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Reaching law for DSMC systems with relative degree 2 switching variable

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ABSTRACT

This paper presents the first attempt to design a reaching law-based discrete-time sliding mode controller with a relative degree 2 switching variable. The current value of this variable is only affected by the control signal and disturbance generated two time instants ago. It is demonstrated that the new reaching law-based strategy offers a smaller quasi-sliding mode band width when compared to a similar control scheme with relative degree 1 switching variable. This in turn leads to reduced system output error in the control system proposed in the paper.

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1. Introduction

Continuous time variable structure systems have been a subject of extensive research since their introduction in late 1950s (Emelyanov, 1967; Utkin, 1977). They possess several advantageous properties, such as computational efficiency and insensitivity with respect to matched disturbance (Draženović, 1969). As a result, they have attracted the attention of various authors (Bartoszewicz, & Nowacka-Leverton, 2009; DeCarlo, Żak, & Mathews, 1988; Edwards & Spurgeon, 1998; Gao & Hung, 1993; Šabanovic, 2011). An important issue in continuous time sliding mode control is the presence of high frequency oscillations called 'chattering.' These undesirable oscillations can potentially damage the plant or cause energy loss. Therefore, several approaches aiming at elimination of chattering have been proposed, one of which is higher order sliding mode control (Bartolini, Ferrara, & Usai, 1998; Levant, 1993). This approach aims at bringing the value of the sliding variable and one or more of its derivatives to zero. Higher order sliding mode control proved to be an effective method of eliminating undesirable oscillations and became an object of further research for various authors (Bartolini, Pisano, Punta, & Usai, 2003; Boiko, Fridman, Pisano, & Usai, 2007; Fridman & Levant, 2002; Laghrouche, Plestan, & Glumineau, 2007; Levant, 2003; Moreno, 2012; Moreno & Osorio, 2008; Moreno & Osorio, 2012).

Another development partly motivated by the problem of chattering was the introduction of discrete-time quasi-sliding mode systems (Milosavljević, 1985; Utkin & Drakunov, 1989). This led to many further advances in the field. Various authors proposed strategies that drive the system state to a cone shaped sector in the state space (Furuta, 1990) or strictly to a certain vicinity of the switching plane (Gao, Wang, & Homaifa, 1995). Discrete-time sliding mode controllers can be divided into switching type ones, which drive the state to cross the switching hyperplane in each step (Gao *et al.*, 1995) and non-switching type, which merely confine the state to a certain band around the hyperplane (Bartolini, Ferrara, & Utkin, 1995; Bartoszewicz, 1998). The width of the band was further investigated by Su, Drakunov, and Ozguner (2000). A significant problem with practical applications of sliding mode control is its need for full information about the system state. Several approaches were developed to address that issue, such as utilisation of time-delay control concept (Corradini & Orlando, 1998), implementation of observers (Chen, Komada, & Fukuda, 2000; Edwards & Spurgeon, 1996; Spurgeon, 2008) or the multirate output feedback method (Janardhahan & Kariwala, 2008; Janardhanan & Bandyopadhyay, 2006; Mehta & Bandyopadhyay, 2008).

The classical approach to sliding mode controller design involves stating the control law and then proving the stability of the sliding motion by selecting an appropriate Lyapunov function. However, in this work an alternative method called the reaching law approach will be considered. This approach was first proposed by Gao & Hung (1993) for continuous time systems and by Gao *et al.* (1995) for discrete-time ones (see also (Bartoszewicz, 1996) for further analysis). It consists of stating the desired evolution of the sliding variable in the form of a reaching law, and then synthesising a feasible control strategy according to the evolution.

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Discrete-time reaching laws became an object of extensive research (Bandyopadhyay & Fulwani, 2009; Bartolini *et al.*, 1995; Bartoszewicz, 1998; Bartoszewicz & Leśniewski, 2014; Bartoszewicz & Żuk, 2009; Chakrabarty, 2014; Chakrabarty & Bandyopadhyay, 2015; Chakrabarty & Bandyopadhyay, 2016; Golo & Milosavljević, 2000; Mija & Susy, 2010; Niu, Ho, & Wang, 2010) and various authors proposed strategies that greatly improved on the original constant plus proportional law proposed by Gao et al.

In this paper, second order sliding mode control will be considered in the context of discrete-time systems. A new reaching law for discrete-time systems will be proposed and applied to design a sliding mode control strategy. The strategy will utilise a relative degree 2 (RD2) sliding variable, which is only affected by disturbance and control signal from two time instants ago. It will be demonstrated that the new reaching law confines the state to a narrower band around the switching hyperplane than a similar strategy for relative degree 1 (RD1) systems and *drives the output* closer to its target value.

The remainder of the paper is organised in the following way. Section 2 describes the considered systems in detail and presents a general procedure of reaching law-based controller design. In Section 3, the proposed reaching laws are presented and their properties are discussed. Section 4 shows the results of a simulation comparing the proposed reaching law-based strategy for RD2 systems with the one for RD1 plants. Section 5 presents the conclusions of the paper.

2. Preliminaries

Let us consider the following class of discrete-time plants:

$$\begin{aligned} \boldsymbol{x}(k+1) &= \boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{b}u(k) + \boldsymbol{b}d(k) \\ y(k) &= \boldsymbol{q}\boldsymbol{x}(k), \end{aligned} \tag{1}$$

where $\boldsymbol{x}$ is an n dimensional state vector, $\boldsymbol{A}$ is the $n \times n$ state matrix, $\boldsymbol{b}$ is the input distribution vector, u is the scalar control signal, d is the scalar disturbance, y is the scalar system output and $\boldsymbol{q}$ is an n dimensional vector. It is assumed that the disturbance affecting the system is matched, which means it affects the plant through the same input channel as the control signal. The disturbance is also assumed to have lower and upper bounds $d_{\min}$ and $d_{\max}$ for all k, i.e.

$$d_{\min} \le d(k) \le d_{\max}. \tag{2}$$

In this paper, a discrete-time sliding mode control strategy will be applied to system (1) in order to drive its output from any initial position to zero. To obtain a feasible sliding mode control strategy, the reaching law approach will be used. Two reaching laws will be considered. The first one is a modified version of the conventional switching type law proposed by Gao *et al.* (1995) for RD1 systems. The second one, which is the main contribution of this paper, leads to a novel controller design procedure for RD2 systems. Properties of the two reaching laws will be investigated and performance for both cases will be compared. It will be demonstrated that the proposed reaching law for RD2 systems drives the system output closer to zero than the law for RD1 plants under the same constraints. Furthermore, it will be shown that both reaching laws ensure the same favourable properties of the system.

2.1. Reaching law-based controller design procedure

The first step of controller design procedure is the selection of an appropriate sliding variable. The variable for RD1 systems is chosen as

$$s_1(k) = \boldsymbol{c}_1{}^{\mathrm{T}}\boldsymbol{x}(k), \tag{3}$$

where $\boldsymbol{c}_1$ is a certain vector, which is selected by the designer and satisfies $\boldsymbol{c}_1{}^{\mathrm{T}}\boldsymbol{b} \neq 0$. Then, a reaching law which ensures the desired evolution of $s_1(k)$ is stated. In the conventional approach (Bartoszewicz, 1998; Gao *et al.*, 1995), the following class of reaching laws

$$s_1(k+1) = f_1[s_1(k), d_{\min}, d_{\max}] + \boldsymbol{c}_1{}^{\mathrm{T}}\boldsymbol{b}d(k) \tag{4}$$

is considered. In Equation (4), f_1 is a certain function selected to ensure the stability of the sliding motion. The control law $u(k)$ obtained from (4) has the following form:

$$u(k) = (\boldsymbol{c}_1{}^{\mathrm{T}}\boldsymbol{b})^{-1}\left\{f_1[s_1(k), d_{\min}, d_{\max}] - \boldsymbol{c}_1{}^{\mathrm{T}}\boldsymbol{A}x(k)\right\}. \tag{5}$$

However, in this paper an alternative approach will be proposed taking RD2 systems into consideration. For discrete-time objects, this property implies that the sliding variable at time k is only affected by the control signal generated at time k – 2. To ensure that (1) is an RD2 system with respect to s, the variable is defined as

$$s_2(k) = \boldsymbol{c}_2{}^{\mathrm{T}}\boldsymbol{x}(k), \tag{6}$$

where vector c_2 is selected so that $c_2{}^T b = 0$ and $c_2{}^T A b \neq 0$. Then, the following form of the reaching law for discrete-time RD2 systems is proposed

$$s_2(k+2) = f_2[s(k), s(k+1), d_{\min}, d_{\max}] + c_2{}^T A b d(k), \tag{7}$$

where f_2 is a certain function, which again is selected to ensure stability of the sliding motion. It must be noted that for RD2 systems, exact value of the sliding variable $s_2(k+1)$ is known at the time instant k, and this is why it can be used in the controller design procedure. Indeed, substitution of $x(k+1)$ from (1) into (6) yields

$$\begin{aligned} s_2(k+1) &= c_2{}^T A x(k) + c_2{}^T b u(k) + c_2{}^T b d(k) \\ &= c_2{}^T A x(k). \end{aligned} \tag{8}$$

Thus, the control signal $u(k)$ and matched disturbance $d(k)$ have no effect on the sliding variable s_2 in the sampling instant $k+1$. In order to derive the control signal from the reaching law (7), $s_2(k+2)$ is first obtained as

$$\begin{aligned} s_2(k+2) &= c_2{}^T A x(k+1) \\ &= c_2{}^T A^2 x(k) + c_2{}^T A b u(k) + c_2{}^T A b d(k). \end{aligned} \tag{9}$$

Substituting $s_2(k+2)$ from (9) into (7) and solving for $u(k)$, we obtain

$$\begin{aligned} u(k) = (c_2{}^T A b)^{-1}\{ &f_2[s_2(k), s_2(k+1), d_{\min}, d_{\max}] \\ &- c_2{}^T A^2 x(k)\}. \end{aligned} \tag{10}$$

Since $c_2{}^T A b \neq 0$, the expression $(c_2{}^T A b)^{-1}$ in relation (10) is properly defined and the strategy is applicable to RD2 systems. This concludes the general design procedure.

3. Proposed control strategy

In this section, reaching laws for RD1 and RD2 systems are presented and proven to ensure several desirable properties of the systems. The considered strategies make the sliding variable cross the switching manifold in finite time and then cross it again in each subsequent sampling instant. It will be further demonstrated that the strategy for RD2 systems, which is the main contribution of this paper, offers a reduced quasi-sliding mode band width and drives the system output closer to zero than a similar strategy for RD1 plants.

3.1. Strategy for relative degree 1 systems

Reaching law-based sliding mode control strategies for discrete-time systems were introduced in the seminal work of Gao *et al.* (1995). However, the constant plus proportional method proposed in that paper resulted in a large sliding variable rate of change at the beginning of the control process, which often leads to unacceptable values of the control signal and state variables. Therefore, a new strategy that limits the sliding variable rate of change will be introduced. In this paper, the following original reaching law is proposed for RD1 systems

$$\begin{aligned} s_1(k+1) = h[s_1(k)] \cdot s_1(k) - \varepsilon \operatorname{sgn}[s_1(k)] - \bar{d}_1 \\ - \delta d_1 \operatorname{sgn}[s_1(k)] + c_1{}^T b d(k), \end{aligned} \tag{11}$$

where

$$\begin{aligned} \bar{d}_1 &= 0.5 \cdot c_1{}^T b \cdot (d_{\max} + d_{\min}), \\ \delta d_1 &= 0.5 \cdot \left| c_1{}^T b \right| \cdot (d_{\max} - d_{\min}) \end{aligned} \tag{12}$$

are the mean of the disturbance and its maximum admissible deviation from the mean. Function

$$h(s) = \begin{cases} 1 & \text{for } |s| \geq s_0 \\ |s|/s_0 & \text{for } |s| < s_0, \end{cases} \tag{13}$$

and $s_0 > 0$, $\varepsilon > 0$ are the design parameters. Furthermore, it is assumed that the sign function $\operatorname{sgn}(s)$ equals 0 for $s = 0$. It is easy to notice that function h belongs to the interval [0, 1]. Furthermore, when the sliding variable is greater than or equal to s_0, h is equal to 1. Consequently, sliding variable rate of change will be limited by $\varepsilon + \delta d_1$ when the variable is far off the switching plane $s_1(k) = 0$. Vector c_1 for the reaching law (11) is selected to ensure that the matrix $A_{c1} = A - b(c_1{}^T b)^{-1} c_1{}^T A$ is nilpotent (Bartoszewicz & Żuk, 2009). In other words, it is chosen to satisfy

$$det\,(\lambda I_n - A_{c1}) = \det\left[\lambda I_n - A + b\left(c_1{}^T b\right)^{-1} c_1{}^T A\right] = \lambda^n. \tag{14}$$

Such a choice of vector c_1 will prove helpful in ensuring the smallest possible tracking error of the system output. The reaching law (11) will now be applied to obtain the control strategy for RD1 systems. Taking (5) into account, the control law can be expressed as

$$\begin{aligned} u(k) = (c_1{}^T b)^{-1}\{ &h[s_1(k)] \cdot s_1(k) - \varepsilon \operatorname{sgn}[s_1(k)] \\ &- \bar{d}_1 - \delta d_1 \operatorname{sgn}[s_1(k)] - c_1{}^T A x(k)\}. \end{aligned} \tag{15}$$

Next, it will be shown that with the proper choice of design parameters s_0 and ε, the proposed strategy ensures that $s_1(k)$ changes its sign in finite time and keeps changing the sign in every step afterwards. Moreover, the system state will always be confined to a band around the switching hyperplane $s_1(k) = 0$. Finally, it will be demonstrated that the proposed reaching law drives the system output to an a priori defined vicinity of zero in finite time. These properties will be formally proven in the following three theorems.

Theorem 3.1: *If the reaching law for system (1) is given by (11), then the system state enters the quasi-sliding mode band*

$$B_1 = \left\{ \boldsymbol{x} : \left| \boldsymbol{c}_1^{\mathrm{T}} \boldsymbol{x} \right| \le \varepsilon + 2\delta d_1 \right\} \tag{16}$$

in finite time and remains inside the band for all subsequent sampling instants.

Proof: It will first be shown that for any state $\boldsymbol{x}(k)$ out of the band (i.e. $\boldsymbol{x}(k) \notin B_1$), the system representative point enters the band in finite time. Relation (12) implies

$$\left| \boldsymbol{c}_1^{\mathrm{T}} \boldsymbol{b} d(k) - \bar{d}_1 \right| \le \delta d_1, \tag{17}$$

Now let $\boldsymbol{x}(k)$ be such a state that $s_1[\boldsymbol{x}(k)] > \varepsilon + \delta d_1$. Relations (11) and (17) give

$$\begin{aligned} s_1(k+1) &= h[s_1(k)] \cdot s_1(k) - \varepsilon - \bar{d}_1 - \delta d_1 + \boldsymbol{c}_1^{\mathrm{T}} \boldsymbol{b} d(k) \\ &\le s_1(k) - \varepsilon + \delta d_1 - \delta d_1 = s_1(k) - \varepsilon. \end{aligned} \tag{18}$$

Therefore, for any positive $s_1(kT)$, the sliding variable will decrease by at least ε in the next step. Furthermore, since function h is always non-negative, one obtains

$$s_1(k+1) \ge -\varepsilon - \bar{d}_1 - \delta d_1 + \boldsymbol{c}_1^{\mathrm{T}} \boldsymbol{b} d(k) \ge -\varepsilon - 2\delta d_1. \tag{19}$$

Consequently, if $s_1(k) > \varepsilon + 2\delta d_1$, the state will always enter the band (16) in finite time. Repeating derivations (18) and (19) for the case of $s_1(k) < -\varepsilon - 2\delta d_1$, one obtains

$$s_1(k) + \varepsilon \le s_1(k+1) \le \varepsilon + 2\delta d_1. \tag{20}$$

Therefore, we conclude that for any initial state, the system representative point will enter the band (16) in finite time. Furthermore, relations (18) and (20) state that the absolute value of the sliding variable will not increase unless the sign of $s_1(k)$ changes in the next step. However, even if the variable changes its sign, relations (19) and (20) imply that $|s_1(k)|$ will not exceed $\varepsilon + 2\delta d_1$. Consequently, once the system representative point enters the quasi-sliding mode band (16), it will remain inside the band for all subsequent sampling instants. ■

It follows from (18) and (20) that from any initial position $\boldsymbol{x}(0)$, the system representative point will reach the band (16) in a finite number of steps not greater than

$$k_0 = \left\lceil \frac{\left| \boldsymbol{c}_1^{\mathrm{T}} \boldsymbol{x}(0) \right| - \varepsilon - 2\delta d_1}{\varepsilon} \right\rceil. \tag{21}$$

It has already been demonstrated that the proposed reaching law drives the system state to a band around the switching hyperplane. However, since switching type strategies are considered in this work, it must also be shown that the reaching law makes the state cross the hyperplane in each sampling instant after crossing it for the first time. For that purpose, the following theorem will be proven.

Theorem 3.2: *If the reaching law for system (1) is defined by (11), $\boldsymbol{x}(k)$ belongs to the band (16), $s_1(k) \neq 0$ and parameter $s_0 > (\varepsilon + 2\delta d_1)^2 \varepsilon^{-1}$, then the system representative point will cross the switching plane in the next sampling instant, i.e. $sgn[s_1(k+1)] = -sgn[s_1(k)]$.*

Proof: First, let $\boldsymbol{x}(k)$ be such a state that $0 < s_1(k) \le \varepsilon + 2\delta d_1$. It will be shown that $s_1(k+1) < 0$. Since $s_0 > (\varepsilon + 2\delta d_1)^2 \varepsilon^{-1} > \varepsilon + 2\delta d_1$, relation (13) gives $h[s_1(k)] = |s_1(k)|/s_0$. Together with the fact that function $h[s_1(k)]$ is non-decreasing for positive $s_1(k)$, relations (11) and (17) give

$$\begin{aligned} s_1(k+1) &= \frac{|s_1(k)|}{s_0} \cdot s_1(k) - \varepsilon - \bar{d}_1 - \delta d_1 + \boldsymbol{c}_1^{\mathrm{T}} \boldsymbol{b} d(k) \\ &\le {s_0}^{-1} (\varepsilon + 2\delta d_1)^2 - \varepsilon. \end{aligned} \tag{22}$$

Since $s_0 > (\varepsilon + 2\delta d_1)^2 \varepsilon^{-1}$ implies ${s_0}^{-1} < (\varepsilon + 2\delta d_1)^{-2} \varepsilon$, one concludes from relation (22) that

$$s_1(k+1) < \frac{\varepsilon}{(\varepsilon + 2\delta d_1)^2} (\varepsilon + 2\delta d_1)^2 - \varepsilon = 0. \tag{23}$$

Now let $\boldsymbol{x}(k)$ be such a state that $0 > s_1(k) \ge -\varepsilon - 2\delta d_1$. Performing derivations analogous to (22) and (23), it can be easily shown that $s_1(k+1) > 0$. In conclusion, for any state inside the quasi-sliding mode band (16) such that $s_1(k) \neq 0$, the sliding variable will change its sign in the next sampling instant, i.e. $\mathrm{sgn}[s_1(k+1)] = -\mathrm{sgn}[s_1(k)]$. ■

Remark 3.1: Theorem 3.2 does not take into account the case where $s_1(k) = 0$, which is possible when the state $\boldsymbol{x}(k)$ enters the band (16) for the first time. In this case, relation (11) gives

$$|s_1(k+1)| = \left| \boldsymbol{c}_1^{\mathrm{T}} \boldsymbol{b} d(k) - \bar{d}_1 \right| \le \delta d_1 < \varepsilon + 2\delta d_1, \tag{24}$$

which means that the state will remain inside the band in the next step. If it happens that the disturbance is constantly equal to $(\boldsymbol{c}_1^{\mathrm{T}} \boldsymbol{b})^{-1} \bar{d}_1$, then the sliding

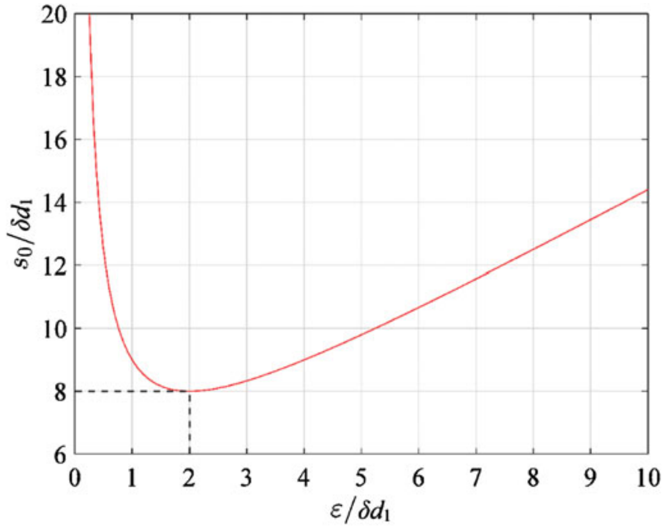

Figure 1. Minimum value of s_0 with respect to ε.

variable remains equal to zero and the ideal sliding motion is achieved. On the other hand, if $d(k)$ assumes any value different from $(\boldsymbol{c}_1{}^{\mathrm{T}}\boldsymbol{b})^{-1}\bar{d}_1$ at time instant k, then the variable $s_1(k+1)$ becomes nonzero and satisfies conditions of Theorem 3.1, ensuring switching for all future steps.

Considering Theorems 3.1, 3.2 and Remark 3.1, one concludes that the system either exhibits the ideal sliding motion or its representative point crosses the switching plane in finite time, crosses it again in each subsequent step and remains inside the quasi-sliding mode band (16). Therefore, the motion ensured by the proposed strategy strictly follows the sliding mode definition given by Gao *et al.* (1995). Furthermore, Theorem 3.2 provides the lower bound on the choice of design parameter s_0 in relation to the freely chosen positive parameter ε. Figure 1 illustrates the minimum value of s_0 for any given ε. Values of both parameters are presented as a multiple of the constant δd_1. Since the lower bound of s_0 tends to infinity as ε approaches zero, small values of ε should be avoided. On the other hand, excessive values of this parameter increase the width of the boundary layer (16). Therefore, it is reasonable to select value of ε in the vicinity of $2\delta d_1$.

It will now be shown that the proposed strategy drives the system output to a strictly specified vicinity of zero in finite time. In other words, it will be demonstrated that for the vector $\boldsymbol{c}_1$ selected to ensure that matrix $\boldsymbol{A}_{c1}$ is nilpotent, there exists a finite k_1 such that for every $k \geq k_1$, the absolute value of the output $y(k)$ has a constant upper bound. This property will be presented in the following theorem.

Theorem 3.3: *If the control strategy for system* (1) *is defined by* (15) *with vector* $\boldsymbol{c}_1$ *selected according to* (14), *then there exists a finite* k_1 *such that for every* $k \geq k_1$

$$\left|y(k)\right| = \left|\boldsymbol{q}\boldsymbol{x}(k)\right| \leq \upsilon_1, \tag{25}$$

where

$$\begin{aligned}\upsilon_1 &= \left|\left(\boldsymbol{c}_1{}^{\mathrm{T}}\boldsymbol{b}\right)^{-1}\right| \cdot (\varepsilon + 2\delta d_1) \cdot \sum_{i=0}^{n-1} \left|\boldsymbol{q}(\boldsymbol{A}_{c1})^i \boldsymbol{b}\right| \\ &= \left|\left(\boldsymbol{c}_1{}^{\mathrm{T}}\boldsymbol{b}\right)^{-1}\right| \cdot (\varepsilon + 2\delta d_1) \\ &\quad \cdot \sum_{i=0}^{n-1} \left|\boldsymbol{q}\left[\boldsymbol{A} - \boldsymbol{b}\left(\boldsymbol{c}_1{}^{\mathrm{T}}\boldsymbol{b}\right)^{-1}\boldsymbol{c}_1{}^{T}\boldsymbol{A}\right]^i \boldsymbol{b}\right|.\end{aligned} \tag{26}$$

Proof: Let k_0 be the first sampling instant such that $|s_1(k_0)| \leq \varepsilon + 2\delta d_1$. It follows from Theorem 3.1 that $|s_1(k)| \leq \varepsilon + 2\delta d_1$ for all $k \geq k_0$. Substitution of (15) into (1) yields

$$\begin{aligned}\boldsymbol{x}(k+1) &= \boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{b}d(k) + \boldsymbol{b}(\boldsymbol{c}_1{}^{\mathrm{T}}\boldsymbol{b})^{-1}\{h[s_1(k)]s_1(k) \\ &\quad - \varepsilon\mathrm{sgn}[s_1(k)] - \bar{d}_1 \\ &\quad - \delta d_1\mathrm{sgn}[s_1(k)] - \boldsymbol{c}_1{}^{T}\boldsymbol{A}\boldsymbol{x}(k)\} \\ &= \left[\boldsymbol{A} - \boldsymbol{b}\left(\boldsymbol{c}_1{}^{\mathrm{T}}\boldsymbol{b}\right)^{-1}\boldsymbol{c}_1{}^{T}\boldsymbol{A}\right]\boldsymbol{x}(k) \\ &\quad + \boldsymbol{b}(\boldsymbol{c}_1{}^{\mathrm{T}}\boldsymbol{b})^{-1}\{h[s_1(k)]s_1(k) - \varepsilon\mathrm{sgn}[s_1(k)] \\ &\quad - \bar{d}_1 - \delta d_1\mathrm{sgn}[s_1(k)] + \boldsymbol{c}_1{}^{T}\boldsymbol{b}d(k)\} \\ &= \left[\boldsymbol{A} - \boldsymbol{b}\left(\boldsymbol{c}_1{}^{\mathrm{T}}\boldsymbol{b}\right)^{-1}\boldsymbol{c}_1{}^{T}\boldsymbol{A}\right]\boldsymbol{x}(k) \\ &\quad + \boldsymbol{b}(\boldsymbol{c}_1{}^{\mathrm{T}}\boldsymbol{b})^{-1}s_1(k+1) \\ &= \boldsymbol{A}_{c1}\boldsymbol{x}(k) + \boldsymbol{b}(\boldsymbol{c}_1{}^{\mathrm{T}}\boldsymbol{b})^{-1}s_1(k+1).\end{aligned} \tag{27}$$

Consequently, for $n = \dim \boldsymbol{A}$

$$\boldsymbol{x}(k+n) = (\boldsymbol{A}_{c1})^n \boldsymbol{x}(k) + \sum_{i=0}^{n-1} (\boldsymbol{A}_{c1})^i \boldsymbol{b} (\boldsymbol{c}_1{}^{\mathrm{T}} \boldsymbol{b})^{-1} s_1(k+n-i). \tag{28}$$

Since vector $\boldsymbol{c}_1$ is selected according to (14), then the following property is ensured:

$$(\boldsymbol{A}_{c1})^n = \left[\boldsymbol{A} - \boldsymbol{b}(\boldsymbol{c}_1{}^{\mathrm{T}}\boldsymbol{b})^{-1}\boldsymbol{c}_1{}^{T}\boldsymbol{A}\right]^n = \boldsymbol{0}_{n\times n}. \tag{29}$$

Substitution of (29) into (28) yields

$$\begin{aligned}\left|\boldsymbol{q}x(k+n)\right| &= \left|\boldsymbol{q}\sum_{i=0}^{n-1} (\boldsymbol{A}_{c1})^i \boldsymbol{b}(\boldsymbol{c}_1{}^{\mathrm{T}}\boldsymbol{b})^{-1} s_1(k+n-i)\right| \\ &\le \sum_{i=0}^{n-1} \left|(\boldsymbol{c}_1{}^{\mathrm{T}}\boldsymbol{b})^{-1} s_1(k+n-i)\right| \cdot \left|\boldsymbol{q}(\boldsymbol{A}_{c1})^i \boldsymbol{b}\right|.\end{aligned} \tag{30}$$

Since $|s_1(k)| \le \varepsilon + 2\delta d_1$ for all $k \ge k_0$, one obtains

$$\begin{aligned}\left|\boldsymbol{q}\boldsymbol{x}(k+n)\right| &\le \left|(\boldsymbol{c}_1{}^{\mathrm{T}}\boldsymbol{b})^{-1}\right| \cdot \sum_{i=0}^{n-1} (\varepsilon + 2\delta d_1) \cdot \left|\boldsymbol{q}(\boldsymbol{A}_{c1})^i \boldsymbol{b}\right| \\ &= \left|(\boldsymbol{c}_1{}^{\mathrm{T}}\boldsymbol{b})^{-1}\right| \cdot (\varepsilon + 2\delta d_1) \\ &\cdot \sum_{i=0}^{n-1} \left|\boldsymbol{q}\left[\boldsymbol{A} - \boldsymbol{b}(\boldsymbol{c}_1{}^{\mathrm{T}}\boldsymbol{b})^{-1}\boldsymbol{c}_1{}^{T}\boldsymbol{A}\right]^i \boldsymbol{b}\right|.\end{aligned} \tag{31}$$

Thus, for every $k \ge k_1 = k_0 + n$, inequality (25) is satisfied. ■

3.2. Strategy for relative degree 2 systems

It has been shown that the reaching law for RD1 systems presented in the previous section confines the system state to a certain band around the switching hyperplane, and therefore ensures some degree of robustness of the considered control system. However, further in this paper, it will be demonstrated that better robustness can be obtained by choosing such a sliding variable that, with respect to this variable, the system has RD2. In this section, a modified strategy will be proposed. In order to obtain an RD2 system, vector $\boldsymbol{c}_2$ is selected so that $\boldsymbol{c}_2{}^{\mathrm{T}}\boldsymbol{b} = 0$ and $\boldsymbol{c}_2{}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{b} \ne 0$. Relation (9) demonstrates that now the sliding variable $s_2(k)$ is only affected by disturbance $d(k-2)$ and control signal $u(k-2)$. The mean of the disturbance affecting the variable and its maximum admissible deviation from the mean are defined as

$$\begin{aligned}\bar{d}_2 &= 0.5 \cdot \boldsymbol{c}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{b} \cdot (d_{\max} + d_{\min}), \\ \delta d_2 &= 0.5 \cdot \left|\boldsymbol{c}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{b}\right| \cdot (d_{\max} - d_{\min}).\end{aligned} \tag{32}$$

It has already been stated that for RD2 systems, $s(k+1)$ is exactly known at the time instant k and its value can be used in the controller design procedure. With this in mind, the reaching law which ensures advantageous properties of the system will be proposed. In the initial stage of the control process, i.e. before the switching plane is crossed for the first time, the reaching law will ensure a limited sliding variable rate of change. Furthermore, the reaching law will guarantee that once the plane is crossed, it will be crossed again in each subsequent time instant. The reaching law is expressed as

$$\begin{aligned}s_2(k+2) = {} & h[s_2(k)] \cdot h[s_2(k+1)] \cdot s_2(k+1) \\ & - 0.5 \cdot \{1 - h[s_2(k)]\} \cdot \varepsilon \mathrm{sgn}[s_2(k)] \\ & - 0.5 \cdot \{1 + h[s_2(k)]\} \cdot \varepsilon \mathrm{sgn}[s_2(k+1)] - \bar{d}_2 \\ & - \delta d_2 \mathrm{sgn}[s_2(k+1)] + \boldsymbol{c}_2{}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{b}d(k),\end{aligned} \tag{33}$$

where h is the function defined by (13) and $s_0 > 0$, $\varepsilon > 0$ are the design parameters. Vector $\boldsymbol{c}_2$ for this strategy is selected to ensure that the matrix $\boldsymbol{A}_{c2} = \boldsymbol{A} - \boldsymbol{b}(\boldsymbol{c}_2{}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{b})^{-1}\boldsymbol{c}_2{}^{\mathrm{T}}\boldsymbol{A}^2$ is nilpotent. In other words, it is chosen so that

$$\begin{aligned}& det\ (\lambda \boldsymbol{I}_n - \boldsymbol{A}_{c2}) \\ & \quad = \det\left[\lambda \boldsymbol{I}_n - \boldsymbol{A} + \boldsymbol{b}(\boldsymbol{c}_2{}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{b})^{-1}\boldsymbol{c}_2{}^{\mathrm{T}}\boldsymbol{A}^2\right] = \lambda^n.\end{aligned} \tag{34}$$

The reaching law (33) will now be applied to design a control strategy for RD2 systems. The design process is described by relations (7)-(10) and the obtained control law has the following form:

$$\begin{aligned}u(k) = (\boldsymbol{c}_2{}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{b})^{-1} \{ & h[s_2(k)] \cdot h[s_2(k+1)] \cdot s_2(k+1) \\ & -0.5 \cdot \{1 - h[s_2(k)]\} \cdot \varepsilon \mathrm{sgn}[s_2(k)] - \\ & 0.5 \cdot \{1 + h[s_2(k)]\} \cdot \varepsilon \mathrm{sgn}[s_2(k+1)] - \bar{d}_2 \\ & -\delta d_2 \mathrm{sgn}[s_2(k+1)] - \boldsymbol{c}_2{}^{\mathrm{T}}\boldsymbol{A}^2\boldsymbol{x}(k)\}.\end{aligned} \tag{35}$$

The proposed reaching law shares many similarities with the one defined by (11). However, before formally proving favourable properties ensured by the strategy for RD2 systems, an important property related to matrices $\boldsymbol{A}_{c1}$ and $\boldsymbol{A}_{c2}$ will be stated in the form of a lemma.

Lemma 3.1: *If $\boldsymbol{c}_1$ is selected according to (14) and $\boldsymbol{c}_2$ according to (34), then $\boldsymbol{A}_{c1} = \boldsymbol{A}_{c2}$.*

Proof: Let us consider the control strategy for RD1 systems obtained from the rule $s_1(k+1) = \boldsymbol{c}_1{}^{\mathrm{T}}\boldsymbol{x}(k+1) = 0$

with vector $\boldsymbol{c}_1$ selected according to (14). The strategy is

$$u(k) = -(\boldsymbol{c}_1{}^{\mathrm{T}}\boldsymbol{b})^{-1}\boldsymbol{c}_1{}^{\mathrm{T}}\boldsymbol{A}x(k) \tag{36}$$

Assuming that the system is not perturbed (i.e. $d(k) = 0$), substitution of (36) into (1) yields

$$\boldsymbol{x}(k+1) = \left[\boldsymbol{A} - \boldsymbol{b}(\boldsymbol{c}_1{}^{\mathrm{T}}\boldsymbol{b})^{-1}\boldsymbol{c}_1{}^{\mathrm{T}}\boldsymbol{A}\right]\boldsymbol{x}(k) = \boldsymbol{A}_{c1}\boldsymbol{x}(k). \tag{37}$$

Then, since vector $\boldsymbol{c}_1$ is chosen to ensure that $\boldsymbol{A}_{c1}$ is nilpotent, relation (29) implies $\boldsymbol{x}(k+n) = \boldsymbol{0}$ for any $\boldsymbol{x}(k)$, where $n = \dim\boldsymbol{A}$. Now let us consider a similar strategy for RD2 plants obtained from the rule $s_2(k+2) = \boldsymbol{c}_2{}^{\mathrm{T}}\boldsymbol{x}(k+2) = 0$, where $\boldsymbol{c}_2$ is selected according to (34). The strategy can be expressed as

$$u(k) = -(\boldsymbol{c}_2{}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{b})^{-1}\boldsymbol{c}_2{}^{\mathrm{T}}\boldsymbol{A}^2\boldsymbol{x}(k). \tag{38}$$

Substituting (38) into (1) under the assumption that the system is unperturbed, one obtains

$$\boldsymbol{x}(k+1) = \left[\boldsymbol{A} - \boldsymbol{b}(\boldsymbol{c}_2{}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{b})^{-1}\boldsymbol{c}_2{}^{\mathrm{T}}\boldsymbol{A}^2\right]\boldsymbol{x}(k) = \boldsymbol{A}_{c2}\boldsymbol{x}(k). \tag{39}$$

Since vector $\boldsymbol{c}_2$ is selected according to (34), the following property is ensured:

$$(\boldsymbol{A}_{c2})^n = \left[\boldsymbol{A} - \boldsymbol{b}(\boldsymbol{c}_2{}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{b})^{-1}\boldsymbol{c}_2{}^{\mathrm{T}}\boldsymbol{A}^2\right]^n = \boldsymbol{0}_{n\times n}. \tag{40}$$

Consequently, (39) implies $\boldsymbol{x}(k+n) = \boldsymbol{0}$ for all $\boldsymbol{x}(k)$, just like in the case of (37). However, there exists exactly one control strategy which can drive the state of an n-dimensional system from its initial conditions to $\boldsymbol{0}$ in n time instants. Therefore, values of the control signal obtained with strategies (36) and (38) must be equal to each other for all k. This gives

$$\boldsymbol{c}_1{}^{\mathrm{T}} = \alpha\boldsymbol{c}_2{}^{\mathrm{T}}\boldsymbol{A} \tag{41}$$

for a certain real $\alpha \neq 0$. Relation (41) further implies

$$\begin{aligned}\boldsymbol{A}_{c1} &= \left[\boldsymbol{A} - \boldsymbol{b}(\boldsymbol{c}_1{}^{\mathrm{T}}\boldsymbol{b})^{-1}\boldsymbol{c}_1{}^{T}\boldsymbol{A}\right]\\ &= \left[\boldsymbol{A} - \boldsymbol{b}\alpha^{-1}(\boldsymbol{c}_2{}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{b})^{-1}\alpha\boldsymbol{c}_2{}^{T}\boldsymbol{A}^2\right]\\ &= \left[\boldsymbol{A} - \boldsymbol{b}(\boldsymbol{c}_2{}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{b})^{-1}\boldsymbol{c}_2{}^{T}\boldsymbol{A}^2\right] = \boldsymbol{A}_{c2}.\end{aligned} \tag{42}$$

Thus, if $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ are selected according to (14) and (34), respectively, then $\boldsymbol{A}_{c1} = \boldsymbol{A}_{c2}$. ■

It should be noted that vectors $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ can be scaled without altering the values of control signals (15) and (35). Therefore, in all future derivations it will be assumed that $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ are selected to ensure that parameter α in relation (41) is equal to 1. This in turn gives $\boldsymbol{c}_1{}^{\mathrm{T}} = \boldsymbol{c}_2{}^{\mathrm{T}}\boldsymbol{A}$, $\bar{d}_1 = \bar{d}_2$ and $\delta d_1 = \delta d_2$. With this in mind, it can be seen that when $|s_2(k)|$ and $|s_2(k+1)|$ are greater than s_0, the sliding variable convergence rate is upper bounded by $\varepsilon + 2\delta d_2$. Therefore, reaching laws (11) and (33) limit the sliding variable rate of change in a similar fashion when the variable is far off the switching plane. It will now be shown that, just like in the case of RD1 systems, the proposed reaching law ensures that the variable changes its sign in each sampling instant after changing it for the first time. It will be further demonstrated that reaching law (33) drives the system state to a narrower band around the switching hyperplane than the reaching law (11) under the same constraints. Finally, it will be shown that the proposed reaching law confines the system output to a smaller vicinity of zero than the law (11) for RD1 systems. These properties will be formally proven in the following theorems.

Theorem 3.4: *If the reaching law for system (1) is defined by (33), then the system representative point will either cross the switching plane or arrive on it in finite time.*

Proof: An inequality similar to (17) is first derived from (32):

$$\left|\boldsymbol{c}_2{}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{b}d(k) - \bar{d}_2\right| \leq \delta d_2 \tag{43}$$

Now let $\boldsymbol{x}(k)$ be any state such that $s_2(k) > 0$ and $s_2(k+1) > 0$. Since function h is upper bounded by 1, relations (33) and (43) imply

$$\begin{aligned}s_2(k+2) &= h[s_2(k)]\cdot h[s_2(k+1)]\cdot s_2(k+1)\\ &\quad - 0.5\cdot 2\cdot\varepsilon - \bar{d}_2 - \delta d_2 + \boldsymbol{c}_2{}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{b}d(k)\\ &\leq s_2(k+1) - \varepsilon + \delta d_2 - \delta d_2 = s_2(k+1) - \varepsilon.\end{aligned} \tag{44}$$

Therefore, if $\boldsymbol{x}(k)$ is such a state that $s_2(k) > 0$ and $s_2(k+1) > 0$, then the sliding variable will decrease by at least ε in the next time instant. Thus, the variable will either become negative or equal to zero in finite time. Now let $\boldsymbol{x}(k)$ be such a state that $s_2(k) < 0$ and $s_2(k+1) < 0$. Similarly as in (44), relations (33) and (43) give $s_2(k+2) \geq s_2(k+1) + \varepsilon$. Consequently, the sliding variable will increase by at least ε in the next sampling instant, which means it is guaranteed to become 0 or positive in finite time. In conclusion, for any state $\boldsymbol{x}(k)$, the variable $s_2(k)$ will be reduced to 0 or change its sign in finite time. ■

Next, it will be shown that after crossing the switching plane for the first time, the system representative point will cross it again in each subsequent time instant and become confined to a layer around the switching surface. A certain helpful property will be first brought up in the form of a lemma.

Lemma 3.2: *If* $\mathit{sgn}[s_2(k+2)] = -\mathit{sgn}[s_2(k+1)]$, *then* $|s_2(k+2)| \leq \varepsilon + 2\delta d_2$.

Proof: Let $\boldsymbol{x}(k)$ be any state such that $s_2(k) > 0$ and $s_2(k+1) > 0$. Since function h is non-negative, relations

(33) and (43) give

$$\begin{aligned} s_2(k+2) &= h[s_2(k)]\cdot h[s_2(k+1)]\cdot s_2(k+1) \\ &\quad -0.5\cdot 2\cdot\varepsilon - \bar{d}_2 - \delta d_2 + \boldsymbol{c}_2{}^{\mathrm{T}}\boldsymbol{Ab}d(k) \\ &\geq 0\cdot s_2(k+1) - \varepsilon - \delta d_2 - \delta d_2 = -\varepsilon - 2\delta d_2. \end{aligned} \tag{45}$$

Furthermore, the assumption of the lemma states that $\mathrm{sgn}[s_2(k+2)] = -\mathrm{sgn}[s_2(k+1)]$, which together with relation (45) gives $0 > s_2(k+2) \geq -\varepsilon - 2\delta d_2$. Through the analogy with (45), if $\boldsymbol{x}(k)$ is such a state that $s_2(k) < 0$ and $s_2(k+1) < 0$, then taking the assumption of the lemma into account one obtains $0 < s_2(k+2) \leq \varepsilon + 2\delta d_2$. Now let $\boldsymbol{x}(k)$ be any state such that $s_2(k) > 0$ and $s_2(k+1) < 0$. Relations (33) and (43) imply

$$\begin{aligned} s_2(k+2) &= h[s_2(k)]\cdot h[s_2(k+1)]\cdot s_2(k+1) \\ &\quad - 0.5\cdot\{1 - h[s_2(k)]\}\cdot\varepsilon + 0.5\cdot\{1 + h[s_2(k)]\} \\ &\quad \cdot\varepsilon - \bar{d}_2 + \delta d_2 + \boldsymbol{c}_2{}^{\mathrm{T}}\boldsymbol{Ab}d(k) \leq h[s_2(k)] \\ &\quad \cdot\{h[s_2(k+1)]\cdot s_2(k+1) + \varepsilon\} + 2\delta d_2 \leq \varepsilon + 2\delta d_2. \end{aligned} \tag{46}$$

The assumption $\mathrm{sgn}[s_2(k+2)] = -\mathrm{sgn}[s_2(k+1)]$, together with relation (46), gives $0 < s_2(k+2) \leq \varepsilon + 2\delta d_2$. Through the analogy with (46) if $\boldsymbol{x}(k)$ is such a state that $s_2(k) < 0$ and $s_2(k+1) > 0$, one obtains $0 > s_2(k+2) \geq -\varepsilon - 2\delta d_2$. In conclusion, if the switching plane has been crossed between time instants $k+1$ and $k+2$, then $|s_2(k+2)| \leq \varepsilon + 2\delta d_2$. ■

Theorem 3.5: *If the reaching law for system (1) is defined by (33), parameter $s_0 > (\varepsilon + 2\delta d_2)^2\varepsilon^{-1}$ and k_0 is the first time instant such that $\mathrm{sgn}[s_2(k_0+1)] = -\mathrm{sgn}[s_2(k_0)]$ then the system representative point will cross the switching plane in each step after k_0, i.e. for all $k \geq k_0$ $\mathrm{sgn}[s_2(k+2)] = -\mathrm{sgn}[s_2(k+1)]$ is ensured.*

Proof: To demonstrate that the representative point will cross the switching plane again after crossing it for the first time, it will be shown that if $s_2(k)$ and $s_2(k+1)$ have opposite signs, then $s_2(k+1)$ and $s_2(k+2)$ will also have opposite signs. First let $\boldsymbol{x}(k)$ be such a state that $s_2(k) > 0$ and $s_2(k+1) < 0$. Then, Lemma 3.2 states that $s_2(k+1) \geq -\varepsilon - 2\delta d_2$. It will be shown that $s_2(k+2) > 0$, which means that the sliding variable changes its sign again after changing it in the previous step. Indeed, relation (33) gives

$$\begin{aligned} s_2(k+2) &= h[s_2(k)]\cdot h[s_2(k+1)]\cdot s_2(k+1) - 0.5 \\ &\quad \cdot\{1 - h[s_2(k)]\}\cdot\varepsilon + 0.5\cdot\{1 + h[s_2(k)]\} \\ &\quad \cdot\varepsilon - \bar{d}_2 + \delta d_2 + \boldsymbol{c}_2{}^{\mathrm{T}}\boldsymbol{Ab}d(k) \\ &\geq h[s_2(k)]\cdot\{h[s_2(k+1)]\cdot s_2(k+1) + \varepsilon\}. \end{aligned} \tag{47}$$

Since $s_0 > (\varepsilon + 2\delta d_2)^2\varepsilon^{-1}$ and function h is non-negative, relation (47) further implies

$$\begin{aligned} s_2(k+2) &\geq h[s_2(k)]\cdot\left[\frac{\varepsilon + 2\delta d_2}{s_0}(-\varepsilon - 2\delta d_2) + \varepsilon\right] \\ &= h[s_2(k)]\cdot\left[-\frac{(\varepsilon + 2\delta d_2)^2}{s_0} + \varepsilon\right] \\ &> h[s_2(k)]\cdot\left[-\frac{(\varepsilon + 2\delta d_2)^2\varepsilon}{(\varepsilon + 2\delta d_2)^2} + \varepsilon\right] = 0. \end{aligned} \tag{48}$$

Therefore, the variable will cross the switching plane again if $s_2(k) > 0$ and $s_2(k+1) < 0$. Now let $\boldsymbol{x}(k)$ be such a state that $s_2(k) < 0$ and $s_2(k+1) > 0$. Then, Lemma 3.2 implies $s_2(k+1) \leq \varepsilon + 2\delta d_2$. Similarly as in relations (47) and (48), one obtains $s_2(k+2) \leq 0$. Thus, the variable will cross the plane again if $s_2(k) < 0$ and $s_2(k+1) > 0$. In conclusion, the system representative point will keep crossing the switching plane in each step after crossing it for the first time. ■

Remark 3.2: Once again, the case where the sliding variable reaches 0 will be analysed separately. First let $s_2(k)$ be any real value and let $s_2(k+1) = 0$. Since $h[s_2(k+1)] = 0$ and $\mathrm{sgn}[s_2(k+1)] = 0$, relation (33) gives

$$\begin{aligned} |s_2(k+2)| &= |-0.5\cdot\{1 - h[s_2(k)]\}\cdot\varepsilon\mathrm{sgn}[s_2(k)] \\ &\quad - \bar{d}_2 + \boldsymbol{c}_2{}^{\mathrm{T}}\boldsymbol{Ab}d(k)| \leq 0.5\cdot\varepsilon + \delta d_2 \\ &< \varepsilon + 2\delta d_2 \end{aligned} \tag{49}$$

Therefore, if $s_2(k+1) = 0$, $|s_2(k+2)|$ is smaller than $\varepsilon + 2\delta d_2$ and can potentially be 0. If the sliding variable equals 0 in two subsequent steps and it happens that the disturbance is constantly equal to $(\boldsymbol{c}_2{}^{\mathrm{T}}\boldsymbol{Ab})^{-1}\bar{d}_2$, then the system achieves the ideal sliding motion. On the other hand, if $d(k)$ assumes a value different from $(\boldsymbol{c}_2{}^{\mathrm{T}}\boldsymbol{Ab})^{-1}\bar{d}_2$ in finite time k, then the sliding variable $s_2(k+2)$ becomes nonzero. Now let $s_2(k+1) = 0$ and $s_2(k+2) > 0$. Since $h[s_2(k+1)] = 0$ and $\mathrm{sgn}[s_2(k+1)] = 0$, relation (33) implies

$$\begin{aligned} s_2(k+3) &= -0.5\cdot\varepsilon\mathrm{sgn}[s_2(k+2)] - \bar{d}_2 \\ &\quad - \delta d_2\mathrm{sgn}[s_2(k+2)] + \boldsymbol{c}_2{}^{\mathrm{T}}\boldsymbol{Ab}d(k) \\ &\leq -0.5\cdot\varepsilon + \delta d_2 - \delta d_2 = -0.5\cdot\varepsilon < 0. \end{aligned} \tag{50}$$

Thus, the representative point will cross the switching hyperplane in the next step. Then, according to Theorem 3.5, the hyperplane will be crossed in each subsequent step afterwards. Likewise, if $s_2(k+1) = 0$ and $s_2(k+2) < 0$, then through the analogy with relation (50), one obtains $s_2(k+3) > 0$, which also ensures that the switching hyperplane has been crossed.

It has been shown that the modified strategy for RD2 plants either makes the system achieve the ideal sliding motion or drives its representative point to cross the switching plane in finite time and to cross it again in each subsequent step. Next it will be demonstrated that the proposed reaching law drives the system state to a band around the switching hyperplane. Furthermore, the width of the band obtained with the reaching law (33) is strictly smaller than the one given by the strategy for RD1 systems (if $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ are selected to ensure $\delta d_1 = \delta d_2$). The reduced width of the quasi-sliding mode band is an essential advantage of the proposed reaching law for RD2 systems, and it will be reflected in a decreased system output error.

Theorem 3.6: *If the reaching law for system (1) is defined by (33), parameter $s_0 > (\varepsilon + 2\delta d_2)^2\varepsilon^{-1}$ and the system state belongs to the quasi-sliding mode band*

$$B_2 = \left\{ \boldsymbol{x} : \left|\boldsymbol{c}_2{}^{\mathrm{T}}\boldsymbol{x}\right| \le \frac{2\delta d_2 s_0}{s_0 - \varepsilon} \right\}, \tag{51}$$

in any two subsequent sampling instants, then the state will remain inside the band for all future steps.

Proof: Let $\boldsymbol{x}(k)$ be such a system state that $|s_2(k)|$ and $|s_2(k+1)|$ are not greater than $2\delta d_2 s_0(s_0 - \varepsilon)^{-1}$. It will be shown that for any $j \ge 2$, $|s_2(k+j)|$ also does not exceed that value. The proof will only be conducted for the case where $s_2(k) > 0$ and $s_2(k+1) < 0$, since the analysis for the reverse case is almost identical. Relation (33) implies

$$\begin{aligned} s_2(k+2) &= h[s_2(k)] \cdot \{h[s_2(k+1)] \cdot s_2(k+1) + \varepsilon\} \\ &\quad - \bar{d}_2 + \delta d_2 + \boldsymbol{c}_2{}^{\mathrm{T}}\boldsymbol{Abd}(k) \\ &\le h[s_2(k)] \cdot \varepsilon + 2\delta d_2 \le \frac{1}{s_0}\left|\frac{2\delta d_2 s_0}{s_0 - \varepsilon}\right| \cdot \varepsilon + 2\delta d_2 \\ &= \frac{2\delta d_2 \varepsilon}{s_0 - \varepsilon} + 2\delta d_2 = \frac{2\delta d_2\varepsilon + 2\delta d_2 s_0 - 2\delta d_2\varepsilon}{s_0 - \varepsilon} \\ &= \frac{2\delta d_2 s_0}{s_0 - \varepsilon}. \end{aligned} \tag{52}$$

Therefore, $s_2(k+2) \le 2\delta d_2 s_0(s_0 - \varepsilon)^{-1}$. Furthermore, Theorem 3.5 implies that $s_2(k+2) > 0$. Consequently, relation (33) gives

$$\begin{aligned} &s_2(k+3) \\ &= h[s_2(k+1)] \cdot \{h[s_2(k+2)] \cdot s_2(k+2) - \varepsilon\} \\ &\quad - \bar{d}_2 + \delta d_2 + \boldsymbol{c}_2{}^{\mathrm{T}}\boldsymbol{Abd}(k) \\ &\ge h[s_2(k)] \cdot (-\varepsilon) - 2\delta d_2 \ge \frac{1}{s_0}\left|\frac{2\delta d_2 s_0}{s_0 - \varepsilon}\right| \\ &\quad \cdot (-\varepsilon) - 2\delta d_2 \\ &= -\frac{2\delta d_2\varepsilon}{s_0 - \varepsilon} - 2\delta d_2 = -\frac{-2\delta d_2\varepsilon - 2\delta d_2 s_0 + 2\delta d_2 \varepsilon}{s_0 - \varepsilon} \\ &= -\frac{2\delta d_2 s_0}{s_0 - \varepsilon}. \end{aligned} \tag{53}$$

Since Theorem 3.5 implies that the variable changes its sign in each step, repeating the derivation (52) for $j = 4, 6, 8, \ldots$ and the derivation (53) for $j = 5, 7, 9, \ldots$, one obtains $|s_2(k+j)| \le 2\delta d_2 s_0(s_0 - \varepsilon)^{-1}$ for any $j \ge 2$. Thus, the state remains confined to the band (51) for all future steps.

It will now be demonstrated that the quasi-sliding mode band (51) is always reached. To that end, the following theorem will be proven. ∎

Theorem 3.7: *If the reaching law for system (1) is defined by (33) and parameter $s_0 > (\varepsilon + 2\delta d_2)^2\varepsilon^{-1}$, then the system state will approach the quasi-sliding mode band (51) at least asymptotically.*

Proof: Let $\boldsymbol{x}(k)$ be such a state out of the band (51) that $|s_2(k)| < \varepsilon + 2\delta d_2$ and $\operatorname{sgn}[s_2(k)] = -\operatorname{sgn}[s_2(k+1)]$. The existence of such a state for a finite k is ensured by Theorem 3.4 and Lemma 3.2. Let p_j denote $|s_2(k+2j)|$ for $j = 0, 1, 2, \ldots$, which means that $p_0 = |s_2(k)|$. Relation (33) implies

$$\begin{aligned} |s_2(k+2)| &= |h[s_2(k)]\{h[s_2(k+1)] \cdot s_2(k+1) \\ &\quad - \varepsilon\operatorname{sgn}[s_2(k+1)]\} - \bar{d}_2 + \delta d_2 \\ &\quad + \boldsymbol{c}_2{}^{\mathrm{T}}\boldsymbol{Abd}(k)| \\ &\le h[s_2(k)] \cdot \varepsilon + \left|-\bar{d}_2 + \delta d_2 + \boldsymbol{c}_2{}^{\mathrm{T}}\boldsymbol{Abd}(k)\right| \\ &\le \frac{p_0}{s_0} \cdot \varepsilon + 2\delta d_2. \end{aligned} \tag{54}$$

Lemma 3.2 and Theorem 3.5 imply that $|s_2(l)| < \varepsilon + 2\delta d_2$ for all $l > k$. Therefore, one obtains

$$\begin{aligned} |s_2(k+2j)| &\le h[s_2(k+2j-2)] \cdot \varepsilon + |\bar{d}_2 + \delta d_2 \\ &\quad - \boldsymbol{c}_2{}^{\mathrm{T}}\boldsymbol{Abd}(k+2j-2)| \\ &\le \frac{p_{j-1}}{s_0} \cdot \varepsilon + 2\delta d_2. \end{aligned} \tag{55}$$

Thus, p_j can be estimated in the following way:

$$\begin{aligned} p_j &\le p_{j-1}\frac{\varepsilon}{s_0} + 2\delta d_2 \le p_{j-2}\frac{\varepsilon^2}{s_0{}^2} + \frac{\varepsilon}{s_0}2\delta d_2 + 2\delta d_2 \le \ldots \le \\ &\le p_0\frac{\varepsilon^j}{s_0{}^j} + \frac{\varepsilon^{j-1}}{s_0{}^{j-1}}2\delta d_2 + \cdots + \frac{\varepsilon}{s_0}2\delta d_2 + 2\delta d_2. \end{aligned} \tag{56}$$

Since p_0 is constant and $\varepsilon < s_0$, one obtains

$$\begin{aligned} \lim_{j\to\infty} |s_2(k+2j)| &= \lim_{j\to\infty} p_j \le \lim_{j\to\infty}\left(p_0\frac{\varepsilon^j}{s_0{}^j} + \sum_{i=0}^{j-1}\frac{\varepsilon^i}{s_0{}^i}2\delta d_2\right) \\ &= \left(1 - \frac{\varepsilon}{s_0}\right)^{-1} 2\delta d_2 = \frac{2\delta d_2 s_0}{s_0 - \varepsilon}. \end{aligned} \tag{57}$$

Therefore, the state $\boldsymbol{x}(k)$ always converges from $\boldsymbol{x}(0)$ to the band (51) at least asymptotically. ∎

It should be noted that the band B_2 presented in Theorem 3.4 is always strictly smaller than the band B_1 ensured by strategy (11) for RD1 systems. Indeed, since $s_0 > \varepsilon + 2\delta d_2$, keeping in mind that $\delta d_1 = \delta d_2$ one obtains the difference between the width of the two bands as

$$(\varepsilon + 2\delta d_1) - \left(\frac{2\delta d_2 s_0}{s_0 - \varepsilon}\right) = \varepsilon - \frac{2\delta d_2 \varepsilon}{s_0 - \varepsilon} > \varepsilon - \frac{2\delta d_2 \varepsilon}{\varepsilon + 2\delta d_2 - \varepsilon} = 0. \tag{58}$$

It has been demonstrated that the proposed strategy for RD2 systems drives the sliding variable into a narrower band than a similar strategy for RD1 plants. It will now be shown that the control strategy (35) confines the system output to a certain vicinity of 0. Furthermore, the size of the vicinity will be strictly smaller than the one given in Theorem 3.3 for RD1 systems. This property will be demonstrated in the following theorem.

Theorem 3.8: *If the control strategy for system (1) is defined by (35) with vector $\mathbf{c}_2$ selected according to (34), then for any $\gamma > 0$ there exists a finite k_2 such that*

$$\begin{aligned}|y(k)| = |\mathbf{q}x(k)| \le \upsilon_2 &= \left|(\mathbf{c}_2{}^{\mathrm{T}}\mathbf{A}\mathbf{b})^{-1}\right| \cdot \left(\frac{2\delta d_2 s_0}{s_0 - \varepsilon} + \gamma\right) \cdot \sum_{i=0}^{n-1} \left|\mathbf{q}(\mathbf{A}_{c2})^i \mathbf{b}\right| \\ &= \left|(\mathbf{c}_2{}^{\mathrm{T}}\mathbf{A}\mathbf{b})^{-1}\right| \cdot \left(\frac{2\delta d_2 s_0}{s_0 - \varepsilon} + \gamma\right) \cdot \sum_{i=0}^{n-1} \left|\mathbf{q}\left[\mathbf{A} - \mathbf{b}(\mathbf{c}_2{}^{\mathrm{T}}\mathbf{A}\mathbf{b})^{-1}\mathbf{c}_2{}^{T}\mathbf{A}^2\right]^i \mathbf{b}\right|. \end{aligned} \tag{59}$$

Proof: Let γ be any arbitrarily small positive number. Theorem 3.7 states that for any initial state, the absolute value of the sliding variable s converges to $2\delta d_2 s_0(s_0 - \varepsilon)^{-1}$ at least asymptotically. Thus, there exists k_0 such that $|s_2(k)| \le 2\delta d_2 s_0(s_0 - \varepsilon)^{-1} + \gamma$ for all $k > k_0$. Substitution of (35) into (1) yields

$$\begin{aligned}\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{b}d(k) + \mathbf{b}\left(\mathbf{c}_2{}^{\mathrm{T}}\mathbf{A}\mathbf{b}\right)^{-1}\{&h[s_2(k)] \cdot h[s_2(k+1)] \cdot s_2(k+1) - \{1 - h[s_2(k)]\} \\ &\cdot \mathrm{sgn}[s_2(k)] - \{1 - h[s_2(k)]\} \cdot \mathrm{sgn}[s_2(k+1)] \\ &- \bar{d}_2 - \delta d_2 \mathrm{sgn}[s_2(k+1)] + \mathbf{c}_2{}^{\mathrm{T}}\mathbf{A}^2\, \mathbf{x}(k)\}. \end{aligned} \tag{60}$$

Taking into account (33), relation (60) can be rewritten as

$$\begin{aligned}\mathbf{x}(k+1) &= \left[\mathbf{A} - \mathbf{b}(\mathbf{c}_2{}^{\mathrm{T}}\mathbf{A}\mathbf{b})^{-1}\mathbf{c}_2{}^{\mathrm{T}}\mathbf{A}^2\right]\mathbf{x}(k) + \mathbf{b}(\mathbf{c}_2{}^{\mathrm{T}}\mathbf{A}\mathbf{b})^{-1}s_2(k+2) \\ &= \mathbf{A}_{c2}\mathbf{x}(k) + \mathbf{b}(\mathbf{c}_2{}^{\mathrm{T}}\mathbf{A}\mathbf{b})^{-1}s_2(k+2). \end{aligned} \tag{61}$$

Consequently, for $n = \dim \mathbf{A}$

$$\mathbf{x}(k+n) = (\mathbf{A}_{c2})^n \mathbf{x}(k) + \sum_{i=0}^{n-1} (\mathbf{A}_{c2})^i \mathbf{b}(\mathbf{c}_2{}^{\mathrm{T}}\mathbf{A}\mathbf{b})^{-1} \times s(k+n-i+1) \tag{62}$$

Substitution of (40) into (62) yields

$$\begin{aligned}\left|\mathbf{q}x(k+n)\right| &= \left|\mathbf{q}\sum_{i=0}^{n-1}(\mathbf{A}_{c2})^i\mathbf{b}(\mathbf{c}_2{}^{\mathrm{T}}\mathbf{A}\mathbf{b})^{-1}s_2(k+n-i+1)\right| \\ &\le \left|(\mathbf{c}_2{}^{\mathrm{T}}\mathbf{A}\mathbf{b})^{-1}\right| \cdot \sum_{i=0}^{n-1}\left|\mathbf{q}(\mathbf{A}_{c2})^i\mathbf{b}s_2(k+n-i+1)\right| \end{aligned} \tag{63}$$

Since $s_2(k) \le 2\delta d_2 s_0(s_0 - \varepsilon)^{-1} + \gamma$ for all $k > k_0$, one obtains

$$|y(k)| = |\mathbf{q}x(k+n)| \le \left|(\mathbf{c}_2{}^{\mathrm{T}}\mathbf{A}\mathbf{b})^{-1}\right| \cdot \left(\frac{2\delta d_2 s_0}{s_0 - \varepsilon} + \gamma\right)\sum_{i=0}^{n-1}\left|\mathbf{q}(\mathbf{A}_{c2})^i\mathbf{b}\right| = \upsilon_2 \tag{64}$$

Therefore, there exists $k_2 = k_0 + n$ such that inequality (59) holds for all $k \ge k_2$. ■

Since vectors $\mathbf{c}_1$ and $\mathbf{c}_2$ are selected to ensure $\delta d_1 = \delta d_2$ and Lemma 3.1 states that $\mathbf{A}_{c1} = \mathbf{A}_{c2}$, the bound (59) can be rewritten as

$$\begin{aligned}\upsilon_2 &= \left|(\mathbf{c}_2{}^{\mathrm{T}}\mathbf{A}\mathbf{b})^{-1}\right| \cdot \left(\frac{2\delta d_2 s_0}{s_0 - \varepsilon} + \gamma\right)\sum_{i=0}^{n-1}\left|\mathbf{q}\mathbf{A}_{c2}{}^i\mathbf{b}\right| \\ &= \left|(\mathbf{c}_1{}^{\mathrm{T}}\mathbf{b})^{-1}\right| \cdot \left(\frac{2\delta d_2 s_0}{s_0 - \varepsilon} + \gamma\right)\sum_{i=0}^{n-1}\left|\mathbf{q}\mathbf{A}_{c1}{}^i\mathbf{b}\right|. \end{aligned} \tag{65}$$

Relation (58) illustrates that $\varepsilon + 2\delta d_1$ is greater than $2\delta d_2 s_0(s_0 - \varepsilon)^{-1}$. Thus, since γ is arbitrarily small, one obtains

$$\begin{aligned}\upsilon_2 &= \left|(\mathbf{c}_1{}^{\mathrm{T}}\mathbf{b})^{-1}\right| \cdot \left(\frac{2\delta d_2 s_0}{s_0 - \varepsilon} + \gamma\right)\sum_{i=0}^{n-1}\left|\mathbf{q}\mathbf{A}_{c1}{}^i\mathbf{b}\right| \\ &< \left|(\mathbf{c}_1{}^{\mathrm{T}}\mathbf{b})^{-1}\right| \cdot (\varepsilon + 2\delta d_1)\sum_{i=0}^{n-1}\left|\mathbf{q}\mathbf{A}_{c1}{}^i\mathbf{b}\right| = \upsilon_1. \end{aligned} \tag{66}$$

On the right hand side of relation (66), one observes the bound (25) formerly presented in Theorem 3.3 for RD1 systems. One concludes from relations (65) and (66) that the bound (59) given in Theorem 3.8 is strictly smaller than the bound (25). Therefore, the proposed strategy for RD2 systems ensures reduced system output error when compared with the strategy for RD1 plants.

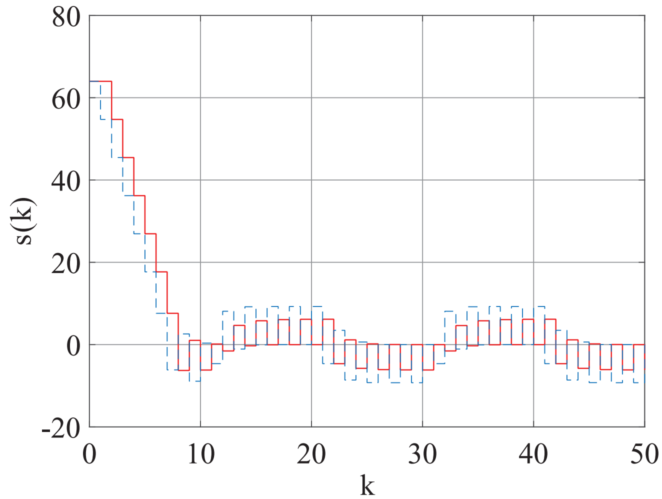

Figure 2. Sliding variable.

4. Simulation example

The strategies for RD1 and RD2 systems will now be compared by means of simulation example. The following continuous time system will be considered

$$\dot{x}(t) = \boldsymbol{A}^c \boldsymbol{x}(t) + \boldsymbol{b}^c u(t) + \boldsymbol{b}^c d(t)$$
$$y(t) = \boldsymbol{q} x(t),$$
$$\boldsymbol{A}^c = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -0.5 \end{bmatrix}, \quad \boldsymbol{b}^c = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \boldsymbol{q}^{\mathrm{T}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \tag{67}$$

The system is subject to disturbance $d(t) = (-1)^{\lfloor t/10 \rfloor + 1}$ and the goal is to drive its state from its initial position $\boldsymbol{x}(0) = [20\ 0\ 0]^{\mathrm{T}}$ to zero. To that end, the continuous time plant (67) is first discretised with a sampling period $T = 1$. Consequently, it can be expressed as a discrete-time system (1) with the following parameters

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0.8592 & 0.3929 \\ 0 & 0.6071 & 0.6627 \\ 0 & -0.6627 & 0.2757 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 0.1408 \\ 0.3929 \\ 0.6627 \end{bmatrix},$$
$$\boldsymbol{q}^{\mathrm{T}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \tag{68}$$

Strategies (15) and (35) will now be applied to the discretised plant. First, vectors $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ for both strategies are selected according to (14) and (34), respectively, which gives $\boldsymbol{c}_1 = [3.199\ 3.06\ 1]$ and $\boldsymbol{c}_2 = [3.199\ -0.139\ -0.597]$. Consequently, relation (12) gives $\bar{d}_1 = 0$ and $\delta d_1 = 2.315$. Parameters for both strategies are selected as

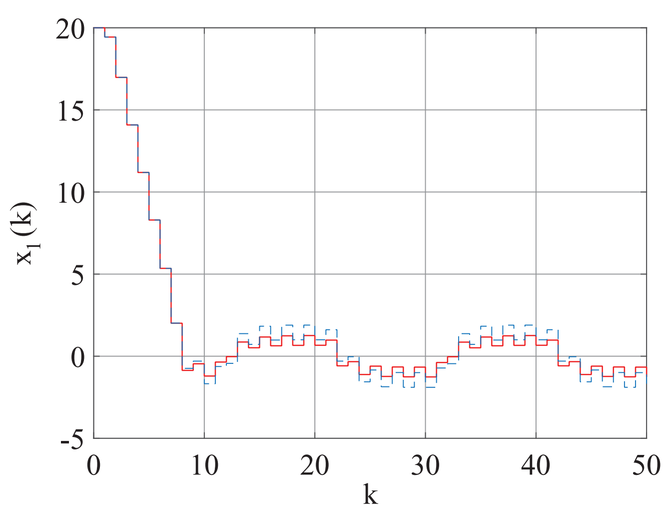

Figure 3. System output.

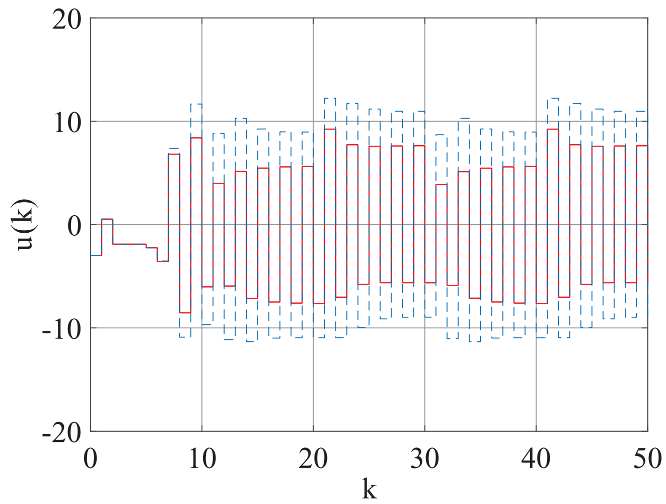

Figure 4. Control signal.

$\varepsilon = 2\delta d_1 = 6.94$ and $s_0 = 27.77$, which satisfies assumptions of Theorems 3.2 and 3.5. Figure 2 illustrates the evolution of the sliding variable for both strategies, Figure 3 shows system output and Figure 4 illustrates the control signal. In all figures, the blue dashed line represents the result for strategy (15) and the solid red line for strategy (35).

It can be seen from Figure 2 that strategy (35) drives the state to a narrower band than the strategy (15). Indeed, strategy (15) confines the state to a band limited by ± 9.26, exactly as stated in Theorem 3.1. On the other hand, strategy (35) drives the state to a band limited by ± 6.173, which is consistent with Theorem 3.6. Figure 3 shows that strategy (35) drives the system output to a narrower vicinity of its target value than strategy (15). For strategy (15), $|y(k)|$ is not greater than 2.895 as stated in Theorem 3.3. For strategy (35) it is smaller than 1.929, which is consistent with Theorem 3.8. Finally, Figure 4 illustrates that the strategy for RD2 systems requires less control effort in the sliding phase.

5. Conclusions

In this paper, switching type reaching law-based control strategies for discrete-time systems have been considered. Rather than following classic sliding mode controller design methodology, the case of RD2 systems has been considered and a design procedure for such plants has been presented. Then, a new reaching law for RD2 systems has been introduced and shown to ensure several desirable properties of the system. The new reaching law has been compared with a similar strategy for RD1 systems. It has been demonstrated that the strategy for RD2 systems offers a reduced quasi-sliding mode band width and system output error.

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