Geophysical & Astrophysical Fluid Dynamics

ISSN: 0309-1929 (Print) 1029-0419 (Online) Journal homepage: https://www.tandfonline.com/loi/ggaf20

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Geophysical and Astrophysical Fluid Dynamics, 2015
Vol. 109, No. 5, 524–537, http://dx.doi.org/10.1080/03091929.2015.1081188

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On the theory of translationally invariant magnetohydrodynamic equilibria with anisotropic pressure and magnetic shear

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(*Received 8 June 2015; accepted 5 August 2015; first published online 2 September 2015*)

We present an improved formalism for translationally invariant magnetohydrodynamic equilibria with anisotropic pressure and currents with a field aligned component. The derivation of a Grad-Shafranov type equation is given along with a constraint which links the shear field to the parallel pressure. The difficulties of the formalism are discussed and various methods of circumventing these difficulties are given. A simple example is then used to highlight the methods and difficulties involved.

Keywords: Magnetohydrodynamics; Magnetohydrodynamic equilibria; Anisotropic pressure; Magnetic shear

1. Introduction

Magnetohydrodynamic (MHD) equilibria are very useful as models of a large number of space and astrophysical plasma systems or as starting points for investigations of plasma activity processes such as waves or instabilities (see, e.g. Schindler 2006, Priest 2014). In many cases the assumption of a scalar pressure is justified, but sometimes the inclusion of an anisotropic pressure tensor is necessary. An anisotropic pressure tensor generally results if the underlying particle distribution functions are gyrotropic, i.e. independent of the gyrophase at a particle level. This can be the result of a difference in collisional timescales along and across the magnetic field or of other microscopic processes maintaining gyrotropy if the plasma is collisionless (for a detailed discussion see, e.g. Chust and Belmont 2006). In this case, fluid descriptions of the plasma can often still be used.

Some examples of space plasma systems for which it has been suggested that pressure anisotropy may play a role are the Jovian magnetosphere (e.g. Kivelson and Southwood 2005), the Earth's magnetosphere (e.g. Cowley 1978, Nötzel *et al*. 1985, Hesse and Birn 1992, Hau 1993, Sonnerup *et al*. 2006) and astrophysical winds and outflows (e.g. Asseo and Beaufils 1983, Tsikarishvili *et al*. 1995, Beskin and Kuznetsova 2000, Kuznetsova 2005). There are also studies on general magnetosphere models with anisotropic pressure (e.g. Cheng 1992, Krasheninnikov and Catto 2000, Zaharia and Cheng 2003, Wu *et al*. 2009) and several studies

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of a more general, theoretical nature (e.g. Heinemann 1990, Heinemann and Pontius 1991, Hau 1996, Cheviakov and Bogoyavlenskij 2004, Cheviakov 2005).

The theory of MHD equilibria with anisotropic pressure seems to have also developed relatively independently in the laboratory plasma community (e.g. Mercier and Cotsaftis 1961, Taylor 1963, Grad 1967, Spies and Nelson 1974, Hall and McNamara 1975, Sestero and Taroni 1977, Nelson *et al*. 1978, Clemente 1993, 1994, Clemente *et al*. 1995, Clemente and Viana 1999, Zwingmann *et al*. 2001, Shi *et al*. 2006, Clemente and Sterzo 2009, Pustovitov 2010, Asahi *et al*. 2011, Asahi *et al*. 2013, Lepikhin and Pustovitov 2013). Hence, a number of different formulations of the MHD equilibrium theory with anisotropic pressure exist, which apart from the specific application the authors had in mind, can depend on the symmetry assumptions made, the inclusion of plasma flows or external forces, the use of particular equations of state, the use of a relativistic formulation of MHD for certain astrophysical systems, or a combination of several of these points. In this paper we will focus exclusively on MHD equilibria with translational invariance, which in the isotropic pressure case would lead to a Grad-Shafranov equation for the magnetic flux function. In the analogous anisotropic pressure case the partial differential equation for the magnetic flux function can be written in a form which is very similar to the Grad-Shafranov equation.

Often, further assumptions are made about the relationship between the shear component of the magnetic field (also referred to as a guide field) and the anisotropy parameter (basically the difference between the parallel and perpendicular pressure), leading to, for example, the magnetic shear field component being constant on flux surfaces as in the isotropic case (e.g Mercier and Cotsaftis 1961, Clemente 1993, Shi *et al*. 2006). While this simplifies the equilibrium equation and makes it more similar to the isotropic Grad-Shafranov equation, it also leads to a restriction in the functional forms allowed for the magnetic pressure tensor. On the other hand, it is still possible to reduce the equilibrium equation to a form similar to a Grad-Shafranov equation even if no simplifying assumptions are made (e.g. Lepikhin and Pustovitov 2013), but this leads to an implicit coupling between the shear component of the magnetic field and the parallel pressure which makes this equilibrium equation very difficult to use in practice. We will discuss these points in more detail in section 2. It is the aim of this paper to provide an alternative formulation of the anisotropic pressure MHD equilibrium problem with translational invariance that does not make use of any simplifying assumptions, but also leads to a Grad-Shafranov type equation which is free of the implicit coupling of previous formulations.

A general framework for the formulation of the MHD equilibrium problem has recently been provided by Schindler (2006) (Chapter 8), using Euler potentials (see, e.g. Stern 1970). This includes, in principle, MHD equilibria with anisotropic pressure in two and in three dimensions and this is one of the cases mentioned in the book. While the formulation using Euler potentials has the advantage of being very general, possible disadvantages are that Euler potentials may not always exist globally (for given magnetic fields), for example in toroidal geometry, and that the equilibrium equations, which the Euler potentials have to satisfy, are generally of mixed type. This, together with the intrinsic non-linearity of the Euler potential approach, makes the application of numerical methods to the solution of the equilibrium problems more difficult. Hence, despite the usefulness of the general theory developed by Schindler (2006), there is still a need for a different formulation for symmetric systems similar to the Grad-Shafranov theory in the case of isotropic pressure equilibria.

The structure of this paper is as follows. In section 2 we discuss the basic theory of static anisotropic MHD equilibria with spatial symmetry. In section 3 we present our new formulation of the anisotropic MHD equilibrium problem using Cartesian coordinates, followed by an illustrative example in section 4. We conclude the paper with a summary and discussion in section 5, followed by an appendix outlining how the new formalism can be used in a spherical geometry with rotational invariance.

2. Basic theory for anisotropic equilibria with translational symmetry

Essentially, one wants to solve the force balance equation and solenoidal constraint,

$$\boldsymbol{j} \times \boldsymbol{B} = \nabla \boldsymbol{\cdot} \mathsf{P} - \rho \nabla \Psi, \quad \nabla \boldsymbol{\cdot} \boldsymbol{B} = 0,$$

where Ψ is some potential, for instance a gravitational force. For now, we set $\Psi = 0$. We assume that the pressure tensor P is of the general form

$$\mathsf{P} = P_\perp \mathsf{I} + \frac{P_\parallel - P_\perp}{B^2} \boldsymbol{B}\boldsymbol{B}. \tag{1}$$

The theory for 2D equilibria with anisotropic pressure, but with vanishing shear field is reasonably well understood. In the following we discuss the general theory using Cartesian coordinates x, y, z, assuming translational symmetry in the y-direction (i.e. $\partial/\partial y = 0$).

The solenoidal constraint is satisfied automatically by writing $\boldsymbol{B}$ in terms of a magnetic flux function $A(x, z)$:

$$\boldsymbol{B} = \nabla A \times \boldsymbol{e}_y.$$

One can then show that using the form of the pressure tensor given in (1), the force balance equation is equivalent to the anisotropic Grad-Shafranov equation

$$-\nabla \boldsymbol{\cdot} \left[\left(\frac{1}{\mu_0} - \frac{1}{B} \frac{\partial P_\parallel}{\partial B} \right) \nabla A \right] = \frac{\partial P_\parallel}{\partial A}, \tag{2}$$

where $P_\parallel(A, B)$ is the pressure parallel to the magnetic field and B is the magnitude of the magnetic field $\boldsymbol{B}$. The perpendicular pressure does not appear in (2), and is given by

$$P_\perp = P_\parallel - B \frac{\partial P_\parallel}{\partial B}. \tag{3}$$

Equation (2) underpins the entire workings of 2D anisotropic equilibria. Once one chooses the parallel pressure function $P_\parallel$ as a function of A and $B = |\nabla A|$, one can go on to solve the Grad-Shafranov equation by whatever means one prefers in order to find A which then determines the magnetic field. Some work has already been done in this area by Cowley (1978), Nötzel *et al.* (1985) and others. One can see from (3) that the isotropic pressure case is recovered if $P_\parallel$ depends only on A, but not on B.

An interesting problem arises if we add a non-vanishing shear field B_y, while keeping the equations translationally invariant (we will call this the 2.5D case from now on). As before, we use a vector potential, $A(x, z)$, in order to satisfy the solenoidal constraint. The magnetic field, $\boldsymbol{B}$, is then given by

$$\boldsymbol{B} = \nabla A \times \boldsymbol{e}_y + B_y \boldsymbol{e}_y, \tag{4}$$

where A and B_y are functions of x and z only. The corresponding current density is given by

$$\boldsymbol{j} = -\nabla^2 A\, \boldsymbol{e}_y + \nabla B_y \times \boldsymbol{e}_y.$$

We therefore have a component of current in the direction of the magnetic field:

$$\boldsymbol{j} \boldsymbol{\cdot} \boldsymbol{B} = \nabla A \boldsymbol{\cdot} \nabla B_y - B_y \nabla^2 A,$$

which is only present with a non-vanishing shear field.

One recovers the same constraint on the pressure as before in (3) (although B now is equal to $\left[(\nabla A)^2 + {B_y}^2\right]^{1/2}$). After a considerable amount of algebra, we arrive at two equations. Firstly, we obtain

$$B_y\left(1 - \frac{\mu_0}{B}\frac{\partial P_\parallel}{\partial B}\right) = F(A), \tag{5a}$$

where $F(A)$ is some arbitrary function of the magnetic flux (Mercier and Cotsaftis 1961). Secondly, we recover a Grad-Shafranov type equation analogous to (2):

$$-\nabla\boldsymbol{\cdot}\left[\left(1 - \frac{\mu_0}{B}\frac{\partial P_\parallel}{\partial B}\right)\nabla A\right] = \mu_0\left[\frac{\partial P_\parallel}{\partial A} + B_y\frac{\partial}{\partial A}\left(B_y\left(\frac{1}{\mu_0} - \frac{1}{B}\frac{\partial P_\parallel}{\partial B}\right)\right)\right]. \tag{5b}$$

These two equations have been found before (see, e.g. Lepikhin and Pustovitov 2013, for the rotationally symmetric case) and are a concise formulation of the problem. As for the 2D case, they also reduce to the isotropic pressure case if $P_\parallel$ does not depend on B. However, on closer inspection, one finds that (5a,b) are implicitly coupled. One must know B_y in order to calculate B and vice versa. In fact, within this formulation of the equilibrium problem it is difficult to decouple B_y and B. Hence, in order to make progress, one requires a formulation that does not have this implicit coupling. We provide such a formulation in the next section.

3. An alternative formulation

We now consider a formulation that uses the flux function A and the modulus of its gradient ($B_p = |\nabla A|$, the magnitude of the magnetic field in the x–z-plane) instead of A and B as in previous papers (e.g. Mercier and Cotsaftis 1961). Throughout this paper we will be using Cartesian coordinates with invariant direction y, i.e. $\partial/\partial y = 0$ for all quantities (the axisymmetric case is outlined in the appendix A). We allow, however, for a non-vanishing B_y component of the magnetic field. For the time being we will also ignore any external forces. Then the force balance equation is

$$\boldsymbol{j} \times \boldsymbol{B} = \nabla\boldsymbol{\cdot}\mathsf{P}. \tag{6}$$

Taking the scalar product of (6) with $\boldsymbol{B}$ we obtain

$$\nabla A \times \left(\nabla P_\parallel - \frac{P_\parallel - P_\perp}{B}\nabla B\right) = \boldsymbol{0}, \tag{7}$$

which implies that the bracketed quantity must be in the same direction as ∇A. If we make the assumption that $\nabla A \times \nabla B_p \neq \boldsymbol{0}$ almost everywhere, the vector fields ∇A and ∇B_p are linearly independent and it is possible to construct any other gradient out of multiples of these vectors.

Applying this idea to the bracketed expression in (7) and using the fact that the ∇B_p components must cancel, we find that

$$\frac{\partial P_\parallel}{\partial B_p} - \frac{P_\parallel - P_\perp}{B}\left(\frac{\partial B}{\partial B_p} + \frac{\partial B}{\partial B_y}\frac{\partial B_y}{\partial B_p}\right) = 0. \tag{8}$$

Using

$$B^2 = {B_y}^2 + {B_p}^2, \tag{9}$$

we find that

$$\frac{\partial P_\parallel}{\partial B_p} - M_y \frac{P_\parallel - P_\perp}{B^2} = 0, \quad \text{where } M_y = B_p + B_y \frac{\partial B_y}{\partial B_p}. \tag{10a,b}$$

We solve (10a) for $P_\perp$:

$$P_\perp = P_\parallel - \frac{B^2}{M_y} \frac{\partial P_\parallel}{\partial B_p}. \tag{11}$$

One can easily see the similarities between (11) and the equivalent (3) in the case without shear field; in fact upon substitution of $B_y = 0$ into (11) we recover (3), remembering that in the limit $B_y \to 0$ we have $B_p \to B$.

To proceed we look at the different components of the expression

$$\boldsymbol{B} \times (\boldsymbol{j} \times \boldsymbol{B}) = \boldsymbol{B} \times \nabla \cdot \mathsf{P}.$$

After some algebra, we obtain the two relations

$$-\nabla \cdot \left[\left(\frac{1}{\mu_0} - \frac{1}{M_y} \frac{\partial P_\parallel}{\partial B_p} \right) \nabla A \right] = \frac{\partial}{\partial A} \left[P_\parallel + \frac{{B_y}^2}{\mu_0} \left(\frac{1}{2} - \frac{\mu_0}{M_y} \frac{\partial P_\parallel}{\partial B_p} \right) \right], \tag{12a}$$

$$B_y \left(1 - \frac{\mu_0}{M_y} \frac{\partial P_\parallel}{\partial B_p} \right) = F(A), \tag{12b}$$

in which M_y is given by (10b).

To simplify further, we introduce an "effective parallel pressure"

$$P_\parallel{}^\star = P_\parallel + \frac{{B_y}^2}{\mu_0} \left(\frac{1}{2} - \frac{\mu_0}{M_y} \frac{\partial P_\parallel}{\partial B_p} \right). \tag{13}$$

Then (12a) can be rewritten as

$$-\nabla \cdot \left[\left(\frac{1}{\mu_0} - \frac{1}{B_p} \frac{\partial P_\parallel{}^\star}{\partial B_p} \right) \nabla A \right] = \frac{\partial P_\parallel{}^\star}{\partial A}, \tag{14}$$

which is completely analogous to the 2-D case given in (2). This is similar to the isotropic case where one can combine the pressure and the shear field into a new quantity, the total pressure. The isotropic problem then becomes equivalent to the problem with no shear field present.

Equation (12b) can also be simplified using $P_\parallel{}^\star$ to obtain

$$B_y \left(1 - \frac{\mu_0}{B_p} \frac{\partial P_\parallel{}^\star}{\partial B_p} \right) = F(A). \tag{15}$$

We also note that (15) can be used to rewrite (13) as

$$P_\parallel = P_\parallel{}^\star - \frac{{B_y}^2}{\mu_0} \left(\frac{1}{2} - \frac{\mu_0}{B_p} \frac{\partial P_\parallel{}^\star}{\partial B_p} \right). \tag{16}$$

With this formalism we have reduced the problem from a coupled set of equations with a difficult entanglement problem, (5a,b), to a single equation which can be solved for the flux function, (14), and secondary equations from which we can recover the shear field and pressure, (15) and (16).

The simplest way to proceed is to specify the effective parallel pressure as a function of A and B_p and then deduce the pressures and shear field *a posteriori*. We show an example of this in the next section. However, occasionally one will wish to specify either the shear field or the pressure functions instead and then work from there. We show how one would proceed in these cases for completeness.

Suppose we want to specify the shear field, B_y. When $B_y = 0$, (12b) is satisfied automatically allowing any choice of pressure function without contradiction. When $B_y \neq 0$, however, only specific pressures are allowed without leading to contradictions. Let B_y be known as a function of A and B_p and non-zero. Then by a rearrangement of (12b) we obtain

$$\frac{\partial P_\parallel}{\partial B_p} = \frac{1}{\mu_0}\left[\left(B_p + B_y\frac{\partial B_y}{\partial B_p}\right) - \left(\frac{\partial B_y}{\partial B_p} + \frac{B_p}{B_y}\right)\right]F(A),$$

which can be integrated with respect to B_p giving a general expression for the parallel pressure and thus the effective parallel pressure:

$$P_\parallel = \frac{{B_p}^2 + {B_y}^2}{2\mu_0} - \left(B_y + \int \frac{B_p}{B_y}\,\mathrm{d}B_p\right)\frac{F(A)}{\mu_0} + G(A), \tag{17a}$$

$${P_\parallel}^\star = \frac{{B_p}^2}{2\mu_0} - \frac{F(A)}{\mu_0}\int \frac{B_p}{B_y}\,\mathrm{d}B_p + G(A), \tag{17b}$$

where $G(A)$ is an arbitrary function of A introduced after integrating and we have used (16) to simplify ${P_\parallel}^\star$. In principle, if we know B_y we can find the allowed forms of the pressure from (17a). The only real difficulty lies in finding the integral of the function B_p/B_y, which is only straightforward for specific choices of B_y.

Now suppose we wish to specify the parallel pressure. When $P_\parallel$ is a function of A alone, the derivative $\partial P_\parallel/\partial B_p$ in (12b) becomes zero implying that any choice of B_y must be simply a function of A (see, e.g. Schindler 2006). When the pressure function depends on B_p, however, only specific forms of B_y are allowed in order to satisfy (12b). Assume $P_\parallel$ is a known function of A and B_p. Then (12b) determines

$$\frac{1}{\mu_0}B_y\left(B_p + B_y\frac{\partial B_y}{\partial B_p}\right) - B_y\frac{\partial P_\parallel}{\partial B_p} = \frac{F(A)}{\mu_0}\left(B_p + B_y\frac{\partial B_y}{\partial B_p}\right),$$

which can be rewritten somewhat clearer as a differential equation of the form

$$\left(\frac{B_y - F}{\mu_0}\right)\frac{\partial\left({B_y}^2\right)}{\partial B_p} + 2B_y\left(\frac{B_p}{\mu_0} - \frac{\partial P_\parallel}{\partial B_p}\right) = 2\frac{B_p F(A)}{\mu_0}. \tag{18}$$

We can only make analytic progress with this nonlinear partial differential equation for B_y in specific cases; in general a numerical approach is required.

Once we have found B_y and $P_\parallel$ from the methods outlined above, we can deduce $P_\perp$ from (11) and (17a). This yields

$$P_\perp = -\frac{{B_p}^2 + {B_y}^2}{2\mu_0} + \left(\frac{{B_p}^2}{B_y} - \int \frac{B_p}{B_y}\,\mathrm{d}B_p\right)\frac{F(A)}{\mu_0} + G(A). \tag{19}$$

Finally, we must ensure that both pressures remain positive. This acts as a constraint on the arbitrary functions $F(A)$ and $G(A)$ introduced earlier, as we will see in our example in the next section.

As an example for an astrophysical/space plasma application of the formalism developed above, we mention that one could model the near-earth magnetotail in the style of Nötzel *et al.* (1985), but now with an added magnetic shear field component. One could specify the effective parallel pressure in the same way as they specify the actual parallel pressure, based on a bi-maxwellian distribution function, ${P_\parallel}^\star = \exp(-2A)B_p/\left(B_p + k\right)$, for some anisotropy parameter k, and then follow the procedure described above.

4. An illustrative example

We now illustrate the theoretical framework given above using a simple example of an equilibrium with anisotropic pressure and currents with a field aligned component. We will use the first approach outlined in section 3 where we specify the effective pressure and then deduce the shear field and pressure profiles *a posteriori*. All quantities used have been normalised (e.g. $\hat{B_p} = B_p/B_0$, where B_0 is a typical magnetic field strength). In the following the hat symbols have been removed for simplicity.

We take the effective parallel pressure to be

$$P_\parallel{}^\star = 1 + \lambda \mathrm{e}^{-B_p},$$

where λ is some positive scalar that describes the anisotropy. Then, from (15), we have

$$B_y\left(1 + \lambda \frac{\mathrm{e}^{-B_p}}{B_p}\right) = F(A),$$

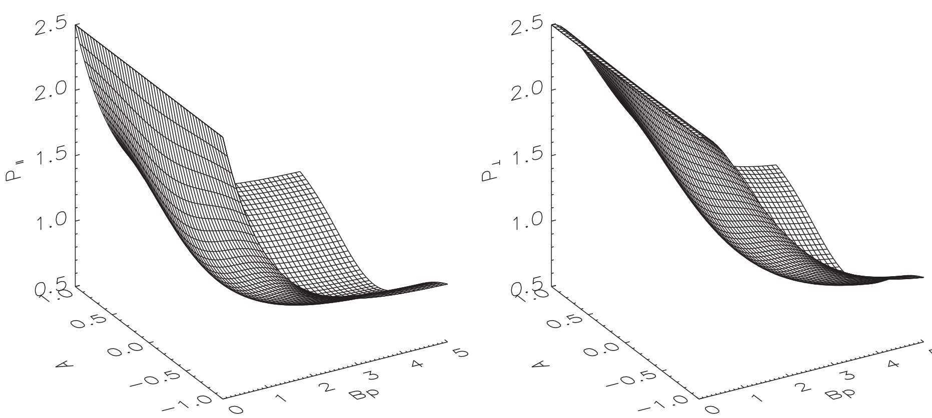
which can be solved for B_y. We also take $F(A) = \mathrm{e}^{-A^2}$ in order to introduce a dependence on the flux function and ensure positivity of the pressures. Then we obtain

$$B_y = \frac{B_p\,\mathrm{e}^{-A^2}}{B_p + \lambda \mathrm{e}^{-B_p}}.$$

In this case the shear grows from zero towards an eventual limit at $F(A)$ (see figure 2 for a surface plot of $B_y(A, B_p)$ at $\lambda = 1.5$).

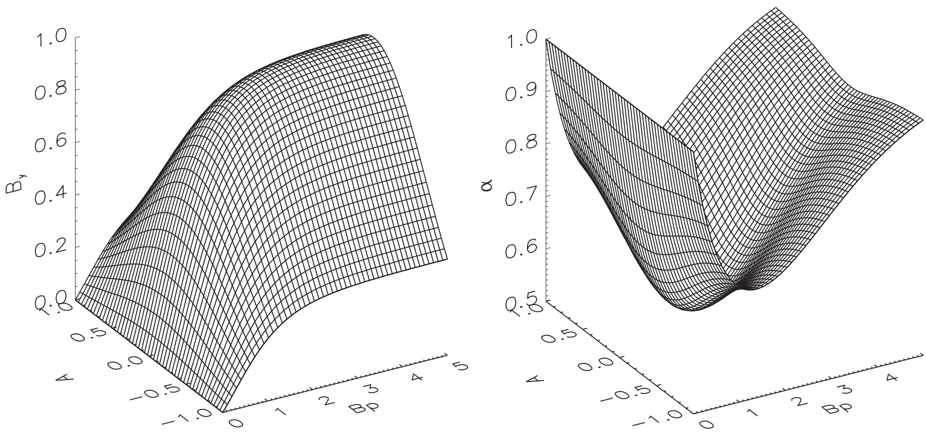
We then find $P_\parallel$ from (16). It is possible to write out $P_\parallel$ explicitly, however the resulting expression does not give much more insight into the problem. We find that the pressure is positive for all values of A and allowed values of B_p and λ. A surface plot of $P_\parallel$ in terms of A and B_p at $\lambda = 1.5$ is shown in figure 1.

The perpendicular pressure is given by (19). Again, we omit expressing the exact form in favour of simply noting some qualities of $P_\perp$. The perpendicular pressure is also positive for all allowed values of B_p, λ and A (see the surface plot in figure 1 which shows $P_\perp(A, B_p)$ at $\lambda = 1.5$).



(a) (b)

Figure 1. Plots of $P_\parallel$ (a) and $P_\perp$ (b) at $\lambda = 1.5$.



(a) (b)

Figure 2. Plot of the shear field, B_y (a) and the anisotropy, $\alpha = P_\parallel/P_\perp$ (b) at $\lambda = 1.5$.

Since we know both pressures, we can determine the anisotropy, $\alpha = P_\parallel/P_\perp$, as a function of A and B_p. In figure 2 we see that the pressures are initially isotropic, then decreasing with B_p until a minimum is reached. The anisotropy then increases again and tends towards isotropy as B_p tends to infinity. This behaviour is typical for all allowed values of A and λ. We then substitute $P_\parallel{}^\star$ into (14). The Grad-Shafranov equation becomes

$$-\nabla\cdot\left[\left(1+\lambda\frac{\mathrm{e}^{-B_p}}{B_p}\right)\nabla A\right] = 0. \tag{20}$$

The equivalent isotropic problem has a constant total pressure, $P^\star = 1$. The actual pressure is

$$P = 1 - \frac{\left(\mathrm{e}^{-A^2}\right)^2}{2},$$

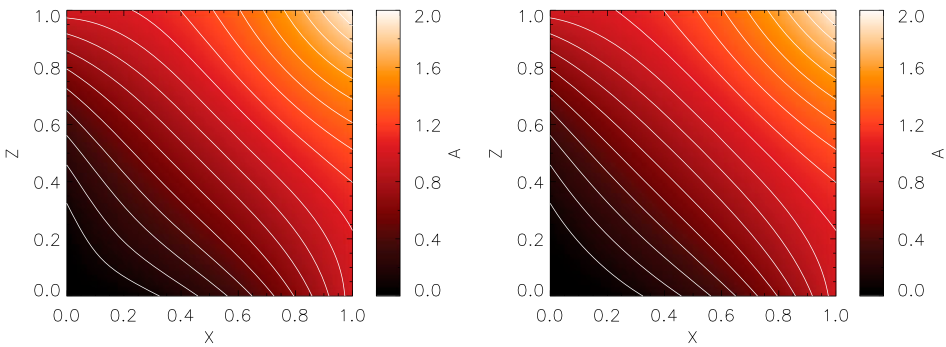
and the shear field is

$$B_y = \mathrm{e}^{-A^2}.$$

The corresponding Grad-Shafranov equation for the isotropic case is then simply the Laplace equation

$$-\nabla^2 A = 0.$$

We solve the above equations with the boundary condition $A = x^2 + y^2$ on the unit square using a numeric continuation code based on Keller's method (see Keller 1977). It has been used previously with a great deal of success to solve various problems in different fields (e.g. Zwingmann 1983, 1987, Neukirch and Hesse 1993, Neukirch 1993a,b, Platt and Neukirch 1994, Schröer *et al*. 1994, Becker *et al*. 1996, 2001, Romeou and Neukirch 1999, 2001, 2002a,b, Kiessling and Neukirch 2003, Neukirch and Romeou 2010). The code uses a finite element discretisation which allows for a flexible grid structure. For more information about the method used here see Neukirch (1993a; 1993b).



(a) $\lambda = 0$ (b) $\lambda = 1.5$

Figure 3. Contour plot of A in the isotropic (a) and anisotropic (b) cases.

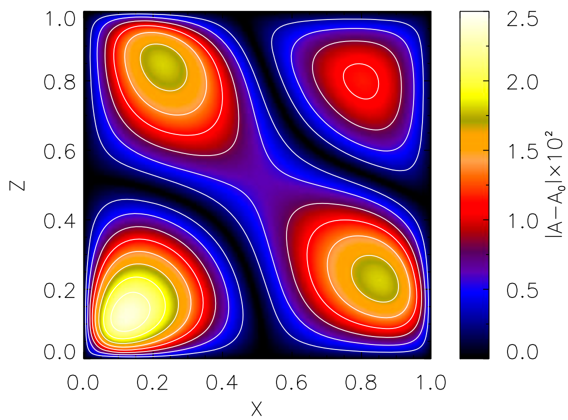


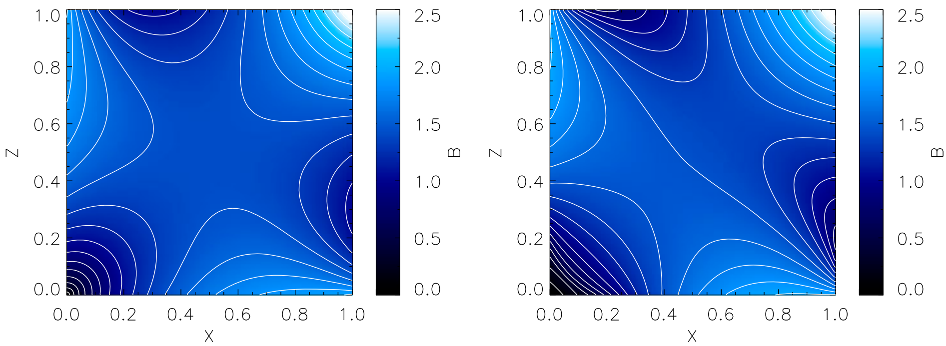
Figure 4. Contour plot of the difference between the isotropic and anisotropic flux functions.

The flux functions for the isotropic and anisotropic cases are shown in figure 3. It is difficult to spot any apparent difference between the two cases. The differences become more apparent in figure 4, which is a contour plot of the difference between the flux function at $\lambda = 1.5$ and $\lambda = 0$. We now see that small differences present themselves, with a maximum difference of ~ 0.025 at the bottom left of the domain.

Indeed, it is more instructive to consider contours of B_p, shown in figure 5 for both the isotropic and anisotropic cases. The structure in the centre has changed somewhat and the contours are slightly more compressed, especially near the origin.

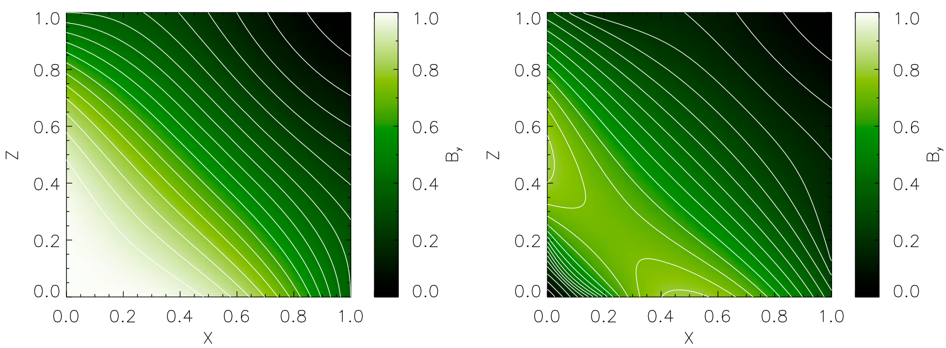
The contours of B_p in figure 5 can be used to map contours of the shear field, B_y, which are shown in figure 6. This is where the most dramatic changes can be seen. We are allowed far more variation in the value of the shear field in the anisotropic case than in the isotropic case where we are restricted to a function of A. This shows that, whilst contours of A can look extremely similar when we introduce anisotropy in 2.5D, the differences are significant when we consider the shear field profile.

We also show the current density in the x-z plane, j_p (see figure 7, where the contours are omitted for clarity). Recall that the current density in the plane is given by the modulus of the gradient of the shear field. We see a large difference in j_p between the two cases. There is a



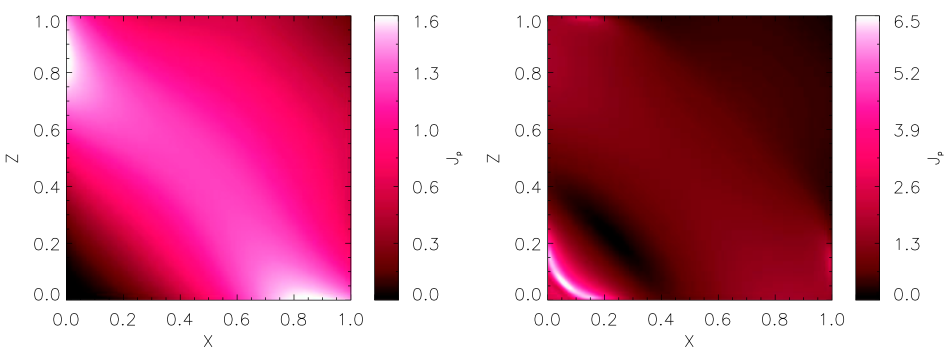
(a) $\lambda = 0$ (b) $\lambda = 1.5$

Figure 5. Contour plot of B_p in the isotropic (a) and anisotropic (b) cases.



(a) $\lambda = 0$ (b) $\lambda = 1.5$

Figure 6. Contour plot of the shear field, B_y, in the isotropic (a) and anisotropic (b) cases.



(a) $\lambda = 0$ (b) $\lambda = 1.5$

Figure 7. Contour plot of the planar current density, j_p, in the isotropic (a) and anisotropic (b) cases. Note the different scales in each plot.

strip near the origin where the planar current density is extremely large in the anisotropic case compared to the isotropic case. Slightly further out from the origin there is another strip where j_p is much less in the anisotropic case than the isotropic case. This test case tells us that even small anisotropies can introduce extremely large currents. The large current density comes from the high gradients of the planar magnetic field, B_p. They are not seen in the isotropic case since they are multiplied by the derivative of the shear field with respect to B_p, which is zero when we have isotropy. As soon as we leave the isotropic regime, the high gradients in B_p are no longer suppressed by the shear field derivatives and we get the large currents that we see in figure 7.

5. Summary and discussion

We have presented an improved formalism for computing translationally invariant MHD equilibria with anisotropic pressure and currents with a field aligned component. This builds on previous works which were limited to 2D fields or significant simplifications in the form of restrictions on either the pressure or shear field. We also show that, by using an effective pressure function, all 2.5D Grad-Shafranov equations can be reduced to an equivalent 2D Grad-Shafranov equation. The formalism outlines a constraint that is not seen in the 2D case: there are restrictions that link the choice of shear field to the choice of pressure. This constraint is satisfied automatically in the 2D case and therefore never appears in the theory. This constraint is more problematic in the current 2.5D formalism since one runs into problems trying to disentangle the shear field from the modulus of the magnetic field. In our formalism this problem is bypassed by considering functions in terms of the modulus of the gradient of the flux function. We no longer need to restrict ourselves on the forms of the pressure or shear field as was required before. Three approaches by which one can now make further progress are discussed. One of these approaches, where we specify the effective parallel pressure, has significant merits: a tractable Grad-Shafranov equation where the pressures and shear field can be easily derived from the effective pressure. An illustrative example is shown which uses this method to solve the 2.5D problem for a shear field that does not simply depend on the flux function alone. The example also shows that anisotropic pressures and isotropic pressures can give similar flux functions but with major differences only showing themselves in the value of the shear field and planar current density. The most important aspect of the new formalism is that it allows the problem of finding MHD equilibria with anisotropic pressure and non-vanishing shear field to be tackled using standard numerical methods due to removing the implicit coupling between the shear field and the magnetic field strength present in current formulations.

Acknowledgements

We would like to thank the referees of this paper for their helpful and insightful comments, and the editor for the great amount of help and guidance provided throughout the publishing process.

Disclosure statement

No potential conflict of interest was reported by the authors.

Funding

This work was supported by the Science and Technology Facilities Council by Doctoral Grant [ST/K502327/1] (JDBH) and Consolidated Grant [ST/K000950/1] (TN).

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Appendix A. Rotational invariance

In this Appendix we show that the same formalism as above can be used in cases with rotational symmetry, however with the same limitations that also apply to the isotropic pressure case with rotational symmetry as discussed later. Specifically we go through the derivation of the formalism in spherical coordinates. This coordinate system would, for example, be the most appropriate for modelling rotating magnetospheres.

Let the magnetic field be given by

$$\boldsymbol{B} = \nabla A \times \nabla \phi + B_\phi \boldsymbol{e}_\phi.$$

We proceed to solve the force balance equation (6) in the same way as previous authors, by considering quantities as functions of A and $B = |\boldsymbol{B}|$, and so arrive at the analogous equations to (5a,b), namely

$$-\nabla \boldsymbol{\cdot} \left[\left(\frac{1}{\mu_0} - \frac{1}{B} \frac{\partial P_\parallel}{\partial B} \right) \frac{\nabla A}{r^2 \sin^2 \theta} \right] = \frac{\partial P_\parallel}{\partial A} + \frac{B_\phi}{\mu_0 r \sin \theta} \frac{\mathrm{d}F}{\mathrm{d}A}, \tag{A.1a}$$

$$B_\phi r \sin \theta \left(1 - \frac{\mu_0}{B} \frac{\partial P_\parallel}{\partial B} \right) = F(A). \tag{A.1b}$$

Again we see the same implicit coupling between B_ϕ and B that we saw with Cartesian coordinates in section 2. We now apply our new formalism where we consider quantities as functions of A and B_p where

$$B_p = \frac{|\nabla A|}{r \sin \theta}.$$

After some algebra, (A.1a,b) are transformed to

$$-\nabla \boldsymbol{\cdot} \left[\left(\frac{1}{\mu_0} - \frac{1}{M_\phi} \frac{\partial P_\parallel}{\partial B_p} \right) \frac{\nabla A}{r^2 \sin^2 \theta} \right] = \frac{\partial P_\parallel}{\partial A} - \frac{B_\phi}{M_\phi} \frac{\partial P_\parallel}{\partial B_p} \frac{\partial B_\phi}{\partial A} + \frac{B_\phi}{\mu_0 r \sin \theta} \frac{\mathrm{d}F}{\mathrm{d}A}, \tag{A.2a}$$

$$B_\phi r \sin \theta \left(1 - \frac{\mu_0}{M_\phi} \frac{\partial P_\parallel}{\partial B_p} \right) = F(A), \tag{A.2b}$$

where

$$M_\phi = B_p + B_\phi \frac{\partial B_\phi}{\partial B_p}. \tag{A.2c}$$

This is the spherical equivalent of the formalism given in (12a,b). It is, however, impossible to combine terms into an effective pressure as we did in the Cartesian case due to the presence of the scale factors. This is completely analogous to the isotropic case where the scale factors prevent the combination of terms into a single total pressure term. To make progress, one must then specify either $P_\parallel$ or B_ϕ as functions of A and B_p.