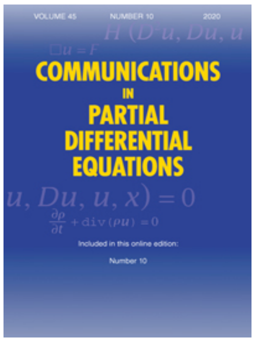

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Inverse source problem for a system of wave equations on a Lorentzian manifold

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ABSTRACT
A system of wave equations on a Lorentzian manifold, the coefficients of which depend on time relates to the Einstein equation in general relativity. We consider inverse source problem for the system in this paper. Having established the Carleman estimate with a second large parameter for the Laplace–Beltrami operator on a Lorentzian manifold under assumptions independent of a choice of local coordinates on a suitable weight function, we consider its application to the inverse source problem for the system and prove local Hölder stability.

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1. Introduction and main result

Let $T>0,\ n\in\mathbb{N},$ and M be a compact oriented n-dimensional smooth manifold with boundary. We set $L:=[-T,T]\times M$ and let (L,g) be a Lorentzian manifold with metric g having signature $(-,+,...,+)$ such that the submanifolds $M^t:=\{t\}\times M$ are spacelike for all $t\in[-T,T]$ and $\partial_t:=\frac{\partial}{\partial t}$ is timelike. The Lorentzian metric is a symmetric non-degenerate covariant 2-tensor field such that for every point $p\in L,$ there is a basis $e_0,...,e_n$ for T_pL such that $g(e_\mu,e_\nu)$ are the components of the standard Minkowski metric $\mathrm{diag}(-1,1,...,1)$. In this paper, we consider an intermediate boundary value problem of a system for a function $h:L\to\mathbb{R}^\ell$ with $\ell\in\mathbb{N}$,

$$\begin{cases} Ph:=\Box_g h+a(t,x)h=H(t,x) & \text{in } L,\\ h=\partial_{\hat{N}}h=0 & \text{on } M^0=\{0\}\times M,\\ h=0 & \text{on } \Sigma_1:=[-T,T]\times\Gamma_1. \end{cases} \tag{1.1}$$

Here let the coefficient a be an $\ell\times\ell$ matrix-valued function on L and the source term H be an ℓ vector-valued function on L. Let $\pi_0:L\to[-T,T]$ be the projection and $\nabla\pi_0$ be the gradient of π_0. $\hat{N}:=-\frac{\nabla\pi_0}{\sqrt{|g(\nabla\pi_0,\nabla\pi_0)|}}$ denotes the future directed unit timelike

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vector field such that for all $p \in M^t$ and $X \in T_pM^t, g(\hat{N}_p, \iota_* X) = 0$, where $\iota : M^t \hookrightarrow L$ is the embedding. We note, in this paper, that summations with respect to Greek indices range from 0 to n, whereas those for Roman indices range from 1 to n. Furthermore, $\Box_g$ is defined by $\Box_g := g^{\mu\nu}(\partial_\mu\partial_\nu - \Gamma^\rho_{\mu\nu}\partial_\rho)$ for functions on L, where $(g^{\mu\nu})$ is the matrix inverse to $(g_{\mu\nu})$, which are components of the metric $g = g_{\mu\nu}dx^\mu \otimes dx^\nu$, and $\Gamma^\rho_{\mu\nu}$ is the Christoffel symbol of the Levi-Civita connection defined by

$$\Gamma^\rho_{\mu\nu} := \frac{1}{2} g^{\rho\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}).$$

$\Gamma_1 \subset \partial M$ denotes a given open submanifold.

The equation in (1.1) relates to general relativity. Because this type of equation having the same principal term is derived from the Einstein equation by choosing a special coordinate system or a suitable gauge function (e.g., [1, Chapter 18.8], [2, Chapter III.11], [3, Part III], [4, 5, Chapter 33]) and then by the linearization of the Einstein equation, we reduce it to the system having the form (1.1). Interested readers are referred to Taylor [1], Choquet-Bruhat [2], and Ringström [3] for a direct derivation of the equation having the same form (1.1).

We assume the source term H is written by $H(t,x) = S(t,x)f(x)$, where S is an $\ell \times \ell$ matrix-valued function on L and f is an ℓ vector-valued function on M. The main focus of this paper is the inverse source problem to determine f from the partial boundary data of the solution: $\partial_N \partial^k_{\hat{N}} h|_{\Sigma_1}$ for $k = 0, 1, 2$, where ∂_N denotes the normal derivative with respect to the metric g. We prove the uniqueness and stability for the local inverse source problem. The argument is based on the Carleman estimate, which was introduced by Carleman in [6], and the Bukhgeim–Klibanov method in [7]. The Carleman estimate was first invented to prove the unique continuation property for elliptic operators for which the coefficients are not necessarily real analytic. Using the Carleman estimate, Bukhgeim and Klibanov proved global uniqueness results for multidimensional coefficient inverse problems. This methodology is widely applicable to not only elliptic equations but also various partial differential equations provided that we can prove the Carleman estimate for the operators we are considering. For hyperbolic equations, Baudouin, De Buhan, and Ervedoza [8] proved the global Carleman estimate for wave equations and considered its applications to controllability, inverse problems, and reconstructions. Imanuvilov and Yamamoto [9] proved the global Lipschitz stability for wave equations by interior observations near the boundary. Bellassoued and Yamamoto [10, 11] considered both local and global inverse source problems, and coefficient inverse problems for wave equations on a compact Riemannian manifold. Jiang, Liu, and Yamamoto [12] considered the local inverse source problems for wave equations, the coefficients of which depend on time t in the Euclidean space under the assumption that the Carleman estimate for such operators exists. In this paper, we prove also the Carleman estimate for the Laplace–Beltrami operator. For time-independent wave equation, to apply the Carleman estimate to consider the inverse source problem, we extend the solution to negative time intervals. However, when the coefficients depend on time, there is a difficulty in extending the solution to negative time intervals when trying to apply the Carleman estimate. For instance, an even extension of the solution with respect to time t no longer satisfies the equation. Hence, we consider the equation in $[-T, T]$ from the beginning.

Because Bellassoued and Yamamoto [11] dealt with the wave equation on a compact Riemannian manifold, we prove the Carleman estimate on a Lorentzian manifold with the help of their tools. Indeed, the assumptions on a weight function (A.1) and (A.2) in the next section are generalizations of the situation for a Riemannian manifold.

To describe our main result, we define the Sobolev space on manifolds, which should be defined so as not to depend on a choice of coordinate systems in general.

Definition 1.1. Let M be a compact oriented n-dimensional smooth manifold, and $\{(U_i, x_i)\}_i$ be a coordinate system. Assume $\{\chi_i\}_i$ is a finite partition of unity subordinate to the covering such that $\mathrm{supp}\chi_i \subset U_i$. Given $u \in C^\infty(M; \mathbb{R}^\ell)$ and integer k, define

$$||u||_{H^k(M;\mathbb{R}^\ell)} := \left(\sum_i \sum_{|\alpha| \le k} \int_{x_i(U_i)} (\chi_i |\partial^\alpha u|^2) \circ x_i^{-1} dx_i^1 \cdots dx_i^n \right)^{\frac{1}{2}},$$

where ∂^α signifies differentiation with respect to x_{i}.

The inner product can be also defined in the same way. By taking the completion of the smooth functions, one obtains a real Hilbert space. Note that different partitions of unity and coordinates yield equivalent norms. (e.g., Ringström [3, Section 15]) Although our integrations and derivatives on compact manifolds should be written using a partition of unity and local coordinates, we omit these representations throughout this paper to avoid notational complexity.

Let $\iota : M^t := \{t\} \times M \hookrightarrow L$ be the embedding and $g_\flat := \iota^* g$ be the induced metric on M^t by the embedding ι. We assume throughout that the Lorentzian metric g is smooth on L such that M^t is spacelike, i.e., $g_\flat$ is Riemannian metric on M^t, and ∂_t is timelike, i.e., $g(\partial_t, \partial_t) < 0$. $\hat{N} := -\frac{\nabla \pi_0}{\sqrt{|g(\nabla \pi_0, \nabla \pi_0)|}}$ denotes the future directed unit timelike vector field such that for all $p \in M^t$ and $X \in T_p M^t, g(\hat{N}_p, \iota_* X) = 0$. We assume the coefficient has enough regularity,

$$a \in W^{2,\infty}(-T, T; L^\infty(M; \mathbb{R}^{\ell \times \ell})).$$

Let $M_\epsilon := \{x \in M | \psi(0, x) > \epsilon\}$ be a level set of ψ, where ψ is the weight function satisfying assumptions (A.1) and (A.2) to be stated in the next section. We are ready to describe the main result of this paper.

Theorem 1.2. *Let $\ell \in \mathbb{N}$, $T > 0$, M be a compact oriented n-dimensional smooth manifold with boundary, and $L := [-T, T] \times M$. Let g be a smooth Lorentzian metric on L such that M^t is spacelike and ∂_t is timelike. Assume $H(t, x) = S(t, x) f(x)$, (A.1), (A.2), (3.2) and (3.3). Furthermore, assume that there exists a unique solution h to (1.1) in the class*

$$h \in \bigcap_{k=0}^{2} H^{4-k}(-T, T; H^k(M; \mathbb{R}^\ell)).$$

Then, there exists $\epsilon^* > 0$ such that for any $\epsilon \in (\epsilon_*, \epsilon^*)$, there exist constants $C > 0$ and $\theta \in (0, 1)$ such that

$$||f||_{L^2(M_\epsilon;\mathbb{R}^\ell)} \le C\mathcal{D} + C\mathcal{F}^{1-\theta}\mathcal{D}^\theta,$$

where $\epsilon_* \ge 0$ is the number in (3.3),

$$\mathcal{F} := ||f||_{L^2(M;\mathbb{R}^\ell)} + \sum_{k=0}^{2} ||h||_{H^{3-k}(-T,T;H^k(M;\mathbb{R}^\ell))},$$

$$\mathcal{D} := \sum_{k=0}^{2} ||\partial_N \partial_{\hat{N}}^k h||_{L^2(-T,T;L^2(\Gamma_1;\mathbb{R}^\ell))},$$

and N denotes the outer unit normal vector field to $\Sigma_1 := [-T,T] \times \Gamma_1$.

(A.1) and (A.2) are the assumptions on a weight function needed for the Carleman estimate. (3.2) and (3.3) are the respective assumptions on the source and coefficient terms, and on a given submanifold Γ_1. Details of these assumptions are explained in subsequent sections.

2. Carleman estimate

Let us fix a local coordinate $(x^1, ..., x^n)$ on M and then, obtain a local coordinate $(x^0 = t, x^1, ..., x^n)$ on L such that

$$g = -dt \otimes dt + g_{ij} dx^i \otimes dx^j.$$

We call the local coordinate semigeodesic coordinate in this paper. Henceforth, if we write statements using a local coordinate, the coordinate is always taken by the semigeodesic coordinate, unless specified otherwise.

Remark 2.1. *There exists the semigeodesic coordinate locally. (e.g., Remark 5.1 in [2, I]) Indeed, for a local coordinate $(y^0(t), y^1, ..., y^n)$ near $(t,x) \in L$, there exists a change of the coordinate into the semigeodesic coordinate $(x^0 = t, x^1, ..., x^n)$ if and only if an inverse transform exists. Then, the components $g'_{\mu\nu}$ of the metric g represented by $(t, x^1, ..., x^n)$ satisfy*

$$g'_{i0} = \frac{\partial y^j}{\partial x^i}\left(g_{j0}\frac{dy^0}{dt} + g_{jk}\frac{\partial y^k}{\partial t}\right), i = 1, ..., n,$$

$$g'_{00} = g_{00}\left(\frac{dy^0}{dt}\right)^2 + 2g_{0j}\frac{dy^0}{dt}\frac{\partial y^j}{\partial t} + g_{jk}\frac{\partial y^j}{\partial t}\frac{\partial y^k}{\partial t}.$$

$g_{j0}\frac{dy^0}{dt} + g_{jk}\frac{\partial y^k}{\partial t} = 0$ for $j = 1, ..., n$ and $g_{00}\left(\frac{dy^0}{dt}\right)^2 + 2g_{0j}\frac{dy^0}{dt}\frac{\partial y^j}{\partial t} + g_{jk}\frac{\partial y^j}{\partial t}\frac{\partial y^k}{\partial t} = -1$ are equivalent to

$$g_{\mu\nu}\frac{\partial y^\nu}{\partial t} = -\delta^0_\mu \left(\frac{dy^0}{dt}\right)^{-1}, \mu = 0, ..., n \Longleftrightarrow \frac{\partial y^\nu}{\partial t} = -g^{0\nu}\left(\frac{dy^0}{dt}\right)^{-1}, \nu = 0, ..., n,$$

which is locally solvable as an initial problem of a first-order system since $g^{00} < 0$ by our assumption that ∂_t is timelike and Lemma 8.5 in [3].

Let $\ell \in \mathbb{N}$, $T > 0$, M be a compact oriented n-dimensional smooth manifold with boundary, and $L := [-T,T] \times M$. Let g be a smooth Lorentzian metric on L such that M^t is spacelike and ∂_t is timelike. In this section, we consider the Carleman estimate for the operator P,

$$\begin{aligned} Ph := & \Box_g h + a(t,x)h \\ & = g^{\mu\nu}(\partial_\mu\partial_\nu - \Gamma^\rho_{\mu\nu}\partial_\rho)h + a(t,x)h. \end{aligned}$$

Let the coefficient a has enough regularity,

$$a \in W^{2,\infty}(-T,T;L^\infty(M;\mathbb{R}^{\ell\times\ell})).$$

To establish the Carleman estimate for the above operator P, we consider first of all Carleman estimate for the Laplace–Beltrami operator for $\mathbb{R}$-valued functions

$$\Box_g = g^{\mu\nu}(\partial_\mu\partial_\nu - \Gamma^\rho_{\mu\nu}\partial_\rho)$$

on an $n+1$-dimensional Lorentzian manifold L. The following method is based on the works by Bellassoued and Yamamoto [10, 11]. Note that angled bracket $\langle\cdot,\cdot\rangle$ denotes the inner product with respect to the metric g, i.e., $\langle X,Y\rangle := g(X,Y) = g_{\mu\nu}X^\mu Y^\nu$ for $X, Y \in T_pL$ and $p \in L$. Let $\pi_0 : L \to [-T,T]$ and $\pi_1 : L \to M$ be the projections, and $d\tau^2$ and $g_\flat$ be the respective induced squared line element and Riemannian metric by the canonical embeddings $[-T,T]\hookrightarrow L$ and $M^t\hookrightarrow L$. $\nabla u = \nabla_g u = \nabla^\mu u \frac{\partial}{\partial x^\mu} = g^{\mu\nu}\partial_\nu u \frac{\partial}{\partial x^\mu}$ denotes the gradient of a function u with respect to the metric g.

(A.1) The Hessian of ψ with respect to g satisfies

$$\begin{gathered} \exists\kappa_1 > 0, \exists\kappa_2 > 0 \text{ s.t. } \forall p \in L, \forall X \in T_pL, \\ \nabla^2\psi(X,X) \geq -2\kappa_2 d\tau^2((d\pi_0)X,(d\pi_0)X) + 2\kappa_1 g_\flat((d\pi_1)X,(d\pi_1)X) \end{gathered}$$

with

$$1 < \frac{\kappa_1}{\kappa_2}.$$

(A.2) ψ has no critical points on L, i.e.,

$$\min_L g_\flat((d\pi_1)\nabla\psi,(d\pi_1)\nabla\psi) > 0,$$

and

$$\psi(0,x) > \psi(t,x) \text{ a.e. } (t,x) \in L.$$

Remark 2.2. These assumptions (A.1) and (A.2) are independent of a choice of local coordinates by their definitions. When we write $X = X^\mu \frac{\partial}{\partial x^\mu} \in T_pL$ by taking the semi-geodesic coordinate, we obtain the representations

$$\begin{aligned} d\tau^2((d\pi_0)X,(d\pi_0)X) &= |X^0|^2 := -g_{00}(X^0)^2 = (X^0)^2, \\ g_\flat((d\pi_1)X,(d\pi_1)X) &= |X|^2 := g_{ij}X^iX^j\left(= \sum_{i,j=1}^n g_{ij}X^iX^j\right). \end{aligned}$$

Example 2.3. We compare these assumptions (A.1) and (A.2) with those used in considering the wave equation on a compact n-dimensional smooth Riemannian manifold $(M,\bar{g})$ by Bellassoued and Yamamoto [10, 11]. We take as a function ψ,

$$\psi(t,x) := \psi_0(x) - \kappa_2 t^2, \quad (t,x) \in [-T,T] \times M,$$

where $\kappa_2 > 0$ is a constant and ψ_0 is a positive smooth function in M. In this case, our considering Lorentzian metric has the form $g = -dt \otimes dt + \bar{g}$ and $g_\flat = \bar{g}$ holds. The assumptions regarding the operator $-\partial_t^2 + \Delta_{\bar{g}}$, where $\Delta_{\bar{g}}$ is the Laplace–Beltrami operator with respect to the metric $\bar{g}$, are the following (B.1) and (B.2).

(B.1) The Hessian of ψ_0 with respect to $\bar{g}$ satisfies

$$\exists \kappa_1 > 0 \text{ s.t. } \forall p \in M, \forall \bar{X} \in T_pM,$$
$$\nabla_{\bar{g}}^2 \psi_0(\bar{X}, \bar{X}) \geq 2\kappa_1 |\bar{X}|_{\bar{g}}^2,$$

where $|\bar{X}|_{\bar{g}} := (\bar{g}_{ij}\bar{X}^i\bar{X}^j)^{\frac{1}{2}}$ with

$$1 < \frac{\kappa_1}{\kappa_2}.$$

(B.2) ψ_0 has no critical points on M,

$$\min_M |\nabla_{\bar{g}} \psi_0|_{\bar{g}} > 0.$$

Clearly, if assumptions (B.1) and (B.2) hold, then our assumptions (A.1) and (A.2) hold. Indeed, for $p \in L$ and $X \in T_pL$, if (B.1) holds, then we have

$$\begin{aligned}\nabla_g^2 \psi(X, X) &= -2\kappa_2 d\tau^2((d\pi_0)X, (d\pi_0)X) + \nabla_{\bar{g}}^2 \psi_0((d\pi_1)X, (d\pi_1)X) \\ &\geq -2\kappa_2 d\tau^2((d\pi_0)X, (d\pi_0)X) + 2\kappa_1 \bar{g}((d\pi_1)X, (d\pi_1)X)\end{aligned}$$

with

$$1 < \frac{\kappa_1}{\kappa_2}.$$

Furthermore, having obtained

$$g_\flat((d\pi_1)\nabla\psi, (d\pi_1)\nabla\psi) = \bar{g}(\nabla_{\bar{g}}\psi_0, \nabla_{\bar{g}}\psi_0) > 0,$$

we find (A.2) holds.

Let us define the weight function using ψ,

$$\varphi(t, x) := e^{\gamma\psi(t,x)}, \quad (t, x) \in L,$$

where $\gamma > 0$ is a parameter. For notational simplicity, we set

$$\sigma(t, x) := s\gamma\varphi(t, x), \quad (t, x) \in L,$$

where $s > 0$ is a parameter. We set $\Sigma := [-T, T] \times \partial M$. Before describing the Carleman estimate, we define a quantity independent of a choice of local coordinates.

Definition 2.4. Let ∇u be the gradient of $u \in C^\infty(L)$ and define the quantity independent of a choice of local coordinates

$$E(u) := d\tau^2((d\pi_0)\nabla u, (d\pi_0)\nabla u) + g_\flat((d\pi_1)\nabla u, (d\pi_1)\nabla u).$$

Remark 2.5. In the same way as Remark 2.2, the quantity has the representation,

$$E(u) = |\nabla^0 u|^2 + |\nabla u|^2,$$

where $\nabla^0 u$ is a component of the gradient $\nabla u = \nabla^\mu u \frac{\partial}{\partial x^\mu}$.

Lemma 2.6. *Assume (A.1) and (A.2). Then, there exists a constant $\gamma_* > 0$ such that for any $\gamma > \gamma_*$, there exist constants $s_* = s_*(\gamma)$ and $C > 0$ such that*

$$\int_L e^{2s\varphi}\sigma\left(E(u) + \sigma^2|u|^2\right)\omega_L \le C\int_L e^{2s\varphi}|\Box_g u|^2\omega_L + C\int_\Sigma e^{2s\varphi}\sigma|\partial_N u|^2\omega_\Sigma$$

holds for all $s > s_*$ and $u \in C^\infty(L)$ satisfying $u = \partial_N u = 0$ on $M^{\pm T}$ and $u = 0$ on Σ. $\partial_N u := \langle \nabla u, N\rangle = Nu$, where N is the outer unit normal vector filed to ∂L with respect to the metric g. ω_L and ω_Σ denote the respective volume elements of L and Σ.

The proof of Lemma 2.6 is presented in Appendix.

Proposition 2.7. *Assume (A.1) and (A.2). Then, there exists a constant $\gamma_* > 0$ such that for any $\gamma > \gamma_*$, there exist constants $s_* = s_*(\gamma)$ and $C > 0$ such that*

$$\sum_{m=1}^{\ell}\int_L e^{2s\varphi}\sigma\left(E(h_m) + \sigma^2|h_m|^2\right)\omega_L \le C\int_L e^{2s\varphi}|Ph|^2\omega_L + C\int_\Sigma e^{2s\varphi}\sigma|\partial_N h|^2\omega_\Sigma$$

holds for all $s > s_*$ and $h \in C^\infty(L;\mathbb{R}^\ell)$ satisfying $h = \partial_N h = 0$ on $M^{\pm T}$ and $h = 0$ on Σ. $\partial_N h := \langle \nabla h, N\rangle = Nh$, where N is the outer unit normal vector field to ∂L.

Proof. With the help of Lemma 2.6, Proposition 2.7 is obtained by addition and absorption by choosing $s > 0$ large enough. □

3 Proof of Theorem 1.2

3.1. Preliminary

Let $T > 0$, M be a compact oriented n-dimensional smooth manifold with boundary, $L := [-T, T] \times M$, and $M^t := \{t\} \times M$. Let (L, g) be a smooth Lorentzian manifold such that M^t is spacelike and ∂_t is timelike with respect to the metric g. Let us fix the semigeodesic coordinate $(x^0 = t, x^1, \dots, x^n)$. We remark that in such a coordinate, we find $\hat{N} = \partial_t$, where $\hat{N} := -\frac{\nabla\pi_0}{\sqrt{|g(\nabla\pi_0, \nabla\pi_0)|}}$ is the future directed unit timelike vector field such that for all $p \in M^t$ and $X \in T_pM^t, g(\hat{N}_p, \iota_* X) = 0$, where $\iota : M^t \hookrightarrow L$ is the embedding. We consider

$$\begin{cases} Ph = g^{\mu\nu}(\partial_\mu\partial_\nu - \Gamma^\rho_{\mu\nu}\partial_\rho)h + a(t,x)h = S(t,x)f(x) & \text{in } L, \\ h = \partial_t h = 0 & \text{on } M^0, \\ h = 0 & \text{on } \Sigma_1 := [-T, T] \times \Gamma_1. \end{cases} \tag{3.1}$$

$\Gamma_1 \subset \partial M$ is an open submanifold. We assume

$$\begin{cases} a \in W^{2,\infty}(-T, T; L^\infty(M;\mathbb{R}^{\ell\times\ell})), \\ S \in W^{2,\infty}(-T, T; L^\infty(M;\mathbb{R}^{\ell\times\ell})), \\ \exists m_0 > 0 \text{ s.t. } \det S(0,\cdot) \ge m_0 \text{ a.e. on } M, \\ f \in L^2(M;\mathbb{R}^\ell). \end{cases} \tag{3.2}$$

This type of inverse source problem having a time-dependent principal part was studied by Jiang, Liu, and Yamamoto [12] for a hyperbolic equation. Furthermore, we assume a unique weak solution h exists to (3.1) in the class

$$h \in \bigcap_{k=0}^{2} H^{4-k}(-T, T; H^k(M; \mathbb{R}^\ell)).$$

We define the level set L_ϵ of ψ for $\epsilon \geq 0$ by

$$L_\epsilon := \{(t,x) \in L | \psi(t,x) > \epsilon\}$$

and

$$M_\epsilon := \{x \in M | \psi(0,x) > \epsilon\}.$$

In regard to a relation between the observation boundary Σ_1 and the level set L_ε, we assume that

$$\exists \epsilon_* \geq 0 \text{ s.t. } \emptyset \neq L_{\epsilon_*} \cap \partial L \subset \Sigma_1. \tag{3.3}$$

On considering the inverse source problem of (3.1) as an application to the Carleman estimate Proposition 2.7, we need a relation in regard to energies.

Lemma 3.1. *Let E be the quantity defined in Definition 2.4. For all $u \in C^\infty(L)$, the identity*

$$E(u) = |\partial_t u|^2 + g^{ij}\partial_i u \partial_j u$$

holds by the semigeodesic coordinate.

Proof. We note here that summations with respect to Greek indices range from 0 to n, whereas those for Roman indices range from 1 to n. We take the semigeodesic coordinate system.

$$\begin{aligned} E(u) &= -g_{00}(g^{0\mu}\partial_\mu u)^2 + g_{ij}(g^{i\mu}\partial_\mu u)(g^{j\nu}\partial_\nu u) \\ &= |\partial_t u|^2 + g_{ij}g^{ip}(\partial_p u)g^{jq}(\partial_q u). \end{aligned}$$

With the help of the semigeodesic coordinate, it follows that $g_\flat^{ij} = g^{ij}$ for all $1 \leq i, j \leq n$. We then obtain by the above formulation,

$$E(u) = |\partial_t u|^2 + g^{jq}\partial_j u \partial_q u.$$

□

Proposition 3.2. *Assume (A.1) and (A.2). Then, there exists a constant $\gamma_* > 0$ such that for any $\gamma > \gamma_*$, there exist constants $s_* = s_*(\gamma)$ and $C > 0$ such that*

$$\begin{aligned} &\sum_{m=1}^{\ell} \int_L e^{2s\varphi}\sigma(|\partial_t h_m|^2 + g^{ij}\partial_i h_m \partial_j h_m + \sigma^2 |h_m|^2)\omega_L \\ &\leq C\int_L e^{2s\varphi}|Ph|^2\omega_L + C\int_\Sigma e^{2s\varphi}\sigma|\partial_N h|^2\omega_\Sigma \end{aligned}$$

holds for all $s > s_*$ and $h \in \bigcap_{k=0}^{2} H^{2-k}(-T, T; H^k(M; \mathbb{R}^\ell))$ satisfying $h = \partial_N h = 0$ on $M^{\pm T}$ and $h = 0$ on Σ.

Proof. We apply Lemma 3.1 to Proposition 2.7 to complete the proof. □

Moreover, in the proof of Theorem 1.2, we shall use the next lemma. Lemma 3.3 plays an important role when we prove inverse source problems with time-dependent coefficients, which was introduced in [12]. Its proof is also presented in Appendix.

Lemma 3.3. *Assume (A.1) and (A.2). Let $\imath : M^t \hookrightarrow L$ be the embedding, $\hat{N}$ be the future directed unit timelike vector field such that $\forall p \in M^t, \forall X \in T_pM^t, g(\hat{N}_p, \imath_* X) = 0$, and $\Delta_{g_\flat}$ be the Laplace–Beltrami operator with respect to the iniduced metric $g_\flat = \imath^* g$. Assume $a \in W^{2,\infty}(-T, T; L^\infty(M))$ and $P = \Box_g + a\cdot$. There exist constants $s_* > 0$ and $C > 0$ such that*

$$\int_L e^{2s\varphi} |\Delta_{g_\flat} v|^2 \omega_L \leq C \int_L e^{2s\varphi} \left(\frac{1}{s} |\partial_{\hat{N}} Pv|^2 + |Pv|^2 \right) \omega_L + Ce^{Cs}\mathcal{E}^2$$

holds for all $s > s_*$ and $v \in \bigcap_{k=0}^{2} H^{3-k}(-T, T; H^k(M))$ satisfying $v = \partial_N v = \partial_N^2 v = 0$ on $M^{\pm T}$ and $v = 0$ on Σ. Note that

$$\mathcal{E} := \sum_{k=0}^{1} ||\partial_N \partial_{\hat{N}}^k v||_{L^2(-T,T;L^2(\partial M))}.$$

To prove Lemma 3.3, we use the global elliptic estimate Lemma 3.4. (e.g., [13, 14] and [15]) Its proof is also presented in Appendix.

Lemma 3.4. *Let M be a compact oriented n-dimensional smooth manifold with boundary and A be an elliptic differential operator on M. Then, there exists a constant $C > 0$ such that*

$$||v||_{H^2(M)} \leq C \Big(||Av||_{L^2(M)} + ||v||_{L^2(M)} \Big)$$

holds for all $v \in H_0^1(M)$ satisfying $Av \in L^2(M)$.

3.2. Proof of Theorem 1.2

Proof of Theorem 1.2. Let $\epsilon_* \geq 0$ be the number in (3.3). We introduce a cutoff function χ,

$$\chi(t,x) := \begin{cases} 1 & \text{in } L_{2\epsilon}, \\ 0 & \text{in } L \setminus L_\epsilon \end{cases}$$

for sufficiently small $\epsilon > \epsilon_*$ so that

$$\emptyset \neq L_{3\epsilon} \cap \partial L (\subset \Sigma_1).$$

Let us fix the semigeodesic coordinate $(x^0 = t, x^1, ..., x^n)$. In such a coordinate, we find $\hat{N} = \partial_t$. For fixed $i = 0$, 1, 2, we set new functions $v^{(i)} := \chi \partial_t^i h$. We calculate $Pv^{(2)}$,

$$\begin{cases} Pv^{(2)} = \chi P \partial_t^2 h + 2\langle \nabla\chi, \nabla\partial_t^2 h\rangle + \partial_t^2 h \Box_g \chi & \text{in } L, \\ v^{(2)} = \partial_t v^{(2)} = 0 & \text{on } M^{\pm T}, \\ v^{(2)} = 0 & \text{on } \Sigma = [-T, T] \times \partial M. \end{cases}$$

Then, we apply Proposition 3.2 to $v^{(2)}$ to obtain

$$\begin{aligned}&\sum_{m=1}^{\ell}\int_L e^{2s\varphi}\left(sE(v_m^{(2)})+s^3|v_m^{(2)}|^2\right)\omega_L\\&\le C\int_L e^{2s\varphi}|\chi(P\partial_t^2 h)|^2\omega_L+C\int_L e^{2s\varphi}|2\langle\nabla\chi,\nabla\partial_t^2 h\rangle+\partial_t^2 h\Box_g\chi|^2\omega_L\\&\quad+Ce^{Cs}\int_{\Sigma_1}|\partial_N v^{(2)}|^2\omega_\Sigma.\end{aligned}\tag{3.4}$$

In regard to the first summand on the right-hand side of (3.4), taking

$$\begin{aligned}&\chi(P\partial_t^2 h)\\&=\chi\partial_t^2 Sf-\partial_t^2 g^{\mu\nu}\partial_\mu\partial_\nu(\chi h)-\partial_t^2 a(\chi h)-2\partial_t g^{\mu\nu}\partial_\mu\partial_\nu(\chi\partial_t h)-2\partial_t a(\chi\partial_t h)\\&\quad+2\partial_t(g^{\mu\nu}\Gamma^\rho_{\mu\nu})\partial_\rho(\chi\partial_t h)+\partial_t^2(g^{\mu\nu}\Gamma^\rho_{\mu\nu})\partial_\rho(\chi h)+[2\partial_t^2 g^{\mu\nu}\partial_\mu\chi\partial_\nu h+\partial_t^2 g^{\mu\nu}(\partial_\mu\partial_\nu\chi)h\\&\quad+4\partial_t g^{\mu\nu}\partial_\mu\chi(\partial_\nu\partial_t h)+2\partial_t g^{\mu\nu}(\partial_\mu\partial_\nu\chi)\partial_t h-\partial_t^2(g^{\mu\nu}\Gamma^\rho_{\mu\nu})(\partial_\rho\chi)h-2\partial_t(g^{\mu\nu}\Gamma^\rho_{\mu\nu})(\partial_\rho\chi)\partial_t h]\end{aligned}$$

into account, and with supp $\partial^\alpha\chi\subset\overline{L_\epsilon\setminus L_{2\epsilon}}$ for $|\alpha|\ge 1$, we apply Lemma 3.4, and then Lemma 3.3 to obtain

$$\begin{aligned}&\int_L e^{2s\varphi}|\chi(P\partial_t^2 h)|^2\omega_L\\&\le C\sum_{i=0}^{1}\sum_{m=1}^{\ell}\left(\int_{-T}^{T}||e^{s\varphi}v_m^{(i)}||^2_{H^2(M)}dt+\int_L e^{2s\varphi}(s^2E(v_m^{(i)})+s^4|v_m^{(i)}|^2)\omega_L\right)\\&\quad+C\int_L e^{2s\varphi}|f|^2\omega_L+Ce^{2\epsilon_2 s}\sum_{k=0}^{1}||h||^2_{H^{2-k}(-T,T;H^k(M;\mathbb{R}^\ell))}\\&\le C\sum_{i=0}^{1}\sum_{m=1}^{\ell}\int_L e^{2s\varphi}(|\Delta_{g_\flat}v_m^{(i)}|^2+s^2E(v_m^{(i)})+s^4|v_m^{(i)}|^2)\omega_L\\&\quad+C\int_L e^{2s\varphi}|f|^2\omega_L+Ce^{2\epsilon_2 s}\sum_{k=0}^{1}||h||^2_{H^{2-k}(-T,T;H^k(M;\mathbb{R}^\ell))}\\&\le C\sum_{i=0}^{1}\int_L e^{2s\varphi}\left(\frac{1}{s}|\partial_t(Pv^{(i)})|^2+s|Pv^{(i)}|^2\right)\omega_L\\&\quad+C\int_L e^{2s\varphi}|f|^2\omega_L+Ce^{2\epsilon_2 s}\sum_{k=0}^{1}||h||^2_{H^{2-k}(-T,T;H^k(M;\mathbb{R}^\ell))}+Ce^{Cs}\mathcal{D}^2,\end{aligned}$$

where $\epsilon_j:=e^{\gamma\cdot j\epsilon}$ for $j\in\{2,3\}$. Furthermore, in regard to the first and second summands on the right-hand side of the above estimate, and because we have

$$\begin{aligned}&Pv^{(0)}=\chi Sf+\left[2g^{\mu\nu}\partial_\mu\chi\partial_\nu h+\Box_g\chi h\right],\\&\partial_t(Pv^{(0)})=\chi\partial_t Sf+\partial_t\chi Sf+\partial_t\left[2g^{\mu\nu}\partial_\mu\chi\partial_\nu h+\Box_g\chi h\right],\\&Pv^{(1)}=\chi\partial_t Sf-\partial_t g^{\mu\nu}\partial_\mu\partial_\nu v^{(0)}+\partial_t(g^{\mu\nu}\Gamma^\rho_{\mu\nu})\partial_\rho v^{(0)}-\partial_t a v^{(0)}+[\partial_t g^{\mu\nu}(\partial_\mu\partial_\nu\chi)h\\&\qquad+2\partial_t g^{\mu\nu}(\partial_\mu\chi)(\partial_\nu h)+2g^{\mu\nu}\partial_\mu\chi\partial_\nu\partial_t h+\Box_g\chi\partial_t h-\partial_t(g^{\mu\nu}\Gamma^\rho_{\mu\nu})(\partial_\rho\chi)h],\end{aligned}$$

and

$$
\begin{aligned}
&\partial_t(Pv^{(1)})\\
&= \chi\partial_t^2 Sf + \partial_t\chi\partial_t Sf - \partial_t^2 g^{\mu\nu}\partial_\mu\partial_\nu v^{(0)} - \partial_t^2 a v^{(0)} + \partial_t^2(g^{\mu\nu}\Gamma^\rho_{\mu\nu})\partial_\rho v^{(0)}\\
&\quad - \partial_t g^{\mu\nu}\partial_\mu\partial_\nu(\partial_t\chi h + v^{(1)}) - \partial_t a(\partial_t\chi h + v^{(1)}) + \partial_t(g^{\mu\nu}\Gamma^\rho_{\mu\nu})\partial_\rho(\partial_t\chi h + v^{(1)})\\
&\quad + \partial_t\big[\partial_t g^{\mu\nu}(\partial_\mu\partial_\nu\chi)h + 2\partial_t g^{\mu\nu}(\partial_\mu\chi)(\partial_\nu h) + 2g^{\mu\nu}\partial_\mu\chi\partial_\nu\partial_t h + \square_g\chi\partial_t h - \partial_t(g^{\mu\nu}\Gamma^\rho_{\mu\nu})(\partial_\rho\chi)h\big],
\end{aligned}
$$

we obtain

$$
\begin{aligned}
&\sum_{i=0}^{1}\int_L e^{2s\varphi}\left(\frac{1}{s}|\partial_t(Pv^{(i)})|^2 + s|Pv^{(i)}|^2\right)\omega_L\\
&\le C\int_L s^2 e^{2s\varphi}|f|^2\omega_L + Cs^2 e^{2\epsilon_2 s}\sum_{k=0}^{2}||h||^2_{H^{3-k}(-T,T;H^k(M;\mathbb{R}^\ell))} + Ce^{Cs}\mathcal{D}^2.
\end{aligned}
\tag{3.5}
$$

Indeed, in regard to the first and second summands on the left-hand side of (3.5), we have

$$
\begin{aligned}
&\int_L e^{2s\varphi}s|Pv^{(0)}|^2\omega_L \le C\int_L se^{2s\varphi}|f|^2\omega_L + Cse^{2\epsilon_2 s}\sum_{k=0}^{1}||h||_{H^{1-k}(-T,T;H^k(M;\mathbb{R}^\ell))},\\
&\int_L e^{2s\varphi}\frac{1}{s}|\partial_t(Pv^{(0)})|^2 d\omega_L \le C\int_L \frac{1}{s}e^{2s\varphi}|f|^2\omega_L + \frac{C}{s}e^{2\epsilon_2 s}\sum_{k=0}^{1}||h||_{H^{2-k}(-T,T;H^k(M;\mathbb{R}^\ell))},\\
&\int_L e^{2s\varphi}s|Pv^{(1)}|^2\omega_L\\
&\le C\int_L se^{2s\varphi}|f|^2\omega_L + C\sum_{m=1}^{\ell}s\left(\int_{-T}^{T}||e^{s\varphi}v_m^{(0)}||^2_{H^2(M)}dt + \int_L e^{2s\varphi}(s^2E(v_m^{(0)}) + s^4|v_m^{(0)}|^2)\omega_L\right)\\
&\quad + Cse^{2\epsilon_2 s}\sum_{k=0}^{1}||h||_{H^{2-k}(-T,T;H^k(M;\mathbb{R}^\ell))}\\
&\le C\int_L se^{2s\varphi}|f|^2\omega_L + C\sum_{m=1}^{\ell}\int_L se^{2s\varphi}\left(|\Delta_{g_b}v_m^{(0)}|^2 + s^2E(v_m^{(0)}) + s^4|v_m^{(0)}|^2\right)\omega_L\\
&\quad + Cse^{2\epsilon_2 s}\sum_{k=0}^{1}||h||_{H^{2-k}(-T,T;H^k(M;\mathbb{R}^\ell))}\\
&\le C\int_L se^{2s\varphi}|f|^2\omega_L + C\int_L e^{2s\varphi}(|\partial_t(Pv^{(0)})|^2 + s^2|Pv^{(0)}|^2)\omega_L\\
&\quad + Cse^{2\epsilon_2 s}\sum_{k=0}^{1}||h||_{H^{2-k}(-T,T;H^k(M;\mathbb{R}^\ell))} + Ce^{Cs}\mathcal{D}^2\\
&\le C\int_L s^2 e^{2s\varphi}|f|^2\omega_L + Cs^2 e^{2\epsilon_2 s}\sum_{k=0}^{1}||h||_{H^{2-k}(-T,T;H^k(M;\mathbb{R}^\ell))} + Ce^{Cs}\mathcal{D}^2,
\end{aligned}
$$

where we used Lemma 3.4 and Lemma 3.3, and

$$
\begin{aligned}
&\int_L e^{2s\varphi}\frac{1}{s}|\partial_t(Pv^{(1)})|^2\omega_L \\
&\le C\int_L \frac{1}{s}e^{2s\varphi}|f|^2\omega_L + C\sum_{i=0}^{1}\sum_{m=1}^{\ell}\frac{1}{s}\left(\int_{-T}^{T}||e^{s\varphi}v_m^{(i)}||_{H^2(M)}^2 dt + \int_L e^{2s\varphi}(s^2E(v_m^{(i)}) + s^4|v_m^{(i)}|^2)\omega_L\right) \\
&\quad + \frac{C}{s}e^{2\epsilon_2 s}\sum_{k=0}^{2}||h||_{H^{3-k}(-T,T;H^k(M;\mathbb{R}^\ell))} \\
&\le C\int_L \frac{1}{s}e^{2s\varphi}|f|^2\omega_L + C\sum_{i=0}^{1}\sum_{m=1}^{\ell}\int_L \frac{1}{s}e^{2s\varphi}(|\Delta_g v_m^{(i)}|^2 + s^2E(v_m^{(i)}) + s^4|v_m^{(i)}|^2)\omega_L \\
&\quad + \frac{C}{s}e^{2\epsilon_2 s}\sum_{k=0}^{2}||h||_{H^{3-k}(-T,T;H^k(M;\mathbb{R}^\ell))} \\
&\le C\int_L \frac{1}{s}e^{2s\varphi}|f|^2\omega_L + C\sum_{i=0}^{1}\int_L e^{2s\varphi}\left(\frac{1}{s^2}|\partial_t(Pv^{(i)})|^2 + |Pv^{(i)}|^2\right)\omega_L \\
&\quad + \frac{C}{s}e^{2\epsilon_2 s}\sum_{k=0}^{2}||h||_{H^{3-k}(-T,T;H^k(M;\mathbb{R}^\ell))} + Ce^{Cs}\mathcal{D}^2 \\
&\le C\int_L se^{2s\varphi}|f|^2\omega_L + C\int_L \frac{1}{s^2}e^{2s\varphi}|\partial_t(Pv^{(1)})|^2\omega_L \\
&\quad + Cse^{2\epsilon_2 s}\sum_{k=0}^{2}||h||_{H^{3-k}(-T,T;H^k(M;\mathbb{R}^\ell))} + Ce^{Cs}\mathcal{D}^2,
\end{aligned}
$$

where we used Lemma 3.4 and Lemma 3.3 again. Taking $s > 0$ sufficiently large yields

$$
\begin{aligned}
&\int_L e^{2s\varphi}\frac{1}{s}|\partial_t(Pv^{(1)})|^2\omega_L \\
&\le C\int_L se^{2s\varphi}|f|^2\omega_L + Cse^{2\epsilon_2 s}\sum_{k=0}^{2}||h||_{H^{3-k}(-T,T;H^k(M;\mathbb{R}^\ell))} + Ce^{Cs}\mathcal{D}^2.
\end{aligned}
$$

Hence, we finally obtain (3.5). Then, applying (3.5) to (3.4) yields

$$
\begin{aligned}
&\sum_{m=1}^{\ell}\int_L e^{2s\varphi}\left(sE(v_m^{(2)}) + s^3|v_m^{(2)}|^2\right)\omega_L \\
&\le C\int_L s^2e^{2s\varphi}|f|^2\omega_L + Cs^2e^{2\epsilon_2 s}\sum_{k=0}^{2}||h||_{H^{3-k}(-T,T;H^k(M;\mathbb{R}^\ell))}^2 + Ce^{Cs}\mathcal{D}^2.
\end{aligned}
\tag{3.6}
$$

Then, using (3.6), we have

$$\begin{aligned}
&||e^{s\varphi(0,\cdot)}\chi(0,\cdot)S(0,\cdot)f||^2_{L^2(M;\mathbb{R}^\ell)} \le C||e^{s\varphi(0,\cdot)}v^{(2)}(0,\cdot)||^2_{L^2(M;\mathbb{R}^\ell)}\\
&= C\int_{-T}^0 \frac{d}{dt}\left(\int_M e^{2s\varphi(t,x)}|v^{(2)}(t,x)|^2\omega_M\right)dt\\
&\le C\int_L e^{2s\varphi}\left(\frac{1}{s}|\partial_t v^{(2)}|^2 + s|v^{(2)}|^2\right)\omega_L\\
&\le C\int_L e^{2s\varphi}|f|^2\omega_L + Ce^{2\epsilon_2 s}\sum_{k=0}^{2}||h||^2_{H^{3-k}(-T,T;H^k(M;\mathbb{R}^\ell))} + Ce^{Cs}\mathcal{D}^2.
\end{aligned} \tag{3.7}$$

Hence, using (3.7), we have

$$\begin{aligned}
&\int_{M_{2\epsilon}} e^{2s\varphi(0,x)}|f|^2\omega_M \le C\int_M e^{2s\varphi(0,x)}|\chi(0,x)|^2|S(0,x)f|^2\omega_M\\
&\le C\int_L e^{2s\varphi}|f|^2\omega_L + Ce^{2\epsilon_2 s}\sum_{k=0}^{2}||h||^2_{H^{3-k}(-T,T;H^k(M);\mathbb{R}^\ell)} + Ce^{Cs}\mathcal{D}^2.
\end{aligned} \tag{3.8}$$

Moreover, we establish

$$\begin{aligned}
&\int_L e^{2s\varphi}|f|^2\omega_L\\
&\le C\int_{-T}^{T}\left(\int_{M_{2\epsilon}} e^{2s\varphi}|f|^2\omega_M\right)dt + C\int_{-T}^{T}\left(\int_{M\setminus M_{2\epsilon}} e^{2s\varphi}|f|^2\omega_M\right)dt\\
&\le C\int_{M_{2\epsilon}} e^{2s\varphi(0,x)}|f|^2\left(\int_{-T}^{T} e^{-2s(\varphi(0,x)-\varphi(t,x))}dt\right)\omega_M + Ce^{2\epsilon_2 s}||f||^2_{L^2(M;\mathbb{R}^\ell)}\\
&\le o(1)\int_{M_{2\epsilon}} e^{2s\varphi(0,x)}|f|^2\omega_M + Ce^{2\epsilon_2 s}||f||^2_{L^2(M;\mathbb{R}^\ell)}
\end{aligned}$$

as $s \to \infty$ by our assumption (A.2) and the Lebesgue dominated convergence theorem. Applying this inequality to (3.8) yields

$$\int_{M_{2\epsilon}} e^{2s\varphi(0,x)}|f|^2\omega_M \le Ce^{2\epsilon_2 s}\left(||f||^2_{L^2(M;\mathbb{R}^\ell)} + \sum_{k=0}^{2}||h||^2_{H^{3-k}(-T,T;H^k(M;\mathbb{R}^\ell))}\right) + Ce^{Cs}\mathcal{D}^2$$

for sufficiently large $s > s_*$. We note that

$$\int_{M_{2\epsilon}} e^{2s\varphi(0,x)}|f|^2\omega_M \ge \int_{M_{3\epsilon}} e^{2s\varphi(0,x)}|f|^2\omega_M \ge e^{2\epsilon_3 s}||f||^2_{L^2(M_{3\epsilon};\mathbb{R}^\ell)}.$$

Hence, we have

$$||f||^2_{L^2(M_{3\epsilon};\mathbb{R}^\ell)} \le Ce^{-2(\epsilon_3-\epsilon_2)s}\left(||f||^2_{L^2(M;\mathbb{R}^\ell)} + \sum_{k=0}^{2}||h||^2_{H^{3-k}(-T,T;H^k(M;\mathbb{R}^\ell))}\right) + Ce^{Cs}\mathcal{D}^2,$$

i.e.,

$$||f||_{L^2(M_{3\epsilon};\mathbb{R}^\ell)} \le Ce^{-(\epsilon_3-\epsilon_2)s}\mathcal{F} + Ce^{Cs}\mathcal{D}, \tag{3.9}$$

for all $s > s_*$. By replacing C by Ce^{Cs_*}, the above estimate holds for all $s > 0$. When $\mathcal{D} \geq \mathcal{F}$, (3.9) implies

$$||f||_{L^2(M_{3\epsilon};\mathbb{R}^\ell)} \leq Ce^{Cs}\mathcal{D}.$$

Moreover, when $\mathcal{D} < \mathcal{F}$, we choose $s > 0$ to minimize the right-hand side of (3.9) such that

$$e^{Cs}\mathcal{D} = e^{-(\epsilon_3-\epsilon_2)s}\mathcal{F},$$

i.e.,

$$s = \frac{1}{C + \epsilon_3 - \epsilon_2}\log\frac{\mathcal{F}}{\mathcal{D}}.$$

We then have

$$||f||_{L^2(M_{3\epsilon};\mathbb{R}^\ell)} \leq 2C\mathcal{F}^{1-\theta}\mathcal{D}^\theta,$$

where

$$\theta := \frac{\epsilon_3 - \epsilon_2}{C + \epsilon_3 - \epsilon_2} \in (0, 1).$$

Hence, there exist constants $C > 0$ and $\theta \in (0, 1)$ such that

$$||f||_{L^2(M_{3\epsilon};\mathbb{R}^\ell)} \leq C(\mathcal{D} + \mathcal{F}^{1-\theta}\mathcal{D}^\theta)$$

holds. □

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A. Appendix

A1. Proof of Lemma 2.6

For the proof of Lemma 2.6, we need the Gauss formula for Lorentzian manifolds. We say the boundary ∂L is spacelike (timelike) if the induced metric to ∂L is Riemannian (Lorentzian). Let N be the outward pointing unit normal vector field to ∂L. If ∂L is spacelike, $\langle N, N\rangle = -1$; otherwise, $\langle N, N\rangle = 1$. We refer to Lemma 10.8 in Ringström [3]. Note that Σ is timelike.

Lemma 4.1 . *Let (L, g) be an $n+1$-dimensional compact oriented Lorentzian manifold with boundary. Assume that the boundary is spacelike or timelike and let X be a smooth vector field. Then if N denotes the outer unit normal to ∂L, it follows that*

$$\int_L \operatorname{div} X\omega_L = \int_{\partial L} \frac{\langle X, N\rangle}{\langle N, N\rangle}\omega_{\partial L}.$$

Proof of Lemma 2.6. First, note that

$$\nabla\varphi = \gamma\varphi\nabla\psi, \quad \Box_g\varphi = \gamma\varphi(\Box_g\psi + \gamma\langle\nabla\psi, \nabla\psi\rangle),$$
$$\nabla^2\varphi(\nabla z, \nabla z) = \gamma\varphi(\nabla^2\psi(\nabla z, \nabla z) + \gamma|\langle\nabla z, \nabla\psi\rangle|^2).$$

We introduce a new function and operator

$$z := e^{s\varphi}u, \quad P_s z := e^{s\varphi}\Box_g(e^{-s\varphi}z).$$

A lengthy calculation yields

$$P_s z = \Box_g z - 2s\langle\nabla\varphi, \nabla z\rangle + s^2\langle\nabla\varphi, \nabla\varphi\rangle z - s\Box_g\varphi z,$$

which decomposes $P_s z$ into $P_s^+ z$ and $P_s^- z$,

$$\begin{cases} P_s^+ z := \Box_g z + s^2 \langle \nabla \varphi, \nabla \varphi \rangle z, \\ P_s^- z := -2s\langle \nabla \varphi, \nabla z \rangle - s\Box_g \varphi z. \end{cases}$$

Note $P_s z = P_s^+ z + P_s^- z$. Because we wish to make a lower bound of $||P_s z||^2_{L^2(L)}$, we calculate the L^2 inner product of $P_s^+ z$ and $P_s^- z$,

$$\begin{aligned}(P_s^+ z, P_s^- z)_{L^2(L)} &= \int_L \Box_g z \cdot (-2s\langle \nabla\varphi, \nabla z\rangle)\omega_L + \int_L \Box_g z \cdot (-s\Box_g \varphi z)\omega_L \\ &+ \int_L s^2 \langle \nabla\varphi, \nabla\varphi\rangle z \cdot (-2s\langle\nabla\varphi, \nabla z\rangle)\omega_L + \int_L s^2\langle\nabla\varphi,\nabla\varphi\rangle z\cdot(-s\Box_g\varphi z)\omega_L \\ &=: \sum_{k=1}^4 I_k.\end{aligned}$$

Let N be the outer unit normal vector field to ∂L. We remark that $z = \partial_N z = \nabla z = 0$ on $M^{\pm T}$. Integration by parts yields

$$\begin{aligned} I_1 &= \int_L 2s\langle \nabla\langle\nabla\varphi,\nabla z\rangle, \nabla z\rangle\omega_L - \int_{\partial L} 2s\frac{\langle \nabla z, N\rangle}{\langle N, N\rangle}\langle\nabla\varphi,\nabla z\rangle\omega_{\partial L} \\ &= \int_L 2s\nabla^2\varphi(\nabla z,\nabla z)\omega_L + \int_L s\langle\nabla\varphi, \nabla\langle\nabla z,\nabla z\rangle\rangle\omega_L - \int_\Sigma 2s\langle\nabla z, N\rangle\langle\nabla\varphi,\nabla z\rangle\omega_\Sigma \\ &= \int_L 2s\nabla^2\varphi(\nabla z,\nabla z)\omega_L - \int_L s\Box_g\varphi\langle\nabla z,\nabla z\rangle\omega_L - \int_\Sigma 2s\langle\nabla z,N\rangle\langle\nabla\varphi,\nabla z\rangle\omega_\Sigma \\ &+ \int_\Sigma s\langle\nabla\varphi, N\rangle\langle\nabla z,\nabla z\rangle\omega_\Sigma,\end{aligned}$$

where we have used the identity

$$\begin{aligned} 2\langle\nabla\langle\nabla\varphi,\nabla z\rangle,\nabla z\rangle &= 2\nabla_\mu(\nabla_\nu\varphi\nabla^\nu z)\nabla^\mu z \\ &= 2(\nabla_\mu\nabla_\nu\varphi)\nabla^\nu z\nabla^\mu z + 2\nabla_\nu\varphi(\nabla_\mu\nabla^\nu z)\nabla^\mu z \\ &= 2\nabla^2\varphi(\nabla z,\nabla z) + \nabla_\nu\varphi\nabla^\nu(\nabla^\mu z\nabla_\mu z) \\ &= 2\nabla^2\varphi(\nabla z,\nabla z) + \langle\nabla\varphi,\nabla\langle\nabla z,\nabla z\rangle\rangle.\end{aligned} \tag{4.1}$$

Furthermore, we obtain

$$\begin{aligned} I_2 &= \int_L s\Box_g\varphi\langle\nabla z,\nabla z\rangle\omega_L + \int_L \frac{s}{2}\langle\nabla\Box_g\varphi, \nabla(|z|^2)\rangle\omega_L - \int_{\partial L} s\frac{\langle\nabla z, N\rangle}{\langle N,N\rangle}\Box_g\varphi z\omega_{\partial L} \\ &= \int_L s\Box_g\varphi\langle\nabla z,\nabla z\rangle\omega_L - \int_L\frac{s}{2}\Box_g^2\varphi|z|^2\omega_L - \int_\Sigma s\langle\nabla z,N\rangle\Box_g\varphi z\omega_\Sigma + \int_\Sigma\frac{s}{2}\langle\nabla\Box_g\varphi,N\rangle|z|^2\omega_\Sigma, \\ I_3 &= -\int_L s^3\langle\nabla\varphi,\nabla\varphi\rangle\langle\nabla\varphi,\nabla(|z|^2)\rangle\omega_L \\ &= \int_L s^3\nabla(\langle\nabla\varphi,\nabla\varphi\rangle\nabla\varphi)|z|^2\omega_L - \int_\Sigma s^3\langle\nabla\varphi,\nabla\varphi\rangle\langle\nabla\varphi,N\rangle|z|^2\omega_\Sigma, \\ I_4 &= -\int_L s^3\langle\nabla\varphi,\nabla\varphi\rangle\Box_g\varphi|z|^2\omega_L.\end{aligned}$$

We remark that the integrand of the first summand of I_3 means

$$\nabla(\langle\nabla\varphi,\nabla\varphi\rangle\nabla\varphi)=\nabla_\mu(\langle\nabla\varphi,\nabla\varphi\rangle\nabla^\mu\varphi).$$

Hence, we have

$$\begin{aligned}\sum_{k=1}^{4}I_k&=\int_L 2s\nabla^2\varphi(\nabla z,\nabla z)\omega_L+\int_L\left(-\frac{s}{2}\square_g^2\varphi+s^3\nabla(\langle\nabla\varphi,\nabla\varphi\rangle\nabla\varphi)-s^3\langle\nabla\varphi,\nabla\varphi\rangle\square_g\varphi\right)|z|^2\omega_L\\&\quad-\int_\Sigma 2s\langle\nabla z,N\rangle\langle\nabla\varphi,\nabla z\rangle\omega_\Sigma+\int_\Sigma s\langle\nabla\varphi,N\rangle\langle\nabla z,\nabla z\rangle\omega_\Sigma-\int_\Sigma s\langle\nabla z,N\rangle\square_g\varphi z\omega_\Sigma\\&\quad+\int_\Sigma\frac{s}{2}\langle\nabla\square_g\varphi,N\rangle|z|^2\omega_\Sigma-\int_\Sigma s^3\langle\nabla\varphi,\nabla\varphi\rangle\langle\nabla\varphi,N\rangle|z|^2\omega_\Sigma\\&=:\mathit{First}+\mathit{Zeroth}+\mathcal{B},\end{aligned}\tag{4.2}$$

where we define

$$\begin{aligned}\mathit{First}&:=\int_L 2s\nabla^2\varphi(\nabla z,\nabla z)\omega_L,\\\mathit{Zeroth}&:=\int_L\left(-\frac{s}{2}\square_g^2\varphi+s^3\nabla(\langle\nabla\varphi,\nabla\varphi\rangle\nabla\varphi)-s^3\langle\nabla\varphi,\nabla\varphi\rangle\square_g\varphi\right)|z|^2\omega_L,\\\mathcal{B}&:=-\int_\Sigma 2s\langle\nabla z,N\rangle\langle\nabla\varphi,\nabla z\rangle\omega_\Sigma+\int_\Sigma s\langle\nabla\varphi,N\rangle\langle\nabla z,\nabla z\rangle\omega_\Sigma-\int_\Sigma s\langle\nabla z,N\rangle\square_g\varphi z\omega_\Sigma\\&\quad+\int_\Sigma\frac{s}{2}\langle\nabla\square_g\varphi,N\rangle|z|^2\omega_\Sigma-\int_\Sigma s^3\langle\nabla\varphi,\nabla\varphi\rangle\langle\nabla\varphi,N\rangle|z|^2\omega_\Sigma.\end{aligned}$$

In regard to the *First*, from our assumption (A.1), we obtain

$$\begin{aligned}\mathit{First}&=\int_L 2s\nabla^2\varphi(\nabla z,\nabla z)\omega_L\\&=\int_L 2\sigma(\nabla^2\psi(\nabla z,\nabla z)+\gamma|\langle\nabla z,\nabla\psi\rangle|^2)\omega_L\\&\geq-4\kappa_2\int_L\sigma|\nabla^0 z|^2\omega_L+4\kappa_1\int_L\sigma|\nabla z|^2\omega_L,\end{aligned}$$

where we remark that $\sigma:=s\gamma\varphi$. Therefore, we need the second estimate,

$$\begin{aligned}(P_s^+z,\sigma z)_{L^2(L)}&=\int_L\square_g z\cdot(\sigma z)\omega_L+\int_L s^2\langle\nabla\varphi,\nabla\varphi\rangle z\cdot(\sigma z)\omega_L\\&=-\int_L\sigma\langle\nabla z,\nabla z\rangle\omega_L-\int_L\frac{s\gamma}{2}\langle\nabla\varphi,\nabla(|z|^2)\rangle\omega_L+\int_L s^2\sigma\langle\nabla\varphi,\nabla\varphi\rangle|z|^2\omega_L\\&\quad+\int_{\partial L}\sigma\frac{\langle\nabla z,N\rangle}{\langle N,N\rangle}z\omega_{\partial L}\\&=-\int_L\sigma\langle\nabla z,\nabla z\rangle\omega_L+\int_L\left(\frac{s\gamma}{2}\square_g\varphi+s^2\sigma\langle\nabla\varphi,\nabla\varphi\rangle\right)|z|^2\omega_L+\int_\Sigma\sigma\langle\nabla z,N\rangle z\omega_\Sigma\\&\quad-\int_\Sigma\frac{s\gamma}{2}\langle\nabla\varphi,N\rangle|z|^2\omega_\Sigma\\&=\int_L\sigma|\nabla^0 z|^2\omega_L-\int_L\sigma|\nabla z|^2\omega_L+\int_L\left(\frac{s\gamma}{2}\square_g\varphi+s^2\sigma\langle\nabla\varphi,\nabla\varphi\rangle\right)|z|^2\omega_L\\&\quad+\int_\Sigma\sigma\langle\nabla z,N\rangle z\omega_\Sigma-\int_\Sigma\frac{s\gamma}{2}\langle\nabla\varphi,N\rangle|z|^2\omega_\Sigma\\&=:\mathit{First}_2+\mathit{Zeroth}_2+\mathcal{B}_2,\end{aligned}\tag{4.3}$$

where we define

$$
\begin{aligned}
First_2 &:= \int_L \sigma|\nabla^0 z|^2\omega_L - \int_L \sigma|\nabla z|^2\omega_L,\\
Zeroth_2 &:= \int_L \left(\frac{s\gamma}{2}\Box_g\varphi + s^2\sigma\langle\nabla\varphi,\nabla\varphi\rangle\right)|z|^2\omega_L,\\
\mathcal{B}_2 &:= \int_\Sigma \sigma\langle\nabla z, N\rangle z\omega_\Sigma - \int_\Sigma \frac{s\gamma}{2}\langle\nabla\varphi, N\rangle|z|^2\omega_\Sigma.
\end{aligned}
$$

We remark that the last equality is obtained by the fact that for all $p\in L$ and $X\in T_pL$,

$$
\begin{aligned}
\langle X,X\rangle &= -d\tau^2((d\pi_0)X,(d\pi_0)X) + g_\flat((d\pi_1)X,(d\pi_1)X)\\
&= -|X^0|^2 + |X|^2
\end{aligned}
$$

holds. Multiplying (4.3) by 4δ for $\delta>0$ to be determined later and adding it to (4.2) yield

$$
\begin{aligned}
\sum_{k=1}^4 I_k + 4\delta(P_s^+z,\sigma z)_{L^2(L)} \ge{}& 4(\delta-\kappa_2)\int_L \sigma|\nabla^0 z|^2\omega_L + 4(\kappa_1-\delta)\int_L\sigma|\nabla z|^2\omega_L\\
&+ Zeroth + 4\delta Zeroth_2 + \mathcal{B} + 4\delta\mathcal{B}_2.
\end{aligned}
$$

From our assumption (A.1), there exists $\delta>0$ such that

$$
\begin{cases}
\delta-\kappa_2>0,\\
\kappa_1-\delta>0.
\end{cases}
$$

Next, we consider the zeroth-order terms $Zeroth + 4\delta Zeroth_2$.

$$
\begin{aligned}
Zeroth &= \int_L\left(-\frac{s}{2}\Box_g^2\varphi + s^3\nabla(\langle\nabla\varphi,\nabla\varphi\rangle\nabla\varphi) - s^3\langle\nabla\varphi,\nabla\varphi\rangle\Box_g\varphi\right)|z|^2\omega_L\\
&= \int_L\left[2\sigma^3\gamma|\langle\nabla\psi,\nabla\psi\rangle|^2 + 2\sigma^3\nabla^2\psi(\nabla\psi,\nabla\psi) + O(s\gamma^4\varphi)\right]|z|^2\omega_L\\
&\ge \int_L\left[2\sigma^3\gamma|\langle\nabla\psi,\nabla\psi\rangle|^2 + 2\sigma^3(-2\kappa_2|\nabla^0\psi|^2 + 2\kappa_1|\nabla\psi|^2) + O(s\gamma^4\varphi)\right]|z|^2\omega_L
\end{aligned}
$$

as $\gamma\to\infty$, where the second equality holds by (4.1). Indeed, we obtain from (4.1)

$$
\begin{aligned}
\nabla(\langle\nabla\varphi,\nabla\varphi\rangle\nabla\varphi) - \langle\nabla\varphi,\nabla\varphi\rangle\Box_g\varphi &= \langle\nabla\langle\nabla\varphi,\nabla\varphi\rangle,\nabla\varphi\rangle\\
&= 2\nabla^2\varphi(\nabla\varphi,\nabla\varphi)\\
&= 2(\gamma\varphi)^3(\nabla^2\psi(\nabla\psi,\nabla\psi) + \gamma|\langle\nabla\psi,\nabla\psi\rangle|^2).
\end{aligned}
$$

Moreover, we get

$$
4\delta Zeroth_2 = \int_L\left[4\delta\sigma^3\langle\nabla\psi,\nabla\psi\rangle + O(s\gamma^3\varphi))\right]|z|^2\omega_L
$$

as $\gamma\to\infty$. We then have

$$
\begin{aligned}
&Zeroth + 4\delta Zeroth_2\\
&\geq \int_L [\sigma^3(2\gamma|\langle\nabla\psi,\nabla\psi\rangle|^2 - 4\kappa_2|\nabla^0\psi|^2 + 4\kappa_1|\nabla\psi|^2 + 4\delta\langle\nabla\psi,\nabla\psi\rangle) + O(s\gamma^4\varphi)]|z|^2\omega_L\\
&= \int_L [\sigma^3(2\gamma|\langle\nabla\psi,\nabla\psi\rangle|^2 - 4\kappa_2|\nabla^0\psi|^2 + 4\kappa_1|\nabla\psi|^2 + 8\delta\langle\nabla\psi,\nabla\psi\rangle - 4\delta\langle\nabla\psi,\nabla\psi\rangle)\\
&\quad + O(s\gamma^4\varphi)]|z|^2\omega_L\\
&= \int_L [\sigma^3(2\gamma|\langle\nabla\psi,\nabla\psi\rangle|^2 - 4\kappa_2|\nabla^0\psi|^2 + 4\kappa_1|\nabla\psi|^2 + 8\delta\langle\nabla\psi,\nabla\psi\rangle\\
&\quad - 4\delta(-|\nabla^0\psi|^2 + |\nabla\psi|^2)) + O(s\gamma^4\varphi)]|z|^2\omega_L\\
&= \int_L \left[\sigma^3\left(2\gamma(\langle\nabla\psi,\nabla\psi\rangle + \frac{2\delta}{\gamma})^2 + 4(\delta-\kappa_2)|\nabla^0\psi|^2 + 4(\kappa_1-\delta)|\nabla\psi|^2 - \frac{8\delta^2}{\gamma}\right)\right.\\
&\quad \left. + O(s\gamma^4\varphi)\right]|z|^2\omega_L\\
&\geq \int_L \left[\sigma^3\left(4(\kappa_1-\delta)|\nabla\psi|^2 - \frac{8\delta^2}{\gamma}\right) + O(s\gamma^4\varphi)\right]|z|^2\omega_L\\
&\geq C\int_L [\sigma^3 + O(s\gamma^4\varphi)]|z|^2\omega_L
\end{aligned}
$$

as $\gamma \to \infty$. Note that we used assumptions (A.1) and (A.2). Therefore, for sufficiently large $\gamma > 0$ there exists a constant C such that

$$
\begin{aligned}
&(P_s^+z, P_s^-z)_{L^2(L)} + 4\delta(P_s^+z,\sigma z)_{L^2(L)} + C\int_L O(s\gamma^4\varphi)|z|^2\omega_L\\
&\geq C\int_L \sigma(|\nabla^0 z|^2 + |\nabla z|^2 + \sigma^2|z|^2)\omega_L + \mathcal{B} + 4\delta\mathcal{B}_2
\end{aligned}
$$

holds for all $z \in C^\infty(L)$ satisfying $z = \partial_N z = 0$ on $M^{\pm T}$. For a sufficiently large fixed $\gamma > 0$, we choose $s > 0$ large enough so that

$$
C\int_L \sigma(|\nabla^0 z|^2 + |\nabla z|^2 + \sigma^2|z|^2)\omega_L \leq ||P_s z||^2_{L^2(L)} - \mathcal{B} - 4\delta\mathcal{B}_2
$$

holds. It remains to estimate the boundary terms $\mathcal{B} + 4\delta\mathcal{B}_2$. Note that N on Σ is a spacelike unit outer normal vector field, i.e., $\langle N, N\rangle = 1$ holds on Σ. We have

$$
\begin{aligned}
&-\mathcal{B} - 4\delta\mathcal{B}_2 = \int_\Sigma 2\sigma\langle\nabla z, N\rangle\langle\nabla\psi,\nabla z\rangle\omega_\Sigma + \int_\Sigma s\Box_g\varphi\langle\nabla z, N\rangle z\omega_\Sigma\\
&\quad + \int_\Sigma \sigma^3\langle\nabla\psi,\nabla\psi\rangle\langle\nabla\psi, N\rangle|z|^2\omega_\Sigma - \int_\Sigma \frac{s}{2}\langle\nabla\Box_g\varphi, N\rangle|z|^2\omega_\Sigma\\
&\quad - \int_\Sigma \sigma\langle\nabla\psi, N\rangle\langle\nabla z,\nabla z\rangle\omega_\Sigma - 4\delta\int_\Sigma \sigma\langle\nabla z, N\rangle z\omega_\Sigma + 2\delta\int_\Sigma \sigma\gamma\langle\nabla\psi, N\rangle|z|^2\omega_\Sigma\\
&= \int_\Sigma 2\sigma\partial_N z\langle\nabla\psi,\nabla z\rangle\omega_\Sigma - \int_\Sigma \sigma\partial_N\psi\langle\nabla z,\nabla z\rangle\omega_\Sigma\\
&= \int_\Sigma \sigma\partial_N\psi|\partial_N z|^2\omega_\Sigma \leq C\int_\Sigma \sigma|\partial_N z|^2\omega_\Sigma,
\end{aligned}
$$

where $\partial_N z := \langle\nabla z, N\rangle$ because we can write $\nabla z = \langle\nabla z, N\rangle N$ as $z = 0$ on Σ, which is proved by taking the semigeodesic coordinate, and then

$$\langle \nabla\psi, \nabla z\rangle = \partial_N\psi\partial_N z, \quad \langle \nabla z, \nabla z\rangle = |\partial_N z|^2$$

holds. Then, after some calculations, we obtain

$$\begin{aligned} e^{2s\varphi}|\nabla^0 u|^2 &= (\nabla^0 z + sz\nabla^0\varphi)^2 \le C(|\nabla^0 z|^2 + \sigma^2|z|^2), \\ e^{2s\varphi}|\nabla u|^2 &= g_{ij}(\nabla^i z + sz\nabla^i\varphi)(\nabla^j z + sz\nabla^j\varphi) \le C(|\nabla z|^2 + \sigma^2|z|^2). \end{aligned}$$

Hence, we finally obtain

$$\int_L e^{2s\varphi}\sigma(|\nabla^0 u|^2 + |\nabla u|^2 + \sigma^2|u|^2)\omega_L \le C\int_L e^{2s\varphi}|\Box_g u|^2\omega_L + C\int_\Sigma e^{2s\varphi}\sigma|\partial_N u|^2\omega_\Sigma$$

for sufficiently large $s > 0$. The proof is completed. □

A2. Proof of Lemma 3.3

Proof of Lemma 3.3. Let us fix the semigeodesic coordinate $(x^0 = t, x^1, \dots, x^n)$. We then find $\hat{N} = \partial_t$. For large $\gamma > 0$ large, we apply Proposition 3.2 to $\partial_t v$ to derive

$$\begin{aligned} &\int_L e^{2s\varphi}|\partial_t^2 v|^2\omega_L \\ &\le \frac{C}{s}\int_L e^{2s\varphi}|P\partial_t v|^2\omega_L + C\int_\Sigma e^{2s\varphi}|\partial_N\partial_t v|^2\omega_\Sigma \\ &\le \frac{C}{s}\int_L e^{2s\varphi}|\partial_t Pv - \partial_t g^{\mu\nu}\partial_\mu\partial_\nu v + \partial_t(g^{\mu\nu}\Gamma^\rho_{\mu\nu})\partial_\rho v - \partial_t av|^2\omega_L + C\int_\Sigma e^{2s\varphi}|\partial_N\partial_t v|^2\omega_\Sigma \\ &\le \frac{C}{s}\int_L e^{2s\varphi}|\partial_t Pv|^2\omega_L + \frac{C}{s}\left(\int_{-T}^T ||e^{s\varphi}v||^2_{H^2(M)}dt + \int_L e^{2s\varphi}(s^2E(v) + s^4|v|^2)\omega_L\right) \\ &\quad + C\int_\Sigma e^{2s\varphi}|\partial_N\partial_t v|^2\omega_\Sigma \\ &\le \frac{C}{s}\int_L e^{2s\varphi}|\partial_t Pv|^2\omega_L + \frac{C}{s}\int_L e^{2s\varphi}\left(|\Delta_{g_\flat} v|^2 + s^2E(v) + s^4|v|^2\right)\omega_L + C\int_\Sigma e^{2s\varphi}|\partial_N\partial_t v|^2\omega_\Sigma \\ &\le \frac{C}{s}\int_L e^{2s\varphi}|\partial_t Pv|^2\omega_L + C\int_L e^{2s\varphi}|Pv|^2\omega_L + \frac{C}{s}\int_L e^{2s\varphi}|\Delta_{g_\flat} v|^2\omega_L \\ &\quad + C\int_\Sigma e^{2s\varphi}|\partial_N\partial_t v|^2\omega_\Sigma + Cs\int_\Sigma e^{2s\varphi}|\partial_N v|^2\omega_\Sigma, \end{aligned}$$

where we use Lemma 3.4 to obtain the fourth inequality. Since $g^{ij}\partial_i\partial_j v = Pv + \partial_t^2 v + g^{\mu\nu}\Gamma^\rho_{\mu\nu}\partial_\rho v - av$ and $g_\flat^{ij} = g^{ij}$ by the semigeodesic coordinate, we obtain

$$\begin{aligned} \int_L e^{2s\varphi}|\Delta_{g_\flat} v|^2\omega_L &\le C\int_L e^{2s\varphi}\left(|\partial_t^2 v|^2 + E(v) + |v|^2 + |Pv|^2\right)\omega_L \\ &\le C\int_L e^{2s\varphi}\left(\frac{1}{s}|\partial_t Pv|^2 + |Pv|^2\right)\omega_L + \frac{C}{s}\int_L e^{2s\varphi}|\Delta_{g_\flat} v|^2\omega_L \\ &\quad + C\int_\Sigma e^{2s\varphi}|\partial_N\partial_t v|^2\omega_\Sigma + Cs\int_\Sigma e^{2s\varphi}|\partial_N v|^2\omega_\Sigma. \end{aligned}$$

Choosing $s > 0$ sufficiently large, we absorb the second term on the right-hand side into the left-hand side to obtain

$$\int_L e^{2s\varphi}|\Delta_{g_\flat} v|^2 \omega_L \leq C\int_L e^{2s\varphi}\left(\frac{1}{s}|\partial_t Pv|^2 + |Pv|^2\right)\omega_L + Ce^{Cs}\mathcal{E}^2.$$

A3. Proof of Lemma 3.4

Proof of Lemma 3.4. *Let $\{(U_i, x_i)\}_i$ be a local coordinate system of M. If $\{\chi_i\}_i$ is a finite partition of unity subordinate to the open covering and χ_i' are chosen with $\chi_i' = 1$ in a neighborhood of* $\mathrm{supp}\chi_i$ *and* $\mathrm{supp}\chi_i' \subset U_i$, *then*

$$\begin{aligned} ||(\chi_i v)\circ x_i^{-1}||_{H^2(U_i)} &\leq C(||(A(\chi_i v))\circ x_i^{-1}||_{L^2(U_i)} + ||(\chi_i v)\circ x_i^{-1}||_{L^2(U_i)}) \\ &\leq C(||(\chi_i' Av)\circ x_i^{-1}||_{L^2(U_i)} + ||(\chi_i' v)\circ x_i^{-1}||_{L^2(U_i)}) \\ &\leq C(||(\eta_i Av)\circ x_i^{-1}||_{L^2(U_i)} + ||(\eta_i v)\circ x_i^{-1}||_{L^2(U_i)}), \end{aligned}$$

where $\eta_i := \frac{\chi_i'}{\sum_i \chi_i'}$. With $1 = \sum_i \chi_i \leq \sum_i \chi_i'$, η_i is determined. Furthermore, because $\mathrm{supp}\eta_i = \mathrm{supp}\chi_i' \subset U_i$, $\{\eta_i\}_i$ is a partition of unity subordinate to the covering $\{U_i\}_i$. Summing up with respect to i yields

$$||v||_{H^2(M)} \leq C\Big(||Av||_{L^2(M)} + ||v||_{L^2(M)}\Big).$$