ψ_x we find that $\|\psi_x'\|_{L^\infty}$ is uniformly bounded for all $\tau > 1$. Finally, we notice that $\psi_x(x - t) = J(x, t)$, so the above estimate also holds for J instead of ψ_x, which proves Equation (B.3). ■

Appendix 3. Deferred proofs

Proof of Lemma 2.5: For all $z \in D(A)$ we have

$$\begin{aligned}
&\mathrm{Re}\,\langle z, Az\rangle_{\mathcal{H}} \\
&= \mathrm{Re}\Big[\alpha_1\gamma \int_0^L (Pw')'(P\bar{v}')'\,\mathrm{d}x + \alpha_1 \int_0^L Pw'\bar{v}'\,\mathrm{d}x \\
&\quad + \alpha_1\gamma P(L)w'(L)\bar{v}'(L) + \alpha_2 w(0)\bar{v}(0) \\
&\quad + \alpha_1\gamma \int_0^L Pv'(P\bar{w}')''\,\mathrm{d}x + \alpha_1 \int_0^L v(P\bar{w}')'\,\mathrm{d}x \\
&\quad - \alpha_1 P(L)v(L)\bar{w}'(L) + \alpha_2\gamma v(0)\bar{F} \\
&\quad + \frac{1}{2}\left[v(0) - 2\alpha_1 P(0)w'(0) + 2\alpha_2 w(0)\right] \\
&\quad \times \left[\bar{F} - 2\alpha_1 P(0)\bar{v}'(0) + 2\alpha_2\bar{v}(0)\right]\Big] \\
&= \mathrm{Re}\Big[\alpha_1\gamma \int_0^L \left[Pv'(P\bar{w}')'\right]'\,\mathrm{d}x + \alpha_1 \int_0^L \left[Pv\bar{w}'\right]'\,\mathrm{d}x \\
&\quad + \alpha_1\gamma P(L)w'(L)\bar{v}'(L) + \alpha_2 w(0)\bar{v}(0) \\
&\quad - \alpha_1 P(L)v(L)\bar{w}'(L) + \alpha_2\gamma v(0)\bar{F} \\
&\quad + \frac{1}{2}\left[v(0) - 2\alpha_1 P(0)w'(0) + 2\alpha_2 w(0)\right] \\
&\quad \times \left[\bar{F} - 2\alpha_1 P(0)\bar{v}'(0) + 2\alpha_2\bar{v}(0)\right]\Big].
\end{aligned} \tag{C.1}$$

Using the boundary conditions in $D(A)$ to evaluate the term $Pv'(P\bar{w}')'|_0^L$, we find that the real parts of all terms at $x = L$ cancel against the real part of the third term of

Equation (C.1). The remaining terms are

$$\begin{aligned}\operatorname{Re}\langle z, Az\rangle_{\mathcal{H}} = \operatorname{Re}\Big[&\bar{v}(0)[-\alpha_1 P(0)w'(0)+\alpha_2 w(0)]\\ &+\gamma\bar{F}[-\alpha_1 P(0)v'(0)+\alpha_2 v(0)]\\ &+\frac{1}{2}\big[v(0)-2\alpha_1 P(0)w'(0)+2\alpha_2 w(0)\big]\\ &\times\big[\bar{F}-2\alpha_1 P(0)\bar{v}'(0)+2\alpha_2\bar{v}(0)\big]\Big].\end{aligned}$$

By introducing the functional J: $w \mapsto -2\alpha_1 P(0)w'(0) + 2\alpha_2 w(0)$, we simplify the expression:

$$\begin{aligned}\operatorname{Re}\langle z, Az\rangle_{\mathcal{H}} = \frac{1}{2}\operatorname{Re}\Big[&\bar{v}(0)J(w)+\gamma\bar{F}J(v)\\ &+\big[v(0)+J(w)\big]\big[\bar{F}+J(\bar{v})\big]\Big].\end{aligned} \tag{C.2}$$

Assuming the relations (2.8) and (2.22) we can write $F = -av(0) - bJ(w) - J(v)$ with $a, b > 0$. Then, the right-hand side of Equation (C.2) only depends on the three independent values $v(0)$, $J(w)$, and $J(v)$. Introducing the new variables $y_1 = \sqrt{a}\,v(0)$, $y_2 = \sqrt{b}\,J(w)$ and $y_3 = \sqrt{\gamma}\,J(v)$ yields

$$\begin{aligned}\operatorname{Re}\langle z, Az\rangle_{\mathcal{H}} = \frac{1}{2}\big(&P_3(\operatorname{Re} y_1, \operatorname{Re} y_2, \operatorname{Re} y_3)\\ &+P_3(\operatorname{Im} y_1, \operatorname{Im} y_2, \operatorname{Im} y_3)\big),\end{aligned} \tag{C.3}$$

where P_3 is the polynomial defined by

$$\begin{aligned}P_3(x_1, x_2, x_3) := -x_1^2 - x_2^2 - x_3^2 - 2x_1x_2\Big(\frac{a+b-1}{2\sqrt{ab}}\Big)\\ -2x_2x_3\Big(\frac{\sqrt{b\gamma}}{2}\Big) - 2x_1x_3\Big(\frac{\sqrt{a\gamma}}{2}\Big).\end{aligned} \tag{C.4}$$

Hence, A is dissipative if

$$\begin{aligned}x_1^2 + x_2^2 + x_3^2 + 2x_1x_2\Big(\frac{a+b-1}{2\sqrt{ab}}\Big) + 2x_2x_3\Big(\frac{\sqrt{b\gamma}}{2}\Big)\\ + 2x_1x_3\Big(\frac{\sqrt{a\gamma}}{2}\Big) \geq 0, \quad \forall x_1, x_2, x_3 \in \mathbb{R}.\end{aligned} \tag{C.5}$$

According to Lemma A.2 this inequality is satisfied if there holds

$$a, b \leq \frac{4}{\gamma}, \qquad \frac{(a+b-1)^2}{4ab} \leq 1, \qquad \frac{(a+b-1)^2}{4ab} \leq 1 - \frac{\gamma}{4}.$$

Since $\gamma > 0$ has not yet been specified, we can choose γ arbitrarily small, so that the above conditions reduce to the single condition:

$$\frac{(a+b-1)^2}{4ab} < 1. \tag{C.6}$$

So, the relation (2.8) on the ϑ_i together with the condition (C.6) on the $a, b > 0$ is sufficient for the dissipativity of A in $\mathcal{H}$ with respect to the inner product (2.6), with the choice (2.22) for α_1 and α_2, and $\gamma > 0$ sufficiently small. ■