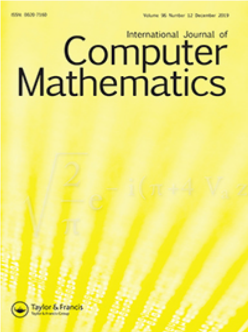

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ARTICLE

Solvability of a non-linear Cauchy problem for an elliptic equation

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ABSTRACT
We study a non-linear operator equation originating from a Cauchy problem for an elliptic equation. The problem appears in applications where surface measurements are used to calculate the temperature below the earth surface. The Cauchy problem is ill-posed and small perturbations to the used data can result in large changes in the solution. Since the problem is non-linear certain assumptions on the coefficients are needed. We reformulate the problem as an non-linear operator equation and show that under suitable assumptions the operator is well-defined. The proof is based on making a change of variables and removing the non-linearity from the leading term of the equation. As a part of the proof we obtain an iterative procedure that is convergent and can be used for evaluating the operator. Numerical results show that the proposed procedure converges faster than a simple fixed point iteration for the equation in the the original variables.

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1. Introduction

In many cases inverse problems can be formulated as operator equations, where the operator is evaluated by solving a boundary value problem with certain data prescribed on the boundary. Examples include the Cauchy problem for the Laplace equation [3,13,17], the Cauchy problem for the Helmholtz equation [4,9,19,20], corrosion detection [1,15], inverse scattering problems [10,23] and in electrical impedance tomography [11,12]. See also [22] and 6 where genetic algorithms were used for solving ill-posed Cauchy problems.

In our previous work we studied the inverse geothermal problem [5], where measurements at the surface level, are used to estimate the stationary temperature profile below the earths surface, see also [7], by solving a Cauchy problem for the heat equation.

Let $x = (x_1, x_2) \in \mathbb{R}^2$ and $\Omega \subset \mathbb{R}^2$ be a two dimensional domain whose boundary $\partial\Omega$ consists of $\Gamma_0, \Gamma_1, \Gamma_2$ and Γ_3 such that $\partial\Omega = \overline{\Gamma_0} \cup \overline{\Gamma_1} \cup \overline{\Gamma_2} \cup \overline{\Gamma_3}$. The boundaries Γ_0, Γ_1 are assumed to be Lipschitz continuous curves and Γ_2, Γ_3 are parallel straight lines of equal length and ℓ distance apart, see Figure 1. The non-linear Cauchy problem under consideration is

$$\begin{aligned} &-\nabla \cdot (k(x,T)\nabla T) + b(x,T) = 0 && \text{in } \Omega \\ &T = \phi_0 && \text{on } \Gamma_0, \\ &\vec{n} \cdot k\nabla T = \phi_1 && \text{on } \Gamma_0, \end{aligned} \tag{1}$$

where $b = b(x,T) > 0$ is the heat production term, $k = k(x,T)$ is the thermal conductivity, $\vec{n}$ is the unit normal to the boundary $\partial\Omega$ and $T = T(x_1, x_2)$ is the sought temperature distribution. In this

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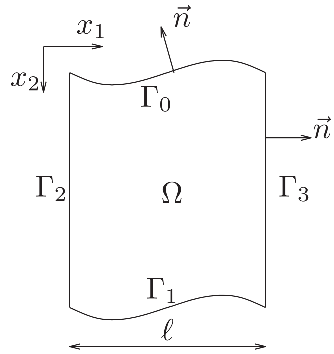

Figure 1. The domain Ω and its boundary.

work we will always assume that the equation is elliptic, i.e. there exists constants k_0 and k_1 such that

$$0 < k_0 \leq k(x, T) \leq k_1 < \infty, \text{ for } x \in \overline{\Omega},\ T \in \mathbb{R}. \tag{2}$$

In order to simplify the analysis of the Cauchy problem (1) we assume that the solution is periodic in x_1 with period ℓ, and that the boundary curves Γ_1 and Γ_0 can be extended as periodic functions for $x_1 \in \mathbb{R}$. There are two ways to think about periodic solutions for an equation. The first option is that points (x_1, x_2) on the boundaries Γ_2 and Γ_3 are considered interior points, where the differential equation is valid. The second option is to use the periodicity assumption to supply boundary conditions to the curves Γ_2 and Γ_3. In particular, we require that

$$T(x_1, x_2) = T(x_1 + \ell, x_2). \tag{3}$$

The second option represents a weaker assumption and is sufficient for our study. Thus, we introduce a space $H^1_{per}(\Omega)$ consisting of all functions in $H^1(\Omega)$ that satisfies the periodicity condition (3).

In our previous work [5] we solved the Cauchy problem (1) by reformulating it as an operator equation. If we require the solution to be a function in $H^1_{per}(\Omega)$ then we we have the following direct problem:

$$\begin{array}{ll} -\nabla \cdot (k(x, T)\nabla T) + b(x, T) = 0, & \text{in } \Omega \\ T = \phi_0 & \text{on } \Gamma_0, \\ \vec{n} \cdot k\nabla T = \psi_1 & \text{on } \Gamma_1, \end{array} \tag{4}$$

with an arbitrary heat flux ψ_1 on Γ_1. The Cauchy problem (1) is then replaced by the operator equation

$$K(\phi_0, \psi_1) := \vec{n} \cdot k\nabla T|_{\Gamma_0} = \phi_1, \tag{5}$$

where T is a solution to the direct problem and ϕ_0 is known boundary data.

Our primary interest is in solving the inverse problem, e.g. (1), and in order to prove solvability for the inverse problem we need to prove existence, and uniqueness for the direct problem (4). Since the problem is non-linear this is difficult unless the coefficients $k(x, T)$ and $b(x, T)$ satisfy certain conditions. In [18], uniqueness is proved for a similar equation in the case of a separable conductivity. See also [14], where uniqueness for a semilinear elliptic equation is investigated, and [2] for additional uniqueness results. In this work we specifically attempt to find the weakest possible restrictions on k and b that still allows us to prove that the operator K is well-defined. The main result is that the operator K, cf. (5), is well-defined if certain bounds for $\nabla_x k(x, T)$ holds, while the coefficient $k(x, T)$ may be discontinuous with respect to T. As a part of the proof we develop a convergent iterative procedure that lets us solve the non-linear problem (4).

This paper is organized as follows: In Subsection 2.1, we outline the function spaces used. In Subsection 2.2, we perform a change of variables and reformulate the problem as Poisson's equation with the non-linearity in the right hand side. Therein we also give the assumptions on the coefficients k and b. In Subsection 2.3 we present results on the existence and uniqueness of a solution to the resulting problem after a change of variables. Also estimates of the solution are given in the appropriate norms. In Section 3, we show that the operator equation proposed is well-defined and also continuous. We also discuss solvability of the original problem before the change of variables. In Section 4, we present numerical experiments related to problem. Finally, in Section 5 we summarize our results and draw some conclusions.

2. The operator equation and its properties

Our strategy for the theoretical analysis of the operator equation is to apply a change of variables and investigate the solvability of the resulting simpler problem. In particular, we study whether the resulting operator is well-defined and also continuous.

2.1. Function spaces

We introduce the function spaces used in this paper. We denote by $L^2(\Omega)$ the space of square integrable real-valued functions in Ω. The Sobolev space $H^1(\Omega)$ consists of all functions in $L^2(\Omega)$ whose first order derivatives belong to $L^2(\Omega)$. The subspace $H^1_{per}(\Omega)$ of $H^1(\Omega)$ denotes functions that are periodic in the x_1 direction i.e $T(0,x_2) = T(\ell,x_2)$. The norm in this space is

$$||u||_{H^1_{per}(\Omega)} = \left(\int_\Omega |u|^2 \,\mathrm{d}x + \int_\Omega |\nabla u|^2 \,\mathrm{d}x\right)^{1/2}.$$

We also denote by $H^1_{per}(\Omega)^*$ the dual space of $H^1_{per}(\Omega)$.

The space $H^1_{0,per}$ is a subspace of H^1_{per} that consists of all functions which are zero on the boundaries Γ_0 and Γ_1. Let $H^{1/2}_{per}(\Gamma_k)$, for k=0,1, be the space of traces of functions from $H^1_{per}(\Omega)$ on the boundaries Γ_0 and Γ_1. This space of functions are of equivalent norm

$$\|u\|_{H^{1/2}_{per}(\Gamma_j)} = \left(\int_{\Gamma_j} |u(x)|^2 \,\mathrm{d}\Gamma_j + \int_{\Gamma_j}\int_{\Gamma_j} \frac{|u(x)-u(y)|^2}{|x-y|^2} \,\mathrm{d}\Gamma_j \,\mathrm{d}\Gamma_j\right)^{1/2} \quad \text{for } j = 0, 1,$$

where $d\Gamma_i$ and $d\Gamma_j$ are arc length on Γ_j. We denote by $H^{-1/2}_{per}(\Gamma_j)$ the dual space of $H^{1/2}_{per}(\Gamma_j)$.

2.2. Assumptions and the change of variables

In this section, we formulate the non-linear Cauchy problem as a non-linear operator equation on Hilbert spaces. In what follows, we assume that the thermal conductivity k and the heat source b in (1) satisfy the following assumptions:

(a) The function k is ℓ-periodic in the x_1 direction, i.e $k(x_1,x_2,T) = k(x_1+\ell,x_2,T)$, satisfying (2), and

$$|\nabla_x k(x,T)| \le A_0, \quad \forall x \in \overline{\Omega}, T \in \mathbb{R}$$

where A_0 is a constant.

(b) The function b is Lipschitz continuous with respect to T, i.e.

$$|b(x,T_1) - b(x,T_2)| \le B_0|T_1 - T_2|, \quad \forall x \in \overline{\Omega}, T_1, T_2 \in \mathbb{R},$$

for some constant B_0, and $b(x,0) \in L^2(\Omega)$.

To remove the non-linearity from the leading order term in the Equation (1), we use the change of variables

$$Q(x) = N(x, T(x)) \quad \text{and } N(x, T) = \int_0^T k(x, \tau)\, \mathrm{d}\tau. \tag{6}$$

Clearly,

$$Q_2(x) - Q_1(x) = \int_{T_1(x)}^{T_2(x)} k(x, \tau)\, \mathrm{d}\tau,$$

and by (a) we have

$$k_0|T_2(x) - T_1(x)| \le |Q_2(x) - Q_1(x)| \le k_1|T_2(x) - T_1(x)|, \tag{7}$$

where $T_i(x), i = 1, 2$ are two different functions that satisfy the eqution (1) and $Q_i = N(x, T_i(x)), i = 1, 2$.

The function $N(x, T)$ is strictly monotonically increasing with respect to T and for a fixed x. Thus, if $Q(x)$ is known we can undo the change of variables (6) and compute $T(x)$. If $T \in H^1_{per}(\Omega)$, Q is also in $H^1_{per}(\Omega)$ and satisfies

$$Q_{x_i} = k(x, T)T_{x_i} + \int_0^{T(x)} k_{x_i}(x, \tau)\, \mathrm{d}\tau.$$

Using the change of variables in equation (1), we attain

$$\Delta Q = \nabla \cdot \vec{g}(x, T) + b(x, T), \tag{8}$$

where

$$\vec{g}(x, T) = \int_0^{T(x)} \nabla k(x, \tau)\, \mathrm{d}\tau \tag{9}$$

and T is considered as a function of Q. In a similar way, we change variables in the boundary conditions (1) and obtain

$$\begin{array}{ll} Q = \psi_0 & \text{on } \Gamma_0 \\ \vec{n} \cdot \nabla Q = h & \text{on } \Gamma_0 \end{array} \tag{10}$$

where

$$\begin{array}{ll} \psi_0(x) & = \int_0^{\phi_0(x)} k(x, \tau)\, \mathrm{d}\tau, \\ h = \vec{n} \cdot \nabla Q & = \phi_1 + \int_0^{\phi_0} \nabla k(x, \tau)\, \mathrm{d}\tau. \end{array} \tag{11}$$

Therefore, Q solves the following Cauchy problem

$$\begin{array}{lr} \Delta Q = \nabla \cdot \vec{g}(x, T) + b(x, T) & \text{in } \Omega \\ Q = \psi_0 & \text{on } \Gamma_0 \\ \vec{n} \cdot \nabla Q = h & \text{on } \Gamma_0. \end{array} \tag{12}$$

The strategy of solving the Cauchy problem (12) is to solve the boundary value problem

$$\begin{array}{lr} \Delta Q = \nabla \cdot \vec{g}(x, T) + b(x, T) & \text{in } \Omega \\ Q = \psi_0 & \text{on } \Gamma_0 \\ \vec{n} \cdot \nabla Q = \eta & \text{on } \Gamma_1, \end{array} \tag{13}$$

with arbitrary data η on Γ_1 and try to match the resulting solutions to the given Cauchy data on Γ_0 in (11).

Assuming that $\psi_0 \in H^{1/2}(\Gamma_0)$, then by the trace theorem [8], the trace $\gamma(H^1(\Omega)) = H^{1/2}(\Gamma_0)$, and we have the existence of a function $G \in H^1(\Omega)$ such that $\gamma(G) = \psi_0$. Thus we set $Q = w + G$, where w solves the problem

$$\begin{array}{ll} \Delta w = \nabla \cdot \vec{g}(x,T) + b(x,T) - \Delta G & \text{in } \Omega \\ w = 0 & \text{on } \Gamma_0 \\ \vec{n} \cdot \nabla w = \eta - \vec{n} \cdot \nabla G & \text{on } \Gamma_1. \end{array} \tag{14}$$

The corresponding weak formulation of (14) is

$$\int_\Omega \nabla w \cdot \nabla v \,\mathrm{d}x = \int_\Omega \vec{g}(x,T) \cdot \nabla v \,\mathrm{d}x - \int_{\Gamma_2} (\eta - \vec{n} \cdot \nabla G - \vec{g}(x,T)) v \,\mathrm{d}x + \int_\Omega b(x,T) v \,\mathrm{d}x, \tag{15}$$

where $w \in H^1(\Omega)$, $v \in H^1(\Omega)$ with $v = 0$ on Γ_0.

We define a new non-linear operator equation mapping heat flux on Γ_1 onto the heat flux on boundary Γ_0, that is,

$$L(\psi_0, \eta) := \vec{n} \cdot \nabla Q|_{\Gamma_0} = h, \tag{16}$$

where Q is a solution to the problem (13). Note that ψ_0 is a known boundary data on Γ_0 and therefore, the operator L depends only on η to give h.

2.3. Solvability and stability results

In what follows, we need the Poincaré inequality which says that there exists a constant Λ such that

$$\|u\|_{L^2(\Omega)} \le \Lambda \|\nabla u\|_{L^2(\Omega)}, \quad \forall u \in H^1_{per}(\Omega), \quad u = 0 \quad \text{on } \Gamma_0 \tag{17}$$

where the constant Λ depends only on the diameter of Ω.

Lemma 2.1: *Assume that*

$$\sigma = \frac{\Lambda^2 (A_0 + \Lambda B_0)^2}{k_0^2} < 1. \tag{18}$$

Then for each $\psi_0 \in H^{1/2}(\Gamma_0)$ and $\eta \in H^{-1/2}(\Gamma_1)$, problem (13) has a unique weak solution $Q \in H^1_{per}(\Omega)$ and this solution satisfies

$$\|Q\|_{H^1_{per}(\Omega)} \le \frac{C}{1 - \sqrt{\sigma}} \left(\|\psi_0\|_{H^{1/2}_{per}(\Gamma_0)} + \|\eta\|_{H^{-1/2}_{per}(\Gamma_1)} + \|b\|_{L^2(\Omega)} \right), \tag{19}$$

where the constant C is independent of ψ_0, η and $b(x,0)$.

Proof: We begin by proving the existence of such a solution. Define a sequence of functions $\{Q_j\}_{j=0}^\infty$, where $Q_0 = 0$ and Q_{j+1} weakly solves the problem

$$\begin{array}{ll} \Delta Q_{j+1} = \nabla \cdot \vec{g}(x,T_j) + b(x,T_j) & \text{in } \Omega, \\ Q_{j+1} = \psi_0 & \text{on } \Gamma_0, \\ \vec{n} \cdot \nabla Q_{j+1} = \eta & \text{on } \Gamma_1, \end{array} \tag{20}$$

for $j = 0, 1, 2, \ldots$, where Q_j and T_j are connected by (6). Let us prove that the sequence converges to an element $Q \in H^1_{per}(\Omega)$ that solves problem (13). First Q_1 satisfies

$$\begin{array}{ll} \Delta Q_1 = b(x,0) & \text{in } \Omega, \\ Q_1 = \psi_0 & \text{on } \Gamma_0, \\ \vec{n} \cdot \nabla Q_1 = \eta & \text{on } \Gamma_1. \end{array} \tag{21}$$

■

The problem is uniquely solvable and the solution Q_1 can be estimated by

$$\|Q_1\|_{H^1_{per}(\Omega)} \leq C(\|\psi_0\|_{H^{1/2}_{per}(\Gamma_0)} + \|\eta\|_{H^{-1/2}_{per}(\Gamma_1)} + \|b(x,0)\|_{L^2(\Omega)}). \tag{22}$$

Next let $v_{j+1} = Q_{j+1} - Q_j$, taking the difference in (20) we get

$$\begin{array}{ll} \Delta v_{j+1} = \nabla\cdot(\vec{g}(x,T_j) - \vec{g}(x,T_{j-1})) + b(x,T_j) - b(x,T_{j-1}) & \text{on } \Omega, \\ v_{j+1} = 0 & \text{on } \Gamma_0, \\ \vec{n}\cdot\nabla v_{j+1} = 0 & \text{on } \Gamma_1. \end{array} \tag{23}$$

Multiplying both sides by v_{j+1} and integrating by parts, we obtain

$$\int |\nabla v_{j+1}|^2 \,\mathrm{d}x = \int (\vec{g}(x,T_j) - \vec{g}(x,T_{j-1}))\nabla v_{j+1} - (b(x,T_j) - b(x,T_{j-1}))v_{j+1} \,\mathrm{d}x. \tag{24}$$

Since

$$\vec{g}(x,T_j) - \vec{g}(x,T_{j-1}) = \int_{T_{j-1}}^{T_j} \nabla_x k(x,\tau)\,\mathrm{d}\tau,$$

then by the assumption (a) and (7), we have

$$|\vec{g}(x,T_j) - \vec{g}(x,T_{j-1})| \leq A_0|T_j - T_{j-1}| \leq \frac{A_0}{k_0}|Q_j - Q_{j-1}|, \tag{25}$$

and similarly by the assumption (b) and (7), we get

$$|b(x,T_j) - b(x,T_{j-1})| \leq B_0|T_j - T_{j-1}| \leq \frac{B_0}{k_0}|Q_j - Q_{j-1}|. \tag{26}$$

Using (25) and (26) in (24), we arrive at the relation

$$\int |\nabla v_{j+1}|^2 \,\mathrm{d}x \leq \frac{A_0}{k_0}\int |v_j||\nabla v_{j+1}| \,\mathrm{d}x + \frac{B_0}{k_0}\int |v_{j+1}||v_j| \,\mathrm{d}x. \tag{27}$$

Using the Cauchy inequality, we get

$$\int_\Omega |\nabla v_{j+1}|^2 \,\mathrm{d}x \leq \frac{\epsilon A_0}{k_0}\int_\Omega |\nabla v_{j+1}|^2 \,\mathrm{d}x + \frac{\rho B_0}{k_0}\int_\Omega |v_{j+1}|^2 \,\mathrm{d}x + \frac{A_0 + B_0}{4\epsilon k_0 + 4\rho k_0}\int_\Omega |v_j|^2 \,\mathrm{d}x.$$

Next we use Poincaré's inequality (17) to obtain

$$\int_\Omega |\nabla v_{j+1}|^2 \,\mathrm{d}x \leq \left(\frac{\epsilon A_0}{k_0} + \frac{\rho\Lambda^2 B_0}{k_0}\right)\int_\Omega |\nabla v_{j+1}|^2 \,\mathrm{d}x + \left(\frac{\Lambda^2 A_0}{4\epsilon k_0} + \frac{\Lambda^2 B_0}{4\rho k_0}\right)\int_\Omega |\nabla v_j|^2 \,\mathrm{d}x.$$

Finally, we obtain

$$\int_\Omega |\nabla v_{j+1}|^2 \,\mathrm{d}x \leq \gamma(\epsilon,\rho)\int_\Omega |\nabla v_j|^2 \,\mathrm{d}x, \tag{28}$$

where

$$\gamma(\epsilon,\rho) = \frac{\frac{\Lambda^2 A_0}{4\epsilon} + \frac{\Lambda^2 B_0}{4\rho}}{k_0 - \epsilon A_0 - \rho\Lambda B_0}.$$

The function $\gamma(\epsilon,\rho)$ attains its minimum value when $\rho = \epsilon/\Lambda$ and $\epsilon = k_0/(2(A_0 + \Lambda B_0))$ and this minimum is given by (18). Therefore,

$$\int_\Omega |\nabla v_{j+1}|^2 \,\mathrm{d}x \leq \sigma\int_\Omega |\nabla v_j|^2 \,\mathrm{d}x. \tag{29}$$

Let us now show that the sequence $\{Q_j\}_{j=0}^{\infty}$ is a Cauchy sequence in $H^1_{per}(\Omega)$. From (29), it is clear to see that, for all $k > j$,

$$\left(\int_\Omega |\nabla(Q_k - Q_j)|^2\,\mathrm{d}x\right)^{\frac{1}{2}} \le \sqrt{\sigma}^{k-1}\left(\int_\Omega |\nabla v^1|^2\,\mathrm{d}x\right)^{\frac{1}{2}} + \cdots + \sqrt{\sigma}^{j}\left(\int_\Omega |\nabla v^1|^2\,\mathrm{d}x\right)^{\frac{1}{2}},$$
$$= \sqrt{\sigma}^{j}(1 + \sqrt{\sigma} + \sqrt{\sigma}^2 + \cdots + \sqrt{\sigma}^{k-j-1})\left(\int_\Omega |\nabla v^1|^2\,\mathrm{d}x\right)^{\frac{1}{2}},$$

which implies

$$\left(\int_\Omega |\nabla(Q_k - Q_j)|^2\,\mathrm{d}x\right)^{\frac{1}{2}} \le \frac{\sqrt{\sigma}^j}{1-\sqrt{\sigma}}\left(\int_\Omega |\nabla Q_1|^2\,\mathrm{d}x\right)^{\frac{1}{2}}. \tag{30}$$

Thus, it follows that $\{Q_j\}_{j=0}^{\infty}$ is a Cauchy sequence with a limit Q in $H^1(\Omega)$. Taking the limit $k \to \infty$ and $j = 0$ in (30), we obtain

$$\left(\int_\Omega |\nabla Q|^2\,\mathrm{d}x\right)^{\frac{1}{2}} \le \frac{1}{1-\sqrt{\sigma}}\left(\int_\Omega |\nabla Q_1|^2\,\mathrm{d}x\right)^{\frac{1}{2}}. \tag{31}$$

Using the estimate (22) for Q_1 in (31), we obtain the estimate (19) for the solution to problem (13).

To prove uniqueness of the solution Q, let us suppose that Q_1 and Q_2 be two solutions to the problem (13). The function $v = Q_2 - Q_1$ satisfies

$$\begin{array}{ll} \Delta v = \nabla \cdot [\vec{g}(x,T_2) - \vec{g}(x,T_1)] + [b(x,T_2) - b(x,T_1)] & \text{in } \Omega, \\ v = 0 & \text{on } \Gamma_0, \\ \vec{n} \cdot v = 0 & \text{on } \Gamma_1. \end{array} \tag{32}$$

Multiplying both sides of the equation (32) by v and integrating by parts results in

$$\int_\Omega |\nabla v|^2\,\mathrm{d}x = \int_\Omega [\vec{g}(x,T_2) - \vec{g}(x,T_1)] \cdot \nabla v\,\mathrm{d}x + \int_\Omega v[b(x,T_2) - b(x,T_1)]\,\mathrm{d}x. \tag{33}$$

But

$$|\vec{g}(x,T_2) - \vec{g}(x,T_1)| = |\int_{T_1}^{T_2} \nabla_x k(x,\tau)\,\mathrm{d}\tau| \le A_0|T_2 - T_1| \le \frac{A_0}{k_0}|Q_2 - Q_1|, \tag{34}$$

where we used assumption (a) and (7). Similarly, by assumption (b) and (7) we obtain

$$|b(x,T_2) - b(x,T-1)| \le B_0|T_2 - T - 1| \le \frac{B_0}{k_0}|Q_2 - Q_1|. \tag{35}$$

Using (34) and (35) in (33) gives

$$\int_\Omega |\nabla v|^2\,\mathrm{d}x \le \int_\Omega \left(\frac{A_0}{k_0}|v||\nabla v| + \frac{B_0}{k_0}|v|^2\right)\mathrm{d}x, \tag{36}$$

and by using the Cauchy inequality we obtain

$$\int_\Omega |\nabla v|^2\,\mathrm{d}x \le \frac{A_0\epsilon}{k_0}\int_\Omega |\nabla v|^2\,\mathrm{d}x + \frac{A_0}{4\epsilon k_0}\int_\Omega |v|^2\,\mathrm{d}x + \frac{B_0}{k_0}\int_\Omega |v|^2\,\mathrm{d}x. \tag{37}$$

Finally, by using the Poincaré inequality we obtain

$$\int_\Omega |\nabla v|^2\,\mathrm{d}x \le \left(\frac{A_0\epsilon}{k_0} + \Lambda^2\left(\frac{A_0}{4\epsilon k_0} + \frac{B_0}{k_0}\right)\right)\int_\Omega |\nabla v|^2\,\mathrm{d}x. \tag{38}$$

The value $\epsilon = \Lambda/2$ minimizes the expression $\frac{A_0\epsilon}{k_0} + \Lambda^2(\frac{A_0}{4\epsilon k_0} + \frac{B_0}{k_0})$ and the minimum is given by $\frac{\Lambda}{k_0}(A_0 + \Lambda B_0)$. Hence

$$\int_\Omega |\nabla v|^2 \,\mathrm{d}x \le \sqrt{\sigma} \int_\Omega |\nabla v|^2 \,\mathrm{d}x. \tag{39}$$

Since $\sigma < 1$ then, also $\sqrt{\sigma} < 1$ and therefore v is a constant. Due to the homogeneous Dirichlet condition on Γ_0 we find that $v = 0$ which proves uniqueness.

In the next lemma, we present a somewhat different approach that gives better estimates of the solution to (13) provided $b(x, T)$ is monotonic increasing in T.

Lemma 2.2: *Assume that*

$$b(x, T_2) \ge b(x, T_1) \quad \text{if } T_2 \ge T_1 \quad \text{and } \widehat{\sigma} = \frac{\Lambda^2 A_0^2}{k_0^2} < 1. \tag{40}$$

Then, for each $\psi_0 \in H^{1/2}(\Gamma_0)$ and $\eta \in H^{-1/2}(\Gamma_1)$, (13) has a unique weak solution $Q \in H^1_{per}(\Omega)$ and this solution satisfies

$$\|Q\|_{H^1_{per}(\Omega)} \le \frac{C}{1 - \sqrt{\widehat{\sigma}}} \left(\|\psi_0\|_{H^{1/2}_{per}(\Gamma_0)} + \|\eta\|_{H^{-1/2}_{per}(\Gamma_1)} + (\|b\|_{L^2(\Omega)} \right), \tag{41}$$

where C is independent of ψ_0, η and $b(x, 0)$.

Proof: In this case we consider a sequence different from the sequence constructed in Lemma 2.1. We put $Q_0 = 0$ and Q_{j+1} solves the problem

$$\begin{aligned} &\Delta Q_{j+1} = \nabla \cdot \vec{g}(x, T_j) + b(x, T_{j+1}) && \text{in } \Omega, \\ &Q_{j+1} = \psi_0 && \text{on } \Gamma_0, \\ &\partial_{x_2} Q_{j+1} = \eta && \text{on } \Gamma_1, \end{aligned} \tag{42}$$

for $j = 0, 1, ..$First, we note that Q_1 solves

$$\begin{aligned} &-\Delta Q_1 + b(x, T_1) = 0 && \text{in } \Omega, \\ &Q_1 = \psi_0 && \text{on } \Gamma_0, \\ &\partial_{x_2} Q_1 = \eta && \text{on } \Gamma_1. \end{aligned} \tag{43}$$

■

This problem is uniquely solvable and the solution Q_1 is estimated by

$$\|Q_1\|_{H^1_{per}} \le C(\|\psi_0\|_{H^{1/2}_{per}(\Gamma_0)} + \|\eta\|_{H^{-1/2}_{per}(\Gamma_1)} + \|b\|_{L^2(\Omega)}) \tag{44}$$

Next let $v_{j+1} = Q_{j+1} - Q_j$, taking the difference in (42), we get

$$\begin{aligned} &\Delta v_{j+1} = \nabla \cdot (\vec{g}(x, T_j) - \vec{g}(x, T_{j-1})) + b(x, T_{j+1}) - b(x, T_j) && \text{in } \Omega, \\ &v_{j+1} = 0 && \text{on } \Gamma_0, \\ &\partial_{x_2} v_{j+1} = 0 && \text{on } \Gamma_1. \end{aligned} \tag{45}$$

Multiplying both sides by v_{j+1} and integrating by parts, we obtain

$$\int_\Omega |\nabla v_{j+1}|^2 \,\mathrm{d}x = \int_\Omega (\vec{g}(x, T_j) - \vec{g}(x, T_{j-1}))\nabla v_{j+1} - (b(x, T_{j+1}) - b(x, T_j))v_{j+1} \,\mathrm{d}x. \tag{46}$$

By the monotonicity of $b(x, T)$, we have

$$\int_\Omega |\nabla v_{j+1}|^2 \, \mathrm{d}x \le \int_\Omega (\vec{g}(x, T_j) - \vec{g}(x, T_{j-1}))\nabla v_{j+1} \, \mathrm{d}x \tag{47}$$

Using estimate (25), we transform (47) to

$$\int_\Omega |\nabla v_{j+1}|^2 \, \mathrm{d}x \le \frac{A_0}{k_0} \int_\Omega |v_j||\nabla v_{j+1}| \, \mathrm{d}x \tag{48}$$

Next we use Cauchy and Poincaré inequalities to show that

$$\int_\Omega |\nabla v_{j+1}|^2 \, \mathrm{d}x \le \widehat{\sigma} \int_\Omega |\nabla v_j|^2 \, \mathrm{d}x. \tag{49}$$

Following the same procedure as in the proof of Lemma 2.1 and using the estimate (44), we obtain the estimate in (41).

Lemma 2.3: *Let (18) be valid. If $Q_k \in H^1_{per}(\Omega)$ is a solution to the problem (13) with $\psi_{0_k} \in H^{1/2}_{per}(\Gamma_0)$ and $\eta_k \in H^{-1/2}_{per}(\Gamma_1)$, where k = 1,2. Then*

$$\|Q_2 - Q_1\|^2_{H^1_{per}(\Omega)} \le C\left(\|\psi_{0_2} - \psi_{0_1}\|^2_{H^{1/2}_{per}(\Gamma_0)} + \|\eta_2 - \eta_1\|^2_{H^{-1/2}_{per}(\Gamma_1)} \right), \tag{50}$$

where C is a positive constant.

Proof: The function $v = Q_2 - Q_1$ solves

$$\begin{aligned} &\Delta v = \nabla \cdot [\vec{g}(x, T_2) - \vec{g}(x, T_1)] + [b(x, T_2) - b(x, T_1)] && \text{in } \Omega \\ &v = \psi_{0_1} - \psi_{0_2} && \text{on } \Gamma_0 \\ &\partial_{x_2} v = \eta_1 - \eta_2 && \text{on } \Gamma_1. \end{aligned} \tag{51}$$

The solution v can be split into $v_1 + v_2 + v_3$ where v_1 solves

$$\begin{aligned} &\Delta v_1 = \nabla \cdot [\vec{g}(x, T_2) - \vec{g}(x, T_1)] + [b(x, T_2) - b(x, T_1)] && \text{in } \Omega \\ &v_1 = 0 && \text{on } \Gamma_0 \\ &\partial_{x_2} v_1 = 0 && \text{on } \Gamma_1, \end{aligned} \tag{52}$$

and v_2 and v_3 satisfies

$$\Delta v_2 = 0 \quad \text{in } \Omega, \quad v_2 = \psi_{0_1} - \psi_{0_2} \quad \text{on } \Gamma_0, \quad \partial_{x_2} v_2 = 0 \quad \text{on } \Gamma_1, \tag{53}$$

and

$$\Delta v_3 = 0 \quad \text{in } \Omega, \quad v_3 = 0 \text{ on } \Gamma_0, \quad \partial_{x_2} v_3 = \eta_2 - \eta_1 \text{ on } \Gamma_1, \tag{54}$$

respectively. To find the estimate for v_1, multiply both sides of (52) by v_1 and integrate by parts to get

$$\int_\Omega |\nabla v_1|^2 \, \mathrm{d}x = \int_\Omega [\vec{g}(x, T_2) - \vec{g}(x, T_1)]\nabla v_1 - [b(x, T_2) - b(x, T_1)]v_1 \, \mathrm{d}x. \tag{55}$$

∎

Using (25) and (26) in (55) we obtain

$$\int_\Omega |\nabla v_1|^2\,\mathrm{d}x \le \int_\Omega \frac{A_0}{k_0}|v||\nabla v_1|\,\mathrm{d}x + \frac{B_0}{k_0}|v||v_1|\,\mathrm{d}x$$
$$\le \frac{A_0}{k_0}\int_\Omega (|v_1|+|v_2|+|v_3|)|\nabla v_1| + \frac{B_0}{k_0}(|v_1|+|v_2|+|v_3|)|v_1|\,\mathrm{d}x. \tag{56}$$

Next we use Poincaré's inequality (17) and obtain

$$\int_\Omega |\nabla v_1|^2 \le \frac{(\Lambda A_0 + \Lambda^2 B_0)}{k_0}\left(\int_\Omega |\nabla v_1|^2\,\mathrm{d}x + \int_\Omega |\nabla v_1||\nabla v_2|\,\mathrm{d}x + \int_\Omega |\nabla v_1||\nabla v_3|\,\mathrm{d}x\right).$$

We next use Cauchy inequality to obtain

$$\int_\Omega |\nabla v_1|^2\,\mathrm{d}x \le C\left(\frac{1}{\epsilon}\int_\Omega |\nabla v_2|^2\,\mathrm{d}x + \frac{1}{\rho}\int_\Omega |\nabla v_3|^2\,\mathrm{d}x\right),$$

where

$$C = \frac{\Lambda A_0 + \Lambda^2 B_0}{4(k_0 - (A_0\Lambda + B_0\Lambda^2)[1+\epsilon+\rho])}.$$

Using $\rho = \frac{\epsilon}{\Lambda}$ as in the proof of existence of a solution, we arrive at the estimate

$$\|v_1\|^2_{H^1_{per}(\Omega)} \le C(\|v_2\|^2_{H^1_{per}(\Omega)} + \Lambda\|v_3\|^2_{H^1_{per}(\Omega)}), \tag{57}$$

where

$$C = \frac{\Lambda A_0 + \Lambda^2 B_0}{4\epsilon(k_0 - (\Lambda A_0 + \Lambda^2 B_0)(1+\epsilon(1+\frac{1}{\Lambda}))}.$$

The functions v_2 and v_3 are estimated by

$$\|v_2\|_{H^1_{per}(\Omega)} \le C_1\|\psi_{0_2} - \psi_{0_1}\|_{H^{1/2}_{per}(\Gamma_0)} \quad \text{and } \|v_3\|_{H^1_{per}(\Omega)} \le C_2\|\eta_2 - \eta_1\|_{H^{-1/2}_{per}(\Gamma_1)},$$

for some positive constants C_1, C_2 [8]. Combining the estimates for v_1, v_2 and v_3 we obtain the estimate (50).

Remark 2.4: Assume that (40) is valid. Then if $Q_k \in H^1_{per}(\Omega)$ is a solution to the problem (13) satisfying $\psi_{0_k} \in H^{1/2}_{per}(\Gamma_0)$ and $\eta_k \in H^{-1/2}_{per}(\Gamma_1)$ for $k = 1,2$, we obtain the estimate

$$\|Q_2 - Q_1\|^2_{H^1_{per}(\Omega)} \le C\left(\|\psi_{0_2} - \psi_{0_1}\|^2_{H^{1/2}_{per}(\Gamma_0)} + \|\eta_2 - \eta_1\|^2_{H^{-1/2}_{per}(\Gamma_1)}\right), \tag{58}$$

where C is a constant depending on only the geometry of the domain, k_0, A_0 and $\epsilon > 0$. The proof of this remark follows the same procedure as in the proof of Lemma 2.3 taking into consideration the assumed monotonicity of b.

3. Definition and properties of the operator *L*

In this section, we prove boundedness and continuity of the non-linear operator $L : L^2(\Gamma_1) \to L^2(\Gamma_0)$. To do this, we first study the properties of the derivatives of the solutions to (13) in appropriate function spaces in the following Lemma.

Lemma 3.1: *Let* (18) *be valid. Let* $\psi_0 \in H^1_{per}(\Gamma_0)$, $\eta \in L^2(\Gamma_1)$ *and let* $Q \in H^1_{per}(\Omega)$ *be a solution to the problem* (13). *Then* $\partial_{x_2} Q|_{\Gamma_0}$ *is well-defined and belongs to* $L^2(\Gamma_0)$ *and*

$$\|\partial_{x_2} Q|_{\Gamma_0}\|_{L^2(\Gamma_0)} \leq C(\|\psi_0\|_{H^1_{per}(\Gamma_0)} + \|\eta\|_{L^2(\Gamma_1)} + \|b(x,0)\|_{L^2(\Omega)}). \tag{59}$$

Proof: To prove this theorem, we split the solution of the problem into two parts $Q = Q_1 + Q_2$. The function Q_2 solves the problem

$$\begin{array}{ll} \Delta Q_2 = f(x) & \text{in } \Omega, \\ Q_2 = 0 & \text{on } \Gamma_0, \\ \partial_{x_2} Q_2 = \eta & \text{on } \Gamma_1, \end{array} \tag{60}$$

where $f(x) = \nabla \cdot \vec{g}(x,T) + b(x,T)$ and Q_1 solves the problem

$$\begin{array}{ll} \Delta Q_1 = 0 & \text{in } \Omega, \\ Q_1 = \psi_0 & \text{on } \Gamma_0, \\ \partial_{x_2} Q_1 = 0 & \text{on } \Gamma_1. \end{array} \tag{61}$$

The function $f(x)$ in (60) can be estimated by

$$\|f(x)\|_{L^2(\Omega)} \leq C(\|Q\|_{H^1_{per}(\Omega)} + \|b(x,0)\|_{L^2(\Omega)})$$

and due to Lemma 2.1,

$$\|f(x)\|_{L^2(\Omega)} \leq C(\|\psi_0\|_{H^{1/2}_{per}(\Gamma_0)} + \|\eta\|_{H^{-1/2}_{per}(\Gamma_1)} + \|b(x,0)\|_{L^2(\Omega)}).$$

From the regularity theory of elliptic problems [8], the solution to (60) belongs to $H^2(\Omega \setminus \Gamma_{1\epsilon})$ in the outside $\epsilon-$ neighbourhood $\Gamma_{1\epsilon}$ of Γ_1 and

$$\|Q_2\|_{H^2_{per}(\Omega \setminus \Gamma_{1\epsilon})} \leq C(\|f(x)\|_{L^2(\Omega)} + \|\eta\|_{L^2(\Gamma_1)}). \qquad \blacksquare$$

Therefore,

$$\|\partial_{x_2} Q_2|_{\Gamma_0}\|_{L^2(\Gamma_0)} \leq C(\|\psi_0\|_{H^1_{per}(\Gamma_0)} + \|\eta\|_{L^2(\Gamma_1)} + \|b(x,0)\|_{L^2(\Omega)}). \tag{62}$$

By Theorem 1.8.2 in [16], the solution Q_1 has a derivative $\partial_{x_2} Q_1$ on Γ_0 that can be estimated by

$$\|\partial_{x_2} Q_1\|_{L^2(\Gamma_0)} \leq C\|\psi_0\|_{H^1_{per}(\Gamma_0)}. \tag{63}$$

Therefore, combining the estimates in (62) and (63), we get the estimate for the derivative of the solution Q in (59).

Remark 3.2: When (40) is valid, then we obtain similar results as in (59). The constant C in this case is independent of the constant B_0.

Lemma 3.3: *Let (18) be valid. Given* $\psi_{0_k} \in H^1_{per}(\Gamma_0)$ *and* $\eta_k \in L^2(\Gamma_1)$, *let* $Q_k \in H^1_{per}(\Omega)$ *be a solution to the problem* (13), *where k = 1,2. Then*

$$\|\partial_{x_2}(Q_1 - Q_2)\|_{L^2(\Gamma_0)} \leq C\left(\|\psi_{0_2} - \psi_{0_1}\|_{H^1_{per}(\Gamma_0)} + \|\eta_1 - \eta_2\|_{L^2(\Gamma_1)}\right). \tag{64}$$

where C is a positive constant.

Proof: The function $v = Q_2 - Q_1$ solves the problem

$$
\begin{aligned}
\Delta v &= p(x) && \text{in } \Omega \\
v &= \psi_{0_1} - \psi_{0_2} && \text{on } \Gamma_0 \\
\partial_{x_2} v &= \eta_1 - \eta_2 && \text{on } \Gamma_1,
\end{aligned}
\tag{65}
$$

where $p(x) = \nabla \cdot [\vec{g}(x, T_2) - \vec{g}(x, T_1)] + [b(x, T_2) - b(x, T_1)]$. The solution v can be split into two; $v = v_1 + v_2$. The function v_1 solves the problem

$$
\begin{aligned}
\Delta v_1 &= p(x) && \text{in } \Omega, \\
v_1 &= 0 && \text{on } \Gamma_0, \\
\partial_{x_2} v_1 &= \eta_2 - \eta_1 && \text{on } \Gamma_1,
\end{aligned}
\tag{66}
$$

and the function v_2 solves the problem

$$
\begin{aligned}
\Delta v_2 &= 0 && \text{in } \Omega, \\
v_2 &= \psi_{0_2} - \psi_{0_1} && \text{on } \Gamma_0, \\
\partial_{x_2} v_2 &= 0 && \text{on } \Gamma_1.
\end{aligned}
\tag{67}
$$

■

The function $p(x)$ can be estimated by $\|p(x)\|_{L^2(\Omega)} \le C\|v\|_{H^1_{per}(\Omega)}$ and due to Lemma 2.3

$$
\|p(x)\|_{L^2(\Omega)} \le C(\|\psi_{0_2} - \psi_{0_1}\|_{H^1_{per}(\Gamma_0)} + \|\eta_2 - \eta_1\|_{L^2(\Gamma_1)}).
$$

Using the same arguments as in the Lemma 3.1, the solution v_1 belongs to $H^2(\Omega \setminus \Gamma_{1\epsilon})$ in the ϵ-neighbourhood of Γ_1 and therefore,

$$
\|v_1\|_{H^2_{per}(\Omega\setminus\Gamma_{1\epsilon})} \le C(\|\psi_{0_2} - \psi_{0_1}\|_{H^1_{per}(\Gamma_0)} + \|\eta_2 - \eta_1\|_{L^2(\Gamma_1)}).
\tag{68}
$$

The derivative $\partial_{x_2} v_1|_{\Gamma_0}$ can then be estimated by

$$
\|\partial_{x_2}|_{\Gamma_0}\|_{L^2(\Gamma_0)} \le C(\|\psi_{0_2} - \psi_{0_1}\|_{H^1_{per}(\Gamma_0)} + \|\eta_2 - \eta_1\|_{L^2(\Gamma_1)}),
\tag{69}
$$

and by Theorem 1.8.2 in [16], $\partial_{x_2} v_2$ is estimated by

$$
\|\partial_{x_2} v_2|_{\Gamma_0}\|_{L^2(\Gamma_0)} \le C\|\psi_{0_2} - \psi_{0_1}\|_{H^1_{per}(\Gamma_0)}.
\tag{70}
$$

Thus, combining the estimates in (69) and (70) gives the required estimate for the derivative of the solution Q in (64).

Remark 3.4: Assuming that (40) holds, let $Q_k \in H^1_{per}(\Omega)$ be a solution to the problem (13) for $\psi_{0_k} \in H^1_{per}(\Gamma_0)$ and $\eta_k \in L^2(\Gamma_1)$, where $k = 1,2$. Then through the same procedure and arguments as in Lemma 3.3 it can be shown that

$$
\|\partial_{x_2}(Q_1 - Q_2)\|_{L^2(\Gamma_0)} \le C\left(\|\psi_{0_2} - \psi_{0_1}\|_{H^1_{per}(\Gamma_0)} + \|\eta_1 - \eta_2\|_{L^2(\Gamma_1)}\right).
\tag{71}
$$

where C is a positive constant independent of B_0.

From results in the Lemmas 3.1 and 3.3, we see that the derivative $\partial_{x_2} Q$ is well-defined and is Lipschitz continuous on the boundary Γ_0. Therefore using the definition of the operator L, see (16), and the Lipschitz continuity of $\partial_{x_2} Q$, we state the following result about the properties of the operator in appropriate function spaces.

Theorem 3.5: *Let (18) be valid. The operator $L : L^2(\Gamma) \to L^2(\Gamma_0)$ is a well-defined Lipschitz continuous map satisfying*

$$\|L(\psi_0, \eta)\|_{L^2(\Gamma_0)} \leq C(\|\psi_0\|_{H^1_{per}(\Gamma_0)} + \|\eta\|_{L^2(\Gamma_1)} + \|b(x, 0)\|_{L^2(\Omega)}) \tag{72}$$

and

$$\|L(\psi_{0_2}, \eta_2) - L(\psi_{0_2}, \eta_1)\|_{L^2(\Gamma_0)} \leq C\left(\|\psi_{0_2} - \psi_{0_1}\|_{H^1_{per}(\Gamma_0)} + \|\eta_1 - \eta_2\|_{L^2(\Gamma_1)}\right), \tag{73}$$

where the constant C is independent of b, ψ_0 and η.

The proof of the boundedness and the continuity of the operator L follows from the discussions in Lemmas 3.1 and 3.3, respectively.

We conclude this section by noting that since the the problem (13) was obtained from the original problem by a change of variables then we also have the following result:

Corollary 3.6: *Provided the assumptions on k and b holds there exists a unique solution to the original problem* (4).

4. Numerical convergence verification

The main topic of discussion in this paper is the solvability of the operator equation (5). However, since the operator is non-linear calculating $K(\phi_0, \psi_1)$ for a given set of boundary data (ϕ_0, ψ_1) is also non-trivial.

In our previous work [5], we implemented a simple fixed point iteration as follows: Let $T_0 \in H^1_{per}(\Omega)$ be a starting guess and iterate by solving

$$\nabla(k(x, T_j)\nabla T_{j+1}) = b(x, T_j), \tag{74}$$

with the boundary conditions as specified in (4). The algorithm was found to converge rapidly for the specific tests we conducted but the convergence was never formally proved.

The solvability result in this paper was based on first making the change of variables $Q = N(x, T(x))$, see (6), and then proving that the iterative procedure

$$\Delta Q_{j+1} = \nabla \cdot \vec{g}(x, T_j) + b(x, T_j), \quad T_j = N^{-1}(x, Q_j), \tag{75}$$

with boundary conditions as in (20), does converge to the unique solution. This offers an alternative method for evaluating the non-linear operator which has some advantages over the previous algorithm. First there is a proof of convergence and second, the boundary value problems at each step are simpler to solve.

4.1. Numerical implementation

For the implementation of our iterative algorithms we need to solve the boundary value problems (74) and (75). For this purpose we introduce a uniform computational grid, of size $N \times M$, on the domain Ω. The functions $T(x_1, x_2)$ and $Q(x_1, x_2)$ are represented by their values at the grid points, e.g. the unknowns are T_j and Q_j during the iterations are represented by matrices.

In our work, we use second order accurate finite differences to discretize the boundary value problems. The code for solving (74) is described in detail in [5] and a similar code is used for solving the Poisson equation (75).

For our new algorithm, implementing the change of variables $Q_j = N(x, T_j)$, and its inverse, represent an additional challenge. Here, additional assumptions are needed for an efficient implementation. In our work we assume that

$$k(x, T) = k_1(x)k_2(T), \tag{76}$$

with both $k_1, k_2 > 0$. We first compute the minimum and the maximum of the temperature and introduce a set of n equidistant values $\{T^i\}$, $T^{(1)} = \min T_j$ and $T^n = \max T_j$. The second step is computing the integrals

$$q_2^i = \int_0^{T^1} k_2(\tau)\, d\tau, \quad q_2^{i+1} = q_2^i + \int_{T^i}^{T^{i+1}} k_1(\tau)\, d\tau, \quad i = 1, 2, \ldots, n-1,$$

using a numerical quadrature rule, and finding the cubic spline $q_2(T)$ that interpolates $\{T^i, q_2^i\}$, with natural endpoint conditions. The change of variables can then be computed pointwise as

$$Q(x) = k_1(x)q_2(T). \tag{77}$$

The procedure is rather efficient and only a small number of integrals n need to be evaluated. In our experiments we use $n = 100$. Note that we can calculate the vector valued function $\vec{g}(x, T)$, see (9), at the same time since $\nabla k(x, T) = (\nabla k_1(x))k_2(T)$.

The inverse $T = N^{-1}(x, Q)$ is computed using a similar algorithm. Now we start by computing $\widetilde{Q} = k_1^{-1}Q$ on the grid, introduce an equidistant discretization $\{q^i\}$ of the interval $[\min \widetilde{Q}, \max \widetilde{Q}]$, and find the values $\{T^i\}$ by solving equations of the type

$$f_{i+1}(T) = q^{i+1} - q^i - \int_{T^i}^{T} k_2(\tau)\, d\tau = 0, \quad f_1(T) = q^1 - \int_0^T k_2(\tau)\, d\tau = 0,$$

using Newton's method. The final T values are computed pointwise by evaluating spline interpolating $\{q^i, T^i\}$ pointwise for the values given by $\widetilde{Q}$.

4.2. Numerical examples

In this section we present a few concrete examples and verify that our iterative procedure is convergent. In all cases we consider a problem in the domain $\Omega = [0, 400] \times [0, 80]$ km and use the boundary conditions $T = 10\,^oC$, for x on the surface Γ_0, and $\vec{n} \cdot (kT) = Q_m$, for x on the lower boundary Γ_2. The solution and the flux Q_m are illustrated in Figure 2.

Example 4.1: In the first test we use the coefficients

$$k(x, T) = \frac{2.5}{1 - 2 \cdot 10^{-4}T} \text{ and } b(x, T) = \frac{2.0 \cdot 10^{-6}}{1 + 1.7 \cdot 10^{-5}T} \tag{78}$$

and demonstrate the convergence of the iterative procedure. The initial guess is $T_0 = Q_0 = 0$. This is a rather poor starting guess and thus the error is initially very large. The convergence is measured using the Frobenius norm. In order to test if the convergence rate depends on the grid size we use both $(N, M) = (500, 250)$ and $(N, M) = (300, 150)$. The results are displayed in Figure 3 and show that the iterative algorithm using the original variable needs 19 iterations to reach full accuracy; while the new algorithm only needs 9 iterations. Also, the convergence rate does not depend significantly on the grid size.

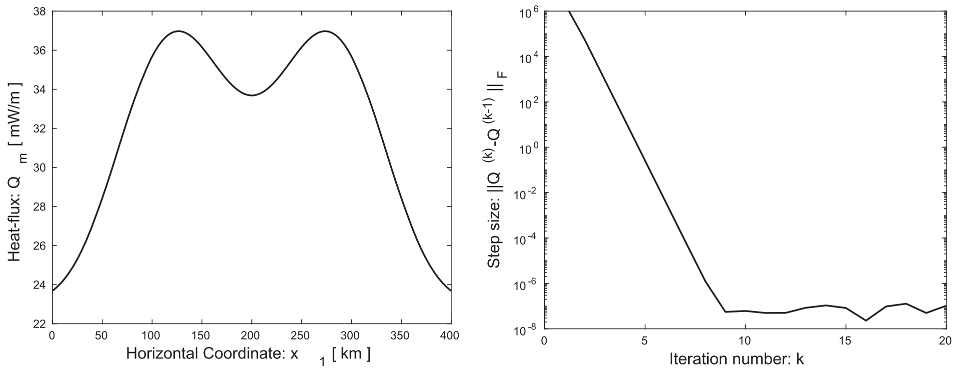

Figure 2. We display the heat flux at the lower boundary Q_m in the left graph. The exact solution $T(x_1, x_2)$ for the test problem used in Example 4.1 is shown to the right.

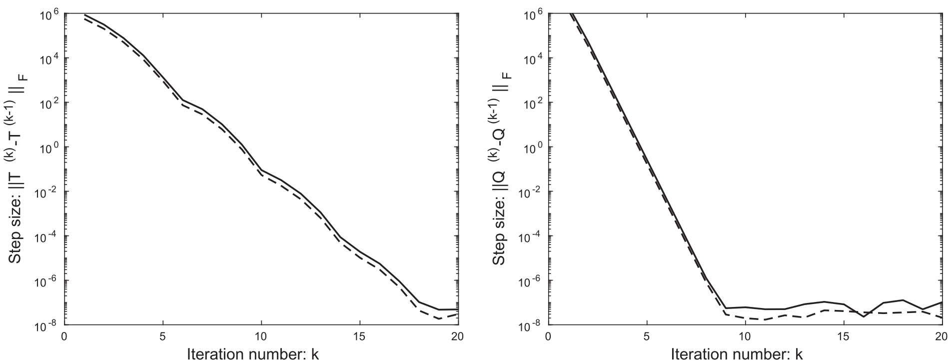

Figure 3. Example 4.1. The convergence rate for both algorithms measured using the Frobenius norm. To the left we display $\|T_{j+1} - T_j\|_F$ for the original algorithm and to the right we display $\|Q_{j+1} - Q_j\|_F$ for our new proposed algorithm. In both cases the results for $N = 500$ and $M = 250$ corresponds to the solid curve and the results for $N = 300$ and $M = 150$ are shown as a dashed curve.

Example 4.2: In our second test we use the grid size $(N, M) = (500, 250)$ and let the coefficient k depend on the x_1 variable. More precisely,

$$k(x,t) = k_1(x)k_2(T), \quad k_1(x_1,x_2) = 1 + \frac{1}{5}\sin\left(\frac{\pi}{\ell}x_1\right), \quad k_2(T) = \frac{2.5}{1 - 2\cdot 10^{-4}T}, \tag{79}$$

where $\ell = 400\,km$ is the width of the domain Ω. The coefficient $b(T)$ is the same as in Example 4.1. Note that for this case we can calculate the gradient of k_1 analytically and thus we can calculate

$$\nabla \cdot \vec{g} = \Delta k_1(x) \int_0^T k_2(\tau)\,\mathrm{d}\tau,$$

using the same technique as for the change of variables $T = N(x, T(x))$. The convergence of the two algorithms is illustrated in Figure 4. For this particular example the new algorithm converges approximately twice as fast as the original one.

5. Conclusions

In this paper we have demonstrated the unique solvability of the auxiliary problem of the original problem. Existence and uniqueness of a solution is proved by using an iterative procedure which is

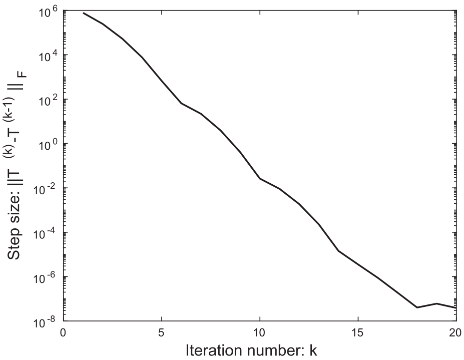

Figure 4. Example 4.2. The convergence rate for both algorithms measured using the Frobenius norm. To the left we display $\|T_{j+1} - T_j\|_F$ for the original algorithm and to the right we display $\|Q_{j+1} - Q_j\|_F$ for our new proposed algorithm. The results are for the case $N = 500$ and $M = 250$.

shown to converge to a unique fixed point. The solution is also bounded within appropriate function spaces.

In our previous work [5] we used fixed point iteration to solve the steady state heat conduction problem (4). The fixed point iterations converged rapidly but we did not have a proof of convergence for general coefficients $k(x, T)$ and $b(x, T)$.

The existence and uniqueness result in this paper was proved by using the change of variables $Q = N(x, T(x))$, see (6), that lets us rewrite the problem in the form $\Delta Q = \nabla \cdot \vec{g}(x, T) + b(x, T)$, i.e. we have Poisson's equation. We also formulate an iterative procedure for finding Q that can be proved to converge. The new iterative procedure has several advantages. First convergence can be proved in advance, also the Poisson equation is simpler to solve and several fast algorithms are available, see e.g. [21]. Thus, in comparison to the previous algorithm, solving the well-posed boundary value problem during each iteration step is potentially simpler for the new algorithm. Though we also need to implement the change of variables $Q = N(x, T(x))$ and its inverse. This can be done efficiently if the coefficients k and b have sufficiently simple analytic expressions. Finally, our experiments indicate that the new algorithm converges roughly twice as fast as the previous one. For the new algorithm the convergence speed is determined by the constant σ, see Lemma 2.1. For the original iterative algorithm, i.e. the scheme (74) in the original variables, we do not yet have a convergence proof. Thus we cannot make any precise comparison between the two methods with regards to the speed of convergence. This is something we hope to do in the future.

Our original interest was solving a Cauchy problem for the non-linear steady state heat conduction problem. By using the above change of variables we instead obtain a Cauchy problem for the simpler Poisson equation. In our future work we will investigate efficient iterative implementations of Tikhonov's regularization and where the sub problems are solved using fast Poisson solvers.

Disclosure statement

No potential conflict of interest was reported by the authors.

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